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ABSTRACT.   The Kasparov groups $KK_*(A, B)$ have a natural structure as pseudo-polonais groups. In this paper we analyze how this topology interacts with the terms of the Universal Coefficient Theorem (UCT) and the splittings of the UCT constructed by J. Rosenberg and the author, as well as its canonical three term decomposition which exists under bootstrap hypotheses. We show that the various topologies on $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$ and other related groups mostly coincide. Then we focus attention on the Milnor sequence and the fine structure subgroup of $KK_*(A, B)$. An important consequence of our work is that under bootstrap hypotheses the closure of zero of $KK_*(A, B)$ is isomorphic to the group $Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))$. Finally, we introduce new splitting obstructions for the Milnor and Jensen sequences and prove that these sequences split if $K_*(A)$ or $K_*(B)$ is torsionfree.

Typeset by $\mathcal{AMS}$-TEX

Introduction

This is the second in a series of papers devoted to the analysis of the topological structure of the Kasparov groups. In the first paper in this series [13] we demonstrated the following facts:

(1) There is a natural structure of a pseudopolonais[1] group on $KK_*(A, B)$. [13, 2.3]

(2) The Kasparov pairing is jointly continuous with respect to this topology. [13, 3.8]

(3) The index map
$$\gamma : KK_*(A, B) \to Hom_{\mathbb{Z}}(K_*(A), K_*(B))$$
is continuous. If $Im(\gamma)$ is closed (e.g., if γ is onto), then γ is an open map. If γ is an algebraic isomorphism then it is an isomorphism of topological groups. [13, 4.6]

In this paper we will study the various purely algebraic and also analytic topologies which naturally occur on the components of $KK_*(A, B)$ within the context of the UCT. Special attention will be paid to $Z_*(A, B)$, the closure of zero in $KK_*(A, B)$, which we call the fine structure subgroup.

Section 1 briefly summarizes various results on the structure of $KK_*(A, B)$ provided that $A \in \mathcal{N}$, the bootstrap category. The canonical KK-filtration diagram (1.4) is introduced. This diagram is used heavily in the remainder of the paper.

Section 2 deals with several "algebraic" topologies on $Ext^1_{\mathbb{Z}}(G, H)$ where G and H are (usually countable) abelian groups. We show that the Jensen isomorphism
$$Pext^1_{\mathbb{Z}}(G, H) \cong \varprojlim{}^1 Hom_{\mathbb{Z}}(G_i, H)$$
(which holds for G written as an increasing union of finitely generated subgroups G_i) is an isomorphism of topological groups.

Section 3 deals with topologizing most of the terms in the KK-filtration diagram. The key result is the natural isomorphism
$$Z_*(A, B) \cong Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))$$
where $Z_*(A, B)$ denotes the closure of zero in $KK_*(A, B)$.

In Section 4 we complete our analysis of the topological structure of the KK-filtration diagram. We show that each of the algebraic splittings of the UCT sequence is continuous.

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[1]A topological space is *polonais* if it is separable, complete, and metric. It is *pseudopolonais* if it is separable and it has a pseudometric (all axioms for a metric space except that if $d(x, y) = 0$ then perhaps $x \neq y$) and if its Hausdorff quotient metric space is polonais. If it is a topological group then we insist that the metric be invariant.

In Section 5 we introduce a new type of invariant for pairs of C^*-algebras A and B, namely, splitting invariants

$$m(A, B) \in Ext^1_{\mathbb{Z}}(\varprojlim KK_*(A_i, B), \varprojlim{}^1 KK_*(A_i, B))$$

and

$$j(A, B) \in Ext^1_{\mathbb{Z}}(\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)), Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))).$$

The obstruction $m(A, B)$ vanishes iff the Milnor sequence splits, and the obstruction $j(A, B)$ vanishes iff the Jensen sequence splits. We prove that

$$j(A, B) = (\varprojlim \delta_i)^* m(A, B)$$

up to isomorphism so that the splitting of the Milnor sequence implies the splitting of the Jensen sequence. If $K_*(A)$ or $K_*(B)$ is torsionfree then both invariants are zero and hence both sequences split. To find non-splitting examples one must assume that both $K_*(A)$ and $K_*(B)$ have p-torsion for some prime p. We briefly examine this situation.

In this paper all C^*-algebras are separable. All C^*-algebras appearing in the first variable of KK are assumed to be nuclear. Whenever $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$ is taken to be a subgroup of $KK_*(A, B)$ it is understood that $A \in \mathcal{N}$ and that the inclusion is via the UCT. All pseudopolonais and polonais groups are understood to be abelian.

Note: Marius Dadarlat recently has discovered [4, Cor. 4.6] a different and very interesting proof of Theorem 3.3 as well as some of its consequences for quasi-diagonality which we first proved in this paper and in [14]. It is a pleasure to thank him as well as his former student Nathaniel Brown for helpful advice and assistance in this series of papers.

1. Algebraic Facts

In this section we briefly summarize various results concerning $KK_*(A, B)$. We assume that A is in the bootstrap category $\mathcal{N}$ so that the UCT [9] holds. We introduce the canonical KK-filtration diagram (1.4) which is central to our analysis.

The first result is purely algebraic. We recall that for abelian groups G and H, $Pext_{\mathbb{Z}}^1(G, H)$ denotes the subgroup of $Ext_{\mathbb{Z}}^1(G, H)$ consisting of pure extensions.[2]

Theorem 1.1, C. U. Jensen [6]**.** *Let G be a countable group written as a union of a sequence of finitely generated subgroups G_i and let H be any group. Then there is a natural isomorphism*

$$Pext_{\mathbb{Z}}^1(G, H) \cong \varprojlim{}^1 Hom_{\mathbb{Z}}(G_i, H).$$

$\square$

We recall the definition of a KK-filtration from [11].

Definition 1.2. *A KK-filtration of a separable C^*-algebra A is an increasing sequence of commutative C^*-algebras*

$$A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \dots$$

which satisfies the following conditions:

(1) $A_i^+ \cong C(X_i)$ *for some finite CW-complex X_i.*

(2) *Each map $K_*(A_i) \to K_*(A_{i+1})$ is an inclusion.*

(3) $\varinjlim A_i$ *is KK-equivalent to A.*

It follows that each $K_*(A_i)$ is finitely generated and that

$$\varinjlim K_*(A_i) \cong K_*(A)$$

so that the sequence $\{K_*(A_i)\}$ is an increasing sequence of finitely generated subgroups with limit $K_*(A)$. Since the UCT is preserved under KK-equivalence, it follows that any KK-filtered C^*-algebra A satisfies the UCT for all B. We show [11, 1.5] that each $A \in \mathcal{N}$ has a KK-filtration and that the filtration is unique in the sense that groups such as

$$\varprojlim{}^1 Hom_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

which *a priori* depend upon a choice of KK-filtration, in fact depend only upon $K_*(A)$ and are independent of choice of KK-filtration.

---

[2]An extension of abelian groups

$$0 \to H \to E \to G \to 0$$

is *pure* if $H \cap nE = nH$ for each positive integer n. Equivalently, the extension is pure if it splits when restricted to every finitely generated subgroup of G. For further information on $Pext$ please see "A Pext Primer" [16].

Theorem 1.3 [11, Theorem 1.6]. *Suppose that A has KK-filtration $\{A_i\}$. Then the following diagram*

(1.4)

$$
\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & 0 & & & \underleftarrow{lim}\, Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) & & \\
 & \downarrow & & & \downarrow {\scriptstyle \underleftarrow{lim}\, \delta_i} & & \\
0 \to & \underleftarrow{lim}{}^1\, KK_*(A_i, B) & \xrightarrow{\sigma} & KK_*(A, B) & \xrightarrow{\rho} & \underleftarrow{lim}\, KK_*(A_i, B) & \to 0 \\
 & \downarrow {\scriptstyle \psi} & & \downarrow {\scriptstyle id} & & \downarrow {\scriptstyle \tilde{\gamma}} & \\
0 \to & Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) & \xrightarrow{\delta} & KK_*(A, B) & \xrightarrow{\gamma} & Hom_{\mathbb{Z}}(K_*(A), K_*(B)) & \to 0 \\
 & \downarrow {\scriptstyle \varphi} & & & & \downarrow & \\
 & \underleftarrow{lim}\, Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) & & & & 0 & \\
 & \downarrow & & & & & \\
 & 0 & & & & &
\end{array}
$$

is commutative, is natural with respect to A and B and has exact rows and columns. Each of the groups is independent of choice of KK-filtration and depends only upon $K_*(A)$ and $K_*(B)$. Further, the UCT map γ splits unnaturally .

Note that the lower row is simply the UCT for $KK_*(A, B)$ and the row above it is the Milnor sequence associated to the given KK-filtration. The right column arises as the inverse limit of UCT sequences for (A_i, B). The map $\tilde{\gamma}$ is onto since by a result of Roos, (cf. [8])

$$
\underleftarrow{lim}{}^1\, Ext^1_{\mathbb{Z}}(G_i, H) = 0.
$$

The exactness of the left column is also a consequence of a theorem of Roos [8].

An immediate consequence of the UCT is the following corollary.

Proposition 1.5. *If $K_*(A)$ is finitely generated then the group $KK_*(A, B)$ is countable.*

Proof. In light of the UCT, it suffices to demonstrate that if G is finitely generated and H is countable then both $Hom_{\mathbb{Z}}(G, H)$ and $Ext^1_{\mathbb{Z}}(G, H)$ are countable, and these are elementary.

$\square$

2. Algebraic topologies on Ext.

In this section we introduce several topologies on $Ext^1_{\mathbb{Z}}(G, H)$ and determine how they are related. We introduce a topology on $\varprojlim^1$ and show that the Jensen isomorphism is an isomorphism of topological groups.

Definition 2.1. *Suppose that G and H are countable abelian groups. We consider three natural topologies on the group $Ext^1_{\mathbb{Z}}(G, H)$. They are:*

    (1) *The $\mathbb{Z}$-adic topology, where the subgroups $nExt^1_{\mathbb{Z}}(G, H)$ are taken as a system of neighborhoods of the identity. This will be denoted $Ext^1_{\mathbb{Z}}(G, H)_{\mathbb{Z}}$.*

    (2) *The[3] quotient topology obtained from regarding Ext as a quotient group of a Hom group resulting from an injective resolution of H. This will be denoted $Ext^1_{\mathbb{Z}}(G, H)_I$.*

    (3) *The Jensen topology, taking the subgroups*

$$Ker\left[Ext^1_{\mathbb{Z}}(G, H) \longrightarrow Ext^1_{\mathbb{Z}}(G_i, H)\right]$$

    as neighborhoods of the identity. This will be denoted $Ext^1_{\mathbb{Z}}(G, H)_J$.

    The second topology will be defined precisely as it arises.

    We begin by some elementary observations. If G is a pseudopolonais group, let G_o denote the closure of zero in G and $\overline{G} = G/G_o$ denote the quotient group. Note that any algebraic isomorphism $G_o \cong H_o$ is an isomorphism of topological groups, since these groups are indiscrete.

    We topologize $Hom_{\mathbb{Z}}(G, H)$ by the topology of pointwise convergence.

Proposition 2.2.

    (1) *$Hom_{\mathbb{Z}}(-, -)$ is a bifunctor from discrete abelian groups to polonais groups and continuous homomorphisms in each variable.*

    (2) *$Ext^1_{\mathbb{Z}}(-, -)_I$ is a bifunctor from discrete abelian groups to pseudopolonais groups and continuous homomorphisms.*

    (3) *The boundary homomorphisms in each variable in the respective Hom-Ext_I sequences are continuous.*

    (4) *If G is finitely generated then $Ext^1_{\mathbb{Z}}(G, H)$ is discrete in the I and J topologies.*

Proof. Part 1) is immediate.
    For Part 2), let

$$0 \to H \to I \to I' \to 0$$

---

[3]Actually there is one topology for every choice of injective resolution, but it will be clear from the proof that these all yield homeomorphic topologies.

be an injective resolution of H. Then $Ext^1_{\mathbb{Z}}(G, H)_I$ is the quotient of the polonais group $Hom_{\mathbb{Z}}(G, I')$ by the image of the polonais group $Hom_{\mathbb{Z}}(G, I)$. The image may not be a closed subgroup, and hence the quotient $Ext^1_{\mathbb{Z}}(G, H)$ is pseudopolonais but not necessarily polonais.

Suppose that $r : H_1 \to H_2$. Choose an injective resolution for H_1 and another for H_2 so there is a commuting diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_1 & \longrightarrow & I_1 & \longrightarrow & I'_1 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \\
0 & \longrightarrow & H_2 & \longrightarrow & I_2 & \longrightarrow & I'_2 & \longrightarrow & 0
\end{array}
$$

(The maps s and t exist since I_2 is injective). This induces a commuting diagram

$$
\begin{array}{ccc}
Hom_{\mathbb{Z}}(G, I'_1) & \xrightarrow{\;t_*\;} & Hom_{\mathbb{Z}}(G, I'_2) \\
{\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow \\
Ext^1_{\mathbb{Z}}(G, H_1)_I & \xrightarrow{\;r_*\;} & Ext^1_{\mathbb{Z}}(G, H_2)_I
\end{array}
$$

The maps t_*, p_1 and p_2 are continuous, and p_1 and p_2 are quotient maps. This implies that r_* is continuous.

For Part 3) we argue as follows. Suppose that G is fixed and

$$0 \to H' \to H \to H'' \to 0$$

is a short exact sequence. We are asked to prove that the boundary homomorphism

$$\delta^H : Hom_{\mathbb{Z}}(G, H'') \to Ext^1_{\mathbb{Z}}(G, H')$$

is continuous. Let

$$0 \to H' \to I \to I' \to 0$$

be an injective resolution of H'. Then there is a commuting diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H' & \longrightarrow & H & \longrightarrow & H'' & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & H' & \longrightarrow & I & \longrightarrow & I' & \longrightarrow & 0
\end{array}
$$

(this exists since I is injective) and by the naturality of the long exact sequence there is a commutative square

$$
\begin{array}{ccc}
Hom_{\mathbb{Z}}(G, H'') & \xrightarrow{\;\delta^H\;} & Ext^1_{\mathbb{Z}}(G, H')_I \\
{\scriptstyle g_*}\downarrow & & {\scriptstyle id}\downarrow \\
Hom_{\mathbb{Z}}(G, I') & \xrightarrow{\;\delta^I\;} & Ext^1_{\mathbb{Z}}(G, H')_I
\end{array}
$$

so that $\delta^H = \delta^I g_*$. The map g_* is continuous by part 1) and the map δ^I is continuous since it is the map which is used to define the quotient topology to $Ext^1_{\mathbb{Z}}(G, H)_I$. So δ^H is continuous as required. The continuity of the functor $Ext^1_{\mathbb{Z}}(-, H)_I$ is immediate from definitions.

Part 4) holds since the group $Ext^1_{\mathbb{Z}}(G, H)$ is countable and complete in each of the topologies, hence discrete. □

Theorem 2.3. *For any abelian groups G and H,*

$$\overline{\{0\}}_I = \overline{\{0\}}_J = \overline{\{0\}}_{\mathbb{Z}} = Pext^1_{\mathbb{Z}}(G, H) \subseteq Ext^1_{\mathbb{Z}}(G, H).$$

Proof. The identity

$$\overline{\{0\}}_{\mathbb{Z}} \cong Pext^1_{\mathbb{Z}}(G, H)$$

is more or less by definition. Roos's Theorem [8] gives us the isomorphism

$$\overline{\{0\}}_J \cong \varprojlim{}^1 Hom_{\mathbb{Z}}(G_i, H)$$

and then applying Jensen's Theorem (1.1) yields the identity

$$Pext^1_{\mathbb{Z}}(G, H) \cong \overline{\{0\}}_J.$$

It remains to identify $\overline{\{0\}}_I$. Suppose that $a \in \overline{\{0\}}_I$. Then for each i,

$$\varphi_i(a) = 0 \in Ext^1_{\mathbb{Z}}(G_i, H)_I \cong Ext^1_{\mathbb{Z}}(G_i, H)_J$$

since this group is Hausdorff, and hence

$$a \in Ker\left[\varphi : Ext^1_{\mathbb{Z}}(G, H) \longrightarrow \varprojlim Ext^1_{\mathbb{Z}}(G_i, H)\right] \cong \overline{\{0\}}_J.$$

Thus $\overline{\{0\}}_I \subseteq \overline{\{0\}}_J$.

In the other direction, suppose that $a \in \overline{\{0\}}_J$. Let

$$0 \to G \to I \xrightarrow{\zeta} I' \to 0$$

be an injective resolution of G. Then $\varphi(a) = 0$, and hence $a|_{G_i} =$ for each i. Represent $a = [f]$ for some $f : G \to I'$. Then

$$[f|_{G_i}] = 0 \in Ext^1_{\mathbb{Z}}(G_i, H)$$

for each i. Thus there exists functions $h_i : G_i \to I$ such that the diagram

$$\begin{array}{ccc} G_i & \longrightarrow & G \\ \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle f} \\ I & \xrightarrow{\zeta} & I' \end{array}$$

commutes. Since I is injective we may extend the map h_i to a map $\hat{h}_i : G \to I$. Then $\{\hat{h}_i\}$ is a sequence in $Hom_{\mathbb{Z}}(G, I)$.

We claim that $\hat{h}_i \to f$ in the topology of pointwise convergence. This is easy. We must show that for each $x \in G$ that $\hat{h}_i(x) \to f(x)$. This is true since as soon as i is large enough so that $x \in G_i$ we have $\hat{h}_i(x) = f(x)$.

This proves that $\overline{\{0\}}_J \subseteq \overline{\{0\}}_I$. Combining with the first part of the proof yields $\overline{\{0\}}_J = \overline{\{0\}}_I$ which completes the proof of the theorem. $\square$

Remark 2.4. In general the topological groups $Ext^1_{\mathbb{Z}}(G, H)_{\mathbb{Z}}$ and $Ext^1_{\mathbb{Z}}(G, H)_S$ are not homeomorphic. Here is an example, courtesy of C. U. Jensen. Let G_i be the direct sum of i copies of the group $\mathbb{Z}/2$ and $G_i \to G_{i+1}$ be the map $x \to (x, 0)$. Then $G = \varinjlim G_i$ is the direct sum of countably many copies of $\mathbb{Z}/2$, and so

$$Ext^1_{\mathbb{Z}}(G, \mathbb{Z}) \cong \prod_{i=1}^{\infty} Ext^1_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) \cong \prod_{i=1}^{\infty} \mathbb{Z}/2.$$

The $\mathbb{Z}$-adic topology is discrete in this case, since $2 Ext^1_{\mathbb{Z}}(G, H) = 0$. However,

$$Ker\left[Ext^1_{\mathbb{Z}}(G, \mathbb{Z}) \to Ext^1_{\mathbb{Z}}(G_n, \mathbb{Z}) \right] \cong \prod_{i=n+1}^{\infty} \mathbb{Z}/2$$

and so $Ext^1_{\mathbb{Z}}(G, H)_S$ is not discrete.

To complete our algebraic discussion, we consider the groups that arise from writing G as a union of an increasing family of finitely generated subgroups G_i. The induced inverse sequence $\{ Hom_{\mathbb{Z}}(G_i, H) \}$ has associated to it the canonical Eilenberg exact sequence which we may use to define $\varprojlim^1$.

$$0 \to Hom_{\mathbb{Z}}(G, H) \to \prod_i Hom_{\mathbb{Z}}(G_i, H) \xrightarrow{\Psi}$$

(2.5)

$$\xrightarrow{\Psi} \prod_i Hom_{\mathbb{Z}}(G_i, H) \to \varprojlim{}^1 Hom_{\mathbb{Z}}(G_i, H) \to 0.$$

This definition is independent of choice of subgroups. We give $\prod_i Hom_{\mathbb{Z}}(G_i, H)$ the product topology and give $\varprojlim^1 Hom_{\mathbb{Z}}(G_i, H)$ the quotient topology.

Proposition 2.6. *The Jensen isomorphism (1.1) is an isomorphism of topological groups*

$$Pext^1_{\mathbb{Z}}(G, H) \xrightarrow{\cong} \varprojlim{}^1 Hom_{\mathbb{Z}}(G_i, H).$$

Proof. We are given a direct sequence of abelian groups

$$G_0 \to G_1 \to G_2 \to \dots$$

and this yields the canonical pure short exact sequence

$$0 \to \oplus G_i \xrightarrow{\psi} \oplus G_i \to G \to 0.$$

Apply the functor $Hom_{\mathbb{Z}}(-, H)$ and one obtains the following commutative diagram with exact columns:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
Hom_{\mathbb{Z}}(G, H) & \xrightarrow{\cong} & \varprojlim Hom_{\mathbb{Z}}(G_i, H) \\
\downarrow & & \downarrow \\
Hom_{\mathbb{Z}}(\oplus G_i, H) & \xrightarrow{H} & \prod Hom_{\mathbb{Z}}(G_i, H) \\
\downarrow{\psi^*} & & \downarrow{\Psi} \\
Hom_{\mathbb{Z}}(\oplus G_i, H) & \xrightarrow{H} & \prod Hom_{\mathbb{Z}}(G_i, H) \\
\downarrow & & \downarrow \\
Pext^1_{\mathbb{Z}}(G, H) & \xrightarrow{h} & \varprojlim{}^1 Hom_{\mathbb{Z}}(G_i, H) \\
& & \downarrow \\
& & 0
\end{array}
$$

where Ψ is the canonical Eilenberg map and H is the evident isomorphism of topological groups. Then the maps h and h^{-1} are continuous bijections, hence isomorphisms of topological groups.

$\square$

3. Topologizing the KK-filtration diagram

In this section we topologize most of the terms in the KK-filtration diagram (1.4), show that each of the natural maps is continuous, that some are open, and as a consequence obtain a deeper understanding of the fine structure subgroup. All maps in this section not otherwise identified are identified in the KK-filtration diagram (1.4). The most important result for applications is Theorem 3.3, where we identify $Z_*(A,B)$, the closure of zero in the Kasparov groups, as $Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))$.

We begin by topologizing $KK_*(A,B)$ with its natural analytic topology, as described in detail in [13]. (This arises from giving extensions the topology of pointwise convergence.) With respect to this topology $KK_*(A,B)$ is a pseudopolonais topological group.

Let $Z_*(A,B)$ denote the closure of zero in $KK_*(A,B)$. Any KK-equivalence of A to A' induces an algebraic isomorphism

$$KK_*(A', B) \cong KK_*(A, B).$$

Since KK-pairings are continuous (this is the principal result of [13]), it follows [13, Theorem 5.1] that a KK-equivalence induces an isomorphism of topological groups. The closure of zero is of course preserved under such an isomorphism, and hence

$$Z_*(A, B) \cong Z_*(A', B).$$

The analogous result holds in the second variable.

Every C^*-algebra in the bootstrap category $\mathcal{N}$ has a KK-filtration which is unique up to KK-equivalence. A KK-equivalence induces an isomorphism of topological groups, by [13, 5.1] and hence up to isomorphism of topological groups we may assume without loss of generality that each A has a KK-filtration, and the entire KK-filtration diagram depends only upon A and B and is natural in each variable.

The group $Hom_{\mathbb{Z}}(K_*(A), K_*(B))$ is topologized as before by first declaring $K_*(A)$ and $K_*(B)$ to be discrete and then by using the topology of pointwise convergence on Hom. We may regard it as a subgroup of the countable (one for each element of $K_*(A)$) product of copies of the (discrete) group $K_*(B)$. Thus $Hom_{\mathbb{Z}}(K_*(A), K_*(B))$ has the structure of a polonais group.

We recall from [13, 7.4] that the natural index map

$$\gamma : KK_*(A, B) \longrightarrow Hom_{\mathbb{Z}}(K_*(A), K_*(B))$$

is continuous. Thus

$$ker(\gamma) = Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{rel}$$

is a closed subgroup of $KK_*(A, B)$ with the relative topology.

Next we topologize the group $\varprojlim KK_*(A_i, B)$ by giving it the relative topology with respect to the natural inclusion

$$\varprojlim KK_*(A_i, B) \longrightarrow \prod KK_*(A_i, B).$$

Note that each $KK_*(A_i, B)$ is countable, complete, and hence discrete. Then it is easy to show that the maps ρ and $\bar{\gamma}$ are continuous. Further, the map $\bar{\gamma}$ is open since γ is open.

There are two reasonable topologies on the group $\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))$. One possibility is to give it the relative topology with respect to the inclusion $\varprojlim \delta_i$, or, equivalently, by giving it the relative topology in the group

$$\prod_i Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)).$$

This topology we denote by $\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))_{rel}$.

Alternately, we may topologize $\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))$ as in Section 2 by giving it the quotient topology as a quotient of the group $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_I$ via the map φ. Denote this option by

$$\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))_I.$$

Under this option the map φ is continuous and open.

Proposition 3.1. *The natural map*

$$\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))_I \longrightarrow \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))_{rel}$$

is a homeomorphism.

Proof. We refer to diagram (1.4) for notation. The map $\rho\delta$ is continuous (essentially by definition, in both cases) and the composition $\tilde{\gamma}\rho\delta = 0$, so that the image of $\rho\delta$ lies in the image of the map $\varprojlim \delta_i$. This implies that the natural map

$$\rho\delta : Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_I \longrightarrow \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))_{rel}$$

is continuous. It vanishes, of course, on the image of ψ and hence produces a continuous map I (the identity!)

$$I : \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))_I \longrightarrow \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))_{rel}$$

which is obviously an algebraic isomorphism.

We may explicitly construct the inverse to I as follows. Given any commuting diagram of abelian groups with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & G'' & \longrightarrow & 0 \\
& & \downarrow{\gamma'} & & \downarrow{\gamma} & & \downarrow{\gamma} & & '' \\
0 & \longrightarrow & H' & \longrightarrow & H & \longrightarrow & H'' & \longrightarrow & 0
\end{array}
$$

the Snake Lemma asserts that there is a long exact sequence

$$
0 \to Ker(\gamma') \to Ker(\gamma) \to Ker(\gamma'') \xrightarrow{\delta} Cok(\gamma') \to Cok(\gamma) \to Cok(\gamma'') \to 0
$$

We show [15] that in the context of topological groups the map δ is a continuous algebraic isomorphism

$$
\delta : \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))_{rel} \longrightarrow \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))_I.
$$

Since $\delta^{-1} = I$ our results imply that I is an open map and thus is an isomorphism of topological groups. $\quad\square$

Henceforth we shall use the notation $\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))$ without subscript to indicate the topological group.

Thus we conclude:

Proposition 3.2. *The right column of the KK-diagram is a short exact sequence of topological groups. The map $\bar{\gamma}$ is continuous and open, hence a quotient map, and $\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))$ is a closed subgroup of the group $\varprojlim KK_*(A_i, B)$.*

$\quad\square$

Theorem 3.3. *Suppose that $A \in \mathcal{N}$. Then the closure of zero in $KK_*(A, B)$ is the group $Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))$. That is,*

$$
Z_*(A, B) = Pext^1_{\mathbb{Z}}(K_*(A), K_*(B)).
$$

Proof. In light of Theorem 2.3, it suffices to show that $Z_*(A, B) = \overline{\{0\}}_I$.

First we show that the identity map

$$
Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_I \longrightarrow Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{rel}
$$

is continuous. For this we must recall parts of the proof of the UCT.

Recall that a geometric injective resolution of $K_*(B)$ is a sequence

$$
0 \to I_0 \to I_1 \to SB \to 0
$$

where I_0 is an ideal in I_1, $K_*(I_j)$ is injective (that is, divisible) for each j, and the resulting K-theory long exact sequence degenerates into an injective resolution of $K_*(B)$ of the form

$$0 \to K_*(B) \to K_*(I_0) \to K_*(I_1) \to 0.$$

These exist and are key to our proof of the UCT [9].

Consider the natural commuting diagram

(3.4)

$$
\begin{array}{ccc}
& & 0 \\
& & \downarrow \\
KK_*(A,B) & \xrightarrow{\ \gamma\ } & Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \\
\downarrow{\hat{f}_*} & & \downarrow{f_*} \\
KK_*(A,I_0) & \xrightarrow{\ \gamma_0\ } & Hom_{\mathbb{Z}}(K_*(A), K_*(I_0)) \\
\downarrow{\hat{g}_*} & & \downarrow{g_*} \\
KK_*(A,I_1) & \xrightarrow{\ \gamma_1\ } & Hom_{\mathbb{Z}}(K_*(A), K_*(I_1)) \\
\downarrow{\hat{\delta}_*} & & \downarrow{\delta_*} \\
KK_*(A,B) & & Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \\
\downarrow{\hat{f}_*} & & \downarrow \\
& & 0
\end{array}
$$

The maps γ_1 and γ_0 are isomorphisms by the UCT, since the $K_*(I_j)$ are injective, and thus isomorphisms of topological groups, by [13, 4.6]. The left column is exact by exactness properties of the Kasparov groups, and the right column is exact by the usual Hom-Ext exact sequence.

Both of the topologies of interest to us arise in this diagram. Specifically,

$$Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{rel} \cong Im(\hat{\delta}_*)_{rel}$$

by the proof of the UCT [9], and

$$Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_I \cong Im(\delta_*)_{quot}$$

by definition.

It is clear from the diagram that the identity map

$$Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_I \longrightarrow Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{rel}$$

is continuous. This immediately implies that $\overline{\{0\}}_I \subseteq \overline{\{0\}}_{rel}$.

Now consider the canonical map

$$\varphi : Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i, K_*(B)).$$

This map is continuous in both topologies, and the two induced topologies coincide on $Ext^1_{\mathbb{Z}}(K_*(A_i, K_*(B))$ by Proposition 3.1. Thus the map φ_{rel} is a continuous homomorphism from $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{rel}$ to a Hausdorff topological group. So $Z_*A, B)$ must be contained in $Ker(\varphi)$. We know that

$$Ker(\varphi) \cong Pext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \cong \overline{\{0\}}_I$$

and hence $Z_*(A, B) \subseteq \overline{\{0\}}_I$. Putting together the information from the previous paragraph yields $Z_*(A, B) = \overline{\{0\}}_I$ as claimed.

□

Next we look carefully at the first row of the KK-diagram, the Milnor $\varprojlim^1$ sequence. As usual we assume that $A \in \mathcal{N}$.

Proposition 3.5. *The natural map*

$$\rho : KK_*(A, B) \to \varprojlim KK_*(A_i, B)$$

is continuous and open.

Proof. Continuity is obvious. We must show that ρ is open. Note that

$$Ker(\rho) \cong Z_*(A, B)$$

by Theorem 3.3. The proof then comes down to an easy lemma.

Lemma 3.6. *Suppose that G is a pseudopolonais group with Hausdorff quotient group $\overline{G}$. Then the natural map $\rho : G \to \overline{G}$ is open.*

Proof. Suppose that U is an open neighborhood of $0 \in G$. It suffices to show that $\rho^{-1}\rho(U)$ is open in G. But $\rho^{-1}\rho(U) = U$.

□

Proposition 3.7. *The right column of diagram 1.4 is a short exact sequence of polonais groups. The group $Im(\varprojlim \delta_i)$ is a closed subgroup. The function $\varprojlim \delta_i$ is an open map onto its image. If $\varprojlim \delta_i$ is a bijection then it is an isomorphism of topological groups.*

Proof. Each of the groups in the sequence is polonais by earlier results. Then

$$Im(\varprojlim \delta_i) = Ker(\bar{\gamma})$$

is a closed subgroup. The map $\varprojlim \delta_i$ is an open map onto its image by [13, 6.4(1)], and if it is a bijection then it is continuous, open, and hence a homeomorphism.   □

4. Topologizing the UCT

In this section we complete our topological results and show how these results fit together in the UCT. We show that each of the algebraic splittings of the UCT produced by [9] is continuous. Finally, we demonstrate how to realize Jensen's counterexample (2.4) geometrically.

Our first task is to study the various $\varprojlim^1$ terms that arise.

The first topology that we consider on the group $\varprojlim^1 KK_*(A_i, B)$ is the relative topology as a subspace of $KK_*(A, B)$ via the map σ. We denote this by $\varprojlim^1 KK_*(A_i, B)_{rel}$. It is clear that the Jensen isomorphism induces a natural isomorphism of topological groups

$$\varprojlim^1 KK_*(A_i, B)_{rel} \cong Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{rel}$$

and we know by (3.3) that

$$Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{rel} \cong Z_*(A, B).$$

We summarize:

Proposition 4.1. *The Milnor sequence*

$$0 \to \varprojlim^1 KK_*(A_i, B)_{rel} \xrightarrow{\sigma} KK_*(A, B) \xrightarrow{\rho} \varprojlim KK_*(A_i, B) \to 0$$

is a short exact sequence of topological groups, the map ρ is open and hence a quotient map, and the group $\varprojlim^1 KK_*(A_i, B) \cong Z_*(A, B)$ is the closure of zero in the analytic topology on KK.

We note that the group $\varprojlim KK_*(A_i, B)$ appears in the work of Rørdam, Dadarlat and Loring and is there denoted $KL_*(A, B)$. Proposition 4.1 implies that this group is the Hausdorff quotient of $KK_*(A, B)$:

$$KL_*(A, B) \cong \frac{KK_*(A, B)}{Z_*(A, B)}.$$

The group $\varprojlim^1 KK_*(A_i, B)$ may also have the quotient topology obtained as a quotient of the group

$$\prod KK_*(A_i, B)$$

as in diagram (2.5), but this topology coincides with relative topology by Proposition 2.6. Similarly there is a natural isomorphism of topological groups

$$\varprojlim^1 Hom_{\mathbb{Z}}(K_*(A_i), K_*(B))_{quot} \cong \varprojlim^1 Hom_{\mathbb{Z}}(K_*(A_i), K_*(B))_{rel}$$

and so we write this group henceforth without subscript.

Theorem 4.2. *Suppose that $\{A_i\}$ is a KK-filtration of $A \in \mathcal{N}$. Then the natural isomorphism*

$$\varprojlim{}^1 \gamma_i : \varprojlim{}^1 KK_*(A_i, B) \longrightarrow \varprojlim{}^1 Hom_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

is an isomorphism of topological groups.

Proof. For each i there is a UCT sequence

$$0 \to Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) \to KK_*(A_i, B) \to Hom_{\mathbb{Z}}(K_*(A_i), K_*(B)) \to 0$$

and these form an inverse system of short exact sequences. This yields a six term $\varprojlim$ - $\varprojlim^1$ sequence. However,

$$\varprojlim{}^1 Ext^1_{\mathbb{Z}}(G_i, H) = 0.$$

by the results of Roos [8]. Then one obtains the algebraic isomorphism

$$\varprojlim{}^1 \gamma_i : \varprojlim{}^1 KK_*(A_i, B) \longrightarrow \varprojlim{}^1 Hom_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

The isomorphism is continuous, by construction, so it suffices to prove that the map $\varprojlim^1 \gamma_i$ is an open map. Each map

$$\gamma_i : KK_*(A_i, B) \to Hom_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

is open (since the UCT splits topologically) and thus the map $\prod \gamma_i$ is open in the commutative diagram

$$
\begin{array}{ccc}
\prod KK_*(A_i, B) & \xrightarrow{\prod \gamma_i} & \prod Hom_{\mathbb{Z}}(K_*(A_i), K_*(B)) \\
\downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi'} \\
\prod KK_*(A_i, B) & \xrightarrow{\prod \gamma_i} & \prod Hom_{\mathbb{Z}}(K_*(A_i), K_*(B)) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\
\varprojlim{}^1 KK_*(A_i, B) & \xrightarrow{\varprojlim{}^1 \gamma_i} & \varprojlim{}^1 Hom_{\mathbb{Z}}(K_*(A_i), K_*(B)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

The maps π and π' are continuous and open quotient maps. This shows that $(\varprojlim^1 \gamma_i)\pi$ is open, which implies that $\varprojlim^1 \gamma_i$ is open. $\square$

We note that there is another possible topology on $KK_*(A, B)$, namely that obtained by taking the sequence of subgroups

$$Ker\left[KK_*(A, B)] \longrightarrow KK_*(A_i, B)\right]$$

and the induced topology on $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$ obtained by the associated sequence of subgroups

$$Ker\left[Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))\right].$$

We denote this topology by using M as a subscript.

The following Proposition will be used in the proof of Proposition 4.4. Recall that for any topological group G, G_o denotes the closure of zero, and $\overline{G} = G/G_o$, denotes the maximal Hausdorff quotient group of G.

Proposition 4.3. *Suppose given a commutative diagram of pseudopolonais groups*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G_o & \longrightarrow & G & \xrightarrow{\pi_G} & \overline{G} & \longrightarrow & 0 \\
& & \downarrow{\lambda_o} & & \downarrow{\lambda} & & \downarrow{\bar{\lambda}} & & \\
0 & \longrightarrow & H_o & \longrightarrow & H & \xrightarrow{\pi_H} & \overline{H} & \longrightarrow & 0
\end{array}
$$

with $\bar{\lambda}$ an isomorphism of topological groups and λ_o and λ algebraic isomorphisms. Then λ and λ_o are isomorphisms of topological groups. Consequently, any algebraic splitting $\overline{G} \to G$ is continuous.

Proof. The map λ_o is an algebraic isomorphism of topological groups with the indiscrete topology and is automatically a homeomorphism. So we concentrate upon λ.

Let U be an open neighborhood of the origin in H. Then $U = U + H_o = \pi_H^{-1}\pi_H U$ is saturated and hence $\pi_H(U)$ is open in $\overline{H}$. Then

$$\lambda^{-1}U = \pi_G^{-1}\bar{\lambda}^{-1}\pi_H U = (\bar{\lambda}\pi_G)^{-1}\pi_H U.$$

The map $\bar{\lambda}\pi_G$ is continuous, $\pi_H C$ is open, and hence $\lambda^{-1}C$ is open in G. Thus λ is continuous.

Reversing the roles of G and H by symmetry, λ^{-1} is also continuous, and hence λ is an isomorphism of topological groups. $\square$

Proposition 4.4. *There are natural isomorphisms of topological groups*

$$
\begin{aligned}
Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_M &\cong Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_J \\
&\cong Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{rel} \\
&\cong Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_I
\end{aligned}
$$

Proof. The first isomorphism is a consequence of the technique of KK-filtration, since any increasing sequence of subgroups of $K_*(A)$ may be realized as a sequence of the form $\{K_*(A_i)\}$. For the second we argue as follows. The natural map

$$\iota : Ext_{\mathbb{Z}}^1(K_*(A), K_*(B))_J \rightarrow Ext_{\mathbb{Z}}^1(K_*(A), K_*(B))_{rel}$$

is continuous and induces a commuting diagram

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \overline{\{0\}}_J & \rightarrow & Ext_{\mathbb{Z}}^1(K_*(A), K_*(B))_J & \rightarrow & \varprojlim Ext_{\mathbb{Z}}^1(K_*(A_i), K_*(B)) & \rightarrow & 0 \\
& & \downarrow{\iota'} & & \downarrow{\iota} & & \downarrow{\iota''} & & \\
0 & \rightarrow & \overline{\{0\}}_{rel} & \rightarrow & Ext_{\mathbb{Z}}^1(K_*(A), K_*(B))_{rel} & \rightarrow & \varprojlim Ext_{\mathbb{Z}}^1(K_*(A_i), K_*(B)) & \rightarrow & 0.
\end{array}
$$

We deliberately do not put subscripts on the term $\varprojlim Ext_{\mathbb{Z}}^1(K_*(A_i), K_*(B))$ since we know that the topologies on this term are homeomorphic. Now ι' is an algebraic and hence a topological isomorphism, and ι'' is an isomorphism of topological groups as well. So we apply Proposition 4.3 to conclude that ι is also an isomorphism of topological groups.

A similar argument shows that the natural map

$$\iota : Ext_{\mathbb{Z}}^1(K_*(A), K_*(B))_I \rightarrow Ext_{\mathbb{Z}}^1(K_*(A), K_*(B))_{rel}$$

is an isomorphism of topological groups. The key point is that ι is continuous, and this follows as in (3.4) $\square$

We assume as usual that $A \in \mathcal{N}$, so that the UCT holds. This takes the form of a natural short exact sequence [7]

$$0 \rightarrow Ext_{\mathbb{Z}}^1(K_*(A), K_*(B)) \longrightarrow KK_*(A, B) \rightarrow Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \rightarrow 0$$

and $KK_*(A, B)$ has a natural structure as a pseudopolonais topological group,

Theorem 4.5. *Suppose that $A \in \mathcal{N}$. Then each of the splittings of the Universal Coefficient Theorem constructed in* [9] *is continuous, and for each splitting the resulting (unnatural) algebraic isomorphism*

$$KK_*(A, B) \cong Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \oplus Ext_{\mathbb{Z}}^1(K_*(A), K_*(B))_{rel}$$

is an isomorphism of pseudopolonais groups. If $K_*(A)$ is finitely generated then the group $KK_*(A, B)$ is polonais.

Proof. The splittings are constructed as follows. Choose KK-equivalences

$$A \approx A_0 \oplus A_1 \qquad\qquad B \approx B_0 \oplus B_1.$$

with
$$K_i(A_j) \;=\; K_i(B_j) \;=\; 0 \qquad i \neq j$$

Then the UCT breaks down into the direct sum of four sequences and since a KK-equivalence is a homeomorphism it is enough to consider two of these cases, say A_0, B_0 and A_0, B_1, and furthermore we may assume that each A_i is KK-filtered.

First assume that $A = A_0$ and $B = B_0$. Then there is only one UCT map which is nontrivial, namely

$$\gamma : KK_0(A,B) \longrightarrow Hom_{\mathbb{Z}}(K_*(A), K_*(B))_0 \;\cong\; Hom_{\mathbb{Z}}(K_0(A), K_0(B)).$$

This map is an isomorphism, and it is an isomorphism of topological groups by [13, 4.6]. So $\Gamma = \gamma^{-1}$ is a continuous splitting.

In the case $A = A_0$ and $B = B_1$ the map

$$\delta : Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{rel} \longrightarrow KK_*(A,B)$$

is an isomorphism, by the UCT, and it is continuous. To show that it is an isomorphism we consider the commuting diagram with exact columns:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\underleftarrow{lim}^1\, Hom_{\mathbb{Z}}(K_*(A_i), K_*(B)) & \xrightarrow{\;\delta'\;} & \underleftarrow{lim}^1\, Hom_{\mathbb{Z}}(K_*(A_i), K_*(B)) \\
\downarrow & & \downarrow \\
Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) & \xrightarrow{\;\delta\;} & Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \\
\downarrow & & \downarrow \\
\underleftarrow{lim}\, Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) & \xrightarrow{\;\delta''\;} & \underleftarrow{lim}\, KK_*(A_i, B) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

The horizontal maps are algebraic isomorphisms. The map δ' is the identity map of topological groups. The map δ'' is an isomorphism by Proposition 3.2. Thus we may apply [13, Th. 6.5] to conclude that the map δ is an isomorphism of topological groups.

Finally, if $K_*(A)$ is finitely generated then Ext is countable, by 1.5, and the result follows. $\square$

Remark 4.6. We may realize the Jensen example (2.4) as follows. Choose C^*-algebras $A, B \in \mathcal{N}$ with

$$K_0(A) = \oplus_1^{\infty} \mathbb{Z}/2 \qquad K_0(B) = \mathbb{Z}/2 \qquad K_1(A) = K_1(B) = 0.$$

(Such C^*-algebras exist and are unique up to KK-equivalence by the proof of the UCT.) Then

$$KK_1(A, B) \cong Ext_{\mathbb{Z}}^1(K_0(A), K_0(B)) \cong Ext_{\mathbb{Z}}^1(\oplus_1^\infty \mathbb{Z}/2, \mathbb{Z}/2) \cong \prod_1^\infty \mathbb{Z}/2$$

and applying the analysis of 2.4 we see that $Ext_{\mathbb{Z}}^1(K_*(A), K_*(B))_{\mathbb{Z}}$ is not homeomorphic to $Ext_{\mathbb{Z}}^1(K_*(A), K_*(B))_J$.

5. Splitting Obstructions

In this section we introduce splitting obstructions $m(A,B)$ and $j(A,B)$ associated to the Milnor and Jensen sequences respectively and show how they are related. We demonstrate that if either $K_*(A)$ or $K_*(B)$ is torsionfree then both sequences split. We also give an example due to Christensen-Strickland in which the obstructions do not vanish.

We suppose as usual that $A \in \mathcal{N}$ with associated KK-filtration diagram (1.4). Define

$$m(A,B) \in Ext^1_{\mathbb{Z}}(\varprojlim KK_*(A_i,B)\,,\,\varprojlim{}^1 KK_*(A_i,B)\,)$$

to be the class of the Milnor sequence

$$0 \to \varprojlim{}^1 KK_*(A_i,B) \to KK_*(A,B) \xrightarrow{\rho} \varprojlim KK_*(A_i,B) \to 0$$

and let

$$j(A,B) \in Ext^1_{\mathbb{Z}}(\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i),K_*(B)), Pext^1_{\mathbb{Z}}(K_*(A),K_*(B))).$$

be the class of the Jensen sequence

$$0 \to Pext^1_{\mathbb{Z}}(K_*(A),K_*(B)) \to Ext^1_{\mathbb{Z}}(K_*(A),K_*(B)) \xrightarrow{\varphi} \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i),K_*(B)) \to 0$$

Thus the Milnor sequence splits iff $m(A,B) = 0$ and similarly the Jensen sequence splits iff $j(A,B) = 0$.[4]

Note that we have shown that

$$\varprojlim{}^1 KK_*(A_i,B) \cong Pext^1_{\mathbb{Z}}(K_*(A),K_*(B))$$

and so the subgroups in the two short exact sequences are canonically isomorphic. We make this identification in the following proposition.

Here is how the two splitting obstructions are related.

Proposition 5.1. *With the notation above, and up to the Jensen isomorphism,*

$$j(A,B) = (\varprojlim \delta_i)^* m(A,B).$$

---

[4]The maps ρ and φ are continuous and open. For each map an algebraic splitting is automatically continuous, by Proposition 4.5. No claim is made for the naturality of such splittings.

Proof. In light of the Jensen isomorphism it suffices to demonstrate that the right square of the commutative diagram

$$
\begin{array}{ccccccc}
0 & \to & Pext^1_{\mathbb{Z}}(K_*(A), K_*(B)) & \xrightarrow{\psi} & Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) & \xrightarrow{\varphi} & \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) & \to & 0 \\
& & \downarrow{\cong} & & \downarrow{\delta} & & \downarrow{\varprojlim \delta_i} & & \\
0 & \to & \varprojlim^1 KK_*(A_i, B) & \xrightarrow{\sigma} & KK_*(A, B) & \xrightarrow{\rho} & \varprojlim KK_*(A_i, B) & \to & 0
\end{array}
$$

is a pullback diagram.

So suppose that $x \in KK_*(A, B)$, $y \in \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))$, and that

$$
\rho(x) = (\varprojlim \delta_i)(y).
$$

Then

$$
\begin{aligned}
\gamma(x) &= (\varprojlim \gamma_i)\rho(x) \\
&= (\varprojlim \gamma_i)(\varprojlim \delta_i)(y) \\
&= \varprojlim (\gamma_i \delta_i)(y) \\
&= 0
\end{aligned}
$$

since $\gamma_i \delta_i = 0$ for each i, and hence

$$
x \in Ker(\gamma) = Im(\delta).
$$

As δ is mono, there is a unique choice $z \in Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$ with $\delta(z) = x$. Then

$$
\begin{aligned}
(\varprojlim \delta_i)(y - \varphi(z)) &= \rho(x) - (\varprojlim \delta_i)\varphi(z) \\
&= \rho(x) - \rho\delta(z) \\
&= \rho(x) - \rho(x) = 0
\end{aligned}
$$

and since $\varprojlim \delta_i$ is mono, this implies that $y = \varphi(z)$. Thus the right square is a pullback and the proof is complete. $\square$

Here is another result along similar lines.

Theorem 5.2.

(1) *If $K_*(A)$ is a direct sum of cyclic groups or if $K_*(B)$ is algebraically compact then $Pext^1_{\mathbb{Z}}(K_*(A), K_*(B)) = 0$ and both obstructions vanish.*

(2) If $K_*(A)$ or $K_*(B)$ is torsionfree then $Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))$ is divisible and both obstructions vanish.

Proof. These are both purely algebraic results. Let $G = K_*(A)$ and $H = K_*(B)$. If G is a direct sum of cyclic groups or if H is algebraically compact then the group $Pext^1_{\mathbb{Z}}(G, H) = 0$ by classical results (cf. [16], 5.4).

If G is torsionfree then for any abelian group H the group $Ext^1_{\mathbb{Z}}(G, H)$ is divisible. Then

$$Pext^1_{\mathbb{Z}}(G, H) = \cap_n n Ext^1_{\mathbb{Z}}(G, H) = Ext^1_{\mathbb{Z}}(G, H)$$

and so $Pext^1_{\mathbb{Z}}(G, H)$ is divisible. If H is torsionfree then for any G the group $Pext^1_{\mathbb{Z}}(G, H)$ is the maximal divisible subgroup of $Ext^1_{\mathbb{Z}}(G, H)$ by [16, 8.5]. This completes the proof. $\square$

To look for cases where the sequences do not split, then, one must have torsion (for the same prime) in both $K_*(A)$ and $K_*(B)$.

Example 5.3. (Christensen-Strickland [3])

Here is an example where both invariants are non-vanishing. Fix a prime p. Now take $A, B \in \mathcal{N}$ with

$$K_0(A) = \mathbb{Z}(p^\infty) \qquad K_1(A) = 0$$

$$K_0(B) = 0 \qquad K_1(B) = \bigoplus_n \mathbb{Z}/p^n$$

Then $Hom_{\mathbb{Z}}(K_*(A), K_*(B)) = 0$ and the Milnor and Jensen sequences both reduce to the short exact sequence

$$0 \to Pext^1_{\mathbb{Z}}(K_0(A), K_1(B)) \to KK_0(A, B) \to \varprojlim K_1(B)/p^n \to 0.$$

Making algebraic identifications, this sequence is isomorphic to the sequence

$$0 \to Pext^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \bigoplus_n \mathbb{Z}/p^n) \to Ext^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \bigoplus_n \mathbb{Z}/p^n) \to \widehat{\bigoplus_n \mathbb{Z}/p^n} \to 0$$

where $\widehat{\bigoplus_n \mathbb{Z}/p^n}$ denotes the p-adic completion of $\bigoplus_n \mathbb{Z}/p^n$. Christensen and Strickland [3, 6.6, 6.7] demonstrate that this sequence does not split. Thus $j(A, B) \neq 0$[5] and by Prop. 5.1, $m(A, B) \neq 0$ as well.

---

[5]In fact they show that the element $j(A, B)$ has infinite order.

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