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CONSTRAINTS**

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Optimal Boundary Control of Hyperbolic Equations with Pointwise State Constraints

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In this paper we consider dynamic optimization problems for hyperbolic systems with boundary controls and pointwise state constraints. In contrast to parabolic dynamics, such systems have not been sufficiently studied in the literature. The reason is the lack of regularity in the case of hyperbolic dynamics. We present necessary optimality conditions for both Neumann and Dirichlet boundary control problems and discuss differences and relationships between them.

1. INTRODUCTION

This paper is devoted to optimal control of hyperbolic equations. As well known, hyperbolic control systems are very interesting and challenging from both viewpoints of the theory and applications being essentially different and much less investigated in comparison with optimal control of parabolic and elliptic partial differential equations. Major issues in the optimal control theory revolve around necessary optimality conditions. While such conditions have been largely investigated for hyperbolic systems with *distributed control* (see, e.g., the books [3,6] and the references therein), not much has been known for *boundary control* problems, especially in the presence of *pointwise state constraints*. The reason is the *lack of regularity* for hyperbolic boundary control systems, in both cases of the Neumann and Dirichlet boundary conditions. This is essentially different from parabolic systems, where the amount of regularity is sufficient to build a satisfactory theory of state-constrained boundary optimal control; see, e.g., [2,10,11] and their references.

In this paper, which is mainly based on our recent developments in [8,9], we present a unified overview of necessary optimality conditions for both Neumann and Dirichlet boundary control problems, with the discussion of differences between them and the comparison with similar problems for parabolic systems. We begin with Neumann boundary controls and then consider Dirichlet ones; these two cases happen to be dramatically different requiring completely different methods for their study. In what follows we use the standard notation in the PDE control theory.

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2. NEUMANN BOUNDARY CONTROL OF STATE-CONSTRAINED SEMILINEAR HYPERBOLIC EQUATIONS

Given an open bounded domain $\Omega \subset \mathbb{R}^n$ with a boundary Γ of class $\mathcal{C}^2$ and given a time $T > 0$, concern the following optimal control problem governed by the *semilinear wave equation*: minimize

$$J(y, u) = \int_{\Omega} f(x, y(T)) dx + \int_Q g(x, t, y) dx dt + \int_{\Sigma} h(s, t, u) ds dt \quad (1)$$

over admissible pairs $\{y(\cdot), u(\cdot)\}$ satisfying

$$\begin{cases} y_{tt} - \Delta y + \vartheta(\cdot, y) = 0 & \text{in } Q := \Omega \times (0, T), \\ \partial_{\nu} y = u & \text{in } \Sigma := \Gamma \times (0, T), \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in } \Omega \end{cases} \quad (2)$$

under the *pointwise constraints* on control and state functions

$$u(\cdot) \in U_{ad} \subset L^2(\Sigma), \quad y(\cdot) \in \Theta \subset \mathcal{C}([0, T]; L^2(\Omega)),$$

where the operator Δ stands for the classical *Laplacian*. Denote this *Neumann boundary control* problem by (NP) and shortly write it as

$$\inf \left\{ J(y, u) \mid \{y(\cdot), u(\cdot)\} \text{ satisfies (1), } u(\cdot) \in U_{ad}, y(\cdot) \in \Theta \right\}.$$

The initial state $(y_0, y_1) \in H^1(\Omega) \times L^2(\Omega)$ in (2) is fixed. Note that, although we focus on the dynamics governed by the semilinear wave equation (2), the results presented below can be extended to hyperbolic equations governed by more general *strongly elliptic operators* in (2)– not just by the Laplacian Δ –with *time-independent and regular* coefficients.

Suppose in what follows that the functions f, g, h , and ϑ in (NP) satisfy standard *standing assumptions* on differentiability and sublinear growth with respect to state and control variable and on measurability and integrable boundedness with respect to t . Note that we do *not* impose any *convexity* assumptions on these functions and on the admissible control set U_{ad} given in the form

$$U_{ad} := \{u \in L^2(\Sigma) \mid u(s, t) \in K(s, t) \text{ a.e. } (s, t) \in \Sigma\},$$

where $K(\cdot)$ is a measurable multifunction whose values are nonempty and closed subsets of $\mathbb{R}$. Regarding the state constraint set $\Theta \subset \mathcal{C}([0, T]; L^2(\Omega))$, we may assume for simplicity that it is closed and *convex* with $\text{int } \Theta \neq \emptyset$, although the results obtained hold true under the more relaxed *sequential normal compactness* property of Ω , which is automatic when Θ is *finite-codimensional* with $\text{ri } \Omega \neq \emptyset$; see [7] for more details.

Solutions to the state system (2) are understood of the following *weak* sense.

Definition 1 (weak solutions to the Neumann system). A function $y(\cdot)$ with $(y, y_t) \in \mathcal{C}([0, T]; L^2(\Omega)) \times \mathcal{C}([0, T]; H^1(\Omega)^*)$ is a WEAK SOLUTION to (1) if

$$\begin{aligned} \int_Q -\vartheta(\cdot, y) z \, dx dt &= \int_Q y \varphi \, dx dt - \langle y_t(0), z(0) \rangle_{H^1(\Omega)^*, H^1(\Omega)} \\ &+ \int_{\Omega} y(0) z_t(0) \, dx + \int_{\Sigma} z u \, ds dt \end{aligned}$$

for all $\varphi \in L^1(0, T; L^2(\Omega))$, where $z(\cdot)$ solves the homogeneous Neumann boundary value problem

$$\begin{cases} z_{tt} - \Delta z = \varphi & \text{in } Q, \\ \partial_\nu z = 0 & \text{in } \Sigma, \\ z(T) = 0, \quad z_t(T) = 0 & \text{in } \Omega. \end{cases}$$

To formulate the main necessary optimality conditions for the Neumann boundary control problem (NP), define the (analog of) *Hamilton-Pontryagin* function

$$H(s, t, p, \lambda) := pu + \lambda h(s, t, u).$$

Theorem 2 (pointwise necessary optimality conditions for Neumann boundary controls). Let $\{\bar{y}(\cdot), \bar{u}(\cdot)\}$ be an optimal solution to problem (NP) under the standing assumptions made. Then there exist $\lambda \geq 0$, $\mu \in \mathcal{M}_b([0, T]; L^2(\Omega))$, and a measurable subset $\tilde{\Sigma} \subset \Sigma$ such that $\mathcal{L}^n(\Sigma \setminus \tilde{\Sigma}) = 0$,

$$(\lambda, \mu) \neq 0, \quad \langle \mu, z - \bar{y} \rangle \leq 0 \quad \text{for all } z \in \Theta, \quad \text{and}$$

$$H(s, t, \bar{u}(s, t), p(s, t), \lambda) = \min_{u \in K(s, t)} H(s, t, u, p(s, t), \lambda) \quad (3)$$

for all $(s, t) \in \tilde{\Sigma}$, where $\mathcal{L}^n$ denotes the n -dimensional Lebesgue measure, and where $p(\cdot)$ is the corresponding weak solution to the adjoint system

$$\begin{cases} p_{tt} - \Delta p + \vartheta'_y(\cdot, \bar{y})p = \lambda g'_y(x, t, \bar{y}) + \mu|_Q & \text{in } Q, \\ \partial_\nu p = 0 & \text{in } \Sigma, \\ p(T) = y_0, \quad p_t(T) = -\lambda f'_y(x, \bar{y}(T)) - \mu|_{\Omega \times \{T\}} & \text{in } \Omega. \end{cases} \quad (4)$$

The proof of Theorem 2 involves the following major components:

1. Regularity analysis of weak solutions to the state system (2) and the adjoint system (4), which includes the proof of the existence and uniqueness results as well as the continuous dependence of such solutions on boundary controls and the initial data in the corresponding functional spaces. This part strongly employs the basic regularity results for the classical linear Neumann boundary value problem [5] together with delicate

methods of regularity analysis for nonlinear hyperbolic partial differential equations with possible *nonsmooth* data; cf. [9].

2. Variational analysis of hyperbolic systems with no state constraints. This is based on the so-called *increment formula* for cost functionals of type (1) for hyperbolic systems (2) with pointwise control constraints but no state constraints by using *needle-type variations* of optimal controls. Although such a device is well-developed in the cases of ordinary control systems and parabolic equations, it requires more careful analysis and estimates in the case of hyperbolic systems under consideration, with the usage of the regularity results discussed above in Item 1. As a conclusion, necessary optimality conditions the *Pontryagin Maximum Principle* type (actually of the minimum type (3)) are obtained for hyperbolic Neumann problems with no state constraints, which are used in the subsequent variational analysis of the original problem; see below.

3. Perturbations and variational analysis of the state-constrained hyperbolic problem. This involves the construction of well-posed approximations of the original Neumann problem (*NP*) via a penalization procedure based on the *Ekeland variational principle*. The approximating problems are of the type considered in Item 2, i.e., they do not have state constraints. The final result of Theorem 2 is now derived by passing to the limit from the approximating problems employing again the regularity analysis and estimates established in Item 1.

For brevity we don't address in this section the *existence* issue for optimal solutions to the Neumann boundary control problem under consideration. However, a general existence theorem for this problem can be derived from the regularity results discussed above in Item 1 via the application of the classical Weierstrass theorem on the existence of optimal solutions to abstract problems of minimizing l.s.c. functions over compact sets in suitable topologies. What we need, however, is to impose additional *convexity* assumptions on the integrand h with respect to the *control variable*, as well as the convexity of the control sets $K(s, t)$. Such a convexity, which is not needed for deriving the pointwise necessary optimality conditions, is required for the existence theorem in order to ensure the lower semicontinuity of the cost functional and the closedness of the feasible control set with respect to the corresponding *weak* convergence of controls that implies, by *regularity*, the *strong* convergence of trajectories. Note that we have to impose essentially more restrictive assumptions to handle the Dirichlet boundary control problem in the next section, where the convexity in both *control and state* variables is needed not only for the existence of optimal solutions but also for deriving necessary optimality conditions. The main reason is that Dirichlet boundary control problems, for hyperbolic as well as for parabolic systems, exhibit much *less regularity* in comparison for their Neumann counterparts, and thus they require different methods for their variational analysis.

3. DIRICHLET BOUNDARY CONTROL OF LINEAR CONSTRAINED HYPERBOLIC EQUATIONS

Next we consider a *Dirichlet* counterpart of the Neumann boundary control problem for hyperbolic equations with pointwise state constraints from Section 2. As mentioned, there are essential differences between Neumann and Dirichlet boundary conditions for

hyperbolic equations; so the methods and results developed in this section are considerably distinguished from those in the preceding one. Roughly speaking, the requirements imposed on the initial data in the Dirichlet problem are *stronger*, while the results we are able to obtain are *weaker* in comparison with the above case of Neumann boundary controls. This is due to the *lack of regularity* in the Dirichlet hyperbolic case (much less regularity than for Neumann hyperbolic systems as well as for Dirichlet parabolic systems), which forces us to develop a different approach to the variational analysis of the state-constrained Dirichlet boundary control problem in what follows. In particular, necessary optimality conditions are derived in this approach by reducing the Dirichlet control problem of *dynamic optimization* to a problem of *mathematical programming* in infinite dimensions with *geometric* and *operator constraints* of special types.

Formulate the *Dirichlet boundary control problem* (*DP*) as follows: minimize the same cost functional (1) over admissible pairs $\{y(\cdot), u(\cdot)\}$ satisfying the multidimensional *linear wave equation* with control functions acting in the Dirichlet boundary conditions

$$\begin{cases} y_{tt} - \Delta y = \vartheta & \text{in } Q := \Omega \times (0, T), \\ y = u & \text{in } \Sigma := \Gamma \times (0, T), \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in } \Omega \end{cases} \quad (5)$$

subject to the *pointwise control and state constraints*

$$u(\cdot) \in U_{ad} \subset L^2(\Sigma), \quad y(\cdot) \in \Theta \subset \mathcal{C}([0, T]; L^2(\Omega)),$$

where $\vartheta \in L^1(0, T; H^{-1}(\Omega))$, $y_0 \in L^2(\Omega)$, and $y_1 \in H^{-1}(\Omega)$ are given functions.

Our primary goal is to present necessary optimality conditions for the Dirichlet state-constrained problem (*DP*) under consideration; the same goal as for the Neumann problem (*NP*) studied in Section 2. However, we have to impose essentially more restrictive assumptions on the initial data of (*DP*), in comparison with those for (*NP*), to achieve even weaker results; see below. Observe that the hyperbolic dynamics in (*DP*) is described by the *linear* wave equation with ϑ independent of y , in comparison with the *semilinear* one in (*NP*). On the other hand, we impose *milder* requirements on the initial state $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ for the Dirichlet problem in comparison with $(y_0, y_1) \in H^1(\Omega) \times L^2(\Omega)$ for the Neumann case. In fact, the results obtained below for (*DP*) can be extended to more general linear hyperbolic equations with a *strongly elliptic operator* instead of the Laplacian Δ .

The main difference between the assumptions on the initial data of (*DP*) in comparison with the Neumann counterpart (*NP*) in Section 1 is that now we impose the *convexity* assumptions on the integrands in (1) in *both state and control* variables, together with the convexity of the control set U_{ad} , while *no* convexity is required for the Neumann problem. As mentioned, it is due to the *lack of regularity* for the Dirichlet system (5) in comparison with the Neumann one. In fact, the extra convexity assumptions allow us to *compensate*, in a sense, the lack of regularity. Note similarly to the Neumann case that the nonempty interiority assumption on the state constraint set Θ can be replaced by the much less restrictive assumption on its *sequential normal compactness*; see [7].

Based on the full convexity and the available regularity, we reduce the Dirichlet control problem under consideration to a special problem of *mathematical programming* with geometric and operator constraints in Banach spaces and then deduce necessary optimality conditions for (DP) from an appropriate version of the (abstract) Lagrange multiplier rule for mathematical programming. The necessary optimality conditions for the Dirichlet problem derived in this way are given in the *integral form* of the Pontryagin maximum principle, in contrast to the pointwise form for the Neumann problem in Section 2. Furthermore, the assumptions made allow us to establish a general *existence theorem* for optimal controls in problem (DP), which are not considered here; see [8] for the exact statement and more details.

Before formulating the necessary optimality conditions for the Dirichlet problem (DP), let us define the appropriate concept of *weak solutions* to (5).

Definition 3 (weak solutions to the Dirichlet system). A function $y(\cdot)$ with $(y, y_t) \in \mathcal{C}([0, T]; L^2(\Omega)) \times \mathcal{C}([0, T]; H^{-1}(\Omega))$ is a WEAK SOLUTION to (5) if one has

$$\begin{aligned} \int_Q f z \, dx dt &= \int_Q y \varphi \, dx dt + \langle y_t(T), z^0 \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \\ &\quad - \langle y_t(0), z(0) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} - \int_\Omega y(T) z^1 \, dx + \int_\Omega y(0) z_t(0) \, dx + \int_\Sigma \frac{\partial z}{\partial \nu_u} \, ds dt \end{aligned}$$

for all $(\varphi, z^0, z^1) \in L^1(0, T; L^2(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega)$, where z solves the homogeneous Dirichlet problem

$$\begin{cases} z_{tt} - \Delta z = \varphi & \text{in } Q, \\ z = 0 & \text{in } \Sigma, \\ z(T) = z^0, \quad z_t(T) = z^1 & \text{in } \Omega. \end{cases}$$

Now we are ready to present the main necessary optimality conditions for the Dirichlet boundary control problem.

Theorem 4 (necessary optimality conditions for the Dirichlet problem). Under the assumptions made, for every optimal solution $\{\bar{y}(\cdot), \bar{u}(\cdot)\}$ to problem (DP) the following conditions hold: there are $\lambda \geq 0$ and $\mu \in \mathcal{M}_b([0, T]; L^2(\Omega))$ such that

$$(\lambda, \mu) \neq 0, \quad \langle \mu, y - \bar{y} \rangle \leq 0 \text{ for all } y \in \Theta \text{ and}$$

$$\int_\Sigma \left(\frac{\partial p}{\partial \nu} + \lambda h'_u(s, t, \bar{u}) \right) (u - \bar{u}) \, ds dt \geq 0 \text{ for all } u \in U_{ad}, \quad (6)$$

where p is the corresponding weak solution to the adjoint system

$$\begin{cases} p_{tt} - \Delta p = \lambda g'_y(x, t, \bar{y}) + \mu|_Q & \text{in } Q, \\ p = 0 & \text{in } \Sigma, \\ p(T) = y_0, \quad p_t(T) = -\lambda f'_u(x, \bar{y}(T)) - \mu|_{\Omega \times \{T\}} & \text{in } \Omega. \end{cases} \quad (7)$$

Moreover, if there exists $\{y(\cdot), u(\cdot)\} \in Y \times (U_{ad} - \bar{u})$ satisfying

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in } Q, \quad y = u & \text{in } \Sigma, \\ y(0) = 0, \quad y_t(0) = 0 & \text{in } \Omega, \quad \bar{y} + y \in \text{int } \Theta \end{cases}$$

with the state space Y defined by

$$Y := \left\{ y \in \mathcal{C}([0, T]; L^2(\Omega)) \mid y_t \in \mathcal{C}([0, T]; H^{-1}(\Omega)), \right.$$

$$\left. y_{tt} - \Delta y \in L^1(0, T; H^{-1}(\Omega)), \quad y|_{\Sigma} \in L^2(\Sigma) \right\},$$

then one can take $\lambda = 1$ in the above optimality conditions.

Observe that (6) is the *integral minimum* condition, which is generally weaker than its *pointwise* counterpart (3) in the Neumann case.

The proof of Theorem 4 involves the following major steps:

1. Regularity analysis. In the case of the *state system* (5) it is based on the fundamental regularity results of [4] related to Dirichlet boundary value problems for linear hyperbolic equations. Besides this, the regularity analysis of the *adjoint system* (7) requires additional limiting procedures involving weak* compactness properties of topological duals to the Banach spaces in question.

2. Reduction to mathematical programming. The dynamic optimization problem (*DP*) can be reduced to the following infinite-dimensional problem (*MP*) of mathematical programming with geometric and operator constraints:

$$\varphi(z, w) \text{ subject to } z \in Z, \quad w \in W_{ad}, \quad f_1(z, w) = 0, \quad f_2(w) \in \Xi,$$

where $\varphi: Z \times W \rightarrow \mathbb{R}$, $f_1: Z \times W \rightarrow Z_1$, $f_2: Z \rightarrow Z_2$, where the sets $W_{ad} \subset W$ and $\Xi \subset Z_2$ are closed and convex, and where the spaces Z, Z_1, Z_2, W are Banach with W being separable. Necessary optimality conditions for problem (*MP*) are derived in [1] under the convexity assumption on W_{ad} and the smoothness assumptions on the mappings involved with the surjective partial derivative f'_{1z} at the optimal point. We refer the reader to [7] for more general results in this direction with no smoothness and convexity assumptions.

3. Necessary conditions for the Dirichlet problem in terms of the initial data. The final step in the proof of Theorem 4 consists of expressing the necessary optimality conditions obtained for (*MP*) in terms of the initial data of problem (*DP*). The Dirichlet problem (*DP*) is written in the form of (*MP*) with

$$Z := Y, \quad (z, w) := (y, u), \quad W := L^2(\Sigma), \quad W_{ad} := U_{ad}, \quad \Xi := \Theta,$$

$$Z_1 := L^1(0, T; H^{-1}(\Omega)) \times L^2(\Sigma) \times L^2(\Omega) \times H^{-1}(\Omega), \quad Z_2 = \mathcal{C}([0, T]; L^2(\Omega)),$$

$$\varphi(y, u) := J(y, u), \quad f_1(y, u) := (y_{tt} - \Delta y - \vartheta, y|_{\Sigma} - u, y(0) - y_0, y_t(0) - y_1), \quad f_2(y) := y.$$

Taking into account the specific form of the above operators, we can deduce the assumptions needed for the optimality conditions in Item 2 via those imposed in Theorem 4 and then to transform the Lagrange-type conditions for problem (MP) in Item 2 into the Pontryagin-type conditions of Theorem 4. Note that the major *surjectivity* assumption on the derivative $f'_1(\bar{z}, \bar{w})$ of the operator constraint in (MP) holds due to the basic regularity results discussed in Item 1.

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