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# THE FINE STRUCTURE OF THE KASPAROV GROUPS I: CONTINUITY OF THE $KK$ -PAIRING

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*to Calvin C. Moore*

**ABSTRACT.** In this paper it is demonstrated that the Kasparov pairing is continuous with respect to the natural topology on the Kasparov groups, so that a  $KK$ -equivalence is an isomorphism of topological groups. In addition, we demonstrate that the groups have a natural pseudopolonais structure, and we prove that various  $KK$ -structural maps are continuous.

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## 1. Introduction

This is the first of a series of papers in which the topological structure of the Kasparov  $KK$ -groups is developed and put to use. The Kasparov groups  $KK_*(A, B)$ , defined for separable  $C^*$ -algebras  $A$  and  $B$ , have been shown to be powerful tools in the analysis of a wide variety of problems in functional analysis and in topology. It is our hope that the “fine structure” of these groups will be of additional help and, particularly, that it will be of use in the classification of separable nuclear simple  $C^*$ -algebras.

The topological structure of the Kasparov groups was first studied in depth by N. Salinas [19]. We shall demonstrate that  $KK_*(-, -)$  is a bifunctor to graded pseudopolonais groups. The key result in this paper, Theorem 6.8, asserts that the  $KK$ -product is jointly continuous with respect to this topology, provided that the  $C^*$ -algebras that appear in the first variable are  $K$ -nuclear. This theorem implies that a  $KK$ -equivalence is a homeomorphism. These theorems are applied in [25] to the study of relative quasidiagonality. Central to that study is the fine structure subgroup of  $KK_*(A, B)$ , namely the group

$$Pext_{\mathbb{Z}}^1(K_*(A), K_*(B)) \cong \varprojlim^1 Hom_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

which under bootstrap hypotheses is the closure of 0 in  $KK_*(A, B)$  [24, 22, 23].

The paper is organized as follows. In §2 we review the topology of function spaces of  $*$ -homomorphisms and their quotients modulo homotopy. Along the way we introduce K. Thomsen’s useful notion of *quasi-unital* maps, and we show how to use these maps to better understand the functorial nature of  $KK$ . Section 3 is devoted to showing that the  $KK$ -pairing is continuous with respect to the topology on the  $KK$ -groups inherited from the Cuntz representation of  $KK$  as  $KK_0(A, B) \cong [qA, B \otimes \mathcal{K}]$ . In §4 we introduce the topology used by Salinas as well as a natural topology associated to the Zekri picture of  $KK_1$  and we show that for  $A$   $K$ -nuclear these all agree. As a consequence we show that the  $KK$ -pairing is separately continuous with respect to the topology used by Salinas. Section 5 deals with certain consequences of the main result. We prove that a  $KK$ -equivalence must be a homeomorphism. In §6 we introduce polonais and pseudopolonais groups, observe that our results demonstrate that the  $KK$ -groups are pseudopolonais, and then state some powerful results that we learned from C.C. Moore which demonstrate why it is useful to know the polonais properties. We use the fact that the  $KK$ -groups are pseudopolonais to demonstrate our primary result, that the  $KK$ -pairing is jointly continuous in the natural Salinas topology. In §7 we conclude by demonstrating that various structural maps are also continuous, and finally by showing that the index map is also continuous.

It is a pleasure to acknowledge our dependence upon Salinas’s work [19]. Without it this paper would not exist. We are very grateful to Larry Brown, Marius Dadarlat, Gert Pedersen, Chris Phillips, Jonathan Rosenberg, Norberto Salinas, Bert Schreiber, Klaus Thomsen, and Richard Zekri for helpful comments.

In this paper all  $C^*$ -algebras are assumed separable with the exception of those that obviously are not (namely multiplier algebras  $\mathcal{M}B$  and their quotients the corona algebras  $\mathcal{Q}B = \mathcal{M}B/B$ ). If  $A$  is not nuclear then when speaking of extensions we always require that they be semi-split, so that equivalence classes of extensions coincide with the relevant Kasparov group. All  $C^*$ -algebras are assumed

to be trivially graded. Tensor products  $A \otimes B$  are always understood to be the minimal tensor product. Isomorphisms of topological groups are isomorphisms of groups which are homeomorphisms as spaces.

## 2. Function Spaces

In this section we review certain aspects of the topologies of the function space of  $*$ -homomorphisms for separable  $C^*$ -algebras  $A$  and  $B$ .

Let  $A$  and  $B$  be  $C^*$ -algebras and let  $Hom(A, B)$  denote the set of all  $*$ -homomorphisms from  $A$  to  $B$ . There are several possible topologies on  $Hom(A, B)$ :

- (1) The topology of pointwise convergence;
- (2) The compact-open topology;
- (3) The topology of uniform convergence on compact sets;
- (4) If  $A$  is separable with countable dense set  $\{a_i\}$  (with each  $a_i \neq 0$ ), the metric topology obtained from the metric

$$\mu(f_1, f_2) = \sum \frac{|(f_1 - f_2)(a_i)|}{2^i |a_i|}.$$

**Proposition 2.1.** *The first three topologies on  $Hom(A, B)$  are homeomorphic. If  $A$  is separable then all four topologies are homeomorphic.*

*Proof.* Topologies 2), 3), and 4) agree by standard arguments, and of course 3) implies 1), so the only statement requiring proof is that 1) implies 3): pointwise convergence implies uniform convergence on compact sets. This is a folklore statement: we insert a proof for convenience.

Suppose that  $f_\alpha \rightarrow f$  pointwise in  $Hom(A, B)$ ,  $K$  is a compact set, and  $\delta > 0$ . The set  $K$  is a subset of a separable space, hence separable. Let  $\{a_i\}$  be a countable dense set for  $K$ . Cover  $K$  with balls of radius  $\delta$  centered at the points  $a_i$ . Since  $K$  is compact, a finite number of these balls cover  $K$ , say corresponding to the finite sequence  $a_1, \dots, a_n$ . Now fix any  $a \in K$ . Then  $|a - a_i| < \delta$  for some  $i$ , with  $1 \leq i \leq n$ . Pick one. Then

$$\begin{aligned} |(f_\alpha - f)(a)| &= |(f_\alpha - f)(a - a_i) + (f_\alpha - f)(a_i)| \\ &\leq 2|a - a_i| + |(f_\alpha - f)(a_i)| \\ &\leq 2\delta + |(f_\alpha - f)(a_i)|. \end{aligned}$$

Now  $f_\alpha \rightarrow f$  uniformly on the (finite!) set  $\{a_1, \dots, a_n\}$ , and hence  $f_\alpha \rightarrow f$  uniformly on  $K$ .  $\square$

Henceforth when we refer to  $Hom(A, B)$  as a topological space the topology described in 1), 2), 3) (or 4) if  $A$  is separable) above is intended and we use their properties interchangeably without further remark. Given a compact set  $K \subset A$  and an open set  $U \subset B$  we let

$$(K, U) = \{f : A \rightarrow B : f(K) \subset U\}$$

These sets form a subbasis for the topology of  $Hom(A, B)$  and in fact we may require  $K$  to be restricted to single points.

Suppose that  $f : X \rightarrow Y$  is a map of topological spaces, and that  $Im(f)$  denotes the image of the map  $f$ . There are two obvious topologies on this space. One may regard  $Im(f)$  as a quotient space of  $X$ , in which case the map  $X \rightarrow Im(f)$  is continuous and a quotient map, or one may regard  $Im(f)$  as a subspace of  $Y$  with

the relative topology. We refer to these as  $Im(f)_{quot}$  and  $Im(f)_{rel}$  respectively. The identity map

$$Im(f)_{quot} \longrightarrow Im(f)_{rel}$$

is of course a continuous bijection. If it is a homeomorphism then we say that the map  $f$  is *relatively open*. If  $f$  is actually an open map then it is relatively open, but not conversely (for instance, a linear inclusion of a line into the plane with the standard metric on both is relatively open but not open.) If  $f : X \rightarrow Y$  is surjective then open is of course equivalent to relatively open. If  $f : X \rightarrow Y$  is injective then relatively open is equivalent to  $f$  being a homeomorphism onto its image.

Suppose that  $f : A \rightarrow B$  is a  $*$ -homomorphism. Then  $f(A)$  is closed in  $B$  and hence may be regarded as a  $C^*$ -algebra in two ways: either as  $A/Ker(f)$  or as  $f(A) \subseteq B$ . These ways are algebraically isomorphic and (by the uniqueness of the topology on a  $C^*$ -algebra) must be topologically the same as well. Thus any  $*$ -homomorphism  $f : A \rightarrow B$  must be relatively open.

**Proposition 2.2.** *Let  $A$ ,  $B$ , and  $C$  be  $C^*$ -algebras. Then the composition map*

$$Hom(A, B) \times Hom(B, C) \xrightarrow{T} Hom(A, C)$$

*is jointly continuous.*

*Proof.* This is essentially identical to [10, p. 259]. Let  $f : A \rightarrow B$ ,  $g : B \rightarrow C$ , choose some  $a \in A$ , and let  $W$  be an open neighborhood of  $gf(a) \in C$ . Then  $g^{-1}W$  is open in  $B$  and contains  $f(a)$ . Choose an open set  $V \in B$  with  $f(a) \in V \subset \overline{V} \subset g^{-1}W$ . Then  $f \in (a, V)$ ,  $g \in (\overline{V}, W)$  and

$$T((a, V), (\overline{V}, W)) \subset (a, W)$$

as required.

□

**Proposition 2.3.** *Suppose that  $h : B \rightarrow B'$  is a map of  $C^*$ -algebras. Then the induced map*

$$h_* : Hom(A, B) \rightarrow Hom(A, B')$$

*is continuous. If  $h$  is mono then  $h_*$  is relatively open.*

*Proof.* Continuity is obvious from the previous proposition, since joint continuity implies separate continuity. We must show that  $h_*$  is relatively open provided that  $h$  is mono. Let  $a \in A$  and let  $U$  be an open set in  $B$ . Then

$$\begin{aligned} h_*(a, U) &= \{hf \in Hom(A, B') : f(a) \in U\} \\ &= \{hf \in Hom(A, B') : hf(a) \in h(U)\} \quad \text{since } h \text{ is mono} \\ &= (a, h(U)) \cap Im(h_*) \\ &= (a, V) \cap Im(h_*) \end{aligned}$$

where  $V$  is some open set in  $B'$  with

$$h(U) = Im(h_*) \cap V.$$

(Such a set  $V$  exists since  $h$  is a relatively open map, as remarked above.) Hence  $h_*(a, U)$  is relatively open in  $Hom(A, B')$  as required. □

**Proposition 2.4.** *Suppose that  $h : A' \rightarrow A$  is a map of  $C^*$ -algebras. Then the induced map*

$$h^* : \text{Hom}(A, B) \longrightarrow \text{Hom}(A', B)$$

*is continuous and relatively open.*

*Proof.* Again, continuity is clear from Proposition 2.2. To show that the map is relatively open, let  $(a, U) \subset \text{Hom}(A, B)$  be a subbasic open set. Then

$$h^*(a, U) = h^*\{f : A \rightarrow B : f(a) \in U\} = \bigcup_x [(x, U) \cap \text{Im}(h^*)]$$

where the union is over all  $x \in h^{-1}(a)$ . Each of the sets  $(x, U) \cap \text{Im}(h^*)$  is relatively open and hence their union is also relatively open in  $\text{Im}(h^*)$ .  $\square$

**Proposition 2.5.** *The natural map*

$$\Psi_C : \text{Hom}(A, B) \longrightarrow \text{Hom}(C \otimes A, C \otimes B)$$

*is continuous. In particular, the suspension map*

$$S = \Psi_{C_o(\mathbb{R})} : \text{Hom}(A, B) \longrightarrow \text{Hom}(SA, SB)$$

*is continuous.*

*Proof.* This is immediate from definitions.

$\square$

**Proposition 2.6.** *Suppose that  $A, B,$  and  $C$  are  $C^*$ -algebras. Then the composition map*

$$[A, B] \times [B, C] \xrightarrow{\overline{T}} [A, C]$$

*is separately continuous.*

*Proof.* This is immediate from the definition of the quotient topology.

$\square$

Note that we do not claim that  $\overline{T}$  is jointly continuous. This would be true if the map

$$\begin{array}{c} \text{Hom}(A, B) \times \text{Hom}(B, C) \\ \downarrow \pi_1 \times \pi_2 \\ [A, B] \times [B, C] \end{array}$$

were a quotient map. In general the product of quotient maps is not necessarily itself a quotient map. Eventually we will show that the  $KK$ -pairing is jointly continuous, but this result will use the fact that the  $KK$ -groups are pseudopolonais.

Next we wish to consider the natural map

$$\text{Hom}(A, B) \rightarrow \text{Hom}(\mathcal{M}A, \mathcal{M}B)$$

where  $\mathcal{M}$  denotes the multiplier algebra of a  $C^*$ -algebra. Unfortunately there is no such functor in general, since not every  $C^*$  map from  $A$  to  $B$  extends to the



multiplier algebras. Fortunately enough maps do, and this leads us to a way to proceed.

The following ideas are due to K. Thomsen [28], and we are deeply grateful to him for his assistance. He cites Nigel Higson [12] as the first to consider such maps. A  $C^*$ -algebra map  $h : A \rightarrow B$  is said to be *quasi-unital* if the closed linear span of  $h(A)B$  is of the form  $pB$  where  $p \in \mathcal{M}B$  is some projection. A quasi-unital map extends to the multiplier algebra level via a  $*$ -homomorphism

$$\mathcal{M}h : \mathcal{M}A \rightarrow \mathcal{M}B$$

and hence induces a map at the level of corona algebras denoted  $\mathcal{Q}h : \mathcal{Q}C \rightarrow \mathcal{Q}D$ .

**Proposition 2.7.** *K. Thomsen [28, Prop. 2.8 ] If  $A$  and  $B$  are  $\sigma$ -unital and  $B$  is stable then any  $C^*$ -map  $A \rightarrow B$  is homotopic to a quasi-unital map. If two quasi-unital maps are homotopic then they are homotopic via a homotopy through quasi-unital maps.*

□

Let  $Hom(A, B)_{qu}$  denote the quasi-unital maps from  $A$  to  $B$ , topologized as a subspace of  $Hom(A, B)$ . Note that  $Hom(A, B)_{qu}$  is *not* an open subspace of  $Hom(A, B)$  in general<sup>1</sup> and hence the inclusion map is not an open map in general. We let  $[A, B]$  denote homotopy classes of  $*$ -homomorphisms from  $A$  to  $B$ , topologized as the quotient of  $Hom(A, B)$ . Similarly, let  $[A, B]_{qu}$  denote quasi-unital homotopy classes of quasi-unital  $*$ -homomorphisms from  $A$  to  $B$ , topologized as the quotient of  $Hom(A, B)_{qu}$ . Then Proposition 2.7 implies that if  $B$  is stable then there is a natural bijection

$$[A, B]_{qu} \longrightarrow [A, B]$$

It is obviously continuous. However, it is probably not a homeomorphism in general, by the above remarks.

**Proposition 2.8.** *Suppose that  $f : A \rightarrow B$  is quasi-unital. Then so is the map  $1 \otimes f : SA \rightarrow SB$ . Conversely, if  $f : A \rightarrow B$  is a  $*$ -homomorphism and  $1 \otimes f$  is quasi-unital then so is  $f$ .*

*Proof.* I am indebted to Klaus Thomsen for the following proof.

If  $f$  is quasi-unital there is some projection  $p \in \mathcal{M}B$  with

$$\overline{f(A)B} = pB$$

where  $\overline{f(A)B}$  denotes the closed linear span of  $f(A)B$ . Then a direct calculation shows that

$$\overline{(1 \otimes f)(SA)SB} = (1 \otimes p)SB$$

and  $1 \otimes p \in \mathcal{M}(SB)$ .

In the other direction, suppose that  $1 \otimes f$  is quasi-unital with

$$\overline{f(SA)SB} = eSB$$

for some projection  $e \in \mathcal{M}(SB)$ . Let

$$q : SB = C_o(\mathbb{R}) \otimes B \rightarrow B$$

<sup>1</sup>For example, if  $A = B = C_0(\mathbb{R})$  and  $f(x) = x^2$ , then  $f(A)B = C_0(\mathbb{R})$  but  $\overline{f(A)B} = C_0(\mathbb{R}) \oplus C_0(\mathbb{R})$ .

be evaluation at zero. Then  $q$  is surjective, and

$$f(A)B = q(1 \otimes f)(SA)(SB)$$

as required.  $\square$

**Remark 2.9** Quasi-unital maps give a nice insight into the general theory of extensions of  $C^*$ -algebras. An essential extension

$$\tau : \quad 0 \rightarrow B \otimes \mathcal{K} \rightarrow E \rightarrow A \rightarrow 0$$

with Busby classifying map

$$\tau : A \longrightarrow \mathcal{Q}(B \otimes \mathcal{K})$$

lies in the group  $KK_1(A, B)$ . If  $f : B \rightarrow B'$  is a  $*$ -homomorphism then  $f_*[\tau] \in KK_1(A, B')$ . However, it is not so easy to see directly how to construct the corresponding extension or its classifying map

$$\hat{\tau} : A \longrightarrow \mathcal{Q}(B' \otimes \mathcal{K}).$$

The natural thing to do is to take a pushout construction, as one would do in abelian groups, but the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & B \otimes \mathcal{K} & \longrightarrow & E & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow f \otimes 1 & & & & \\ & & B' \otimes \mathcal{K} & & & & \end{array}$$

may not be completed without some assumptions on  $f$ . Eilers, Loring, and Pedersen [11] show that if  $f$  is *proper* then it is possible to complete the pushout and thereby obtain  $\hat{\tau}$ . However, not every map is proper.

Using our knowledge of quasi-unital maps, there is a nice solution. Given  $f$  as above, replace it by a quasi-unital map  $g$  homotopic to it. This then extends to

$$\mathcal{M}g : \mathcal{M}(B \otimes \mathcal{K}) \rightarrow \mathcal{M}(B' \otimes \mathcal{K})$$

and hence induces a map

$$\mathcal{Q}g : \mathcal{Q}(B \otimes \mathcal{K}) \rightarrow \mathcal{Q}(B' \otimes \mathcal{K})$$

Then we have

$$A \xrightarrow{\tau} \mathcal{Q}(B \otimes \mathcal{K}) \xrightarrow{\mathcal{Q}g} \mathcal{Q}(B' \otimes \mathcal{K})$$

If this map happens to be mono then it classifies an extension which is in the class  $f_*[\tau]$ . If not (as is likely), simply add on a map of the form

$$A \xrightarrow{\pi\sigma} \mathcal{Q}(B' \otimes \mathcal{K})$$

where

$$\sigma : A \rightarrow \mathcal{M}(B' \otimes \mathcal{K})$$

is some trivial extension. Then the composite

$$(\mathcal{Q}g)\tau \oplus \pi\sigma : A \longrightarrow \mathcal{Q}(B' \otimes \mathcal{K}) \oplus \mathcal{Q}(B' \otimes \mathcal{K}) \rightarrow \mathcal{Q}(B' \otimes \mathcal{K})$$

is mono and it represents the desired element:

$$f_*[\tau] = [(\mathcal{Q}g)\tau \oplus \pi\sigma] \in KK_1(A, B').$$

### 3. Continuity of the Cuntz pairing

In this section we introduce the Cuntz quasi-homomorphism picture of the Kasparov groups and verify that the  $KK$ -pairing is continuous in the corresponding topology.

Let  $A$  be a  $C^*$ -algebra and let  $A \star A$  denote the  $C^*$ -algebra free product. Then there are two canonical maps  $\alpha_A$  and  $\bar{\alpha}_A$  mapping  $A \rightarrow A \star A$ . (When the context is clear we omit the subscript.) Following J.Cuntz [6, 7, 8], let  $qA$  denote the closed ideal of  $A \star A$  generated by all elements of the form  $\alpha(a) - \bar{\alpha}(a)$ . The identity map  $A \rightarrow A$  induces a canonical map  $A \star A \rightarrow A$  and a natural short exact sequence

$$0 \rightarrow qA \rightarrow A \star A \rightarrow A \rightarrow 0.$$

It is easy to see that  $q$  is a covariant functor and that if  $A$  is separable then so too is  $qA$ .

Cuntz has shown that there is a natural isomorphism

$$(3.1) \quad [qA, B \otimes \mathcal{K}] \xrightarrow{\cong} KK_0(A, B).$$

We give  $KK_0(A, B)$  a topology by declaring the map (3.1) to be a homeomorphism. We shall refer to this topology as the *Cuntz topology* on  $KK_0(A, B)$ . Similarly we topologize  $KK_1(A, B)$  by declaring the isomorphism

$$KK_1(A, B) \cong KK_0(SA, B) \cong [q(SA), B \otimes \mathcal{K}]$$

to be a homeomorphism. (In Section 4 we will discuss other possibilities for the topology on  $KK_*(A, B)$  and at that time we will refer to the present topology as  $KK_*(A, B)_q$ .)

Next we need some elementary facts.

**Proposition 3.2.** *Suppose that  $A$  is separable. Then the natural map*

$$q : Hom(A, B) \rightarrow Hom(qA, qB)$$

*and the induced map*

$$q_* : [A, B] \rightarrow [qA, qB]$$

*are both continuous.*

*Proof.* It is clear that the natural map

$$Hom(A, B) \longrightarrow Hom(A \star A, B \star B)$$

given by  $f \rightarrow f \star f$  is continuous. Composing with the natural map  $qA \rightarrow A \star A$  yields a continuous map

$$Hom(A, B) \longrightarrow Hom(qA, B \star B)$$

which in fact factors through  $Hom(qA, qB)$ . As  $Hom(qA, qB)$  has the relative topology in  $Hom(qA, B \star B)$ , this implies that the natural map  $q : Hom(A, B) \rightarrow Hom(qA, qB)$  is continuous.

Since the quotient maps  $Hom(A, B) \rightarrow [A, B]$  and  $Hom(qA, qB) \rightarrow [qA, qB]$  are continuous, it follows immediately that the induced map  $q_* : [A, B] \rightarrow [qA, qB]$  is also continuous.

□

Next we introduce a key map

$$\Phi_{AB} : [qA, B \otimes \mathcal{K}] \longrightarrow [qA, qB \otimes \mathcal{K}].$$

We follow the notation of [13, Theorem 5.1.12]. There are canonical (at least up to unitary equivalence)  $*$ -homomorphisms

$$\rho_B : q(B \otimes \mathcal{K}) \rightarrow qB \otimes \mathcal{K}, \quad \phi_B : qB \rightarrow q^2B \otimes M_2(\mathbb{C}), \quad \theta : \mathcal{K} \otimes M_2(\mathbb{C}) \rightarrow \mathcal{K}.$$

Using these, we define  $\Phi_{AB}$  to be the composite

$$\begin{aligned} [qA, B \otimes \mathcal{K}] &\xrightarrow{q} [q^2A, q(B \otimes \mathcal{K})] \\ &\xrightarrow{(\rho_B)_*} [q^2A, qB \otimes \mathcal{K}] \\ &\xrightarrow{\Psi_{M_2(\mathbb{C})}} [q^2A \otimes M_2(\mathbb{C}), B \otimes \mathcal{K} \otimes M_2(\mathbb{C})] \\ &\xrightarrow{(\phi_A)_*} [qA, B \otimes \mathcal{K} \otimes M_2(\mathbb{C})] \\ &\xrightarrow{\theta_*} [qA, B \otimes \mathcal{K}]. \end{aligned}$$

It is immediate from Proposition 3.2 and results of §2 that  $\Phi_{AB}$  is continuous. In fact more is true:

**Proposition 3.3.** *The natural map*

$$\Phi_{AB} : [qA, B \otimes \mathcal{K}] \longrightarrow [qA, qB \otimes \mathcal{K}].$$

*is an isomorphism of topological groups.*

*Proof.* We have shown in the discussion above that the map  $\Phi_{AB}$  is continuous. There is a natural inverse map which is constructed as follows. The natural maps  $1_B : B \rightarrow B$  and  $0 : B \rightarrow B$  induce a natural map  $1_B \star 0 : B \star B \rightarrow B$ . Let

$$\gamma^B : qB \rightarrow B$$

be the restriction of this map to  $qB$ . Then  $\gamma^B$  induces a continuous map

$$(\gamma^B \otimes 1)_* : [qA, qB \otimes \mathcal{K}] \longrightarrow [qA, B \otimes \mathcal{K}]$$

which is shown in [13] to be the map inverse to  $\Phi_{AB}$ . The map  $(\gamma^B \otimes 1)_*$  is continuous by Propositions 2.3 and 2.5.

□

**Theorem 3.4.** *The Kasparov pairing*

$$KK_*(A, B) \times KK_*(B, C) \xrightarrow{\otimes_B} KK_*(A, C)$$

*is separately continuous in the Cuntz topology.*

*Proof.* Suppose first that  $*$  = 0 so that we are looking at the pairing

$$[qA, B \otimes \mathcal{K}] \times [qB, C \otimes \mathcal{K}] \longrightarrow [qA, C \otimes \mathcal{K}].$$

This pairing is simply the composite

$$\begin{array}{c}
[qA, B \otimes \mathcal{K}] \times [qB, C \otimes \mathcal{K}] \\
\downarrow \Phi_{AB \otimes \Psi_{\mathcal{K}}} \\
[qA, qB \otimes \mathcal{K}] \times [qB \otimes \mathcal{K}, C \otimes \mathcal{K} \otimes \mathcal{K}] \\
\downarrow \text{compose} \\
[qA, C \otimes \mathcal{K} \otimes \mathcal{K}] \\
\downarrow \cong \\
[qA, C \otimes \mathcal{K}]
\end{array}$$

and hence is continuous in each variable. Since Bott periodicity (in either variable) may be regarded as pairing with  $KK$ -elements of degree zero, it follows at once that the natural Bott map

$$KK_0(A, B) \rightarrow KK_0(A, S^2B)$$

is a homeomorphism, and similarly in the  $A$  variable.

Next, we recall that the Cuntz topology on  $KK_1(A, B)$  was given by insisting that the natural connecting isomorphism

$$KK_0(SA, B) \xrightarrow{\cong} KK_1(A, B)$$

which arises from the canonical short exact sequence

$$0 \rightarrow SA \rightarrow CA \rightarrow A \rightarrow 0$$

be a homeomorphism. (This choice is of course consistent with the Bott maps.) Then it is an easy exercise to prove that the pairing is continuous as stated.

□

The most general case of the Kasparov pairing is built from the previous pairing and the following map.

**Proposition 3.5.** *The external mapping*

$$\tau_C : KK_*(A, B) \rightarrow KK_*(A \otimes C, B \otimes C)$$

*is continuous.*

*Proof.* There is a natural map  $\nu : q(A \otimes C) \rightarrow qA \otimes C$ , which arises as the restriction of the natural map

$$(\alpha_A \otimes 1_C) \star (\bar{\alpha}_A \otimes 1_C) : (A \otimes C) \star (A \otimes C) \rightarrow (A \star A) \otimes C$$

to  $qA$  and the mapping above is the composite

$$[qA, B \otimes \mathcal{K}] \xrightarrow{\Psi_C} [qA \otimes C, B \otimes C \otimes \mathcal{K}] \xrightarrow{\nu^*} [q(A \otimes C), B \otimes C \otimes \mathcal{K}]$$

hence continuous.

□

Combining results we have the most general theorem of this section.

**Theorem 3.6.** *The Kasparov pairing*

$$KK_*(A_1, B_1 \otimes D) \times KK_*(D \otimes A_2, B_2) \xrightarrow{\otimes_D} KK_*(A_1 \otimes A_2, B_1 \otimes B_2)$$

*is separately continuous in the topology associated to the Cuntz picture of*

$$KK_0(A, B) \cong [qA, B \otimes \mathcal{K}].$$

*Proof.* Let  $E = B_1 \otimes D \otimes A_2$ . Then

$$x \otimes_D y = \tau_{A_2}(x) \otimes_E \tau_{B_1}(y)$$

so that 3.4 and 3.5 together imply the result.

□

We shall improve this theorem to show joint continuity in Theorem 6.8.

#### 4. The Topologies on $KK_1(A, B)$

In this section we show that four topologies on  $KK_1(A, B)$  coincide. Of course the topology that we are interested in is the topology used by Salinas, which we shall denote  $KK_1(A, B)_S$  in this section. The main consequence of this section is that this topology coincides with the natural topology on  $[q(SA), B \otimes \mathcal{K}]$  in the Cuntz picture of  $KK$ . Since we have already shown that the Kasparov pairing is continuous in the Cuntz picture, this will imply that the pairing is continuous in the Salinas topology. This is what we need in applications.

Here are the four topologies that we shall consider:

##### 1. The Salinas Topology: $KK_1(A, B)_S$

We regard  $KK_1(A, B)$  as the quotient of the space  $\mathcal{E}(A, B) \subseteq \text{Hom}(A, \mathcal{Q}(B \otimes \mathcal{K}))$  of associated semisplit extensions, with natural quotient map

$$\Upsilon_S : \mathcal{E}(A, B) \longrightarrow KK_1(A, B).$$

We topologize  $\mathcal{E}(A, B)$  by giving it the relative topology in  $\text{Hom}(A, \mathcal{Q}(B \otimes \mathcal{K}))$  or equivalently by giving it a metric topology as in (2.1). Fix one such metric  $\mu$  on  $\mathcal{E}(A, B)$ . Given  $x, x' \in KK_1(A, B)$ , define

$$\hat{\mu}(x, x') = \inf \mu(\tau, \tau')$$

where the *inf* is taken over all  $\tau \in x, \tau' \in x'$ . Salinas shows [19, 3.1] that one obtains a pseudometric<sup>2</sup> on  $KK_1(A, B)$ . The associated topology is of course independent of the choice of the sequence  $\{a_i\}$ . With respect to the pseudometric  $KK_1(A, B)$  is a topological group, and the pseudometric is invariant under the group action. The quotient topology induced upon  $KK_1(A, B)$  from  $\mathcal{E}(A, B)$  via  $\Upsilon_S$  coincides with this topology. We shall denote  $KK_1(A, B)$  with this topology as  $KK_1(A, B)_S$ .

Salinas shows that this construction defines a functor to the category of topological groups and continuous homomorphisms in each variable. One special case is easy to describe: the group  $K_1(B) = KK_1(\mathbb{C}, B)_S$  is countable, complete, and hence discrete. However in general the topology may be highly non-trivial.

We note that in Salinas's paper, he has a standing assumption that  $A$  must be unital. Dadarlat ([9] and private communication) has verified that this assumption is not essential. All of Salinas's results remain true in the nonunital case, including the equality between the closure of zero and the quasidiagonal extensions (cf. [25]).

##### 2. The Zekri topology $KK_1(A, B)_Z$

Following Zekri [29, 30], let  $EA$  denote the universal  $C^*$ -algebra generated by a separable  $C^*$ -algebra  $A$  and an element  $F$  with  $F^* = F$  and  $F^2 = 1$ , with 1 acting on  $A$  as the identity. Let  $[A, F]$  denote the set of commutators and let  $\epsilon A$  denote the smallest closed ideal of  $EA$  which contains  $[A, F]$ . Then there is a canonical commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \epsilon A & \longrightarrow & EA & \longrightarrow & A^+ \oplus A^+ \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow d \\ 0 & \longrightarrow & \epsilon A & \longrightarrow & \tilde{E}A & \longrightarrow & A \longrightarrow 0 \end{array}$$

<sup>2</sup>A *pseudometric* is a function  $\mu : X \times X \rightarrow \mathbb{R}$  satisfying the usual metric axioms except that if  $\mu(x, y) = 0$  it need not be the case that  $x = y$ . If  $X$  is a topological group we insist that the

where  $d(a) = (a, a)$  is the diagonal map,  $A^+$  is the unitalization of  $A$ , and  $\tilde{E}A$  is the pullback.

Zekri defines a  $K$ -extension  $e$  of  $B$  by  $A$  to be a diagram

$$(4.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & C^*(A, U) & \longrightarrow & A^+ \oplus A^+ \longrightarrow 0 \\ & & & & \uparrow & & \\ & & & & J & \xrightarrow{\mu} & B \otimes \mathcal{K} \end{array}$$

where  $U = U^* = U^{-1}$  is a self-adjoint unitary,  $J$  is the ideal generated by  $[A, U]$ , the row is exact, and  $\mu$  is a monomorphism. Diagram (4.1) yields a universal example of the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & C^*(A, U) & \longrightarrow & A^+ \oplus A^+ \longrightarrow 0 \\ & & & & \uparrow & & \\ & & & & \epsilon A & \xrightarrow{1} & \epsilon A \end{array}$$

(after stabilizing  $A$ ). Let  $KExt(A, B)$  denote the set of  $K$ -extensions.

There is a natural map

$$\zeta : KExt(A, B) \longrightarrow Hom(\epsilon A, B \otimes \mathcal{K})$$

defined as follows. Given a  $K$ -extension  $e$  as in (4.1), there is a canonical map  $EA \rightarrow C^*(A, U)$ . Its restriction to  $\epsilon A$  factors uniquely through  $J$ . Then we define  $\zeta(e)$  to be the composite

$$\epsilon A \rightarrow J \rightarrow B \otimes \mathcal{K}.$$

**Proposition 4.2.** (Zekri [29, Theorem 2.4]) *The natural map*

$$\zeta : KExt(A, B) \longrightarrow Hom(\epsilon A, B \otimes \mathcal{K})$$

*is a bijection on isomorphism classes.*  $\square$

Zekri shows [29, Theorem 4.4] that if  $A$  is  $K$ -nuclear then there is a natural isomorphism

$$[\epsilon A, B \otimes \mathcal{K}] \cong KK_1(A, B)$$

and since the left hand side has a natural topology as a quotient of the space  $Hom(\epsilon A, B \otimes \mathcal{K})$ , the right hand side inherits a topology via this isomorphism which we denote  $KK_1(A, B)_Z$ .

### 3. Another Cuntz topology: $KK_1(A, B)_C$

This topology arises from the Cuntz picture of  $KK_1(A, B)$ . (It is a priori different from the topology that we discussed in §3.)

Let  $H_B$  denote the universal Hilbert  $B$ -module, and let

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}(H_B \oplus H_B).$$

Let  $\mathbb{E}_1(A, B)$  denote the collection of maps



such that the commutator  $[\sigma(A), P] \subset B \otimes \mathcal{K}$ , and the resulting  $*$ -homomorphism

$$\Upsilon_C(\sigma) : A \rightarrow \mathcal{Q}(B \otimes K)$$

given by

$$\Upsilon_C(\sigma)(a) = \pi(P\sigma(a)P)$$

is a monomorphism. This gives a natural map

$$\Upsilon_C : \mathbb{E}_1(A, B) \rightarrow KK_1(A, B)$$

We topologize the set  $\mathbb{E}_1(A, B)$  as follows. Let  $\bar{\mu}$  be a standard metric on the space  $Hom(A, \mathcal{M}(H_B))$  as in (2.1). Define a pseudometric on  $\mathbb{E}_1(A, B)$  by

$$\mu((\sigma, P), (\sigma', P)) = \max\{\bar{\mu}(P\sigma P, P\sigma' P), \bar{\mu}((1-P)\sigma(1-P), (1-P)\sigma'(1-P))\}.$$

Then  $KK_1(A, B)$  inherits a topology as the quotient of  $\mathbb{E}_1(A, B)$  which we denote  $KK_1(A, B)_C$ .

#### 4. The $qA$ topology: $KK_1(A, B)_q$

This is the topology that we used in Section 3, for which the  $KK$ -product is separately continuous. Recall that this topology is simply described. Cuntz has shown that there is a natural isomorphism  $[qA, B \otimes \mathcal{K}] \cong KK_0(A, B)$  and hence a natural isomorphism

$$KK_1(A, B) \cong KK_0(SA, B) \cong [q(SA), B \otimes \mathcal{K}].$$

We denote the resulting topology by  $KK_1(A, B)_q$ .

We must show that these four topologies coincide. Here is the first step:

**Proposition 4.3.** *There is a canonical map*

$$h_* : \mathbb{E}_1(A, B) \longrightarrow \mathcal{E}(A, B)$$

*which is continuous and which induces an isomorphism of topological groups*

$$\tilde{h} : KK_1(A, B)_C \longrightarrow KK_1(A, B)_S.$$

If we ignore topologies, then  $\tilde{h}$  is just the identity map.

*Proof.* There is a natural map  $\mathcal{M}(H_B \oplus H_B) \rightarrow \mathcal{M}(H_B)$  given by

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \longrightarrow PTP = T_{11}.$$

This is not a  $*$ -homomorphism, of course, but it certainly is continuous. Composing with the projection  $\mathcal{M}(H_B) \rightarrow \mathcal{Q}(H_B)$  we obtain a map

$$h : \mathcal{M}(H_B \oplus H_B) \rightarrow \mathcal{Q}(H_B)$$

which is continuous. Then  $h_*$  is defined<sup>3</sup> by

$$h_*(\sigma, P) = h\sigma$$

and it induces a continuous map

$$\tilde{h} : KK_1(A, B)_C \longrightarrow KK_1(A, B)_S.$$

Of course this is the identity map at the level of sets, but we must keep track of the topologies as well. To prove that  $\tilde{h}$  is an isomorphism of topological groups, it suffices then to prove that the map  $\tilde{h}$  is open. By homogeneity, it suffices to show this in a neighborhood of zero.

So fix  $\epsilon > 0$  and let  $B_\epsilon^C$  be an open ball about zero in  $KK_1(A, B)_C$ . We claim that

$$\tilde{h}(B_\epsilon^C) = B_\epsilon^S$$

where  $B_\epsilon^S$  is the corresponding open ball in  $KK_1(A, B)_S$ . (This of course implies that  $\tilde{h}$  is open.)

Suppose first that  $[\tau] \in B_\epsilon^C$ . Then there is some trivial extension

$$(\sigma, P) \in \mathbb{E}(A, B)$$

and some extension

$$(\hat{\tau}, P) \in \mathbb{E}(A, B)$$

with  $\pi(P\hat{\tau}P) = \tau$  and

$$\mu((\sigma, P), (\hat{\tau}, P)) < \epsilon.$$

Then  $\bar{\mu}(P\sigma P, P\hat{\tau}P) < \epsilon$  in  $Hom(A, \mathcal{M}(H_B))$  and hence  $\mu([P\sigma P], [\tau]) < \epsilon$ . Thus  $\tilde{h}[\tau] \in B_\epsilon^S$ . So  $\tilde{h}(B_\epsilon^C) \subseteq B_\epsilon^S$ .

Conversely, suppose that  $[\tau] \in B_\epsilon^S$ . Then also  $-[\tau] \in B_\epsilon^S$ , since the pseudometric is invariant the group action. By abuse of language, we write  $-\tau$  for some representative of  $-[\tau]$ . There is some trivial extension  $\sigma : A \rightarrow \mathcal{M}(H_B)$  with  $\mu(\tau, \pi\sigma) < \epsilon$ , and similarly, some trivial extension  $\sigma' : A \rightarrow \mathcal{M}(H_B)$  with  $\mu(-\tau, \pi\sigma') < \epsilon$ . Let

$$\bar{\sigma} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma' \end{bmatrix}.$$

---

<sup>3</sup>Although we do not need this, we note that  $h_*$  is onto. Any semisplit  $C^*$ -monomorphism

$$\tau : A \rightarrow \mathcal{Q}(H_B)$$

lifts to a completely positive contraction  $A \rightarrow \mathcal{M}(H_B)$ . Then this completely positive map may be dilated to a  $C^*$ -homomorphism

$$\sigma : A \rightarrow \mathcal{M}(H_B \oplus H_B)$$

by the generalized Stinespring theorem, due to Kasparov [14]. Then  $(\sigma, P) \in \mathbb{E}(A, B)$  and  $h_*(\sigma, P) = \tau$ . Thus the canonical map

$$h_* : \mathbb{E}_1(A, B) \longrightarrow \mathcal{E}(A, B)$$

The extension  $\tau \oplus (-\tau)$  is a trivial extension and thus lifts to some

$$\tilde{\tau} : A \rightarrow \mathcal{M}(H_B \oplus H_B).$$

Then  $(\tilde{\tau}, P) \in \mathbb{E}(A, B)$ . Finally, it is easy to see that

$$\mu((\tilde{\tau}, P), (\bar{\sigma}, P)) < \epsilon$$

with respect to the pseudometric on  $\mathbb{E}(A, B)$  so that

$$\tilde{h}[\tilde{\tau}] = [\tau] \in B_\epsilon^C$$

as required. This implies that  $\tilde{h}(B_\epsilon^C) = B_\epsilon^S$  and hence that  $\tilde{h}$  is open. As  $\tilde{h}$  is known to be a continuous bijection, it must be an isomorphism of topological groups, as desired.  $\square$

Next we compare the Cuntz and Zekri topologies, and for this we assume that  $A$  is  $K$ -nuclear.

Define

$$\hat{\alpha} : \mathbb{E}_1(A, B) \rightarrow KExt(A, B)$$

as follows. Given  $\sigma \in \mathbb{E}_1(A, B)$ , there is an associated  $K$ -extension

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & C^*(A, 2P-1) \\ & & & & \uparrow \\ & & & & B \otimes \mathcal{K} & \xrightarrow{1} & B \otimes \mathcal{K} \end{array}$$

which we denote  $\hat{\alpha}(\sigma, P)$ .

The composite

$$\mathbb{E}_1(A, B) \xrightarrow{\hat{\alpha}} KExt(A, B) \xrightarrow{\tilde{\pi}} [\epsilon A, B \otimes \mathcal{K}]$$

is given as follows. Given  $(\sigma, P)$ , the universal map

$$EA \rightarrow C^*(A, 2P-1)$$

induces a commutative diagram

$$\begin{array}{ccccc} EA & \longrightarrow & C^*(A, 2P-1) & & \\ \uparrow & & \uparrow & & \\ \epsilon A & \xrightarrow{\eta} & B \otimes K & \xrightarrow{1} & B \otimes K \end{array}$$

and the homotopy class of  $\eta$  is defined to be  $\alpha(\sigma, P) \in [\epsilon A, B \otimes \mathcal{K}]$ .

The map  $\tilde{\pi}\hat{\alpha}$  is continuous, for if  $\sigma_j$  converges to  $\sigma$  then it is clear that the associated maps

$$\hat{\alpha}(\sigma_j) : \epsilon A \rightarrow B \otimes \mathcal{K}$$

converge to  $\hat{\alpha}(e)$ . Further, the map descends to homotopy classes, giving a commuting diagram

$$\begin{array}{ccccc} \mathbb{E}_1(A, B) & \xrightarrow{\hat{\alpha}} & KExt(A, B) & & \\ \downarrow \pi & & \downarrow \tilde{\pi} & & \\ KK_1(A, B)_C & \xrightarrow{\alpha} & [\epsilon A, B \otimes \mathcal{K}] & \xrightarrow{\cong} & KK_1(A, B)_Z \end{array}$$

We may describe the map  $\alpha$  in a more universal fashion. Recall that there is a universal  $K$ -extension which we denote  $\hat{e}$ . It is immediate that  $\hat{\alpha}(\hat{e}) : \epsilon A \rightarrow \epsilon A$  is the identity element  $1_{\epsilon A}$ . Then we have

$$\alpha(e) = \hat{\alpha}(e)_* 1_{\epsilon A}$$

for any extension  $A$ , where

$$\hat{\alpha}(e)_* : [\epsilon A, \epsilon A \otimes \mathcal{K}] \rightarrow [\epsilon A, \epsilon B \otimes \mathcal{K}].$$

It is clear from this description that  $\alpha$  is continuous.

Next we define

$$\beta : KK_1(A, B)_Z \rightarrow KK_1(A, B)_C.$$

Recall that the universal  $K$ -extension has the form

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & EA \\ & & & & \uparrow \\ & & & & \epsilon A \xrightarrow{1} \epsilon A. \end{array}$$

Expand the diagram to

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & EA & \longrightarrow & \mathcal{M}(\epsilon A) \\ & & & & \uparrow & & \\ & & & & \epsilon A & \xrightarrow{1} & \epsilon A. \end{array}$$

Let  $\sigma_u : A \rightarrow \mathcal{M}(\epsilon A)$  be the composite

$$A \rightarrow EA \rightarrow \mathcal{M}(\epsilon A)$$

in the diagram, and let  $P_u = \sigma_u((F+1)/2)$ . Then  $(\sigma_u, P_u)$  defines a universal element  $u_A \in \mathbb{E}_1(A, \epsilon A)$  and a corresponding class  $[u_A] \in KK_1(A, \epsilon A)_C$ . If  $h : \epsilon A \rightarrow B \otimes \mathcal{K}$  then we define

$$\beta(h) = h_*([u_A]).$$

Alternately we may construct this class explicitly using quasi-unital approximations. Given  $[h] \in [\epsilon A, B \otimes \mathcal{K}]$ , choose a quasi-unital representative  $h : \epsilon A \rightarrow B \otimes \mathcal{K}$ . Then we may define  $\beta(h) = [(\sigma_h, P_h)]$  where  $\sigma_h$  is the composite

$$A \xrightarrow{\sigma_u} \mathcal{M}(\epsilon A) \xrightarrow{\mathcal{M}(h)} \mathcal{M}(B \otimes \mathcal{K})$$

and  $P_h = \mathcal{M}(h)(P_u)$ .

It is clear from the definitions that  $\beta$  is continuous.

**Theorem 4.4.** *Suppose that  $A$  is  $K$ -nuclear. Then the map*

$$\alpha : KK_1(A, B)_Z \cong [\epsilon A, B] \rightarrow KK_1(A, B)_C$$

*is an isomorphism of topological groups with inverse  $\beta$ .*

*Proof.* Zekri has shown that the map  $\alpha$  is a bijection with inverse  $\beta$ , so the only point at issue is whether these maps are continuous. They are indeed, as we have shown above.

□

We note that Higson has shown (cf. [2, 22.1]) that for any choice of a rank 1 projection  $e$ , the natural map  $A \rightarrow A \otimes \mathcal{K}$  sending  $a$  to  $a \otimes e$  gives rise to a natural isomorphism

$$[A \otimes \mathcal{K}, B \otimes \mathcal{K}] \xrightarrow{\cong} [A, B \otimes \mathcal{K}].$$

We use this isomorphism below without further comment.

Next we show that  $KK_1(A, B)_q \cong KK_1(A, B)_Z$ .

**Theorem 4.5.** *Suppose that  $A$  is  $K$ -nuclear. Then there is a natural homeomorphism*

$$[\epsilon A, B \otimes \mathcal{K}] \xrightarrow{\cong} [qSA, B \otimes \mathcal{K}]$$

*such that the diagram*

$$\begin{array}{ccc} [\epsilon A, B \otimes \mathcal{K}] & \longrightarrow & [qSA, B \otimes \mathcal{K}] \\ \downarrow \cong & & \downarrow \cong \\ KK_1(A, B)_Z & \longrightarrow & KK_1(A, B)_q \end{array}$$

*commutes.*

The  $K$ -nuclearity assumption (or something like it) apparently is needed since under that hypothesis there is a natural homotopy equivalence  $\epsilon A \rightarrow q\epsilon A$  which we use. This is due to Zekri [29], who notes that the  $\epsilon A$  is not necessarily homotopy equivalent to  $q\epsilon A$  for  $A$  not  $K$ -nuclear.

*Proof.* Zekri shows [30] that  $\epsilon A$  is  $KK$ -equivalent to  $SA$ . Let

$$u : q\epsilon A \rightarrow SA \otimes \mathcal{K} \quad \text{and} \quad v : qSA \rightarrow \epsilon A \otimes \mathcal{K}$$

represent classes  $[u] \in KK_0(\epsilon A, SA)$  and  $[v] \in KK_0(SA, \epsilon A)$  in the Cuntz picture which implement the  $KK$ -equivalence.

Define a map

$$\eta : [\epsilon A, B \otimes \mathcal{K}] \longrightarrow [qSA, B \otimes \mathcal{K}]$$

to be the composite

$$[\epsilon A, B \otimes \mathcal{K}] \cong [\epsilon A \otimes \mathcal{K}, B \otimes \mathcal{K}] \xrightarrow{v^*} [qSA, B \otimes \mathcal{K}]$$

In the other direction, define a map

$$\theta : [qSA, B \otimes \mathcal{K}] \longrightarrow [\epsilon A, B \otimes \mathcal{K}]$$

as follows. If  $g : qSA \rightarrow B \otimes \mathcal{K}$  then define  $\theta([g])$  to be the homotopy class of the composition

$$\epsilon A \xrightarrow{\cong} q\epsilon A \xrightarrow{qu} q(SA \otimes \mathcal{K}) \longrightarrow q(SA) \otimes \mathcal{K} \xrightarrow{g \otimes 1} B \otimes \mathcal{K} \otimes \mathcal{K} \rightarrow B \otimes \mathcal{K}$$

where  $\epsilon A \xrightarrow{\cong} q\epsilon A$  is the homotopy equivalence of Zekri. (This is where we use the  $K$ -nuclearity assumption.) These maps are clearly homomorphisms which are natural in  $B$  for fixed  $A$ . (They do depend upon the choice of  $KK$ -equivalence, of course.) We must show that  $\eta$  and  $\theta$  are inverse to one another.

The map  $\eta\theta([g])$  is the composite

$$qSA \xrightarrow{v} \epsilon A \otimes \mathcal{K} \simeq q(\epsilon A) \otimes \mathcal{K} \simeq q^2(\epsilon A) \otimes \mathcal{K} \xrightarrow{qu \otimes 1} qSA \otimes \mathcal{K} \xrightarrow{g} B \otimes \mathcal{K} \otimes \mathcal{K} \cong B \otimes \mathcal{K}$$

which is homotopic to  $qSA \xrightarrow{g} B \otimes \mathcal{K}$  since the composite

$$qSA \xrightarrow{v} \epsilon A \otimes \mathcal{K} \simeq q(\epsilon A) \otimes \mathcal{K} \simeq q^2(\epsilon A) \otimes \mathcal{K} \xrightarrow{qu \otimes 1} qSA \otimes \mathcal{K}$$

is essentially the  $KK$ -product

$$v \otimes_{\epsilon A} u = 1_{KK_0(SA, SA)}.$$

Thus  $\Psi\Theta([g]) = [g]$ . Similarly, in the other direction  $\theta\eta([f]) = [f]$  via the fact that

$$u \otimes_{SA} v = 1_{KK_0(\epsilon A, \epsilon A)}.$$

The maps  $\eta$  and  $\theta$  are visibly continuous, and hence they are homeomorphisms.

□

We conclude this section by summarizing our principal conclusions.

**Theorem 4.6.** *Suppose that  $A$  and  $B$  are separable  $C^*$ -algebras and that  $A$  is  $K$ -nuclear. Then there are natural isomorphisms of topological groups*

$$KK_1(A, B)_S \cong KK_1(A, B)_Z \cong KK_1(A, B)_C \cong KK_1(A, B)_q.$$

*Thus if all  $C^*$ -algebras appearing in the first variable of  $KK$  are  $K$ -nuclear, then the Kasparov  $KK$ -pairing is separately continuous in the Salinas topology.*

□

**Remark 4.7** In our applications we normally assume that  $A$  is  $K$ -nuclear since our applications deal with Salinas's work on quasidiagonality and hence require that the Kasparov pairing be continuous in the topology that he uses. So **we shall take all  $C^*$ -algebras appearing in the first variable of  $KK$  to be  $K$ -nuclear for the rest of the paper.** Generally speaking the results still hold without this assumption, but they have to be interpreted as holding for the topology  $KK_*(A, B)_q$ .

## 5. Consequences of the continuity of the $KK$ -product

In this section we begin our application of the continuity of the  $KK$ -pairing in order to show that other structural maps are continuous. The first result is an immediate and very useful consequence.

**Theorem 5.1.** *Suppose that the  $C^*$ -algebra  $D$  is  $KK$ -equivalent to  $D'$  via an invertible  $KK$ -class  $y \in KK_0(D, D')$ . Then for any  $C^*$ -algebra  $B$  the induced isomorphism*

$$y \otimes_{D'} (-) : KK_*(D', B) \longrightarrow KK_*(D, B)$$

*is an isomorphism of topological groups and for any  $C^*$ -algebra  $A$  the induced isomorphism*

$$(-) \otimes_D y : KK_*(A, D) \rightarrow KK_*(A, D')$$

*is an isomorphism of topological groups.*

*Proof.* Let  $y' \in KK_0(D', D)$  be the  $KK$ -inverse of  $y$ . Then the map

$$y' \otimes_D (-) : KK_*(D, B) \longrightarrow KK_*(D', B)$$

is the inverse to the map  $y \otimes_{D'} (-)$ . Both of these maps are continuous, by Theorem 4.6, and that implies that the map  $y \otimes_{D'} (-)$  is an isomorphism of topological groups. The second part of the theorem is proved similarly.  $\square$

Suppose given a short exact sequence

$$0 \rightarrow J \xrightarrow{i} A \xrightarrow{p} A/J \rightarrow 0$$

which is split with splitting  $s : A/J \rightarrow A$ . There is a natural  $KK$ -class

$$[1_A] - [sp] \in KK_0(A, A)$$

and this class lies in the image of the injection

$$KK_0(A, J) \xrightarrow{i_*} KK_0(A, A).$$

We denote by  $t^s \in KK_0(A, J)$  the unique class satisfying

$$i_*(t^s) = [1_A] - [sp].$$

### Proposition 5.2.

(1) *Suppose that*

$$0 \rightarrow J \xrightarrow{i} A \xrightarrow{p} A/J \rightarrow 0$$

*is a split short exact sequence with splitting  $s : A/J \rightarrow A$ . Then there is a canonical<sup>4</sup> isomorphism of topological groups*

$$KK_*(A, B) \cong KK_*(J, B) \oplus KK_*(A/J, B).$$

<sup>4</sup>It is not clear why the  $KK$ -class of the splitting  $s : A/J \rightarrow A$

The continuous structural maps are given by

$$i^* : KK_*(A, B) \longrightarrow KK_*(J, B),$$

$$s^* : KK_*(A, B) \longrightarrow KK_*(A/J, B),$$

$$p^* : KK_*(A/J, B) \longrightarrow KK_*(A, B),$$

and

$$t^s : KK_*(J, B) \longrightarrow KK_*(A, B).$$

(2) Suppose that

$$0 \rightarrow J \xrightarrow{i} B \xrightarrow{p} B/J \rightarrow 0$$

is a split short exact sequence with splitting  $s : B/J \rightarrow B$ . Then there is a canonical isomorphism of topological groups

$$KK_*(A, B) \cong KK_*(A, J) \oplus KK_*(A, B/J).$$

with continuous structural maps which are analogous to the maps of Part (1).

*Proof.* (1): The algebraic isomorphism follows immediately from exactness properties of the Kasparov groups, and three of the four structural maps are obviously continuous. The real point is that the map  $t^s$  also is continuous, which follows from Theorem 4.6. The proof of (2) is similar.  $\square$

Note that whenever there is an isomorphism  $G \cong G_1 \oplus G_2$  of topological groups then the projection maps are open. We use this fact in the following theorem.

**Theorem 5.3.** *Suppose that  $\{A_j\}$  is a countable family of  $C^*$ -algebras. Then the natural isomorphism of groups*

$$\varphi : KK_*(\oplus A_j, B) \longrightarrow \prod KK_*(A_j, B)$$

is an isomorphism of topological groups, where the right hand side is topologized as the product of the topological groups  $KK_*(A_j, B)$ .

*Proof.* The map  $\varphi$  is the product of the maps

$$\varphi_k : KK_*(\oplus A_j, B) \longrightarrow KK_*(A_k, B)$$

induced by the canonical inclusions  $A_k \rightarrow \oplus A_j$ , and the map  $\varphi$  is known to be an isomorphism, by work of J. Rosenberg [18, 1.12]. Each map  $\varphi_k$  is continuous, and this implies that the map  $\varphi$  is continuous.

It remains to show that the map  $\varphi$  is open, and for this it suffices to show that each  $\varphi_k$  is open. This is indeed the case, by Proposition 5.2, since each sequence

$$0 \rightarrow A_k \rightarrow \oplus A_j \rightarrow (\oplus A_j)/A_k \rightarrow 0$$

is split exact.  $\square$



## 6. Polonais and Pseudopolonais groups

Recall from C.C. Moore [16, 17] that a topological space  $X$  is said to be *polonais* if it is complete, separable, and metric. If  $X$  is a topological group then we also insist that the metric be invariant under the group action.

The motivating example is the space  $Hom(A, B)$ , where  $A$  and  $B$  are separable. Then this is a polonais space.

**Definition 6.1.** *A topological group  $G$  is said to be pseudopolonais if it is separable, pseudometric, and if the quotient group  $\overline{G} = G/G_o$  of the group  $G$  by  $G_o$ , the closure of zero, with the quotient topology is a polonais group.*

We note that this is in the direction but not as general as the weakening of the polonais hypothesis considered by C. C. Moore [17, p. 10].

If  $A$  and  $B$  are separable then the space  $[A, B]$  with its natural topology is separable, complete, and has a pseudometric  $\hat{\mu}$  given as in Section 4.

**Theorem 6.2.** *The groups  $KK_*(A, B)$  have a natural structure as pseudopolonais topological groups given by the isomorphism of topological groups*

$$KK_0(A, B) \cong [qA, B \otimes \mathcal{K}]$$

$$KK_1(A, B) \cong [q(SA), B \otimes \mathcal{K}].$$

*Proof.* Theorem 4.6 gives us the isomorphisms as indicated. So it suffices to show that  $[qA, B \otimes \mathcal{K}]$  has the structure of a pseudopolonais topological group. Given the observation above, the group is complete, separable, and pseudopolonais. It remains to check that the group action is invariant under the metric. The group addition operation is given as follows: if  $f, g : qA \rightarrow B \otimes \mathcal{K}$  then  $[f] + [g]$  is the homotopy class of the composite

$$qA \xrightarrow{(f,g)} (B \otimes \mathcal{K}) \oplus (B \otimes \mathcal{K}) \subseteq B \otimes \mathcal{K} \otimes M_2(\mathbb{C}) \cong B \otimes \mathcal{K}$$

and it is easy to check that this respects the pseudometric.

□

Next we recall some important facts about polonais groups which are all found in [17, 2.3].

### Theorem 6.3.

- (1) *If  $\{G_i\}$  is a sequence of polonais groups then their product  $\prod_i G_i$  with the product topology is polonais.*
- (2) *Suppose that*

$$0 \rightarrow G' \xrightarrow{i} G \xrightarrow{j} G'' \rightarrow 0$$

*is a short exact sequence of Hausdorff topological groups with  $i$  a homeomorphism onto its image and  $j$  continuous and open. Then  $G$  is polonais if and only if  $G'$  and  $G''$  are polonais.*

- (3) *Let  $G_1$  and  $G_2$  be separable metric groups with  $G_1$  polonais, and let  $\phi : G_1 \rightarrow G_2$  be a Borel homomorphism. Then  $\phi$  is continuous.*
- (4) *Let  $G_1$  and  $G_2$  be polonais and let  $\phi : G_1 \rightarrow G_2$  be a continuous bijection. Then  $\phi$  is open and hence an isomorphism of topological groups.*

*Proof.* Part 1 is [17, Prop. 2] and Part 2 is [10, Prop. 3]. Parts 3 and 4 appear as [17, Prop. 5]; Moore attributes them to Banach [1] and to Kuratowski [15] respectively.  $\square$

Here is an easy consequence.

**Proposition 6.4.** *Suppose that  $G$  and  $H$  are polonais groups and that  $f : G \rightarrow H$  is a continuous homomorphism.*

- (1) *Suppose that  $Im(f)$  is a closed subgroup of  $H$ . Give  $G/Ker(f)$  the quotient topology from  $G$  and  $Im(f)$  the relative topology from  $H$ . Then the natural bijection*

$$\hat{f} : G/Ker(f) \rightarrow Im(f)$$

*is an isomorphism of topological groups and hence  $f$  itself is relatively open.*

- (2) *If  $f$  is onto then  $\hat{f} : G/Ker(f) \rightarrow H$  is an isomorphism of topological groups.*  
 (3) *If*

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3$$

*is an exact sequence of polonais groups then  $Im(f_1) \cong Ker(f_2)$  is closed, and hence, topologizing as in 1), the natural map is an isomorphism*

$$G_1/Ker(f_1) \cong Im(f_1)$$

*of topological groups.*

*Proof.* Both  $G/Ker(f)$  and  $Im(f)$  are polonais in their specified topologies, and  $\hat{f}$  is a continuous bijection. So 1) is immediate from Theorem 6.3 (4). Parts 2) and 3) follow at once from 1).  $\square$

**Proposition 6.5.** *Suppose that  $G$  and  $H$  are pseudopolonais groups and that  $\beta : G \rightarrow H$  is a continuous homomorphism inducing  $\beta' : G_o \rightarrow H_o$ . Suppose that  $\bar{\beta} : \bar{G} \rightarrow \bar{H}$  is an open map.*

- (1) *If both  $\beta$  and  $\beta'$  are onto, then  $\beta$  is an open map.*  
 (2) *If both  $\beta$  and  $\beta'$  are bijections, then  $\beta$  is an isomorphism of topological groups.*

*Proof.*

We first consider 1). Let  $U$  be an open neighborhood of 0 in  $G$ . Then of course  $G_o \subseteq U$ . We wish to show that  $\beta(U)$  is open in  $H$ . Note that

$$\beta^{-1}\beta U = \{u + z : u \in U, z \in Z\} = \bigcup_{z \in Z} (U + z)$$

and this is a union of open sets, hence open. Thus without loss of generality, we may assume that  $U$  is open and saturated with respect to  $\beta$ ; i.e.,  $U = \beta^{-1}\beta U$ . Consider the commuting diagram

$$\begin{array}{ccc} G & \xrightarrow{\beta} & H \\ \downarrow \pi & & \downarrow \pi \\ \bar{G} & \xrightarrow{\bar{\beta}} & \bar{H} \end{array}$$

The map  $\bar{\beta}$  is open by assumption, and hence  $\pi\beta U$  is open in  $\bar{H}$ . This implies that the  $\pi$ -saturation of  $\beta U$ , is open in  $H$ . However,

$$\begin{aligned} \text{Sat}(\beta U) &= \beta(U) + H_0 \\ &= \beta(U) + \beta(G_o) \\ &= \beta(U + G_o) \\ &= \beta(U) \end{aligned}$$

and this completes the proof of 1). Part 2) follows from 1).  $\square$

**Corollary 6.6.** *Suppose that  $G$  is a pseudopolonais group with closed subgroup  $G'$ . Let  $G''$  be a polonais group and suppose that there is a short exact sequence*

$$0 \rightarrow G' \rightarrow G \xrightarrow{f} G'' \rightarrow 0$$

*with  $f$  continuous. Give  $G/G'$  the quotient polonais topology. Then the natural algebraic isomorphism*

$$\hat{f} : G/G' \xrightarrow{\cong} G''$$

*is an isomorphism of topological groups.*

*Proof.* This is immediate from 6.3 and the fact that polonais groups are Hausdorff.  $\square$

Our proof of the following result from folklore is based entirely upon ideas of C. C. Moore, particularly [17, Prop. 11] and [16, Prop. 1.4].

**Theorem 6.7.** *Suppose that  $G$ ,  $H$ , and  $K$  are polonais groups and that*

$$m : G \times H \longrightarrow K$$

*is a bilinear pairing which is separately continuous. Then  $m$  is jointly continuous.*

*Proof.* The fact that  $m$  is separately continuous implies that the map  $m$  is jointly measurable, by a result of Kuratowski [15, p. 285]. A theorem of Banach [1, p. 25] implies that there exists a set  $P \subseteq G \times H$  of first category such that the function

$$m : (G \times H) - P \longrightarrow K$$

is continuous.

We write  $m(g, h) = gh$  for convenience. Suppose that  $(g_n, h_n) \rightarrow (g_o, h_o)$  in  $G \times H$ . We must prove that  $g_n h_n \rightarrow g_o h_o$  in  $K$ . Consider the set

$$\tilde{P} = \cup_{n=1}^{\infty} P(-g_n, -h_n).$$

This is a set of first category in  $G \times H$ , and so  $\tilde{P} \neq G \times H$ . Choose some  $(g, h) \in (G \times H) - \tilde{P}$ . Then

$$(g + g_n, h + h_n) \in (G \times H) - P \quad \forall n$$

and since  $m$  is continuous on that set, we have

$$(g + g_n)(h + h_n) \rightarrow (g + g_o)(h + h_o).$$

We know that  $g_n h \rightarrow g_o h$  and that  $g h_n \rightarrow g h_o$  and so subtracting these terms and the constant term yields  $g_n h_n \rightarrow g_o h_o$  as required.  $\square$

At least we can complete the proof of the main theorem of this paper

**Theorem 6.8.** *The Kasparov pairing*

$$KK_*(A_1, B_1 \otimes D) \times KK_*(D \otimes A_2, B_2) \xrightarrow{\otimes_D} KK_*(A_1 \otimes A_2, B_1 \otimes B_2)$$

is jointly continuous in the Salinas topology, provided that all  $C^*$ -algebras appearing in the first variable are  $K$ -nuclear.

*Proof.* For simplicity, let

$$\begin{aligned} G &= KK_*(A_1, B_1 \otimes D) \\ H &= KK_*(D \otimes A_2, B_2) \\ K &= KK_*(A_1 \otimes A_2, B_1 \otimes B_2) \\ m(x, y) &= x \otimes_D y \end{aligned}$$

Then we must show that  $m : G \times H \rightarrow K$  is jointly continuous. We know that  $m$  is separately continuous, by Theorem 4.6. Recall that  $G_o$  denotes the closure of zero in  $G$ ,  $\overline{G} = G/G_o$ , and similarly for  $H$  and  $K$ . The map  $m$  induces a natural map

$$\overline{m} : \overline{G} \times \overline{H} \rightarrow \overline{K}$$

and it is easy to see that this map is also separately continuous. Now the groups  $\overline{G}$ ,  $\overline{H}$ , and  $\overline{K}$  are polonais, by Theorem 6.2. Thus we may apply Theorem 6.7 to conclude that the map  $\overline{m}$  is jointly continuous.

Let  $\pi$  generically denote the map from a group to its Hausdorff quotient. Since the diagram

$$\begin{array}{ccc} G \times H & \xrightarrow{m} & K \\ \downarrow \pi \times \pi & & \downarrow \pi \\ \overline{G} \times \overline{H} & \xrightarrow{\overline{m}} & \overline{K} \end{array}$$

commutes, the composite

$$G \times H \xrightarrow{m} K \xrightarrow{\pi} \overline{K}$$

is jointly continuous. Finally, let  $U$  be an open neighborhood of  $0 \in K$ . The map

$$\pi : K \rightarrow \overline{K}$$

is open for any pseudopolonais group, and hence  $\pi U$  is open in  $\overline{K}$ . Then

$$m^{-1}(U) = (\pi m)^{-1}(\pi U) = (\pi \times \pi)^{-1} \overline{m}^{-1}(\pi U)$$

which is open since both  $\overline{m}$  and  $\pi \times \pi$  are continuous. Thus  $m$  is jointly continuous and so the proof is complete.  $\square$

**Remark 6.9** The study of the topological structure of  $KK_*(A, B)$  was initiated by L. G. Brown, R.G. Douglas, and P.A. Fillmore in their 1973 [5] announcement.<sup>5</sup>

<sup>5</sup>At that time it was not known that

$$\mathcal{E}xt_*(A) \cong KK_*(A, \mathbb{C}) \equiv K^*(A),$$

They considered the natural topology on  $K^*(A)$  (usually they concentrated upon  $K^*(C(X))$ ); but it was understood that commutativity was inessential), noted that  $0$  was not necessarily closed in this topology, and used the fact that  $0$  *was* closed for  $X \subset \mathbb{C}$  to demonstrate that the set of bounded operators of the form (*normal*) + (*compact*) was norm-closed.

They also introduced the group  $PExt(X)$ , defined as

$$PExt(X) = \{x \in K^1(C(X)) : f_*x = 0 \ \forall f\}$$

where  $f$  ranges over all continuous functions from  $X$  to an ANR.<sup>6</sup> Writing  $X = \varprojlim X_n$  where the  $X_n$  are finite complexes, they announced that

$$PExt(X) = Ker [K^1(C(X)) \rightarrow \varprojlim K^1(C(X_n))].$$

Given Bott periodicity, the Milnor  $\varprojlim^1$  sequence, and the UCT (none of these were completely established in 1973), this was tantamount to proving that

$$PExt(X) \cong Pext_{\mathbb{Z}}^1(K^0(X), \mathbb{Z}).$$

Finally, they announced that

$$Ker [\gamma_{\infty} : K^1(C(X)) \rightarrow Hom_{\mathbb{Z}}(K^1(X), \mathbb{Z})]$$

was the maximal compact subgroup of  $K^1(C(X))$ . All of these results were announced in [5] but were not discussed in subsequent publication, except for the case  $X \subset \mathbb{C}$ .

It is particularly gratifying to finally establish that the Kasparov groups are pseudopolonais. L.G. Brown asked about this matter in the early days of the BDF groups, and he has asked the author regularly every few years since. His persistence is appreciated. We should note that his original question asked if it was possible to write the groups as the quotient of one polonais group by another. This question is still open.

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<sup>6</sup>For “D” instead of “mathbb{C}” and “mathbb{R}” instead of “mathbb{C}” and “mathbb{R}” instead of “mathbb{C}”.

## 7. Suspensions, Images, Boundary Maps, and the Index Map

In this section we show that various suspension and boundary maps are continuous. We also show that the boundary homomorphisms in  $KK$ -long exact sequences are given by instances of the  $KK$ -pairing and hence are continuous in each variable. Finally, we demonstrate that the index map

$$\gamma : KK_*(A, B) \longrightarrow Hom_{\mathbb{Z}}(K_*A, K_*(B))$$

is continuous in the natural topology on  $Hom$ .

The following proposition is well-known and we include a proof only for convenience.

**Proposition 7.1.** *Suppose given a short exact sequence of  $C^*$ -algebras*

$$0 \rightarrow J \xrightarrow{i} C \xrightarrow{p} C/J \rightarrow 0.$$

*Then:*

- (1) *If  $C/J$  contractible, then  $[i] \in KK_0(J, C)$  is  $KK$ -invertible.*
- (2) *If  $J$  is contractible, then  $[p] \in KK_0(C, C/J)$  is  $KK$ -invertible.*

*Proof.* Suppose that  $C/J$  is contractible. Then the map

$$i_* : KK_0(C, J) \rightarrow KK_0(C, C)$$

is an isomorphism, so define  $\sigma \in KK_0(C, J)$  to be the unique element with

$$\sigma \otimes_J [i] = i_*\sigma = [1_C].$$

Consider the commutative diagram

$$\begin{array}{ccc} KK_0(C, J) & \xrightarrow{i^*} & KK_0(J, J) \\ \downarrow i_* & & \downarrow i_* \\ KK_0(C, C) & \xrightarrow{i^*} & KK_0(J, C) \end{array}$$

Computing, we have

$$i_*i^*\sigma = i^*i_*\sigma = i^*[1_C] = [i] = i_*[1_J]$$

and since  $i_*$  is an isomorphism, it follows that

$$[i] \otimes_C \sigma = i^*\sigma = [1_J]$$

so that  $\sigma \in KK_0(C, J)$  is a  $KK$ -inverse to  $[i] \in KK_0(J, C)$ . This demonstrates Part (1).

The proof of Part (2) is similar and uses  $\sigma \in KK_0(C/J, C)$  with  $p^*\sigma = [1_C]$ .  $\square$

For any separable  $C^*$ -algebra  $E$ , let

$$b_E \in KK_1(E, SE)$$

be the Bott periodicity element. This element is the image of the universal Bott element

$$b \in KK_1(\mathbb{C}, S\mathbb{C}) \cong KK_0(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$$

under the canonical structural map

$$KK_1(\mathbb{C}, S\mathbb{C}) \rightarrow KK_1(\mathbb{C} \otimes E, S\mathbb{C} \otimes E) \cong KK_1(E, SE).$$

Much of the following proposition is due to Kasparov [14, p. 566]

**Proposition 7.2.** *Suppose that*

$$(*) \quad 0 \rightarrow D' \rightarrow D \xrightarrow{f} D'' \rightarrow 0$$

*is a short exact sequence of  $K$ -nuclear  $C^*$ -algebras. Then there exists a canonical class*

$$\Delta \in KK_1(D'', D')$$

*such that for all  $C^*$ -algebras  $A$  the boundary homomorphism*

$$\delta_* : KK_*(A, D'') \rightarrow KK_{*-1}(A, D')$$

*is given by*

$$\delta_*(x) = x \otimes_{D''} \Delta$$

*and for all  $C^*$ -algebras  $B$  the boundary homomorphism*

$$\delta^* : KK_*(D', B) \rightarrow KK_{*-1}(D'', B)$$

*is given by*

$$\delta^*(y) = \Delta \otimes_{D'} y.$$

*Both boundary maps are continuous. If  $(*)$  is an essential extension then the class  $\Delta$  corresponds to the class of the extension  $(*)$  under the identification*

$$\mathcal{E}xt(D'', D') \xrightarrow{\cong} KK_1(D'', D').$$

*Finally, if  $D$  is contractible then the class  $\Delta$  is  $KK$ -invertible.*

*Proof.* The map  $f : D \rightarrow D''$  has mapping cone sequence

$$0 \rightarrow SD'' \xrightarrow{\zeta} Cf \rightarrow D \rightarrow 0$$

and there is an associated homotopy-commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & D' & & \\
 & & & & \downarrow \eta & & \\
 0 & \longrightarrow & SD'' & \xrightarrow{\zeta} & Cf & \longrightarrow & D \longrightarrow 0 \\
 & & \downarrow id & & \downarrow & & \downarrow f \\
 0 & \longrightarrow & SD'' & \longrightarrow & CD'' & \longrightarrow & D'' \longrightarrow 0 \\
 & & & & \downarrow & & 
 \end{array}$$

by [20, 2.3], where  $CD''$ , the cone of  $D''$ , is contractible, and hence the natural inclusion  $\eta : D' \rightarrow Cf$  is  $KK$ -invertible, by Proposition 5.1. Let  $\eta^{-1} \in KK_0(Cf, D')$  denote its  $KK$ -inverse. There is then a commuting diagram

$$\begin{array}{ccc} KK_*(A, D'') & \xrightarrow{\delta_*} & KK_{*-1}(A, D') \\ \downarrow (-) \otimes_{D''} b_{D''} & & \uparrow (-) \otimes_{Cf} (\eta^{-1}) \\ KK_{*-1}(A, SD'') & \xrightarrow{(-) \otimes_{SD''} [\zeta]} & KK_{*-1}(A, Cf) \end{array}$$

and each vertical map is an isomorphism of topological groups. Define

$$\Delta = b_{D''} \otimes_{SD''} [\zeta] \otimes_{Cf} (\eta^{-1}) \in KK_1(D', D'').$$

Note that  $\Delta$  is determined uniquely by the class  $[f] \in KK_0(D, D'')$ . The commutativity of the diagram implies that  $\delta_*(x) = x \otimes_{D''} \Delta$  as required. Since  $KK$ -pairing with any element is continuous, by Theorem 6.8, it follows that the boundary homomorphism is continuous. This establishes the proposition in the second variable.

The argument in the first variable is similar with an additional twist. The map  $\delta^*$  may be represented as the composite

$$\begin{array}{ccc} KK_*(D', B) & \xrightarrow{\delta^*} & KK_{*-1}(D'', B) \\ \downarrow (\eta^{-1}) \otimes_{D'} ( ) & & \uparrow b_{D''} \otimes_{SD''} ( ) \\ KK_*(Cf, B) & \xrightarrow{[\zeta] \otimes_{Cf} ( )} & KK_*(SD'', B) \end{array}$$

so that

$$\delta^*(y) = \Delta \otimes_{D'} y$$

and so  $\delta^*$  is continuous as claimed.

Finally, suppose that  $D$  is contractible. Then  $\zeta$  is  $KK$ -invertible, by Proposition 7.1(1). Since the Bott element and  $\eta^{-1}$  are also  $KK$ -invertible and the  $KK$ -pairing is associative, this implies that  $\Delta$  is  $KK$ -invertible.  $\square$

**Remark 7.3** We note as a general statement that all  $KK$ -homology and cohomology operations are given by appropriate  $KK$ -products and hence are continuous. For instance, fix a positive integer  $n$  and let  $n : S\mathbb{C} \rightarrow S\mathbb{C}$  be the standard map of degree  $n$ . Then the mapping cone sequence has the form

$$0 \rightarrow S^2\mathbb{C} \rightarrow Cn \rightarrow S\mathbb{C} \rightarrow 0$$

and applying the functor  $K_*(A \otimes (-))$  to this sequence gives rise in a manner that we have described elsewhere [15] to the Bockstein operation

$$\beta_n : K_j(A; \mathbb{Z}/n) \rightarrow K_{j-1}(A).$$

Now this map is of course continuous, since the groups themselves are discrete. A more interesting operation arises by applying the functor  $KK_*(A, B \otimes (-))$ . Define

$$KK_*(A, B; \mathbb{Z}/n) = KK_*(A, B \otimes Cn).$$

Then there is a Bockstein operation

$$\beta_n : KK_j(A, B; \mathbb{Z}/n) \rightarrow KK_{j-1}(A, B)$$

and it is a non-trivial fact that this map is also continuous. Its continuity follows directly from 6.8



The Kasparov pairing gives a natural index map

$$\gamma : KK_*(A, B) \rightarrow Hom_{\mathbb{Z}}(K_*(A), K_*(B))$$

defined by

$$\gamma(y)(x) = x \otimes_A y$$

where the natural identifications

$$KK_*(\mathbb{C}, A) \cong K_*(A) \quad KK_*(\mathbb{C}, B) \cong K_*(B)$$

are made without further comment. We wish to topologize  $Hom_{\mathbb{Z}}(K_*(A), K_*(B))$  so as to be consistent with the Salinas topology, which is based upon the point-norm topology for extensions, and so the natural way to do this is to regard  $K_*(A)$  and  $K_*(B)$  as discrete and to use the topology of pointwise convergence on  $Hom_{\mathbb{Z}}(K_*(A), K_*(B))$ . Assume this topology as given henceforth. Note that under this topology the group

$$Hom_{\mathbb{Z}}(K_*(A), K_*(B))$$

is polonais: it is a closed subset of the polonais, totally disconnected group

$$\prod_1^{\infty} K_*(B)$$

where  $K_*(B)$  is given the discrete topology and the product topology is used on the product. If  $K_*(A)$  is finitely generated then  $Hom_{\mathbb{Z}}(K_*(A), K_*(B))$  is discrete, but in general this is not the case. For instance,

$$Hom_{\mathbb{Z}}(\oplus_1^{\infty} (\mathbb{Z}/2), \mathbb{Z}/2) \cong \prod_1^{\infty} Hom_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \prod_1^{\infty} \mathbb{Z}/2$$

is topologically a Cantor set. In general,  $Hom_{\mathbb{Z}}(G, H)$  is totally disconnected (I am indebted to George Elliott for this point).

**Proposition 7.4.** *The natural map*

$$\gamma : KK_*(A, B) \rightarrow Hom_{\mathbb{Z}}(K_*(A), K_*(B))$$

*is continuous. If  $Im(\gamma)$  is closed then  $\gamma$  is open onto its image. If  $\gamma$  is an algebraic isomorphism then  $\gamma$  is an isomorphism of topological groups.*

*Proof.* Suppose that  $y^\alpha$  is a net in  $KK_*(A, B)$  which converges to  $y \in KK_*(A, B)$ . Then for each  $x \in K_*(A)$ ,

$$\gamma(y^\alpha)(x) = x \otimes_A y^\alpha$$

is a net in  $K_*(B)$  which converges to

$$\gamma(y)x = x \otimes_B y \in K_*(B)$$

since the map  $x \otimes_B (-)$  is continuous by Theorem 4.6. Thus  $\gamma(y^\alpha)$  converges pointwise to  $\gamma(y)$  as desired. So  $\gamma$  is continuous.

Now suppose that  $Im(\gamma)$  is closed, so that it is polonais. It suffices to prove that  $\gamma$  is open. Factor the map  $\gamma$  as

$$KK_*(A, B) \xrightarrow{\pi} \overline{KK}_*(A, B) \xrightarrow{\overline{\gamma}} Im(\gamma).$$

Since  $\pi$  is an open map, it suffices to show that the map  $\overline{\gamma}$  is open. However the group  $\overline{KK}_*(A, B)$  is polonais, and  $Ker(\overline{\gamma})$  is a closed subgroup, so the quotient group

$$KK_*(A, B)/Ker(\gamma) = \overline{KK}_*(A, B)/Ker(\overline{\gamma})$$

is also polonais. Thus the induced map

$$\hat{\gamma} : KK_*(A, B)/Ker(\gamma) \rightarrow Im(\gamma)$$

is a continuous bijection of polonais groups, and hence a homeomorphism, by Theorem 6.3(4), and of course this implies that  $\overline{\gamma}$  is open.  $\square$

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