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Cover Page Footnote

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On Some Properties and Estimation of Size-Biased Polya-Eggenberger Distribution

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A size-biased version of Polya-Eggenberger distribution is introduced explicitly and by a mixture model. The proposed distribution is unimodal with positive integer moments. The recurrence relation between moments (about the origin) of the proposed distribution is established and its relationship with other distributions is discussed. Different estimation techniques are proposed to estimate the parameters of the distribution.

Key words: Polya-Eggenberger distribution, size-biased Polya-Eggenberger distribution, recurrence relation between moments, estimation techniques.

Introduction

The Polya-Eggenberger distribution (PED), introduced by Polya and Eggenberger (1923) through an urn model and further analyzed by Polya (1930), is a discrete frequency distribution that was originally considered in connection with contagious distributions. The genesis of this distribution is expressed in terms of random drawings of colored balls from an urn: Initially, it is supposed that there are a white balls and b black balls in the urn; one ball is drawn at random and then replaced together with s balls of the same color. If this procedure is repeated n times and x represents the total number of times a white ball is drawn, then the distribution of x is given by:

$$P(x = k) = \binom{n}{x} \frac{a(a+s) \dots (a+x-1)s b(b+s) \dots (b+n-x-1)s}{(a+b)(a+b+s) \dots (a+b+n-1)s} \quad (1.1)$$

Where $x = 0, 1, 2, \dots, n$ and n, a, b and s are parameters of the distribution. The distribution is known as the Polya-Eggenberger distribution with parameters (n, a, b, s) .

Taking $\alpha = (a/s)$; $\gamma = (b/s)$, results in an alternative form of (1.1) in ascending factorials as:

$$P(x = k) = \binom{n}{k} \frac{\alpha^{[x]} \gamma^{[n-x]}}{(\alpha + \gamma)^{[n]}}, \quad (1.2)$$

$x = 0, 1, 2, \dots, n$

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Distribution (1.2) is the most convenient form of PED for computational purposes. Another way to represent (1.1) is

$$P(X = x) = \frac{\binom{-a/s}{x} \binom{-b/s}{n-x}}{\binom{-(a+b)/s}{n}}, \quad (1.3)$$

$x = 0, 1, 2, \dots, n,$

and an alternative form of (1.1) in terms of parameters n ; $P = a/(a+b)$; $Q = 1 - P = b/(a+b)$ and $\delta = s/(a+b)$ is

$$P(x=x) = \binom{n}{x} \frac{\prod_{j=0}^{x-1} (P+j\delta) \prod_{j=0}^{n-x-1} (Q+j\delta)}{\prod_{j=0}^{n-1} (1+j\delta)},$$

$$x = 0, 1, 2, \dots, n. \quad (1.4)$$

It is possible for s (and therefore δ) to be negative, however s must satisfy the inequality $(a+b)+s(n-1) > 0$.

Srodka (1964) gave the recurrence relation among the moments about zero of the Polya-Eggenberger distribution (1.3) as:

$$\mu'_{r+1} = (c+b+rs)^{-1} \sum_{j=0}^r \left[\begin{matrix} an \binom{r}{j} - (a-sn) \binom{r}{j+1} \\ -s \binom{r}{j+2} \end{matrix} \right] \mu'_{r-j}.$$

$$r = 0, 1, 2, \dots$$

The r^{th} factorial moment is given by

$$\mu_{(r)} = n^{(r)} \prod_{j=0}^{r-1} \left[\frac{p+j\delta}{1+j\delta} \right] \sum_{k=r-0}^{n-r} \binom{n-r}{k-r} \frac{\left[\prod_{j=0}^{k-r-1} (P'+j\delta') \right] \times \left[\prod_{j=0}^{n-k-1} (Q'+j\delta') \right]}{\prod_{j=0}^{n-r-1} (1+j\delta')}$$

$$(1.5)$$

where

$$P' = (P+r\delta)(1+r\delta)^{-1};$$

$$Q' = Q(1+r\delta)^{-1};$$

and

$$\delta' = \delta(1+r\delta)^{-1}.$$

Specifically, the first four central moments of the Polya-Eggenberger distribution are

$$\mu'_1 = nP = \frac{na}{a+b} \quad (1.6)$$

$$\mu_2 = nPQ(1+n\delta)(1+\delta)^{-1}$$

$$\mu_3 = nPQ(Q-P)(1+n\delta)(1+2n\delta)(1+\delta)^{-1}(1+2\delta)^{-1}$$

and

$$\mu_4 = \frac{nPQ(1+n\delta) \left[(1+2n\delta)(1+3n\delta)(1-3PQ) + (n-1)\{\delta+3PQ(1+n\delta)\} \right]}{(1+\delta)(1+2\delta)(1+3\delta)}$$

Models of Size-Biased Polya-Eggenberger Distribution (SBPED)

Size-biased distributions are a special case of the more general form known as weighted distributions. First introduced by Fisher (1934) to model ascertainment bias, weighted distributions were later formalized in a unifying theory by Rao (1965). Such distributions occur naturally in practice when observations from a sample are recorded with unequal probability, such as from probability proportional to size (PPS) designs. Briefly, if a random variable X has distribution $f(x, \theta)$, with unknown parameters θ , then the corresponding weighted distribution is of the form

$$f^w(x, \theta) = \frac{w(x)f(x, \theta)}{E[w(x)]} \quad (2.1)$$

where $w(x)$ is a non-negative weight function such that $E[w(x)]$ exists.

A special case arises when the weight function is of the form $w(x) = x^\beta$. Such distributions are known as size-biased distributions of order β (Patil&Ord, 1976; Patil, 1981; Mahfoud&Patil, 1982) and are written as:

$$f^*_{\beta}(x, \theta) = \frac{x^{\beta} f(x, \theta)}{\mu'_{\beta}} \quad (2.2)$$

where $\mu'_{\beta} = \sum_x x^{\beta} f(x, \theta)$ is the β^{th} raw moment of $f(x, \theta)$.

If X is a Polya-Eggenberger variate with a probability mass function as given by (1.2), then its mean is given by (1.6). The size-biased version of X , known as the size-biased Polya-Eggenberger distribution (SBPED), can be obtained directly by taking $\beta = 1$ in (2.2) and using (1.6). The resulting equation is:

$$p(x) = \frac{x}{n\alpha} \binom{n}{x} \frac{\alpha^{[x]} \gamma^{[n-x]}}{(\alpha + \gamma)^{[n]}}$$

which can be put into the form:

$$p(x) = \binom{n-1}{x-1} \frac{(\alpha+1)^{[x-1]} \gamma^{[n-x]}}{(\alpha + \gamma + 1)^{[n-1]}}, \quad x = 1, 2, \dots, n. \quad (2.3)$$

Equation (2.3) gives the probability mass function (PMF) of the size-biased Polya-Eggenberger distribution (SBPED).

The Mixture Model

The size-biased Polya-Eggenberger distribution can also be regarded as a beta mixture of size-biased binomial distribution. The PMF of the size-biased binomial distribution is given by

$$P(x) = \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x}, \quad (2.4)$$

$$0 < p < 1, \quad x = 1, 2, \dots, n$$

and the PDF of the beta distribution of first kind is

$$f(p) = \frac{1}{\beta(\alpha, \gamma)} p^{\alpha-1} (1-p)^{\gamma-1}, \quad (2.5)$$

$$0 < p < 1.$$

Compounding (2.4) with (2.5) through the values of p results in

$$P(x) = \binom{n-1}{x-1} \frac{1}{\beta(\alpha, \gamma)} \int_0^1 p^{\alpha+x-2} (1-p)^{\gamma+n-x-1} dp$$

$$= \binom{n-1}{x-1} \frac{\beta(\alpha+x-1, \gamma+n-x)}{\beta(\alpha, \gamma)}$$

which, after simplification, gives:

$$p(x) = \binom{n-1}{x-1} \frac{\alpha^{[x]} \gamma^{[n-x]}}{(\alpha + \gamma)^{[n-1]}}, \quad (2.6)$$

$$x = 1, 2, \dots, n.$$

Replacing α with $\alpha+1$, (2.6) coincides with (2.3) and the PMF of the size-biased Polya-Eggenberger distribution can be obtained.

Structural Properties of SBPED: Unimodality

The proposed model (2.3) is unimodal according to the results of Holgate (1970).

Lemma: If the mixing distribution is non-negative, continuous and unimodal, then the resulting distribution is unimodal.

The proposed model (2.3) is unimodal because the mixing distribution is a beta distribution, which is unimodal for $\alpha > 1$ and $\gamma > 1$.

Structural Properties of SBPED: Recurrence Relation between Probabilities

Taking $x=x+1$ in (2.3) and dividing the resulting equation by (2.3), results in the ratio

$$\frac{P(x+1)}{P(x)} = \frac{(n-x)}{x} \frac{(\alpha+x)}{(\gamma+n-x-1)}$$

which gives the recurrence relation between probabilities as:

$$P(x+1) = \left[\frac{(n-x)}{x} \frac{(\alpha+x)}{(\gamma+n-x-1)} \right] P(x). \quad (3.1)$$

Structural Properties of SBPED: Recurrence Relation between Moments

Multiplying (3.1) by x^k and summing the resulting equation over x results in

$$\begin{aligned} \mu'_{k+1}(n, \alpha, \gamma) &= \sum_{x=1}^n x^k \frac{(n-1)!}{(n-x-1)!(x-1)!} \frac{(\alpha+1)^{[x]} \gamma^{[n-x]}}{(\alpha+\gamma+1)^{[n-1]}} \\ &= \frac{(n-1)(\alpha+1)}{(\alpha+\gamma+1)} \left[\sum_{x=1}^{n-1} x^k \frac{(\overline{n-1-1})!}{(\overline{n-1-x})!(x-1)!} \right. \\ &\quad \left. \times \frac{(\alpha+2)^{[x-1]} \gamma^{[\overline{n-1-x}]}}{(\alpha+\gamma+2)^{[n-1-1]}} \right] \end{aligned} \quad (3.2)$$

which reduces to

$$\mu'_{k+1}(n, \alpha, \gamma) = \frac{(n-1)(\alpha+1)}{(\alpha+\gamma+1)} \mu'_{k+1}(n-1, \alpha+1, \gamma). \quad (3.3)$$

Equation (3.3) represents the recurrence relation between moments of proposed model (2.3). In particular

$$\mu'_1 = \frac{(n-1)(\alpha+1)}{(\alpha+\gamma+1)} \quad (3.4)$$

$$\mu'_2 = \frac{(n-1)(n-2)(\alpha+1)(\alpha+2)}{(\alpha+\gamma+1)(\alpha+\gamma+2)}$$

$$\mu'_3 =$$

$$\frac{(n-1)(n-2)(n-3)(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+\gamma+1)(\alpha+\gamma+2)(\alpha+\gamma+3)}$$

and

$$\mu'_4 =$$

$$\frac{(n-1)(n-2)(n-3)(n-4)(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}{(\alpha+\gamma+1)(\alpha+\gamma+2)(\alpha+\gamma+3)(\alpha+\gamma+4)}$$

Structural Properties of SBPED: Factorial Moments

Suppose $\mu'_{(k)}(n, \alpha, \gamma)$ denotes the k^{th} factorial moments of proposed model (2.3), then by definition

$$\begin{aligned} \mu'_{(k)}(n, \alpha, \gamma) &= \sum_{x=1}^n x^{(k)} \frac{(n-1)!}{(n-x)!(x-1)!} \frac{(\alpha+1)^{[x]} \gamma^{[n-x]}}{(\alpha+\gamma+1)^{[n-1]}} \\ &= \sum_{x=k}^n x \frac{(n-1)!}{(n-x)!(x-k)!} \frac{(\alpha+1)^{[x-1]} \gamma^{[n-x]}}{(\alpha+\gamma+1)^{[n-1]}} \end{aligned} \quad (3.5)$$

Taking $x = x+k-1$ in (3.5) results in

$$\begin{aligned} \mu'_{(k)} &= \sum_{x=1}^{n-k+1} (x+k-1) \\ &\quad \frac{(n-1)!}{(\overline{n-k+1-x})!(x-1)!} \frac{(\alpha+1)^{[x+k-2]} \gamma^{[\overline{n-k+1-x}]}}{(\alpha+\gamma+1)^{[n-1]}} \end{aligned}$$

and, after simplifying, gives:

$$\begin{aligned} \mu'_{(k)} &= \frac{(n-1)^{(k-1)} (\alpha+1)^{[k-1]}}{(\alpha+\gamma+1)^{[k-1]}} \\ &\quad \sum_{x=1}^{n-k+1} \left[\frac{(x+k-1) \times \frac{(\overline{n-k+1-1})!}{(\overline{n-k+1-x})!(x-1)!}}{\frac{(\alpha+k-1+1)^{[x-1]} \gamma^{[\overline{n-k+1-x}]} }{(\alpha+k-1+\gamma+1)^{[n-k+1-1]}}} \right] \end{aligned}$$

This equation, together with (3.2), gives

$$\begin{aligned} \mu'_{(k)} &= \frac{(n-1)^{(k-1)} (\alpha+1)^{[k-1]}}{(\alpha+\gamma+1)^{[k-1]}} \times \\ &\quad [\mu'_1(n-k+1, \alpha+k-1, \gamma) + (k-1)] \end{aligned}$$

Using (3.4), the k^{th} factorial moment is obtained as

$$\mu'_{(k)} = \frac{(n-1)^{(k-1)}(\alpha+1)^{[k-1]}}{(\alpha+\gamma+1)^{[k-1]}} \times \left[\frac{(n-k)(\alpha+k)}{(\alpha+\gamma+k)} + (k-1) \right] \quad (3.6)$$

and the first four factorial moments are

$$\mu'_1 = \frac{(n-1)(\alpha+1)}{(\alpha+\gamma+1)}$$

$$\mu'_{(2)} = \frac{(n-1)(n-2)(\alpha+1)(\alpha+2)}{(\alpha+\gamma+1)(\alpha+\gamma+2)} + \frac{(n-1)(\alpha+1)}{(\alpha+\gamma+1)}$$

$$\mu'_{(3)} = \frac{(n-1)(n-2)(n-3)(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+\gamma+1)(\alpha+\gamma+2)(\alpha+\gamma+3)} + \frac{2(n-1)(n-2)(\alpha+1)(\alpha+2)}{(\alpha+\gamma+1)(\alpha+\gamma+2)}$$

and

$$\mu'_{(4)} = \frac{(n-1)(n-2)(n-3)(n-4)(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}{(\alpha+\gamma+1)(\alpha+\gamma+2)(\alpha+\gamma+3)(\alpha+\gamma+4)} + \frac{3(n-1)(n-2)(n-3)(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+\gamma+1)(\alpha+\gamma+2)(\alpha+\gamma+3)}.$$

Relation with Other Distributions

Theorem 4.1

Let X be a size-biased Polya-Eggenberger variate with PMF (2.3). If $\gamma \rightarrow \infty$ such that $\alpha\gamma^{-1} = \theta$, and $n \rightarrow \infty$ such that $n\theta = \lambda$, then X tends to a size-biased Poisson distribution with parameter λ .

Theorem 4.1 Proof

The PMF of proposed model (2.3) is

$$P(x) = \left[\frac{(n-1) \dots (n-x+1)}{(x-1)!} \times \frac{(\alpha+1)(\alpha+2) \dots (\alpha+x-1)\gamma(\gamma+1) \dots (\gamma+n-x-1)}{(\alpha+\gamma+1)(\alpha+\gamma+2) \dots (\alpha+\gamma+n-1)} \right]$$

Taking the limit $\gamma \rightarrow \infty$ such that $\frac{\alpha}{\gamma} = \theta$ results in

$$P(x) = \frac{(n-1) \dots (n-x+1)}{(x-1)!} \frac{\theta^{x-1}}{(1+\theta)^{n-1}}.$$

Proceeding to limit $n \rightarrow \infty$, such that $n\theta = \lambda$, the equation reduces to a size-biased Poisson distribution with parameter λ .

Estimation

Different estimation techniques are now put forth to estimate the parameters of the proposed model. The model (2.3) has three parameters n , α and γ . The parameter n is known, whereas the remaining two parameters α and γ must be estimated.

The Moment Method

Let m'_1 , m'_2 be the sample moments (about origin) of a size-biased Polya-Eggenberger distribution (2.3). The method of moments consists in comparing the sample moments with the population moments of the distribution. The two equations thus obtained are

$$m'_1 = \frac{(n-1)(\alpha+1)}{(\alpha+\gamma+1)} \quad (5.1)$$

and

$$m'_2 = \frac{(n-1)(n-2)(\alpha+1)(\alpha+2)}{(\alpha+\gamma+1)(\alpha+\gamma+2)}. \quad (5.2)$$

Dividing (5.2) by (5.1) gives:

$$(t+1) \frac{m'_2}{m'_1} = n\alpha - 2\alpha + 2n - 4 \quad (5.3)$$

where

$$\alpha + \gamma + 1 = t. \quad (5.4)$$

From equation (5.1),

$$\alpha = \frac{m'_1 t + 1 - n}{n - 1} \quad (5.5)$$

and, from (5.3),

$$\alpha = \frac{(t+1) \frac{m'_2}{m'_1} + 4 - 2n}{n - 2}. \quad (5.6)$$

Eliminating α between (5.5) and (5.6) results in:

$$\frac{m'_1 t + 1 - n}{n - 1} = \frac{(t+1) \frac{m'_2}{m'_1} + 4 - 2n}{n - 2},$$

which, after simplification, gives the value of t as

$$t = \frac{m'_2(n-1) - n + 1}{m'_1(n-2) - n(m'_1 + m'_2) - 2m'_1 + m'_2}.$$

Substituting the value of t from this equation into (5.5), the value of α can be obtained, and after substituting the values of t and α into (5.4) the value of γ can be obtained.

Using the Mean and First Two Cell Frequencies

Taking $x=1, 2$ in the size-biased Polya-Eggenberger distribution (2.3) and equating these probabilities with their corresponding relative frequencies $\frac{f_1}{N}, \frac{f_2}{N}$ results in:

$$\frac{\gamma^{[n-1]}}{(\alpha + \gamma + 1)^{[n-1]}} = \frac{f_1}{N} \quad (5.7)$$

and

$$\frac{(\alpha + 1)\gamma^{[n-2]}}{(\alpha + \gamma + 1)^{[n-1]}} = \frac{f_2}{N},$$

where

$$N = \sum f_i. \quad (5.8)$$

Dividing (5.8) by (5.7) the ratio

$$\frac{f_2}{f_1} = \frac{(n-1)(\alpha+1)}{(\gamma+n-2)} \quad (5.9)$$

is obtained. Eliminating α between (5.9) and (5.1), the value of γ results as

$$\gamma = \frac{(n-\bar{x}-1)(n-2)(f_1+f_2)}{\bar{x}f_2 + (n-1)(\bar{x}f_1 - f_2)}.$$

Substituting the value of γ from this equation into (5.1) the value of α can be obtained.

The Method of Maximum-Likelihood

The log likelihood function of size-biased Polya-Eggenberger distribution is given by

$$\begin{aligned} \log L &= N \log \Gamma(n) - \sum f_x \log \Gamma(n-x+1) \\ &\quad - \sum f_x \log \Gamma(x+1) + \sum f_x \log \Gamma(\alpha+x) \\ &\quad - N \log \Gamma(\alpha+1) + \sum f_x \log \Gamma(\gamma+n-x) \\ &\quad - N \log \Gamma(\gamma) + N \log \Gamma(\alpha+\gamma+1) \\ &\quad - \sum f_x \log \Gamma(\alpha+\gamma+n), \end{aligned}$$

where f_x is the observed frequency for the variate value x and $N = \sum f_x$.

The proposed model has two unknown parameters, namely, (α, γ) . The log likelihood equations for estimating α and γ are

$$\frac{\partial \log L}{\partial \alpha} = 0 = -N \sum_{k=1}^{n-1} \frac{1}{\alpha + \gamma + k} + \sum f_x \sum_{k=1}^{x-1} \frac{1}{\alpha + k} \quad (5.10)$$

and

$$\frac{\partial \log L}{\partial \gamma} = 0 = \sum f_x \sum_{k=0}^{n-x-1} \frac{1}{\gamma+k} + \sum f_x \sum_{k=1}^{n-1} \frac{1}{\alpha+\gamma+k} \quad (5.11)$$

These equations do not provide direct solutions, thus an iterative solution method, such as Newton-Rampson or Fisher's scoring method, are required to solve these equations. The following system of equations may also be solved:

$$(\hat{\theta} - \theta_0) \left[\frac{\partial^2 \log L}{\partial \theta^2} \right]_{\theta_0} = \left[\frac{-\partial \log L}{\partial \theta} \right]_{\theta_0}$$

where $\hat{\theta} = (\alpha, \gamma)$ is a parameter vector, the ML estimate of θ and θ_0 is the trial value of θ which may first be obtained by equating the theoretical frequencies with the observed frequencies.

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