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The Dixmier-Douady Invariant for Dummies

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The Dixmier-Douady invariant is the primary tool in the classification of continuous trace C^*-algebras. These algebras have come to the fore in recent years because of their relationship to twisted K-theory and via twisted K-theory to branes, gerbes, and string theory.

This note sets forth the basic properties of the Dixmier-Douady invariant using only classical homotopy and bundle theory. Algebraic topology enters the scene at once since the algebras in question are algebras of sections of certain fibre bundles.

The results stated are all contained in the original papers of Dixmier and Douady [5], Donovan and Karoubi [7], and Rosenberg [23]. Our treatment is novel in that it avoids the sheaf-theoretic techniques of the original proofs and substitutes more classical algebraic topology. Some of the proofs are borrowed directly from the recent paper of Atiyah and Segal [1]. Those interested in more detail and especially in the connections with analysis should consult Rosenberg [23], the definitive work of Raeburn and Williams [21], as well as the recent paper of Karoubi [13] and the book by Cuntz, Meyer, and Rosenberg [4]. We briefly discuss twisted K-theory itself, mostly in order to direct the interested reader to some of the (exponentially-growing) literature on the subject.

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Fibre Bundles

Suppose that G is a topological group and $G \to T \to X$ is a principal G-bundle over the compact space X. Then up to equivalence it is classified by a map f to the classifying space BG and there is a pullback diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G \arrow[d] \\ T \arrow[r] \arrow[d] & EG \arrow[d] \\ X \arrow[r, "f"] & BG \end{tikzcd}$$

where the right column is the universal principal G-bundle.

Suppose further that F is some G-space. Then following Steenrod [24] we may form the associated fibre bundle

$$F \longrightarrow T \times_G F \longrightarrow X$$

with fibre F and structural group G. Pullbacks commute with taking associated bundles, so there is a pullback diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F \arrow[d] \\ T \times_G F \arrow[r] \arrow[d] & EG \times_G F \arrow[d] \\ X \arrow[r, "f"] & BG. \end{tikzcd}$$

Now suppose that M is some fixed C^*-algebra, soon to be either the matrix ring $M_n = M_n(\mathbb{C})$ for some $n < \infty$ or the compact operators $\mathcal{K}$ on some separable Hilbert space $\mathcal{H}$. Take $G = U(M)$, the group of unitaries of the C^*-algebra. (If M is not unital then we modify by first adjoining a unit canonically to form the unital algebra M^+ and then define $U(M)$ to be the kernel of the natural homomorphism $U(M^+) \to U(M^+/M) \cong S^1$.) Then $U(M)$ acts naturally on M by conjugation; denote M with this action as M^{ad}. The center $ZU(M)$ of $U(M)$ acts trivially, and so the action descends to an action of the *projective unitary*

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group $PU(M) = U(M)/ZU(M)$ on M, denoted M^{ad}. Note that if M is simple then its center is just $\mathbb{C}$ and $ZU(M) \cong S^1$ so that $PU(M)$ is just the quotient group $(U(M))/S^1$.

Having fixed M, let

$$\zeta : PU(M) \longrightarrow T \to X$$

be a principal $PU(M)$-bundle over a compact space X. Form the associated fibre bundle

$$\mathbb{P}\zeta : M^{ad} \longrightarrow T \times_{PU(M)} M^{ad} \xrightarrow{p} X.$$

This fibre bundle always has non-trivial sections. Define A_ζ to be the space of sections:

$$A_\zeta = \Gamma(\mathbb{P}\zeta) = \{s : X \longrightarrow T \times_{PU(M)} M^{ad} \mid ps = 1\}.$$

This is a C^*-algebra with pointwise operations that are well defined because we are using the adjoint action. It is unital if M is unital. If $M = M_n$ or $M = \mathcal{K}$ then this is a continuous trace C^*-algebra. If X is locally compact but not compact then A_ζ is still defined by using sections that vanish at infinity and it is not unital.

Note that if $\mathbb{P}\zeta$ is a trivial fibre bundle then sections correspond to functions $X \to M$ and hence

$$A_\zeta \cong C(X) \otimes M,$$

where $C(X)$ denotes the C^*-algebra of continuous complex-valued functions on X. (If X is only locally compact then we use C_o to denote continuous functions vanishing at infinity.)

Continuous trace C^*-algebras may be defined intrinsically, of course. Here is one approach. If A is a (complex) C^*-algebra, then let $\hat{A}$ denote the set of unitary equivalence classes of irreducible $*$-representations of A with the Fell topology (cf. [21]).

Definition. Let X be a second countable locally compact Hausdorff space. A *continuous trace C^*-algebra with $\hat{A} = X$* is a C^*-algebra A with $\hat{A} = X$ such that the set

$$\{x \in A \mid \text{the map } \pi \to tr(\pi(a)\pi(a)^*) \text{ is finite and continuous on } \hat{A}\}$$

is dense in A.

From the definition it is easy to see that commutative C^*-algebras $C_o(X)$ as well as stable commutative C^*-algebras $C_o(X, M_n)$ and $C_o(X, \mathcal{K})$ are continuous trace. In fact every continuous trace algebra arises as a bundle of sections of the type we have been discussing.

Products

Vector spaces come equipped with natural direct sum and tensor product operations, and these pass over to vector bundles. Thus if $E_1 \to X$ and $E_2 \to X$ are complex vector bundles of dimension r and s respectively then we may form bundles $E_1 \oplus E_2 \to X$ of dimension $r + s$ and $E_1 \otimes E_2 \to X$ of dimension rs. There are two corresponding operations on classifying spaces. The one that concerns us is the tensor product operation. Fix some unitary isomorphism of vector spaces

$$\mathbb{C}^r \otimes \mathbb{C}^s \cong \mathbb{C}^{rs}.$$

(This isomorphism is unique up to homotopy, since the various unitary groups are connected.) Let $U_n = U(M_n(\mathbb{C}))$. This determines a homomorphism

$$U_r \times U_s \xrightarrow{\otimes} U_{rs}$$

and the associated map on classifying spaces

$$BU_r \times BU_s \xrightarrow{\otimes} BU_{rs}$$

given by the composite

$$BU_r \times BU_s \cong B(U_r \times U_s) \xrightarrow{B(\otimes)} BU_{rs}.$$

Let $[X, Y]$ denote homotopy classes of maps and recall that if X is compact and connected then isomorphism classes of complex n-plane vector bundles over X correspond to elements of $[X, BU_n]$. Then this construction induces an operation

$$[X, BU_r] \times [X, BU_s] \xrightarrow{\otimes} [X, BU_{rs}],$$

which does indeed correspond to the tensor product operation on bundles. Precisely, if $E_1 \to X$ and $E_2 \to X$ are represented by f_1 and f_2 respectively, then the tensor product bundle $E_1 \otimes E_2 \to X$ is represented by $f_1 \otimes f_2$. (This holds at once for compact connected spaces. If X is not connected then one checks this on each component.)

The inclusion

$$U_r \cong U_r \times \{1\} \to U_r \times U_s \to U_{rs}$$

is denoted

$$\alpha_{rs} : U_r \to U_{rs}.$$

The center of U_k is the group S^1 regarded as matrices of the form zI, where z is a complex number of norm 1. The quotient group PU_k is the *projective unitary group*. The fibration $S^1 \to U_k \to PU_k$ induces the sequence

$$0 \to \mathbb{Z} \xrightarrow{k} \mathbb{Z} \to \mathbb{Z}/k \to 0$$

on fundamental groups, and, in particular, $\pi_1(PU_k) \cong \mathbb{Z}/k$.

There is a natural induced map and commuting diagram

$$\begin{array}{ccc}
S^1 \times S^1 & \longrightarrow & S^1 \\
\downarrow & & \downarrow \\
U_r \times U_s & \longrightarrow & U_{rs} \\
\downarrow & & \downarrow \\
PU_r \times PU_s & \longrightarrow & PU_{rs}
\end{array}$$

and this induces a tensor product operation and a commuting diagram

$$\begin{array}{ccc} BU_r \times BU_s & \longrightarrow & BU_{rs} \\ \downarrow & & \downarrow \\ BPU_r \times BPU_s & \longrightarrow & BPU_{rs}. \end{array}$$

It is easy to see that

$$\pi_2(BPU_k) \cong \pi_1(PU_k) \cong \mathbb{Z}/k$$

and that the natural map $\alpha_{rs} : PU_r \to PU_{rs}$ induces a commuting diagram

$$\begin{array}{ccc} \pi_2(BP_r) & \xrightarrow{(\alpha_{rs})_*} & \pi_2(BP_{rs}) \\ \downarrow\cong & & \downarrow\cong \\ \mathbb{Z}/r & \xrightarrow{s} & \mathbb{Z}/rs. \end{array}$$

There is a similar structure in infinite dimensions. Fix some separable Hilbert space $\mathcal{H}$ with associated group of unitaries $\mathcal{U}$ on which we impose the strong operator topology. The group $\mathcal{U}$ is contractible in this topology (cf. [21], Lemma 4.72). Fix some unitary isomorphism $\mathcal{H} \otimes \mathcal{H} \cong \mathcal{H}$. This is unique up to homotopy since $\mathcal{U}$ is path-connected. Then there is a canonical homomorphism

$$\mathcal{U} \times \mathcal{U} \xrightarrow{\otimes} \mathcal{U}$$

and associated maps on classifying spaces

$$\begin{array}{ccc} B\mathcal{U} \times B\mathcal{U} & \xrightarrow{\otimes} & B\mathcal{U} \\ \downarrow & & \downarrow \\ BP\mathcal{U} \times BP\mathcal{U} & \xrightarrow{\otimes} & BP\mathcal{U} \end{array}$$

where $P\mathcal{U}$ denotes the infinite projective unitary group.

The action of S^1 on $\mathcal{U}$ is free and thus

$$P\mathcal{U} \simeq BS^1 \simeq K(\mathbb{Z}, 2).$$

This implies that

$$BP\mathcal{U} \simeq K(\mathbb{Z}, 3).$$

It is simpler to separate the discussion of finite and infinite dimensional bundles at this point.

A Note on Cohomology for Compact Spaces

If X is a finite complex then the Eilenberg-Steenrod uniqueness theorem guarantees for us that singular, simplicial, representable, and Čech cohomology theories all coincide. Moving up to compact spaces one must pause to reconsider the question. The natural choice in the classical Dixmier-Douady context is Čech cohomology, as this relates best to sheaf theories, and so the Dixmier-Douady invariant was originally defined to take values in $\check{H}^3(X;\mathbb{Z})$. However, a classical homotopy approach dictates defining $H^3(X;\mathbb{Z}) = [X, K(\mathbb{Z}, 3)]$. Fortunately these two functors agree on compact spaces; the result is again due to Eilenberg and Steenrod.

Proposition ([8]). *On the category of compact spaces, Čech cohomology is representable. That is, there is a natural isomorphism*

$$\check{H}^n(X;\mathbb{Z}) \cong [X, K(\mathbb{Z}, n)].$$

Proof. The natural isomorphism is well known for X a finite complex, by the Eilenberg-Steenrod uniqueness theorem. Suppose that X is a compact space. Then write $X = \varprojlim X_j$ for some inverse system of finite complexes. (See [8], Chapters IX, X, and XI for open covers, nerves, and inverse limits.) Continuity of Čech theory implies that

$$\check{H}^n(X;\mathbb{Z}) \cong \varinjlim H^n(X_j;\mathbb{Z}).$$

The maps $X \to X_j$ induce natural maps

$$[X_j, K(\mathbb{Z}, n)] \to [X, K(\mathbb{Z}, n)]$$

and these coalesce to form

$$\Phi : \varinjlim [X_j, K(\mathbb{Z}, n)] \to [X, K(\mathbb{Z}, n)].$$

Claim: the map Φ is a bijection. The key fact needed is the following result of Eilenberg-Steenrod ([8], p. 287, Theorem 11.9): if $X = \varprojlim X_j$ is compact, Y is a simplicial complex, and $f : X \to Y$, then up to homotopy f factors through one of the X_j. This implies immediately that Φ is onto. On the other hand, if $g : X_j \to Y$ and the composite $X \to X_j \to Y$ is null-homotopic then the null-homotopy factors through some $X_k \times [0, 1]$ and hence $[g] = 0$. □

Bundles with Fibre $\mathcal{K}$

Recall that $\mathcal{K} = \mathcal{K}(\mathcal{H})$ denotes the algebra of compact operators on a separable Hilbert space $\mathcal{H}$. Let

$$\zeta : P\mathcal{U} \longrightarrow T \longrightarrow X$$

be a principal $P\mathcal{U}$-bundle with associated C^*-algebra A_ζ. All automorphisms of $\mathcal{K} = \mathcal{K}(\mathcal{H})$ are given by conjugation by unitary operators on the Hilbert space $\mathcal{H}$, so the group of unitaries $\mathcal{U}$ acts on $\mathcal{K}$ by the adjoint action. The center of the group is just S^1, and it acts trivially, of course, and so

$$\mathrm{Aut}(\mathcal{K}) \cong \mathcal{U}/S^1 = P\mathcal{U},$$

the infinite projective unitary group. Thus

$$[X, BP\mathcal{U}] \cong [X, K(\mathbb{Z}, 3)] \cong H^3(X;\mathbb{Z}).$$

We may regard maps $X \to BP\mathcal{U}$ as *projective vector bundles* in analogy with projective representations.

The resulting C^*-algebras A_ζ are *stable* in the sense that $A_\zeta \otimes \mathcal{K} \cong A_\zeta$.

Define the *Dixmier-Douady* invariant $\delta(A_\zeta)$ of the C^*-algebra A_ζ to be the homotopy class of a map

$$f : \hat{A} \to BP\mathcal{U} \cong K(\mathbb{Z}, 3)$$

that classifies the bundle $E \to X$.

We note that given A_ζ then its Dixmier-Douady invariant lies naturally in the group $H^3(\hat{A}_\zeta;\mathbb{Z})$. The identification of $\hat{A}_\zeta$ with X is only given mod the group of homeomorphisms of X, and hence the Dixmier-Douady invariant is only defined modulo the action of the homeomorphism group of X on $H^3(X;\mathbb{Z})$. Of course this action preserves the order of the element $\delta(A_\zeta)$.

So we have established the first parts of the following Dixmier-Douady result:

Theorem ([5], [23]). *Let X be a compact space. Then:*

(1) *There is a natural isomorphism*

$$\delta : [X, BP\mathcal{U}] \xrightarrow{\cong} \check{H}^3(X,\mathbb{Z}).$$

(2) *Suppose we are given a principal $P\mathcal{U}$-bundle ζ, associated fibre bundle $\mathbb{P}\zeta$, and associated C^*-algebra A_ζ. Then $\delta(A_\zeta) = 0$ if and only if $\mathbb{P}\zeta$ is equivalent to a trivial matrix bundle, and in that case*

$$A_\zeta \cong C(X) \otimes \mathcal{K}.$$

(3) *The Dixmier-Douady invariant is additive, in the sense that*

$$\delta(A_{\zeta_1\otimes\zeta_2}) = \delta(A_{\zeta_1}) + \delta(A_{\zeta_2}).$$

(4) *The invariant respects conjugation:*

$$\delta(A_{\zeta^*}) = -\delta(A_\zeta).$$

(5) *Every element of $\check{H}^3(X;\mathbb{Z})$ may be realized as the Dixmier-Douady invariant of some infinite-dimensional bundle and associated C^*-algebra.*

Proof. Only (3) and (4) remain to be demonstrated. Part (3) comes down to an analysis of the commutative diagram

$$\begin{array}{ccc} BP\mathcal{U}\times BP\mathcal{U} & \xrightarrow{\simeq} & K(\mathbb{Z},3)\times K(\mathbb{Z},3) \\ \downarrow\otimes & & \downarrow \\ BP\mathcal{U} & \xrightarrow{\simeq} & K(\mathbb{Z},3) \end{array}$$

which deloops to

$$\begin{array}{ccc} P\mathcal{U}\times P\mathcal{U} & \xrightarrow{\simeq} & K(\mathbb{Z},2)\times K(\mathbb{Z},2) \\ \downarrow\otimes & & \downarrow m \\ P\mathcal{U} & \xrightarrow{\simeq} & K(\mathbb{Z},2). \end{array}$$

The map m is determined up to homotopy by its representative in

$$\check{H}^2(K(\mathbb{Z},2)\times K(\mathbb{Z},2);\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$$

and this class is $(1,1)$ so that m is indeed the map inducing addition in $H^2(-;\mathbb{Z})$.

Part (4) is a similar argument, which we omit. □

This result may be viewed in a more bundle-theoretic manner. The fibration

$$S^1 \to \mathcal{U} \to P\mathcal{U}$$

induces an exact sequence

$$[X,K(\mathbb{Z},2)] \to [X,B\mathcal{U}] \xrightarrow{\epsilon} [X,BP\mathcal{U}] \xrightarrow{\delta} [X,K(\mathbb{Z},3)]$$

which, after identifications, becomes

$$Vect_1(X) \to Vect_\infty(X) \xrightarrow{\epsilon} PVect_\infty(X) \xrightarrow{\delta} \check{H}^3(X;\mathbb{Z})$$

where $Vect_k(X)$ denotes isomorphism classes of vector bundles over X of dimension k, $PVect_\infty(X)$ denotes isomorphism classes of projective vector bundles over X, and we have identified

$$[X,K(\mathbb{Z},2)] \cong \check{H}^2(X;\mathbb{Z}) \cong Vect_1(X)$$

using the first Chern class of the bundle.

The map ϵ takes an infinite-dimensional vector bundle $V \to X$ and associates to it the matrix bundle

$$\epsilon(V \to X) \quad = \quad End(V) \to X$$

where $End(V)_x = End(V_x)$, and so if $\delta(A_\zeta) = 0$ then the bundle $\mathbb{P}\zeta$ is isomorphic to a bundle of endomorphisms: $\mathbb{P}\zeta \cong End(V)$ as bundles. Then we use the fact that every vector bundle over X with infinite-dimensional fibres is trivial as a vector bundle (since $\mathcal{U}$ is contractible), and thus there are bundle isomorphisms

$$\mathbb{P}\zeta \cong End(V) \cong End(X\times\mathcal{H}) \cong X\times\mathcal{K}$$

so that

$$A_\zeta \cong C(X)\otimes\mathcal{K}$$

as C^*-algebras. In fact, recalling that $\mathcal{U}$ is contractible, $\mathcal{U} \simeq *$, we may extend the sequence above to read

$$[X,K(\mathbb{Z},2)] \to [X,*] \xrightarrow{\epsilon} [X,BP\mathcal{U}]$$
$$\xrightarrow{\delta} [X,K(\mathbb{Z},3)] \to [X,*]$$

and then deduce that δ is also onto.

Bundles with Fibre $M_n(\mathbb{C})$

The inclusion of S^1 as the center of U_n gives rise to a fibration sequence

$$S^1 \to U_n \to PU_n \to K(\mathbb{Z},2)$$
$$\to BU_n \to BPU_n \xrightarrow{\delta} K(\mathbb{Z},3).$$

For $n \geq 2$ the map $U_n \to PU_n$ induces an isomorphism on homotopy. Passing to classifying spaces, this yields

$$\pi_2(BPU_n) \cong \mathbb{Z}/n$$

as previously noted, and

$$\pi_j(BU_n) \cong \pi_j(BPU_n), \qquad j > 2.$$

The *Dixmier-Douady invariant* is defined to be the induced map

$$\delta : [X,BPU_n] \to [X,K(\mathbb{Z},3)] \cong \check{H}^3(X;\mathbb{Z}).$$

There is a long exact sequence

$$[X,K(\mathbb{Z},2)] \to [X,BU_n]$$
$$\xrightarrow{\epsilon} [X,BPU_n] \xrightarrow{\delta} [X,K(\mathbb{Z},3)]$$

which translates into

$$\check{H}^2(X;\mathbb{Z}) \to Vect_n(X) \xrightarrow{\epsilon} PVect_n(X) \xrightarrow{\delta} \check{H}^3(X;\mathbb{Z}).$$

The map ϵ is defined as follows. Given a complex vector bundle $V \to X$, then

$$\epsilon(V \to X) \;=\; \mathrm{End}(V) \to X$$

where $\mathrm{End}(V) \to X$ is the endomorphism bundle of V; the fibre over a point x is just $\mathrm{End}(V_x)$. This yields the third Dixmier-Douady result:

Proposition. *Suppose that X is compact. Let ζ be a principal PU_n-bundle over X with associated bundle $\mathbb{P}\zeta$ and C^*-algebra A_ζ. Suppose that $\delta(A_\zeta) = 0$. Then there is a complex vector bundle $V \to X$ of dimension n over X and a bundle isomorphism*

$$\begin{array}{ccc} \mathbb{P}\zeta & \xrightarrow{\cong} & \mathrm{End}(V) \\ \downarrow & & \downarrow \\ X & \xrightarrow{1} & X. \end{array}$$

Note that, in contrast to the infinite-dimensional case, endomorphism bundles need not be trivial bundles. There is one improvement possible, and we are indebted to Peter Gilkey for this explicit construction.

Corollary. *The vector bundle $V \to X$ in the Proposition may be taken to have trivial first Chern class, so that its structural bundle may be reduced to an SU_n-bundle.*

Proof. Suppose that $\mathbb{P}\zeta \cong \mathrm{End}(V)$ as in the Proposition. Let L be a complex line bundle over X with $c_1(L) = -c_1(V)$. Let $V' = V \otimes L$. Then using the fact that $L \otimes L^*$ is a trivial line bundle we have

$$\begin{aligned} \mathrm{End}(V') &\cong (V')^* \otimes V' \cong V \otimes V^* \otimes L \otimes L^* \\ &\cong V \otimes V^* \cong \mathrm{End}(V) \end{aligned}$$

so we may replace V by V' and obtain the same endomorphism bundle. □

Note that even though V and V' have isomorphic endomorphism bundles, in general they will not themselves be isomorphic. In fact, $End(V) \cong End(V')$ if and only if $V' \cong V \otimes L$ for some line bundle L.

We can refine this observation as follows. The diagram above expands to a natural commuting diagram (below)

The Dixmier-Douady map $\delta : BPU_n \to K(\mathbb{Z}, 3)$ factors as

$$BPU_n \xrightarrow{\gamma} K(\mathbb{Z}/n, 2) \xrightarrow{\beta} K(\mathbb{Z}, 3).$$

The map β induces the Bockstein homomorphism

$$\beta : \check{H}^2(X;\mathbb{Z}/n) \to \check{H}^3(X;\mathbb{Z})$$

with

$$Ker(\beta) = Image[\check{H}^2(X;\mathbb{Z}) \longrightarrow \check{H}^2(X;\mathbb{Z}/n)]$$

and

$$Image(\beta) = \{x \in \check{H}^3(X;\mathbb{Z}) \;:\; nx = 0\}$$

whose image lies in the torsion subgroup of $\check{H}^3(X;\mathbb{Z})$.

Theorem. *Let X be a compact space and let $n \in \mathbb{N}$. Then:*

(1) *There is a natural exact sequence*

$$\begin{aligned} 0 &\to \check{H}^2(X;\mathbb{Z}) \xrightarrow{\sigma} Vect_n(X) \\ &\xrightarrow{\epsilon} [X, BPU_n] \xrightarrow{\delta} \check{H}^3(X;\mathbb{Z}). \end{aligned}$$

(2) *Suppose we are given a principal PU_n-bundle ζ over a compact space X and associated C^*-algebra A_ζ. Then*
   (a) *If $\gamma(A_\zeta) \neq 0$ but $\delta(A_\zeta) = 0$ then $\gamma(A_\zeta)$ lifts to an integral class in $\check{H}^2(X;\mathbb{Z})$.*
   (b) *If $\gamma(A_\zeta) = 0$ then $\mathbb{P}\zeta \cong End(V)$ with $c_1(V) = 0$.*

(3) *The Dixmier-Douady invariant is additive, in the sense that*

$$\delta_{\zeta_1 \otimes \zeta_2} = \delta_{\zeta_1} + \delta_{\zeta_2}.$$

(4) *The invariant respects conjugation:*

$$\delta_{\zeta^*} = -\delta_\zeta.$$

(5) *For any M_n-bundle ζ, it is the case that $n\delta_\zeta = 0$.*

Proof. Most of this result has already been established. The map σ takes a class $c \in \check{H}^2(X;\mathbb{Z})$ and associates to it a vector bundle of the form $L \oplus \theta^{n-1}$ where L is a line bundle with first Chern class c and θ^{n-1} is a trivial bundle of dimension $n-1$. This map is one-to-one since it is split by the first Chern class map

$$c_1 : \mathrm{Vect}_n(X) \to \check{H}^2(X;\mathbb{Z}).$$

Parts (3) and (4) follow as in the infinite-dimensional case. □

The various maps $\alpha_{rs} : PU_r \to PU_{rs}$ induce maps on classifying spaces that by abuse of language are also denoted $\alpha_{rs} : BPU_r \to BPU_{rs}$. These maps

$$\begin{array}{ccccccccccc} SU_n & \longrightarrow & PU_n & \longrightarrow & K(\mathbb{Z}/n,1) & \longrightarrow & BSU_n & \longrightarrow & BPU_n & \xrightarrow{\gamma} & K(\mathbb{Z}/n,2) \\ \downarrow & & \downarrow 1 & & \downarrow & & \downarrow & & \downarrow 1 & & \downarrow \beta \\ U_n & \longrightarrow & PU_n & \longrightarrow & K(\mathbb{Z},2) & \longrightarrow & BU_n & \xrightarrow{\epsilon} & BPU_n & \xrightarrow{\delta} & K(\mathbb{Z},3). \end{array}$$

form a directed system $\{BPU_r, \alpha_{rs}\}$. Write BPU_∞ for the colimit. Note that this is *not* the same as $BP\mathcal{U} = K(\mathbb{Z}, 3)$.

Proposition (Serre, [9], pp. 228–229).

(1) *The natural map*

$$\varinjlim \pi_j(BPU_n) \to \pi_j(BPU_\infty)$$

is an isomorphism.

(2) $\pi_2(BPU_\infty) \cong \mathbb{Q}/\mathbb{Z}$.

(3) If $j \geq 2$ *then* $\pi_{2j}(BPU_\infty) \cong \mathbb{Q}$.

(4) If $j \geq 2$ *then* $\pi_{2j-1}(BPU_\infty) = 0$.

(5) *There is a natural splitting*

$$BPU_\infty \simeq K(\mathbb{Q}/\mathbb{Z}, 2) \times F$$

with $\pi_j(F) = \mathbb{Q}$ for $j \geq 4$ *and even and* $\pi_j(F) = 0$ *otherwise.*

Proof. Each map $\alpha_{rs} : BPU_r \to BPU_{rs}$ is a cofibration and so (1) is immediate. We showed previously that $\pi_2(BPU_r) \cong \mathbb{Z}/r$. The map induced by α_{rs} takes the generator of $\pi_2(BPU_r)$ to s times the generator of $\pi_2(BPU_{rs})$ and so

$$\pi_2(BPU_\infty) = \varinjlim(\mathbb{Z}/r, \alpha_*) = \mathbb{Q}/\mathbb{Z}.$$

For $n >> j > 2$, $\pi_{2j}(BPU_n) \cong \pi_{2j}(BU_n) \cong \mathbb{Z}$ by Bott periodicity, and it follows easily that

$$\pi_{2j}(BPU_\infty) \cong \varinjlim(\pi_{2j}(BPU_n), \alpha_*) = \mathbb{Q}.$$

Similarly, in odd degrees homotopy groups vanish, and calculation yields the result. There results a fibration $BPU_\infty \to K(\mathbb{Q}/\mathbb{Z}, 2)$; call the fibre F. Then the homotopy of F is as stated, and as the base space has trivial rational cohomology this implies that the fibration is trivial. □

The various Dixmier-Douady maps

$$\delta : [X, BPU_n] \longrightarrow \check{H}^3(X; \mathbb{Z})$$

are coherent and hence pass to the limit to produce an induced Dixmier-Douady map

$$\delta^\infty : [X, BPU_\infty] \longrightarrow \check{H}^3(X; \mathbb{Z}).$$

It is obvious that δ^∞ takes values in the torsion subgroup of $\check{H}^3(X; \mathbb{Z})$. In fact, more is true:

Proposition. *Let X be a compact space. Then:*

(1) *The image of the map δ^∞ is the whole torsion subgroup of $\check{H}^3(X; \mathbb{Z})$.*

(2) *Let $x \in \check{H}^3(X; \mathbb{Z})$ be a torsion class. Then there is some finite n and some principal PU_n-bundle ζ over X such that $\delta(A_\zeta) = x$.*

Proof. The lattice of cyclic subgroups of $\mathbb{Q}/\mathbb{Z}$ induces an equivalence

$$\varinjlim K(\mathbb{Z}/n, 2) \longrightarrow K(\mathbb{Q}/\mathbb{Z}, 2).$$

Furthermore, the various Bockstein maps $K(\mathbb{Z}/n, 2) \longrightarrow K(\mathbb{Z}, 3)$ all factor as

$$K(\mathbb{Z}/n, 2) \longrightarrow K(\mathbb{Q}/\mathbb{Z}, 2) \xrightarrow{\tilde{\beta}} K(\mathbb{Z}, 3).$$

The exactness of the coefficient sequence

$$\check{H}^2(X; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\tilde{\beta}} \check{H}^3(X; \mathbb{Z}) \longrightarrow \check{H}^3(X; \mathbb{Q})$$

implies that the image of $\tilde{\beta}$ is exactly the torsion subgroup of $\check{H}^3(X; \mathbb{Z})$. This shows (1).

For (2), let x be a torsion class. Then it is in the image of the Bockstein map

$$\tilde{\beta}' : \check{H}^2(X; \mathbb{Q}/\mathbb{Z}) \to \check{H}^3(X; \mathbb{Z})$$

and thus may be represented as $x = \tilde{\beta}(y)$ where $y \in [X, K(\mathbb{Q}/\mathbb{Z}, 2)]$. The map

$$[X, BPU_\infty] \longrightarrow [X, K(\mathbb{Q}/\mathbb{Z}, 2)]$$

is onto, and so the class y lifts to some class

$$z \in [X, BPU_\infty] \cong \varinjlim[X, BPU_n].$$

Choose some $\zeta \in [X, BPU_k]$ representing z. (Note that if x has order n then n divides k but in general $n \neq k$.) Then $\delta(A_\zeta) = x$ as required. □

Twisted K-theory

Twisted K-theory was first introduced by Donovan and Karoubi [7] for finite-dimensional bundles and then by Rosenberg [23] in the general case. In our context the point is to look at the $\mathbb{Z}/2$-graded group $K_*(A_\zeta)$. Let A_ζ denote a continuous trace algebra over X. Recall that K_0 is defined for any unital ring as the Grothendieck group of finitely projective modules. For our purposes a topological definition is cleaner and so we may simply define

$$K_j(A_\zeta) \cong \pi_{j+1}(U(A_\zeta \otimes \mathcal{K})), \qquad j \in \mathbb{Z}/2,$$

where in all cases we grade as $K_j = K^{-j}$ (and then note that by periodicity there are only two groups anyhow). If the bundle is infinite dimensional then it is not necessary to tensor with $\mathcal{K}$ since the algebra is already stable. These groups are denoted in the literature by (for instance)

$$K^*(X; \zeta) \text{ or } K^*(X; \delta(\zeta)) \text{ or } K^*_{\delta(\zeta)}(X) \text{ or } K^*_\Delta(X).$$

The point is that once one specifies X and $\Delta = \delta(\zeta) \in \check{H}^3(X; \mathbb{Z})$ then A_ζ is specified up to equivalence, and hence $K^*_\Delta(X)$ makes sense as notation for $K_j(A_\zeta)$. Here are the basic properties:

Proposition.

(1) Domain: *The groups $K^*_\Delta(X)$ are defined for locally compact spaces X and principal PU_n or $P\mathcal{U}$-bundles ζ over X with associated Dixmier-Douady class $\Delta = \delta(\zeta) \in H^3(X; \mathbb{Z})$.*

(2) Naturality: *Given (X, Δ) together with a continuous function $f : Y \to X$ then there is an induced map*

$$f^* : K^*_\Delta(X) \longrightarrow K^*_{f^*\Delta}(Y)$$

and twisted K-theory is natural with respect to these maps.

(3) Periodicity: *The groups $K^*_\Delta(X)$ are periodic of period* 2.

(4) Product: *There is a cup product operation*

$$K^*_{\Delta_1}(X) \times K^*_{\Delta_2}(X) \longrightarrow K^*_{\Delta_1+\Delta_2}(X).$$

(5) Relation to untwisted K-theory: *There is a natural isomorphism*

$$K_0^*(X) \cong K^*(X),$$

where if X is locally compact but not compact then K-theory with compact support is intended.

$\square$

Karoubi notes that the cup product is not canonically defined at the level of cohomology classes. For instance, in the finite-dimensional case, one must choose representatives from among the various algebra bundles; i.e. choose Morita equivalences that are not canonical in general.

Rosenberg [22] points out the simplest case where twisted K-theory actually does something interesting. Take $X = S^3$. Then the Dixmier-Douady invariant takes values in $H^3(S^3;\mathbb{Z}) \cong \mathbb{Z}$ and hence is determined by an integer m. Rosenberg shows that

$$K_m^0(S^3) = 0, \qquad K_m^{-1}(S^3) = \mathbb{Z}/m.$$

He takes us further by introducing a twisted Atiyah-Hirzebruch spectral sequence (for X a finite complex) converging to $K_\Delta^*(X)$ and with

$$E_2^* = H^*(X; K_*(\mathbb{C})).$$

Just as with the classical Atiyah-Hirzebruch spectral sequence we have $d_2 = 0$. The differential d_3 is determined by the integral Steenrod operation $Sq_{\mathbb{Z}}^3$ (as is the case classically) and (the first mention of the twist) the class Δ:

$$d_3(x) = Sq_{\mathbb{Z}}^3(x) - \Delta x.$$

This spectral sequence is developed further by Atiyah and Segal [1], [2] who show, for instance, that the spectral sequence does not collapse after rationalization.

There are other ways to twist K-theory and to make it equivariant, to make it real (rather than complex) or both, and the reader should consult the papers of Atiyah and Segal [1], [2], Freed, Hopkins, and Teleman [11], and especially the fine survey paper of Karoubi [13] before burying oneself in the physics literature. That physics comes into the picture goes back to the observation of Witten that D-brane charges in type IIB string theory over a space M are elements of $K_\delta^0(M)$—see for instance [3], where the role of the Dixmier-Douady invariant in the classification of bundle gerbes is summarized along with a differential geometric model for twisted K-theory. See [10], [16], [17], [19] for further developments.

Rational Homotopy

For stable continuous trace algebras, the groups $\pi_*(UA_\zeta)$ are periodic of period 2 and in fact correspond to the twisted K-theory groups. However, if ζ is a principal PU_n-bundle where n is finite then the natural map

$$\pi_j(UA_\zeta) \to K_{j-1}(A_\zeta)$$

is neither injective nor surjective in general. Furthermore, K-theory obscures the geometric dimension of the space X, since $K^0(S^{2n})$ doesn't depend on n and hence cannot detect it. In this situation a more natural question is to calculate $\pi_*(UA_\zeta)$ itself. This is impossible even in very elementary cases (e.g., when $A = M_2(\mathbb{C})$). A more reasonable project is to calculate the rational homotopy groups

$$\pi_j(UA_\zeta) \otimes \mathbb{Q},$$

and this has been done in general for X compact by [15], [14]. The answer depends upon the individual groups $H^j(X;\mathbb{Q})$ and upon n. It turns out to be independent of the principal bundle. This is to be expected, at least after the fact, since the Dixmier-Douady invariant is finite when the bundle is finite dimensional and hence is trivial in the world of rational homotopy.

Generalizations

What if X is assumed to be a CW-complex or, more generally, a compactly generated space that is not necessarily compact? First, the definition of A_ζ does not lead to a C^*-algebra since infinite CW-complexes are not locally compact. These would be pro-C^*-algebras such as those studied by N. C. Phillips [20]. The good news is that some of the proofs in this note generalize. The bundle classification results require restriction to Dold's numerable bundles [6], [12]. These are bundles that are trivial with respect to a locally finite cover, and so one can assume, for instance, that X is paracompact. In the infinite-dimensional case the isomorphism

$$[X, BP\mathcal{U}] \cong [X, K(\mathbb{Z},3)]$$

is tautological, since $BP\mathcal{U} \simeq K(\mathbb{Z},3)$.

In the finite-dimensional situation we obtain maps

$$\mathrm{Vect}_n^{num}(X) \xrightarrow{\epsilon} [X, BPU_n] \xrightarrow{\delta} H^3_{\mathrm{sing}}(X;\mathbb{Z}),$$

where $\mathrm{Vect}_n^{num}(X)$ denotes isomorphism classes of numerable vector bundles, and the Dixmier-Douady results still hold when singular cohomology is understood throughout.

What if X is assumed to be locally compact but not necessarily compact? In that case the definition of A_ζ is modified to include only those sections that vanish at infinity, so that the sup norm is defined and then A_ζ is a C^*-algebra again. The Dixmier-Douady results still hold, but it is probably better in this setting to shift back to the sheaf-theoretic setting of the original proofs, since the classification of vector bundles over locally compact spaces is somewhat awkward.

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