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ON THE OPTIMAL IMPULSE CONTROL PROBLEM FOR DEGENERATE DIFFUSIONS*

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Abstract. In this paper, we give a characterization of the optimal cost of an impulse control problem as the maximum solution of a quasi-variational inequality without assuming nondegeneracy. An estimate of the velocity of uniform convergence of the sequence of stopping time problems associated with the impulse control problem is given.

Introduction. Summary of main results. In this article, we develop the proofs of results announced in Note [5].

The impulse control problem has been studied by several authors. A. Bensoussan and J. L. Lions [2] treated nondegenerate diffusions, M. Robin [11] developed the case of Feller processes, and J. P. Lepeltier and B. Marchal [4] investigated a similar problem for a more general kind of Markov processes. In a purely analytical framework, L. Tartar [13] considered an abstract coercive quasi-variational inequality and F. Mignot and J. P. Puel [10] a first order quasi-variational inequality.

We study here the case of degenerate diffusions which lead to a second order noncoercive quasi-variational inequality. The deterministic case leading to a first order quasi-variational inequality is treated in [6].

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}^t\}_{t \geqq 0}$ be a nondecreasing right-continuous family of completed sub-σ-fields of $\mathcal{F}$.

Let ν be any admissible[1] impulse control and $y(t) = y_x(t, \nu, \omega)$, $t \geqq 0$, $\omega \in \Omega$ be the diffusion with jumps on $\mathbb{R}^N$ starting at x, with Lipschitz continuous coefficients $g(\cdot)$ and $\sigma(\cdot)$.

Suppose $\mathcal{O}$ is an open subset of $\mathbb{R}^N$, and $\tau = \tau_x(\nu, \omega)$ the first exit time of process $y(t)$ from $\bar{\mathcal{O}}$.

Next, let $f(x)$ be a bounded upper semicontinuous nonnegative real function on $\bar{\mathcal{O}}$, and $k(\xi)$ be a continuous real function on $\mathbb{R}^N_+$ such that

$$k(\xi) \geqq k_0 > 0 \ \forall \xi \geqq 0, \quad \text{and} \quad k(\xi) \to \infty \text{ if } |\xi| \to \infty. \tag{0.1}$$

Given $x \in \bar{\mathcal{O}}$ and an admissible impulse control $\nu = \{\theta_1, \xi_1; \cdots; \theta_i, \xi_i; \cdots\}$, the functional cost is defined by

$$J_x(\nu) = E\left\{\int_0^\tau f(y(t))\, e^{-\alpha t}\, dt + \sum_{i=1}^{\infty} k(\xi_i) 1_{\theta_i < \infty}\, e^{-\alpha \theta_i}\right\}, \tag{0.2}$$

where α is a positive constant.

Our purpose is to characterize the optimal cost

$$\hat{u}(x) = \inf \{J_x(\nu)/\nu \text{ an admissible impulse control}\}, \tag{0.3}$$

and to obtain an optimal admissible impulse control.

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[1] See Def. (1.7).

We denote by A_0 the second order differential operator associated with the Ito equation[2]

$$(0.4)\qquad A_0=-\tfrac{1}{2}\operatorname{tr}\left(\sigma\sigma^*\frac{\partial^2}{\partial x^2}\right)-g\frac{\partial}{\partial x}$$

and $A=A_0+\alpha$.

Let $\Gamma_0\subset\partial\mathcal{O}$ be the set of regular points, and let us use the integral formulation of A[3].

We define by M the operator

$$(0.5)\qquad [M\phi](x)=\inf\{k(\xi)+\phi(x+\xi)/\xi\geqq 0,\ x+\xi\in\bar{\mathcal{O}}\}.$$

Assume that $\mathcal{O}$ is sufficiently smooth such that M maps continuous functions ϕ into continuous functions $M\phi$. We will give conditions below (Lemma 1.3), so that M has the proposed regularity.

Finally, we introduce the problem: To find a real bounded measurable function on $\bar{\mathcal{O}}$, $u(x)$ such that

$$(0.6)\qquad \begin{aligned} &u=0 && \text{on } \Gamma_0,\\ &u\leqq Mu && \text{in } \bar{\mathcal{O}}\backslash\Gamma_0,\\ &Au\leqq f && \text{in the martingale sense on } \bar{\mathcal{O}}\backslash\Gamma_0.\end{aligned}$$

Now, we consider the following sequence of variational inequalities corresponding to optimal stopping time problems (cf. [7]).

Let $\hat{u}^0(x)$ be the bounded upper semicontinuous nonnegative real function on $\bar{\mathcal{O}}$ such that

$$(0.7)\qquad \begin{aligned} &\hat{u}^0=0 && \text{on } \Gamma_0,\\ &A\hat{u}^0=f && \text{in the martingale sense on } \bar{\mathcal{O}}\backslash\Gamma_0,\end{aligned}$$

and given $\hat{u}^{n-1}(x)$, let $\hat{u}^n(x)$ be the bounded upper semicontinuous nonnegative real function on $\bar{\mathcal{O}}$ which is the maximum solution of

$$(0.8)\qquad \begin{aligned} &u^n=0 && \text{on } \Gamma_0,\\ &u^n\leqq M\hat{u}^{n-1} && \text{in } \bar{\mathcal{O}}\backslash\Gamma_0,\\ &Au^n\leqq f && \text{in the martingale sense on } \bar{\mathcal{O}}\backslash\Gamma_0.\end{aligned}$$

We have the following characterization.

THEOREM 0.1. *Assume that g,σ are Lipschitz continuous,* (0.1), *and that f is bounded upper semicontinuous and nonnegative. Then problem* (0.6) *admits a maximum solution $\hat{u}$ which is upper semicontinuous and given as the optimal cost* (0.3). *Moreover, the following assertions are true.*

$$(0.9)^4\qquad \|\hat{u}\|\leqq\frac{1}{\alpha}\|f\|,$$

$$(0.10)\qquad \hat{u}^n(x)\to\hat{u}(x)(n\to\infty)\quad \textit{uniformly in } x\in\bar{\mathcal{O}}.$$

[2] If B is a matrix, then B^* denotes the transpose of B and tr (B) the trace of B.

[3] See Def. (1.13).

[4] $\|\cdot\|$ denotes the supremum norm on $\bar{\mathcal{O}}$.

Furthermore, if Γ_0 is closed and f continuous, the function $\hat{u}$ is also continuous on $\bar{\mathcal{O}}$ and there exists an optimal admissible impulse control.

Regarding $\hat{u}$ as a distribution in $\mathcal{O}$, we have

THEOREM 0.2. *Let the assumptions be the same as in Theorem* 0.1. *Suppose*

$$\frac{\partial^2}{\partial x^2}\sigma\sigma^* \in L^1_{\text{loc}}(\mathcal{O}). \tag{0.11}$$

Then the optimal cost $\hat{u}$ verifies

$$A\hat{u} \leqq f \quad \text{in } \mathcal{D}'(\mathcal{O}). \tag{0.12}$$

Moreover, if Γ_0 is closed and f continuous, the following equation

$$A\hat{u} = f \quad \text{in } \mathcal{D}'([\hat{u} < M\hat{u}]) \tag{0.13}$$

is also true.

Now, a quasi-variational formulation is given.

Let $\beta_0(x)$, $\beta_1(x)$ be the weight functions $(1+|x|^2)^{-(\lambda+1)/2}$, $(1+|x|^2)^{-\lambda/2}$, $\lambda > N/2$ respectively. Introduce the following Hilbert spaces, $H = \{v/\beta_0 v \in L^2(\mathcal{O})\}$ with scalar product $(\cdot\,,\cdot)$, and $V = \{v \in H/\beta_1(\partial v/\partial x_i) \in L^2(\mathcal{O}),\ \forall i = 1, \cdots, N \text{ and } v = 0 \text{ on } \Gamma\}$. The space V' is the dual of V, and $\langle\cdot\,,\cdot\rangle$ denotes the duality between V' and V.

Consider the following quasi-variational inequality:

$$\begin{aligned} &u \in V, \qquad u \leqq Mu, \\ &\langle Au, v-u\rangle \geqq (f, v-u) \quad \forall v \in V, \quad v \leqq Mu. \end{aligned} \tag{0.14}$$

Assume

$$\frac{\partial^2}{\partial x^2}\sigma\sigma^* \in L^\infty(\mathcal{O}), \tag{0.15}$$

and that there exists a Lipschitz continuous subsolution $\bar{w}$, i.e.,

(0.16)[5] $\qquad \bar{w} \in W_0^{1,\infty}(\mathcal{O})$ and $A\bar{w} \leqq -f$ in $\mathcal{D}'(\mathcal{O})$,

where the constant α is assumed large enough.

For instance, if $\mathcal{O} = \mathbb{R}^N$ or $\sigma\sigma^*$ is coercive on Γ, then the assumption (0.16) is satisfied.

THEOREM 0.3. *Let the conditions of Theorem* 0.1, (0.15), *and* (0.16) *hold. Suppose that f is Lipschitz continuous; then the quasi-variational inequality* (0.14) *has a maximum solution $\hat{u}$ which is Lipschitz continuous and explicitly given as the optimal cost* (0.2).

This work is divided into three sections. The first section establishes several useful lemmas. In § 2, the integral formulation of the impulse control problem is studied, and in the last section, the associated quasi-variational inequality is treated.

In this paper, we will use extensively the results of [7].

1. Preliminary results. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{\mathcal{F}^t\}_{t\geqq 0}$ a nondecreasing right-continuous family of completed sub-σ-fields of $\mathcal{F}$, and $w(t)$ a standard Brownian motion in $\mathbb{R}^N$ with respect to $\mathcal{F}^t$.

[5] Also in the martingale sense.

Suppose we are given two Lipschitz continuous functions $g(x)$ and $\sigma(x)$ on $\mathbb{R}^N$, taking values in $\mathbb{R}^N$ and $\mathbb{R}^N \otimes \mathbb{R}^N$, respectively, $g=(g_i)$, $\sigma=(\sigma_{ij})$,

(1.1)[6]
$$\frac{\partial g_i}{\partial x_k}, \frac{\partial \sigma_{ij}}{\partial x_k} \in B(\mathbb{R}^N), \qquad i, j, k = 1, \cdots, N.$$

We consider the diffusion $y^0(t) = y_x^0(t, \omega)$, $t \geqq 0$, $\omega \in \Omega$ and $x \in \mathbb{R}^N$, described by the Ito equation

$$\begin{aligned} dy^0(t) &= g(y^0(t))\,dt + \sigma(y^0(t))\,dw(t), \qquad t \geqq 0, \\ y^0(0) &= x. \end{aligned} \tag{1.2}$$

Let Λ be a closed subset of $\mathbb{R}^N$, convex with respect to zero[7]. An impulse control ν is a set $\{\theta_1, \xi_1; \cdots; \theta_i, \xi_i; \cdots\}$ where $\{\theta_i\}_{i=1}^{\infty}$ is an increasing sequence of stopping times with respect to $\mathscr{F}^t$ convergent to infinity ($\theta_i \leqq \theta_{i+1}$, $\theta_i \to \infty$) and $\{\xi_i\}_{i=1}^{\infty}$ is a sequence of random variables taking values on Λ, adapted with respect to $\{\theta_i\}_{i=1}^{\infty}$ ($\xi_i: \Omega \to \Lambda$, $\mathscr{F}^{\theta_i}$ measurable).

Now, we define the sequence of diffusions with jumps $\{y^n(t)\}_{n=1}^{\infty}$, $y^n(t) = y_x^n(t, \nu, \omega)$, $t \geqq 0$, $\omega \in \Omega$, $x \in \mathbb{R}^N$, and ν any impulse control, by the Ito equation

$$\begin{aligned} dy^n(t) &= g(y^n(t))\,dt + \sigma(y^n(t))\,dw(t), \qquad t \geqq \theta_n, \\ y^n(t) &= y^{n-1}(t) + 1_{\theta_n = t}\xi_n, \qquad t \leqq \theta_n. \end{aligned} \tag{1.3}$$

We have

$$y^n(t) = y^i(t) \text{ on } [0, \theta_n] \quad \forall_i \geqq n. \tag{1.4}$$

So, if we define

$$y(t, \nu) = \lim_{n \to \infty} y^n(t), \qquad t \geqq 0, \tag{1.5}$$

the process $y(t) = y_x(t, \nu, \omega)$, which is right-continuous[8], satisfies the stochastic equation,

$$\begin{aligned} dy(t) &= g(y(t))\,dt + \sigma(y(t))\,dw(t) + \sum_{i=1}^{\infty} \xi_i \delta(t - \theta_i)\,dt, \qquad t \geqq 0, \\ y(0) &= x, \end{aligned} \tag{1.6}$$

where $\delta(t)$ is the Dirac measure.

Suppose $\mathcal{O}$ an open subset of $\mathbb{R}^N$, and $\tau = \tau_x(\nu, \omega)$, $\tau^0 = \tau_x^0(\omega)$ the first exit time of processes $y(t)$, $y^0(t)$ respectively, from $\bar{\mathcal{O}}$.

We call $\nu = \{\theta_1, \xi_1; \cdots; \theta_i, \xi_i; \cdots\}$ an admissible impulse control if it satisfies

$$y(\tau) \in \bar{\mathcal{O}} \quad \text{a.s. on } [\tau < \infty]; \tag{1.7}$$

that is, no jump of the process $y(t)$ is outside of $\bar{\mathcal{O}}$ before τ.

Denote by Γ_0 the set of regular points (cf. D. W. Stroock and S. R. S. Varadhan [12]),

$$\Gamma_0 = \{x \in \Gamma = \partial\mathcal{O} / P(\tau_x^0 > 0) = 0\}. \tag{1.8}$$

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[6] $B(\mathbb{R}^N)$ denotes the set of all Borel measurable and bounded functions on $\mathbb{R}^N$ taking values in $\mathbb{R}$.

[7] i.e., $\lambda\xi \in \Lambda$, $\forall \lambda \in [0, 1]$, $\forall \xi \in \Lambda$. Generally, we take $\Lambda = \mathbb{R}_+^N$.

[8] $y(t)$ has also left limits.

LEMMA 1.1. *Assume* (1.1). *Let* ν *be any admissible impulse control, and* θ *be any stopping time; then the following assertions are true.*

$$P(y(\tau, \nu) \notin \Gamma_0, \tau < \infty) = 0, \tag{1.9}$$

$$E\{|y_x(\theta) - y_{x'}(\theta)|^2 e^{-\gamma\theta}\} \leqq |x - x'|^2 \quad \forall x, x' \in \mathbb{R}^N, \tag{1.10}$$

where the positive constant γ *depends on the Lipschitz constant of functions* g *and* σ.

Proof. Setting

$$\gamma = \sup\left\{ \operatorname{tr}\left[\frac{(\sigma(x)-\sigma(x'))(\sigma(x)-\sigma(x'))^*}{|x-x'|^2}\right] + \frac{2(x-x')(g(x)-g(x'))}{|x-x'|^2} \Big/ x, x' \in \mathbb{R}^N\right\}, \tag{1.11}$$

and recalling that the process $y_x(t) - y_{x'}(t)$ is a diffusion (from Ito's formula) to the function $|x|^2 e^{-\gamma t}$, we obtain (1.10) as Lemma 1.1 in [7].

Finally, using (1.7) from Markov's property we get

$$P(y^n(\tau^n) \notin \Gamma_0, \tau^n < \infty) = 0, \tag{1.12}$$

where τ^n is the first exit time of process $y^n(t)$ from $\bar{\mathcal{O}}$. So regarding (1.4), we deduce (1.9). □

Let u, v be real bounded[9] upper semicontinuous functions on $\bar{\mathcal{O}}$. Then the integral formulation of operation A (cf. [7]) is given by

$Au \leqq v$ in $\bar{\mathcal{O}} \backslash \Gamma_0$ if the process

$$X_t = \int_0^{\theta \wedge \tau^0} v(y^0(s))\, e^{-\alpha s}\, ds + u(y^0(t \wedge \tau^0))\, e^{-\alpha(t \wedge \tau^0)} \tag{1.13[10]}$$

is a submartingale for each $x \in \bar{\mathcal{O}} \backslash \Gamma_0$.

LEMMA 1.2. *Assume* (1.1) *and* $\mathcal{O}$ *smooth*[11]. *Let* $f(x)$ *be a real bounded continuous function on* $\bar{\mathcal{O}}$. *Suppose that there exists* $\bar{w}$ *such that*

$$\begin{aligned} &\bar{w} \in C(\bar{\mathcal{O}}), \qquad \bar{w}, \frac{\partial \bar{w}}{\partial x_i} \in B(\bar{\mathcal{O}}), \qquad i = 1, \cdots, N, \\ &A\bar{w} \leqq -f \quad \text{in } \mathcal{D}'(\mathcal{O}), \qquad \bar{w}(x) = 0 \quad \forall x \in \Gamma. \end{aligned} \tag{1.14}$$

Then, for any admissible[12] *impulse control* $\nu = \{\theta_1, \xi_1; \cdots; \theta_i, \xi_i; \cdots\}$ *such that*

$$\theta_i \notin [\tau_x \wedge \tau_{x'}, \tau_x[\quad \forall i = 1, 2, \cdots, \tag{1.15[13]}$$

the following estimation is true:

$$E\left\{\int_{\tau_x \wedge \tau_{x'}}^{\tau_x} f(y_x(t))\, e^{-\alpha t}\, dt\right\} \leqq \left\|\frac{\partial \bar{w}}{\partial x}\right\| |x - x'| \quad \forall x, x' \in \bar{\mathcal{O}}, \tag{1.16}$$

where $\|\partial \bar{w}/\partial x\|$ *denotes the smallest Lipschitz continuous constant of* $\bar{w}$.

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[9] u and v may have polynomial growth if $\mathcal{O}$ is not bounded.

[10] We say $Au \leqq v$ in the martingale sense.

[11] We also assume α large enough.

[12] Clearly, admissible for x.

[13] $\tau_x \wedge \tau_{x'}$ denotes the minimum between τ_x and $\tau_{x'}$.

Proof. First, assume $\bar{w} \in C^2(\mathbb{R}^N)$; $\bar{w}, \partial\bar{w}/\partial x_i \in B(\mathbb{R}^N)$, $i=1,\cdots,N$. Ito's formula applied to function $\bar{w}(x)$ and process $y_x(t)$ gives

$$E\{\bar{w}(y_x(\tau_x))\,e^{-\alpha\tau_x}-\bar{w}(y_x(\tau_x\wedge\tau_{x'}))\,e^{-\alpha(\tau_x\wedge\tau_{x'})}\} = -E\left\{\int_{\tau_x\wedge\tau_{x'}}^{\tau_x} A\bar{w}(y_x(t))\,e^{-\alpha t}\,dt\right\}. \tag{1.17}$$

Since

$$\bar{w}(y_x(\tau_x))=0=\bar{w}(y_{x'}(\tau_x\wedge\tau_{x'}))\quad \text{a.s. in } (\tau_{x'}\leqq\tau_x<\infty],$$

from (1.17), we deduce

$$E\left\{\int_{\tau_x\wedge\tau_{x'}}^{\tau_x} f(y_x(t))\,e^{-\alpha t}\,dt\right\} \leqq E\{|\bar{w}(y_x(\tau_x\wedge\tau_{x'}))-\bar{w}(y_{x'}(\tau_x\wedge\tau_{x'}))|\,e^{-\alpha(\tau_x\wedge\tau_{x'})}\}. \tag{1.18}$$

Next, defining

$$\gamma_0=\sup\left\{\frac{1}{2}\operatorname{tr}\left[\frac{(\sigma(x)-\sigma(x'))(\sigma(x)-\sigma(x'))^*}{|x-x|^2}\right] + \frac{(x-x')(g(x)-g(x'))}{|x-x'|^2}\Big/ x, x'\in\mathbb{R}^N\right\}, \tag{1.19}$$

and assuming $\alpha\geqq\gamma_0$, from Lemma 1.1 and (1.18) we obtain (1.16). Finally, if $\bar{w}\notin C^2(\mathcal{O})$, by approximating $\bar{w}$ by regular functions the lemma is proved. □

Remark 1.1. Assume $\bar{w}\in W^{1,\infty}(\mathcal{O})$, $f\in C(\bar{\mathcal{O}})\cap B(\bar{\mathcal{O}})$. Approximating $\bar{w}$ by regular functions, we deduce that $[A\bar{w}\leqq f$ in $\mathcal{D}'(\mathcal{O})]$ is equivalent to $[A\bar{w}\leqq f$ in the martingale sense of (1.13)]. This fact will be used several times.

Suppose we are given a continuous real function $k(\xi)$ on Λ, such that

$$\begin{aligned} &k(\xi)\geqq k_0>0\quad \forall\xi\in\Lambda,\\ &k(\xi)\to\infty\quad \text{if } |\xi|\to\infty\quad \text{with } \xi\in\Lambda.\end{aligned} \tag{1.20}$$

We define the operator $M: B(\bar{\mathcal{O}})\to B(\bar{\mathcal{O}})$ by

$$[M\phi](x)=\inf\{k(\xi)+\phi(x+\xi)/\xi\in\Lambda,\ x+\xi\in\bar{\mathcal{O}}\}. \tag{1.21}$$

We always assume $\mathcal{O}$ and Λ smooth enough, such that

There exists $P\colon \bar{\mathcal{O}}\times\Lambda\to\Lambda$ measurable and uniformly continuous in $x\in\bar{\mathcal{O}}$ verifying

$$\begin{aligned} &x+P(x,\xi)\in\bar{\mathcal{O}}\quad \forall x\in\bar{\mathcal{O}},\quad \forall\xi\in\Lambda,\\ &P(x,\xi)=\xi\quad \text{if } x+\xi\in\bar{\mathcal{O}}.\end{aligned} \tag{1.22}$$

For instance, if $\Lambda=\mathbb{R}_+^N$ and $\mathcal{O}$ convex with regular boundary, we can take $P(x,\xi)$ as the projection of ξ on $\Lambda\cap(\bar{\mathcal{O}}-x)$.

LEMMA 1.3. *Assume* (1.20) *and* (1.22). *Then if ϕ is upper semicontinuous* (*or continuous*) *on $\bar{\mathcal{O}}$, so is $M\phi$.*

Proof. Starting at

$$[M\phi](x)-[M\phi](x')=\sup_{\xi'}\inf_{\xi}[(k(\xi)-k(\xi')+(\phi(x+\xi)-\phi(x'+\xi'))],$$

and choosing $\xi = P(x, \xi')$, we get

$$(1.23)\qquad \begin{aligned}[M\phi](x)-[M\phi](x') \leqq \sup_{\xi'} [k(P(x, \xi')) - k(P(x', \xi'))] \\ + \sup_{\xi'} [\phi(x+P(x, \xi')) - \phi(x'+P(x', \xi'))].\end{aligned}$$

So, from (1.23) and the uniform continuity of function $P(x, \xi)$, the lemma is proved. □

LEMMA 1.4. *Suppose* (1.20), (1.22) *and*

$$(1.24)\qquad \phi \text{ bounded and upper semicontinuous on } \bar{\mathcal{O}}.$$

Then, for each $\varepsilon > 0$ *there exists a function* $\xi_\varepsilon(x)$ *such that*

$$(1.25)\qquad \begin{aligned}&\xi_\varepsilon : \bar{\mathcal{O}} \to \Lambda \text{ bounded and Borel measurable,} \\ &x + \hat{\xi}_\varepsilon(x) \in \bar{\mathcal{O}} \quad \forall x \in \bar{\mathcal{O}},\end{aligned}$$

$$(1.26)\qquad [M\phi](x) + \varepsilon \geqq [k(\hat{\xi}_\varepsilon(x)) + \phi(x + \hat{\xi}_\varepsilon(x))] \quad \forall x \in \bar{\mathcal{O}}.$$

Moreover, if ϕ *is continuous, there exists* $\hat{\xi}(x)$ *verifying* (1.25) *and* (1.26) *with* $\varepsilon = 0$.

Proof. First, if ϕ is continuous, the classical theorems of selection imply the result.

Next, if ϕ is only upper semicontinuous, there exists a decreasing sequence $\{\phi_n\}_{n=1}^\infty$ of continuous functions convergent to ϕ. So, we also have $M\phi_n$ decreasing to $M\phi$.

Let $\hat{\xi}^n(x)$ be a function which satisfies (1.25) and

$$[M\phi_n](x) = [k(\hat{\xi}^n(x)) + \phi_n(x + \hat{\xi}^n(x))] \quad \forall x \in \bar{\mathcal{O}},$$

and let $n_\varepsilon(x)$ be the function

$$n_\varepsilon(x) = \min \{n \geqq 1/[M\phi_n](x) \leqq [M\phi](x) + \varepsilon\}.$$

Then, if we set

$$(1.27)\qquad \hat{\xi}_\varepsilon(x) = \xi^n(x) \quad \text{if } n = n_\varepsilon(x),$$

the lemma is proved. □

2. Integral formulation. Let Γ_0 be the set of regular points (1.8) and A be the operator given by (1.13). Assume $f(x)$ an upper semicontinuous function on $\bar{\mathcal{O}}$ such that

$$(2.1)\qquad f \in B(\bar{\mathcal{O}}), \qquad f \geqq 0.$$

Consider the following problem: To find $u(x)$ such that

$$(2.2)\qquad u \in B(\bar{\mathcal{O}}), \qquad u(x) = 0 \quad \forall x \in \Gamma_0,$$

$$(2.3)\qquad Au \leqq f \text{ in } \bar{\mathcal{O}} \backslash \Gamma_0 \quad [\text{martingale sense (1.13)}],$$

$$(2.4)\qquad u \leqq Mu \quad \text{on } \bar{\mathcal{O}} \backslash \Gamma_0.$$

Let us define the sequence $\{\hat{u}^n\}_{n=1}^\infty$ of solutions to variational inequalities corresponding to optimal stopping time problems (cf. [7]). Starting with $\hat{u}^0(x)$ verifying (2.2) and

$$(2.5)\qquad A\hat{u}^0 = f \text{ in } \bar{\mathcal{O}} \backslash \Gamma_0 \,[\text{martingale sense (1.13)}],$$

we set $\hat{u}^n(x)$ as the maximum solution of problem (2.2), (2.3) and

$$(2.6)\qquad u^n \leqq M\hat{u}^{n-1} \quad \text{on } \bar{\mathcal{O}} \backslash \Gamma_0,$$

This section is divided into two parts. First we solve problem (2.2), (2.3), (2.4) and consider the case where the set of regular points Γ_0 is closed. Next we study the general case and give some complementary results

2.1. Regular case.

THEOREM 2.1. *Let the assumptions* (1.1), (1.20), (1.22) *and* (2.1) *hold. Then the problem* (2.2), (2.3), (2.4) *admits a maximum solution* $\hat{u}$ *which is given by the decreasing limit*

$$\hat{u}(x) = \lim_{n\to\infty} \hat{u}^n(x) \quad \forall x \in \bar{\mathcal{O}}. \tag{2.7}$$

Moreover, the function $\hat{u}(x)$ *is upper semicontinuous and the following estimate is true:*

$$\|u\| \leqq \frac{1}{\alpha}\|f\|, \tag{2.8}$$

where $\|\cdot\|$ *denotes the supremum norm on* $\bar{\mathcal{O}}$.

Proof. Using the monotone property of operator M,

$$\phi \leqq \psi \text{ implies } M\phi \leqq M\psi, \tag{2.9}$$

and knowing that $0 \leqq \hat{u}^1 \leqq \hat{u}^0$, we deduce

$$0 \leqq \hat{u}^{n+1} \leqq \hat{u}^n \leqq \hat{u}^0, \qquad n = 1, 2, \cdots. \tag{2.10}$$

Then, for any solution u of problem (2.2), (2.3), the trivial maximum principle in the martingale formulation implies $u \leqq \hat{u}^0$. Because of (2.4) and (2.9), we obtain

$$u \leqq \hat{u}^n, \qquad n = 1, 2, \cdots. \tag{2.11}$$

So, the function $\hat{u}$ defined by (2.7) is the maximum solution of problem (2.2), (2.3), and (2.4). Since $\hat{u}^n$ is upper semicontinuous (cf. [7]), we conclude the theorem. □

Remark 2.1. If we set $\psi = M\hat{u}$, the maximum solution $\hat{u}$ can also be considered as an optimal stopping time cost, i.e., the maximum solution of problem (2.2), (2.3) and $u \leqq \psi$.

We can also define the sequence $\{\hat{u}^n\}_{n=1}^{\infty}$ as the optimal costs

$$\hat{u}^0(x) = E\left\{\int_0^{\tau^0} f(y^0(t))\, e^{-\alpha t}\, dt\right\}, \tag{2.12}$$

and given $\hat{u}^{n-1}$ we obtain $\hat{u}^n$ by

$$\hat{u}^n(x) = \inf_\theta E\left\{\int_0^{\theta\wedge\tau^0} f(y^0(t))\, e^{-\alpha t}\, dt + M\hat{u}^{n-1}(y^0(\theta)) 1_{\theta<\tau^0}\, e^{-\alpha\theta}\right\} \tag{2.13}$$

where θ is any stopping time of $\mathcal{F}^t$.

THEOREM 2.2. *Let the conditions* (1.1), (1.20), (1.22), (2.1), *and*

$$f \in C(\bar{\mathcal{O}}), \tag{2.14}$$

$$\Gamma_0 \text{ closed}, \tag{2.15}$$

hold. Then the maximum solution $\hat{u}$ *of problem* (2.2), (2.3), (2.4) *is continuous. Moreover,* $\hat{u}$ *is given as the optimal cost* (0.3), *and the following estimate is true:*

$$\|\hat{u}^n - \hat{u}\| \leqq \frac{\|f\|^2}{k_0\alpha^2(n+1)}, \qquad n = 0, 1, 2, \cdots. \tag{2.16}$$

Proof. Recalling that, from [7] and Lemma 1.3, $\hat{u}^n$ is continuous, we need only to show the estimate (2.16). Since Γ_0 is closed, we are in the case of Feller processes (cf. A. Bensoussan [1] and M. Robin [11]).

First, we are going to prove that

(2.17)[14] $$\hat{u}^n(x)=\inf\{J_x(\nu)/\nu \text{ admissible impulse control such that } \theta_i=\infty\ \forall i\geqq n+1\},$$

where the functional cost $J_x(\nu)$ is given by (0.2).

Indeed, from Lemma 1.4, there exist functions $\hat{\xi}^i(x)$, $i=1,\cdots,n$ verifying (1.25) and

$$[Mu^{n-i}](x)=k(\hat{\xi}^i(x))+\hat{u}^{n-i}(x+\hat{\xi}^i(x))\quad \forall x\in\bar{\mathcal{O}}. \tag{2.18}$$

Thus, we define $\hat{\nu}^n=\{\hat{\theta}_i,\hat{\xi}_i\}_{i=1}^{\infty}$ as follows.

$$\tilde{\theta}^0=0; \tag{2.19}$$

$$\begin{aligned}&d\hat{y}^0(t)=g(\hat{y}^0(t))\,dt+\sigma(\hat{y}^0(t))\,dw(t),\qquad t\geqq 0,\\&\hat{y}^0(0)=x;\end{aligned} \tag{2.20}$$

(2.21)[15] $$\hat{\tau}^i=\inf\{t\geqq 0/\hat{y}^i(t)\notin\bar{\mathcal{O}}\},\qquad i=0,1,\cdots,n;$$

(2.22)[16] $$\tilde{\theta}^{i+1}=\inf\{t\in[\tilde{\theta}^i,\hat{\tau}^i]/\hat{u}^{n-i}(\hat{y}^i(t))=[M\hat{u}^{n-i-1}](\hat{y}^i(t))\},\qquad i=0,1,\cdots,n-1;$$

(2.23)[17] $$\hat{\xi}_{i+1}=\hat{\xi}^i(\hat{y}^i(\tilde{\theta}^{i+1})),\qquad i=0,1,\cdots,n-1;$$

$$\begin{aligned}&d\hat{y}^i(t)=g(\hat{y}^i(t))\,dt+\sigma(\hat{y}^i(t))\,dw(t),\qquad t\geqq\tilde{\theta}^i,\\&\hat{y}^i(\tilde{\theta}^i)=\hat{y}^{i-1}(\tilde{\theta}^i)+\hat{\xi}_i,\\&\hat{y}^i(t)=\hat{y}^{i-1}(t),\qquad t<\tilde{\theta}^i,\quad i=1,2,\cdots,n;\end{aligned} \tag{2.24}$$

and next

$$\hat{\theta}_i=\begin{cases}\tilde{\theta}_i & \text{if } i\leqq n \text{ and } \tilde{\theta}^i<\hat{\tau}^{i-1}\\ \infty & \text{otherwise,}\end{cases}\qquad i=1,2,\cdots, \tag{2.25}$$

$$\hat{\xi}_i=0\quad\text{if } i\geqq n+1. \tag{2.26}$$

We have

$$y(t,\hat{\nu}^n)=\hat{y}^n(t),\qquad t\geqq 0, \tag{2.27}$$

and from Markov's property

$$\hat{u}^n(x)=J_x(\hat{\nu}^n), \tag{2.28}$$

$$\hat{u}^n(x)\leqq J_x(\nu)\quad\text{if } \nu \text{ has at most } n \text{ impulses.} \tag{2.29}$$

Then, (2.28) and (2.29) imply (2.17).

Now we are going to show the estimate (2.16).

Let $\nu=\{\theta_i,\xi_i\}_{i=1}^{\infty}$ be any admissible impulse control; setting $\nu^n=\{\theta_1,\xi_1;\cdots;\theta_n,\xi_n;\infty,\xi_{n+1};\cdots\}$ we have

$$y(t,\nu)=y(t,\nu^n)=y^n(t)\quad\text{if } t<\theta_n\wedge\tau^n.$$

[14] i.e., ν has at the most n impulses.

[15] We set $\hat{\tau}^i=\infty$ if $\hat{y}^i(t)\in\bar{\mathcal{O}}\ \forall t\geqq 0$.

[16] We set $\tilde{\theta}^{i+1}=\hat{\tau}^i$ if the subset is empty.

[17] If $\tilde{\theta}^{i+1}=\infty$ we set $\hat{\xi}_{i+1}=0$.

Hence, if $\hat{u}$ is given by (0.3), we obtain

$$0 \leqq \hat{u}^n - \hat{u} \leqq \sup_{\nu} E\left\{ \int_{\theta_n \wedge \tau^n}^{\tau^n} f(y^n(t))\, e^{-\alpha t}\, dt \right\}. \tag{2.30}$$

Since

$$e^{-\alpha\theta_n} \leqq \frac{1}{k_0(n+1)} \sum_{i=1}^{\infty} k(\xi_i) 1_{\theta_i < \infty}\, e^{-\alpha\theta_i},$$

and since it is possible to take the supremum only over all admissible impulse controls such that

$$E\left\{ \sum_{i=1}^{\infty} k(\xi_i) 1_{\theta_i < \infty}\, e^{-\alpha\theta_i} \right\} \leqq \frac{1}{\alpha} \|f\|,$$

the estimate (2.16) follows from (2.30). □

Remark 2.2. The estimate (2.16) can be improved using a probabilistic version of results in B. Hanouzet and J. L. Joly [3]. We have

$$\|\hat{u}^n - \hat{u}\| \leqq Cq^n, \qquad n = 0, 1, 2, \cdots, \tag{2.31}$$

where constants $C > 0$ and $q \in [0, 1[$ depend only on $\|f\|$, α, and k_0. Indeed, we define the operator $S: C(\bar{\mathcal{O}}) \to C(\bar{\mathcal{O}})$ by

$$Sv = \inf_{\theta} E\left\{ \int_0^{\theta \wedge \tau^0} f(y^0(t))\, e^{-\alpha t}\, dt + Mv(y^0(\theta)) 1_{\theta < \tau^0}\, e^{-\alpha\theta} \right\}, \tag{2.32}$$

where θ is any stopping time of $\mathcal{F}^t$.

Let $\hat{u}^0$ be the function given by (2.12), so using estimate (2.8) and the fact that $k_0 \leqq M(0)$, we deduce

(2.33)[18]
$$\lambda \hat{u}^0 \leqq S(0) \quad \text{if } 0 \leqq \lambda \leqq \frac{\alpha k_0}{\|f\|}.$$

Clearly, the operator S is increasing and concave, hence it is easy to prove from (2.33) the following property:

(2.34)
$\forall u, v \in C(\bar{\mathcal{O}})$, $0 \leqq u, v \leqq \hat{u}^0$ and satisfying

$$-rv \leqq u - v \leqq pu, \qquad r, p \in [0, 1],$$

we have

$$-(1-\lambda) rSv \leqq Su - Sv \leqq (1-\lambda) pSu.$$

Next, we obtain from (2.34)

$$\|S^n \hat{u}^0 - S^m \hat{u}^0\| \leqq (1-\lambda)^{m-n} \|\hat{u}^0\|, \qquad m > n, \tag{2.35}$$

and recalling that $\hat{u}^n = S^n \hat{u}^0$, we have the estimate (2.31) with $C = \|\hat{u}^0\|$ and $q = 1 - \lambda$. □

COROLLARY 2.1. *Let the assumptions be as in Theorem* 2.2. *Then there exists an optimal admissible impulse control* $\hat{\nu} = \hat{\nu}_x$,

$$\hat{u}(x) = J_x(\hat{\nu}), \tag{2.36}$$

where $\hat{u}$ *is given by* (0.3).

[18] We assume that $\lambda \leqq 1$.

Proof. From Theorem 2.2, the function $\hat{u}(x)$ is continuous. Then, from Lemma 1.4, there exists a function $\hat{\xi}(x)$ verifying (1.25) and

$$[M\hat{u}](x) = k(\hat{\xi}(x)) + \hat{u}(x + \hat{\xi}(x)) \quad \forall x \in \bar{\mathcal{O}}. \tag{2.37}$$

Then, we define $\hat{\nu} = \{\hat{\theta}_i, \hat{\xi}_i\}_{i=1}^{\infty}$ by

$$\tilde{\theta}^0 = 0; \tag{2.38}$$

$$\begin{aligned} &d\hat{y}^0(t) = g(\hat{y}^0(t))\,dt + \sigma(\hat{y}^0(t))\,dw(t), \qquad t \geqq 0,\\ &\hat{y}^0(0) = x; \end{aligned} \tag{2.39}$$

$$\hat{\tau}^i = \inf\{t \geqq 0 / \hat{y}^i(t) \notin \bar{\mathcal{O}}\}, \qquad i = 0, 1, 2, \cdots; \tag{2.40}$$

$$\tilde{\theta}^{i+1} = \inf\{t \in [\tilde{\theta}^i, \hat{\tau}^i] / \hat{u}(\hat{y}^i(t)) = [M\hat{u}](\hat{y}^i(t))\}, \qquad i = 0, 1, 2, \cdots, \tag{2.41}$$

$$\hat{\xi}_{i+1} = \hat{\xi}(\hat{y}^i(\tilde{\theta}^{i+1})), \qquad i = 0, 1, 2, \cdots; \tag{2.42}$$

$$\begin{aligned} &d\hat{y}^i(t) = g(\hat{y}^i(t))\,dt + \sigma(\hat{y}^i(t))\,dw(t), && t \geqq \tilde{\theta}^i,\\ &\hat{y}^i(\tilde{\theta}^i) = \hat{y}^{i-1}(\tilde{\theta}^i) + \hat{\xi}_i, && i = 1, 2, \cdots,\\ &\hat{y}^i(t) = \hat{y}^{i-1}(t), \qquad t < \tilde{\theta}^i, \end{aligned} \tag{2.43}$$

and later on,

$$\hat{\theta}_i = \begin{cases} \tilde{\theta}^i & \text{if } \tilde{\theta}^i < \hat{\tau}^{i-1},\\ \infty & \text{otherwise,} \end{cases} \qquad i = 1, 2, \cdots. \tag{2.44}$$

We have

$$y(t, \hat{\nu}) = \hat{y}^n(t) \quad \text{if } 0 \leqq t < \hat{\theta}_n, \tag{2.45}$$

and from Markov's property

$$\begin{aligned} \hat{u}(x) = E\left\{\int_0^{\hat{\theta}_n \wedge \hat{\tau}^{n-1}} f(\hat{y}^n(t))\,e^{-\alpha t}\,dt + \sum_{i=1}^{n} k(\hat{\xi}_i) 1_{\hat{\theta}_i < \infty}\, e^{-\alpha \hat{\theta}_i}\right\}\\ + E\{1_{\hat{\theta}_n < \hat{\tau}^{n-1}} \hat{u}(\hat{y}^n(\hat{\theta}_n))\,e^{-\alpha \hat{\theta}_n}\}. \end{aligned} \tag{2.46}$$

Hence, letting $n \to \infty$ in (2.46) and, using (2.45) and (1.9), we obtain (2.36). □

2.2. Complementary results. Now we omit assumptions (2.14) and (2.15).

THEOREM 2.3. *Let the conditions* (1.1), (1.20), (1.22), *and* (2.1) *hold. Then the maximum solution $\hat{u}$ of problem* (2.2), (2.3), (2.4) *is given as the optimal cost* (0.3), *and the estimate* (2.16) *is true.*

Proof. As in Theorem 2.2, we just need to prove (2.17). Moreover, we will only show that

(2.47) $\forall \varepsilon > 0$ there exists $\hat{\nu}^{\varepsilon}$, an admissible impulse control which has at most n impulses, such that

$$\hat{u}^n(x) + \varepsilon \geqq J_x(\hat{\nu}^{\varepsilon}).$$

Indeed, given $\varepsilon > 0$, from Theorem 3.4 in [7], we can choose a stopping time which is ε-optimal and depends measurably on the initial point, so there exist functions $\hat{\theta}^i_{\varepsilon}(x)$, $i = 1, 2, \cdots, n$, such that

$$\begin{aligned} &\hat{\theta}^i_{\varepsilon} : \bar{\mathcal{O}} \times \Omega \to [0, \infty] \text{ is Borel measurable,}\\ &\forall x \in \bar{\mathcal{O}}, \quad \hat{\theta}^i_{\varepsilon}(x) \text{ is a stopping time;} \end{aligned} \tag{2.48}$$

$$\hat{u}^{n-i+1}+\varepsilon 2^{-n-1} \geqq E\left\{\int_0^{\hat{\theta}_\varepsilon^i \wedge \tau^0} f(y^0(t))\, e^{-\alpha t}\, dt + 1_{\hat{\theta}_\varepsilon^i<\tau^0}[M\hat{u}^{n-i}] \cdot (y^0(\hat{\theta}_\varepsilon^i)) \exp(-\alpha \hat{\theta}_\varepsilon^i)\right\}. \tag{2.49}$$

Also from Lemma 1.4, there exist functions $\hat{\xi}_\varepsilon^i(x)$, $i=1, 2, \cdots, n$, verifying (1.25), and

$$[M\hat{u}^{n-i}](x)+\varepsilon 2^{-n-1} \geqq k(\hat{\xi}_\varepsilon^i(x)) + \hat{u}^{n-i}(x+\hat{\xi}_\varepsilon^i(x)) \quad \forall x \in \bar{\mathcal{O}}. \tag{2.50}$$

Thus, defining the admissible impulse control $\hat{\nu}^\varepsilon = \{\hat{\theta}_i, \hat{\xi}_i\}_{i=1}^\infty$ by (2.19), (2.20), and

$$\hat{\tau}^i = \inf\{t \geqq 0 / \hat{y}^i(t) \notin \bar{\mathcal{O}}\}, \qquad i=0, 1, \cdots, n, \tag{2.51}$$

$$\tilde{\theta}^i = [\tilde{\theta}^{i-1} + \hat{\theta}_\varepsilon^i(\hat{y}^{i-1}(\tilde{\theta}^{i-1}))] \wedge \hat{\tau}^{i-1}, \qquad i=1, \cdots, n, \tag{2.52}$$

$$\hat{\xi}_i = \hat{\xi}_\varepsilon^{i-1}(\hat{y}^{i-1}(\tilde{\theta}^i)), \qquad i=1, \cdots, n, \tag{2.53}$$

and (2.24), (2.25), (2.26) we deduce assertion (2.47) using Markov's property. □

COROLLARY 2.2. *Let the assumptions be as in Theorem* 2.3. *Then given* $\varepsilon>0$ *there exists a function* $\hat{\nu}_\varepsilon(x)=\{\hat{\theta}_i(x), \hat{\xi}_i(x)\}_{i=1}^\infty$ *such that* $\hat{\theta}_i$ *and* $\hat{\xi}_i$ *verify* (2.48) *and* (1.25) *respectively, and*

$$\hat{u}(x)+\varepsilon \geqq J_x(\hat{\nu}_\varepsilon(x)) \quad \forall x \in \bar{\mathcal{O}}, \tag{2.54}$$

where $\hat{u}$ *is the optimal cost given by* (0.3).

Proof. We just need to combine the methods of Theorem 2.3 and Corollary 2.1. □

Finally, the function $\hat{u}$ is regarded as a distribution in $\mathcal{O}$. Notice that Theorem 0.1 is completely proved.

Recalling that A is the differential operator (0.4) and assuming

$$\frac{\partial^2}{\partial x^2}\sigma\sigma^* \in L^1_{\mathrm{loc}}(\mathcal{O}) \tag{2.55}$$

we can define Au, for any $u \in B(\bar{\mathcal{O}})$, as the following distribution,

$$\langle Au, \phi\rangle = \int_{\mathcal{O}} uA^*\phi\, dx \quad \forall \phi \in \mathcal{D}(\mathcal{O}), \tag{2.56}$$

where A^* is the adjoint of A,

$$A^*\phi = -\frac{1}{2}\operatorname{tr}\left[\frac{\partial^2}{\partial x^2}\sigma\sigma^*\phi\right] + \frac{\partial}{\partial x} g\phi + \alpha\phi. \tag{2.57}$$

THEOREM 2.4. *Assume the boundary* Γ *is smooth, and conditions* (1.1), (1.20), (1.22), (2.1), *and* (2.55) *hold. Then the optimal cost* $\hat{u}$ *given by* (0.3) *satisfies*

$$A\hat{u} \leqq f \quad \text{in } \mathcal{D}'(\mathcal{O}). \tag{2.58}$$

Moreover, if (2.14) *and* (2.15) *are true, we also have*

(2.59)[19]
$$A\hat{u} = f \quad \text{in } \mathcal{D}'([\hat{u}<M\hat{u}]).$$

Proof. We need only to use Theorem 3.6 in [7] and Remark 2.1. □

[19] $[\hat{u}<M\hat{u}]$ denotes the subset of $\mathcal{O}$ such that $\hat{u}(x)<M\hat{u}(x)$.

3. Quasi-variational inequality. Let $a_{ij}(x)$, $a_i(x)$ be functions for $i, j = 1, \cdots, N$ such that

$$(a_{ij})_{ij} \text{ is a nonnegative symmetric matrix and}$$

$$a_{ij} \in C^1(\mathbb{R}^N), \qquad \frac{\partial^2 a_{ij}}{\partial x_k\, \partial x_l} \in L^\infty(\mathbb{R}^N) \quad \forall i, j, k, l = 1, \cdots, N, \tag{3.1}$$

$$a_i \in C(\mathbb{R}^N), \qquad \frac{\partial a_i}{\partial x_k} \in L^\infty(\mathbb{R}^N) \quad \forall i, k = 1, \cdots, N. \tag{3.2}$$

Define the following differential operator A,

$$A = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} + \sum_{i=1}^{N} a_i \frac{\partial}{\partial x_i} + \alpha, \tag{3.3}$$

where α is a positive constant.

We always identify g and σ given by (1.1) as

$$(a_{ij})_{ij} = \tfrac{1}{2}\sigma\sigma^*, \qquad a_i = \sum_{j=1}^{N} \frac{\partial a_{ij}}{\partial x_j} - g_i. \tag{3.4}$$

Let $\beta_0(x)$ and $\beta_1(x)$ be the weight functions $(1+|x|^2)^{-(\lambda+1)/2}$ and $(1+|x|^2)^{-\lambda/2}$, $\lambda > N/2$, respectively.

Introduce the Hilbert spaces

$$H = \{v/\beta_0 v \in L^2(\mathcal{O})\}, \tag{3.5}$$

with the inner product

$$(u, v) = \int_{\mathcal{O}} (\beta_0 u)(\beta_0 v)\, dx \tag{3.6}$$

and the norm $|\cdot|$;

$$V = \left\{ v \in H/\beta_1 \frac{\partial v}{\partial x_k} \in L^2(\mathcal{O}) \ \ \forall k = 1, \cdots, N \right\} \tag{3.7}$$

with the norm

$$\|v\| = \left(|v|^2 + \sum_{k=1}^{N} \int_{\mathcal{O}} \left| \beta_1 \frac{\partial v}{\partial x_k} \right|^2 dx \right)^{1/2}. \tag{3.8}$$

V' denotes the dual space of V and $\langle \cdot, \cdot \rangle$ the duality between V' and V.

We have

$$V \subset H \subset V', \quad L^\infty(\mathcal{O}) \subset H, \quad \left\{ v \middle/ \frac{\partial v}{\partial x_i} \in L^\infty(\mathcal{O}) \quad \forall i = 1, \cdots, N \right\} \subset V. \tag{3.9}$$

Let $a(\cdot, \cdot)$ be the bilinear form associated with the operator A,

$$\begin{aligned} a(u, v) = &\sum_{i,j=1}^{N} \int_{\mathcal{O}} \tilde{a}_{ij} \left(\beta_1 \frac{\partial u}{\partial x_i} \right) \left(\beta_1 \frac{\partial v}{\partial x_j} \right) dx \\ &+ \sum_{i=1}^{N} \int_{\mathcal{O}} \tilde{a}_i \left(\beta_1 \frac{\partial u}{\partial x_i} \right) (\beta_0 v)\, dx + \alpha(u, v), \end{aligned} \tag{3.10}$$

where

$$\tilde{a}_{ij}(x) = (1+|x|^2)^{-1} a_{ij}(x),$$

(3.11)
$$\tilde{a}_i(x) = (1+|x|^2)^{-1/2} a_i(x) - 2(\lambda+1)(1+|x|^2)^{-3/2} \sum_{j=1}^{N} a_{ij}(x) x_j.$$

Notice that a_{ij}, a_i are not supposed to be bounded, but a_{ij} is at most of quadratic growth, and a_i of linear growth. Then, $\tilde{a}_{ij}$, $\tilde{a}_i$ in (3.11) are bounded.

This section is divided into two parts. First, we consider the case where $\mathcal{O} = \mathbb{R}^N$. Next, we study the general case.

3.1. Case $\mathcal{O} = \mathbb{R}^N$. Assume $\mathcal{O} = \mathbb{R}^N$. After some calculation, we deduce

(3.12) $$a(u, v) = (Au, v) \quad \forall u, v \in V, Au \in H,$$

(3.13)[20] $$|a(u, v)| \leqq C\|u\|\|v\| \quad \forall u, v \in V,$$

and if α is large enough there exists $\alpha_0 > 0$ such that

(3.14) $$a(u, v) \geqq \alpha_0(u, u) \quad \forall u \in V.$$

Next, from (3.12) and (3.13), it follows that

(3.15) $$a(u, v) = \langle Au, v\rangle \quad \forall u, v \in V.$$

We recall that M denotes the operator given by (1.21). We define, for any $u \in V \cap L^\infty(\mathbb{R}^N)$, the closed cone $K(u)$ in V by

(3.16) $$K(u) = \{v \in V / v(x) \leqq [Mu](x) \text{ a.e. in } \mathbb{R}^N\}.$$

Let us consider the following quasi-variational inequality,

(3.17) Find $u \in V \cap L^\infty(\mathbb{R}^N)$ such that $u \in K(u)$ and
$$a(u, v-u) \geqq (f, v-u) \quad \forall v \in K(u),$$

and also the sequence of variational inequalities

(3.18) Find $u^0 \in V$ such that $a(u^0, v) = (f, v) \quad \forall v \in V$.

(3.19) Find $u^n \in V \cap L^\infty(\mathbb{R}^N)$ such that $u^n \in K(u^{n-1})$ and
$$a(u^n, v-u^n) \geqq (f, v-u^n) \quad \forall v \in K(u^{n-1}).$$

We have

THEOREM 3.1. *Let the assumptions* (3.1), (3.2), (1.20), (2.1), *and*

(3.20) $$\frac{\partial f}{\partial x_k} \in L^\infty(\mathbb{R}^N), \qquad k = 1, \cdots, N$$

hold. Then the quasi-variational inequality (3.17) *admits a maximum solution $\hat{u}$ which is given as the optimal cost* (0.3). *Moreover, $\hat{u}$ is Lipschitz continuous and the following estimates are true.*

(3.21)[21] $$\left\|\frac{\partial \hat{u}}{\partial x}\right\|_{L^\infty} \leqq \frac{1}{\alpha - \gamma_0}\left\|\frac{\partial f}{\partial x}\right\|_{L^\infty},$$

[20] C denotes a constant.

[21] $\|\partial \hat{u}/\partial x\|_{L^\infty}$ denotes the smallest Lipschitz continuous constant of $\hat{u}$, and γ_0 is given by (1.19).

$$0 \leqq u^n - \hat{u} \leqq C(n+1)^{-1}, \qquad n = 0, 1, \cdots, \tag{3.22}$$

where the constant C depends only on the supremum norm of f and α, k_0.[22]

Proof. First, from Theorem 4.1 in [7], the sequence defined by (3.18), (3.19) coincides with that defined by (2.12), (2.13).

Then, from (2.17), we have

$$|u^n(x) - u^n(x')| \leqq \sup \{|J_x(\nu) - J_{x'}(\nu)|/\nu \text{ an impulse control such that } \theta_i = \infty \ \forall\, i \geqq n+1\}.$$

Hence, Lemma 1.1 and (3.20) imply

$$\left\| \frac{\partial u^n}{\partial x} \right\|_{L^\infty} \leqq \frac{1}{\alpha - \gamma_0} \left\| \frac{\partial f}{\partial x} \right\|_{L^\infty} \qquad \forall n = 0, 1, 2, \cdots. \tag{3.23}$$

Thus, using Theorem 2.2 and classical technique, the proof is completed. □

Remark 3.1. Clearly, using only analytic methods, like B. Hanouzet and J. L. Joly [3], we can prove that (Remark 2.2)

$$0 \leqq u^n - \hat{u} \leqq cq^n, \qquad n = 0, 1, \cdots, \quad \text{with } 0 < q < 1. \quad □ \tag{3.24}$$

3.2. General case. Now, we come back to the general case, $\mathcal{O}$ an open subset of $\mathbb{R}^N$ with boundary Γ sufficiently smooth.

Define the closed subspace of V,

$$V_0 = \{v \in V / v = 0 \text{ on } \Gamma\}. \tag{3.25}$$

Then, as in the case $\mathcal{O} = \mathbb{R}^N$, if α is large enough there exists a constant $\alpha_0 > 0$ such that

$$a(u, u) \geqq \alpha_0(u, u) \quad \forall u \in V_0, \tag{3.26}$$

and we also have

$$a(u, v) = \langle Au, v \rangle \quad \forall u, v \in V_0. \tag{3.27}$$

For any $u \in V_0 \cap L^\infty(\mathcal{O})$, we define $K_0(u)$, the following closed cone in V_0 by

$$K_0(u) = \{v \in V_0 / v \leqq Mu, \text{ a.e. in } \mathcal{O}\}. \tag{3.28}$$

Let us consider the quasi-variational inequality

$$\begin{gathered} \text{Find } u \in V_0 \cap L^\infty(\mathcal{O}) \text{ such that } u \in K_0(u) \text{ and} \\ a(u, v-u) \geqq (f, v-u) \quad \forall v \in K_0(u), \end{gathered} \tag{3.29}$$

and the associated sequence of variational inequalities,

$$\text{Find } u^0 \in V_0 \text{ such that } a(u^0, v) = (f, v) \quad \forall v \in V_0. \tag{3.30}$$

$$\begin{gathered} \text{Find } u^n \in V_0 \cap L^\infty(\mathcal{O}) \text{ such that } u^n \in K_0(u^{n-1}) \text{ and} \\ a(u^n, v-u^n) \geqq (f, v-u^n) \quad \forall v \in K_0(u^{n-1}). \end{gathered} \tag{3.31}$$

[22] k_0 is given in (1.20).

Remark 3.2. Assume (2.1). Suppose that $\mathcal{O}$ is bounded and satisfies the uniform exterior sphere condition of radius $\rho > 0$, and that

$$\Gamma = \{x \in \Gamma / |\sigma(x)n(x)| > 0\}$$

$$\cup \{x \in \Gamma / 2g(x)n(x) < -\operatorname{tr}(\sigma(x)\sigma^*(x))\}, \tag{3.32}$$

$n(x)$ is the inner normal with modulus ρ.

Then, there exists a Lipschitz continuous subsolution

(3.33)[23]
$$\bar{w} \in C(\bar{\mathcal{O}});\ \bar{w}, \frac{\partial \bar{w}}{\partial x_i} \in L^\infty(\mathcal{O}), \qquad i = 1, \cdots, N,$$

$$A\bar{w} \leqq -f \text{ in } \mathcal{O}, \qquad \bar{w}(x) = 0 \quad \forall x \in \Gamma.$$

Indeed, we only need to use Lemma 1.5 in [7].

THEOREM 3.2. *Let the conditions* (3.1), (3.2), (1.20), (1.22), (2.1), (3.33) *and*[24]

$$\frac{\partial f}{\partial x_k} \in L^\infty(\mathcal{O}), \qquad k = 1, \cdots, N, \tag{3.34}$$

hold. Then the quasi-variational inequality (3.29) *admits a maximum solution $\hat{u}$ which is given as the optimal cost* (0.3). *Moreover, $\hat{u}$ is Lipschitz continuous and the estimates* (3.22) *and*

$$\left\|\frac{\partial \hat{u}}{\partial x}\right\|_{L^\infty} \leqq \frac{1}{\alpha - \gamma_0}\left\|\frac{\partial f}{\partial x}\right\|_{L^\infty} + \left\|\frac{\partial \bar{w}}{\partial x}\right\|_{L^\infty} \tag{3.35}$$

are true.

Proof. As for Theorem 3.1, we just need to prove the following estimate,

$$\left\|\frac{\partial u^n}{\partial x}\right\|_{L^\infty} \leqq \frac{1}{\alpha - \gamma_0}\left\|\frac{\partial f}{\partial x}\right\|_{L^\infty} + \left\|\frac{\partial \bar{w}}{\partial x}\right\|_{L^\infty}, \qquad n = 0, 1, \cdots. \tag{3.36}$$

Indeed, starting at

$$u^n(x) - u^n(x') = \sup_{\nu'} \inf_{\nu} [J_x(\nu) - J_{x'}(\nu')], \tag{3.37}$$

we set, for any $\nu' = \{\theta'_i, \xi'_i\}_{i=1}^\infty$, the impulse control $\nu = \{\theta_i, \xi_i\}_{i=1}^\infty$ defined by (1.2) and

$$\tau_x^i = \inf\{t \geqq 0 / y_x^i(t) \notin \bar{\mathcal{O}}\}, \qquad i = 0, 1, \cdots; \tag{3.38}$$

(3.39)[25]
$$\theta_i = \begin{cases} \theta'_i & \text{if } \theta'_i < \tau_x^{i-1} \wedge \tau_{x'}^{i-1}, \\ \infty & \text{otherwise;} \end{cases}$$

$$\xi_i = \begin{cases} \xi'_i & \text{if } \theta_i < \infty \quad \text{and} \quad \xi'_i + y_x^{i-1}(\theta_i) \in \bar{\mathcal{O}}, \\ 0 & \text{if } \theta_i = \infty, \\ \lambda\xi'_i & \text{if } \theta_i < \infty \quad \text{and} \quad \lambda\xi'_i + y_x^{i-1}(\theta_i) \in \Gamma; \end{cases} \tag{3.40}$$

$$dy^i(t) = g(y^i(t))\,dt + \sigma(y^i(t))\,dw(t), \qquad t \geqq \theta_i,$$

$$y^i(\theta_i) = y^{i-1}(\theta_i) + \xi_i, \tag{3.41}$$

$$y^i(t) = y^{i-1}(t), \qquad t < \theta_i.$$

[23] In the martingale sense with α large enough.

[24] We also assume α large enough and $k(\lambda\xi) \leqq k(\xi)$, $\forall \xi \in \Lambda$, $\lambda \in [0, 1]$.

[25] $\tau_{x'}^i$ is given as τ_x^i in (3.38).

Notice that ξ_i is well defined, because if $\xi_i' + y_x^{i-1}(\theta_i) \notin \bar{\mathcal{O}}$ and $\theta_i < \infty$ we have $y^{i-1}(\theta_i) \in \bar{\mathcal{O}}$, and so there exists $\lambda \in [0, 1]$ such that $\lambda \xi_i' + y_x^{i-1}(\theta_i) \in \Gamma = \Gamma_0$.

Thus, ν is an admissible impulse control for x, and choosing ν as above in (3.37), we deduce

$$
\begin{aligned}
u^n(x) - u^n(x') \leqq \sup_{\nu'} E\left\{ \int_{\tau_x \wedge \tau_{x'}}^{\tau_x} f(y_x(t, \nu))\, e^{-\alpha t}\, dt \right\} \\
+ \sup_{\nu'} E\left\{ \int_0^{\tau_x \wedge \tau_{x'}} |f(y_x(t, \nu)) - f(y_{x'}(t, \nu'))|\, e^{-\alpha t}\, dt \right\},
\end{aligned}
\tag{3.42}
$$

where the supremum is taken over all admissible impulse controls ν'.

Finally, from Lemma 1.2 and the fact that

$$
y_x(t, \nu) = y_x(t, \nu'), \quad \text{a.s. in } [0, \tau_x \wedge \tau_{x'}[, \tag{3.43}
$$

the estimate (3.36) follows from (3.42). □

THEOREM 3.3. *Under the conditions of Theorem* 3.2, *the following quasi-variational inequality*

$$
\begin{aligned}
&\hat{u} \in W_0^{1,\infty}(\mathcal{O}), \qquad \hat{u} \leqq M\hat{u} \text{ in } \mathcal{O}, \\
&A\hat{u} \leqq f \text{ in } \mathcal{D}'(\mathcal{O}), \qquad A\hat{u} = f \text{ in } \mathcal{D}'([\hat{u} < M\hat{u}]),
\end{aligned}
\tag{3.44}
$$

has one and only one solution $\hat{u}$. Moreover, $\hat{u}$ is given as the optimal cost (0.3).

Proof. We only need to prove the uniqueness of problem (3.44). Moreover, it suffices to show that any solution of (3.44) is a solution of (2.46).

Indeed, using a classical technique (cf. D. W. Stroock and S. R. S. Varadhan [12]), we can prove that if $\hat{u}$ verifies

$$
\hat{u} \in W_0^{1,\infty}(\mathcal{O}), \qquad A\hat{u} = f \quad \text{in } \mathcal{D}'([\hat{u} < M\hat{u}]),
$$

then we also have

$$
A\hat{u} = f \text{ in the martingale sense on } [\hat{u} < M\hat{u}].
$$

Therefore, as in Corollary 2.1, we obtain the equality (2.46) and the theorem is established. □

Remark 3.3. It is possible to consider a function $a_0(x)$ instead of the constant α for the definition of cost (0.2). Moreover, we can also consider f not necessarily bounded and $k = k(x, \xi)$.

Remark 3.4. All these results can be extended to the parabolic case.

Remark 3.5. In [9], we give an application to the impulse control problems with partial information.

Final Remark. In a separate paper (cf. [8]) the stopping time and impulse control problems for degenerate diffusions with boundary conditions will be studied.

REFERENCES

[1] A. BENSOUSSAN, *On the semigroup formulation of variational inequalities and quasi-variational inequalities*, First Franco-Southeast Asian Mathematical Conference, Singapore, May-1979.

[2] A. BENSOUSSAN AND J. L. LIONS, *Contrôle impulsionnel et inéquations quasi-variationnelles*, Dunod, Paris, to be published.

[3] B. HANOUZET AND J. L. JOLY, *Convergence uniforme des itérés définissant la solution d'une inéquation quasi-variationnelle abstraite*, C.R. Acad. Sci. Paris Sér. A, 286 (1978), pp. 735–738.

[4] J. P. LEPELTIER AND B. MARCHAL, *Techniques probabilistes dans le contrôle impulsionnel*, Stochastics, 2 (1979), pp. 243–286.

[5] J. L. MENALDI, *Sur le problème de contrôle impulsionnel et l'inéquation quasi-variationnelle dégénérée associée*, C.R. Acad. Sci. Paris Sér. A, 284 (1977), pp. 1499–1502.

[6] ———, *Le problème de contrôle impulsionnel déterministe et l'inéquation quasi-variationnelle de premier ordre associée*, Appl. Math. Optim., to appear.

[7] ——, *On the optimal stopping time problem for degenerate diffusions*, this Journal, this issue, pp. 697–721.

[8] ———, *On the degenerate variational inequality with Neumann boundary conditions*, submitted to J. Optim. Theory Appl.

[9] ——, *The separation principle for impulse control problems*, Proc. Amer. Math. Soc., to appear.

[10] F. MIGNOT AND J. P. PUEL, *Inéquations variationnelles et quasi-variationnelles hyperboliques du premier ordre*, J. Math. Pures et Appl., 55 (1976), pp. 353–378.

[11] M. ROBIN, *Contrôle impulsionnel des processus de Markov*, Thèse d'Etat, Université de Paris IX, Paris, 1977.

[12] D. W. STROOCK AND S. R. S. VARADHAN, *On degenerate elliptic-parabolic operators of second order and their associated diffusions*, Comm. Pure Appl. Math., 25 (1972), pp. 651–713.

[13] L. TARTAR, *Inéquations quasi-variationnelles abstraites*, C.R. Acad. Sci. Paris Sér. A, 278 (1974), pp. 1193–1196.