5-1-2001

Geometric Realization and *K*-Theoretic Decomposition of C^*-Algebras

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GEOMETRIC REALIZATION AND K-THEORETIC DECOMPOSITION OF C*-ALGEBRAS

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Submitted November 25, 1999

Abstract. Suppose that A is a separable C^*-algebra and that G_* is a (graded) subgroup of the $\mathbb{Z}/2$-graded group $K_*(A)$. Then there is a natural short exact sequence

$$(*) \qquad 0 \to G_* \longrightarrow K_*(A) \longrightarrow K_*(A)/G_* \to 0.$$

In this note we demonstrate how to geometrically realize this sequence at the level of C^*-algebras. As a result, we KK-theoretically decompose A as

$$0 \to A \otimes \mathcal{K} \longrightarrow A_f \longrightarrow SA_t \to 0$$

where $K_*(A_t)$ is the torsion subgroup of $K_*(A)$ and $K_*(A_f)$ is its torsionfree quotient. Then we further decompose A_t: it is KK-equivalent to $\oplus_p A_p$ where $K_*(A_p)$ is the p-primary subgroup of the torsion subgroup of $K_*(A)$. We then apply this realization to study the Kasparov group $K^*(A)$ and related objects.

In Section 1 we produce the basic geometric realization. For any separable C^*-algebra A and group G_* we produce associated C^*-algebras A_s (s for subgroup) and A_q (q for quotient group) and, most importantly, a short exact sequence of C^*-algebras

$$0 \to A \otimes \mathcal{K} \to A_q \to SA_s \to 0$$

whose associated K_*-long exact sequence is (*). In the case where G_* is the torsion subgroup of $K_*(A)$ we use the notation A_t (t for torsion) and A_f (f for torsionfree) respectively. We further decompose A_t into its p-primary summands A_p for each prime p.

Section 2 deals with the following question: may calculations of the Kasparov groups $KK_*(A, B)$ be reduced down to the four cases (A_t, B_t), (A_t, B_f), (A_f, B_t) and (A_f, B_f) ? We show that this is indeed possible in a wide variety of situations. Sections 3 and 4 deal with these special cases.

1991 *Mathematics Subject Classification.* Primary 46L80, 19K35, 46L85.

Key words and phrases. K-theory for C^*-algebras, geometric realization, Kasparov theory.

Geometric realization as a general technique was introduced to topological K-theory of spaces by M. F. Atiyah [1] in his proof of the Künneth theorem for $K^*(X \times Y)$. We adapted the technique [6] to prove the corresponding theorem for the K-theory for C^*-algebras and used it with J. Rosenberg in our proof of the Universal Coefficient Theorem (UCT) [4].

1. Geometric Realization

In this section we produce the main geometric realization and we extend the result to give a p-primary decomposition for a C^*-algebra.

Let $\mathcal{N}$ denote the bootstrap category [6, 4].

Theorem 1.1. *Suppose that A is a separable C^*-algebra. Let G_* be some subgroup of $K_*(A)$. Then there is an associated C^*-algebra $A_s \in \mathcal{N}$, a separable C^*-algebra A_q, and a short exact sequence*

$$0 \to A \otimes \mathcal{K} \to A_q \to SA_s \to 0 \tag{1.2}$$

whose induced K-theory long exact sequence fits into the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_*(A_s) & \longrightarrow & K_*(A \otimes \mathcal{K}) & \longrightarrow & K_*(A_q) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & G_* & \longrightarrow & K_*(A) & \longrightarrow & K_*(A)/G_* & \longrightarrow & 0. \end{array} \tag{1.3}$$

If A is nuclear then so is A_q. If $A \in \mathcal{N}$ then so is A_q. If $A \in \mathcal{N}$ and if G_* is a direct summand of $K_*(A)$ then A is KK-equivalent to $A_s \oplus A_q$.

Note that we think of A_s as realizing the subgroup G_* and A_q as realizing the quotient group $K_*(A)/G_*$, hence the notation.

Proof. Let A_s denote any C^*-algebra in $\mathcal{N}$ with

$$K_*(A_s) \cong G_*.$$

Such C^*-algebras exist and are unique up to KK-equivalence by the UCT [4]. Let

$$\theta : K_*(A_s) \to K_*(A)$$

be the corresponding homomorphism. Since $A_s \in \mathcal{N}$, the UCT holds for the pair (A_s, A), and so θ is in the image of the index map

$$\gamma : KK_*(A_s, A) \to Hom_{\mathbb{Z}}(K_*(A_s), K_*(A)).$$

Say that

$$\theta = \gamma(\tau)$$

for some

$$\tau \in KK_0(A_s, A).$$

As A_s is nuclear,

$$KK_*(A_s, A) \cong \mathcal{E}xt(SA_s, A)$$

and hence τ corresponds to an equivalence class of extensions of C^*-algebras of the form

$$0 \to A \otimes \mathcal{K} \to E \to SA_s \to 0.$$

Define $A_q = E$. (This choice depends upon the choice of A_s among its KK-equivalence class and the choice of τ modulo the kernel of γ). Note that E is nuclear/bootstrap if and only if A is nuclear/bootstrap. Then the diagram

$$\begin{array}{ccccccc} K_j(A_q) & \longrightarrow & K_j(SA_s) & \xrightarrow{\delta} & K_{j-1}(A \otimes \mathcal{K}) & \longrightarrow & K_{j-1}(A_q) \\ & & \downarrow \cong & & \downarrow \cong & & \\ & & K_{j-1}(A_s) & \xrightarrow{\theta} & K_{j-1}(A) & & \end{array}$$

commutes, and thus δ is mono and the long exact K_*-sequence breaks apart as shown.

If G_* is a direct summand of $K_*(A)$ then

$$K_*(A) \cong G_* \oplus K_*(A)/G_* \cong K_*(A_s) \oplus K_*(A_q) \cong K_*(A_s \oplus A_q)$$

and, replacing algebras by their suspensions as needed, the KK-equivalence is obtained.

□

Henceforth we shall regard A_s and A_q as C^*-algebras associated to A and G_*, with the understanding that these are well-defined only up to KK-equivalence modulo the kernel of γ, as explained above.

The next step is to decompose A_t into its p-primary components.

Theorem 1.4. *Let $A \in \mathcal{N}$ and suppose that $K_*(A)$ is a torsion group, so that $A = A_t$. Then A is KK-equivalent to a C^*-algebra $\oplus A_p$, where*

$$K_*(A_p) \cong K_*(A)_p$$

the p-primary torsion subgroup of $K_*(A)$.

Proof. For each prime p, choose $N_{(p)} \in \mathcal{N}$ with $K_1(N_{(p)}) = 0$ and

$$K_0(N_{(p)}) \cong \mathbb{Z}_{(p)}$$

the integers localized at p. Define

$$A_p = A_t \otimes N_{(p)}.$$

The Künneth formula [6] implies that

$$K_*(A_p) \cong K_*(A_t \otimes N_{(p)}) \cong K_*(A_t) \otimes K_*(N_{(p)}) \cong K_*(A_t) \otimes \mathbb{Z}_{(p)} \cong K_*(A)_p$$

as desired. Then

$$K_*(\oplus_p A_p) \cong \oplus_p K_*(A_p) \cong \oplus_p K_*(A)_p \cong K_*(A_t)$$

and another use of the UCT implies that A_t is KK-equivalent to $\oplus_p A_p$.

□

We summarize:

Theorem 1.5. *Suppose that A is a separable C^*-algebra. Then there is an associated C^*-algebra $A_t \in \mathcal{N}$, a separable C^*-algebra A_f, and a short exact sequence*

$$0 \to A \otimes \mathcal{K} \to A_f \to SA_t \to 0 \tag{1.6}$$

whose induced K-theory long exact sequence fits into the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_*(A_t) & \longrightarrow & K_*(A \otimes \mathcal{K}) & \longrightarrow & K_*(A_f) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & K_*(A)_t & \longrightarrow & K_*(A \otimes \mathcal{K}) & \longrightarrow & K_*(A)_f & \longrightarrow & 0. \end{array} \tag{1.7}$$

If A is nuclear then so is A_f. If $A \in \mathcal{N}$ then so is A_f. Further, the C^*-algebra A_t has a p-primary decomposition: it is KK-equivalent to a C^*-algebra $\oplus_p A_p$, where $A_p \in \mathcal{N}$ for all p and

$$K_*(A_p) \cong K_*(A)_p$$

the p-primary torsion subgroup of $K_*(A)$. Finally, if $A \in \mathcal{N}$ and $K_*(A)_t$ is a direct summand of $K_*(A)$ then A may be replaced by the KK-equivalent C^*-algebra $A_t \oplus A_f$

□

2. Splitting the Kasparov Groups

If A and B are in $\mathcal{N}$ and their K-theory torsion subgroups $K_*(A)_t$ and $K_*(B)_t$ are direct summands then the final conclusion of Theorem 1.5 implies that we may reduce the computation of $KK_*(A, B)$ to the calculation of the four groups, namely

(1) $KK_*(A_t, B_t)$

(2) $KK_*(A_t, B_f)$

(3) $KK_*(A_f, B_t)$

(4) $KK_*(A_f, B_f)$.

We discuss the calculation of those groups in subsequent sections. In this section we see what can be done *without* assuming that the torsion subgroups are direct summands.

Theorem 2.1. *Suppose that $A \in \mathcal{N}$ and $K_*(B)$ is torsionfree. Then there is a short exact sequence*

$$0 \to KK_*(A_f, B) \to KK_*(A, B) \to KK_*(A_t, B) \to 0. \tag{2.2}$$

In particular, letting $K^*(A) = KK_*(A, \mathbb{C},)$ there is a short exact sequence

$$0 \to K^*(A_f) \to K^*(A) \to K^*(A_t) \to 0. \tag{2.3}$$

If $K_*(B)$ is not necessarily torsionfree, then sequence 2.2 is exact if and only if the natural map

$$\theta_h^* : Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \to Hom_{\mathbb{Z}}(K_*(A_t), K_*(B)) \tag{2.4}$$

is onto, where $\theta : K_*(A_t) \to K_*(A)$ *is the canonical inclusion.*

Note that the map θ_h^* in (2.4) is frequently onto. This is the case, for instance, if $K_*(A_t)$ is a direct summand of $K_*(A)$.

The map θ is, up to isomorphism, the boundary homomorphism in the K_*-sequence associated to the short exact sequence

$$0 \to A \otimes \mathcal{K} \to A_f \to SA_t \to 0$$

and hence

$$\theta(x) = x \otimes_{A_t} \delta$$

where $\delta \in KK_1(A_t, A)$ by [9]. Thus the map θ_h^* of (2.4) is induced from a KK-pairing.

Proof. Consider the commuting diagram

$$\begin{array}{ccccccccc}
 & & Hom_{\mathbb{Z}}(K_*(A_t), K_*(B)) & & KK_{*-1}(A_t, B) & & 0 & & \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Ext^1_{\mathbb{Z}}(K_*(A_f), K_*(B)) & \longrightarrow & KK_*(A_f, B) & \longrightarrow & Hom_{\mathbb{Z}}(K_*(A_f), K_*(B)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) & \longrightarrow & KK_*(A, B) & \longrightarrow & Hom_{\mathbb{Z}}(K_*(A), K_*(B)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_e^*} & & \downarrow{\scriptstyle \theta^*} & & \downarrow{\scriptstyle \theta_h^*} & & \\
0 & \longrightarrow & Ext^1_{\mathbb{Z}}(K_*(A_t), K_*(B)) & \longrightarrow & KK_*(A_t, B) & \longrightarrow & Hom_{\mathbb{Z}}(K_*(A_t), K_*(B)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} & & \\
 & & 0 & & KK_{*+1}(A_f, B) & & Ext^1_{\mathbb{Z}}(K_*(A_f), K_*(B)) & &
\end{array}$$

The three middle rows are exact by the UCT, the middle column is exact by the exactness of KK, and the two outer columns are exact by the standard *Hom-Ext*-sequence.

Suppose that $K_*(B)$ is torsionfree. Then

$$Hom_{\mathbb{Z}}(K_*(A_t), K_*(B)) = 0 \tag{2.5}$$

since $K_*(A_t)$ is a torsion group, and the surjectivity of θ_e^* implies the surjectivity of θ^*.

If $K_*(B)$ is not necessarily torsionfree, then the Snake Lemma [11] implies that there is an exact sequence

$$0 = Coker(\theta_e^*) \longrightarrow Coker(\theta^*) \longrightarrow Coker(\theta_h^*) \to 0$$

and hence θ_h^* is onto if and only if θ^* is onto. The theorem then follows immediately, for the middle column of the diagram degenerates to (2.2) if and only if θ^* is onto. $\square$

Next we consider the dual situation, when $K_*(A)$ is a torsion group.

Theorem 2.6. *Suppose that $A \in \mathcal{N}$ and that $K_*(A)$ is a torsion group. Then there is a natural exact sequence*

$$0 \to KK_*(A, B_t) \to KK_*(A, B) \to KK_*(A, B_f) \to 0. \tag{2.7}$$

If $K_*(A)$ is not a torsion group then sequence (*) is exact if and only if the natural map

$$\pi_* : Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \to Hom_{\mathbb{Z}}(K_*(A), K_*(B_f))$$

is onto, where $\pi : B \otimes \mathcal{K} \to B_f$ is the natural map.

The proof of this result is dual to that of Theorem 2.1 and is omitted for brevity. □

3. Computing $KK_*(A_f, B)$

In this section we consider the case where $K_*(A)$ is torsionfree (so that $A = A_f$). Recall [2, 12] that a subgroup H of an abelian group K is *pure* if for each positive integer n,

$$nH = H \cap nG,$$

and an extension of groups

$$0 \to H \to K \to G \to 0$$

is *pure* if H is a pure subgroup of K. For abelian groups G and H, $Pext^1_{\mathbb{Z}}(G, H)$ is the subgroup of $Ext^1_{\mathbb{Z}}(G, H)$ consisting of pure extensions.

Recall [5, 8] that there is a natural topology on the Kasparov groups and that with respect to this topology the UCT sequence splittings constructed in [4] are continuous, so that the splitting is a splitting of topological groups [9].

Theorem 3.1. *Suppose that $A \in \mathcal{N}$ and that $K_*(A)$ is torsionfree. Then there is a natural sequence of topological groups*

$$0 \to Pext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to KK_*(A, B) \to Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0$$

The group $Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))$ is the closure of zero in the natural topology on the group $KK_*(A, B)$ and thus the group
$Hom_{\mathbb{Z}}(K_*(A), K_*(B)$) is the Hausdorff quotient of $KK_*(A, B)$.

Proof. The UCT gives us the sequence

$$0 \to Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to KK_*(A, B) \to Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0$$

which splits unnaturally. If $K_*(A)$ is torsionfree then

$$Pext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \cong Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)).$$

The remaining part of the theorem holds since we have shown in general [10] that the group $Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))$ is the closure of zero in the natural topology on $KK_*(A, B)$ in the presence of the UCT.

□

We note that the resulting algebraic problems are frequently very difficult. If G is a torsionfree abelian group then $Hom_{\mathbb{Z}}(G, H)$ is unknown in general, though there is much known in special cases (cf. [2, 3]). The group $Pext^1_{\mathbb{Z}}(G, H)$ is also difficult, though the case $Pext^1_{\mathbb{Z}}(G, \mathbb{Z})$ is known (cf. [3]). We discuss $Pext$ in detail in [12].

4. Computing $KK_*(A_t, B)$

In this section we concentrate upon the situation when $K_*(A)$ is a torsion group. Before beginning, we digress slightly to recall [7] in more detail how one introduces coefficients into K-theory.

Given a countable abelian group G, select some C^*-algebra $N_G \in \mathcal{N}$ with

$$K_0(N_G) = G \qquad\qquad K_1(N_G) = 0.$$

The C^*-algebra N_G is unique up to KK-equivalence, by the UCT. Then for any C^*-algebra A, define

$$K_j(A; G) \cong K_j(A \otimes N_G). \tag{4.1}$$

The Künneth Theorem [6] implies that there is a natural short exact sequence

$$0 \to K_j(A) \otimes G \xrightarrow{\alpha} K_j(A; G) \to Tor_1^{\mathbb{Z}}(K_{j-1}(A), G) \to 0 \tag{4.2}$$

which splits unnaturally. If G is torsionfree then α is an isomorphism

$$\alpha : K_j(A) \otimes G \xrightarrow{\cong} K_j(A; G).$$

Let $\mathbf{X}(G) = Hom(G, \mathbb{R}/\mathbb{Z})$ denote the Pontryagin dual of the group G.

Theorem 4.3. *Suppose that $A \in \mathcal{N}$ with $K_*(A)$ a torsion group and suppose that $K_*(B)$ is torsionfree, so that $A = A_t$ and $B = B_f$. Then:*

(1)
$$KK_*(A, B) \cong Ext^1_{\mathbb{Z}}(K_*(A), K_{*-1}(B)).$$

(2)
$$KK_*(A, B) \cong Hom_{\mathbb{Z}}(K_*(A), K_{*-1}(B) \otimes \mathbb{Q}/\mathbb{Z}).$$

(3) *The group $KK_*(A, B)$ is reduced and algebraically compact.*

(4)
$$K^j(A) \cong \mathbf{X}(K_{j-1}(A))$$

.

(5) *More generally, if $K_*(B)$ is finitely generated free, then*

$$KK_j(A, B) \cong \oplus_n \mathbf{X}(K_{j-1}(A))$$

where n is the number of generators of $K_*(B)$.

Proof. Part 1) follows at once from the UCT and the fact that there are no non-trivial homomorphisms from a torsion group to a torsionfree group. Part 2) follows from Part 1) by elementary homological algebra. Part 3) follows easily from a deep result of Fuchs and Harrison [cf. 2, 46.1]: if G is a torsion group then any group of the form $Hom_{\mathbb{Z}}(G, H)$ is reduced and algebraically compact. Part 4) follows from part 3) by setting $B = \mathbb{C}$ and observing that for any torsion group G, we have

$$\mathbf{X}(G) = Hom_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z}.)$$

□

There is one additional case that fits into the present discussion and which partially overlaps with the result above.

Theorem 4.4. *Suppose that $A \in \mathcal{N}$ and that $K_*(A)$ has no free direct summand. Then there is a natural short exact sequence of topological groups*

$$0 \to Hom_{\mathbb{Z}}(K_*(A), \mathbb{R}) \to \mathbf{X}(K_*(A)) \xrightarrow{\chi} K^*(A) \to 0. \tag{4.5}$$

The map $\chi : \mathbf{X}(K_*(A)) \to K^*(A)$ is a degree one continuous open surjection. It is a homeomorphism if and only if $K_*(A)$ is a torsion group.

To be explicit about the grading,

$$\chi : \mathbf{X}(K_j(A)) \to K^{j-1}(A)$$

which is the usual parity shift as torsion phenomena move from homology to cohomology.

Proof. The UCT for $K^*(A)$ has the form

$$0 \to Ext^1_{\mathbb{Z}}(K_*(A), \mathbb{Z}) \xrightarrow{\delta} K^*(A) \to Hom_{\mathbb{Z}}(K_*(A), \mathbb{Z}) \to 0$$

with δ of degree one, so it suffices to compute Ext. In general the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

yields a long exact sequence

$$Hom_{\mathbb{Z}}(K_*(A), \mathbb{Z}) \to Hom_{\mathbb{Z}}(K_*(A), \mathbb{R}) \to \mathbf{X}(K_*(A)) \to Ext^1_{\mathbb{Z}}(K_*(A), \mathbb{Z}) \to 0.$$

The fact that $K_*(A)$ has no free direct summand implies that $Hom_{\mathbb{Z}}(K_*(A), \mathbb{Z}) = 0$, so the sequence degenerates to

$$0 \to Hom_{\mathbb{Z}}(K_*(A), \mathbb{R}) \to \mathbf{X}(K_*(A)) \to Ext^1_{\mathbb{Z}}(K_*(A), \mathbb{Z}) \to 0.$$

Applying the UCT one obtains the sequence 4.5 as desired. The map χ is the composite of the UCT map and a natural homeomorphism. The rest of the Theorem is immediate.

□

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