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K_1 OF THE COMPACT OPERATORS IS ZERO[1]

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ABSTRACT. We prove that K_1 of the compact operators is zero. This theorem has the following operator-theoretic formulation: *any invertible operator of the form (identity) + (compact) is the product of (at most eight) multiplicative commutators* $(A_jB_jA_j^{-1}B_j^{-1})^{\pm 1}$, *where each* B_j *is of the form (identity) + (compact)*. The proof uses results of L. G. Brown, R. G. Douglas, and P. A. Fillmore on essentially normal operators and a theorem of A. Brown and C. Pearcy on multiplicative commutators.

1. **Statement of results.** Let $\mathcal{L}$ be the bounded operators on a separable, infinite dimensional Hilbert space, $\mathcal{K}$ the closed two-sided ideal of compact operators, and $\mathfrak{A} = \mathcal{L}/\mathcal{K}$ the Calkin algebra.

THEOREM. $K_1(\mathcal{K}) = 0$.

That is, the "algebraic K_1" of $\mathcal{K}$, regarded as an ideal in $\mathcal{L}$, is zero. The result may be interpreted as follows. Let G be the set of invertible operators in $\mathcal{L}$ of the form $I + K$, where $K \in \mathcal{K}$. Let H denote the subgroup of G generated by all multiplicative commutators $(u,g) = ugu^{-1}g^{-1}$ where $u \in \mathcal{L}$ is invertible and $g \in G$. Then $G = H$. (This uses the definition of K_1 [8, p. 36] and the fact that the matrix rings $M_n(\mathcal{K})$ and $M_n(\mathcal{L})$ are isomorphic to $\mathcal{K}$ and $\mathcal{L}$ respectively.) So the theorem is equivalent to the following operator-theoretic proposition.

PROPOSITION. *Let $I + K$ be invertible with $K \in \mathcal{K}$. Then there exist invertible operators A_j and $B_j = I + K_j$ $(j = 1, \ldots, n)$ with $K_j \in \mathcal{K}$ such that*

$$I + K = \prod_{j=1}^{n} (A_j, B_j)^{\pm 1}.$$

In fact $n \leqslant 8$. This proposition is proved in §2.

Combining the theorem with known information yields the first six terms of the Milnor long exact sequence in algebraic K-theory associated to $\mathcal{K} \hookrightarrow \mathcal{L} \to \mathfrak{A}$. It reads:

$$\begin{array}{ccccccccccccc} K_1(\mathcal{K}) & \longrightarrow & K_1(\mathcal{L}) & \longrightarrow & K_1(\mathfrak{A}) & \longrightarrow & K_0(\mathcal{K}) & \longrightarrow & K_0(\mathcal{L}) & \longrightarrow & K_0(\mathfrak{A}) & \longrightarrow & 0 \\ \| & & \| & & \| & & \| & & \| & & \| & & \\ 0 & & 0 & & \mathbf{Z} & & \mathbf{Z} & & 0 & & 0 & & \end{array}$$

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2. **Proof of the proposition.** Let $I + K$ be invertible with $K \in \mathcal{K}$. Write $I + K = UP$ in polar decomposition. Then each of the operators U and P is of the form $I + N$ where N is compact normal. We may assume that N has an infinite dimensional null-space $\mathcal{K}_0$. So it suffices to show that any invertible operator of the form $I + N$ with N compact normal with infinite dimensional null-space $\mathcal{K}_0$ is a product of (two) multiplicative commutators of the correct sort.

A. Brown and C. Pearcy show [1, Theorem 3] that $I + N = (G_1, G_2)$ where G_1 and G_2 are invertible and G_1 is a bilateral shift. An examination of their proof shows that G_2 is also normal. Then the G_j live on $\mathcal{K}_0^\perp$. Let H_1 and H_2 be commuting normals living on $\mathcal{K}_0$ such that $\sigma(G_j) \subset \sigma_e(H_j) =$ an annulus (where σ_e denotes essential spectrum), and $\sigma_e(H_1) \times \sigma_e(H_2) =$ joint $\sigma_e\{H_1, H_2\} \equiv X$. Then $I + N = (G_1 \oplus H_1, G_2 \oplus H_2)$. The operators $G_1 \oplus H_1$ and $G_2 \oplus H_2$ thus essentially commute.

Let $\tau \in \operatorname{Ext}(X)$ be the extension

$$0 \to \mathcal{K} \to C^*\{I, \mathcal{K}, G_1 \oplus H_1, G_2 \oplus H_2\} \to C(X) \to 0,$$

where $C^*\{T_j\}$ denotes the C^*-algebra generated by $\{T_j\}$ [4], [5]. We claim that $\tau = 0$; the extension splits. The proof is as follows. The space X is homotopy equivalent to a torus, hence $\operatorname{Ext}(X) \cong \mathbf{Z} \oplus \mathbf{Z}$ via the index map [5]. A direct check shows that the index of $\pi(G_j \oplus H_j)$ is zero for $j = 1, 2$, hence $\tau = 0$. (A more economical choice of H_1 using the fact that G_1 is unitary would yield $X \subset R^3$ and allow avoidance of a homotopy argument.)

By the basic Brown-Douglas-Fillmore theorem [5], there exist commuting normals N_1, N_2 and compact operators C_j such that

$$G_j \oplus H_j = N_j(I + C_j), \qquad j = 1, 2.$$

Then

$$\begin{aligned} I + N &= (G_1 \oplus H_1, G_2 \oplus H_2) = (N_1(I + C_1), N_2(I + C_2)) \\ &= (B_1, A_1)(A_2, B_2) = (A_1, B_1)^{-1}(A_2, B_2) \end{aligned}$$

by direct computation, where

$$\begin{aligned} A_1 &= N_1 N_2 (I + C_2) N_1^{-1}, \qquad A_2 = N_1, \\ B_1 &= N_1(I + C_1)N_1^{-1} = I + N_1 C_1 N_1^{-1} \in I + \mathcal{K}, \\ B_2 &= N_2(I + C_2)N_2^{-1} = I + N_2 C_2 N_2^{-1} \in I + \mathcal{K}. \end{aligned}$$

This completes the proof.

3. **Remarks.**

REMARK 1. Our interest in $K_1(\mathcal{K})$ arose from the following considerations (inspired by Helton and Howe [6]; see also [2], [3]). Let $\mathcal{A}$ be a *-subalgebra of $\mathcal{L}$ containing the trace class $\mathcal{T}$ and suppose that $\mathcal{A}/\mathcal{T}$ is commutative. Let $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{A}$. An invertible operator T in $I + \mathcal{K}_0$ represents zero in $K_1(\mathcal{K}_0)$ if T can be represented as a product of commutators $(A_j, B_j)^{\pm 1}$ as in the proposition, but with $A_j, B_j \in \mathfrak{A}$. If this is so, then $\det T = 1$. The same holds true if $\mathfrak{A}$ is replaced by $M_n(\mathfrak{A})$ provided T is also in the determinant class $I + \mathfrak{T}$.

Which hypotheses on $\mathfrak{A}$ are really necessary for these conclusions to hold? Our proof shows that if K is compact normal then $I + K$ is a product of commutators $(A_j, B_j)^{\pm 1}$ where the A_j and B_j lie in a *-algebra which is commutative mod $\mathfrak{K}$. Thus the assumption that $\mathfrak{A}$ be commutative mod $\mathfrak{T}$ is necessary. A more interesting and difficult question is whether the hypothesis that $\mathfrak{A}$ be closed under * can be eliminated. The special case where T is a single commutator is equivalent to the corresponding question for traces of additive commutators. In the case $T \in \mathfrak{A}$, rather than $T \in M_n(\mathfrak{A})$, this special case is equivalent to the general case.

REMARK 2. The inequality $n \leqslant 8$ of the proposition can be improved to $n \leqslant 6$ by means of a trick used by Radjavi:

$$\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & ST \end{pmatrix}.$$

Also

$$\begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix} = (A, B), \quad \text{where } A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} S^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

REMARK 3. A more constructive proof of the proposition (which yields $n \leqslant 8$) can be given by using an idea from [9]. At the cost of increasing n to 24, one can also require that both A_j and B_j be in $I + \mathfrak{K}$.

REMARK 4. The fact that $K_0(\mathfrak{K}) = \mathbf{Z}$ has a number of generalizations. If $\mathfrak{b}$ is any proper two sided ideal of $\mathfrak{L}$, then $K_0(\mathfrak{b}) = \mathbf{Z}$. Similarly, $K_0(\mathfrak{c}) = \mathbf{Z}$ for a large class of dense *-subalgebras of $\mathfrak{K}$–in particular for $\mathfrak{c} = \mathfrak{A} \cap \mathfrak{K}$ where $\mathfrak{A}/\mathfrak{T}$ is commutative as before and $\mathfrak{A}$ is maximal in a certain sense. The situation for K_1 is quite different. Our result that $K_1(\mathfrak{K}) = 0$ contrasts with the fact that $K_1(\mathfrak{T}) \neq 0$ (by a determinant argument [2]). However, the methods alluded to in Remark 3 do apply to the Schatten classes $\mathcal{C}_p$ for some p. The fact that $K_1(\mathfrak{b})$ depends upon the ring in which $\mathfrak{b}$ is an ideal complicates such considerations.

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