12-3-2006

Generalized Differentiation of Parameter-Dependent Sets and Mappings

Boris S. Mordukhovich

Wayne State University, boris@math.wayne.edu

Bingwu Wang

Eastern Michigan University, bwang@emunix.emich.edu

Recommended Citation

Mordukhovich, Boris S. and Wang, Bingwu, "Generalized Differentiation of Parameter-Dependent Sets and Mappings" (2006). *Mathematics Research Reports*. Paper 43.
http://digitalcommons.wayne.edu/math_reports/43

This Technical Report is brought to you for free and open access by the Mathematics at DigitalCommons@WayneState. It has been accepted for inclusion in Mathematics Research Reports by an authorized administrator of DigitalCommons@WayneState.

**GENERALIZED DIFFERENTIATION OF PARAMETER-
DEPENDENT SETS AND MAPPINGS**

BORIS. S. MORDUKHOVICH and BINGWU WANG

**WAYNE STATE
UNIVERSITY**

Detroit, MI 48202

**Department of Mathematics
Research Report**

**2006 Series
#14**

*This research was partly supported by the National Science Foundation and the Australian
Research Council*

GENERALIZED DIFFERENTIATION OF PARAMETER-DEPENDENT SETS AND MAPPINGS

BORIS S. MORDUKHOVICH¹

Department of Mathematics, Wayne State University
Detroit, Michigan 48202, USA
Email: boris@math.wayne.edu

and

BINGWU WANG

Department of Mathematics, Eastern Michigan University
Ypsilanti, Michigan 48197, USA
Email: bwang@emunix.emich.edu

Dedicated to Bert Jongen in honor of his 60th birthday

The paper concerns new aspects of generalized differentiation theory that plays a crucial role in many areas of modern variational analysis, optimization, and their applications. In contrast to the majority of previous developments, we focus here on generalized differentiation of parameter-dependent objects (sets, set-valued mappings, and nonsmooth functions), which naturally appear, e.g., in parametric optimization and related topics. The basic generalized differential constructions needed in this case are different for those known in parameter-independent settings, while they still enjoy comprehensive calculus rules developed in this paper.

Keywords: Nonsmooth and parametric optimization; Variational analysis; Generalized differentiation; Calculus rules; Normal compactness

Mathematical Subject Classification 2000: Primary: 49J52, 49J53; Secondary: 90C29

1 Introduction

During the recent years, modern variational analysis has been well recognized as a rapidly growing and fruitful area of mathematics with numerous applications; see particularly the books [4, 10, 11, 19] and the references therein. One of the major motivations for developing basic tools of variational analysis came from optimization-related problems, although nowadays variational methods play a crucial role in the study of a broad spectrum of theoretical and applied problems of non-variational nature. Since advanced variational principles and optimization techniques naturally generate *nonsmooth* behavior of the corresponding functions/mappings and sets, generalized differentiation theory lies at the very heart of variational analysis and its applications; see, e.g., the books mentioned above.

Previous developments on generalized differentiation mainly concerned nonsmooth objects that do not depend on parameters. However, *parameter-dependent* (or *moving*) objects

¹Research was partially supported by the National Science Foundation under grants DMS-0304989 and DMS-0603846 and by the Australian Research Council under grant DP-0451168.

naturally appear, in particular, the framework of *parametric optimization* (see, e.g., [7] with the references therein) while requiring a special attention from the viewpoint of generalized differentiation. Some attempts in this direction are undertaken in [2, 12, 15, 16] (with particular applications to multiobjective optimization, optimal control, and economics), and the main results obtained are summarized in [11, Section 5.3].

In this paper we present a *systematic study* of the basic generalized differentiation constructions for sets, set-valued mappings, and extended-real-valued functions and develop for them new *calculus rules* in both finite-dimensional and infinite-dimensional settings. Furthermore, we establish new results on the so-called *normal compactness* properties for moving objects that are automatic in finite dimensions while playing a very significant role in infinite-dimensional variational analysis and generalized differentiation. In particular, this paper contains new sufficient conditions for the fulfillment of the *extended* normal compactness properties and develops general results on the preservation of these properties under various operations; such calculus results are especially important for applications. Our main *driving force* for developing calculus rules for both generalized differentiation and normal compactness is the *extremal principle* of variational analysis; see [10, Chapter 2] and Section 3 below for its limiting version in the parameter-dependent setting.

The rest of the paper is organized as follows. In Section 2 we present the basic definitions and also some preliminaries needed for the main results of the paper. Section 3 contains a new version of the exact/limiting extremal principle for moving sets. Section 4 is devoted to the basic calculus rules for extended normals, coderivatives, and subgradients of parameter-dependent sets, set-valued mappings, and nonsmooth functions. The final Section 5 presents new verification and calculus results for extended normal compactness.

Our notation mainly follows the book [10]. Recall that, given a set-valued mapping (multifunction) $F: X \rightrightarrows X^*$ between a Banach space X and its topological dual X^* , the *sequential Painlevé-Kuratowski upper/outer limit* of F as $x \rightarrow \bar{x}$ with respect to the norm topology of X and the weak* topology w^* of X^* is

$$\begin{aligned} \limsup_{x \rightarrow \bar{x}} F(x) := \left\{ x^* \in X^* \mid \begin{array}{l} \exists \text{ sequences } x_k \rightarrow \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \\ \text{with } x_k^* \in F(x_k) \text{ for all } k \in \mathbb{N} \end{array} \right\}, \end{aligned} \quad (1.1)$$

where $\mathbb{N} := \{1, 2, \dots\}$. Recall also the symbol $x \xrightarrow{\Omega} \bar{x}$ signifies that $x \rightarrow \bar{x}$ with $x \in \Omega$ for the set $\Omega \subset X$. Unless otherwise stated, all the spaces under consideration are Banach with the norm $\|\cdot\|$ and the canonical pairing $\langle \cdot, \cdot \rangle$ between the space in question and its dual. We use B_X to denote the closed unit ball of X , where the subindex “ X ” is omitted when there is no confusion; B^* stands for the closed unit ball of the dual space in question.

2 Basic Definitions and Preliminaries

Developing a *geometric dual-space* approach to variational analysis and generalized differentiation as in [10, 11], we start with normals to arbitrary sets and proceed with generalized derivatives (coderivatives) for set-valued mappings/multifunctions, and then with subdifferentials of extended-real-valued functions.

Consider a nonempty subset $\Omega \subset X$ of a Banach space and recall the construction of ε -normals to Ω at $\bar{x} \in X$ defined by

$$\hat{N}_\varepsilon(\bar{x}; \Omega) := \left\{ x^* \in X^* \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq \varepsilon \right\} \text{ for } \bar{x} \in \Omega, \varepsilon \geq 0 \quad (2.1)$$

and by $\hat{N}_\varepsilon(\bar{x}; \Omega) := \emptyset$ for $\bar{x} \notin \Omega, \varepsilon \geq 0$. When $\varepsilon = 0$, the set (2.1) is a cone called *prenormal cone* or *Fréchet normal cone* to Ω at $\bar{x}$ and denoted by $\hat{N}(\bar{x}; \Omega)$.

Throughout the paper, $\mathcal{T}$ stands for our underlying *index/parameter set*, which is a metric space with the distance function $d_{\mathcal{T}}$.

Let $\bar{t} \in \mathcal{T}$, and let $\{\Omega_t\}_{t \in \mathcal{T}}$ be a collection of subsets in X with $\bar{x} \in \Omega_{\bar{t}}$. The *extended normal cone* $\bar{N}(\bar{x}; \Omega_{\bar{t}})$ is defined by

$$\bar{N}(\bar{x}; \Omega_{\bar{t}}) := \limsup_{\substack{x \rightarrow \bar{x}, t \rightarrow \bar{t} \\ \varepsilon \downarrow 0}} \hat{N}_\varepsilon(x; \Omega_t), \quad (2.2)$$

where the Painlevé-Kuratowski limit (1.1) is *sequential*. Observe that the *extended normal cone* (2.2) is different from the *basic normal cone* to $\Omega_{\bar{t}}$ at $\bar{x}$, which corresponds to the construction (2.2) with $t \equiv \bar{t}$ in the limiting procedure. Similarly to the case of the basic normal cone $N(\bar{x}; \Omega)$ as in [10, Theorem 2.35], we can equivalently put $\varepsilon = 0$ in (2.2) if Ω_t is locally closed around $\bar{x}$ (as $x \in \Omega_t$) for all $t \in \mathcal{T}$ near $\bar{t}$ and if the space X is *Asplund*, i.e., each of its separable subspace has a separable dual. Recall that the class of Asplund spaces is sufficiently broad containing, in particular, all reflexive Banach spaces and all spaces with separable duals; see [17] for more information and references on the geometric theory of Asplund spaces and [10, 11] for the extensive usage of Asplund spaces in variational analysis, generalized differentiation, and their applications.

Let $F: X \rightrightarrows Y$ be a set-valued mapping between Banach spaces with the *graph*

$$\text{gph } F := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

The ε -coderivative $\hat{D}_\varepsilon^* F(\bar{x}, \bar{y}): Y^* \rightrightarrows X^*$ of F at $(\bar{x}, \bar{y}) \in \text{gph } F$ is constructed as

$$\hat{D}_\varepsilon^* F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in \hat{N}_\varepsilon((\bar{x}, \bar{y}); \text{gph } F)\}, \quad y^* \in Y^*. \quad (2.3)$$

Next given a parametric family of set-valued mappings $\{F_t\}_{t \in \mathcal{T}}: X \rightrightarrows Y$ between Banach spaces, we define two kinds of extended limiting coderivatives of F_t at $(\bar{x}, \bar{y}) \in \text{gph } F_{\bar{t}}$, which are generally different when $\dim Y = \infty$. The first one called the *extended normal coderivative* is defined by

$$\bar{D}^* F_{\bar{t}}(\bar{x}, \bar{y})(y^*) := \limsup_{\substack{(x, y) \rightarrow (\bar{x}, \bar{y}), t \rightarrow \bar{t} \\ y^* \xrightarrow{w^*} \bar{y}^* \\ \varepsilon \downarrow 0}} D_\varepsilon^* F_t(x, y)(y^*), \quad y^* \in Y^*. \quad (2.4)$$

Observe directly from the definitions that

$$\bar{D}^* F_{\bar{t}}(\bar{x}, \bar{y})(y^*) = \{x^* \in X^* \mid (x^*, -y^*) \in \bar{N}((\bar{x}, \bar{y}); \text{gph } F_{\bar{t}})\} \text{ whenever } y^* \in Y^*. \quad (2.5)$$

The second limiting coderivative is called the *extended mixed coderivative* of $F_{\bar{t}}$ at $(\bar{x}, \bar{y})$ and is defined by

$$\overline{D}_M^* F_{\bar{t}}(\bar{x}, \bar{y})(\bar{y}^*) := \limsup_{\substack{(x,y) \rightarrow (\bar{x}, \bar{y}), t \rightarrow \bar{t} \\ y^* \rightarrow \bar{y}^* \\ \varepsilon \downarrow 0}} D_\varepsilon^* F_t(x, y)(y^*), \quad \bar{y}^* \in Y^*, \quad (2.6)$$

where in contrast to (2.4) the *strong* convergence $y^* \rightarrow \bar{y}^*$ is used on Y^* , while the weak* convergence on X^* is used in both cases (2.4) and (2.6). As in the case of (2.2), we can equivalently put $\varepsilon = 0$ in (2.4) and (2.6) if both spaces X and Y are Asplund and if the graph of F_t is locally closed around $(\bar{x}, \bar{y})$ for all $t \in T$ near $\bar{t}$. Observe in this respect that the *product* of Asplund spaces is also *Asplund* [17]; this fundamental property of Asplund spaces is often used in the sequel.

Let us associate with any extended-real-valued function $\varphi: X \rightarrow \overline{\mathbb{R}} := (-\infty, \infty]$ the corresponding *epigraphical multifunction* $E_\varphi: X \rightrightarrows \mathbb{R}$ defined by

$$E_\varphi(x) := \{\mu \in \mathbb{R} \mid \mu \geq \varphi(x)\} \quad \text{with} \quad \text{gph } E_\varphi = \text{epi } \varphi.$$

Given the parametric family of functions $\{\varphi_t\}_{t \in T}$ from X to $\overline{\mathbb{R}}$ with $\varphi_{\bar{t}}(\bar{x}) < \infty$ for some $\bar{t} \in T$ and $\bar{x} \in X$, we define the *extended subdifferential* $\overline{\partial} \varphi_{\bar{t}}(\bar{x})$ and the *extended singular subdifferential* $\overline{\partial}^\infty \varphi_{\bar{t}}(\bar{x})$ of $\varphi_{\bar{t}}$ at $\bar{x}$ —or of φ at $(\bar{t}, \bar{x})$ —by, respectively,

$$\overline{\partial} \varphi_{\bar{t}}(\bar{x}) := \overline{D}^* E_{\varphi_{\bar{t}}}(\bar{x}, \varphi_{\bar{t}}(\bar{x}))(1) \quad \text{and} \quad \overline{\partial}^\infty \varphi_{\bar{t}}(\bar{x}) := \overline{D}^* E_{\varphi_{\bar{t}}}(\bar{x}, \varphi_{\bar{t}}(\bar{x}))(0). \quad (2.7)$$

Based on (2.4) and similarly to [10, Theorem 1.89], we can represent the extended subdifferential $\overline{\partial} \varphi_{\bar{t}}(\bar{x})$ in (2.7) via sequential limits of Fréchet-like ε -subgradients of locally lower semicontinuous (l.s.c.) functions φ_t around $\bar{x}$ in Banach spaces, with $\varepsilon = 0$ in the Asplund space setting. Likewise, it is possible to do it for the extended singular subdifferential $\overline{\partial}^\infty \varphi_{\bar{t}}(\bar{x})$ in Asplund spaces; cf. [10, Theorem 2.38]. We are not going to use these representations in the paper and leave this to the reader.

Denote by δ_Ω the *indicator function* of the set Ω equal 0 for $x \in \Omega$ and ∞ otherwise. Then we easily get from (2.5) and (2.7) that

$$\overline{\partial} \delta_{\Omega_{\bar{t}}}(\bar{x}) = \overline{\partial}^\infty \delta_{\Omega_{\bar{t}}}(\bar{x}) = \overline{N}(\bar{x}; \Omega_{\bar{t}}) \quad \text{whenever} \quad \bar{x} \in \Omega_{\bar{t}}. \quad (2.8)$$

It has been well recognized that the main results of infinite-dimensional variational analysis and generalized differentiation require some “normal compactness” properties of sets and mappings whose main purpose is to compensate, in the framework of variational analysis, the natural lack of compactness in infinite dimensions. Properties of this type are comprehensively studied and applied in [10, 11] (see also the references therein) for the case of parameter-independent objects. The main attention in [10, 11] is paid to the so-called *sequential normal compactness* (SNC) properties, which seem to be the least restrictive among all the known properties of this type. Let us now define appropriate counterparts of the SNC properties for the case of *parameter-dependent* sets that naturally generate the corresponding properties for (set-valued and single-valued) mappings and extended-real-valued functions. These *extended SNC* properties are studied in more detail in Section 5. Note that we can equivalently put $\varepsilon = 0$ in both parts of the following Definition 2.1 in the case of locally closed sets in Asplund spaces.

Definition 2.1 (extended sequential normal compactness). Let $\mathcal{I} := \{1, \dots, p\}$ be an index set and let X_i , $i \in \mathcal{I}$, be Banach spaces with $\Omega_t \subset X_1 \times \dots \times X_p$ for all $t \in \mathcal{T}$. Suppose $\bar{x} = (\bar{x}_1, \dots, \bar{x}_p) \in \Omega_{\bar{t}}$, and let $\mathcal{I}_1 \subset \mathcal{I}$, $\mathcal{I}_2 := \mathcal{I} \setminus \mathcal{I}_1$. Then we say that:

(i) The family $\{\Omega_t\}_{t \in \mathcal{T}}$ is **PARTIALLY EXTENDEDLY SNC (PESNC)** at $(\bar{t}, \bar{x})$ with respect to $\{X_i \mid i \in \mathcal{I}_1\}$ if for any sequences $\varepsilon_k \downarrow 0$, $t_k \rightarrow \bar{t}$, and $x_k \rightarrow \bar{x}$ with $x_k \in \Omega_{t_k}$ and $x_k^* = (x_{1,k}^*, \dots, x_{p,k}^*) \in \widehat{N}_{\varepsilon_k}(x_k; \Omega_{t_k})$ one has

$$\left[\begin{array}{l} x_{i,k}^* \xrightarrow{w^*} 0 \text{ for all } i \in \mathcal{I}_1, \\ x_{i,k}^* \xrightarrow{\|\cdot\|} 0 \text{ for all } i \in \mathcal{I}_2 \end{array} \right] \implies [x_{i,k}^* \xrightarrow{\|\cdot\|} 0 \text{ for all } i \in \mathcal{I}_1].$$

In particular, we say that $\{\Omega_t\}_{t \in \mathcal{T}}$ is **EXTENDEDLY SNC (ESNC)** at $(\bar{t}, \bar{x})$ if $\mathcal{I}_1 = \mathcal{I}$; in this case no product structure is needed.

(ii) The family $\{\Omega_t\}_{t \in \mathcal{T}}$ is **STRONGLY PESNC** at $(\bar{t}, \bar{x})$ with respect to $\{X_i \mid i \in \mathcal{I}_1\}$ if for any $\varepsilon_k \downarrow 0$, $t_k \rightarrow \bar{t}$, $x_k \rightarrow \bar{x}$ with $x_k \in \Omega_{t_k}$, $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega_{t_k})$ one has

$$[x_{i,k}^* \xrightarrow{w^*} 0 \text{ for all } i \in \mathcal{I}] \implies [x_{i,k}^* \xrightarrow{\|\cdot\|} 0 \text{ for all } i \in \mathcal{I}_1].$$

Based on Definition 2.1, we can define the corresponding ESNC notions for parametric families of mappings and extended-real-valued functions $\{F_t\}_{t \in \mathcal{T}}$ their graphs and epigraphs, respectively. In particular, we say that the family of set-valued mappings $\{F_t\}_{t \in \mathcal{T}}$ is **PESNC** (resp. **strongly PESNC**) at $(\bar{t}, \bar{x}, \bar{y}) \in \text{gph } F_{\bar{t}}$ if the family of sets $\{\text{gph } F_t\}_{t \in \mathcal{T}}$ is PESNC (resp. strongly PESNC) at $(\bar{t}, \bar{x}, \bar{y})$ with respect to X ; The family of extended-real-valued functions $\{\varphi_t\}_{t \in \mathcal{T}}$ is said to be **ESNEC** (i.e., extendedly sequentially normally epi-compact) at $(\bar{t}, \bar{x})$ if the epigraphical family $\{\text{epi } \varphi_t\}_{t \in \mathcal{T}}$ is ESNC at $(\bar{t}, \bar{x}, \varphi_{\bar{t}}(\bar{x}))$.

It is clear that all the above ESNC properties are automatic in finite dimensions. In Section 5 we discuss these properties in more detail, present efficient conditions implying their validity in infinite-dimensional spaces and their relationships with other properties of this kind, and also derive calculus rules ensuring their preservations under various operations. In the next two sections we establish several results on the extremal principle for systems of moving sets and on calculus rules for generalized differentiation of parameter-dependent objects that involve the ESNC assumptions in their formulations and proofs.

3 Exact Extremal Principle for Moving Sets

It has been well recognized that the so-called *extremal principle* for systems of sets is one of the cornerstones in modern variational analysis and their applications; see, e.g., the books [10, 11], which revolve to a large extent around the extremal principle. In this paper we use the following *fuzzy intersection rule* for systems of sets, which is derived in [13] from the *approximate* extremal principle (see also [10, Lemma 3.1]) and then is shown [20] to be *equivalent* to as yet another *characterization* of Asplund spaces.

Lemma 3.1 (fuzzy intersection rule). Let Ω_1, Ω_2 be subsets of the Asplund space X . Assume that Ω_1, Ω_2 are locally closed around $\bar{x} \in \Omega_1 \cap \Omega_2$ and that $x^* \in \widehat{N}(\bar{x}, \Omega_1 \cap \Omega_2)$. Then for any $\varepsilon > 0$ there are

$$\lambda \geq 0, \quad x_i \in \Omega_i \cap (\bar{x} + \varepsilon B), \quad \text{and} \quad x_i^* \in \widehat{N}(x_i; \Omega_i) + \varepsilon B^*, \quad i = 1, 2,$$

satisfying the conditions

$$\lambda x^* = x_1^* + x_2^*, \quad \max\{\lambda, \|x_1^*\|\} = 1.$$

The afore-mentioned approximate extremal principle for systems of *fixed sets* is given in terms of Fréchet normals (2.1). Its *exact/limiting* counterpart formulated via basic normals (2.2) at the point in question is given in [10, Theorem 2.22] under the SNC assumptions on the (all but one) sets involved. In a number of applications (in particular, to *multiobjective optimization* problems; see, [11, Section 5.3]) we need a better version of the limiting extremal principle for systems of fixed sets in *product spaces* established in [13].

On the other hand, some versions of the *extended extremal principle*, in both approximate and limiting forms, are established in [12] for systems of *moving sets*; see also [11, Subsection 5.3.3]. The main goal of this section is to derive a new refined version of the *exact* extended extremal principle for systems of *moving sets in product spaces*. The new version obtained below exploits the *partial* ESNC properties introduced in Definition 2.1(i,ii) that take into account the product structure of the space in question.

First we recall some notions and results from [11, 12] used in what follows. Let T_i be a metric space with the distance d_i and let $\{S_{i,t}\}_{t \in T_i}$, $i = 1, 2$, be a collection of subsets in X . We say that $\bar{x}$ is an *extended local extremal point* of the system $\{S_{1,t}, S_{2,t}\}$ at $(\bar{t}_1, \bar{t}_2)$ provided that $\bar{x} \in S_{1,\bar{t}_1} \cap S_{2,\bar{t}_2}$ and there exists a neighborhood U of $\bar{x}$ such that for every $\varepsilon > 0$ there is $(t_1, t_2) \in T_1 \times T_2$ with

$$d_i(t_i, \bar{t}_i) \leq \varepsilon, \quad \text{dist}(\bar{x}; S_{i,t_i}) \leq \varepsilon \text{ for } i = 1, 2, \text{ and } S_{1,t_1} \cap S_{2,t_2} \cap U = \emptyset.$$

The following versions of extended extremal principle hold:

Versions of the Extremal Principle for Moving Sets [11, 12]. *Suppose that X is an Asplund space, that $\bar{x}$ is an extended local extremal point of the system $\{S_{1,t}, S_{2,t}\}$ at $(\bar{t}_1, \bar{t}_2)$, and that the sets $S_{i,t}$ are locally closed around $\bar{x}$ for all $t_i \in T_i$ around $\bar{t}_i$, $i = 1, 2$. Then for every $\varepsilon > 0$ there are elements*

$$t_i \in T_i, \quad x_i \in S_{i,t_i}, \quad \text{and} \quad x_i^* \in \widehat{N}(x_i; S_{i,t_i}) + \varepsilon B^*, \quad i = 1, 2, \quad (3.1)$$

satisfying the conditions

$$d_i(t_i, \bar{t}_i) \leq \varepsilon, \quad \|x_i - \bar{x}\| \leq \varepsilon, \quad \|x_1^*\| + \|x_2^*\| = 1, \quad \text{and} \quad x_1^* + x_2^* = 0. \quad (3.2)$$

If in addition one of families $\{S_{i,t}\}_{t \in T_i}$ is ESNC at the corresponding point $(\bar{t}_i, \bar{x})$ as $i = 1, 2$, then there is a dual vector $x^ \in X^*$ satisfying*

$$0 \neq x^* \in \overline{N}(\bar{x}; S_{1,\bar{t}_1}) \cap (-\overline{N}(\bar{x}; S_{2,\bar{t}_2})). \quad (3.3)$$

The next result establishes a new version of the exact extended extremal principle (3.3) for moving sets that takes into account the *product structure* of the space in question.

Theorem 3.2 (exact extremal principle for moving sets in product spaces). *Let $X_1, \dots, X_p$ be Asplund spaces with $X := X_1 \times \dots \times X_p$. Suppose that $\bar{x}$ is an extended local extremal point of the system $\{S_{1,t}, S_{2,t}\}$ at $(\bar{t}_1, \bar{t}_2)$ and that the sets $S_{i,t}$ are locally closed around $\bar{x}$ for all $t \in T_i$ near $\bar{t}_i$, $i = 1, 2$. Given $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{I} := \{1, \dots, p\}$ with $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}$, assume that one of the families $\{S_{1,t}\}_{t \in \mathcal{I}_1}$ and $\{S_{2,t}\}_{t \in \mathcal{I}_2}$ is PESNC at $(\bar{t}_1, \bar{x})$ with respect to $\{X_i \mid i \in \mathcal{I}_1\}$ and the other is strongly PESNC at $(\bar{t}_1, \bar{x})$ with respect to $\{X_i \mid i \in \mathcal{I}_2\}$. Then there is $x^* \in X^*$ satisfying relationships (3.3) of the exact extremal principle.*

Proof. Take an arbitrary sequence $\varepsilon_k \downarrow 0$ and find, by the approximate version (3.1) and (3.2) of the extended extremal principle, sequences $\mu_k \xrightarrow{\mathcal{I}_1} \bar{t}_1$, $\eta_k \xrightarrow{\mathcal{I}_2} \bar{t}_2$, $u_k \rightarrow \bar{x}$, $v_k \rightarrow \bar{x}$ with $(u_k, v_k) \in S_{1,\mu_k} \times S_{2,\eta_k}$, and $(u_k^*, v_k^*) \in \hat{N}(u_k; S_{1,\mu_k}) \times \hat{N}(v_k; S_{2,\eta_k})$ satisfying

$$\|u_k^* + v_k^*\| \leq 2\varepsilon_k \text{ and } 1/2 - \varepsilon_k \leq \|u_k^*\|, \|v_k^*\| \leq 1/2 + \varepsilon_k. \quad (3.4)$$

Since X is an Asplund space, its dual unit ball is *sequentially compact*; see, e.g., [17]. Therefore we can find w^* -convergent subsequences of $\{u_k^*\}$ and $\{v_k^*\}$. Without loss of generality, assume that $(u_k^*, v_k^*) \xrightarrow{w^*} (u^*, v^*)$ as $k \rightarrow \infty$. Now passing to the limit in the first relationship of (3.4) as $k \rightarrow \infty$ and taking into account the lower semicontinuity of the norm functions in the sequential weak* topology of X^* , we get $u^* + v^* = 0$. Thus letting $x^* := u^* = -v^*$ and using the definition of the extended normal cone (2.2), we arrive at

$$x^* = u^* \in \overline{N}(\bar{x}; S_{1,\bar{t}_1}) \cap (-\overline{N}(\bar{x}; S_{2,\bar{t}_2})).$$

It remains to show that $u^* \neq 0$ under the PESNC assumptions imposed in the theorem. To proceed, suppose the contrary, i.e., that $u^* = v^* = 0$. Without loss of generality, assume that the family $\{S_{1,t}\}_{t \in \mathcal{I}_1}$ is PESNC at $\{S_{1,t}\}_{t \in \mathcal{I}_1}$ while the family $\{S_{2,t}\}_{t \in \mathcal{I}_2}$ is *strongly* PESNC with respect to $\{X_i \mid i \in \mathcal{I}_2\}$. Taking into account that $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}$, we represent u_k^* and v_k^* componentwisely in the product structure:

$$u_k^* = ((u_{i,k}^* \mid i \in \mathcal{I}_1), (u_{i,k}^* \mid i \in \mathcal{I}_2)), \quad v_k^* = ((v_{i,k}^* \mid i \in \mathcal{I}_1), (v_{i,k}^* \mid i \in \mathcal{I}_2)). \quad (3.5)$$

The strong PESNC property of $\{S_{2,t}\}_{t \in \mathcal{I}_2}$ with respect to $\{X_i \mid i \in \mathcal{I}_2\}$ yields that $\|v_{i,k}^*\| \rightarrow 0$ as $k \rightarrow \infty$ for all $i \in \mathcal{I}_2$. By the first relationship of (3.3) we have that $\|u_{i,k}^*\| \rightarrow 0$ as $k \rightarrow \infty$ for all $i \in \mathcal{I}_2$. Then using the assumed PESNC property of the family $\{S_{1,t}\}_{t \in \mathcal{I}_1}$, we get that $\|u_{i,k}^*\| \rightarrow 0$ as $k \rightarrow \infty$ for all $i \in \mathcal{I}_1$. Since $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}$ by the assumption of the theorem and the first relationship in (3.4), this gives that

$$\|u_k^*\|, \|v_k^*\| \rightarrow 0 \text{ as } k \rightarrow \infty$$

for the whole sequences $\{u_k^*\}$ and $\{v_k^*\}$ in (3.5), which clearly contradicts the second relationship in (3.4) and thus completes the proof of the theorem. $\triangle$

4 Extended Calculus of Generalized Differentiation

In this section we derive extensive *calculus rules* dealing with the extended generalized differential constructions for *parameter-dependent* objects introduced in Section 2. Our

geometric approach is similar to that developed in [11, 13] for parameter-independent sets, set-valued mappings, and extended-real-valued functions. We start with the *intersection rule* for the extended normal cone (2.2) that requires new *normal qualification conditions* (in both finite and infinite dimensions) together with the PESNC conditions in infinite dimensions introduced in Section 2.

The following basic qualification conditions extend those from [13] and [10, Subsection 3.1.1] to the case of parameter-dependent systems of sets. Since in this paper we apply these conditions only to locally closed sets in *Asplund spaces*, we avoid sequences $\varepsilon_k \downarrow 0$ in their limiting formulations and representations.

Definition 4.1 (extended qualification conditions for parametric families of sets). *Let $\{\Omega_{j,t}\}_{t \in T}$, $j = 1, 2$, be two parametric families of subsets in X , and let $\bar{x} \in \Omega_{1,\bar{t}} \cap \Omega_{2,\bar{t}}$. We say that:*

(i) *The system $\{\Omega_{j,t}\}_{t \in T}$, $j = 1, 2$, satisfies the EXTENDED NORMAL QUALIFICATION CONDITION at $(\bar{t}, \bar{x})$ if*

$$\bar{N}(\bar{x}; \Omega_{1,\bar{t}}) \cap (-\bar{N}(\bar{x}; \Omega_{2,\bar{t}})) = \{0\}. \quad (4.1)$$

(ii) *The system $\{\Omega_{j,t}\}_{t \in T}$, $j = 1, 2$, satisfies the EXTENDED LIMITING QUALIFICATION CONDITION at $(\bar{t}, \bar{x})$ if for any sequences $t_k \rightarrow \bar{t}$, $x_{j,k} \in \Omega_{j,t_k}$, and $x_{j,k}^* \xrightarrow{w^*} x_j^*$ such that $x_{j,k}^* \in \hat{N}(x_{j,k}; \Omega_{j,t_k})$ as $j = 1, 2$ and $k \rightarrow \infty$ one has*

$$\|x_{1,k}^* + x_{2,k}^*\| \rightarrow 0 \implies x_1^* = x_2^* = 0. \quad (4.2)$$

It follows from the afore-mentioned representation of the extended normal cone (2.2) to locally closed sets in *Asplund spaces* that the extended normal qualification condition (4.1) can be equivalently presented in the limiting form: for any sequences $t_k \rightarrow \bar{t}$, $x_{j,k} \in \Omega_{j,t_k}$, and $x_{j,k}^* \xrightarrow{w^*} x_j^*$ such that $x_{j,k}^* \in \hat{N}(x_{j,k}; \Omega_{j,t_k})$ as $j = 1, 2$ and $k \rightarrow \infty$ one has

$$x_{1,k}^* + x_{2,k}^* \xrightarrow{w^*} 0 \implies x_1^* = x_2^* = 0,$$

which shows that (4.1) is more restrictive than (4.2), although the former is expressed in the more convenient pointbased form. In what follows we will see significant advantages of (4.2) in comparison with (4.1) in the case of extended *coderivative calculus* for mappings between infinite dimensions, where the strong convergence in (4.2) leads to a better *point-based* qualification condition in term of *mixed coderivatives* generated by (4.2) in spaces with natural *product structures* of graphical sets.

The next result gives the *basic intersection rule* for the extended normal cone (2.2) in products of *Asplund spaces*.

Theorem 4.2 (basic intersection rule for extended normals in product spaces). *Let $X_1, \dots, X_p$ be Asplund spaces, and let $\{\Omega_{j,t}\}_{t \in T}$, $j = 1, 2$, be two parametric families of subsets in $X_1 \times \dots \times X_p$ with $\bar{x} \in \Omega_{1,\bar{t}} \cap \Omega_{2,\bar{t}}$ such that each $\Omega_{j,t}$ is locally closed around $\bar{x}$ for all $t \in T$ near $\bar{t}$. Given $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{I} := \{1, \dots, p\}$ with $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}$, assume that:*

(i) *One of the families $\{\Omega_{1,t}\}_{t \in T}$ and $\{\Omega_{2,t}\}_{t \in T}$ is PESNC with respect to $\{X_i \mid i \in \mathcal{I}_1\}$ at $(\bar{t}, \bar{x})$ while the other is strongly PESNC with respect to $\{X_i \mid i \in \mathcal{I}_2\}$ at $(\bar{t}, \bar{x})$.*

(ii) The system $\{\Omega_{j,t}\}_{t \in T}$, $j = 1, 2$, satisfies the extended limiting qualification condition (4.2) at $(\bar{t}, \bar{x})$.

Then the following intersection rule holds:

$$\overline{N}(\bar{x}; \Omega_{1,\bar{t}} \cap \Omega_{2,\bar{t}}) \subset \overline{N}(\bar{x}; \Omega_{1,\bar{t}}) + \overline{N}(\bar{x}; \Omega_{2,\bar{t}}). \quad (4.3)$$

Proof. Pick an arbitrary element $x^* \in \overline{N}(\bar{x}; \Omega_{1,\bar{t}} \cap \Omega_{2,\bar{t}})$. Since the intersection set $\Omega_{1,t} \cap \Omega_{2,t}$ is locally closed around $\bar{x}$ for all $t \in T$ near $\bar{t}$, we use the afore-mentioned representation of the extended normal cone (2.2) to it in Asplund spaces and find sequence $(t_k, x_k) \rightarrow (\bar{t}, \bar{x})$ and $x_k^* \xrightarrow{w^*} x^*$ such that

$$x_k \in \Omega_{1,t_k} \cap \Omega_{2,t_k} \text{ and } x_k^* \in \widehat{N}(x_k; \Omega_{1,t_k} \cap \Omega_{2,t_k}), \quad k \in \mathbb{N}. \quad (4.4)$$

Taking now an arbitrary sequence $\varepsilon_k \downarrow 0$ and applying the fuzzy intersection rule from Lemma 3.1 to (4.4), we get elements

$$(u_k, v_k) \in \Omega_{1,t_k} \times \Omega_{2,t_k}, \quad \lambda_k \geq 0, \quad (u_k^*, v_k^*) \in \widehat{N}(u_k; \Omega_{1,t_k}) \times \widehat{N}(v_k; \Omega_{2,t_k}) \quad (4.5)$$

satisfying the relationships $\|u_k - x_k\| \leq \varepsilon_k$, $\|v_k - x_k\| \leq \varepsilon_k$, and

$$\|u_k^* + v_k^* - \lambda_k x_k^*\| \leq 2\varepsilon_k, \quad 1 - \varepsilon_k \leq \max\{\lambda_k, \|u_k^*\|\} \leq 1 + \varepsilon_k \quad (4.6)$$

for all $k \in \mathbb{N}$. By the classical uniform boundedness principle, the sequence $\{x_k^*\}$ is bounded in X^* , and so are the sequences $\{u_k^*\}$ and $\{v_k^*\}$ due to (4.6). Since the unit ball in duals to Asplund spaces is *weak* sequentially compact*, the sequences $\{u_k^*\}$ and $\{v_k^*\}$ contains w^* -convergent subsequences. Without loss of generality, suppose that $u_k^* \xrightarrow{w^*} u^* \in X^*$, $v_k^* \xrightarrow{w^*} v^* \in X^*$, and $\lambda_k \rightarrow \lambda \geq 0$ for all $k \rightarrow \infty$. Then we have by passing to the limit in (4.5) and (4.6) as $k \rightarrow \infty$ and using the extended normal cone definition (2.2) and the lower semicontinuity of the norm function in the sequential weak* topology of X^* that

$$u^* \in \overline{N}(\bar{x}; \Omega_{1,\bar{t}}), \quad v^* \in \overline{N}(\bar{x}; \Omega_{2,\bar{t}}), \quad \text{and } \lambda x^* = u^* + v^*.$$

The latter immediately implies the required relationship (4.3) provided that $\lambda \neq 0$.

Suppose on the contrary that $\lambda = 0$ and then arrive at contradiction with the qualification and PESNC conditions assumed. Indeed, in this case (4.6) implies that $\|u_k^* + v_k^*\| \rightarrow 0$ as $k \rightarrow \infty$, and hence $u^* = v^* = 0$ by the extended limiting qualification condition (4.2). Thus we have the componentwise limiting relationships

$$u_k^* = (u_{1,k}^*, \dots, u_{p,k}^*) \xrightarrow{w^*} 0 \quad \text{and} \quad v_k^* = (v_{1,k}^*, \dots, v_{p,k}^*) \xrightarrow{w^*} 0 \quad \text{as } k \rightarrow \infty. \quad (4.7)$$

By the strong PESNC condition imposed on $\{\Omega_{2,t}\}_{t \in T}$ in the theorem, we have $\|v_{i,k}^*\| \rightarrow 0$ for all $i \in \mathcal{I}_2$. By the first relationship in (4.6) this implies that $\|u_{i,k}^*\| \rightarrow 0$ for all $i \in \mathcal{I}_2$. Consequently, the assumed PESNC property of $\{\Omega_{1,t}\}_{t \in T}$ yields that $\|u_{i,k}^*\| \rightarrow 0$ for all $i \in \mathcal{I}_1$. Therefore $\|u_k^*\| \rightarrow 0$ as $k \rightarrow \infty$ for the whole sequence $\{u_k^*\}$ in (4.7). This clearly contradicts the second relationship in (4.6) and completes the proof of the theorem. $\triangle$

If the product structure is not imposed on X , the obtained Theorem 4.2 admits the following efficient simplification, which however is less precise in comparison with the full statement of the theorem.

Corollary 4.3 (intersection rule for extended normals with no product structure). *Let $\{\Omega_{j,t}\}_{t \in T}$, $j = 1, 2$, be two parametric families of subsets in the Asplund space X such that $\bar{x} \in \Omega_{1,\bar{t}} \cap \Omega_{2,\bar{t}}$ and each of the sets $\Omega_{j,t}$ is locally closed around $\bar{x}$ for all $t \in T$ near $\bar{t}$. Suppose that this system satisfies the extended limiting qualification condition (4.2) at $(\bar{t}, \bar{x})$ and that one of these families is ESNC at the reference point. Then the intersection rule (4.3) holds for extended normals.*

Proof. Follows from Theorem 4.2 with $p = 1$ and $\mathcal{I}_1 = \{1\}$. $\triangle$

To proceed, we need to extend the inner semicontinuity and inner semicompactness notions from [10, Definition 1.63] to the case of parameter-dependent families of sets.

Definition 4.4 (extended inner semicontinuity and inner semicompactness of moving sets). *Let $S_t: X \rightrightarrows Y$, $t \in T$, be a parametric family of set-valued mappings. We say that:*

- (i) *The family $\{S_t\}_{t \in T}$ is EXTENDEDLY INNER SEMICOMPACT at $(\bar{t}, \bar{x})$ if for any sequence $(t_k, x_k) \rightarrow (\bar{t}, \bar{x})$ with $S_{t_k}(x_k) \neq \emptyset$ there is a sequence $y_k \in S_{t_k}(x_k)$ that contains a subsequence converging to some $\bar{y} \in S_{\bar{t}}(\bar{x})$ as $k \rightarrow \infty$.*
- (ii) *The family $\{S_t\}_{t \in T}$ is EXTENDEDLY INNER SEMICONTINUOUS at $(\bar{t}, \bar{x}, \bar{y})$ for some fixed $\bar{y} \in S_{\bar{t}}(\bar{x})$ if for any sequence $(t_k, x_k) \rightarrow (\bar{t}, \bar{x})$ with $S_{t_k}(x_k) \neq \emptyset$ there is a sequence $y_k \in S_{t_k}$ that contains a subsequence converging to $\bar{y}$ as $k \rightarrow \infty$.*

Observe that extended inner semicompactness is an essentially less restrictive assumption in comparison with extended inner semicontinuity; in particular, the former automatically holds for any family $\{S_t\}_{t \in T}$ of set-valued mappings with (locally) *uniformly bounded* values in finite-dimensional spaces. On the other hand, imposing extended inner semicontinuity allows us to get better (more precise) calculus rules.

The next calculus result for extended normals gives two independent versions of the *summation rule* under the extended inner semicompactness and inner semicontinuity assumptions, respectively. Note that we do *not impose* any qualification and/or ESNC conditions as in the intersection rule of Theorem 4.2.

Theorem 4.5 (summation rule for extended normals). *Let $\{\Omega_{j,t}\}_{t \in T}$, $j = 1, 2$, be two parametric families of subsets in the Asplund space X such that $\bar{x} \in \Omega_{1,\bar{t}} + \Omega_{2,\bar{t}}$ and the sets Ω_t are locally closed around $\bar{x}$ for all $t \in T$ near $\bar{t}$. Consider the family of set-valued mappings $S_t: X \rightrightarrows X \times X$ defined by*

$$S_t(x) := \{(u, v) \mid u + v = x, u \in \Omega_{1,t}, v \in \Omega_{2,t}\}$$

and assume that this family is extendedly inner semicompact at $(\bar{t}, \bar{x})$. Then

$$\overline{N}(\bar{x}; \Omega_{1,\bar{t}} + \Omega_{2,\bar{t}}) \subset \bigcup_{(u,v) \in S_{\bar{t}}(\bar{x})} \overline{N}(u; \Omega_{1,\bar{t}}) \cap \overline{N}(v; \Omega_{2,\bar{t}}).$$

If furthermore $\{S_t\}_{t \in T}$ is assumed to be extendedly inner semicontinuous at $(\bar{t}, \bar{x}, \bar{u}, \bar{v})$ for some $(\bar{u}, \bar{v}) \in S_{\bar{t}}(\bar{x})$, then

$$\overline{N}(\bar{x}; \Omega_{1,\bar{t}} + \Omega_{2,\bar{t}}) \subset \overline{N}(\bar{u}; \Omega_{1,\bar{t}}) \cap \overline{N}(\bar{v}; \Omega_{2,\bar{t}}).$$

Proof. Take $x^* \in \overline{N}(\bar{x}; \Omega_{1,\bar{t}} + \Omega_{2,\bar{t}})$ and find, by the representation of extended normals in Asplund spaces, sequences $(t_k, x_k) \rightarrow (\bar{t}, \bar{x})$ with $x_k \in \Omega_{1,t_k} + \Omega_{2,t_k}$ and $x_k^* \xrightarrow{w^*} x^*$ with $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega_{1,t_k} + \Omega_{2,t_k})$. If the family $\{S_t\}_{t \in \mathcal{T}}$ is assumed to be extendedly inner semi-compact at $(\bar{t}, \bar{x})$, then there exists a sequence $(u_k, v_k) \in S_{t_k}(x_k)$ that contains a subsequence converging to some $(u, v) \in S_{\bar{t}}(\bar{x})$. Define

$$\widetilde{\Omega}_{1,t} := \Omega_{1,t} \times X, \quad \widetilde{\Omega}_{2,t} := X \times \Omega_{2,t} \quad \text{whenever } t \in \mathcal{T}. \quad (4.8)$$

Then we can easily check that

$$(x_k^*, x_k^*) \in \widehat{N}_{\varepsilon_k}((u_k, v_k); \widetilde{\Omega}_{1,t_k} \cap \widetilde{\Omega}_{2,t_k}), \quad k \in \mathbb{N},$$

and, by passing to the limit as $k \rightarrow \infty$,

$$(x^*, x^*) \in \overline{N}((u, v); \widetilde{\Omega}_{1,\bar{t}} \cap \widetilde{\Omega}_{2,\bar{t}}). \quad (4.9)$$

Apply the intersection rule of Theorem 4.2 to (4.9) checking that the qualification and ESNC conditions imposed therein are *automatically fulfilled* for the set systems (4.8). Thus there exist extended normals

$$(u^*, 0) \in \overline{N}((u, v); \widetilde{\Omega}_{1,\bar{t}}) \quad \text{and} \quad (0, v^*) \in \overline{N}((u, v); \widetilde{\Omega}_{2,\bar{t}})$$

satisfying $(x^*, x^*) = (u^*, 0) + (0, v^*)$, which gives $u^* = v^* = x^*$. Observing finally that $u^* \in \overline{N}(u; \Omega_{1,\bar{t}})$ and $v^* \in \overline{N}(v; \Omega_{2,\bar{t}})$, we arrive at $x^* \in \overline{N}(\bar{u}; \Omega_{1,\bar{t}}) \cap \overline{N}(\bar{v}; \Omega_{2,\bar{t}})$ and complete the proof of the theorem under the extended inner semicontinuity assumption. The proof in the case of extended inner semicontinuity is similar to the above. $\triangle$

The next calculus result ensures an efficient representation of extended normals to *inverse images/preimages*

$$F^{-1}(\Theta) := \{x \in X \mid F(x) \cap \Theta \neq \emptyset\} \quad (4.10)$$

of sets $\Theta \subset Y$ under set-valued mappings $F: X \rightrightarrows Y$ via their *extended normal coderivatives* (2.4). Such moving sets are particularly important in applications to *parametric optimization*. Observe that the smaller *extended mixed coderivatives* (2.6) are used to formulate the refined qualification condition in what follows. For brevity, we present this and subsequent calculus rules only in the case of extended semicontinuity; the case of extended inner semicontinuity can be considered similarly.

Theorem 4.6 (extended normals to inverse images). *Let X, Y be Asplund spaces, let $\{F_t\}_{t \in \mathcal{T}}$ is a parametric family of set-valued mappings from X into Y , and let $\{\Theta_t\}_{t \in \mathcal{T}}$ be a family of subsets in Y . Given $\bar{x} \in F_{\bar{t}}^{-1}(\Theta_{\bar{t}})$, suppose that the parametric family of set-valued mappings*

$$x \mapsto F_t(x) \cap \Theta_t, \quad t \in \mathcal{T},$$

is extendedly inner semicompact at $(\bar{t}, \bar{x})$ and that for each $\bar{y} \in F_{\bar{t}}(\bar{x}) \cap \Theta_{\bar{t}}$ the set Θ_t is locally closed around $\bar{y}$, $\text{gph } F_t$ is locally closed around $(\bar{x}, \bar{t})$ whenever t is near $\bar{t}$, and the following assumptions hold:

- (i) Either $\{F_t^{-1}\}_{t \in \mathcal{T}}$ is PESNC at $(\bar{t}, \bar{y}, \bar{x})$, or $\{\Theta_t\}_{t \in \mathcal{T}}$ is ESNC at $(\bar{t}, \bar{y})$.
- (ii) The system $\{F_t, \Theta_t\}_{t \in \mathcal{T}}$ satisfies the qualification condition

$$\overline{D}_M^* F_{\bar{t}}^{-1}(\bar{y}, \bar{x})(0) \cap (-\overline{N}(\bar{y}; \Theta_{\bar{t}})) = \{0\}. \quad (4.11)$$

Then one has the inclusion

$$\overline{N}(\bar{x}; F_{\bar{t}}^{-1}(\Theta_{\bar{t}})) \subset \bigcup \left\{ \overline{D}^* F_{\bar{t}}(\bar{x}, \bar{y})(y^*) \mid y^* \in \overline{N}(\bar{y}; \Theta_{\bar{t}}), \bar{y} \in F_{\bar{t}}(\bar{x}) \cap \Theta_{\bar{t}} \right\}. \quad (4.12)$$

Proof. Take $x^* \in \overline{N}(\bar{x}; F_{\bar{t}}^{-1}(\Theta_{\bar{t}}))$ and, by definition (2.2) of extended normals to the set under consideration, find sequences $\varepsilon_k \downarrow 0$, $(t_k, x_k) \rightarrow (\bar{t}, \bar{x})$ with $x_k \in F_{t_k}^{-1}(\Theta_{t_k})$, and $x_k^* \xrightarrow{w^*} x^*$ with $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; F_{t_k}^{-1}(\Theta_{t_k}))$, $k \in \mathbb{N}$. (Note that $F_{\bar{t}}^{-1}(\Theta)$ may not be locally closed under the assumptions made; so we need to use $\varepsilon_k \downarrow 0$ even in the Asplund space setting.) By the inner semicontinuity assumption of the theorem, there is a sequence of $y_k \in F_{t_k}(x_k) \cap \Theta_{t_k}$ that contains a subsequence converging to some $\bar{y} \in F_{\bar{t}}(\bar{x}) \cap \Theta_{\bar{t}}$. Without loss of generality, assume that $y_k \rightarrow \bar{y}$ for all $k \rightarrow \infty$. Now construct the two families of locally closed (around the reference points) subsets of $X \times Y$ by

$$\Omega_{1,t} := \text{gph } F_t \text{ and } \Omega_{2,t} := X \times \Theta_t, \quad t \in \mathcal{T}. \quad (4.13)$$

It is easy to see that $(x_k, y_k) \in \Omega_{1,t_k} \cap \Omega_{2,t_k}$ for all $k \in \mathbb{N}$. Furthermore, observe from the construction of x_k^* and the structures of (4.13) that

$$(x_k^*, 0) \in \widehat{N}_{\varepsilon_k}((x_k, y_k); \Omega_{1,t_k} \cap \Omega_{2,t_k}) \text{ for all } k \in \mathbb{N}.$$

Thus, by passing to the limit as $k \rightarrow \infty$, we get

$$(x^*, 0) \in N((\bar{x}, \bar{y}); \Omega_{1,\bar{t}} \cap \Omega_{2,\bar{t}}). \quad (4.14)$$

It follows from the structure of the set collections in (4.13) that all the assumptions of Theorem 4.2 are satisfied for the sets under consideration. Thus (4.3) applied to (4.14) ensures the existence of $(x_1^*, -y_1^*) \in \overline{N}((\bar{x}, \bar{y}); \text{gph } F_{\bar{t}})$ and $y_2^* \in \overline{N}(\bar{y}; \Theta_{\bar{t}})$ such that

$$(x^*, 0) = (x_1^*, -y_1^*) + (0, y_2^*),$$

which implies that $x^* = x_1^*$ and $y_1^* = y_2^*$ and completes the proof of the theorem. $\triangle$

Theorem 4.6 allows us, in particular, to obtain useful representations of extended normals to parametric families of sets given by the *inequality and/or equality constraints*, which are especially important in applications to *parametric mathematical programming*. In the next result we consider the two types of such sets given separately by

$$\Omega_t := \{x \in X \mid \varphi_t(x) \leq 0\} \text{ and } \Omega_t := \{x \in X \mid \varphi_t(x) = 0\}, \quad (4.15)$$

where $\{\varphi_t\}_{t \in \mathcal{T}}$ is a parametric family of extended-real-valued functions. The notation $\mathcal{R}_+ C$ in what follows signifies as usual the set $\{\alpha c \mid \alpha \geq 0, c \in C\}$. Recall that $\bar{\partial}\varphi_{\bar{t}}(\bar{x})$ and $\bar{\partial}^\infty \varphi_{\bar{t}}(\bar{x})$ stands, respectively, for the *extended subdifferential* and *extended singular subdifferential* of φ at $(\bar{t}, \bar{x})$ defined in (2.7).

Corollary 4.7 (extended normals to parametric inequality and equality constraints). *In the notation above, the following assertions hold:*

(i) *Let Ω_t be given by the inequality constraint in (4.15), and let $(\bar{t}, \bar{x}) \in T \times X$ be such that $\varphi_{\bar{t}}(\bar{x}) = 0$. Assume that X is Asplund, that φ_t is l.s.c. around $\bar{x}$ for all $t \in T$ near $\bar{t}$, that the function of two variables $(t, x) \mapsto \varphi_t(x)$ is l.s.c. at $(\bar{t}, \bar{x})$, and that the qualification condition $0 \notin \bar{\partial}\varphi_{\bar{t}}(\bar{x})$ is satisfied. Then*

$$\bar{N}(\bar{x}; \Omega_{\bar{t}}) \subset \bar{\partial}^\infty \varphi_{\bar{t}}(\bar{x}) \cup \mathbb{R}^+ \bar{\partial} \varphi_{\bar{t}}(\bar{x}).$$

(ii) *Let Ω_t be given by the equality constraint in (4.15), and let $(\bar{t}, \bar{x}) \in T \times \Omega_{\bar{t}}$. Assume that X is Asplund, that φ_t is continuous around $\bar{x}$ for all $t \in T$ near $\bar{t}$, that the function of two variables $(t, x) \mapsto \varphi_t(x)$ is continuous at $(\bar{t}, \bar{x})$, and that the qualification condition $0 \notin \bar{\partial}\varphi_{\bar{t}}(\bar{x}) \cup \bar{\partial}(-\varphi_{\bar{t}})(\bar{x})$ is satisfied. Then*

$$\bar{N}(\bar{x}; \Omega_{\bar{t}}) \subset \bar{\partial}^\infty \varphi_{\bar{t}}(\bar{x}) \cup \bar{\partial}^\infty (-\varphi_{\bar{t}})(\bar{x}) \cup \mathbb{R}^+ \bar{\partial} \varphi_{\bar{t}}(\bar{x}) \cup \mathbb{R}^+ \bar{\partial} (-\varphi_{\bar{t}})(\bar{x}).$$

Proof. Assertion (i) follows from Theorem 4.6 with $F_t = E_{\varphi_t}$ and $\Theta_t = (-\infty, 0]$. In this case, the qualification condition (4.11) and inclusion (4.12) of the theorem reduce to the corresponding statements in (i) due to relationships (2.7). The extended inner semicompactness and local closedness assumptions of the theorem obviously correspond to those imposed in the corollary for the case under consideration.

Assertion (ii) follows from Theorem 4.6 with $F_t = \varphi_t$ and $\Theta_t = \{0\}$. To check this, observe the extended coderivative-subdifferential relationships

$$\bar{D}^* \varphi_{\bar{t}}(\bar{x})(1) = \bar{\partial} \varphi_{\bar{t}}(\bar{x}) \text{ and } \bar{D}^* \varphi_{\bar{t}}(\bar{x})(-1) = \bar{\partial}(-\varphi_{\bar{t}})(\bar{x}),$$

which are proved for continuous parametric functions φ_t in general Banach spaces similarly to [10, Theorem 1.80], and the one with the extended singular subdifferential

$$\bar{D}^* \varphi_{\bar{t}}(\bar{x})(0) = \bar{\partial}^\infty \varphi_{\bar{t}}(\bar{x}) \cup \bar{\partial}^\infty (-\varphi_{\bar{t}})(\bar{x})$$

justified in Asplund spaces similarly to [10, Theorem 2.40]. $\triangle$

The next theorem presents general *sum rules* for both extended normal and mixed coderivatives of set-valued mappings. Observe that the qualification condition (4.17) in both cases is formulated in terms of the extended *mixed* coderivative; it actually follows from the extended limiting qualification condition from Definition 4.1 for parametric systems of sets.

Given two parametric families of set-valued mappings $F_{j,t}: X \rightrightarrows Y$, $t \in T$, $j = 1, 2$, define the auxiliary family $S_t: X \times Y \rightrightarrows Y \times Y$, $t \in T$, by

$$S_t(x, y) := \{(y_1, y_2) \in Y^2 \mid y_1 \in F_{1,t}(x), y_2 \in F_{2,t}(x), y_1 + y_2 = y\}. \quad (4.16)$$

Theorem 4.8 (sum rules for extended coderivatives). *Let $F_{j,t}: X \rightrightarrows Y$, $t \in T$, $j = 1, 2$, be two parametric families of set-valued mappings between Asplund spaces X and Y , and let $(\bar{x}, \bar{y}) \in \text{gph}(F_{1,\bar{t}} + F_{2,\bar{t}})$. Assume that the family $\{S_t\}_{t \in T}$ from (4.16) is extendedly inner semicompact at $(\bar{t}, \bar{x}, \bar{y})$ and that for every $(\bar{y}_1, \bar{y}_2) \in S_{\bar{t}}(\bar{x}, \bar{y})$ the graphs of F_1 and F_2 are locally closed around $(\bar{x}, \bar{y}_1)$ and $(\bar{x}, \bar{y}_2)$, respectively, for all $t \in T$ near $\bar{t}$ and that*

- (i) Either $\{F_{1,t}\}_{t \in \mathcal{T}}$ is PESNC at $(\bar{t}, \bar{x}, \bar{y}_1)$, or $\{F_{2,t}\}_{t \in \mathcal{T}}$ is PESNC at $(\bar{t}, \bar{x}, \bar{y}_2)$.
(ii) The following qualification condition holds:

$$\overline{D}_M^* F_{1,\bar{t}}(\bar{x}, \bar{y}_1)(0) \cap (-\overline{D}_M^* F_{2,\bar{t}}(\bar{x}, \bar{y}_2)(0)) = \{0\}. \quad (4.17)$$

Then for all $y^* \in Y^*$ one has the inclusions

$$\overline{D}^*(F_{1,\bar{t}} + F_{2,\bar{t}})(\bar{x}, \bar{y})(y^*) \subset \bigcup_{(\bar{y}_1, \bar{y}_2) \in S_{\bar{t}}(\bar{x}, \bar{y})} [\overline{D}^* F_{1,\bar{t}}(\bar{x}, \bar{y}_1)(y^*) + \overline{D}^* F_{2,\bar{t}}(\bar{x}, \bar{y}_2)(y^*)], \quad (4.18)$$

$$\overline{D}_M^*(F_{1,\bar{t}} + F_{2,\bar{t}})(\bar{x}, \bar{y})(y^*) \subset \bigcup_{(\bar{y}_1, \bar{y}_2) \in S_{\bar{t}}(\bar{x}, \bar{y})} [\overline{D}_M^* F_{1,\bar{t}}(\bar{x}, \bar{y}_1)(y^*) + \overline{D}_M^* F_{2,\bar{t}}(\bar{x}, \bar{y}_2)(y^*)]. \quad (4.19)$$

Proof. First we prove (4.18). Take $(x^*, y^*) \in X^* \times Y^*$ with $x^* \in \overline{D}^*(F_{1,\bar{t}} + F_{2,\bar{t}})(\bar{x}, \bar{y})(y^*)$ and find sequences $\varepsilon_k \downarrow 0$, $(t_k, x_k, y_k) \rightarrow (\bar{t}, \bar{x}, \bar{y})$ as $k \rightarrow \infty$ with $y_k \in (F_{1,t_k} + F_{2,t_k})(x_k)$, and $(x_k^*, y_k^*) \xrightarrow{w^*} (x^*, y^*)$ with $(x_k^*, -y_k^*) \in \widehat{N}_{\varepsilon_k}((x_k, y_k); \text{gph}(F_{1,t_k} + F_{2,t_k}))$. Due to the extended inner compactness assumption on (4.16), there is a sequence of $(y_{1k}, y_{2k}) \in S_{t_k}(x_k, y_k)$ that contains a subsequence converging to some point $(\bar{y}_1, \bar{y}_2) \in S_{\bar{t}}(\bar{x}, \bar{y})$. Without loss of generality, assume that $(y_{1k}, y_{2k}) \rightarrow (\bar{y}_1, \bar{y}_2)$ as $k \rightarrow \infty$.

Define the two parametric families of sets in $X \times Y \times Y$ by

$$\Omega_{j,t} := \{(x, y_1, y_2) \in X \times Y \times Y \mid (x, y_j) \in \text{gph } F_{j,t}\}, \quad j = 1, 2. \quad (4.20)$$

Both sets $\Omega_{j,t}$ are locally closed-graph around $(\bar{x}, \bar{y}_1, \bar{y}_2)$ for each $t \in \mathcal{T}$ near $\bar{t}$ by the assumptions imposed on $F_{j,t}$. It is easy to see that $(x_k, y_{1k}, y_{2k}) \in \Omega_{1,t_k} \cap \Omega_{2,t_k}$ and

$$(x_k^*, -y_k^*, -y_k^*) \in \widehat{N}_{\varepsilon_k}((x_k, y_{1k}, y_{2k}); \Omega_{1,t_k} \cap \Omega_{2,t_k}), \quad k \in \mathbb{N}, \quad (4.21)$$

so by passing to the limit as $k \rightarrow \infty$ we have

$$(x^*, -y^*, -y^*) \in \overline{N}((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_{1,\bar{t}} \cap \Omega_{2,\bar{t}}). \quad (4.22)$$

Apply the intersection rule of Theorem 4.2 to (4.22) taking into account the structures of the sets $\Omega_{j,t}$ in (4.20) and checking that the assumptions made in this theorem ensure the fulfillment of both conditions (i) and (ii) in Theorem 4.2. Thus we get

$$(x^*, -y^*, -y^*) = (\bar{x}_1^*, -\bar{y}_1^*, 0) + (\bar{x}_2^*, 0, -\bar{y}_2^*), \quad (4.23)$$

which implies that $\bar{y}_1^* = \bar{y}_2^* = y^*$, $x^* = \bar{x}_1^* + \bar{x}_2^*$ and so justifies (4.18).

To prove (4.19), take $(x^*, y^*) \in X^* \times Y^*$ with $x^* \in \overline{D}_M^*(F_{1,\bar{t}} + F_{2,\bar{t}})(\bar{x}, \bar{y})(y^*)$ and by definition (2.6) find sequences $\varepsilon_k \downarrow 0$, $(t_k, x_k, y_k) \rightarrow (\bar{t}, \bar{x}, \bar{y})$ with $y_k \in (F_{1,t_k} + F_{2,t_k})(x_k)$ and $x_k^* \xrightarrow{w^*} x^*$, $y_k^* \rightarrow y^*$ with

$$(x_k^*, -y_k^*) \in \widehat{N}_{\varepsilon_k}((x_k, y_k); \text{gph}(F_{1,t_k} + F_{2,t_k})), \quad k \in \mathbb{N}.$$

Due to the extended inner compactness assumption imposed on (4.16), there exists a sequence of $(y_{1k}, y_{2k}) \in S_{t_k}(x_k, y_k)$ that contains a subsequence converging to some point

$(\bar{y}_1, \bar{y}_2) \in S_{\bar{t}}(\bar{x}, \bar{y})$; let it happen for all $k \rightarrow \infty$. Since the space $X \times Y \times Y$ is Asplund and the sets $\Omega_{j,t}$ in (4.20) are locally closed around the reference points, we may assume without loss of generality that $\varepsilon_k = 0$ in (4.21). Now applying Lemma 3.1 to (4.21), arguing as in the proof of Theorem 4.2, and taking into account the special structures of the sets $\Omega_{j,t}$ in (4.20), we find $(\bar{x}_j^*, \bar{y}_j^*) \in X^* \times Y^*$ with $\bar{x}_j^* \in \bar{D}_M^* F_{j,\bar{t}}(\bar{x}, \bar{y}_j)(\bar{y}_j^*)$ for $j = 1, 2$ such that (4.23) holds. This implies (4.19) and completes the proof of the theorem. $\triangle$

As a consequence of Theorem 4.8, we derive the following *extended subdifferential sum rules* involving the extended subdifferential and extended singular subdifferential (2.7) of l.s.c. functions on Asplund spaces.

Corollary 4.9 (extended subdifferential sum rules). *Let X be an Asplund space, and let $\{\varphi_{j,t}\}_{t \in T}$, $j = 1, 2$, be two parametric families of l.s.c. extended-real-valued functions on X . Assume that one of these families is ESNEC at $(\bar{t}, \bar{x})$ and that functions $(t, x) \mapsto \varphi_{j,t}(x)$ are l.s.c. at $(\bar{t}, \bar{x})$ for both $j = 1, 2$. Impose the singular subdifferential qualification condition*

$$\bar{\partial}^\infty \varphi_{1,\bar{t}}(\bar{x}) \cap (-\bar{\partial}^\infty \varphi_{2,\bar{t}}(\bar{x})) = \{0\}. \quad (4.24)$$

Then we have the extended subdifferential sum rules

$$\bar{\partial}(\varphi_{1,\bar{t}} + \varphi_{2,\bar{t}})(\bar{x}) \subset \bar{\partial}\varphi_{1,\bar{t}}(\bar{x}) + \bar{\partial}\varphi_{2,\bar{t}}(\bar{x}), \quad \bar{\partial}^\infty(\varphi_{1,\bar{t}} + \varphi_{2,\bar{t}})(\bar{x}) \subset \bar{\partial}^\infty \varphi_{1,\bar{t}}(\bar{x}) + \bar{\partial}^\infty \varphi_{2,\bar{t}}(\bar{x}).$$

Proof. Both subdifferential sum rules follow from Theorem 4.8 applied to the families of epigraphical multifunctions $F_{j,t} = E_{\varphi_{j,t}}: X \rightrightarrows \mathbb{R}$, $t \in T$, $j = 1, 2$. $\triangle$

The next theorem establishes general *chain rules* for both extended limiting coderivatives (2.4) and (2.6) of parametric families of *compositions*

$$(F \circ G)_t(x) := F_t(G_t(x)) = \bigcup \left\{ F_t(y) \mid y \in G_t(x) \right\} \quad (4.25)$$

involving families of set-valued mappings $G_t: X \rightrightarrows Y$ and $F_t: Y \rightrightarrows Z$, $t \in T$. Define the family $S_t: X \times Z \rightrightarrows Y$, $t \in T$, by

$$S_t(x, z) := G_t(x) \cap F_t^{-1}(z) = \{y \in Y \mid y \in G_t(x), z \in F_t(y)\}. \quad (4.26)$$

Theorem 4.10 (chain rules for extended coderivatives). *Let X, Y and Z be Asplund spaces, and let $G_t: X \rightrightarrows Y$ and $F_t: Y \rightrightarrows Z$, $t \in T$, be two families of set-valued mappings with locally closed graphs around the reference points. Given $(\bar{x}, \bar{z}) \in \text{gph}(F_{\bar{t}} \circ G_{\bar{t}})$, assume that the family of S_t in (4.26) is extendedly inner semicompact at $(\bar{t}, \bar{x}, \bar{z})$ and that for every $\bar{y} \in S_{\bar{t}}(\bar{x}, \bar{z})$ the following hold:*

- (i) *Either $\{G_t^{-1}\}_{t \in T}$ is PESNC at $(\bar{t}, \bar{y}, \bar{x})$, or $\{F_t\}_{t \in T}$ is PESNC at $(\bar{t}, \bar{y}, \bar{z})$.*
- (ii) *The mixed coderivative qualification condition is satisfied:*

$$\bar{D}_M^* G_{\bar{t}}^{-1}(\bar{y}, \bar{x})(0) \cap (-\bar{D}_M^* F_{\bar{t}}(\bar{y}, \bar{z})(0)) = \{0\}.$$

The one has the extended coderivatives chain rules

$$\bar{D}^*(F_{\bar{t}} \circ G_{\bar{t}})(\bar{x}, \bar{z}) \subset \bigcup_{\bar{y} \in G_{\bar{t}}(\bar{x}) \cap F_{\bar{t}}^{-1}(\bar{z})} \bar{D}^* G_{\bar{t}}(\bar{x}, \bar{y}) \circ \bar{D}^* F_{\bar{t}}(\bar{y}, \bar{z}), \quad (4.27)$$

$$\overline{D}_M^*(F_{\bar{t}} \circ G_{\bar{t}})(\bar{x}, \bar{z}) \subset \bigcup_{\bar{y} \in G_{\bar{t}}(\bar{x}) \cap F_{\bar{t}}^{-1}(\bar{z})} \overline{D}^* G_{\bar{t}}(\bar{x}, \bar{y}) \circ \overline{D}_M^* F_{\bar{t}}(\bar{y}, \bar{z}), \quad (4.28)$$

where the inclusions hold for all argument $z^* \in Z^*$ on both sides.

Proof. We only prove the normal coderivative chain rule (4.27) observing that the proof of (4.28) can be furnished by combining this procedure with the arguments used in the proof of the mixed coderivative sum rule (4.19) in Theorem 4.8.

Take $(x^*, z^*) \in X^* \times Z^*$ with $x^* \in \overline{D}^*(F_{\bar{t}} \circ G_{\bar{t}})(\bar{x}, \bar{z})(z^*)$ and find by definition sequences $\varepsilon_k \downarrow 0$, $(x_k, z_k) \in \text{gph}(F_{t_k} \circ G_{t_k})$, and $(x_k^*, -z_k^*) \in \widehat{N}_{\varepsilon_k}((x_k, z_k); \text{gph}(F_{t_k} \circ G_{t_k}))$ such that

$$(t_k, x_k, z_k) \rightarrow (\bar{t}, \bar{x}, \bar{z}) \text{ and } (x_k^*, z_k^*) \xrightarrow{w^*} (x^*, z^*) \text{ as } k \rightarrow \infty.$$

By the extended lower semicompactness assumption imposed on (4.26), we find a sequence of $y_k \in S_{t_k}(x_k, z_k)$ that contains a subsequence converging to $\bar{y} \in S_{\bar{t}}(\bar{x}, \bar{z})$. Without loss of generality, assume that $y_k \rightarrow \bar{y}$ as $k \rightarrow \infty$. Construct the sets

$$\Omega_{1,t} := \text{gph } G_t \times Z \text{ and } \Omega_{2,t} := X \times \text{gph } F_t, \quad t \in T, \quad (4.29)$$

which are locally closed around the points of interest with $(x_k, y_k, z_k) \in \Omega_{1,t_k} \cap \Omega_{2,t_k}$ for each $k \in \mathbb{N}$. Then check that

$$(x_k^*, 0, -z_k^*) \in \widehat{N}_{\varepsilon_k}((x_k, y_k, z_k); \Omega_{1,t_k} \cap \Omega_{2,t_k}) \text{ for all } k \in \mathbb{N},$$

and hence $(x^*, 0, z^*) \in \overline{N}((\bar{x}, \bar{y}, \bar{z}); \Omega_{1,\bar{t}} \cap \Omega_{2,\bar{t}})$ by passing to the limit as $k \rightarrow \infty$. Now taking into account assumptions (i) and (ii) of the theorem and applying Theorem 4.2 to the above intersection, we find extended normals

$$(u^*, y_1^*, 0) \in \overline{N}((\bar{x}, \bar{y}, \bar{z}); \Omega_{1,\bar{t}}) \text{ and } (0, y_2^*, -v^*) \in \overline{N}((\bar{x}, \bar{y}, \bar{z}); \Omega_{2,\bar{t}})$$

satisfying the relationship

$$(x^*, 0, -z^*) = (u^*, y_1^*, 0) + (0, y_2^*, -v^*).$$

Thus $u^* = x^*$, $v^* = z^*$, $-y_1^* = y_2^* := y^*$, and

$$x^* \in \overline{D}^* G_{\bar{t}}(\bar{x}, \bar{y})(y^*), \quad y^* \in \overline{D}^* F_{\bar{t}}(\bar{y}, \bar{z})(z^*),$$

which justify (4.27) and complete the proof of the theorem. $\triangle$

Similarly to the case of parameter-independent objects [10, Sections 3.1 and 3.2], we can derive from Theorem 4.10 *other calculus rules* for extended coderivatives and subdifferentials of parameter-dependent mappings and functions.

5 Extended Normal Compactness in Variational Analysis

In this section we study in detail the *extended sequential normal compactness* (ESNC) properties, with their *partial* modifications, introduced in Section 2. As seen in Sections 3 and 4, these properties are present in the major calculus results for extended normals, coderivatives, and subdifferentials of parameter-dependent objects. Thus the sufficient conditions for the fulfillment of the ESNC properties obtained in what follows ensure the validity of the generalized calculus results derived above. Furthermore, in this section we develop basic results of *ESNC calculus*, which ensure the preservation of ESNC properties under various operations performed on sets, mappings, and functions. The latter calculus is one of the most important ingredients of infinite-dimensional variational analysis and its applications.

Let us start with a simple while important observation showing that the extended generalized differential and sequential normal compactness properties introduced in Section 2 are *invariant* with respect to “sequentially null perturbations.” Given a parametric family $\{x_t\}_{t \in T}$ of elements in a Banach space X , we say that it is *sequentially null* at $\bar{t} \in T$ if $x_{t_k} \rightarrow 0$ whenever $t_k \rightarrow 0$ as $k \rightarrow \infty$.

Proposition 5.1 (invariance with respect to sequentially null perturbations). *Let $\{\Omega_t\}_{t \in T}$ be a parametric family of subsets in X , and let $\{x_t\}_{t \in T}$ be sequentially null at $\bar{t} \in T$. Then $\{\Omega_t\}_{t \in T}$ and the perturbed family $\{\Omega_t + x_t\}_{t \in T}$ share the same extended normal (2.2) and ESNC compactness properties from Definition 2.1.*

Proof. Follows directly from the definitions. $\triangle$

The above observation illuminates an important fact on the extended differential constructions and sequential normal compactness properties of Section 2: these constructions and properties are *not essentially related to the graph*

$$\text{gph } \Omega = \{(t, x) \in T \times X \mid x \in \Omega_t, t \in T\}, \quad (5.1)$$

since the graph can be altered without affecting the underlying constructions and properties. This implies, in particular, that it *does not* make much sense to seek relationships between the ESNC properties of parameter-dependent objects and the corresponding SNC properties of their (parameter-independent) *graphs* (5.1). Let us present some simple examples that illustrate this observation. Recall that the SNC and PSNC properties of sets in these examples are understood in the sense of [10], i.e., as in Definition 2.1 with $\{\Omega_t\} \equiv \Omega$.

Example 5.2 (difference between ESNC properties of parameter-dependent sets and SNC properties of their graphs).

(i) Let $T = \mathbb{R}$, and let X be an infinite-dimensional Banach space. Define $\{\Omega_t\}_{t \in T}$ by

$$\Omega_t := \begin{cases} X & \text{if } t = 0, \\ |t|B & \text{otherwise.} \end{cases}$$

We can directly check that $\text{gph } \Omega$ is SNC at $(0, 0)$ and $\Omega_1 = X$ is also SNC at $x = 0$. However, the family $\{\Omega(t)\}_{t \in T}$ is not ESNC at $(0, 0)$.

(ii) Let T be an infinite-dimensional Banach space, and let $X = \mathbb{R}$. Define Ω_t to be 0 when $t = 0$, and the empty set otherwise. It is obvious that $\{\Omega_t\}_{t \in T}$ is ESNC at $(0, 0)$, while $\text{gph } F$ is neither SNC nor PSNC at this point.

It happens that the ESNC properties for moving objects are implied by certain *uniform* counterparts of *Lipschitzian* properties of sets and mappings. The following definition postulates appropriate parametric uniform versions of the *epi-Lipschitzian* [18] and *compactly epi-Lipschitzian* (CEL) [3] properties of parameter-independent sets. In [5, 6, 10] the reader can find more information about the latter properties and relationships between them.

Definition 5.3 (uniform epi-Lipschitzian and CEL properties). *We say that the parametric family of sets $\{\Omega_t\}_{t \in T}$ with $\bar{x} \in \Omega_{\bar{t}}$ is UNIFORMLY COMPACTLY EPI-LIPSCHITZIAN (uniformly CEL) around $(\bar{t}, \bar{x})$ if there exist a neighborhood U of $\bar{x}$, a neighborhood O of the origin in X , a neighborhood V of $\bar{t}$, a compact set $C \subset X$, and a number $\gamma_t > 0$ for each $t \in V$ such that*

$$\Omega_t \cap U + \gamma O \subset \Omega_t + \gamma C \text{ whenever } \gamma \in (0, \gamma_t) \text{ and } t \in V. \quad (5.2)$$

The family $\{\Omega_t\}_{t \in T}$ is said to be UNIFORMLY EPI-LIPSCHITZIAN around $(\bar{t}, \bar{x})$ if C in (5.2) can be selected as a singleton.

The next proposition establishes the relationship between the uniform CEL and ESNC properties of parametric families of sets in arbitrary Banach spaces and also justifies sufficient conditions for the fulfillment of the uniformly epi-Lipschitzian and hence all the other properties under consideration.

Proposition 5.4 (sufficient conditions for the ESNC property). *Let $\{\Omega_t\}_{t \in T}$ be a parametric family of sets in the Banach space X . The following assertions hold:*

- (i) *$\{\Omega_t\}_{t \in T}$ is ESNC at $(\bar{t}, \bar{x})$ if it is uniformly CEL around this point.*
- (ii) *Let each Ω_t be convex, and let there exist a neighborhood V of $\bar{t}$ such that*

$$\text{int} \bigcap_{t \in V} \Omega_t \neq \emptyset.$$

Then the family $\{\Omega_t\}_{t \in T}$ is uniformly epi-Lipschitzian (and hence uniformly CEL) around any $(\bar{t}, \bar{x})$ with $\bar{x} \in \Omega_{\bar{t}}$.

Proof. The proof of (i) is similar to that given in [10, Theorem 1.26] for the case of nonparametric sets; see also [9] for a somewhat simplified version.

To justify (ii), take $v \in X$ with $v + rB \subset \bigcap_{t \in V} \Omega_t$ for some $r > 0$. Then for each $t \in V$, $\gamma \in [0, 1]$, and $x \in \Omega_t$ with $\|x - \bar{x}\| \leq r/2$ we have

$$(1 - \gamma)x + \gamma(v + rh) \in \Omega_t \text{ whenever } h \in B$$

due to the convexity of Ω_t . This yields

$$x + \gamma(\bar{x} - x + rh) \in \Omega_t + \gamma(\bar{x} - v) \text{ for all } h \in B.$$

Furthermore, since the obvious estimate

$$\|u/2 - (\bar{x} - x)/2\| \leq 1/2 + (1/r)(r/2) = 1 \text{ for } u \in B,$$

we get the relationships

$$x + \gamma(r/2)u = x + \gamma(\bar{x} - x + r(u/2 - (\bar{x} - x)/r)) \in \Omega_t + \gamma(\bar{x} - v)$$

with $\gamma \in [0, 1]$. This allows us to conclude that

$$\Omega_t \cap (\bar{x} + (r/2)B(X)) + \gamma(r/2)B \subset \Omega_t + \gamma(\bar{x} - v)$$

whenever $t \in V$ and $\gamma \in [0, 1]$, which completes the proof of (ii). $\triangle$

Observe that the nonempty interior condition in Proposition 5.4(ii) is sufficient but *not necessary* for the uniform epi-Lipschitzian property of the family of convex sets $\{\Omega_t\}_{t \in T}$, even in *finite dimensions*. Indeed, let $T = X = \mathbb{R}$ and define $\{\Omega_t\}_{t \in T}$ by

$$\Omega_t := \begin{cases} X & \text{if } t = 0, \\ [-|t|, |t|] & \text{otherwise.} \end{cases}$$

We can easily check that this family is uniformly epi-Lipschitzian around $(0, 0)$, while the intersection set in Proposition 5.4(ii) has no interior points.

Let us next present some sufficient conditions ensuring the fulfillment of *partial* (and *strong partial*) ESNC properties from Definition 2.1. These properties take into account the *product structure* of the space in question and, as shown in Section 4, are the most efficient in the case of (graphs of) set-valued mappings $F: X \rightrightarrows Y$, which are naturally associated with to the product space $X \times Y$. We now formulate appropriate *uniform* counterparts of certain Lipschitzian properties of set-valued mappings that imply the validity of the partial and strong partial versions of ESNC.

We say that a parametric family $\{F_t\}_{t \in T}$ of set-valued mappings from X to Y is *uniformly Lipschitz-like* around $(\bar{t}, \bar{x}, \bar{y})$ with $(\bar{x}, \bar{y}) \in \text{gph } F_{\bar{t}}$ if there exist $\ell \geq 0$, a neighborhood U of $\bar{x}$, and a neighborhood V of $\bar{y}$ such that

$$F_t(x) \cap V \subset F_t(u) + \ell\|x - u\|B \text{ for all } x, u \in U \text{ and } t \in T \text{ near } \bar{t}.$$

This property reduces to Aubin's Lipschitz-like (or "pseudo-Lipschitzian") property of $F: X \rightrightarrows Y$ around $(\bar{x}, \bar{y})$ for parameter-independent mappings; see [1, 10, 19].

Further, we say that the family of set-valued mappings $\{F_t\}_{t \in T}$ *uniformly partial CEL* around $(\bar{t}, \bar{x}, \bar{y})$ with $(\bar{x}, \bar{y}) \in \text{gph } F_{\bar{t}}$ if there exist a neighborhood U of $(\bar{x}, \bar{y})$, a neighborhood O of the origin in X , and a neighborhood V of $\bar{t}$, as well as a compact set $C \subset X$ and a number $\gamma_t > 0$ for each $t \in V$ such that

$$(\text{gph } F_t) \cap U + \gamma(0 \times \{0\}) \subset \text{gph } F_t + \gamma C \text{ whenever } \gamma \in (0, \gamma_t) \text{ and } t \in V.$$

This property is a uniform extension of the partial CEL property of [8] to parametric families of set-valued mappings. The following proposition establishes relationships between the partial ESNC and above Lipschitzian properties of multifunctions.

Proposition 5.5 (partial ESNC from Lipschitzian properties of set-valued mappings). *Let $\{F_t\}_{t \in T}$ be a parametric family of set-valued mappings between Banach spaces X and Y , and let $(\bar{x}, \bar{y}) \in \text{gph } F_{\bar{t}}$. The following assertions holds:*

- (i) *The family $\{F_t\}_{t \in T}$ is PESNC at $(\bar{t}, \bar{x}, \bar{y})$ provided that it is uniformly Lipschitz-like around this point.*
- (ii) *The family $\{F_t\}_{t \in T}$ is strongly PESNC at $(\bar{t}, \bar{x}, \bar{y})$ if it is uniformly partially CEL around this point.*

Proof. To establish (i), we proceed similarly to the proof of [10, Theorem 1.43] given in the case of parameter-independent mappings. The proof of (ii) is similar to that of [10, Theorem 1.75] in the parameter-independent case; cf. also [8] for the latter result. $\triangle$

Next we establish the principal rules of *ESNC calculus* that give efficient conditions ensuring the preservation of these properties under various operations. It happens that the major conditions of ESNC calculus are *extended qualification conditions* similar to (while generally different from) those developed in Section 4 for calculus rules of extended generalized differentiation. The reader can compare the results and proofs given in this section with SNC calculus rules derived in [14] and [10, Section 3.3] for nonparametric objects.

As usual, we start with considering properties of sets and formulate the basic qualification condition of ESNC calculus. Since the results obtained in this section concern only Asplund spaces, we avoid $\varepsilon_k \downarrow 0$ in all the formulations. Observe that the following *mixed* qualification condition, in contrast to those from Definition 4.1, essentially exploits the *product structure* of the spaces in question.

Definition 5.6 (extended mixed qualification condition for parametric systems of sets). *Let $\{\Omega_{j,t}\}_{t \in T}$, $j = 1, 2$, be two parametric families of subsets in the product space $X \times Y$, and let $(\bar{x}, \bar{y}) \in \Omega_{1,\bar{t}} \cap \Omega_{2,\bar{t}}$. We say that the system $\{\Omega_{j,t}\}_{t \in T}$, $j = 1, 2$, satisfies the EXTENDED MIXED QUALIFICATION CONDITION at $(\bar{t}, \bar{x}, \bar{y})$ with respect to Y if for any sequences $t_k \rightarrow \bar{t}$, $(x_{j,k}, y_{j,k}) \in \Omega_{j,t_k}$, and $(x_{j,k}^*, y_{j,k}^*) \xrightarrow{w^*} (x_j^*, y_j^*)$ as $k \rightarrow \infty$ such that $(x_{j,k}^*, y_{j,k}^*) \in \hat{N}((x_{j,k}, y_{j,k}); \Omega_{j,t_k})$ as $j = 1, 2$ one has*

$$\left[x_{1,k}^* + x_{2,k}^* \xrightarrow{w^*} 0, \|y_{1,k}^* + y_{2,k}^*\| \rightarrow 0 \right] \implies (x_1^*, y_1^*) = (x_2^*, y_2^*) = 0. \quad (5.3)$$

The extended mixed qualification condition (5.3) is clearly implied by the normal qualification condition of Definition 4.1(i) in the space $X \times Y$:

$$N((\bar{x}, \bar{y}); \Omega_{1,\bar{t}}) \cap [-N((\bar{x}, \bar{y}); \Omega_{2,\bar{t}})] = \{0\}.$$

On the other hand, (5.3) is *more restrictive* than the extended limiting qualification condition (4.2) in $X \times Y$, while the latter is *not sufficient* for ESNC calculus.

The next theorem, ensuring the preservation of the *PESNC property* of set intersections under the extended mixed qualification condition of Definition 5.6 is the *basic result* of the whole ESNC calculus.

Theorem 5.7 (PESNC property of set intersection). *Let $X_1, \dots, X_p$ be Asplund spaces, and let $\{\Omega_{j,t}\}_{t \in T}$, $j = 1, 2$, be two parametric families of locally closed subsets in*

$X_1 \times \dots \times X_p$ with $\bar{x} \in \Omega_{1,\bar{t}} \cap \Omega_{2,\bar{t}}$. Given $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{I} := \{1, \dots, p\}$ with $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}$, impose the following assumptions:

- (i) For each $j = 1, 2$ the family of sets $\{\Omega_{j,t}\}_{t \in \mathcal{T}}$ is EPSNC with respect to $\{X_i \mid i \in \mathcal{I}_j\}$ at the point $(\bar{t}, \bar{x})$.
- (ii) Either $\{\Omega_{1,t}\}$ is strongly EPSNC at $(\bar{t}, \bar{x})$ with respect to $\{X_i \mid i \in \mathcal{I}_1 \setminus \mathcal{I}_2\}$, or $\{\Omega_{2,t}\}$ is strongly EPSNC at $(\bar{t}, \bar{x})$ with respect to $\{X_i \mid i \in \mathcal{I}_2 \setminus \mathcal{I}_1\}$.
- (iii) The system of sets $\{\Omega_{j,t}\}_{t \in \mathcal{T}}$, $j = 1, 2$, satisfies the extended mixed qualification condition at $(\bar{t}, \bar{x})$ with respect to $\{X_i \mid i \in (\mathcal{I}_1 \setminus \mathcal{I}_2) \cup (\mathcal{I}_2 \setminus \mathcal{I}_1)\}$.

Then the family $\{\Omega_{1,t} \cap \Omega_{2,t}\}_{t \in \mathcal{T}}$ is EPSNC at $(\bar{t}, \bar{x})$ with respect to $\{X_i \mid i \in \mathcal{I}_1 \cap \mathcal{I}_2\}$.

Proof. It follows the way of proving [10, Theorem 3.79] (see also [14]) by employing the arguments to deal with extended normals and qualification conditions developed in Section 4. Note that the driving force here is again the *extremal principle* via the usage of the equivalent Lemma 3.1. $\triangle$

Let us present two important consequences of Theorem 5.7. The first one provides an efficient specification of the general result in the case of *two space products*.

Corollary 5.8 (PESNC property in products of two spaces). *Let $\{\Omega_{j,t}\}_{t \in \mathcal{T}}$, $j = 1, 2$, be two parametric families of locally closed sets in the product of Asplund spaces $X \times Y$. Assume that one of these families is ESNC at $(\bar{t}, \bar{x}, \bar{y})$ while the other is PESNC at $(\bar{t}, \bar{x}, \bar{y})$ with respect to X . Assume also that the extended mixed qualification condition holds for the system $\{\Omega_{j,t}\}_{t \in \mathcal{T}}$, $j = 1, 2$, at $(\bar{t}, \bar{x}, \bar{y})$ with respect to Y . Then $\{\Omega_{1,t} \cap \Omega_{2,t}\}_{t \in \mathcal{T}}$ is PESNC at $(\bar{t}, \bar{x})$ with respect to X .*

Proof. Suppose that the first family $\{\Omega_{1,t}\}_{t \in \mathcal{T}}$ is ESNC at $(\bar{t}, \bar{x})$. Then the result follows from Theorem 5.7 with $p = 2$, $X_1 = X$, $X_2 = Y$, $\mathcal{I}_1 = \{1, 2\}$, and $\mathcal{I}_2 = \{1\}$. $\triangle$

The next corollary of Theorem 5.7 concerns the preservation of the *ESNC property* under intersections of *finitely many* parametric families of sets in Asplund spaces without imposing any product structure of the space in question.

Corollary 5.9 (ESNC property for intersections of finitely many sets). *Let $\{\Omega_{j,t}\}_{t \in \mathcal{T}}$, $j = 1, \dots, n$, be parametric families of locally closed sets in the Asplund space X , and let $\bar{x} \in \Omega_{1,\bar{t}} \cap \dots \cap \Omega_{n,\bar{t}}$. Assume that each family is ESNC at $(\bar{t}, \bar{x})$ and that the following qualification condition is satisfied:*

$$\left[x_1^* + \dots + x_n^* = 0, x_j^* \in \bar{N}(\bar{x}; \Omega_{j,\bar{t}}) \right] \implies x_j^* = 0 \text{ for all } j = 1, \dots, n.$$

Then the intersection family $\{\Omega_{1,t} \cap \dots \cap \Omega_{n,t}\}_{t \in \mathcal{T}}$ is ESNC at $(\bar{t}, \bar{x})$.

Proof. For $n = 2$ this follows from Corollary 5.9 by putting $Y = \{0\}$, i.e., with no product structure. The general case is justified by induction. $\triangle$

Similarly to the above results on the preservation of the PESNC and ESNC properties under set intersections, we get the following intersection rule for preserving the *strong PESNC* property under the *normal qualification condition* of Definition 4.1(i). We present the result in products of two Asplund spaces.

Theorem 5.10 (strong PESNC property in products of two spaces). *Let $\{\Omega_{j,t}\}_{t \in T}$, $j = 1, 2$, be two parametric families of locally closed sets in the product of Asplund spaces $X \times Y$. Assume that one of these families is ESNC at $(\bar{t}, \bar{x}, \bar{y})$ while the other is strongly PESNC at $(\bar{t}, \bar{x}, \bar{y})$ with respect to X . Assume also that the extended normal qualification condition holds for the system $\{\Omega_{j,t}\}_{t \in T}$, $j = 1, 2$, at $(\bar{t}, \bar{x}, \bar{y})$ with respect to Y . Then $\{\Omega_{1,t} \cap \Omega_{2,t}\}_{t \in T}$ is strongly PESNC at $(\bar{t}, \bar{x})$ with respect to X .*

Proof. It follows the procedures in the proofs of Theorem 5.7 and Corollary 5.8. $\triangle$

The next theorem ensures the preservation of the ESNC property under *summation* of sets. Observe that, in contrast to the previous results, it does *not* require *any qualification condition*. For brevity we formulate this theorem only for the case of extended inner semicompactness of the auxiliary mapping below; the case of extended inner semicontinuity is formulated and treated similarly to Theorem 4.5.

Theorem 5.11 (ESNC property under set summations). *Let $\{\Omega_{j,t}\}_{t \in T}$, $j = 1, 2$, be two families of locally closed subsets of the Asplund space X , and let $\bar{x} \in \Omega_{1,\bar{t}} \cap \Omega_{2,\bar{t}}$. Define the family of set-valued mappings $S_t: X \rightrightarrows X \times X$, $t \in T$, by*

$$S_t(x) := \{(x_1, x_2) \mid x_1 + x_2 = x, x_j \in \Omega_{j,t}, j = 1, 2\}$$

and assume that $\{S_t\}_{t \in T}$ is extendedly inner semicompact at $(\bar{t}, \bar{x})$ and that for each point $(x_1, x_2) \in S_{\bar{t}}(\bar{x})$ one of the families $\{\Omega_{j,t}\}_{t \in T}$, $j = 1, 2$, is ESNC at $(\bar{t}, x_1)$ and $(\bar{t}, x_2)$, respectively. Then the summation family $\{\Omega_{1,t} + \Omega_{2,t}\}_{t \in T}$ is ESNC at $(\bar{t}, \bar{x})$.

Proof. We start with the ESNC definition and then proceed similarly to the proof of Theorem 4.5 while applying the PESNC intersection rule from Corollary 5.8 to the sets

$$\tilde{\Omega}_{1,t} := \Omega_{1,t} \times X, \quad \tilde{\Omega}_{2,t} := X \times \Omega_{2,t}, \quad t \in T,$$

in the product space $X \times X$. $\triangle$

Let us now present several ESNC calculus results involving the associated ESNC properties for set-valued mappings and extended-real-valued functions under qualification conditions in terms of the extended coderivatives and singular subdifferentials of Section 2. Due to the space limitation, we omit proof details referring the reader to the corresponding arguments in [14] and [10, Section 3.3] for parameter-independent objects and to the procedures to deal with parameter-dependent objects developed in Section 4.

The next theorem provides sufficient conditions ensuring the ESNC property of *inverse images* (4.10). Observe that the qualification condition in this result is formulated via the extended *normal coderivative* (2.4), in contrast to the mixed one in (4.11).

Theorem 5.12 (ESNC property of inverse images). *Let $\{F_t\}_{t \in T}$ be a parametric family of set-valued mappings between Asplund spaces X and Y , and let $\{\Theta_t\}_{t \in T}$ be a family of subsets in Y . Given $\bar{x} \in F_{\bar{t}}^{-1}(\Theta_{\bar{t}})$, suppose as that the sets $\text{gph } F_t$ and Θ_t are locally closed-graph around the reference points and that the family of set-valued mappings*

$$S_t(x) := F_t(x) \cap \Theta_t, \quad t \in T,$$

is extendedly inner semicompact at $(\bar{t}, \bar{x})$. Assume also that for every $\bar{y} \in S_{\bar{t}}(\bar{x})$ we have:

- (i) Either $\{F_t\}_{t \in T}$ is EPSNC at $(\bar{t}, \bar{x}, \bar{y})$ and $\{\Theta_t\}_{t \in T}$ is ESNC at $(\bar{t}, \bar{y})$, or $\{F_t\}_{t \in T}$ is ESNC at $(\bar{t}, \bar{x}, \bar{y})$.
- (ii) The normal qualification condition holds:

$$\ker \bar{D}^* F_{\bar{t}}(\bar{x}, \bar{y}) \cap \bar{N}(\bar{y}; \Theta_{\bar{t}}) = \{0\}.$$

Then the inverse image family $\{F_t^{-1}(\Theta_t)\}_{t \in T}$ is ESNC at $(\bar{t}, \bar{x})$.

Proof. Follows the scheme in the proof of [10, Theorem 3.84] with taking into account the parametric situation similarly to the proof of Theorem 4.6 above. $\triangle$

Corollary 5.13 (ESNC property of parametric functional constraints). *In the notation of Corollary 4.7, the following assertions hold:*

- (i) *Let the parametric family of sets $\{\Omega_t\}_{t \in T}$ be given by the inequality constraint in (4.16), and let all the assumptions of Corollary 4.7(i) are satisfied. Suppose in addition that the family $\{\varphi_t\}_{t \in T}$ is ESNEC at $(\bar{t}, \bar{x})$. Then the family $\{\Omega_t\}_{t \in T}$ is ESNC at this point.*
- (ii) *Let the parametric family of sets $\{\Omega_t\}_{t \in T}$ be given by the equality constraint in (4.16), and let all the assumptions of Corollary 4.7(ii) are satisfied. Suppose in addition that the family $\{\varphi_t\}_{t \in T}$ is ESNC at $(\bar{t}, \bar{x})$. Then the family $\{\Omega_t\}_{t \in T}$ is also ESNC at $(\bar{t}, \bar{x})$.*

Proof. Follows from Theorem 5.12 in the way of proving Corollary 4.7. $\triangle$

Finally in this section, we obtain calculus results on the preservation of ESNC properties under various compositions involving parametric families of set-valued mappings and extended-real-valued functions. Let us start with the PESNC property for sums of general multifunctions between Asplund spaces.

Theorem 5.14 (PESNC and ESNC properties for sums of set-valued mappings).

Let $F_{j,t}: X \rightrightarrows Y$, $t \in T$, $j = 1, 2$, be families of set-valued mappings between Asplund spaces, and let $(\bar{x}, \bar{y}) \in \text{gph}(F_{1,\bar{t}} + F_{2,\bar{t}})$. Assume that the family $\{S_t\}_{t \in T}$ from (4.16) is extendedly inner semicompact at $(\bar{t}, \bar{x}, \bar{y})$ and that for every $(\bar{y}_1, \bar{y}_2) \in S_{\bar{t}}(\bar{x}, \bar{y})$ the graphs of F_1 and F_2 are locally closed around $(\bar{x}, \bar{y}_1)$ and $(\bar{x}, \bar{y}_2)$, respectively, for all $t \in T$ near $\bar{t}$ and that

- (i) *Each $\{F_{i,t}\}_{t \in T}$ is PESNC at $(\bar{t}, \bar{x}, \bar{y}_i)$, $j = 1, 2$.*
- (ii) *The qualification condition (4.17) holds.*

Then $\{F_{1,t} + F_{2,t}\}_{t \in T}$ is PESNC at $(\bar{t}, \bar{x}, \bar{y})$. Furthermore, if each $\{F_{i,t}\}_{t \in T}$ is ESNC at $(\bar{t}, \bar{x}, \bar{y}_i)$, $j = 1, 2$, and (4.17) is replaced by the normal coderivative qualification condition

$$\bar{D}^* F_{1,\bar{t}}(\bar{x}, \bar{y}_1)(0) \cap (-\bar{D}^* F_{2,\bar{t}}(\bar{x}, \bar{y}_2)(0)) = \{0\},$$

then $\{F_{1,t} + F_{2,t}\}_{t \in T}$ is ESNC at $(\bar{t}, \bar{x}, \bar{y})$.

Proof. Follows the proof of [10, Theorems 3.88 and 3.90] for nonparametric objects with taking into account the parametric structures under consideration similarly to the proof of Theorem 4.8 above. $\triangle$

Corollary 5.15 (ESNEC property of functions under summation). *Let X be an Asplund space, and let $\{\varphi_{j,t}\}_{t \in T}$, $j = 1, 2$, be two parametric families of l.s.c. extended-real-valued functions on X . Assume that both of them are ESNEC at $(\bar{t}, \bar{x})$ and that functions $(t, x) \mapsto \varphi_{j,t}(x)$, $j = 1, 2$, are l.s.c. at $(\bar{t}, \bar{x})$. Then the sum family $\{\varphi_{1,t} + \varphi_{2,t}\}_{t \in T}$ is ESNEC at $(\bar{t}, \bar{x})$ provided that the singular subdifferential qualification condition (4.24) is satisfied.*

Proof. Follows from Theorem 5.14 applied to the parametric families of multifunctions $F_{j,t} = E_{\varphi_{j,t}}$, $t \in T$, $j = 1, 2$. $\triangle$

The last theorem of this paper establishes efficient conditions for preservations of PESNC and ESNC properties for parametric families of compositions $F_t \circ G_t$ (4.25) of general set-valued mappings between Asplund spaces.

Theorem 5.16 (PESNC and ESNC properties under general composition of set-valued mappings). *Let X, Y and Z be Asplund spaces, and let $G_t: X \rightrightarrows Y$ and $F_t: Y \rightrightarrows Z$, $t \in T$, be two parametric families of set-valued mappings with locally closed graphs around the reference points. Given $(\bar{x}, \bar{z}) \in \text{gph}(F_{\bar{t}} \circ G_{\bar{t}})$, assume that the family of S_t in (4.26) is extendedly inner semicompact at $(\bar{t}, \bar{x}, \bar{z})$ and that for every $\bar{y} \in S_{\bar{t}}(\bar{x}, \bar{z})$ the following hold:*

- (i) *Either $\{G_t\}_{t \in T}$ is PESNC at $(\bar{t}, \bar{y}, \bar{x})$ and $\{F_t\}_{t \in T}$ is PESNC at $(\bar{t}, \bar{y}, \bar{z})$, or $\{G_t\}_{t \in T}$ is ESNC at $(\bar{t}, \bar{x}, \bar{y})$.*
- (ii) *One has the qualification condition:*

$$\ker \bar{D}^* G_{\bar{t}}(\bar{x}, \bar{y}) \cap (-\bar{D}_M^* F_{\bar{t}}(\bar{y}, \bar{z})(0)) = \{0\}.$$

Then $\{F_t \circ G_t\}_{t \in T}$ is PESNC at $(\bar{t}, \bar{x}, \bar{z})$. If furthermore either $\{G_t\}_{t \in T}$ is PESNC at $(\bar{t}, \bar{x}, \bar{y})$ and $\{F_t\}_{t \in T}$ is ESNC at $(\bar{t}, \bar{y}, \bar{z})$, or $\{G_t\}_{t \in T}$ is ESNC at $(\bar{t}, \bar{x}, \bar{y})$ and $\{F_t^{-1}\}_{t \in T}$ is PESNC at $(\bar{t}, \bar{z}, \bar{y})$ and if the qualification condition

$$\ker \bar{D}^* G_{\bar{t}}(\bar{x}, \bar{y}) \cap (-\bar{D}^* F_{\bar{t}}(\bar{y}, \bar{z})(0)) = \{0\}$$

is satisfied, then the family $\{F_t \circ G_t\}_{t \in T}$ is ESNC at $(\bar{t}, \bar{x}, \bar{z})$.

Proof. Follows the proof lines of [10, Theorems 3.95 and 3.98] for parameter-independent objects with adapting the arguments of Theorem 4.10 to deal with the dependence on parameters. $\triangle$

The results obtained in Theorems 5.14 and 5.16 can be applied to derive other calculus rules ensuring the preservation of the ESNC properties under consideration for parametric families of mappings and functions, similarly to the corresponding results of [10, Section 3.3] for parameter-independent counterparts.

References

- [1] J.-P. Aubin (1984). Lipschitz behavior of solutions to convex minimization problems. *Math. Oper. Res.* 9, 87–111.

- [2] S. Bellaassali and A. Jourani (1994). Necessary optimality conditions in multiobjective dynamic optimization. *SIAM J. Control Optim.* **42**, 2043–2061.
- [3] J. M. Borwein and H. M. Strójas (1985). Tangential approximations. *Nonlinear Anal.* **9**, 1347–1366.
- [4] J. M. Borwein and Q. J. Zhu (2005). *Techniques of Variational Analysis*. CMS Books in Mathematics, Springer-Verlag, New York.
- [5] M. Fabian and B. S. Mordukhovich (2003). Sequential normal compactness versus topological normal compactness in variational analysis. *Nonlinear Anal.* **54**, 1057–1067.
- [6] A. D. Ioffe (2000). Codirectional compactness, metric regularity and subdifferential calculus. In *Constructive, Experimental and Nonlinear Analysis*, edited by M. Théra. *Canad. Math. Soc. Conf. Proc.* **27**, 123–164.
- [7] H. Th. Jongen, K. Meers and E. Triesch (2004). *Optimization Theory*. Kluwer Academic Publishers, Boston, MA.
- [8] A. Jourani and L. Thibault (1995). Verifiable conditions for openness, metric regularity of multivalued mappings. *Trans. Amer. Math. Soc.* **347**, 1225–1268.
- [9] P. D. Loewen (1992). Limits of Fréchet normals in nonsmooth analysis. In *Optimization and Nonlinear Analysis*, edited by A. Ioffe, L. Marcus and S. Reich. *Pitman Research Notes Math. Ser.* **244**, pp. 178–188. Longman, Harlow, Essex, UK.
- [10] B. S. Mordukhovich (2006). *Variational Analysis and Generalized Differentiation, I: Basic Theory*. Grundlehren Series (Fundamental Principles of Mathematical Sciences) **330**, Springer-Verlag, Berlin.
- [11] B. S. Mordukhovich (2006). *Variational Analysis and Generalized Differentiation, II: Applications*. Grundlehren Series (Fundamental Principles of Mathematical Sciences) **331**, Springer-Verlag, Berlin.
- [12] B. S. Mordukhovich, J. S. Treiman and Q. J. Zhu (2003). An extended extremal principle with applications to multiobjective optimization. *SIAM J. Optim.* **14**, 359–379.
- [13] B. S. Mordukhovich and B. Wang (2002). Extensions of generalized differential calculus in Asplund spaces. *J. Math. Anal. Appl.* **272**, 164–186.
- [14] B. S. Mordukhovich and B. Wang (2003). Calculus of sequential normal compactness in variational analysis. *J. Math. Anal. Appl.* **282**, 63–84.
- [15] B. S. Mordukhovich and B. Wang (2004). Generalized differentiation for moving objects. In *Optimal Control, Stabilization and Nonsmooth Analysis*, edited by M. de Queiroz, M. Malisoff and P. Wolenski. *Lecture Notes Cont. Inf. Sci.* **301**, pp. 351–361, Springer-Verlag, New York.
- [16] B. S. Mordukhovich and L. Wang (2004). Optimal control of neutral functional-differential inclusions. *SIAM J. Control Optim.* **43**, 111–136.
- [17] R. R. Phelps (1993). *Convex Functions, Monotone Operators and Differentiability*, 2nd edition. Lectures Notes in Mathematics **1364**, Springer-Verlag, Berlin.
- [18] R. T. Rockafellar (1979). Directional Lipschitzian functions and subdifferential calculus. *Proc. London Math. Soc.* **39**, 331–355.
- [19] R. T. Rockafellar and R. J.-B. Wets (1998). *Variational Analysis*. Grundlehren Series (Fundamental Principles of Mathematical Sciences) **317**, Springer-Verlag, Berlin.
- [20] B. Wang (2006). The fuzzy intersection rule in variational analysis with applications. *J. Math. Anal. Appl.*, to appear.