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A TWO-STAGE POSTNIKOV SYSTEM WHERE $E_2 \neq E_\infty$ IN THE EILENBERG-MOORE SPECTRAL SEQUENCE

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Abstract. Let $\Omega B \to PB \to B$ be the path fibration over the simply-connected space B, let $\Omega B \to E \to X$ be the induced fibration via the map $f: X \to B$, and let X and B be generalized Eilenberg-MacLane spaces. G. Hirsch has conjectured that H^*E is additively isomorphic to $\mathrm{Tor}_{H^*B}(Z_2, H^*X)$, where cohomology is with Z_2 coefficients. Since the Eilenberg-Moore spectral sequence which converges to H^*E has $E_2 = \mathrm{Tor}_{H^*B}(Z_2, H^*X)$, the conjecture is equivalent to saying $E_2 = E_\infty$. In the present paper we set $X = K(Z_2 + Z_2, 2)$, $B = K(Z_2, 4)$ and $f^*i =$ the product of the two fundamental classes, and we prove that $E_2 \neq E_3$, disproving Hirsch's conjecture. The proof involves the use of homology isomorphisms $C^*X \xrightarrow{g} \bar{C}(H^*\Omega X) \xrightarrow{h} H^*X$ developed by J. P. May, where $\bar{C}$ is the reduced cobar construction. The map g commutes with cup-1 products. Since the cup-1 product in $\bar{C}(H^*\Omega X)$ is well known, and since differentials in the spectral sequence correspond to certain cup-1 products, we may compute d_2 on specific elements of E_2.

Introduction. A two-stage Postnikov system $\mathscr{P}$ is a diagram

$$\begin{array}{ccc} \Omega B & \longrightarrow & \Omega B \\ \downarrow & & \downarrow \\ \mathscr{P}\colon E & \longrightarrow & PB \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow{f} & B \end{array}$$

where p' is the path fibration over the simply-connected space B, p is the induced fibration, and X and B are generalized Eilenberg-MacLane spaces. Throughout this paper the coefficient group for homology and cohomology is Z_2.

In [3], G. Hirsch conjectures that for any $\mathscr{P}$ (with Z_2 coefficients), H^*E is additively isomorphic to $\mathrm{Tor}_{H^*B}(Z_2, H^*X)$. The Eilenberg-Moore spectral sequence $E_r(\mathscr{P})$ which converges to H^*E has $E_2(\mathscr{P}) = \mathrm{Tor}_{H^*B}(Z_2, H^*X)$, and so Hirsch's conjecture is equivalent to the assertion that $E_2(\mathscr{P}) = E_\infty(\mathscr{P})$. A counterexample to Hirsch's conjecture is given by

THEOREM. *In the Z_2 Eilenberg-Moore spectral sequence $E_r(\mathscr{P})$ for $B = K(Z_2, 4)$ with fundamental class i, $X = K(Z_2 + Z_2, 2)$ with fundamental classes j, k, and $f^*i = jk$, there is some c in $E_2^{-2,17}$ such that $d_2(c) \neq 0$, and hence $E_2 \neq E_3$.*

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The proof of this theorem has two stages. We first describe some machinery (due to John Moore and to J. Peter May, [5], [7]) for computing differentials in the spectral sequence, and we then exhibit c and show that $d_2(c) \neq 0$.

The author would like to express his deep gratitude to J. Peter May, who suggested this method of evaluating differentials. The author is also grateful to Ib Madsen for his valuable suggestions.

1. **Differentials and cup-1 products.** Let $\mathscr{P}$ be a two-stage Postnikov system, $X=K(\pi, n)$ where $n \geqq 2$ (and if $n=2$ and $r>1$ then no factors of the form Z_2r appear in π) and let $B(-,-,-)$ be the two-sided bar construction. (See [4] for details.) May asserts in [7] (and we shall sketch a proof in §2) that there exists a homology isomorphism $\alpha\colon C^*X \to H^*X$ of differential Hopf algebras (regarding H^*X as having zero differential). Granting that fact, we shall show in this section how May's method works to compute certain differentials in $E_r(\mathscr{P})$.

We give H^*X the structure of a left C^*B module by the composite

$$C^*B \otimes H^*X \xrightarrow{f^\# \otimes 1} C^*X \otimes H^*X \xrightarrow{\alpha \otimes 1} H^*X \otimes H^*X \xrightarrow{\text{mult.}} H^*X.$$

Then α is a morphism of C^*B modules, and

$$B(1, 1, \alpha)\colon B(Z_2, C^*B, C^*X) \to B(Z_2, C^*B, H^*X)$$

induces a morphism of algebraic Eilenberg-Moore spectral sequences

$$E_r(\mathscr{P}) = E_r(Z_2, C^*B, C^*X) \to E_r(Z_2, C^*B, H^*X)$$

which is an isomorphism for $r \geqq 1$.

Suppose that $\bar{c} = \sum_i ([a_i]_*[b_i])m_i$ is a cycle in $E_1(\mathscr{P}) = B(Z_2, H^*B, H^*X)$, where $[a]_*[b] = [a|b] + [b|a]$ is the shuffle product. Let d' and d'' denote respectively the external and internal differentials in $B(Z_2, C^*B, H^*X)$, and let a_i' and b_i' be cocycles in C^*B representing a_i and b_i. Then we have the following:

$$\begin{array}{ccc} \sum ([a_i']_*[b_i'])m_i & \xrightarrow{d'} & \sum [a_i'b_i' + b_i'a_i']m_i \\ & & \Big\uparrow{\scriptstyle d''} \\ & & \sum [a_i' \cup_1 b_i']m_i \xrightarrow{d'} \sum \alpha((f^\# a_i') \cup_1 (f^\# b_i'))m_i. \end{array}$$

(The term $\sum ([a_i'](f^*b_i) + [b_i'](f^*a_i))m_i$ must be zero in order for $\bar{c}$ to be a cycle, since H^*X has zero differential.) By the definition of d_2, we have

$$d_2(\operatorname{cls} \bar{c}) = \sum \alpha((f^\# a_i') \cup_1 (f^\# b_i'))m_i \quad \text{in } E_2^{0,*}(\mathscr{P}) \tag{1.1}$$

which is the result we shall use. We must first, however, know how to evaluate terms of the form $\alpha(c \cup_1 d)$ when c and d are cocycles in C^*X. This depends upon additional properties of α which we shall explore in the following section.

2. **The map** $\alpha: C^*X \to H^*X$. We fix the following notation:

$X = K(\pi, n)$ where $n \geqq 2$, and if $n=2$ and $r>1$ then no factors of the form Z_{2^r} appear in π.

$Z_2(\pi)$ = the group ring of π with coefficients in Z_2.

$\bar{B}$ = the reduced bar construction $(B(Z_2, -, Z_2))$.

$\bar{W}$ = the reduced "W-construction" (see [2], [6] for definitions and details).

C_* = normalized chain functor on simplicial abelian groups.

$Y^n = Y(Y^{n-1})$ where $Y^1 = Y$ and $Y = \bar{B}$ or $\bar{W}$.

Abbreviate "chain-homotopy cocommutative differential graded Hopf algebra" by "D_1-algebra" and denote the chain homotopy by D_1. Finally, let $\tilde{\pi}$ be the simplicial group given by $\tilde{\pi}_q = \pi$ for all q and taking all face and degeneracy operators to be the identity.

We first note that $\bar{B}(Z_2(\pi))$ and $C_*(\bar{W}\tilde{\pi})$ are D_1-algebras, that there is an isomorphism of D_1-algebras $c^1: \bar{B}(Z_2(\pi)) \to C_*(\bar{W}\tilde{\pi})$, and that C_*X may be taken to be $C_*(\bar{W}^n\tilde{\pi})$. (See [6, §§9 and 23].) The following key theorem is due to May [7], generalizing a result of J. Moore [2].

THEOREM. *Let G be a connected simplicial abelian group with $G_0 = e_0$, let U be a connected D_1-algebra over Z_2 such that $D_1 U \subset IU \otimes IU$, and suppose that $f: U \to C_*G$ is a homology isomorphism of differential Hopf algebras. Then there is a homology isomorphism of D_1-algebras $f': \bar{B}U \to C_*(\bar{W}G)$, i.e., f' is a morphism of D_1-algebras which induces an isomorphism on the homology level. In particular, there is a homology isomorphism of D_1-algebras $c^n: \bar{B}^n(Z_2(\pi)) \to C_*X$.*

By Cartan [2, Théorème 5, p. 4.07], there are homology isomorphisms $e^n: H_*X \to \bar{B}^n(Z_2(\pi))$. Each e^n is a morphism of differential Hopf algebras. We define $\alpha: C^*X \to H^*X$ to be the composite

$$C^*X \xrightarrow{c^{n*}} \bar{B}^n(Z_2(\pi))^* \xrightarrow{e^{n*}} H^*X.$$

Then α is a homology isomorphism of differential Hopf algebras.

Now e^{n*} factors as

$$\bar{B}^n(Z_2(\pi))^* \xrightarrow{\bar{B}(e^{n-1})^*} \bar{B}(H_*\Omega X)^* = \bar{C}(H^*\Omega X) \xrightarrow{h} H^*X$$

where $e^{n-1}: H_*\Omega X \to \bar{B}^{n-1}(Z_2(\pi))$, $\bar{C} = \bar{B}^*$ is the reduced cobar construction, and h is defined as the factor. We may thus rewrite α to be the composite

$$C^*X \xrightarrow{g} \bar{C}(H^*\Omega X) \xrightarrow{h} H^*X.$$

It is important to note that since c^n is not natural, g is *not* natural in spaces X, and so α is not natural.

We recall from J. F. Adams [1] that $\bar{C}(H^*\Omega X)$ has a cup-1 product (dual to the appropriate chain homotopy D_1). If each a_i is primitive, then the cup-1 product is given by

$$\begin{aligned}[t] &[a_1|\dots|a_r] \cup_1 [b_1|\dots|b_s] \\ &\qquad = \sum_{j=1}^{r} \sum_{i=1}^{s} [a_1|\dots|a_{j-1}|b_1|\cdots|b_{i-1}|a_j b_i|b_{i+1}|\cdots|b_s|a_{j+1}|\dots a_r]. \end{aligned} \tag{2.1}$$

By its definition, g preserves cup-1 products. We may thus improve (1.1) to read $d_2(\text{cls } \bar{c}) = \sum h((gf^{\#}a_i') \cup_1 (gf^{\#}b_i'))m_i$. We define a "cup-1" product in H^*X by $a \cup_1 b = h((ga') \cup_1 (gb'))$, where a' and b' are cocycles representing a and b respectively. This is well defined, and (1.1) becomes

$$d_2(\text{cls } \bar{c}) = \sum ((f^*a_i) \cup_1 (f^*b_i))m_i. \tag{2.2}$$

We are now completely off the cochain level. To use (2.2) we need only compute cup-1 products in $\bar{C}(H^*\Omega X)$ with (2.1), and then apply the map h.

We now describe the map h more explicitly. Let $H^*\Omega X = P(y_i)$. Then $H_*\Omega X = \Gamma(x_i) = E(\gamma_{2^j}(x_i))$, where Γ is the divided polynomial algebra functor and x_i is dual to y_i in the basis of monomials of the generators. There is an Eilenberg-Moore spectral sequence converging to H^*X, and with $E_2 = \text{Ext}_{H_*\Omega X}(Z_2, Z_2)$. It is routine that $E_2 = P(v(i,j))$, where $v(i,j)$ corresponds to $\langle \gamma_{2^j}(x_i) \rangle$. Cartan proves [2] that $E_2 = E_\infty$ and $E^0H^*X = P(v(i,j))$. This is free, and hence $H^*X = P(v(i,j))$. The cohomology suspension σ: $QH^*X \to H^*\Omega X$ is given by $\sigma v(i,j) = y_i^{2^j}$. The map h: $\bar{C}(H^*\Omega X) \to H^*X$ is given on algebra generators by

$$\begin{aligned} h[y] &= v(i,j) && \text{if } y = y_i^{2^j} \text{ for some } i, j, \\ &= 0 && \text{if } y \text{ is some other monomial.} \end{aligned} \tag{2.3}$$

The above formulas will be used in the next section.

3. **Computation of the differential.** Let $B = K(Z_2, 4)$ with fundamental class i, let $X = K(Z_2 + Z_2, 2)$ with fundamental classes j and k, and define f by $f^*i = jk$. For future use we record the following information:

$$\begin{aligned} f^*i &= jk, \\ f^*(\text{Sq}^1 i) &= (\text{Sq}^1 j)k + j(\text{Sq}^1 k), \\ f^*(\text{Sq}^2 i) &= j^2k + (\text{Sq}^1 j)(\text{Sq}^1 k) + jk^2, \\ f^*(\text{Sq}^3 i) &= j^2(\text{Sq}^1 k) + (\text{Sq}^1 j)k^2. \end{aligned} \tag{1}$$

(2) $\quad \text{Im} f^*$ is invariant under the automorphism of H^*X which permutes $(\text{Sq}^I j)$ and $(\text{Sq}^I k)$.

Define $\bar{c}$ in $E_1^{-2,17}(\mathscr{P})$ by

$$\begin{aligned} \bar{c} = {} & [\text{Sq}^1 i]_*[\text{Sq}^3 i]k(\text{Sq}^1 k) + [i]_*[\text{Sq}^1 i](j(\text{Sq}^1 k)^2 + jk^3 + k^4) \\ & + [i]_*[\text{Sq}^2 i](jk(\text{Sq}^1 k) + k^2(\text{Sq}^1 k)) + [\text{Sq}^1 i]_*[\text{Sq}^2 i]k^3 \\ & + [i]_*[\text{Sq}^3 i]((\text{Sq}^1 k)^2 + jk^2 + k^3) + [\text{Sq}^2 i]_*[\text{Sq}^3 i]k^2. \end{aligned}$$

Using $d_1([a]_*[b])m=[a](f^*b)m+[b](f^*a)m$ one may verify that $d_1(\bar{c})=0$ and hence $\bar{c}$ represents an element c in $E_2^{-2,17}(\mathscr{P})$. The element c is nonzero since no boundary can have $[\mathrm{Sq}^1\, i]_*[\mathrm{Sq}^3\, i]k(\mathrm{Sq}^1\, k)$ as a summand—the internal degree is too low. To prove the theorem, it suffices to prove

LEMMA. *In* $E_2(\mathscr{P})$, $d_2(c)=(\mathrm{Sq}^2\,\mathrm{Sq}^1\, j)k^4(\mathrm{Sq}^1\, k)\neq 0$.

By (2) above, the element $(\mathrm{Sq}^2\,\mathrm{Sq}^1\, j)k^4(\mathrm{Sq}^1\, k)\neq 0$ in $E_2^{0,*}(\mathscr{P})=Z_2\otimes_{H^*B} H^*X$. We claim that, after projection into $E_2^{0,*}$,

(1) $(f^*\,\mathrm{Sq}^1\, i)\cup_1(f^*\,\mathrm{Sq}^3\, i)=j^3(\mathrm{Sq}^2\,\mathrm{Sq}^1\, k)+(\mathrm{Sq}^2\,\mathrm{Sq}^1\, j)k^3$,

(2) $(f^*i)\cup_1(f^*\,\mathrm{Sq}^1\, i)=0$,

(3) $(f^*i)\cup_1(f^*\,\mathrm{Sq}^2\, i)=j^3(\mathrm{Sq}^1\, k)+(\mathrm{Sq}^1\, j)k^3$,

(4) $(f^*\,\mathrm{Sq}^1\, i)\cup_1(f^*\,\mathrm{Sq}^2\, i)=(\mathrm{Sq}^2\,\mathrm{Sq}^1\, j)k(\mathrm{Sq}^1\, k)+j(\mathrm{Sq}^1\, j)(\mathrm{Sq}^2\,\mathrm{Sq}^1\, k)$,

(5) $(f^*i)\cup_1(f^*\,\mathrm{Sq}^3\, i)=0$,

(6) $(f^*\,\mathrm{Sq}^2\, i)\cup_1(f^*\,\mathrm{Sq}^3\, i)=j^2(\mathrm{Sq}^1\, j)(\mathrm{Sq}^2\,\mathrm{Sq}^1\, k)+(\mathrm{Sq}^2\,\mathrm{Sq}^1\, j)k^2(\mathrm{Sq}^1\, k)$.

Given that (1) through (6) are true, we shall complete the proof of the lemma by applying (2.2):

$$\begin{aligned} d_2(c) &= d_2(\mathrm{cls}\,\bar{c}) = (j^3(\mathrm{Sq}^2\,\mathrm{Sq}^1\, k)+(\mathrm{Sq}^2\,\mathrm{Sq}^1\, j)k^3)k(\mathrm{Sq}^1\, k) \\ &\quad +(j^3(\mathrm{Sq}^1\, k)+(\mathrm{Sq}^1\, j)k^3)jk(\mathrm{Sq}^1\, k) \\ &\quad +((\mathrm{Sq}^2\,\mathrm{Sq}^1\, j)k(\mathrm{Sq}^1\, k)+j(\mathrm{Sq}^1\, j)(\mathrm{Sq}^2\,\mathrm{Sq}^1\, k))k^3 \\ &\quad +(j^2(\mathrm{Sq}^1\, j)(\mathrm{Sq}^2\,\mathrm{Sq}^1\, k)+(\mathrm{Sq}^2\,\mathrm{Sq}^1\, j)k^2(\mathrm{Sq}^1\, k))k^2 \\ &= 3(\mathrm{Sq}^2\,\mathrm{Sq}^1\, j)k^4(\mathrm{Sq}^1\, k) = (\mathrm{Sq}^2\,\mathrm{Sq}^1\, j)k^4(\mathrm{Sq}^1\, k) \end{aligned}$$

as claimed in the lemma. This proves the theorem stated in the introduction, modulo proving (1) through (6). We shall do (1) and (2) in detail, the others being similar. Let $\sigma i=s$ and $\sigma j=t$. We compute modulo $\ker h$:

$$\text{(1)}\quad \begin{aligned} (f^*\,\mathrm{Sq}^1\, i)\cup_1(f^*\,\mathrm{Sq}^3\, i) &= h(([\mathrm{Sq}^1\, s|t]+[s|\mathrm{Sq}^1\, t])\cup_1([s|s|\mathrm{Sq}^1\, t]+[\mathrm{Sq}^1\, s|t|t])) \\ &= h([(\mathrm{Sq}^1\, s)^2|t|t|t]+[s|s|s|(\mathrm{Sq}^1\, t)^2]) \quad \text{by (2.1)} \\ &= h([\mathrm{Sq}^2\,\mathrm{Sq}^1\, s|t|t|t]+[s|s|s|\mathrm{Sq}^2\,\mathrm{Sq}^1\, t]) \\ &= (\mathrm{Sq}^2\,\mathrm{Sq}^1\, j)k^3+j^3(\mathrm{Sq}^2\,\mathrm{Sq}^1\, k) \quad \text{by (2.3)}, \end{aligned}$$

$$\text{(2)}\quad \begin{aligned} (f^*i)\cup_1(f^*\,\mathrm{Sq}^1\, i) &= h([s|t]\cup_1([\mathrm{Sq}^1\, s|t]+[s|\mathrm{Sq}^1\, t])) \\ &= h([s|\mathrm{Sq}^1\, s|t^2]+[s^2|t|\mathrm{Sq}^1\, t]) \\ &= h([s|\mathrm{Sq}^1\, s|\mathrm{Sq}^1\, t]+[\mathrm{Sq}^1\, s|t|\mathrm{Sq}^1\, t]) \\ &= (j+k)(\mathrm{Sq}^1\, j)(\mathrm{Sq}^1\, k) \\ &= 0, \quad \text{since } (\mathrm{Sq}^1\, j)(\mathrm{Sq}^1\, k) = 0 \text{ in } E_2(\mathscr{P}). \end{aligned}$$

REMARK. In any two-stage Postnikov system $\mathscr{P}$, the first chance for a nonzero differential is d_2 on $QE_2^{-2,*}$. In the system above we picked c, the element with lowest total degree in $QE_2^{-2,*}$, and we proved that $d_2(c)\neq 0$. This would tend to indicate that $E_2 \neq E_\infty$ is usual. For more details, and in particular a description of the methods used in studying the E_2 term of the spectral sequence and in finding c, see [8].

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