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Multilinear And Multiparameter Pseudo-Differential Operators And Trudinger-Moser Inequalities

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MULTIPARAMETER AND MULTILINEAR PSEUDO-DIFFERENTIAL OPERATORS AND SHARP TRUDINGER-MOSER INEQUALITIES

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To my grandmother

and my adorable Tuanzi.

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CHAPTER 1 INTRODUCTION

1.1 Brief Background and Introduction

Pseudo-differential operators play important roles in harmonic analysis, several complex variables, partial differential equations and other branches of modern mathematics. We studied some types of multilinear and multiparameter Pseudo-differential operators. They include a class of trilinear Pseudo-differential operators, where the symbols are in the form of products of Hörmader symbols defined on lower dimensions, and we established the Hölder type L^p estimates for such operators. Such operators derive from the trilinear Coifman-Meyer type operators with flag singularities. And we also studied a class of bilinear bi-parameter Pseudo-differential operators, where the symbols are taken from the general Hörmander class, and we studied the restriction for the order of the symbols which could imply the Hölder type L^p estimates. Such types of operators are motivated by the Calderón-Vaillancourt theorem in single parameter setting.

Trudinger-Moser inequalities can be treated as the limiting case of the Sobolev embeddings. Sharp Trudinger-Moser inequalities on the first order Sobolev spaces and their analogous Adams inequalities on high order Sobolev spaces play an important role in geometric analysis, partial differential equations and other branches of modern mathematics. Such geometric inequalities have been studied extensively by many authors in recent years and there is a vast literature. There are two types of such optimal inequalities: critical and subcritical sharp inequalities, both are with best constants. Critical sharp inequalities are under the restriction of the full Sobolev norms for the functions under consideration, while the subcritical inequalities are under the restriction of the partial Sobolev norms for the functions under consideration. There are subtle differences between these two type of inequalities. Surprisingly, we proved that these critical and subcritical Trudinger-Moser and Adams inequalities are actually equivalent.

1.2 Trilinear Pseudo-differential Operators with Flag Symbols

Definition 1.1. For $n \geq 1$ we denote by $\mathcal{M}(\mathbb{R}^n)$ the set of all bounded symbols $m \in L^\infty(\mathbb{R}^n)$, smooth away from the origin and satisfying the classical Marcinkiewcz-Mikhlin-Hörmander condition

$$|\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi|^\alpha}$$

for every $\xi \in \mathbb{R}^n \backslash \{0\}$ and sufficiently many multi-indices α.

Definition 1.2. We define the Fourier transform of a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$ to be

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{2\pi x \cdot \xi} dx.$$

Definition 1.3. For $m > 0$, $0 \leq \rho, \delta \leq 1$, we say that a smooth function $\sigma(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ belongs to the Hörmander class $S^m_{\rho,\delta}$ if

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m+\delta|\alpha|-\rho|\beta|}$$

for all multi-indices α, β and some positive constants $C_{\alpha,\beta}$ depending on α, β.

Definition 1.4. The classical linear Pseudo-differential operators are defined to consist of operators in the form

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(x, \xi) \cdot \widehat{f}(\xi) \cdot e^{2\pi i x \xi} d\xi$$

initially defined for Schwartz class $\mathcal{S}(\mathbb{R}^n)$, where $\sigma(x,\xi) \in S^m_{\rho,\delta}$.

Definition 1.5. For $d \in \mathbb{N}$, $m > 0$, $0 \le \rho, \delta \le 1$, we say that a smooth function $\sigma(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^{dn}$ belongs to the multilinear Hörmander class $BS^m_{\rho,\delta}$ if

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}$$

for all multi-indices α, β and some positive constants $C_{\alpha,\beta}$ depending on α, β.

Definition 1.6. The classical trilinear Pseudo-differential operators are initially defined for Schwartz functions $f, g, h \in \mathcal{S}(\mathbb{R}^n)$ as

$$T_\sigma(f,g,h)(x) = \int_{\mathbb{R}^{3n}} \sigma(x,\xi,\eta,\zeta) \cdot \widehat{f}(\xi)\widehat{g}(\eta)\widehat{h}(\zeta) \cdot e^{2\pi i x(\xi+\eta+\zeta)} d\xi d\eta d\zeta$$

for $\sigma(x,\xi,\eta) \in BS^0_{1,0}$, where $x, \xi, \eta, \zeta \in \mathbb{R}^n$.

We study the following type of trilinear Pseudo-differential operators with flag type symbols. Let $a(x,\xi,\eta), b(x,\eta,\zeta) \in BS^0_{1,0}$ be symbols satisfying the conditions

$$|\partial_x^l \partial_\xi^\alpha \partial_\eta^\beta a(x,\xi,\eta)| \lesssim \frac{1}{(1+|\xi|+|\eta|)^{\alpha+\beta}}$$
$$|\partial_x^l \partial_\eta^\beta \partial_\zeta^\gamma b(x,\eta,\zeta)| \lesssim \frac{1}{(1+|\eta|+|\zeta|)^{\beta+\gamma}}$$

for every $x, \xi, \eta, \zeta \in \mathbb{R}$ and sufficiently many indices α, β and γ, define the operator

$$T_{ab}(f,g,h)(x) := \int_{\mathbb{R}^3} a(x,\xi,\eta) b(x,\eta,\zeta) \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta) e^{2\pi i x(\xi+\eta+\zeta)} d\xi d\eta d\zeta.$$

We established its Hölder's type L^p estimate for such operators $T_{ab}(f,g,h)$.

Theorem 1.7. *The operator T_{ab} defined as* (2.1) *is bounded from $L^{p_1} \times L^{p_2} \times L^{p_3}$ to L^r for $1 < p_1, p_2, p_3 \leq \infty$ with $1/p_1 + 1/p_2 + 1/p_3 = 1/r$ and $0 < r < \infty$, provided that $(p_1, p_2) \neq (\infty, \infty)$ and $(p_2, p_3) \neq (\infty, \infty)$.*

The idea of the proof is to reduce the trilinear Pseudo-differential operator with the symbol of flag type to a localized version and takes advantage of the *flag paraproducts* from Muscalu's work [72] on the L^p estimates for the Fourier multipliers with symbols of flag singularities.

The work of such types of operators are motivated by the following trilinear Coifman-Meyer type operator with flag singularities studied by C. Muscalu [72], where the multiplier involved is a product of two symbols and has *flag singularities*.

1.3 Bi-parameter and Bilinear Calderón-Vaillancourt Theorem

Then we introduce the bi-parameter Pseudo-differential operators with the symbols taken from the Hörmander class $BS^m_{0,0}$. In the single parameter case, the following operator has been studied by Miyachi and Tomita in [70]

Definition 1.8. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and for $\sigma(x, \xi, \eta) \in BS^m_{0,0}$, define

$$T_\sigma(f,g)(x) = \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \sigma(x,\xi,\eta) \cdot \widehat{f}(\xi) \cdot \widehat{g}(\eta) \cdot e^{2\pi i x(\xi+\eta)} d\xi d\eta$$

where $x, \xi, \eta \in \mathbb{R}^n$.

In bi-parameter setting, let $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. We first define the bi-parameter Hörmander class as

Definition 1.9. For $m > 0$, $0 \leq \rho, \delta \leq 1$, the bi-parameter bilinear Hörmander symbols

$BBS^m_{\rho,\delta}$ consist of smooth functions on $\mathbb{R}^{2n}\times\mathbb{R}^{2n}\times\mathbb{R}^{2n}$ that satisfy

$$\begin{aligned}&|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\beta_1}_{\xi_1}\partial^{\gamma_1}_{\eta_1}\partial^{\beta_2}_{\xi_2}\partial^{\gamma_2}_{\eta_2}\sigma(x,\xi,\eta)|\\ \le\, & C_{\alpha,\beta,\gamma}(1+|\xi_1|+|\eta_1|)^{\frac{m}{2}+\delta|\alpha_1|-\rho(|\beta_1|+|\gamma_1|)}\cdot(1+|\xi_2|+|\eta_2|)^{\frac{m}{2}+\delta|\alpha_2|-\rho(|\beta_2|+|\gamma_2|)} \end{aligned} \tag{1.1}$$

for all multi-indices $\alpha=(\alpha_1,\alpha_2),\beta=(\beta_1,\beta_2),\gamma=(\gamma_1,\gamma_2),$.

We study the following type of bi-parameter bilinear Pseudo-differential operators defined for $f,g\in\mathcal{S}(\mathbb{R}^{2n})$ with $\sigma(x,\xi,\eta)\in BBS^m_{\rho,\delta}$.

$$T_\sigma(f,g)=\int\int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}}\sigma(x,\xi,\eta)\cdot\widehat{f}(\xi)\cdot\widehat{g}(\eta)\cdot e^{2\pi ix(\xi+\eta)}d\xi d\eta$$

where $x=(x_1,x_2),\xi=(\xi_1,\xi_2),\eta=(\eta_1,\eta_2)\in\mathbb{R}^n\times\mathbb{R}^n$ and we denote the class of such operators by $Op(BBS^m_{\rho,\delta})$.

It is clear that the estimates for the bi-parameter and bilinear symbols $\sigma(x,\xi,\eta)$ are weaker than the classical single parameter bilinear symbol. It is these estimates which make the substantial difference between the bilinear Pseudo-differential operators and the bi-parameter and bilinear Pseudo-differential operators. The result is the following:

Theorem 1.10. *Let $m\in\mathbb{R}$, $1\le p,q,r\le\infty$, and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$.*

(a) All the operators of class $Op(BBS^m_{0,0})$ are bounded in $L^p\times L^q\to L^r$ if

$$m<m(p,q)=-2n\left(\max\{\frac{1}{2},\frac{1}{p},\frac{1}{q},1-\frac{1}{r}\}\right)$$

(b) If the operators of class $Op(BBS^m_{0,0})$ are bounded in $L^p \times L^q \to L^r$, then we must have

$$m \leq m(p,q) = -2n \left(\max\{\frac{1}{2}, \frac{1}{p}, \frac{1}{q}, 1 - \frac{1}{r}\} \right)$$

The index $m(p,q)$ in the above theorem can be interpreted as being *subcritical* in the sense that if $m < m(p,q)$ then any operators with symbols in the class $BBS^m_{0,0}$ must be bounded from $L^p(\mathbb{R}^{2n}) \times L^q(\mathbb{R}^{2n})$ to $L^r(\mathbb{R}^{2n})$ for any p, q, r satisfying $p, q, r \geq 1$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, while if $m > m(p,q)$ then there exist operators with symbols in $BBS^m_{0,0}$ such that they fail to be bounded from $L^p(\mathbb{R}^{2n}) \times L^q(\mathbb{R}^{2n})$ to $L^r(\mathbb{R}^{2n})$.

The proof of the theorem mainly consists of two parts: the boundedness of $L^\infty \times L^\infty \to L^\infty$ when $m < -2n$, and the boundedness of $L^2 \times L^2 \to L^1$ when $m < -n$, and then our theorem follows from the duality interpolation argument.

1.4 Sharp Trudinger-Moser Inequalities

The Trudinger-Moser and Adams inequalities are the replacements for the Sobolev embeddings in the limiting case. When $\Omega \subset \mathbb{R}^N$ is a bounded domain and $kp < N$, it is well-known that $W_0^{k,p}(\Omega) \subset L^q(\Omega)$ for all $1 \leq q \leq \frac{Np}{N-kp}$. However, by counterexamples, $W_0^{k,\frac{N}{k}}(\Omega) \nsubseteq L^\infty(\Omega)$. In this situation, Trudinger [90] proved that $W_0^{1,N}(\Omega) \subset L_{\varphi_N}(\Omega)$ where $L_{\varphi_N}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_N(t) = \exp\left(\alpha |t|^{N/(N-1)}\right) - 1$ for some $\alpha > 0$.

Theorem (Trudinger-1967). *Let Ω be a domain with finite measure in Euclidean*

$N-$space $\mathbb{R}^N$, $N \geq 2$. Then there exists a constant $\alpha > 0$, such that

$$\frac{1}{|\Omega|}\int_{\Omega} \exp\left(\alpha |u|^{\frac{N}{N-1}}\right) dx \leq c_0$$

for any $u \in W_0^{1,N}(\Omega)$ with $\int_{\Omega} |\nabla u|^N dx \leq 1$.

We note when the volume of Ω is infinite, there are mainly two types of inequalities: subcritical and critical inequalities.

Theorem (Adachi-Tanaka, 1999 [1]). *For any $\alpha < \alpha_N$, there exists a positive constant $C_{N,\alpha}$ such that $\forall u \in W^{1,N}(\mathbb{R}^N)$, $\|\nabla u\|_N \leq 1$:*

$$\int_{\mathbb{R}^N} \phi_N\left(\alpha |u|^{\frac{N}{N-1}}\right) dx \leq C_{N,\alpha} \|u\|_N^N, \tag{1.2}$$

where

$$\phi_N(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}.$$

The constant α_N is sharp in the sense that the supremum is infinity when $\alpha \geq \alpha_N$.

The above inequality fails at the critical case $\alpha = \alpha_N$. So it is natural to ask when the above can be true when $\alpha = \alpha_N$. This is done in [81], [61]

Theorem (Ruf, 2005 [81]; Li-Ruf, 2008 [61]). *For all $0 \leq \alpha \leq \alpha_N$:*

$$\sup_{\|u\| \leq 1} \int_{\mathbb{R}^N} \phi_N\left(\alpha |u|^{\frac{N}{N-1}}\right) dx < \infty \tag{1.3}$$

where

$$\|u\| = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^N + |u|^N\right) dx\right)^{1/N}.$$

Moreover, this constant α_N *is sharp in the sense that if* $\alpha > \alpha_N$*, then the supremum is infinity.*

For our work related to the equivalence of the above two types of inequalities, we begin with an improved sharp subcritical Trudinger-Moser inequality:

Theorem 1.11. *Let* $N \geq 2$*,* $\alpha_N = N\left(\frac{N\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}\right)^{\frac{1}{N-1}}$*,* $0 \leq \beta < N$ *and* $0 \leq \alpha < \alpha_N$*. Denote*

$$AT\left(\alpha,\beta\right) = \sup_{\|\nabla u\|_N \leq 1} \frac{1}{\|u\|_N^{N-\beta}} \int_{\mathbb{R}^N} \phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta}.$$

Then there exist positive constants $c = c\left(N,\beta\right)$ *and* $C = C\left(N,\beta\right)$ *such that when* α *is close enough to* α_N *:*

$$\frac{c\left(N,\beta\right)}{\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{(N-\beta)/N}} \leq AT\left(\alpha,\beta\right) \leq \frac{C\left(N,\beta\right)}{\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{(N-\beta)/N}}. \tag{1.4}$$

Moreover, the constant α_N *is sharp in the sense that* $AT\left(\alpha_N,\beta\right) = \infty$*.*

Then we can provide another proof to the sharp critical Trudinger-Moser inequality using Theorem 4.1 only.

Theorem 1.12. *Let* $N \geq 2$*,* $0 \leq \beta < N$*,* $0 < a$*,* b*. Denote*

$$MT_{a,b}\left(\beta\right) = \sup_{\|\nabla u\|_N^a + \|u\|_N^b \leq 1} \int_{\mathbb{R}^N} \phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta};$$

$$MT\left(\beta\right) = MT_{N,N}\left(\beta\right).$$

Then $MT_{a,b}\left(\beta\right) < \infty$ *if and only if* $b \leq N$*. The constant* α_N *is sharp. Moreover, we have the* *following identity:*

$$MT_{a,b}(\beta) = \sup_{\alpha\in(0,\alpha_N)} \left(\frac{1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}}{\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}} \right)^{\frac{N-\beta}{b}} AT(\alpha,\beta). \tag{1.5}$$

In particular, $MT(\beta) < \infty$ *and*

$$MT(\beta) = \sup_{\alpha\in(0,\alpha_N)} \left(\frac{1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}}{\left(\frac{\alpha}{\alpha_N}\right)^{N-1}} \right)^{\frac{N-\beta}{N}} AT(\alpha,\beta).$$

Now consider the sharp subcritical and critical Adams inequalities on $W^{2,\frac{N}{2}}\left(\mathbb{R}^N\right),\ N \geq 3$. Our first result is the following sharp subcritical Adams inequality:

Theorem 1.13. *Let* $N \geq 3$, $0 \leq \beta < N$ *and* $0 \leq \alpha < \beta(N,2)$. *Denote*

$$ATA(\alpha,\beta) = \sup_{\|\Delta u\|_{\frac{N}{2}}\leq 1} \frac{1}{\|u\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}} \int_{\mathbb{R}^N} \frac{\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-2}}\right)}{|x|^{\beta}}dx;$$

$$\phi_{N,2}(t) = \sum_{j\in\mathbb{N}:j\geq\frac{N-2}{2}} \frac{t^j}{j!}.$$

Then there exist positive constants $c = c(N,\beta)$ *and* $C = C(N,\beta)$ *such that when* α *is close enough to* $\beta(N,2)$:

$$\frac{c(N,\beta)}{\left[1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}\right]^{1-\frac{\beta}{N}}} \leq ATA(\alpha,\beta) \leq \frac{C(N,\beta)}{\left[1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}\right]^{1-\frac{\beta}{N}}}. \tag{1.6}$$

Moreover, the constant $\beta(N,2)$ *is sharp in the sence that* $AT(\alpha_N,\beta) = \infty$.

Theorem 1.14. *Let $N \geq 3$, $0 \leq \beta < N,\ 0 < a,\ b$. We denote:*

$$A_{a,b}(\beta) = \sup_{\|\Delta u\|_{\frac{N}{2}}^{a} + \|u\|_{\frac{N}{2}}^{b} \leq 1} \int_{\mathbb{R}^N} \frac{\phi_{N,2}\left(\beta(N,2)\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-2}}\right)}{|x|^{\beta}} dx;$$

$$A_{\frac{N}{2},\frac{N}{2}}(\beta) = A(\beta);$$

Then $A_{a,b}(\beta) < \infty$ if and only if $b \leq \frac{N}{2}$. The constant $\beta(N,2)$ is sharp. Moreover, we have the following identity:

$$A_{a,b}(\beta) = \sup_{\alpha \in (0,\beta(N,2))} \left(\frac{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}}{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}} \right)^{\frac{N-\beta}{2b}} ATA(\alpha,\beta). \tag{1.7}$$

In particular, $A(\beta) < \infty$ and

$$A(\beta) = \sup_{\alpha \in (0,\beta(N,2))} \left(\frac{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}}{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}} \right)^{\frac{N-\beta}{N}} ATA(\alpha,\beta).$$

Finally, we study the following improved sharp critical Adams inequality under the assumption that a version of the sharp subcritical Adams inequality holds:

Theorem 1.15. *Let $0 < \gamma < N$ be an arbitrary real positive number, $p = \frac{N}{\gamma}$, $0 \leq \alpha < \beta_0(N,\gamma) = \frac{N}{\omega_{N-1}} \left[\frac{\pi^{\frac{N}{2}} 2^{\gamma} \Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{N-\gamma}{2}\right)} \right]^{\frac{p}{p-1}}$, $0 \leq \beta < N,\ 0 < a,\ b$. We note*

$$GATA(\alpha,\beta) = \sup_{u \in W^{\gamma,p}(\mathbb{R}^N): \left\|(-\Delta)^{\frac{\gamma}{2}} u\right\|_p \leq 1} \frac{1}{\|u\|_p^{p\left(1-\frac{\beta}{N}\right)}} \int_{\mathbb{R}^N} \frac{\phi_{N,\gamma}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{p}{p-1}}\right)}{|x|^{\beta}} dx;$$

$$GA_{a,b}(\beta) = \sup_{u\in W^{\gamma,p}(\mathbb{R}^N):\left\|(-\Delta)^{\frac{\gamma}{2}}u\right\|_p^a+\|u\|_p^b\le 1} \int_{\mathbb{R}^N} \frac{\phi_{N,\gamma}\left(\beta_0(N,\gamma)\left(1-\frac{\beta}{N}\right)|u|^{\frac{p}{p-1}}\right)}{|x|^\beta}dx$$

where

$$\phi_{N,\gamma}(t) = \sum_{j\in\mathbb{N}:j\ge p-1}\frac{t^j}{j!}.$$

Assume that $GATA(\alpha,\beta)<\infty$ *and there exists a constant* $C(N,\gamma,\beta)>0$ *such that*

$$GATA(\alpha,\beta) \le \frac{C(N,\gamma,\beta)}{\left(1-\left(\frac{\alpha}{\beta_0(N,\gamma)}\right)^{\frac{p-1}{p}}\right)} \tag{1.8}$$

Then when $b\le p$*, we have* $GA_{a,b}(\beta)<\infty$*. In particular* $GA_{p,p}(\beta)<\infty$*.*

Though we have to assume a sharp subcritical Adams inequality (4.10), the main idea of Theorem 4.5 is that since $GATA(\alpha,\beta)$ is actually subcritical, i.e. α is strictly less than the critical level $\beta_0(N,\gamma)$, it is easier to study than $GA_{a,b}(\beta)$. Hence, it suggests a new approach in the study of $GA_{a,b}(\beta)$.

To achieve the best constant under the restriction of the semi-norm, we can also study the following Trudinger-Moser inequality with exact growth.

Theorem 1.16. *Let* $\lambda>0,\ 0\le\beta<N,\ q>1,\ 0<\alpha\le\alpha_N$ *and* $p>q$*. Denote*

$$TME_{p,q,N,\alpha,\beta} = \sup_{u\in D^{1,N}(\mathbb{R}^N)\cap L^q(\mathbb{R}^N):\ \|\nabla u\|_N\le 1}\frac{1}{\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}}\int_{\mathbb{R}^N}\frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)u^{\frac{N}{N-1}}\right)}{\left(1+|u|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx.$$

Then $TME_{p,q,N,\alpha,\beta}$ *can be attained in any of the following cases*

(a) $\beta>0$ *and all* $0<\alpha\le\alpha_N$*,*

(b) $\beta=0$*,* $\frac{q(N-1)}{N}\notin\mathbb{N}$ *and all* $0<\alpha\le\alpha_N$*,*

(c) $\beta = 0$, $\frac{q(N-1)}{N} \in \mathbb{N}$, $p > N$ *and all* $0 < \alpha \leq \alpha_N$,

(d) $\beta = 0$, $\frac{q(N-1)}{N} \in \mathbb{N}$, $p \leq N$, $p < \frac{N-1}{N-2}q$ *and* $\alpha = \alpha_N$.

CHAPTER 2 L^p ESTIMATE FOR A TRILINEAR PSEUDO-DIFFERENTIAL OPERATOR

2.1 Introduction

For $n \geq 1$ we denote by $\mathcal{M}(\mathbb{R}^n)$ the set of all bounded symbols $m \in L^\infty(\mathbb{R}^n)$, smooth away from the origin and satisfying the classical Marcinkiewcz-Mikhlin-Hörmander condition

$$|\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi|^\alpha}$$

for every $\xi \in \mathbb{R}^n \backslash \{0\}$ and sufficiently many multi-indices α. Denote by T_m by the n-linear operator

$$T_m(f_1, \ldots, f_n)(x) := \int_{\mathbb{R}^n} m(\xi)\hat{f}_1(\xi_1) \cdots \hat{f}_n(\xi_n) e^{2\pi i(\xi_1 + \cdots + \xi_n)\cdot x} d\xi,$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $f_1, \ldots, f_n$ are Schwartz functions on $\mathbb{R}$, denoted by $\mathcal{S}(\mathbb{R})$. From the classical Coifman-Meyer theorem we know T_m extends to a bounded n-linear operator from $L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_n}(\mathbb{R})$ to $L^r(\mathbb{R})$ for $1 < p_1, \ldots, p_n \leq \infty$ and $1/p_1 + \cdots + 1/p_n = 1/r > 0$. In fact this property holds for the high dimensions when $f_i \in L^{p_i}(\mathbb{R}^d)$, $i = 1, \ldots, n$ and $m \in \mathcal{M}(\mathbb{R}^{nd})$, see [25, 34, 43]. The case $p \geq 1$ was proved by Coifman and Meyer [25] and was extended to $p < 1$ by Grafakos and Torres [34] and Kenig and Stein [43]. Moreover, in the multiparameter setting, the same boundedness property is true, see [73–75], and also see [16] for a weaker restriction for the multiplier.

For the corresponding pseudo-differential variant of the classical Coifman-Meyer theorem, let the symbol $\sigma(x, \xi)$ belong to the bilinear Hörmander symbol class $BS^0_{1,0}$, that is, σ satisfies the condition

$$|\partial_x^l \partial_\xi^\alpha \sigma(x,\xi)| \lesssim \frac{1}{(1+|\xi|)^{|\alpha|}} \tag{2.1}$$

for any $x \in \mathbb{R}$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and sufficiently many indices l, α . We have the following

Theorem 2.1. *The operator*

$$T_\sigma(f_1, \ldots, f_n)(x) := \int_{\mathbb{R}^n} \sigma(x,\xi) \hat{f}_1(\xi_1) \cdots f_n(\xi_n) e^{2\pi i(\xi_1 + \cdots + \xi_n)\cdot x} d\xi \tag{2.2}$$

is bounded from $L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_n}(\mathbb{R})$ *to* $L^r(\mathbb{R})$ *for* $1 < p_1, \ldots, p_n \leq \infty$ *and* $1/p_1 + \cdots + 1/p_n = 1/r > 0$*, where* $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R})$ *and* σ *satisfies* (2.1).

For the proof of the above theorem, see [6] for bilinear, high dimensional case and [73] for one dimensional, n-linear case. Also, this boundedness property holds in the multi-parameter setting, see [26, 73].

For the trilinear Coifman-Meyer type theorem, Muscalu [72] proved the following theorem where the multiplier involved is a product of two symbols and has *flag singularities*, that is, for $m_1, m_2 \in \mathcal{M}(\mathbb{R}^2)$ satisfying

$$\begin{aligned} |\partial_\xi^\alpha \partial_\eta^\beta m_1(\xi,\eta)| &\lesssim \frac{1}{(|\xi|+|\eta|)^{\alpha+\beta}} \\ |\partial_\eta^\beta \partial_\zeta^\gamma m_2(\eta,\zeta)| &\lesssim \frac{1}{(|\eta|+|\zeta|)^{\beta+\gamma}} \end{aligned} \tag{2.3}$$

for every $\xi, \eta, \zeta \in \mathbb{R}$ and sufficiently many indices α, β and γ, we define

$$T_{m_1,m_2}(f_1, f_2, f_3)(x) := \int_{\mathbb{R}^3} m_1(\xi,\eta) m_2(\eta,\zeta) \hat{f}_1(\xi) \hat{f}_2(\eta) \hat{f}_3(\zeta) e^{2\pi i(\xi+\eta+\zeta)\cdot x} d\xi d\eta d\zeta, \tag{2.4}$$

where $f_1, f_2, f_3 \in \mathcal{S}(\mathbb{R})$. Then we have

Theorem 2.2. ([72]) *The operator defined in* (2.4) *maps* $L^{p_1} \times L^{p_2} \times L^{p_3} \to L^r$ *for* $1 < p_1, p_2, p_3 < \infty$ *with* $1/p_1 + 1/p_2 + 1/p_3 = 1/r$ *and* $0 < r < \infty$*. In addition,* T_{m_1,m_2} *also maps* $L^\infty \times L^p \times L^q \to L^s$, $L^p \times L^\infty \times L^q \to L^s$, $L^\infty \times L^t \times L^\infty \to L^t$ *for every* $1 < p, q, t < \infty$ *and* $1/p + 1/q = 1/s$.

Moreover, for the above theorem, the estimates like $L^\infty \times L^\infty \times L^t \to L^t$ or $L^\infty \times L^\infty \times L^\infty \to L^\infty$ are false, and these can be checked if we set f_2 to be identically 1.

Our main purpose is to consider a pseudo-differential operator corresponding to the above theorem, that is, let $a(x, \xi, \eta), b(x, \eta, \zeta) \in BS^0_{1,0}$ be symbols satisfying the conditions

$$\begin{aligned} |\partial_x^l \partial_\xi^\alpha \partial_\eta^\beta a(x, \xi, \eta)| &\lesssim \frac{1}{(1 + |\xi| + |\eta|)^{\alpha+\beta}} \\ |\partial_x^l \partial_\eta^\beta \partial_\zeta^\gamma b(x, \eta, \zeta)| &\lesssim \frac{1}{(1 + |\eta| + |\zeta|)^{\beta+\gamma}} \end{aligned} \tag{2.5}$$

for every $x, \xi, \eta, \zeta \in \mathbb{R}$ and sufficiently many indices α, β and γ, define the operator

$$T_{ab}(f, g, h)(x) := \int_{\mathbb{R}^3} a(x, \xi, \eta) b(x, \eta, \zeta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\zeta) e^{2\pi i x(\xi+\eta+\zeta)} d\xi d\eta d\zeta.$$

It's easy to see that the symbol $a(x, \xi, \eta) \cdot b(x, \eta, \zeta)$ satisfies a less restrictive condition than the condition (2.1) for the symbol σ in Theorem 2.1. Our main result on this is the following

Theorem 2.3. *The operator* T_{ab} *defined as* (2.1) *is bounded from* $L^{p_1} \times L^{p_2} \times L^{p_3}$ *to* L^r *for* $1 < p_1, p_2, p_3 < \infty$ *with* $1/p_1 + 1/p_2 + 1/p_3 = 1/r$ *and* $0 < r < \infty$*. In addition,* T_{ab} *also maps* $L^\infty \times L^p \times L^q \to L^s$, $L^p \times L^\infty \times L^q \to L^s$, $L^\infty \times L^t \times L^\infty \to L^t$ *for every* $1 < p, q, t < \infty$ *and*

$1/p + 1/q = 1/s$.

The proof of Theorem 2.3 is to reduce the trilinear pseudo-differential operator with the symbol of flag singularity to a localized version and takes advantage of the *flag paraproducts* from Muscalu's work [72] on the L^p estimates for the Fourier multipliers with symbols of flag singularity. Namely, we need to prove an equivalent localized version Theorem 2.9 of Theorem 2.3 (see [73], and also [26] for the multi-parameter setting). Moreover, the key to prove the localized result is that, conditions (2.5) allow us to only consider the dyadic intervals with lengths at most 1 in the *flag paraproducts*.

More precisely, in section 2.3 we show that our main theorem can be reduced to an estimate for a localized operator

$$T_{ab}^{0,0}(f,g,h)(x) = (\int_{\mathbb{R}^3} a_0(\xi,\eta) b_0(\eta,\zeta) \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta) e^{2\pi i x(\xi+\eta+\zeta)} d\xi d\eta d\zeta) \varphi_0(x),$$

where $\varphi_0(x)$ is a Schwartz function supported near the origin and a_0, b_0 satisfy a stronger decay condition than the classical Hörmander-Mikhlin condition.

In section 2.4, we will decompose the operator $T_{ab}^{0,0}$ to some operators of different forms. Among these operators, some of them could be reduced to the classical pseudo-differential operator in Theorem 2.1, and the others could be written as *flag paraproducts*, which are used in the proof of Theorem 2.2, in the forms of

$$(T_1(f,g,h)\cdot\varphi_0)(x) = \sum_{I\in\mathcal{I}} \frac{1}{|I|^{\frac{1}{2}}} \langle f, \phi_I^1\rangle \langle B_I^1(g,h), \phi_I^2\rangle \phi_I^3 \varphi_0$$

$$\text{where} \qquad B_I^1(g,h) = \sum_{J\in\mathcal{J}, |\omega_J^3|\le|w_I^2|} \frac{1}{|J|^{\frac{1}{2}}} \langle g, \phi_J^1\rangle \langle h, \phi_J^2\rangle \phi_J^3,$$

but with dyadic intervals have lengths at most 1.

Then by taking advantage of the *flag paraproducts* mentioned above, we will be able to prove the desired estimate for the localized version of our theorem in section 5.

2.2 Notations and Preliminaries

Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing, C^∞ functions in $\mathbb{R}$. Define the Fourier transform of a function f in $\mathcal{S}(\mathbb{R})$ as

$$F(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\cdot\xi}dx$$

extended in the usual way to the space of tempered distribution $\mathcal{S}'(\mathbb{R})$, which is the dual space of $\mathcal{S}(\mathbb{R})$.

We use $A \lesssim B$ to represent that there exists a universal constant $C > 1$ so that $A \leq CB$, and use the notation $A \sim B$ to denote that $A \lesssim B$ and $B \lesssim A$.

We call the intervals in the form of $[2^k n, 2^k(n+1)]$ in $\mathbb{R}$ to be dyadic intervals, where $k, n \in \mathbb{Z}$. We denote by $\mathbb{D}$ the set of all such dyadic intervals.

Definition 2.4. For $I \in \mathbb{D}$, we define the approximate cutoff function as

$$\tilde{\chi}_I(x) := (1 + \frac{\text{dist}(x, I)}{|I|})^{-100} \tag{2.6}$$

Definition 2.5. Let $I \subseteq \mathbb{R}$ be an arbitrary interval. A smooth function φ is said to be a bump adapted to I if and only if one has

$$|\varphi^{(l)}| \leq C_l C_M \frac{1}{|I|^l} \frac{1}{(1 + |x - x_I|/|I|)^M}$$

for every integer $M \in \mathbb{N}$ and sufficiently many derivatives $l \in \mathbb{N}$, where x_I denotes the center of I and $|I|$ is the length of I.

If φ_I is a bump adapted to I, we say that $|I|^{1/p}\varphi_I$ is an L^p-normalized bump adapted to I, for $1 \le p \le \infty$.

Definition 2.6. A sequence of L^2-normalized bumps $(\Phi_I)_{I\in\mathbb{D}}$ adapted to dyadic intervals $I \in \mathbb{D}$ is called a non-lacunary sequence if and only if for each $I \in \mathbb{D}$ there exists an interval $\omega_I = \omega_{|I|}$ symmetric with respect to the origin so that $\operatorname{supp}\widehat{\Phi_I} \subseteq \omega_I$ and $|\omega_I| \sim |I|^{-1}$.

Definition 2.7. A sequence of L^2-normalized bumps $(\Phi_I)_{I\in\mathbb{D}}$ adapted to dyadic intervals $I \in \mathbb{D}$ is called a lacunary sequence if and only if for each $I \in \mathbb{D}$ there exists an interval $\omega_I = \omega_{|I|}$ so that $\operatorname{supp}\widehat{\Phi_I} \subseteq \omega_I$, $|\omega_I| \sim |I|^{-1} \sim dist(0,\omega_I)$ and $0 \notin 5\omega_I$.

Definition 2.8. Let $\mathcal{I}, \mathcal{J} \subseteq \mathbb{D}$ be two families of dyadic intervals with lengths at most 1. Suppose that $(\phi_I^j)_{I\in\mathcal{I}}$ for $j = 1,2,3$ are three families of L^2-normalized bump functions such that the family $(\phi_I^2)_{I\in\mathcal{I}}$ is non-lacunary while the families $(\phi_I^j)_{I\in\mathcal{I}}$ for $j \neq 2$ are both lacunary, and $(\phi_J^j)_{J\in\mathcal{J}}$ for $j = 1,2,3$ are three families of L^2-normalized bump functions, where at least two of the three are lacunary.

We define as in [72] the discrete model operators T_1 and T_{1,k_0} for a positive integer k_0 by

$$T_1(f,g,h) = \sum_{I\in\mathcal{I}} \frac{1}{|I|^{\frac{1}{2}}} \langle f, \phi_I^1\rangle \langle B_I^1(g,h), \phi_I^2\rangle \phi_I^3 \tag{2.7}$$

$$\text{where} \qquad B_I^1(g,h) = \sum_{J\in\mathcal{J}, |\omega_J^3| \le |w_I^2|} \frac{1}{|J|^{\frac{1}{2}}} \langle g, \phi_J^1\rangle \langle h, \phi_J^2\rangle \phi_J^3 \tag{2.8}$$

$$T_{1,k_0}(f,g,h) = \sum_{I\in\mathcal{I}} \frac{1}{|I|^{\frac{1}{2}}} \langle f, \phi_I^1\rangle \langle B_{I,k_0}^1(g,h), \phi_I^2\rangle \phi_I^3 \tag{2.9}$$

$$\text{where} \qquad B_{I,k_0}^1(g,h) = \sum_{J\in\mathcal{J}, 2^{k_0}|\omega_J^3| \sim |w_I^2|} \frac{1}{|J|^{\frac{1}{2}}} \langle g, \phi_J^1\rangle \langle h, \phi_J^2\rangle \phi_J^3 \tag{2.10}$$

2.3 Reduction to A Localized Version

To prove the theorem, we proceed as follows. First pick a sequence of smooth functions $(\varphi_n)_n \in \mathbb{Z}$ such that $\operatorname{supp} \varphi_n \subseteq [n-1, n+1]$ and

$$\sum_{n \in \mathbb{Z}} \varphi_n = 1.$$

Then we can decompose the operator T_{ab} in (2.1) as

$$T_{ab} = \sum_{n \in \mathbb{Z}} T_{ab}^n$$

where

$$T_{ab}^n(f, g, h)(x) := T_{ab}(f, g, h)(x)\varphi_n(x).$$

Suppose we can prove the estimate

$$\|T_{ab}^n(f, g, h)\|_r \lesssim \|f\tilde{\chi}_{I_n}\|_{p_1} \|g\tilde{\chi}_{I_n}\|_{p_2} \|h\tilde{\chi}_{I_n}\|_{p_3}, \tag{2.11}$$

where I_n is the interval $[n, n+1]$, and $\tilde{\chi}_{I_n}$ is defined as in (2.6).

Then our main Theorem 2.3 can be proved by the following estimate

$$\begin{aligned}
\|T_{ab}(f, g, h)\|_r &\lesssim (\sum_{n \in \mathbb{Z}} \|T_{ab}^n(f, g, h)\|_r^r)^{1/r} \lesssim (\sum_{n \in \mathbb{Z}} \|f\tilde{\chi}_{I_n}\|_{p_1}^r \|g\tilde{\chi}_{I_n}\|_{p_2}^r \|h\tilde{\chi}_{I_n}\|_{p_3}^r)^{1/r} \\
&\lesssim (\sum_{n \in \mathbb{Z}} \|f\tilde{\chi}_{I_n}\|_{p_1}^{p_1})^{1/p_1} (\sum_{n \in \mathbb{Z}} \|g\tilde{\chi}_{I_n}\|_{p_2}^{p_2})^{1/p_2} (\sum_{n \in \mathbb{Z}} \|h\tilde{\chi}_{I_n}\|_{p_3}^{p_3})^{1/p_3} \\
&\lesssim \|f\|_{p_1} \|g\|_{p_2} \|h\|_{p_3}.
\end{aligned}$$

Thus, we only need to prove (2.11).

Consider that for a fixed $n_0 \in \mathbb{Z}$, we have

$$T_{ab}^{n_0}(f,g,h)(x) = \int_{\mathbb{R}^3} a(x,\xi,\eta)\tilde{\varphi}_{n_0}(x)b(x,\eta,\zeta)\tilde{\varphi}_{n_0}(x)\varphi_{n_0}(x)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}d\xi d\eta d\zeta,$$

where $\tilde{\varphi}_{n_0}$ is a smooth function supported on the interval $[n_0-2, n_0+2]$ and equals 1 on the support of φ_{n_0}. Then we rewrite the symbols $a(x,\xi,\eta)\tilde{\varphi}_{n_0}(x)$ and $b(x,\eta,\zeta)\tilde{\varphi}_{n_0}(x)$ by using Fourier series with respect to the x variable

$$a(x,\xi,\eta)\tilde{\varphi}_{n_0}(x) = \sum_{l_1\in\mathbb{Z}} a_{l_1}(\xi,\eta)e^{2\pi i x l_1}$$

$$b(x,\eta,\zeta)\tilde{\varphi}_{n_0}(x) = \sum_{l_2\in\mathbb{Z}} b_{l_2}(\xi,\eta)e^{2\pi i x l_2},$$

where by taking advantage of conditions (2.5) we can have

$$|\partial_{\xi,\eta}^{\alpha,\beta} a_{l_1}(\xi,\eta)| \lesssim \frac{1}{(1+|l_1|)^M}\frac{1}{(1+|\xi|+|\eta|)^{\alpha+\beta}}$$

$$|\partial_{\eta,\zeta}^{\beta,\gamma} b_{l_2}(\eta,\zeta)| \lesssim \frac{1}{(1+|l_2|)^M}\frac{1}{(1+|\eta|+|\gamma|)^{\beta+\gamma}}$$

for a large number M and sufficiently many indices α,β,γ. Note the decay in l_1,l_2 means we only need to consider the case for $l_1,l_2=0$, which is given by

$$(T_{ab}^{n_0,0,0}(f,g,h)(x) = (\int_{\mathbb{R}^3} a_0(\xi,\eta)b_0(\eta,\zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}d\xi d\eta d\zeta)\varphi_{n_0}(x),$$

where symbols a_0, b_0 satisfy the following conditions

$$|\partial^{\alpha,\beta}_{\xi,\eta} a_0(\xi,\eta)| \lesssim \frac{1}{(1+|\xi|+|\eta|)^{\alpha+\beta}}$$
$$|\partial^{\beta,\gamma}_{\eta,\zeta} b_0(\eta,\zeta)| \lesssim \frac{1}{(1+|\eta|+|\gamma|)^{\beta+\gamma}}. \tag{2.12}$$

Using the translation invariance, we only need to prove the following localized result for $n_0 = 0$

Theorem 2.9. *The operator*

$$T^{0,0}_{ab}(f,g,h)(x) = (\int_{\mathbb{R}^3} a_0(\xi,\eta) b_0(\eta,\zeta) \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta) e^{2\pi i x(\xi+\eta+\zeta)} d\xi d\eta d\zeta)\varphi_0(x) \tag{2.13}$$

has the following boundedness property

$$\|T^{0,0}_{ab}(f,g,h)\|_r \lesssim \|f\tilde{\chi}_{I_0}\|_{p_1} \|g\tilde{\chi}_{I_0}\|_{p_2} \|h\tilde{\chi}_{I_0}\|_{p_3} \tag{2.14}$$

for $1 < p_1, p_2, p_3 < \infty$ *and* $1/p_1 + 1/p_2 + 1/p_3 = 1/r$, *where* φ_0 *is a smooth function supported within* $[-1,1]$ *and* a_0, b_0 *satisfy the conditions* (2.12).

In addition, this estimate also holds for the cases where at most one $p_i = \infty$ *for* $i = 1,2,3$ *or* $p_1, p_3 = \infty, 1 < p_2 < \infty$.

Now we are ready to do some decompositions to the operator in (2.13).

2.4 Reduction of the Localized Operator

In this section, we will mainly show the problem can be reduced to some operators or paraproducts that we are familiar with.

Let $\varphi \in \mathcal{S}(\mathbb{R})$ be a Schwartz function such that $\operatorname{supp}\hat{\varphi} \subseteq [-1,1]$ and $\hat{\varphi}(\xi) = 1$ on $[-1/2, 1/2]$. Define $\psi \in \mathcal{S}(\mathbb{R})$ be the Schwartz function satisfying

$$\hat{\psi}(\xi) := \hat{\varphi}(\xi/2) - \hat{\varphi}(\xi),$$

and let

$$\widehat{\psi_k}(\cdot) = \widehat{\psi}(\cdot/2^k) \qquad \text{and} \qquad \widehat{\psi_{-1}}(\cdot) = \hat{\varphi}(\cdot).$$

Note that

$$1 = \sum_{k\geq -1} \widehat{\psi_k}, \quad \text{where } \operatorname{supp}\hat{\psi} \subseteq [-2^{k+1}, -2^{k-1}] \cup [2^{k-1}, 2^{k+1}] \text{ for } k \geq 0.$$

Then for any $m, n \in \mathbb{Z}$, we use $m \gg n$ to denote $m - n > 100$ and $m \simeq n$ to denote $|m - n| \leq 100$. Consider the decomposition

$$\begin{aligned} & 1(\xi,\eta,\zeta) \\ &= (\sum_{k_1'\geq -1}\sum_{k_1''\geq -1} \widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta))(\sum_{k_2'\geq -1}\sum_{k_2''\geq -1} \widehat{\psi_{k_2'}}(\eta)\widehat{\psi_{k_2''}}(\zeta)). \end{aligned} \tag{2.15}$$

Without loss of generality, we consider

$$\begin{aligned} & (\sum_{k_1'\geq -1}\sum_{k_1''\geq -1} \widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta)) = \sum_{k_1'\gg k_1''\geq -1} \widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta) \\ + & \sum_{-1\leq k_1'\ll k_1''} \widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta) + \sum_{k_1'\simeq k_1'' k_1'>100, or\, k_1''>100} \widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta) + \sum_{k_1'\simeq k_1'', k_1', k_1''\leq 100} \widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta) \\ := & A + B + C + D, \end{aligned} \tag{2.16}$$

where term D can be written out specifically, which contains finite number of terms:

$$D = \hat{\varphi}(\xi)\hat{\varphi}(\eta) + Others$$

To estimate C, note in this case actually both k_1' and k_1'' are at least 1. Suppose $k_1' > 100$, we have:

$$\sum_{k_1' \simeq k_1'', k_1' > 100} \widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta) = \sum_{k>100} \widehat{\psi_k}(\xi)\widehat{\tilde{\psi}_k}(\eta)$$

and then

$$C = \sum_{k>100} \widehat{\psi_k}(\xi)\widehat{\tilde{\psi}_k}(\eta) + \sum_{k>100} \widehat{\tilde{\psi}_k}(\xi)\widehat{\psi_k}(\eta)$$

where $\operatorname{supp} \widehat{\tilde{\psi}_k} \subseteq [-2^{k+101}, -2^{k-101}] \cup [2^{k-101}, 2^{k+101}]$.

Estimates for A and B are quite similar:

$$A = \sum_{k_1'} \Big(\sum_{-1 \le k_1'' < k_1' - 100} \widehat{\psi_{k_1''}}(\eta) \Big) \widehat{\psi_{k_1'}}(\xi) = \sum_{k \ge 100} \widehat{\psi_k}(\xi)\widehat{\varphi_k}(\eta) \tag{2.17}$$

$$B = \sum_{k_1''} \Big(\sum_{-1 \le k_1' < k_1'' - 100} \widehat{\psi_{k_1'}}(\xi) \Big) \widehat{\psi_{k_1''}}(\eta) = \sum_{k \ge 100} \widehat{\varphi_k}(\xi)\widehat{\psi_k}(\eta), \tag{2.18}$$

where φ_k is a Schwartz function with $\operatorname{supp} \widehat{\varphi_k} \subseteq [-2^{k-100}, 2^{k+100}]$. For $k \ge 0$ we call the families like $(\psi_k)_k$ to be Ψ *type* functions, whose Fourier transform have almost disjoint supports for different scales and call the families like $(\varphi_k)_k$ to be Φ *type* functions, whose Fourier transforms have overlapping supports for different scales. In the rest of work, for convenience purpose we don't distinguish between ψ_k and $\tilde{\psi}_k$, since they are of the same type and have comparative scales for the supports of their Fourier transforms, and we always use ψ_k to represent such Ψ *type* functions. Similarly we always use φ_k to represent a Φ *type* function. With such notations we can write (2.16) as

$$
\begin{aligned}
&(\sum_{k_1'\geq -1}\widehat{\psi_{k_1'}}(\xi))(\sum_{k_1''\geq -1}\widehat{\psi_{k_1''}}(\eta))\\
=&\sum_{k\geq 100}\widehat{\psi_k}(\xi)\widehat{\varphi_k}(\eta)+\sum_{k\geq 100}\widehat{\varphi_k}(\xi)\widehat{\psi_k}(\eta)+\sum_{k>100}\widehat{\psi_k}(\xi)\widehat{\psi_k}(\eta)+D. \qquad (2.19)
\end{aligned}
$$

Later from the proof, we will see in (2.19) the three summations work similarly, since what we really need is at least one lacunary family in each summation. And all the functions in D play a same role as $\hat{\varphi}(\xi)\hat{\varphi}(\xi)$, which means we actually can replace (2.19) by an equivalently version, which is

$$\sum_{k\geq 0}\widehat{\phi_k^1}(\xi)\widehat{\phi_k^2}(\eta)+\hat{\varphi}(\xi)\hat{\varphi}(\xi), \qquad (2.20)$$

where at least one of the families $(\widehat{\phi_k^1}(\xi))_k$ and $(\widehat{\phi_k^2}(\xi))_k$ is Ψ *type*.

Now to deal with (2.15), it's equivalent to consider

$$
\begin{aligned}
1(\xi,\eta,\zeta) &= (\sum_{k_1'\geq -1}\sum_{k_1''\geq -1}\widehat{\psi_{k_1'}}(\xi)\widehat{\psi_{k_1''}}(\eta))(\sum_{k_2'\geq -1}\sum_{k_2''\geq -1}\widehat{\psi_{k_2'}}(\eta)\widehat{\psi_{k_2''}}(\zeta))\\
&\approx (\sum_{k_1}\widehat{\phi_{k_1}^1}(\xi)\widehat{\phi_{k_1}^2}(\eta)+\hat{\varphi}(\xi)\hat{\varphi}(\eta))(\sum_{k_2}\widehat{\phi_{k_2}^1}(\eta)\widehat{\phi_{k_2}^2}(\zeta)+\hat{\varphi}(\eta)\hat{\varphi}(\zeta))\\
&= (\sum_{k_1}\widehat{\phi_{k_1}^1}(\xi)\widehat{\phi_{k_1}^2}(\eta)\sum_{k_2}\widehat{\phi_{k_2}^1}(\eta)\widehat{\phi_{k_2}^2}(\zeta))+(\sum_{k_1}\widehat{\phi_{k_1}^1}(\xi)\widehat{\phi_{k_1}^2}(\eta))\hat{\varphi}(\eta)\hat{\varphi}(\zeta)\\
&\quad +(\sum_{k_2}\widehat{\phi_{k_2}^1}(\eta)\widehat{\phi_{k_2}^2}(\zeta))\hat{\varphi}(\xi)\hat{\varphi}(\eta)+\hat{\varphi}(\xi)\hat{\varphi}(\eta)\hat{\varphi}(\eta)\hat{\varphi}(\zeta)\\
&:= E+F+G+H, \qquad (2.21)
\end{aligned}
$$

where for convenience purpose the symbol "≈" is used to show the equivalence, and we will simply treat $1(\xi,\eta,\zeta)=E+F+G+H$ in the rest of the work.

Then by using the above and (2.13), we can decompose the localized operator as

$$T_{ab}^{0,0}(f,g,h)(x) = (\int_{\mathbb{R}^3} a_0(\xi,\eta)b_0(\eta,\zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi ix(\xi+\eta+\zeta)}d\xi d\eta d\zeta)\varphi_0(x)$$

$$= (\int_{\mathbb{R}^3} a_0(\xi,\eta)b_0(\eta,\zeta)(E+F+G+H)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi ix(\xi+\eta+\zeta)}d\xi d\eta d\zeta)\varphi_0(x)$$

$$:= T_{ab}^{E,0,0} + T_{ab}^{F,0,0} + T_{ab}^{G,0,0} + T_{ab}^{H,0,0}. \tag{2.22}$$

2.4.1 Estimates for $T_{ab}^{H,0,0}$

Recall

$$T_{ab}^{H,0,0}(f,g,h)(x) = (\int_{\mathbb{R}^3} a_0(\xi,\eta)b_0(\eta,\zeta)\hat{\varphi}(\xi)\hat{\varphi}(\eta)\hat{\varphi}(\eta)\hat{\varphi}(\zeta) \cdot\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi ix(\xi+\eta+\zeta)}d\xi d\eta d\zeta)\varphi_0(x),$$

where note that $m_H(\xi,\eta,\zeta) := a_0(\xi,\eta)b_0(\eta,\zeta)\hat{\varphi}(\xi)\hat{\varphi}(\eta)\hat{\varphi}(\eta)\hat{\phi}(\zeta)$ satisfies the condition

$$|\partial_\xi^\alpha \partial_\eta^\beta \partial_\zeta^\gamma m_H(\xi,\eta,\zeta)| \lesssim \frac{1}{(1+|\xi|+|\eta|+|\zeta|)^{\alpha+\beta+\gamma}}$$

for sufficiently many indices α,β,γ. Then our desired localized estimate follows from Theorem 2.1, since we find the operator $T_{ab}^{H,0,0}$ is just the localized operator used in the proof of Theorem 2.1, see [26, 73].

2.4.2 Estimates for $T_{ab}^{F,0,0} + T_{ab}^{G,0,0}$

Recall

$$F = (\sum_{k_1} \widehat{\phi_{k_1}^1}(\xi)\widehat{\phi_{k_1}^2}(\eta))\hat{\varphi}(\eta)\hat{\varphi}(\zeta),$$

where at least one of the families $(\widehat{\phi_{k_1}^1})_{k_1}$ and $(\widehat{\phi_{k_1}^2})_{k_1}$ is Ψ *type*.

When $(\widehat{\phi_{k_1}^2})_{k_1}$ is Ψ *type*, Note that to make $\sum_{k_1} \widehat{\phi_{k_1}^2}(\eta)\hat{\varphi}(\eta) \neq 0$, k_1 will have a upper bound for the summation, say $k_1 \leq 100$. Then desired estimate under this situation can be done by using the same way as in $T_{ab}^{H,0,0}$, since only finite number of terms are involved.

When $(\widehat{\phi_{k_1}^2})_{k_1}$ is Φ *type*, we must have $(\widehat{\phi_{k_1}^1})_{k_1}$ is Ψ *type*. Recall

$$T_{ab}^{F,0,0}(f,g,h)(x) =$$
$$(\sum_{k_1} \int_{\mathbb{R}^3} a_0(\xi,\eta)\widehat{\phi_{k_1}^1}(\xi)\widehat{\phi_{k_1}^2}(\eta)b_0(\eta,\zeta)\hat{\varphi}(\eta)\hat{\varphi}(\zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi i x(\xi+\eta+\zeta)}d\xi d\eta d\zeta)\varphi_0(x), \quad (2.23)$$

then we can use Fourier series to write

$$a_0(\xi,\eta)\widehat{\phi_{k_1}^1}(\xi)\widehat{\phi_{k_1}^2}(\eta) = \sum_{n_1,n_2\in\mathbb{Z}} C_{n_1,n_2}^{k_1} e^{2\pi i n_1 \xi/2^{k_1}} e^{2\pi i n_2 \eta/2^{k_1}}, \quad (2.24)$$

where the Fourier coefficients $C_{n_1,n_2}^{k_1}$ are given by

$$C_{n_1,n_2}^{k_1} = \frac{1}{2^{2k_1}} \int_{\mathbb{R}^2} a_0(\xi,\eta)\widehat{\phi_{k_1}^1}(\xi)\widehat{\phi_{k_1}^2}(\eta)e^{-2\pi i n_1 \xi/2^{k_1}} e^{-2\pi i n_2 \eta/2^{k_1}}.$$

By the decay condition (2.12) and the advantage that $(\widehat{\phi_{k_1}^1})_{k_1}$ is Ψ *type*, we can get the following by integration by parts sufficiently many times

$$|C_{n_1,n_2}^{k_1}| \lesssim \frac{1}{(1+|n_1|+|n_2|)^M}.$$

Note by the decay in n_1, n_2 we only need to consider the case when $n_1, n_2 = 0$, see [73] and the proof in section 2.5 for more details, and similar things can be done for $b_0(\eta,\zeta)\hat{\varphi}(\eta)\hat{\varphi}(\zeta)$. Then, we can use Hölder's inequality and take advantage the fact that φ is a bump function adapted to $[-1,1]$ to prove the localized result for (2.23), that is,

$$
\begin{aligned}
& \|(\sum_{k_1}\int_{\mathbb{R}^3}\widehat{\phi^1_{k_1}}(\xi)\widehat{\phi^2_{k_1}}(\eta)\hat{\varphi}(\eta)\hat{\varphi}(\zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi ix(\xi+\eta+\zeta)}d\xi d\eta d\zeta)\varphi_0(x)\|_r \\
\approx\ & \|(\sum_{k_1}\int_{\mathbb{R}^3}\widehat{\phi^1_{k_1}}(\xi)\hat{\varphi}(\eta)\hat{\varphi}(\zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi ix(\xi+\eta+\zeta)}d\xi d\eta d\zeta)\varphi_0(x)\|_r \\
=\ & \|(\sum_{k_1}\phi^1_{k_1}*f)(x)\varphi_0(x)(\varphi*g)(x)\tilde{\varphi}_0(x)(\varphi*h)(x)\tilde{\varphi}_0(x)\|_r \\
\lesssim\ & \|(\sum_{k_1}\phi^1_{k_1}*f)(x)\varphi_0(x)\|_{p_1}\|(\varphi*g)(x)\tilde{\varphi}_0(x)\|_{p_2}\|(\varphi*h)(x)\tilde{\varphi}_0(x)\|_{p_3} \\
\lesssim\ & \|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3},
\end{aligned}
$$

where we take $\tilde{\phi}_0$ to be 1 on supp ϕ_0 and supported in a slightly larger interval containing supp ϕ_0. The last inequality is true since $(\varphi_{k_1})_{k_1}$ is Ψ *type*. Also, in the above we can simply write $\sum_{k_1}\widehat{\phi^2_{k_1}}(\eta)\hat{\varphi}(\eta)=\hat{\varphi}(\eta)$ in the above since k_1 is positive.

2.4.3 Estimates for $T^{E,0,0}_{ab}$

Recall

$$E=(\sum_{k_1\geq 0}\widehat{\phi^1_{k_1}}(\xi)\widehat{\phi^2_{k_1}}(\eta))(\sum_{k_2\geq 0}\widehat{\phi^1_{k_2}}(\eta)\widehat{\phi^2_{k_2}}(\zeta)),$$

where at least one of the families $(\widehat{\phi^1_{k_1}})_{k_1}$ and $(\widehat{\phi^2_{k_1}})_{k_1}$ is Ψ *type* and at least one of the families $(\widehat{\phi^1_{k_2}})_{k_2}$ and $(\widehat{\phi^2_{k_2}})_{k_2}$ is Ψ *type*.

Also we consider the corresponding localized operator

$$
\begin{aligned}
T^{E,0,0}_{ab}(f,g,h)(x) \quad = \quad & (\int_{\mathbb{R}^3}(\sum_{k_1}\widehat{\phi^1_{k_1}}(\xi)\widehat{\phi^2_{k_1}}(\eta))a_0(\xi,\eta))(\sum_{k_2}\widehat{\phi^1_{k_2}}(\eta)\widehat{\phi^2_{k_2}}(\zeta)b_0(\eta,\zeta)) \\
& \cdot\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi ix(\xi+\eta+\zeta)}d\xi d\eta d\zeta)\varphi_0(x).
\end{aligned}
$$

By using Fourier series as before, we only need to consider the following operator

$$(\int_{\mathbb{R}^3}(\sum_{k_1}\widehat{\phi^1_{k_1}}(\xi)\widehat{\phi^2_{k_1}}(\eta)))(\sum_{k_2}\widehat{\phi^1_{k_2}}(\eta)\widehat{\phi^2_{k_2}}(\zeta))\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi ix(\xi+\eta+\zeta)}d\xi d\eta d\zeta)\varphi_0(x).$$

As usual we consider three cases of E

$$\begin{aligned} E &= (\sum_{k_1\gg k_2}+\sum_{k_1\ll k_2}+\sum_{k_1\simeq k_2})(\widehat{\phi^1_{k_1}}(\xi)\widehat{\phi^2_{k_1}}(\eta))(\widehat{\phi^1_{k_2}}(\eta)\widehat{\phi^2_{k_2}}(\zeta)) \\ &:= I+J+K, \end{aligned}$$

and decompose

$$T^{E,0,0}_{ab} := T^{I,0,0}_{ab}+T^{J,0,0}_{ab}+T^{K,0,0}_{ab}.$$

Note K is actually a symbol in $BS^0_{1,0}$, since k is positive. That is,

$$T^{K,0,0}_{ab}(f,g,h)(x) = (\int_{\mathbb{R}^3} m_K(\xi,\eta,\zeta)\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi ix(\xi+\eta+\zeta)}d\xi d\eta d\zeta)\varphi_0(x),$$

where $m_K(\xi,\eta,\zeta)$ satisfies the condition as (2.12). Thus, the desired localized estimate follows from the proof of Theorem 2.1, just as $T^{H,0,0}_{ab}$.

$T^{I,0,0}_{ab}$ and $T^{J,0,0}_{ab}$ are similar, we define T^I_{ab} by the following equality

$$\begin{gathered} T^I_{ab}(f,g,h)(x)\cdot\varphi_0(x) =: T^{I,0,0}_{ab}(f,g,h)(x) = \\ (\int_{\mathbb{R}^3}(\sum_{k_1}\widehat{\phi^1_{k_1}}(\xi)\widehat{\phi^2_{k_1}}(\eta))(\sum_{k_2}\widehat{\phi^1_{k_2}}(\eta)\widehat{\phi^2_{k_2}}(\zeta))\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\zeta)e^{2\pi ix(\xi+\eta+\zeta)}d\xi d\eta d\zeta)\varphi_0(x). \end{gathered} \tag{2.25}$$

From [72, 73], we know T^I_{ab} can be written by using paraproducts, which is the following lemma.

Lemma 2.10. *Define T^I_{ab} as in* (2.25)*, then we can write*

$$T^I_{ab}(f,g,h)(x) =$$
$$T_1(f,g,h)(x) + \sum_{l=1}^{M-1}\sum_{k_0=100}^{\infty}(2^{-k_0})^l T_{l,k_0}(f,g,h)(x) + \sum_{k_0=100}^{\infty}(2^{-k_0})^M T_{M,k_0}(f,g,h)(x)$$

where

$$T_1(f,g,h) = \sum_{I\in\mathcal{I}}\frac{1}{|I|^{\frac{1}{2}}}\langle f,\phi^1_I\rangle\langle B^1_I(g,h),\phi^2_I\rangle\phi^3_I$$
$$\text{with} \qquad B^1_I(g,h) = \sum_{\substack{J\in\mathcal{J}\\ |\omega^3_J|\le|w^2_I|}}\frac{1}{|J|^{\frac{1}{2}}}\langle g,\phi^1_J\rangle\langle h,\phi^2_J\rangle\phi^3_J$$
$$T_{l,k_0}(f,g,h) = \sum_{I\in\mathcal{I}}\frac{1}{|I|^{\frac{1}{2}}}\langle f,\phi^1_I\rangle\langle B^l_{I,k_0}(g,h),\phi^2_I\rangle\phi^3_I$$
$$\text{with} \qquad B^l_{I,k_0}(g,h) = \sum_{\substack{J\in\mathcal{J}\\ 2^{k_0}|\omega^3_J|\sim|w^2_I|}}\frac{1}{|J|^{\frac{1}{2}}}\langle g,\phi^1_J\rangle\langle h,\phi^2_J\rangle\phi^3_J$$

In the above,

(a) $T_1(f,g,h)$ and $B^1_I(g,h)$ are defined as (2.7) *and* (2.8) *in definition* (2.8)*.*

(b) For each l, $T_l(f,g,h)$ and $B^l_I(g,h)$ are of the type (2.9) *and* (2.10) *in definition* 2.8*. l here is actually involved in the families $(\phi^2_I)_I$ and $(\phi^2_J)_J$, but it won't affect our proof since it does not change the types of those functions.*

(c) M is a large positive integer, and the multiplier $m_{M,k_0}(\xi,\eta,\zeta)$ in T_{M,k_0} satisfies the

condition

$$|\partial_\xi^\alpha \partial_\eta^\beta \partial_\zeta^\gamma m_{M,k_0}(\xi,\eta,\zeta)| \lesssim (2^{k_0})^{\alpha+\beta+\gamma} \frac{1}{(1+|\xi|+|\eta|+|\zeta|)^{\alpha+\beta+\gamma}} \tag{2.26}$$

for sufficiently many indices α, β, γ

(d) *All the dyadic intervals in* T_1 *and* T_{l,k_0} *have lengths at most* 1 *for all* $k_0 \geq 100, 1 \leq l \leq M-1$.

Proof. We follow closely the work [72], where the Fourier expansions of $\widehat{\phi_{k_1}^2}(\eta)$ are used to get the desired forms of paraproducts. The only two statements we need to show are that all the dyadic intervals there have lengths at most one and the decay number 1 in the denominator from (2.26). Actually both of them follow from the fact $k_1, k_2 \geq 0$. □

So far we have reduced Theorem 2.9 to the estimate of the operator $T_{ab}^{I,0,0}$.

2.5 Proof of Theorem 2.9

In this section by using the decomposition in Lemma 2.10, we are able to prove the localized estimate for $T_{ab}^{I,0,0}$, which will complete the proof of Theorem 2.9.

2.5.1 Estimates for $\sum_{k_0=100}^{\infty}(2^{-k_0})^M T_{M,k_0}(f,g,h)(x)$

For this part, note that the condition (2.26) is almost the classical case. Then by repeating the work in [26, 73] we will see this condition can provide an estimate

$$\|T_{M,k_0}(f,g,h)\varphi_0(x)\| \lesssim C2^{10k_0}\|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3}$$

which is accepted since we can choose M large enough.

2.5.2 Estimates for $T_1(f,g,h)(x)$

Taking advantage of that $|I| \leq 1$, we can split

$$\begin{aligned} T_1(f,g,h)(x) &= \sum_{I \subseteq 5I_0} \frac{1}{|I|^{\frac{1}{2}}} \langle f, \phi_I^1 \rangle \langle B_I^1(g,h), \phi_I^2 \rangle \phi_I^3 + \sum_{I \subseteq (5I_0)^c} \frac{1}{|I|^{\frac{1}{2}}} \langle f, \phi_I^1 \rangle \langle B_I^1(g,h), \phi_I^2 \rangle \phi_I^3 \\ &= I + II. \end{aligned} \tag{2.27}$$

For Part I, we do the following decompositions first

$$f = \sum_{n_1} f\chi_{I_{n_1}}, \qquad \sum_{n_2} g\chi_{I_{n_2}}, \qquad \sum_{n_3} h\chi_{I_{n_3}},$$

where $I_{n_i} = [n_i, n_i + 1], i = 1, 2, 3, n_i \in \mathbb{Z}$. Then we can write

$$T_1(f,g,h)(x) = \sum_{n_1} \sum_{n_2} \sum_{n_3} T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x).$$

When $|n_1|, |n_2|, |n_3| \leq 10$, the desired estimate follows from Theorem 2.2

$$\begin{aligned} &\| \sum_{|n_1| \leq 10} \sum_{|n_2| \leq 10} \sum_{|n_3| \leq 10} T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x) \cdot \varphi_0(x) \|_r \\ \lesssim\ &\| \sum_{|n_1| \leq 10} f\chi_{I_{n_1}} \|_{p_1} \| \sum_{|n_2| \leq 10} g\chi_{I_{n_2}} \|_{p_2} \| \sum_{|n_3| \leq 10} h\chi_{I_{n_3}} \|_{p_3} \\ \lesssim\ &\| f\tilde{\chi}_{I_0} \|_{p_1} \| g\tilde{\chi}_{I_0} \|_{p_2} \| h\tilde{\chi}_{I_0} \|_{p_3}, \end{aligned}$$

where the last inequality holds from $\chi_{[-11,11]} \lesssim \tilde{\chi}_{I_0}(x)$.

When $|n_1|, |n_2|, |n_3| > 10$, we write

$$\|T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x)\cdot\varphi_0(x)\|_r$$
$$= \|\sum_{I\in\mathcal{I}}\sum_{\substack{J\in\mathcal{J}\\|\omega_J^3|\le|\omega_I^2|}}\frac{1}{|I|^{\frac{1}{2}}}\frac{1}{|J|^{\frac{1}{2}}}\langle f\chi_{I_{n_1}},\phi_I^1\rangle\langle g\chi_{I_{n_2}},\phi_J^1\rangle\langle h\chi_{I_{n_3}},\phi_J^3\rangle\langle\phi_I^2,\phi_J^3\rangle\phi_I^3(x)\varphi_0(x)\|_r.$$

Then we use Hölder's inequality to get

$$\begin{aligned}
&\|\frac{1}{|I|^{\frac{1}{2}}}\frac{1}{|J|^{\frac{1}{2}}}\langle f\chi_{I_{n_1}},\phi_I^1\rangle\langle g\chi_{I_{n_2}},\phi_J^1\rangle\langle h\chi_{I_{n_3}},\phi_J^3\rangle\langle\phi_I^2,\phi_J^3\rangle\phi_I^3(x)\varphi_0(x)\|_r\\
\lesssim\ &\frac{1}{|I|^2}\frac{1}{|J|^2}(1+\frac{\text{dist}(I_{n_1},I)}{|I|})^{-M_1}(\|f\chi_{I_{n_1}}\|_{p_1}|I|^{\frac{p_1-1}{p_1}})(1+\frac{\text{dist}(I_{n_2},J)}{|J|})^{-N_1}\\
&\cdot(\|g\chi_{I_{n_2}}\|_{p_2}|J|^{\frac{p_2-1}{p_2}})(1+\frac{\text{dist}(I_{n_3},J)}{|J|})^{-N_2}(\|h\chi_{I_{n_3}}\|_{p_3}|J|^{\frac{p_3-1}{p_3}})\\
&\cdot|I|^{\frac{1}{r}}\int_{\mathbb{R}}(1+\frac{\text{dist}(x,I)}{|I|})^{-M_2}(1+\frac{\text{dist}(x,J)}{|J|})^{-N_3}dx\\
\lesssim\ &\frac{1}{|I|}(\frac{|I|}{|J|})^{\frac{1}{p_2}+\frac{1}{p_3}}(1+\frac{\text{dist}(I_{n_1},I)}{|I|})^{-M_1}(1+\frac{\text{dist}(I_{n_2},J)}{|J|})^{-N_1}(1+\frac{\text{dist}(I_{n_3},J)}{|J|})^{-N_2}\\
&\cdot\int_{\mathbb{R}}(1+\frac{\text{dist}(x,I)}{|I|})^{-M_2}(1+\frac{\text{dist}(x,J)}{|J|})^{-N_3}dx\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3}, \quad (2.28)
\end{aligned}$$

where M_j, N_j are sufficiently large integers and ϕ_I^j, ϕ_J^j are L^2-normalized bump functions adapted to I, J for $j = 1, 2, 3$.

We first consider the case when $\text{dist}(I, J) \le 3$. Recall we have the restriction that $|\omega_J^3| \le |\omega_I^2|$, which implies that $|I|/|J| \lesssim 1$. By using the subadditivity of $\|\cdot\|_r^r$ we have

$$\begin{aligned}
&\|T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x)\cdot\varphi_0(x)\|_r^r\\
\lesssim\ &\sum_{i,j\ge0}\sum_{\substack{I\subseteq5I_0,J\subseteq9I_0\\|I|=2^{-i},|J|=2^{-j}}}(\frac{1}{|I|}(1+\frac{\text{dist}(I_{n_1},I)}{|I|})^{-M_1}(1+\frac{\text{dist}(I_{n_2},J)}{|J|})^{-N_1}(1+\frac{\text{dist}(I_{n_3},J)}{|J|})^{-N_2}\\
&\cdot\int_{\mathbb{R}}(1+\frac{\text{dist}(x,I)}{|I|})^{-M_2}(1+\frac{\text{dist}(x,J)}{|J|})^{-N_3}dx\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3})^r
\end{aligned}$$

$$\lesssim \sum_{i,j\geq 0} \sum_{\substack{I\subseteq 5I_0, J\subseteq 9I_0\\ |I|=2^{-i}, |J|=2^{-j}}} (2^i(1+2^i(|n_1|-6))^{-M_1}(1+2^j(|n_2|-9))^{-N_1}(1+2^j(|n_3|-9))^{-N_2}$$
$$\cdot(\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3})^r$$
$$\lesssim ((|n_1|-6)^{-M_1}(|n_2|-9)^{-N_1}(|n_3|-9)^{-N_2}\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3})^r.$$

Observe that for large enough integers M_1, N_1, N_2 we have

$$\chi_{I_{n_1}}(|n_1|-6)^{\frac{-M_1}{2}} \lesssim \tilde{\chi}_{I_0},\ \chi_{I_{n_2}}(|n_2|-9)^{\frac{-N_1}{2}} \lesssim \tilde{\chi}_{I_0},\ \chi_{I_{n_3}}(|n_3|-9)^{\frac{-N_2}{2}} \lesssim \tilde{\chi}_{I_0}.$$

Thus,

$$\|\sum_{|n_1|>10}\sum_{|n_2|>10}\sum_{|n_3|>10} T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x)\cdot\varphi_0(x)\|_r^r$$
$$\lesssim \sum_{|n_1|>10}\sum_{|n_2|>10}\sum_{|n_3|>10} ((|n_1|-6)^{-M_1}(|n_2|-9)^{-N_1}(|n_3|-9)^{-N_2}$$
$$\cdot\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3})^r$$
$$\lesssim \sum_{|n_1|>10}\sum_{|n_2|>10}\sum_{|n_3|>10} ((|n_1|-6)^{\frac{-M_1}{2}}(|n_2|-9)^{\frac{-N_1}{2}}(|n_3|-9)^{\frac{-N_2}{2}}$$
$$\cdot\|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3})^r$$
$$\lesssim (\|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3})^r.$$

For the other possibility, that is , when $\text{dist}(I,J) > 3$, we consider whether J is close to I_{n_2} or I_{n_3}. Without loss of generality, we assume $\text{dist}(J, I_{n_2}) \leq 2, \text{dist}(J, I_{n_3}) > 2$, and other cases will follow in the similar way. Using the notation $J_m = [m, m+1], m \in \mathbb{Z}$ and (2.28) we can get

$$\|T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x)\cdot\varphi_0(x)\|_r^r$$

$$\begin{aligned}
&\lesssim \sum_{i,j\geq 0}\sum_{\substack{I\subseteq 5I_0\\|I|=2^{-i}}}\sum_{|m|>3}\sum_{\substack{J\subseteq J_m,|J|=2^{-j}\\ \mathrm{dist}(J,I_{n_2})\leq 2,\mathrm{dist}(J,I_{n_3})>2}}(\frac{1}{|I|}(1+\frac{\mathrm{dist}(I_{n_1},I)}{|I|})^{-M_1}(1+\frac{\mathrm{dist}(I_{n_2},J)}{|J|})^{-N_1}\\
&\quad (1+\frac{\mathrm{dist}(I_{n_3},J)}{|J|})^{-N_2}\cdot\int_{\mathbb{R}}(1+\frac{\mathrm{dist}(x,I)}{|I|})^{-M_2}(1+\frac{\mathrm{dist}(x,J)}{|J|})^{-N_3}dx\\
&\quad \cdot\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3})^r\\
&\lesssim \sum_{i,j\geq 0}\sum_{\substack{I\subseteq 5I_0\\|I|=2^{-i}}}\sum_{|m|>3}\sum_{\substack{J\subseteq J_m|J|=2^{-j}\\ \mathrm{dist}(J,I_{n_2})\leq 2,\mathrm{dist}(J,I_{n_3})>2}}(2^i(1+2^i(|n_1|-6)^{-M_1})(1+2^j(|m-n_3|))^{-N_2}|m|^{-N_0}\\
&\quad \cdot\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3})^r\\
&\lesssim \sum_{i,j\geq 0}\sum_{\substack{I\subseteq 5I_0\\|I|=2^{-i}}}\sum_{|m|>3}\sum_{\substack{J\subseteq J_m|J|=2^{-j}\\ \mathrm{dist}(J,I_{n_2})\leq 2,\mathrm{dist}(J,I_{n_3})>2}}(2^i(1+2^i(|n_1|-6))^{-M_1}(1+2^j(|m-n_3|))^{-N_2}|n_2|^{-N_0}\\
&\quad \cdot\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3})^r,
\end{aligned}$$

where $N_0=\min\{M_2,N_3\}$ is sufficiently large and we use $m\sim n_2$.

Now we take the sum over n_1,n_2,n_3 and get

$$\begin{aligned}
&\|\sum_{|n_1|>10}\sum_{|n_2|>10}\sum_{|n_3|>10}T_1(f\chi_{I_{n_1}},g\chi_{I_{n_2}},h\chi_{I_{n_3}})(x)\cdot\varphi_0(x)\|_r^r\\
\lesssim\ &\sum_{|n_1|>10}\sum_{|n_2|>10}\sum_{|n_3|>10}((|n_1|-6)^{-\frac{M_1}{2}}|n_2|^{-N_0}(|n_3|-3)^{-\frac{N_2}{2}}\\
&\cdot\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3})^r\\
\lesssim\ &\sum_{|n_1|>10}\sum_{|n_2|>10}\sum_{|n_3|>10}((|n_1|-6)^{-\frac{M_1}{4}}|n_2|^{-\frac{N_0}{2}}(|n_3|-3)^{-\frac{N_2}{4}}\\
&\cdot\|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3})^r\\
\lesssim\ &(\|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3})^r.
\end{aligned}$$

For other possible chooses of n_1,n_2,n_3, they will be treated in different ways. Among these cases, when $|n_1|>10$, we can do similar things as the above to get our desired estimate directly, by considering whether J is close to I or not. Note in the case we are free to take summation over J since we have a decay on i and $j \leq i$.

But when $|n_1| \leq 10$, say $|n_1|, |n_2| \leq 10$, $|n_3| \geq 10$ things are different. In this situation, the term $(1+\frac{\text{dist}(I_{n_1},I)}{|I|})^{-M_1}$ in (2.28) won't give us a decay factor, which means we will have trouble when taking the summation over dyadic intervals I. Actually the decay factors from other terms are with respect to j which can't help since $i > j$. Recall our desired estimate in this case

$$\|\sum_{|n_1|,|n_2|\leq 10}\sum_{|n_3|>10} T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x)\cdot\varphi_0(x)\|_r \lesssim \|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3}. \tag{2.29}$$

Suppose that from the proof of Theorem 2.2 (see [72, 73]) we can get an additional decay with respect to n_3 such like $1/|n_3|^M$ for sufficiently positive integer M, then we only need to apply Theorem 2.2 to get

$$\begin{aligned}&\|\sum_{|n_1|,|n_2|\leq 10}\sum_{|n_3|>10} T_1(f\chi_{I_{n_1}}, g\chi_{I_{n_2}}, h\chi_{I_{n_3}})(x)\cdot\varphi_0(x)\|_r\\ \lesssim\ &\frac{1}{|n_3|^M}\|f\chi_{I_{n_1}}\|_{p_1}\|g\chi_{I_{n_2}}\|_{p_2}\|h\chi_{I_{n_3}}\|_{p_3} \lesssim \|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3}.\end{aligned}$$

Now we will see how to get such a decay $1/|n_3|^M$. As before we consider two possible cases $\text{dist}(I, J) \leq 3$ and $\text{dist}(I, J) > 3$.

When $\text{dist}(I, J) > 3$, as before consider the integral

$$\int_{\mathbb{R}}(1+\frac{\text{dist}(x,I)}{|I|})^{-M_2}(1+\frac{\text{dist}(x,J)}{|J|})^{-N_3}dx.$$

We can get a decay about $|m|^{-M}$ for $J \subseteq J_m, m \in \mathbb{Z}$, and see whether J_m is close n_3 to or not. As before by considering whether J is close to I_{n_3} or not, we will get an additional decay $1/|n_3|^M$.

When $\mathrm{dist}(I,J) \le 3$, as before we have that J is near the origin $J \subseteq 9I_0$. In this case our desired decay comes from the *size* and *energy* estimates used in the proof of Theorem 2.2, see [72, 73]. Those *size* and *engergy* terms corresponding to the function $h\chi_{n_3}$ would be defined based on the inner product terms like $|\langle h\chi_{I_{n_3}}, \phi_J^2\rangle|$. Now since J is close to the origin, such inner product will provide a decay about $1/|n_3|^M$. (Or one can see the proof of Lemma 2.13 or section 8.11 in [73] to see clearly we can actually get such a decay factor for the *size* estimate.) That means we can get an additional decay from the result of Theorem 2.2, since the boundedness there is based on the *size* and *energy* estimates.

So far we have proved Part I in (2.27).

For Part II, using the intervals $I_n = [n, n+1]$, $J_m = [m, m+1]$, $m, n \in \mathbb{Z}$ we can write

$$\begin{aligned}
&\|T_1(f,g,h)(x)\cdot\varphi_0(x)\|_r^r \\
=\ &\| \sum_{I\subseteq(5I_0)^c} \sum_{\substack{J\in\mathcal{J} \\ |\omega_J^3|\le|\omega_I^2|}} \frac{1}{|I|^{\frac12}} \frac{1}{|J|^{\frac12}} \langle f,\phi_I^1\rangle\langle g,\phi_J^1\rangle\langle h,\phi_J^3\rangle\langle \phi_I^2,\phi_J^3\rangle\phi_I^3(x)\varphi_0(x)\|_r^r \\
\lesssim\ & \sum_{|n|\ge5}\sum_{m\in\mathbb{Z}}\sum_{I\subseteq I_n}\sum_{\substack{J\subseteq J_m \\ |\omega_J^3|\le|\omega_I^2|}} \|\frac{1}{|I|^{\frac12}} \frac{1}{|J|^{\frac12}} \langle f,\phi_I^1\rangle\langle g,\phi_J^1\rangle\langle h,\phi_J^3\rangle\langle \phi_I^2,\phi_J^3\rangle\phi_I^3(x)\varphi_0(x)\|_r^r.
\end{aligned}$$

We will use Hölder's inequality and take advantage of the decay factors as before to write the above as

$$\sum_{|n|\ge5}\sum_{m\in\mathbb{Z}}\sum_{i,j\ge0}\sum_{\substack{I\subseteq I_n, J\subseteq J_m \\ |I|=2^{-i},|J|=2^{-j}}} \|\frac{1}{|I|^{\frac12}} \frac{1}{|J|^{\frac12}} \langle f,\phi_I^1\rangle\langle g,\phi_J^1\rangle\langle h,\phi_J^3\rangle\langle \phi_I^2,\phi_J^3\rangle\phi_I^3(x)\varphi_0(x)\|_r^r$$

$$\lesssim \sum_{|n|\geq 5}\sum_{m\in\mathbb{Z}}\sum_{i,j\geq 0}\sum_{\substack{I\subseteq I_n, J\subseteq J_m\\ |I|=2^{-i},|J|=2^{-j}}} (\frac{1}{|I|^2}\frac{1}{|J|^2}(\|f\tilde{\chi}_{I_n}\|_{p_1}|I|^{\frac{p_1-1}{p_1}})(\|g\tilde{\chi}_{J_m}\|_{p_2}|J|^{\frac{p_2-1}{p_2}})\cdot$$

$$(\|h\tilde{\chi}_{J_m}\|_{p_3}|J|^{\frac{p_3-1}{p_3}})|I|^{\frac{1}{r}}(1+\frac{\text{dist}(I,I_0)}{|I|})^{-M_3}\int_{\mathbb{R}}(1+\frac{\text{dist}(x,I)}{|I|})^{-M_2}(1+\frac{\text{dist}(x,J)}{|J|})^{-N_3}dx)^r$$

$$\lesssim \sum_{|n|\geq 5}\sum_{m\in\mathbb{Z}}\sum_{i,j\geq 0}\sum_{\substack{I\subseteq I_n, J\subseteq J_m\\ |I|=2^{-i},|J|=2^{-j}}} (2^i(1+2^i(|n|-2))^{-M_3}\|f\tilde{\chi}_{I_n}\|_{p_1}\|g\tilde{\chi}_{J_m}\|_{p_2}\|h\tilde{\chi}_{J_m}\|_{p_3}$$

$$\cdot\int_{\mathbb{R}}(1+\frac{\text{dist}(x,I)}{|I|})^{-M_2}(1+\frac{\text{dist}(x,J)}{|J|})^{-N_3}dx)^r, \tag{2.30}$$

where again M_j, N_j are sufficiently large integers. Then we consider two possible cases, $\text{dist}(I_n, J_m) \leq 5$ and $\text{dist}(I_n, J_m) > 5$.

When $\text{dist}(I_n, J_m) \leq 5$, we use the same technique as before

$$(|n|-2)^{-\frac{M}{12}}|\tilde{\chi}_{I_n}| \lesssim |\tilde{\chi}_{I_0}| \qquad \text{and} \qquad |\tilde{\chi}_{I_n}| \sim |\tilde{\chi}_{J_m}|,$$

for M sufficiently large. Note that the decay factor for i actually implies a decay for the summation over dyadic intervals J, since $i \geq j$. Then we can estimate (2.30) by

$$\begin{aligned}
&\lesssim \sum_{|n|\geq 5}((|n-2|^{-\frac{M_3}{2}})\|f\tilde{\chi}_{I_n}\|_{p_1}\|g\tilde{\chi}_{J_m}\|_{p_2}\|h\tilde{\chi}_{J_m}\|_{p_3})^r \\
&\lesssim \sum_{|n|\geq 5}((|n-2|^{-\frac{M_3}{4}})\|f\tilde{\chi}_0\|_{p_1}\|g\tilde{\chi}_0\|_{p_2}\|h\tilde{\chi}_0\|_{p_3})^r \\
&\lesssim (\|f\tilde{\chi}_{I_0}\|_{p_1}\|g\tilde{\chi}_{I_0}\|_{p_2}\|h\tilde{\chi}_{I_0}\|_{p_3})^r,
\end{aligned}$$

which is the desired estimate.

When $\mathrm{dist}(I_n, J_m) > 5$, we need to take advantage of the integral in (2.30). That is,

$$\int_{\mathbb{R}} (1 + \frac{\mathrm{dist}(x, I)}{|I|})^{-M_2} (1 + \frac{\mathrm{dist}(x, J)}{|J|})^{-N_3} dx \lesssim |n - m|^{-L},$$

where $L = \min\{M_2, N_3\}$ is large enough. Now (2.30) can be written by

$$\begin{aligned}
&\lesssim \sum_{|n|\geq 5} \sum_{|m-n|>5} \sum_{i,j\geq 0} \sum_{\substack{I\subseteq I_n, J\subseteq J_m \\ |I|=2^{-i}, |J|=2^{-j}}} (2^i(1 + 2^i(|n| - 2))^{-M_3} \\
&\quad \cdot \|f\tilde{\chi}_{I_n}\|_{p_1} \|g\tilde{\chi}_{J_m}\|_{p_2} \|h\tilde{\chi}_{J_m}\|_{p_3} |m - n|^{-L})^r \\
&\lesssim \sum_{|n|\geq 5} ((|n - 2|^{\frac{-M_3}{2}}) \|f\tilde{\chi}_{I_n}\|_{p_1} \|g\tilde{\chi}_{J_n}\|_{p_2} \|h\tilde{\chi}_{J_n}\|_{p_3})^r \\
&\lesssim (\|f\tilde{\chi}_{I_0}\|_{p_1} \|g\tilde{\chi}_{I_0}\|_{p_2} \|h\tilde{\chi}_{I_0}\|_{p_3})^r,
\end{aligned}$$

where as before the decay factor for i allows us to take the summation over dyadic intervals J, since $i \geq j$.

Now are are done with Part II, which means we have proved the desired estimate for $T_1(f, g, h)(x)$.

2.5.3 Estimates for $\sum_{k_0=100}^{\infty} (2^{-k_0})^l T_{l,k_0}(f, g, h)(x)$

There is nothing new in this case, since it will be almost the same as what we did for $T_1(f, g, h)(x)$. Note for $T_{l,k_0}(f, g, h)(x)$, the only difference is that we have $|I|^{-1} \sim |\omega_I^2| \sim 2^{k_0}|J|^{-1} \sim |\omega^3|_J$ instead of $|I|^{-1} \sim |\omega_I^2| \geq |J|^{-1} \sim |\omega_J^3|$ in $T_1(f, g, h)(x)$. That is, let $|I| = 2^{-i}, |J| = 2^{-j}$, we will have $i - k_0 = j \geq 0, k_0 \geq 100$. Recall we only need $i \geq j$ in the proof for $T_1(f, g, h)(x)$, and the method obviously works for $T_{l,k_0}(f, g, h)(x)$ in the setting $i - k_0 = j \geq 0, k_0 \geq 100$, which will give us a bound uniformly with respect to k_0. Then we will be able to take the summation over k_0 by using $l \geq 1$. In this way we can get the estimate for $\sum_{k_0=100}^{\infty}(2^{-k_0})^l T_{l,k_0}(f,g,h)(x)$.

So far we have proved the desired localized estimate for the operator $T_{ab}^{E,0,0}(f,g,h)(x)$ in (2.22), which means Theorem 2.9 has been proved. Then from this localized result, we can conclude that Theorem 2.3 is true.

CHAPTER 3 BI-PARAMETER AND BILINEAR CALDERÓN-VAILLANCOURT THEOREM WITH SUBCRITICAL ORDER

3.1 Introduction

Pseudo-differential operators play important roles in harmonic analysis, several complex variables, partial differential equations and other branches of modern mathematics, see e.g. [31], [79], [44], [85], [83], [87], [89], etc.

We first recall that the Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n)$ of linear pseudo-differential operators are defined to consist of operators in the form

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)\cdot \widehat{f}(\xi)\cdot e^{2\pi i x\xi} d\xi \tag{3.1}$$

where $x, \xi, \eta \in \mathbb{R}^n$ and σ satisfies

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}$$

for all multi-indices α, β and some positive constants $C_{\alpha,\beta}$ depending on α, β. The function f is taken initially from the Schwartz class $\mathcal{S}(\mathbb{R}^n)$.

Hörmander [37, 38] proved the operators with symbols in $S^0_{\rho,\delta}$ are L^2 bounded when $0 \leq \delta < \rho \leq 1$. In a celebrated paper, Calderón and Vaillancourt [10] established the L^2 boundedness when $0 \leq \delta = \rho < 1$. C. Fefferman [29] further extended to the L^p boundedness ($1 < p < \infty$) for operators with symbols in $S^{-m}_{\rho,\delta}$ with $0 \leq \delta < \rho \leq 1$ and $m \geq n|\frac{1}{p} - \frac{1}{2}|(1-\rho)$. The result of C. Fefferman is sharp in the sense that for $m < n|\frac{1}{p} - \frac{1}{2}|(1-\rho)$, then the L^p boundedness fails. Paivarinta and E. Somersalo later considered the critical case of $\delta = \rho$

in [78] by establishing h^p to h^p boundedness for all $0 < p < \infty$, where h^p is the local Hardy space of Goldberg [35]. The result of [78] strengthens the H^1 to L^1 boundedness of Coifman and Meyer [24] when $m = \frac{n}{2}$. We also refer to the more extensive treatment of pseudo-differential operators and their applications in PDEs to [4], [31], [79], [44], [46], [83], [87], [89], etc.

The bilinear analogue of such pseudo-differential operators are defined to be the class $BS^m_{\rho,\delta}(\mathbb{R}^{2n})$ consisting of operators of the following form: Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and define

$$T_\sigma(f,g)(x) = \int\int_{\mathbb{R}^n\times\mathbb{R}^n} \sigma(x,\xi,\eta)\cdot \widehat{f}(\xi)\cdot\widehat{g}(\eta)\cdot e^{2\pi i x(\xi+\eta)} d\xi d\eta \tag{3.2}$$

where $x, \xi, \eta \in \mathbb{R}^n$ and σ satisfies

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq C_{\alpha,\beta,\gamma}(1+|\xi|+|\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)} \tag{3.3}$$

for all multi-indices α, β, γ and some positive constants $C_{\alpha,\beta,\gamma}$ depending on α, β, γ.

The first work of bilinear singular integrals and pseudo-differential operators is due to Coifman and Meyer [24, 25] which originated from specific problems about Calderón's commutators. Subsequently, the symbolic calculus for bilinear pseudo-differential operators was studied, e.g., in the works [6, 68] motivated by the bilinear Calderón-Zygmund theory developed [17, 34, 43], etc. and references therein. In particular, critical order for boundedness of bilinear pseudo-differential operators with symbols $BS^m_{0,0}$ has been considered in [6, 70].

The L^p estimates of multi-parameter and multi-linear Coifman-Meyer type Fourier multipliers were established in [74]. Recently, Chen and the first author [22] gave a different proof of the L^p estimates of [74] and also establish the L^p estimates under the limited smoothness of the Fourier symbol; Dai and the first author [26] proved the same L^p estimates of [74] for multi-parameter and multi-linear pseudo-differential operators. More recently, Hong and the first author [36] carried out a theory of symbolic calculus for multi-parameter and multi-linear pseudo-differential operators.

Let $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. In this article we will study the following type of bi-parameter and bilinear pseudo-differential operators defined for $f, g \in \mathcal{S}(\mathbb{R}^{2n})$:

$$T_\sigma(f,g) = \int\int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\cdot \widehat{f}(\xi)\cdot\widehat{g}(\eta)\cdot e^{2\pi i x(\xi+\eta)} d\xi d\eta$$

where $x=(x_1,x_2), \xi=(\xi_1,\xi_2), \eta=(\eta_1,\eta_2)\in\mathbb{R}^n\times\mathbb{R}^n$ and σ satisfies

$$\begin{aligned} |\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\eta_1}^{\gamma_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_2}^{\gamma_2}\sigma(x,\xi,\eta)| \leq \quad & C_{\alpha,\beta,\gamma}(1+|\xi_1|+|\eta_1|)^{\frac{m}{2}+\delta|\alpha_1|-\rho(|\beta_1|+|\gamma_1|)} \\ & \cdot(1+|\xi_2|+|\eta_2|)^{\frac{m}{2}+\delta|\alpha_2|-\rho(|\beta_2|+|\gamma_2|)} \end{aligned} \tag{3.4}$$

for all multi-indices $\alpha=(\alpha_1,\alpha_2), \beta=(\beta_1,\beta_2), \gamma=(\gamma_1,\gamma_2)$, and some positive constants $C_{\alpha,\beta,\gamma}$ depending on α,β,γ.

We denote the class of such symbols by $BBS^m_{\rho,\delta}$. We also denote by $Op(BBS^m_{\rho,\delta})$ the class of all operators T_σ with $\sigma\in BBS^m_{\rho,\delta}$.

It is clear that the estimates in (3.4) that the bi-parameter and bilinear symbol $\sigma(x,\xi,\eta)$ satisfies are weaker than those in (3.3) satisfied by the bilinear symbol. It is these estimates which make the substantial difference between the bilinear pseudo-differential operators and the bi-parameter and bilinear pseudo-differential operators.

Given the above bi-parameter and bilinear operator $T = T_\sigma$, we can define its adjoints T^{*1} and T^{*2} as follows:

$$\langle T(f,g),h\rangle = \langle T^{*1}(h,g),f\rangle = \langle T^{*2}(f,h),g\rangle \text{ for all } f,g \in \mathcal{S}(\mathbb{R}^n)$$

The main result on this is the following:

Theorem 3.1 (Main Theorem). *Let $m \in \mathbb{R}$, $1 \le p,q,r \le \infty$, and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$.*

(a) All the operators of class $Op(BBS^m_{0,0})$ are bounded in $L^p \times L^q \to L^r$ if

$$m < m(p,q) = -2n\left(\max\{\frac{1}{2},\frac{1}{p},\frac{1}{q},1-\frac{1}{r}\}\right)$$

(b) If the operators of class $Op(BBS^m_{0,0})$ are bounded in $L^p \times L^q \to L^r$, then we must have

$$m \le m(p,q) = -2n\left(\max\{\frac{1}{2},\frac{1}{p},\frac{1}{q},1-\frac{1}{r}\}\right)$$

The index $m(p,q)$ in the above theorem can be interpreted as being *subcritical* in the sense that if $m < m(p,q)$ then any operators with symbols in the class $BBS^m_{0,0}$ must be bounded from $L^p(\mathbb{R}^{2n}) \times L^q(\mathbb{R}^{2n})$ to $L^r(\mathbb{R}^{2n})$ for any p,q,r satisfying $p,q,r \ge 1$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, while if $m > m(p,q)$ then there exist operators with symbols in $BBS^m_{0,0}$ such that they fail to be bounded from $L^p(\mathbb{R}^{2n}) \times L^q(\mathbb{R}^{2n})$ to $L^r(\mathbb{R}^{2n})$ when $p,q,r \ge 1$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. We should mention in the bilinear (one-parameter) case, Bényi, Bernicot, Maldonado, Naibo and Torres [6] established the boundedness for $m < m(p,q)$ and Miyachi and Tomita [70] proved the boundedness at the critical case when $m = m(p,q)$.

The proof of the Main Theorem mainly consists of two parts: the boundedness of $L^\infty \times L^\infty \to L^\infty$ when $m < -2n$, and the boundedness of $L^2 \times L^2 \to L^1$ when $m < -n$, and then our theorem follows from the interpolation argument.

3.2 The Boundedness on $L^\infty \times L^\infty \to L^\infty$

In this section, we will prove the boundedness of the bi-parameter and bilinear operator T_σ on $L^\infty \times L^\infty \to L^\infty$. Actually we can prove the following more general case:

Theorem 3.2. *When $m < -2n(1-\rho)$, for $\sigma \in BBS^m_{\rho,\delta}$ where $0 \le \delta, \rho \le 1, \delta < 1$, we then have that $T_\sigma : L^\infty \times L^\infty \to L^\infty$.*

To prove this theorem, we need the following lemma in the bi-parameter setting (see also [36]). A one-parameter version can be found in [6].

Lemma 3.3. *Let $m \in \mathbb{R}, 0 \le \delta, \rho \le 1, \sigma \in BBS^m_{\rho,\delta}$.*

(a) If $0 < R_1, R_2 \le 1$ and supp$(\sigma) \subseteq \{(x,\xi,\eta) : |\xi_i| + |\eta_i| \le R_i, i = 1,2\}$ then

$$\|T_\sigma(f,g)\|_{L^\infty} \lesssim (R_1R_2)^{2n} \|f\|_{L^\infty} \|g\|_{L^\infty}, \quad f, g \in L^\infty.$$

(b) If $R_1, R_2 \ge 1$ and supp$(\sigma) \subseteq \{(x,\xi,\eta) : R_i \le |\xi_i| + |\eta_i| \le 4R_i, i = 1,2\}$ then

$$\|T_\sigma(f,g)\|_{L^\infty} \lesssim (R_1R_2)^{(1-\rho)n+\frac{m}{2}} \|f\|_{L^\infty} \|g\|_{L^\infty}, \quad f, g \in L^\infty.$$

(c) If $0 < R_1 \le 1, R_2 \ge 1$ and supp$(\sigma) \subseteq \{(x,\xi,\eta) : |\xi_1| + |\eta_1| \le R_1, R_2 \le |\xi_2| + |\eta_2| \le 4R_2\}$, then

$$\|T_\sigma(f,g)\|_{L^\infty} \lesssim (R_1)^n (R_2)^{(1-\rho)n+\frac{m}{2}} \|f\|_{L^\infty} \|g\|_{L^\infty}, \quad f, g \in L^\infty.$$

Proof. Consider

$$T_\sigma(f,g) = \int_{\mathbb{R}^n} \mathcal{K}(x, x-y, x-z) f(y) g(z) dydz,$$

where

$$\mathcal{K}(x,y,z) = \int_{\mathbb{R}^n} \sigma(x,\xi,\eta) e^{2\pi i \xi\cdot y} e^{2\pi i \eta \cdot z} d\xi d\eta = \mathcal{F}_{4n}^{-1}(\sigma(x,\cdot,\cdot))(y,z)$$

and $\mathcal{F}_{4n}^{-1}$ denotes the inverse Fourier transform with respect to $(\xi,\eta) \in (\mathbb{R}^2 \times \mathbb{R}^2)$. Then it suffices to show that

(a) $\sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}^n} |\mathcal{K}(x,y,z)| dydz \lesssim (R_1R_2)^{2n}$,

(b) $\sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}^n} |\mathcal{K}(x,y,z)| dydz \lesssim (R_1R_2)^{(1-\rho)n+\frac{m}{2}}$,

(c) $\sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}^n} |\mathcal{K}(x,y,z)| dydz \lesssim (R_1)^n (R_2)^{(1-\rho)n+\frac{m}{2}}$.

for the corresponding three parts in the lemma.

For part (a), note σ is a smooth function with compact support. For an $N \in \mathbb{N}_0$, we have

$$\begin{aligned} (1+|(y,z)|^2)^N \mathcal{K}(x,y,z) &\approx \int_{\mathbb{R}^n} \sigma(x,\xi,\eta)(1-\triangle_\xi - \triangle_\eta)^N (e^{2\pi i\xi\cdot y} e^{2\pi i \eta\cdot z}) d\xi d\eta \\ &= \int_{\mathbb{R}^n} (1-\triangle_\xi-\triangle_\eta)^N (\sigma(x,\xi,\eta))(e^{i\xi\cdot y} e^{i\eta\cdot z}) d\xi d\eta, \end{aligned}$$

which implies

$$|\mathcal{K}(x,y,z)| \lesssim \frac{(R_1R_2)^{2n}}{(1+|(y,z)|^2)^N},$$

and part(a) is true if we choose $N > 2n$.

For part (b) consider

$$\int_{\mathbb{R}^n} |\mathcal{K}(x,y,z)|dydz = \int_{\substack{|y_1|+|z_1|\leq (R_1)^{-\rho}\\ |y_2|+|z_2|\leq (R_2)^{-\rho}}} |\mathcal{K}(x,y,z)|dydz + \int_{\substack{|y_1|+|z_1|\geq (R_1)^{-\rho}\\ |y_2|+|z_2|\geq (R_2)^{-\rho}}} |\mathcal{K}(x,y,z)|dydz$$
$$+ \int_{\substack{|y_1|+|z_1|\leq (R_1)^{-\rho}\\ |y_2|+|z_2|\geq (R_2)^{-\rho}}} |\mathcal{K}(x,y,z)|dydz + \int_{\substack{|y_1|+|z_1|\geq (R_1)^{-\rho}\\ |y_2|+|z_2|\leq (R_2)^{-\rho}}} |\mathcal{K}(x,y,z)|dydz.$$

By using Cauchy-Schwarz inequality, Plancherel's formula and $R_1, R_2 \geq 1$, we have

$$\Big(\int_{\substack{|y_1|+|z_1|\leq (R_1)^{-\rho}\\ |y_2|+|z_2|\leq (R_2)^{-\rho}}} |\mathcal{K}(x,y,z)|dydz\Big)^2$$
$$\lesssim (R_1R_2)^{-2\rho n} \int_{\substack{|y_1|+|z_1|\leq (R_1)^{-\rho}\\ |y_2|+|z_2|\leq (R_2)^{-\rho}}} |\mathcal{K}(x,y,z)|^2 dydz$$
$$\lesssim (R_1R_2)^{-2\rho n} \int_{\substack{|\xi_1|+|\eta_1|\sim R_1\\ |\xi_2|+|\eta_2|\sim R_2}} |\sigma(x,\xi,\eta)|^2 d\xi d\eta$$
$$\lesssim (R_1R_2)^{-2\rho n} \int_{\substack{|\xi_1|+|\eta_1|\sim R_1\\ |\xi_2|+|\eta_2|\sim R_2}} (1+|\xi_1|+|\eta_1|)^m (1+|\xi_2|+|\eta_2|)^m d\xi d\eta$$
$$\lesssim (R_1R_2)^{-2\rho n}(R_1R_2)^{m+2n} = (R_1R_2)^{(2(1-\rho)n+m)},$$

$$\Big(\int_{\substack{|y_1|+|z_1|\geq (R_1)^{-\rho}\\ |y_2|+|z_2|\geq (R_2)^{-\rho}}} |\mathcal{K}(x,y,z)|dydz\Big)^2 \lesssim \Big(\int_{\substack{|y_1|+|z_1|\geq (R_1)^{-\rho}\\ |y_2|+|z_2|\geq (R_2)^{-\rho}}} ||(y_1,z_1)|^{2N}|(y_2,z_2)|^{2N}\mathcal{K}(x,y,z)|^2 dydz\Big)$$
$$\times \Big(\int_{\substack{|y_1|+|z_1|\geq (R_1)^{-\rho}\\ |y_2|+|z_2|\geq (R_2)^{-\rho}}} \frac{1}{|(y_1,z_1)|^{4N}|(y_2,z_2)|^{4N}} dydz\Big)$$
$$\lesssim (R_1R_2)^{\rho(4N-2n)} \int_{\substack{|\xi_1|+|\eta_1|\sim R_1\\ |\xi_2|+|\eta_2|\sim R_2}} |(-\triangle_{\xi_1}-\triangle_{\eta_1})^N(-\triangle_{\xi_2}-\triangle_{\eta_2})^N\sigma(x,\xi,\eta)|^2 d\xi d\eta$$
$$\lesssim (R_1R_2)^{\rho(4N-2n)} \int_{\substack{|\xi_1|+|\eta_1|\sim R_1\\ |\xi_2|+|\eta_2|\sim R_2}} (1+|\xi_1|+|\eta_1|)^{m-\rho 4N}(1+|\xi_2|+|\eta_2|)^{m-\rho 4N} d\xi d\eta$$
$$\lesssim (R_1R_2)^{\rho(4N-2n)}(R_1R_2)^{m-\rho 4N+2n} = (R_1R_2)^{2(1-\rho)n+m},$$

and

$$(\int_{\substack{|y_1|+|z_1|\le(R_1)^{-\rho}\\|y_2|+|z_2|\ge(R_2)^{-\rho}}}|\mathcal{K}(x,y,z)|dydz)^2 \lesssim (\int_{\substack{|y_1|+|z_1|\le(R_1)^{-\rho}\\|y_2|+|z_2|\ge(R_2)^{-\rho}}}||(y_2,z_2)|^{2N}\mathcal{K}(x,y,z)|^2dydz)$$
$$\times(\int_{\substack{|y_1|+|z_1|\le(R_1)^{-\rho}\\|y_2|+|z_2|\ge(R_2)^{-\rho}}}\frac{1}{|(y_2,z_2)|^{4N}}dydz)$$
$$\lesssim {R_1}^{-2\rho n}{R_2}^{\rho(4N-2n)}\int_{\substack{|\xi_1|+|\eta_1|\sim R_1\\|\xi_2|+|\eta_2|\sim R_2}}|(-\triangle_{\xi_2}-\triangle_{\eta_2})^N\sigma(x,\xi,\eta)|^2d\xi d\eta$$
$$\lesssim {R_1}^{-2\rho n}{R_2}^{\rho(4N-2n)}\int_{\substack{|\xi_1|+|\eta_1|\sim R_1\\|\xi_2|+|\eta_2|\sim R_2}}(1+|\xi_1|+|\eta_1|)^m(1+|\xi_2|+|\eta_2|)^{m-\rho 4N}d\xi d\eta$$
$$\lesssim {R_1}^{-2\rho n}{R_2}^{\rho(4N-2n)}{R_1}^{m+2n}{R_2}^{m-\rho 4N+2n}=(R_1R_2)^{2(1-\rho)n+m}.$$

Thus, we are done with part (b).

For part (c) we consider

$$\int_{\mathbb{R}^n}|\mathcal{K}(x,y,z)|dydz=\int_{|y_2|+|z_2|\le(R_2)^{-\rho}}|\mathcal{K}(x,y,z)|dydz+\int_{|y_2|+|z_2|\ge(R_2)^{-\rho}}|\mathcal{K}(x,y,z)|dydz.$$

Then:

$$(\int_{|y_2|+|z_2|\le(R_2)^{-\rho}}|\mathcal{K}(x,y,z)|dydz)^2\lesssim(\int_{|y_2|+|z_2|\le(R_2)^{-\rho}}|(1+|(y_1,z_1)|^2)^N\mathcal{K}(x,y,z)|^2dydz)$$
$$\times(\int_{|y_2|+|z_2|\le(R_2)^{-\rho}}\frac{1}{(1+|(y_1,z_1)|^2)^{2N}}dydz)$$
$$\lesssim {R_2}^{-2\rho n}\int|(1-\triangle_{\xi_1}-\triangle_{\eta_1})^N\sigma(x,\xi,\eta)|^2d\xi d\eta$$
$$\lesssim {R_2}^{-2\rho n}\int_{\substack{|\xi_2|+|\eta_2|\sim R_2\\|\xi_1|+|\eta_1|\le R_1\le 1}}(1+|\xi_1|+|\eta_1|)^m(1+|\xi_2|+|\eta_2|)^m d\xi d\eta$$
$$\lesssim {R_2}^{-2\rho n}{R_2}^{m+2n}{R_1}^{2n}=(R_1)^{2n}(R_2)^{2(1-\rho)n+m}.$$

$$(\int_{|y_2|+|z_2|\geq (R_2)^{-\rho}} |\mathcal{K}(x,y,z)|dydz)^2 \lesssim (\int_{|y_2|+|z_2|\geq (R_2)^{-\rho}} |(1+|(y_1,z_1)|^2)^N|(y_2,z_2)|^{2N}\mathcal{K}(x,y,z)|^2 dydz)$$
$$\times (\int_{|y_2|+|z_2|\geq (R_2)^{-\rho}} \frac{1}{(1+|(y_1,z_1)|^2)^{2N}|(y_2,z_2)|^{4N}} dydz)$$
$$\lesssim (R_2)^{\rho(4N-2n)} \int_{\substack{|\xi_2|+|\eta_2|\sim R_2\\ |\xi_1|+|\eta_1|\leq R_1\leq 1}} |(1-\triangle_{\xi_1}-\triangle_{\eta_1})^N(-\triangle_{\xi_2}-\triangle_{\eta_2})^N\sigma(x,\xi,\eta)|^2 d\xi d\eta$$
$$\lesssim (R_2)^{\rho(4N-2n)} \int_{\substack{|\xi_2|+|\eta_2|\sim R_2\\ |\xi_1|+|\eta_1|\leq R_1\leq 1}} (1+|\xi_1|+|\eta_1|)^m(1+|\xi_2|+|\eta_2|)^{m-\rho 4N} d\xi d\eta$$
$$\lesssim (R_2)^{\rho(4N-2n)}(R_2)^{m-\rho 4N+2n}(R_1)^{2n} = (R_1)^{2n}(R_2)^{2(1-\rho)n+m},$$

where we choose $N > 2n$. □

Now we use the above lemma to prove the boundedness $T_\sigma : L^\infty \times L^\infty \to L^\infty$:

Proof. We take functions $\psi_0(x,y), \psi(x,y) \in \mathcal{S}(\mathbb{R}^2)$ such that $\operatorname{supp}\psi_0 \subseteq \{|x|+|y| \leq 1\}$, $\operatorname{supp}\psi \subseteq \{1/2 \leq |x|+|y| \leq 2\}$ and $\sum_{j=0}^\infty \psi_j(x,y) = 1, x,y \in \mathbb{R}$, where $\psi_j(x,y) = \psi(2^{-j}x, 2^{-j}y), j \in \mathbb{N}_+$. Then we do the decomposition:

$$\sigma(x,\xi,\eta) = \sum_{j,k=0}^\infty \sigma_{jk}(x,\xi,\eta),$$

where $\sigma_{jk}(x,\xi,\eta) = \sigma(x,\xi,\eta)\psi_j(\xi_1,\eta_1)\psi_k(\xi_2,\eta_2)$. By Lemma 3.3, we have

$$\|T_{\sigma_{jk}}(f,g)\|_\infty \lesssim 2^{\frac{j(m+2n(1-\rho))}{2}} 2^{\frac{k(m+2n(1-\rho))}{2}} \|f\|_\infty \|g\|_\infty, \quad j,k \in \mathbb{N}_0.$$

Then when $m < -2n(1-\rho)$, we have

$$\|T_\sigma(f,g)\|_\infty \;\leq\; \sum_{j,k=0}^\infty \|T_{\sigma_{jk}}(f,g)\|_\infty$$

$$\lesssim \sum_{j,k=0}^{\infty} 2^{\frac{j(m+2n(1-\rho))}{2}} 2^{\frac{k(m+2n(1-\rho))}{2}} \|f\|_\infty \|g\|_\infty \lesssim \|f\|_\infty \|g\|_\infty.$$

□

For $\langle T(f,g),h\rangle = \langle T^{*1}(h,g),f\rangle = \langle T^{*2}(f,h),g\rangle$, the following lemma holds

Lemma 3.4. ([36]) *Assume that* $0 \le \delta \le \rho \le 1, \delta < 1$ *and* $\sigma \in BBS^m_{\rho,\delta}$*, then for* $T^{*j}_\delta = T_{\delta^{*j}}$ *,we have* $\sigma^{*j} \in BBS^m_{\rho,\delta} \quad j = 1,2.$

By these lemmas and the duality argument, we have the following boundedness

Corollary 3.5. For T_σ with $m < 2n(\rho - 1)$ and $\sigma \in BBS^m_{\rho,\delta}$, $\delta < 1$ and $0 \le \delta \le \rho \le 1$ we have:

$$T_\sigma(f,g) : L^\infty \times L^\infty \to L^\infty, \quad L^1 \times L^\infty \to L^1, \quad L^\infty \times L^1 \to L^1.$$

3.3 The Boundedness of the Operator $T_\sigma(f,g) : L^2 \times L^2 \to L^1$

In this section, we consider the boundedness of the operator $T_\sigma(f,g) : L^2 \times L^2 \to L^1$. The proof of this result in our bi-parameter setting is rather involved. To prove this boundedness, we need the following lemmas whose proofs can be given without too much difficulty and we will omit them.

Lemma 3.6. *Let* $r_1, r_2 > 0$*, and let* N *be a sufficiently large integer. Suppose* $\sigma(x,\xi,\eta)$ *satisfies either of the following conditions:*

(a) $|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\beta_1}_{\xi_1}\partial^{\beta_2}_{\xi_2}\partial^{\gamma_1}_{\eta_1}\partial^{\gamma_2}_{\eta_2}\sigma(x,\xi,\eta)| \le (r_1 r_2)^{\frac{-n}{2}} \chi_{\{|\xi_1| \le r_1\}}\chi_{\{|\xi_2| \le r_2\}}$*, for all* $|\alpha|, |\beta|, |\gamma| \le N$ *;*

(b) $|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\beta_1}_{\xi_1}\partial^{\beta_2}_{\xi_2}\partial^{\gamma_1}_{\eta_1}\partial^{\gamma_2}_{\eta_2}\sigma(x,\xi,\eta)| \le (r_1 r_2)^{\frac{-n}{2}} \chi_{\{|\eta_1| \le r_1\}}\chi_{\{|\eta_2| \le r_2\}}$*, for all* $|\alpha|, |\beta|, |\gamma| \le N$*,*

where $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2), \gamma = (\gamma_1, \gamma_2)$ *are multi-indices. Then*

$$\|T_\sigma(f,g)\|_{L^1} \le C\|f\|_{L^2}\|g\|_{L^2}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^2)$*, where* N *and* C *can be taken independent of* r_1, r_2 *and depending only on* n*.*

A one-parameter version of this lemma can be found in [70].

Theorem 3.7. *The operators of class* $Op(BBS^m_{0,0})$ *with* $m < -n$ *are bounded in* $L^2 \times L^2 \to L^1$*.*

Proof. For $\sigma \in BBS^m_{0,0}$, we keep using the decomposition in the proof of Theorem 3.2

$$\sigma(x,\xi,\eta) = \sum_{j,k=0}^{\infty} \sigma(x,\xi,\eta)\psi_j(\xi_1,\eta_1)\psi_k(\xi_2,\eta_2) = \sum_{j,k=0}^{\infty} \sigma_{jk}(x,\xi,\eta),$$

where the symbol σ_{jk} satisfies the condition

$$|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\beta_1}_{\xi_1}\partial^{\beta_2}_{\xi_2}\partial^{\gamma_1}_{\eta_1}\partial^{\gamma_2}_{\eta_2}\sigma_{jk}(x,\xi,\eta)| \lesssim (2^j)^{\frac{m}{2}}(2^k)^{\frac{m}{2}}\chi_{\{|\xi_1|+|\eta_1|\lesssim 2^j\}}\chi_{\{|\xi_2|+|\eta_2|\lesssim 2^k\}}.$$

Then by Lemma 3.6, there holds

$$\|T_{\sigma_{jk}}(f,g)\|_{L^1} \lesssim (2^j)^{\frac{m+n}{2}}(2^k)^{\frac{m+n}{2}}\|f\|_{L^2}\|g\|_{L^2}.$$

When $m < -n$, we have

$$\sum_{j,k=0}^{\infty} \|T_{\sigma_{jk}}(f,g)\|_{L^1} \lesssim \sum_{j,k=0}^{\infty} (2^j)^{\frac{m+n}{2}} (2^k)^{\frac{m+n}{2}} \|f\|_{L^2}\|g\|_{L^2} \approx \|f\|_{L^2}\|g\|_{L^2}.$$

□

Now that we have finished the proof of the boundedness of $L^2 \times L^2 \to L^1$, then by the duality argument, we also have

Corollary 3.8. For $\sigma(x,\xi,\eta) \in BBS^m_{0,0}$ with $m < -n$, we have:

$$T_\sigma(f,g) : L^2 \times L^2 \to L^1, \quad L^2 \times L^\infty \to L^2, \quad L^\infty \times L^2 \to L^2.$$

3.4 Proof of the Main Theorem

The following interpolation result follows from the complex interpolation method of the classes $BBS^m_{0,0}$ (see [78]).

Lemma 3.9. *For $m_0, m_1 \in \mathbb{R}$ and any $\theta \in (0,1)$,*

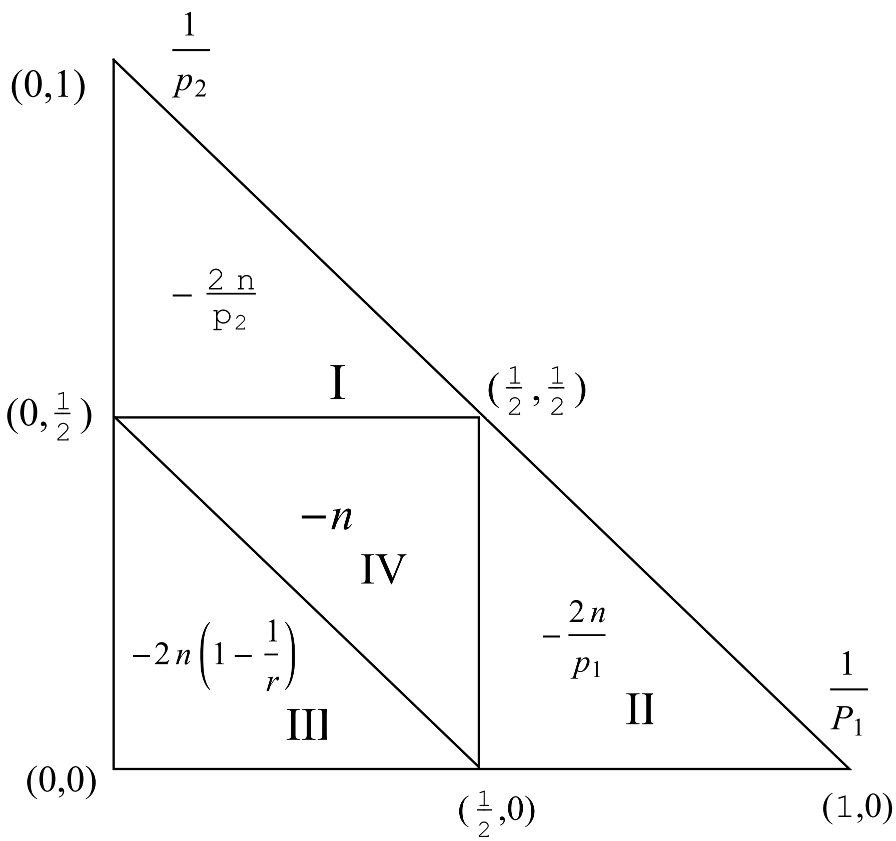
(i) $(BBS^{m_0}_{0,0}, BBS^{m_1}_{0,0})_{[\theta]} = BBS^m_{0,0}$.

(ii) If the operators $Op(BBS^{m_i}_{0,0})$ are bounded in $L^{p_i} \times L^{q_i} \to L^{r_i}$ with $1 \le p_i, q_i, r_i \le \infty$ and $1/p_i + 1/q_i = 1/r_i, i = 0,1$.

Then the operators $Op(BBS^m_{0,0})$ are bounded in $L^p \times L^q \to L^r$, where $(m,p,q,r) = \theta(m_0,p_0,q_0,r_0) + (1-\theta)(m_1,p_1,q_1,r_1)$.

By using the above interpolation lemma, we can complete the proof for our main theorem.

Proof. For (a), use Corollary 3.5 and Corollary 3.8, we have the following interpolation graph:



In graph above, we divide the triangle into four parts. When$(\frac{1}{p_1}, \frac{1}{p_2})$ is taken from each of the four regions, the corresponding upper bound for m is shown there.

For (b): Note that we have $BS^m_{0,0} \subseteq BBS^m_{0,0}$ for $m \leq 0$, so this follows directly from the result when $\sigma(x, \xi, \eta) \in BS^m_{0,0}$in [70]. $\square$

CHAPTER 4 EQUIVALENCE OF TRUDINGER-MOSER INEQUALITIES

4.1 Introduction

In this section, we will begin with giving an overview of the state of affairs of the best constants for sharp Trudinger-Moser and Adams inequalities. section 5.1.1 concerns the sharp Trudinger-Moser inequalities and section 5.1.2 discusses the sharp Adams inequalities involving high order derivatives. In section 5.1.3, we will state our main results on the equivalence between critical and subcritical Trudinger-Moser and Adams inequalities.

4.1.1 Trudinger-Moser Inequalities

Motivated by the applications to the prescribed Gauss curvature problem on two dimensional sphere $\mathbb{S}^2$, J. Moser proved in [71] an exponential type inequality on $\mathbb{S}^2$ with an optimal constant. In the same paper, he sharpened an inequality on any bounded domain Ω in the Euclidean space $\mathbb{R}^N$ studied independently by Pohozaev [80], Trudinger [90] and Yudovich [91], namely the embedding $W_0^{1,N}(\Omega) \subset L_{\varphi_N}(\Omega)$, where $L_{\varphi_N}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_N(t) = \exp\left(\alpha\,|t|^{N/(N-1)}\right) - 1$ for some $\alpha > 0$. More precisely, using the Schwarz rearrangement, Moser first proved the following inequality:

Theorem (Moser, 1971). *Let Ω be a domain with finite measure in Euclidean $N-$space $\mathbb{R}^N$, $n \geq 2$. Then there exists a constant $\alpha_N > 0$, such that*

$$\frac{1}{|\Omega|}\int_{\Omega} \exp\left(\alpha_N\,|u|^{\frac{N}{N-1}}\right) dx \leq c_0 \tag{4.1}$$

for any $u \in W_0^{1,N}(\Omega)$ with $\int_{\Omega} |\nabla u|^N\,dx \leq 1$. The constant $\alpha_N = \omega_{N-1}^{\frac{1}{N-1}}$, where ω_{N-1} is the area of the surface of the unit $N-$ ball, is optimal in the sense that if we replace α_N by any *number* $\alpha > \alpha_N$, *then the above inequality can no longer hold with some* c_0 *independent of* u.

Moser used the following symmetrization argument: every function u is associated to a radially symmetric function u^* such that the sublevel-sets of u^* are balls with the same area as the corresponding sublevel-sets of u. Moreover, u is a positive and non-increasing function defined on $B_R(0)$ where $|B_R(0)| = |\Omega|$. Hence, by the layer cake representation, we can have that

$$\int_\Omega f(u)\,dx = \int_{B_R(0)} f(u^*)\,dx$$

for any function f that is the difference of two monotone functions. In particular, we obtain

$$\|u\|_p = \|u^*\|_p\,;$$
$$\int_\Omega \exp\left(\alpha\,|u|^{\frac{n}{n-1}}\right)dx = \int_{B_R(0)} \exp\left(\alpha\,|u^*|^{\frac{n}{n-1}}\right)dx.$$

Moreover, the well-known Pólya-Szegö inequality

$$\int_{B_R(0)} |\nabla u^*|^p\,dx \leq \int_\Omega |\nabla u|^p\,dx \tag{4.2}$$

plays a crucial role in the approach of J. Moser.

Moser's result has been studied and extended in many directions. For instance, we refer the reader to the sharp Moser inequality with mean value zero by Chang and Yang [14], Lu and Yang [67], Leckband [57], sharp Trudinger-Moser trace inequalities and sharp Trudinger-Moser inequalities without boundary conditions by Cianchi [18,19], Trudinger-Moser inequality for Hessians by Tian and Wang [88], etc. We also refer to the survey articles of Chang and Yang [15] and Lam and Lu [47] for descriptions of applications of such inequalities to nonlinear PDEs.

Recently, using the L^p affine energy $\mathcal{E}_p(f)$ of f instead of the standard L^p energy of gradient $\|\nabla f\|_p$, Cianchi, Lutwak, Yang and Zhang proved in [20] a sharp version of affine Trudinger-Moser inequality by replacing the constraint $||\nabla f||_n \leq 1$ by $\mathcal{E}_p(f) \leq 1$ in Moser's inequality.

Moser's inequality has also been extended to the singular case $0 \leq \beta < N$:

$$\frac{1}{|\Omega|^{1-\frac{\beta}{N}}} \int_{\Omega} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right) dx \leq c_0 \tag{4.3}$$

for any $\alpha \leq \alpha_N$, *any* $u \in W_0^{1,N}(\Omega)$ *with* $\int_{\Omega} |\nabla u|^N dx \leq 1$. *This constant* α_N *is sharp in the sense that if* $\alpha > \alpha_N$, *then the above inequality can no longer hold with some* c_0 *independent of* u.

As far as the existence of extremal functions of Moser's inequality, the first breakthrough was due to the celebrated work of Carleson and Chang [12] in which they proved that the supremum

$$\sup_{u \in W_0^{1,N}(\Omega), \int_{\Omega} |\nabla u|^N dx \leq 1} \frac{1}{|\Omega|} \int_{\Omega} \exp\left(\alpha_N |u|^{\frac{N}{N-1}}\right) dx$$

can be achieved when Ω is an Euclidean ball. This result came as a surprise because it has been known that the Sobolev inequality does not have extremal functions supported on any finite ball. Subsequently, existence of extremal functions has been established on arbitrary domains in [30], [62], and on Riemannian manifolds in [59, 60], etc.

We note when the volume of Ω is infinite, the Trudinger-Moser inequality (4.3) becomes meaningless. Thus, it becomes interesting and nontrivial to extend such inequalities to unbounded domains. Here we state the following two such results in the Euclidean spaces.

We first recall the subcritical Trudinger-Moser inequality in the Euclidean spaces established by Adachi and Tanaka [1].

Theorem (1999, [1]). *For any $\alpha < \alpha_N$, there exists a positive constant $C_{N,\alpha}$ such that* $\forall u \in W^{1,N}\left(\mathbb{R}^N\right),\ \|\nabla u\|_N \leq 1:$

$$\int_{\mathbb{R}^N} \phi_N\left(\alpha\, |u|^{\frac{N}{N-1}}\right) dx \leq C_{N,\alpha}\, \|u\|_N^N\,, \tag{4.4}$$

where

$$\phi_N(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}.$$

The constant α_N is sharp in the sense that the supremum is infinity when $\alpha \geq \alpha_N$.

We note in the above theorem, we only impose the restriction on the norm $\int_{\mathbb{R}^N} |\nabla u|^N$ without restricting the full norm

$$\left[\int_{\mathbb{R}^N} |\nabla u|^N + \tau \int_{\mathbb{R}^N} |u|^N\right]^{1/N} \leq 1.$$

The method in [1] requires a symmetrization argument which is not available in many other non-Euclidean settings. The above inequality fails at the critical case $\alpha = \alpha_N$. So it is natural to ask when the above can be true when $\alpha = \alpha_N$. This is done in [81], [61] by using the restriction of the full norm of the non-isotropic Sobolev space $W^{1,N}\left(\mathbb{R}^N\right)$: $\left[\int_{\mathbb{R}^N} |\nabla u|^N + \tau \int_{\mathbb{R}^N} |u|^N\right]^{1/N}$.

Theorem (2005, [81]; 2008, [61]). *For all* $0 \le \alpha \le \alpha_N$:

$$\sup_{\|u\| \le 1} \int_{\mathbb{R}^N} \phi_N \left(\alpha \left| u \right|^{\frac{N}{N-1}} \right) dx < \infty \tag{4.5}$$

where

$$\|u\| = \left(\int_{\mathbb{R}^N} \left(\left| \nabla u \right|^N + \left| u \right|^N \right) dx \right)^{1/N} .$$

Moreover, this constant α_N *is sharp in the sense that if* $\alpha > \alpha_N$*, then the supremum is infinity.*

More results about the Trudinger-Moser inequalities on the Heisenberg groups could be found in [53, 54, 56]. It is worth noting that the above results on subcritical and critical Trudinger-Moser inequalities were proved by using symmetrization arguments, and later Lam and Lu [51], Lam, Lu and Tang [54] avoided the use of symmetrizationproved to prove such results via level sets, which enabled them to establish such inequalities on more general settings rather than the Euclidean space, such as Heisenberg groups.

The inequality (4.4) uses the seminorm $\|\nabla u\|_N$ and hence fails at the critical case $\alpha = \alpha_N$, the best constant. Thus, it can be considered as a sharp subcritical Trudinger-Moser inequality. In (4.5), when using the full norm of $W^{1,N}\left(\mathbb{R}^N\right)$, the best constant could be attained. Namely, the inequality holds at the critical case $\alpha = \alpha_N$. Hence, (4.5) is the sharp critical Trudinger-Moser inequality.

Nevertheless, our main purpose is to show that in fact, these two versions of critical and subcritical Trudinger-Moser type inequalities are indeed equivalent. Hence, since Theorem C is easier to study than Theorem B, our work suggests a new approach to the critical

Trudinger-Moser type inequality.

Sharp Trudinger-Moser inequalities on unbounded domains of the Heisenberg groups were also established by Lam, Lu and Tang [50,54,56]. We also mention that extremal functions for Trudinger-Moser inequalities on bounded domains were studied by Carleson and Chang [12], de Figueiredo, do Ó, and Ruf [27], Flucher [30], Lin [62], and on Riemannian manifolds by Y. X. Li [59,60], and on unbounded domains by Ruf [81], Li-Ruf [61], Ishiwata [40], Ishiwata, Nakamura and Wadade [41] and Dong, Lu [28].

4.1.2 Adams Inequalities

It is worthy noting that symmetrization has been a very useful and efficient (and almost inevitable) method when dealing with the sharp geometric inequalities. Thus, it is very fascinating to investigate such sharp geometric inequalities, in particular, the Trudinger-Moser type inequalities, in the settings where the symmetrization is not available such as on the higher order Sobolev spaces, the Heisenberg groups, Riemannian manifolds, sub-Riemannian manifolds, etc. Indeed, in these settings, an inequality like (4.2) is not available. In these situations, the first break-through came from the work of D. Adams [2] when he attempted to set up the Trudinger-Moser inequality in the higher order setting in Euclidean spaces. In fact, using a new idea that one can write a smooth function as a convolution of a (Riesz) potential with its derivatives, and then one can use the symmetrization for this convolution, instead of the symmetrization of the higher order derivatives, Adams proved the following inequality with boundary Dirichlet condition [2], and Tarsi extended it to the Navier boundary condition [86] when $\beta = 0$, and then Lam and Lu extended it to the case $0 \leq \beta < N$ [49].

Theorem (Lam-Lu) *Let Ω be an open and bounded set in $\mathbb{R}^N$. If m is a positive integer* *less than* N, $0 \leq \beta < N$, *then there exists a constant* $C_0 = C(N, m, \beta) > 0$ *such that for any* $u \in W_N^{m,\frac{N}{m}}(\Omega)$ *and* $||\nabla^m u||_{L^{\frac{N}{m}}(\Omega)} \leq 1$, *then*

$$\frac{1}{|\Omega|^{1-\frac{\beta}{N}}} \int_\Omega \exp(\alpha \left(1 - \frac{\beta}{N}\right) |u(x)|^{\frac{N}{N-m}}) \frac{dx}{|x|^\beta} \leq C_0$$

for all $\beta \leq \beta(N, m)$ *where*

$$\beta(N,\ m) \ = \begin{cases} \frac{N}{w_{N-1}} \left[\frac{\pi^{N/2} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{N-m+1}{2})} \right]^{\frac{N}{N-m}} & \text{when } m \text{ is odd} \\ \frac{N}{w_{N-1}} \left[\frac{\pi^{N/2} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{N-m}{2})} \right]^{\frac{N}{N-m}} & \text{when } m \text{ is even} \end{cases}.$$

Furthermore, the constant $\beta(N, m)$ *is optimal in the sense that for any* $\alpha > \beta(N, m)$, *the integral can be made as large as possible.*

The Adams inequalities for high order derivatives on domains of infinite volume were studied by Ogawa [76], Ozawa [77], Kozono, Sato and Wadade [45] with non-optimal constants. The sharp constants were recently established by Ruf and Sani [82] in the case of even order derivatives and by Lam and Lu in all order of derivatives including fractional orders [48, 49, 51, 54]. The idea of [82] is to use the comparison principle for polyharmonic equations (thus could deal with the case of even order of derivatives) and thus involves some difficult construction of auxiliary functions. The argument in [48, 49] uses the representation of the Bessel potentials and thus avoids dealing with such a comparison principle. Moreover, the argument in [49] does not use the symmetrization method and thus also works for the sub-Riemannian setting such as the Heisenberg groups [50, 52]. More results in this direction were proved in [8, 21, 50, 54, 56]. The following general version is taken from [51].

Theorem (Lam-Lu, 2013 [51]). *Let* m *be a positive integer less than* N, $0 \leq \beta < N$, *then there exists a constant* $C_0 = C(N, m, \beta) > 0$ *such that for any* $u \in W^{m,\frac{N}{m}}(\mathbb{R}^N)$ *and* $||\,(-I+\Delta)^{\frac{m}{2}}\,u||_{\frac{N}{m}} \leq 1$, *then*

$$\int_{\mathbb{R}^N} \phi_{N,m}(\beta_0\,(N,m)\left(1-\frac{\beta}{N}\right)|u(x)|^{\frac{N}{N-m}})\frac{dx}{|x|^{\beta}} \leq C_0.$$

Here

$$\begin{aligned} \beta_0\,(N,m) \; &= \frac{N}{w_{N-1}}\left[\frac{\pi^{N/2}2^m\Gamma(\frac{m}{2})}{\Gamma(\frac{N-m}{2})}\right]^{\frac{N}{N-m}}, \\ \phi_{N,m}\,(t) &= \sum_{j\in\mathbb{N}:j\geq\frac{N-m}{m}}\frac{t^j}{j!}. \end{aligned}$$

Furthermore the constant $\beta_0\,(N,m)$ *is optimal in the sense that if it is replaced by any number larger than* $\beta_0\,(N,m)$, *then the above inequality no longer holds with a constant* C_0 *independent of* u.

Sharp Trudinger-Moser inequalities were also recently established on hyperbolic spaces by Mancini and Sandeep [69] on conformal discs and by Lu and Tang in all dimensions [63,64] including singular versions of subcritical type inequalities [1] and those of critical type [61,81]. Sharp Trudinger-Moser inequalities on infinite volume domains of the Heisenberg groups were also established by Lam, Lu and Tang [50, 54, 56].

Very little is known for existence of extremals for Adams inequalities. The only known cases are in the second order derivatives on compact Riemannian manifolds and bounded domains in dimension four by Li and Ndiaye [58] and Lu and Yang [66] respectively, and other cases are still widely open. Adams inequalities have been extended to many other settings such as on the compact Riemannian manifolds in [32], spheres in [5], CR spheres in [23], [8], etc.

Our work mainly focus on the equivalence of the subcritical and critical Trudinger-Moser inequalities, as well as the equivalence of some subcritical and critical Adam's inequalties. The paper will appear in Rev. Mat. Iberoam.

4.1.3 Our Main Results

We begin with an improved sharp subcritical Trudinger-Moser inequality:

Theorem 4.1. *Let $N \geq 2$, $\alpha_N = N\left(\frac{N\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}\right)^{\frac{1}{N-1}}$, $0 \leq \beta < N$ and $0 \leq \alpha < \alpha_N$. Denote*

$$AT(\alpha,\beta) = \sup_{\|\nabla u\|_N \leq 1} \frac{1}{\|u\|_N^{N-\beta}} \int_{\mathbb{R}^N} \phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta}.$$

Then there exist positive constants $c = c(N,\beta)$ and $C = C(N,\beta)$ such that when α is close enough to α_N :

$$\frac{c(N,\beta)}{\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{(N-\beta)/N}} \leq AT(\alpha,\beta) \leq \frac{C(N,\beta)}{\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{(N-\beta)/N}}. \tag{4.6}$$

Moreover, the constant α_N is sharp in the sence that $AT(\alpha_N,\beta) = \infty$.

Then we will provide another proof to the sharp critical Trudinger-Moser inequality using Theorem 4.1 only.

Theorem 4.2. *Let $N \geq 2$, $0 \leq \beta < N$, $0 < a$, b. Denote*

$$MT_{a,b}(\beta) = \sup_{\|\nabla u\|_N^a + \|u\|_N^b \leq 1} \int_{\mathbb{R}^N} \phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta};$$

$$MT(\beta) = MT_{N,N}(\beta).$$

Then $MT_{a,b}(\beta) < \infty$ if and only if $b \leq N$. The constant α_N is sharp. Moreover, we have the following identity:

$$MT_{a,b}(\beta) = \sup_{\alpha \in (0,\alpha_N)} \left(\frac{1 - \left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}}{\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}} \right)^{\frac{N-\beta}{b}} AT(\alpha, \beta). \tag{4.7}$$

In particular, $MT(\beta) < \infty$ and

$$MT(\beta) = \sup_{\alpha \in (0,\alpha_N)} \left(\frac{1 - \left(\frac{\alpha}{\alpha_N}\right)^{N-1}}{\left(\frac{\alpha}{\alpha_N}\right)^{N-1}} \right)^{\frac{N-\beta}{N}} AT(\alpha, \beta).$$

We now consider the sharp subcritical and critical Adams inequalities on $W^{2,\frac{N}{2}}\left(\mathbb{R}^N\right)$, $N \geq 3$. Our first result is the following sharp subcritical Adams inequality:

Theorem 4.3. *Let $N \geq 3$, $0 \leq \beta < N$ and $0 \leq \alpha < \beta(N,2)$. Denote*

$$ATA(\alpha, \beta) = \sup_{\|\Delta u\|_{\frac{N}{2}} \leq 1} \frac{1}{\|u\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}} \int_{\mathbb{R}^N} \frac{\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-2}}\right)}{|x|^{\beta}} dx;$$

$$\phi_{N,2}(t) = \sum_{j \in \mathbb{N}: j \geq \frac{N-2}{2}} \frac{t^j}{j!}.$$

Then there exist positive constants $c = c(N,\beta)$ and $C = C(N,\beta)$ such that when α is close enough to $\beta(N,2)$:

$$\frac{c(N,\beta)}{\left[1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}\right]^{1-\frac{\beta}{N}}} \leq ATA(\alpha,\beta) \leq \frac{C(N,\beta)}{\left[1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}\right]^{1-\frac{\beta}{N}}}. \tag{4.8}$$

Moreover, the constant $\beta(N,2)$ is sharp in the sence that $AT(\alpha_N, \beta) = \infty$.

Theorem 4.4. *Let $N \geq 3$, $0 \leq \beta < N$, $0 < a,\ b$. We denote:*

$$A_{a,b}(\beta) = \sup_{\|\Delta u\|_{\frac{N}{2}}^{a} + \|u\|_{\frac{N}{2}}^{b} \leq 1} \int_{\mathbb{R}^N} \frac{\phi_{N,2}\left(\beta(N,2)\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-2}}\right)}{|x|^{\beta}} dx;$$

$$A_{\frac{N}{2},\frac{N}{2}}(\beta) = A(\beta);$$

Then $A_{a,b}(\beta) < \infty$ if and only if $b \leq \frac{N}{2}$. The constant $\beta(N,2)$ is sharp. Moreover, we have the following identity:

$$A_{a,b}(\beta) = \sup_{\alpha \in (0,\beta(N,2))} \left(\frac{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}}{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}} \right)^{\frac{N-\beta}{2b}} ATA(\alpha,\beta). \tag{4.9}$$

In particular, $A(\beta) < \infty$ and

$$A(\beta) = \sup_{\alpha \in (0,\beta(N,2))} \left(\frac{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}}{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}} \right)^{\frac{N-\beta}{N}} ATA(\alpha,\beta).$$

Finally, we will study the following improved sharp critical Adams inequality under the assumption that a version of the sharp subcritical Adams inequality holds:

Theorem 4.5. *Let $0 < \gamma < N$ be an arbitrary real positive number, $p = \frac{N}{\gamma}$, $0 \leq \alpha < \beta_0(N,\gamma) = \frac{N}{\omega_{N-1}}\left[\frac{\pi^{\frac{N}{2}} 2^{\gamma} \Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{N-\gamma}{2}\right)}\right]^{\frac{p}{p-1}}$, $0 \leq \beta < N$, $0 < a,\ b$. We note*

$$GATA(\alpha,\beta) = \sup_{u \in W^{\gamma,p}(\mathbb{R}^N):\left\|(-\Delta)^{\frac{\gamma}{2}}u\right\|_p \leq 1} \frac{1}{\|u\|_p^{p\left(1-\frac{\beta}{N}\right)}} \int_{\mathbb{R}^N} \frac{\phi_{N,\gamma}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{p}{p-1}}\right)}{|x|^{\beta}} dx;$$

$$GA_{a,b}(\beta) = \sup_{u\in W^{\gamma,p}(\mathbb{R}^N):\left\|(-\Delta)^{\frac{\gamma}{2}}u\right\|_p^a+\|u\|_p^b\leq 1} \int_{\mathbb{R}^N} \frac{\phi_{N,\gamma}\left(\beta_0(N,\gamma)\left(1-\frac{\beta}{N}\right)|u|^{\frac{p}{p-1}}\right)}{|x|^\beta}dx$$

where

$$\phi_{N,\gamma}(t) = \sum_{j\in\mathbb{N}:j\geq p-1} \frac{t^j}{j!}.$$

Assume that $GATA(\alpha,\beta) < \infty$ *and there exists a constant* $C(N,\gamma,\beta) > 0$ *such that*

$$GATA(\alpha,\beta) \leq \frac{C(N,\gamma,\beta)}{\left(1-\left(\frac{\alpha}{\beta_0(N,\gamma)}\right)^{\frac{p-1}{p}}\right)} \tag{4.10}$$

Then when $b \leq p$*, we have* $GA_{a,b}(\beta) < \infty$*. In particular* $GA_{p,p}(\beta) < \infty$*.*

Though we have to assume a sharp subcritical Adams inequality (4.10), the main idea of Theorem 4.5 is that since $GATA(\alpha,\beta)$ is actually subcritical, i.e. α is strictly less than the critical level $\beta_0(N,\gamma)$, it is easier to study than $GA_{a,b}(\beta)$. Hence, it suggests a new approach in the study of $GA_{a,b}(\beta)$.

4.2 Some Lemata

Lemma 4.6.

$$AT(\alpha,\beta) = \sup_{\|\nabla u\|_N\leq 1;\|u\|_N=1} \int_{\mathbb{R}^N} \phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta}.$$

Proof. For any $u \in W^{1,N}\left(\mathbb{R}^N\right) : \|\nabla u\|_N \leq 1; \|u\|_N = 1$, we define

$$v(x) = u(\lambda x)$$

$$\lambda = \|u\|_N.$$

Then,

$$\nabla v\left(x\right)=\lambda\nabla u\left(\lambda x\right).$$

Hence

$$\left\Vert \nabla v\right\Vert _{N}=\left\Vert \nabla u\right\Vert _{N}\leq1;\left\Vert v\right\Vert _{N}=1,$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^{N}}\phi_{N}\left(\alpha\left(1-\frac{\beta}{N}\right)\left|v\left(x\right)\right|^{\frac{N}{N-1}}\right)\frac{dx}{\left|x\right|^{\beta}}\\
&=\int_{\mathbb{R}^{N}}\phi_{N}\left(\alpha\left(1-\frac{\beta}{N}\right)\left|u\left(\lambda x\right)\right|^{\frac{N}{N-1}}\right)\frac{dx}{\left|x\right|^{\beta}}\\
&=\frac{1}{\lambda^{N-\beta}}\int_{\mathbb{R}^{N}}\phi_{N}\left(\alpha\left(1-\frac{\beta}{N}\right)\left|u\left(\lambda x\right)\right|^{\frac{N}{N-1}}\right)\frac{d\left(\lambda x\right)}{\left|\lambda x\right|^{\beta}}\\
&=\frac{1}{\left\Vert u\right\Vert _{N}^{N-\beta}}\int_{\mathbb{R}^{N}}\phi_{N}\left(\alpha\left(1-\frac{\beta}{N}\right)\left|u\right|^{\frac{N}{N-1}}\right)\frac{dx}{\left|x\right|^{\beta}}.
\end{aligned}$$

□

By Lemma 4.6, we can always assume $\left\Vert u\right\Vert _{N}=1$ in the sharp subcritical Trudinger-Moser inequality.

Lemma 4.7. *The sharp subcritical Trudinger-Moser inequality is a consequence of the sharp critical Trudinger-Moser inequality. More precisely, if $MT_{a,b}\left(\beta\right)$ is finite, then $AT\left(\alpha,\beta\right)$ is finite. Moreover,*

$$AT\left(\alpha,\beta\right)\leq\left(\frac{\left(\frac{\alpha}{\alpha_{N}}\right)^{\frac{N-1}{N}b}}{1-\left(\frac{\alpha}{\alpha_{N}}\right)^{\frac{N-1}{N}a}}\right)^{\frac{N-\beta}{b}}MT_{a,b}\left(\beta\right). \tag{4.11}$$

In particular,

$$AT(\alpha,\beta) \le \left(\frac{\left(\frac{\alpha}{\alpha_N}\right)^{N-1}}{1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}} \right)^{1-\frac{\beta}{N}} MT(\beta).$$

Proof. Let $u \in W^{1,N}\left(\mathbb{R}^N\right):\ \|\nabla u\|_N \le 1;\ \|u\|_N = 1$. Set

$$v(x) = \left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}} u(\lambda x)$$

$$\lambda = \left(\frac{\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}}{1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}} \right)^{1/b}.$$

then

$$\|\nabla v\|_N^a = \left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a} \|\nabla u\|_N^a \le \left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}$$

$$\|v\|_N^b = \left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b} \frac{1}{\lambda^b} \|u\|_N^b = 1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}.$$

Hence $\|\nabla v\|_N^a + \|v\|_N^b \le 1$. By the definition of $MT_{a,b}(\beta)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \phi_N\left(\alpha(1-\frac{\beta}{N})\,|u|^{N/(N-1)}\right)\frac{dx}{|x|^\beta} \\
&= \int_{\mathbb{R}^N} \phi_N\left(\alpha(1-\frac{\beta}{N})\,|u(\lambda x)|^{N/(N-1)}\right)\frac{d(\lambda x)}{|\lambda x|^\beta} \\
&= \lambda^{N-\beta}\int_{\mathbb{R}^N} \phi_N\left(\alpha_N(1-\frac{\beta}{N})\,|v|^{N/(N-1)}\right)\frac{dx}{|x|^\beta} \\
&\le \left(\frac{\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}}{1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}} \right)^{\frac{N-\beta}{b}} MT_{a,b}(\beta).
\end{aligned}$$

□

Lemma 4.8.

$$ATA\left(\alpha,\beta\right)=\sup_{\|\Delta u\|_{\frac{N}{2}}\leq 1;\|u\|_{\frac{N}{2}}=1}\int_{\mathbb{R}^{N}}\frac{\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-2}}\right)}{|x|^{\beta}}dx.$$

Proof. Let $u\in W^{2,\frac{N}{2}}\left(\mathbb{R}^{N}\right):\|\Delta u\|_{\frac{N}{2}}\leq 1$ and set

$$v\left(x\right)=u\left(\lambda x\right);$$
$$\lambda=\|u\|_{\frac{N}{2}}^{\frac{1}{2}}$$

Then it is easy to check that

$$\Delta v\left(x\right)=\lambda^{2}\Delta u\left(\lambda x\right)$$

and

$$\|\Delta v\|_{\frac{N}{2}}=\|\Delta u\|_{\frac{N}{2}};$$
$$\|v\|_{\frac{N}{2}}^{\frac{N}{2}}=\int_{\mathbb{R}^{N}}|v\left(x\right)|^{\frac{N}{2}}dx=\int_{\mathbb{R}^{N}}|u\left(\lambda x\right)|^{\frac{N}{2}}dx=\frac{1}{\lambda^{N}}\int_{\mathbb{R}^{N}}|u\left(x\right)|^{\frac{N}{2}}dx=1.$$

Moreover

$$\begin{aligned}\int_{\mathbb{R}^{N}}\frac{\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|v|^{\frac{N}{N-2}}\right)}{|x|^{\beta}}dx&=\int_{\mathbb{R}^{N}}\frac{\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|u\left(\lambda x\right)|^{\frac{N}{N-2}}\right)}{|x|^{\beta}}dx\\&=\frac{1}{\lambda^{N-\beta}}\int_{\mathbb{R}^{N}}\frac{\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|u\left(x\right)|^{\frac{N}{N-2}}\right)}{|x|^{\beta}}dx\\&=\frac{1}{\|u\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}}\int_{\mathbb{R}^{N}}\frac{\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-2}}\right)}{|x|^{\beta}}dx.\end{aligned}$$

□

Lemma 4.9. *Assume $A_{a,b}(\beta) < \infty$, then $ATA(\alpha,\beta) < \infty$. Moreover,*

$$ATA(\alpha,\beta) \leq \left(\frac{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}} \right)^{\frac{N-\beta}{2b}} A_{a,b}(\beta). \tag{4.12}$$

In particular, if $A(\beta) < \infty$, then

$$ATA(\alpha,\beta) \leq \left(\frac{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}}{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}} \right)^{\frac{N-\beta}{N}} A(\beta).$$

Proof. Let $u \in W^{2,\frac{N}{2}}\left(\mathbb{R}^N\right) : \|\Delta u\|_{\frac{N}{2}} \leq 1$ and $\|u\|_{\frac{N}{2}} = 1$. We define

$$v(x) = \left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}} u(\lambda x)$$

$$\lambda = \left(\frac{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}} \right)^{\frac{1}{2b}}.$$

then

$$\|\Delta v\|_{\frac{N}{2}} = \left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}} \|\Delta u\|_{\frac{N}{2}} \leq \left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}}$$

$$\|v\|_{\frac{N}{2}}^b = \left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b} \frac{1}{\lambda^{2b}} \|u\|_{\frac{N}{2}}^b = 1 - \left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}.$$

Hence $\|\Delta v\|_{\frac{N}{2}}^{a} + \|v\|_{\frac{N}{2}}^{b} \leq 1$. By the definition of $A_{a,b}(\beta)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \phi_{N,2}\left(\alpha(1-\frac{\beta}{N})\left|u\right|^{N/(N-2)}\right)\frac{dx}{|x|^{\beta}} \\
&= \int_{\mathbb{R}^N} \phi_{N,2}\left(\alpha(1-\frac{\beta}{N})\left|u(\lambda x)\right|^{N/(N-2)}\right)\frac{d(\lambda x)}{|\lambda x|^{\beta}} \\
&= \lambda^{N-\beta}\int_{\mathbb{R}^N} \phi_{N}\left(\alpha_N(1-\frac{\beta}{N})\left|v\right|^{N/(N-2)}\right)\frac{dx}{|x|^{\beta}} \\
&\leq \left(\frac{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}}\right)^{\frac{N-\beta}{2b}} A_{a,b}(\beta).
\end{aligned}$$

□

4.3 Trudinger-Moser Inequalities of Adachi-Tanaka Type

In this section, we will prove the improved sharp subcritical Trudinger-Moser inequality. We would like to note here that we don't assume $MT(\beta) < \infty$ in the proof of Theorem 4.1.

Proof of Theorem 4.1. Suppose that $u \in C_0^{\infty}\left(\mathbb{R}^N\right) \setminus \{0\}$, $u \geq 0$, $\|\nabla u\|_N \leq 1$ and $\|u\|_N = 1$. Let

$$\Omega = \left\{ x : u(x) > \left(1 - \left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{\frac{1}{N}} \right\}.$$

Then the volume of Ω can be estimated as follows:

$$|\Omega| = \int_{\Omega} 1 dx \leq \int_{\Omega} \frac{u(x)^N}{1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}} dx \leq \frac{1}{1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}}.$$

We have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N\setminus\Omega} \frac{\phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{N/(N-1)}\right)}{|x|^\beta}dx \\
&\leq \int\limits_{\{u\leq 1\}} \frac{\phi_N\left(\alpha |u|^{N/(N-1)}\right)}{|x|^\beta}dx \\
&\leq e^\alpha \int\limits_{\{u\leq 1\}} \frac{u^N}{|x|^\beta}dx \\
&\leq e^\alpha \int\limits_{\{u\leq 1;|x|\geq 1\}} \frac{u^N}{|x|^\beta}dx + e^\alpha \int\limits_{\{u\leq 1;|x|<1\}} \frac{u^N}{|x|^\beta}dx \\
&\leq \frac{C(N,\beta)}{\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{(N-\beta)/N}}.
\end{aligned}$$

Now, consider

$$\begin{aligned}
I &= \int\limits_{\Omega} \frac{\phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{N/(N-1)}\right)}{|x|^\beta}dx \\
&\leq \int\limits_{\Omega} \frac{\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{N/(N-1)}\right)}{|x|^\beta}dx.
\end{aligned}$$

On Ω, we set

$$v(x) = u(x) - \left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{\frac{1}{N}}.$$

Then it is clear that $v \in W_0^{1,N}(\Omega)$ and $\|\nabla v\|_N \leq 1$. Also, on Ω, with $\varepsilon = \frac{\alpha_N}{\alpha} - 1$:

$$|u|^{N/(N-1)} \leq \left(|v| + \left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{\frac{1}{N}}\right)^{N/(N-1)}$$

$$\leq (1+\varepsilon)|v|^{N/(N-1)} + (1-\frac{1}{(1+\varepsilon)^{N-1}})^{\frac{1}{1-N}}|\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{\frac{1}{N}}|^{N/(N-1)}$$

$$= \frac{\alpha_N}{\alpha}|v|^{N/(N-1)} + 1.$$

Hence, by Trudinger-Moser inequality on bounded domains:

$$\begin{aligned}
I &\leq \int_{\Omega} \frac{\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{N/(N-1)}\right)}{|x|^{\beta}}dx \\
&\leq \int_{\Omega} \frac{\exp\left(\alpha_N\left(1-\frac{\beta}{N}\right)|v|^{N/(N-1)}+\alpha\right)}{|x|^{\beta}}dx \\
&\leq C(N,\beta)|\Omega|^{1-\frac{\beta}{N}} \\
&\leq \frac{C(N,\beta)}{\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{(N-\beta)/N}}.
\end{aligned}$$

In conclusion, we have

$$AT(\alpha,\beta) \leq \frac{C(N,\beta)}{\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{(N-\beta)/N}}.$$

Next, we will show that $AT(\alpha_N,\beta)=\infty$. Indeed, consider the following sequence:

$$u_n(x) = \begin{cases} 0 \text{ if } |x| \geq 1, \\ \left(\frac{N-\beta}{\omega_{N-1}n}\right)^{1/N}\log\left(\frac{1}{|x|}\right) \text{ if } e^{-\frac{n}{N-\beta}} < |x| < 1 \\ \left(\frac{1}{\omega_{N-1}}\right)^{\frac{1}{N}}\left(\frac{n}{N-\beta}\right)^{\frac{N-1}{N}} \text{ if } 0 \leq |x| \leq e^{-\frac{n}{N-\beta}} \end{cases}.$$

Then we can see easily that

$$\left\|\nabla u_n\right\|_N = 1;\ \left\|u_n\right\|_N = o_n(1).$$

However

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} \frac{\phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u_n|^{N/(N-1)}\right)}{|x|^\beta}dx \\
&\geq \int\limits_{\left\{0\leq|x|\leq e^{-\frac{n}{N-\beta}}\right\}} \frac{\phi_N(n)}{|x|^\beta}dx \\
&= \omega_{N-1}\phi_N(n) \int\limits_0^{e^{-\frac{n}{N-\beta}}} r^{N-1-\beta}dr \\
&= \frac{\omega_{N-1}\phi_N(n)}{e^n(N-\beta)} \to \frac{\omega_{N-1}}{N-\beta} \text{ as } n\to\infty.
\end{aligned}$$

Now, it is clear that there exists a large constant M_1, such that when $n \geq M_1$,

$$\begin{aligned}
\|u_n\|_N^N &= \int_0^{e^{-\frac{n}{N-\beta}}} (\frac{1}{\omega_{N-1}})^{N/N}(\frac{n}{N-\beta})^{\frac{N(N-1)}{N}} r^{N-1}dr + \int_{e^{-\frac{n}{N-\beta}}}^1 (\frac{N-\beta}{\omega_{N-1}n})^{N/N}(\log(\frac{1}{r}))^N r^{N-1}dr \\
&\approx n^{N-1}\int_0^{e^{-\frac{n}{N-\beta}}} r^{N-1}dr + \frac{1}{n}\int_0^{\frac{n}{N-\beta}} y^N e^{-Ny}dy \\
&\approx n^N e^{-\frac{nN}{N-\beta}} + \frac{1}{n} \approx \frac{1}{n}
\end{aligned}$$

So

$$\|u_n\|_N^{N-\beta} \approx \frac{1}{n^{N-\beta}} \quad \text{when} \quad n \geq M_1.$$

Now we consider the following integral

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\phi_N(\alpha(1-\beta/N)|u_n|^{\frac{N}{N-1}})}{|x|^\beta} dx \\
&\gtrsim \int_0^{e^{-\frac{n}{N-\beta}}} \phi_N\left(\alpha(1-\beta/N)(\frac{1}{\omega_{N-1}})^{\frac{1}{N-1}}(\frac{n}{N-\beta})\right) r^{N-1-\beta} dr \\
&\gtrsim \int_0^{e^{-\frac{n}{N-\beta}}} \phi_N\left(\frac{\alpha}{\alpha_N}n\right) r^{N-1-\beta} dr \gtrsim \phi_N\left(\frac{\alpha}{\alpha_N}n\right) e^{-n}
\end{aligned}$$

We note that there exists a large constant M_2 independent of α such that for $n \geq M_2$

$$\phi_N\left(\frac{\alpha}{\alpha_N}n\right) \approx e^{(\frac{\alpha}{\alpha_N})n}$$

as long as $\frac{\alpha}{\alpha_N} \geq \frac{1}{2}$.

Now we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\phi_N(\alpha(1-\beta/N)|u|^{\frac{N}{N-1}})}{|x|^\beta} dx \\
&\gtrsim e^{\left(\frac{\alpha}{\alpha_N}n\right)} e^{-n} = e^{-(1-\frac{\alpha}{\alpha_N})n}
\end{aligned}$$

Now for α that is close enough to α_N we can pick n such that $1 \leq (1-\frac{\alpha}{\alpha_N})n \leq 2$, i.e

$$\alpha \approx (1-\frac{1}{n})\alpha_N \geq \left(1 - \frac{1}{\max(M_1, M_2)}\right)\alpha_N$$

or

$$\max(M_1, M_2) \leq n \approx \frac{1}{1-\frac{\alpha}{\alpha_N}},$$

Then

$$
\begin{aligned}
&\frac{1}{\|u_n\|_N^{N-\beta}} \int_{\mathbb{R}^N} \frac{\Phi_N(\alpha(1-\beta/N)|u_n|^{\frac{N}{N-1}})}{|x|^\beta} dx \\
&\gtrsim n^{N-\beta} e^{-2} \\
&\approx \left(\frac{1}{1-\frac{\alpha}{\alpha_N}} \right)^{N-\beta} \qquad (4.13)
\end{aligned}
$$

And note that when α is close enough to α_N, we have

$$
\frac{1-(\frac{\alpha}{\alpha_N})^{N-1}}{1-\frac{\alpha}{\alpha_N}} \approx 1,
$$

which implies

$$
AT\left(\alpha,\beta\right) \geq \frac{c\left(N,\beta\right)}{\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{(N-\beta)/N}}
$$

when α is close enough to α_N. □

Now, we will provide a proof to Theorem 4.2 using the above improved sharp subcritical Trudinger-Moser inequality (4.6). This suggests a new approach to and another look at the study of the sharp Trudinger-Moser inequality:

Proof of Theorem 4.2. First assume that $b \leq N$. Let $u \in W^{1,N}\left(\mathbb{R}^N\right) \setminus \{0\} : \|\nabla u\|_N^a + \|u\|_N^b \leq 1$. Assume that

$$
\|\nabla u\|_N = \theta \in (0,1)\,; \;\; \|u\|_N^b \leq 1-\theta^a.
$$

If $\frac{1}{2} < \theta < 1$, then we set

$$v(x) = \frac{u(\lambda x)}{\theta}$$
$$\lambda = \frac{(1-\theta^a)^{\frac{1}{b}}}{\theta} > 0.$$

Hence

$$\|\nabla v\|_N = \frac{\|\nabla u\|_N}{\theta} = 1;$$
$$\|v\|_N^N = \int_{\mathbb{R}^N} |v|^N \, dx = \frac{1}{\theta^N} \int_{\mathbb{R}^N} |u(\lambda x)|^N \, dx = \frac{1}{\theta^N \lambda^N} \|u\|_N^N \leq \frac{(1-\theta^a)^{\frac{N}{b}}}{\theta^N \lambda^N} = 1.$$

By Theorem 4.1, we get

$$\int_{\mathbb{R}^N} \frac{\phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)}{|x|^\beta} dx = \int_{\mathbb{R}^N} \frac{\phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u(\lambda x)|^{\frac{N}{N-1}}\right)}{|\lambda x|^\beta} d(\lambda x)$$
$$\leq \lambda^{N-\beta} \int_{\mathbb{R}^N} \frac{\phi_N\left(\theta^{\frac{N}{N-1}}\alpha_N(1-\frac{\beta}{N})\,|v|^{N/(N-1)}\right)}{|x|^\beta} dx$$
$$\leq \lambda^{N-\beta} AT\left(\theta^{\frac{N}{N-1}}\alpha_N, \beta\right) \leq \left(\frac{(1-\theta^a)^{\frac{N}{b}}}{\theta^N}\right)^{1-\frac{\beta}{N}} \frac{C(N,\beta)}{\left(1-\left(\frac{\theta^{\frac{N}{N-1}}\alpha_N}{\alpha_N}\right)^{N-1}\right)^{1-\frac{\beta}{N}}}$$
$$\leq \frac{\left((1-\theta^a)^{\frac{N}{b}}\right)^{1-\frac{\beta}{N}}}{(1-\theta^N)^{1-\frac{\beta}{N}}} C(N,\beta) \leq C(N,\beta,a,b) \text{ since } b \leq N.$$

If $0 < \theta \leq \frac{1}{2}$, then with

$$v(x) = 2u(2x),$$

we have

$$\|\nabla v\|_N = 2\,\|\nabla u\|_N \leq 1$$

$$\|v\|_N \leq 1.$$

By Theorem 4.1:

$$\int_{\mathbb{R}^N} \frac{\phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)}{|x|^\beta}dx \leq 2^N \int_{\mathbb{R}^N} \frac{\phi_N\left(\frac{\alpha_N(1-\frac{\beta}{N})}{2^{\frac{N}{N-1}}}|v|^{N/(N-1)}\right)}{|x|^\beta}dx$$
$$\leq C\left(N,\beta\right).$$

Next, we will verify that the constant $\alpha_N(1-\frac{\beta}{N})$ is our best possible. Indeed, we choose the sequence $\{u_k\}$ as follows

$$u_n(x) = \begin{cases} 0 \text{ if } |x| \geq 1, \\ \left(\frac{N-\beta}{\omega_{N-1}n}\right)^{1/N}\log\left(\frac{1}{|x|}\right) \text{ if } e^{-\frac{n}{N-\beta}} < |x| < 1 \\ \left(\frac{1}{\omega_{N-1}}\right)^{\frac{1}{N}}\left(\frac{n}{N-\beta}\right)^{\frac{N-1}{N}} \text{ if } 0 \leq |x| \leq e^{-\frac{n}{N-\beta}} \end{cases}. \tag{4.14}$$

Then,

$$\|\nabla u_n\|_N = 1;\ \|u_n\|_N = O(\frac{1}{n^{\frac{1}{N}}}).$$

Set

$$w_n(x) = \lambda_n u_n\left(x\right) \text{ where } \lambda_n \in (0,1) \text{ is a solution of } \lambda_n^a + \lambda_n^b \left\|u_n\right\|_N^b = 1.$$
$$\lambda_n = 1 - O\left(\frac{1}{n^{\frac{b}{aN}}}\right) \to_{k\to\infty} 1.$$

Then

$$\|\nabla w_n\|_N^a + \|w_n\|_N^b = 1.$$

Also, for $\alpha > \alpha_N$:

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)}{|x|^\beta}dx \\
&\geq \int_{\left\{0\leq|x|\leq e^{-\frac{n}{N-\beta}}\right\}} \frac{\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right) - \sum_{j=0}^{N-2}\frac{\left[\alpha\left(1-\frac{\beta}{N}\right)\right]^j}{j!}|w_n|^{\frac{N}{N-1}j}}{|x|^\beta}dx \\
&\geq \left[\exp\left(\frac{\alpha n\left(1-O\left(\frac{1}{n^{\frac{b}{a(N-1)}}}\right)\right)}{\alpha_N}\right) - O\left(k^{N-1}\right)\right]\frac{\omega_{N-1}\exp\left(-n\right)}{N-\beta} \\
&\to \infty \text{ as } n\to\infty.
\end{aligned}$$

Now, we will show that

$$MT_{a,b}\left(\beta\right) = \sup_{\alpha\in(0,\alpha_N)}\left(\frac{1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}}{\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}}\right)^{\frac{N-\beta}{b}} AT\left(\alpha,\beta\right)$$

when $MT_{a,b}\left(\beta\right) < \infty$. Indeed, by (4.11), we have

$$\sup_{\alpha\in(0,\alpha_N)}\left(\frac{1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}}{\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}}\right)^{\frac{N-\beta}{b}} AT\left(\alpha,\beta\right) \leq MT_{a,b}\left(\beta\right).$$

Now, let (u_n) be the maximizing sequence of $MT_{a,b}\left(\beta\right)$, i.e., $u_n \in W^{1,N}\left(\mathbb{R}^N\right)\backslash\{0\} : \|\nabla u_n\|_N^a +$

$\|u_n\|_N^b \leq 1$ and

$$\int_{\mathbb{R}^N} \phi_N \left(\alpha_N \left(1 - \frac{\beta}{N} \right) |u_n|^{\frac{N}{N-1}} \right) \frac{dx}{|x|^\beta} \to_{n\to\infty} MT_{a,b}(\beta).$$

We define

$$v_n(x) = \frac{u(\lambda_n x)}{\|\nabla u_n\|_N}$$
$$\lambda_n = \left(\frac{1 - \|\nabla u_n\|_N^a}{\|\nabla u_n\|_N^b} \right)^{1/b} > 0.$$

Hence

$$\|\nabla v_n\|_N = 1 \text{ and } \|v_n\|_N \leq 1.$$

Also,

$$\begin{aligned} &\int_{\mathbb{R}^N} \phi_N \left(\alpha_N \left(1 - \frac{\beta}{N} \right) |u_n|^{\frac{N}{N-1}} \right) \frac{dx}{|x|^\beta} \\ &= \lambda_n^{N-\beta} \int_{\mathbb{R}^N} \frac{\phi_N \left(\|\nabla u_n\|_N^{\frac{N}{N-1}} \alpha_N (1 - \frac{\beta}{N}) |v_n|^{N/(N-1)} \right)}{|x|^\beta} dx \\ &\leq \lambda_n^{N-\beta} AT \left(\|\nabla u_n\|_N^{\frac{N}{N-1}} \alpha_N, \beta \right) \leq \sup_{\alpha \in (0,\alpha_N)} \left(\frac{1 - \left(\frac{\alpha}{\alpha_N} \right)^{\frac{N-1}{N} a}}{\left(\frac{\alpha}{\alpha_N} \right)^{\frac{N-1}{N} b}} \right)^{\frac{N-\beta}{b}} AT(\alpha, \beta). \end{aligned}$$

Hence, we receive

$$MT_{a,b}(\beta) = \sup_{\alpha \in (0,\alpha_N)} \left(\frac{1 - \left(\frac{\alpha}{\alpha_N} \right)^{\frac{N-1}{N} a}}{\left(\frac{\alpha}{\alpha_N} \right)^{\frac{N-1}{N} b}} \right)^{\frac{N-\beta}{b}} AT(\alpha, \beta)$$

when $MT_{a,b}(\beta) < \infty$.

Now, if there exists some $b > N$ such that $MT_{a,b}(\beta) < \infty$. Then we have

$$\overline{\lim}_{\alpha \to \alpha_N^-} \left(1 - \left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}\right)^{\frac{N-\beta}{b}} AT(\alpha, \beta) < \infty.$$

Also, since $MT(\beta) < \infty$:

$$\overline{\lim}_{\alpha \to \alpha_N^-} \left(1 - \left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{\frac{N-\beta}{N}} AT(\alpha, \beta) < \infty.$$

By Theorem 4.1, we can show that

$$\overline{\lim}_{\alpha \to \alpha_N^-} \left(1 - \left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{\frac{N-\beta}{N}} AT(\alpha, \beta) > 0. \tag{4.15}$$

Hence

$$\underline{\lim}_{\alpha \to \alpha_N^-} \frac{\left(1 - \left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}\right)^{\frac{N-\beta}{b}}}{\left(1 - \left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{\frac{N-\beta}{N}}} < \infty$$

which is impossible since $b > N$. The proof is now completed. $\square$

4.4 Adams Inequalities

4.4.1 Sharp Adams Inequalities on $W^{2,\frac{N}{2}}\left(\mathbb{R}^N\right)$

We now prove Theorem 4.3. Again, it is worthy nothing that no version of Theorem 4.4 is assumed in order to prove Theorem 4.3.

Proof of Theorem 4.3. Let $u \in C_0^\infty\left(\mathbb{R}^N\right)\setminus\{0\}$, $u \geq 0$, $\|\Delta u\|_{\frac{N}{2}} \leq 1$ and $\|u\|_{\frac{N}{2}} = 1$. Set

$$\Omega(u) = \left\{ x \in \mathbb{R}^n : u(x) > \left[1 - \left(\frac{\alpha}{\beta\left(N,2\right)}\right)^{\frac{N-2}{2}}\right]^{\frac{2}{N}} \right\}.$$

Since $u \in C_0^\infty\left(\mathbb{R}^n\right)$, we have that $\Omega(u)$ is a bounded set. Moreover, we have

$$|\Omega(u)| \leq \int_{\Omega(u)} \frac{|u|^{\frac{N}{2}}}{1 - \left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}} dx \leq \frac{1}{1 - \left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}}.$$

We have the following estimates:

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N\setminus\Omega(u)} \phi_{N,2}\left(\alpha(1-\frac{\beta}{N})\,|u|^{N/(N-2)}\right)\frac{dx}{|x|^\beta} \\
&\leq \int\limits_{\{u\leq 1\}} \phi_{N,2}\left(\alpha(1-\frac{\beta}{N})\,|u|^{N/(N-2)}\right)\frac{dx}{|x|^\beta} \\
&\leq C\left(N\right)\int\limits_{\{u\leq 1\}} \frac{|u|^{\frac{N}{2}}}{|x|^\beta}dx \\
&\leq C\left(N\right)\left(\int\limits_{\{u\leq 1;|x|\geq 1\}} \frac{|u|^{\frac{N}{2}}}{|x|^\beta}dx + \int\limits_{\{u\leq 1;|x|<1\}} \frac{|u|^{\frac{N}{2}}}{|x|^\beta}dx\right) \\
&\leq C\left(N,\beta\right).
\end{aligned}$$

We now show that $AT\left(\alpha_N,\beta\right) = \infty$. Indeed, let $\psi \in C^\infty\left([0,1]\right)$ be such that

$$\psi\left(0\right) = \psi'\left(0\right) = 0;\ \psi\left(1\right) = \psi'\left(1\right) = 1.$$

For $0<\varepsilon<\frac{1}{2}$ we set

$$H(t)=\begin{cases}\varepsilon\psi\left(\frac{t}{\varepsilon}\right) & 0<t\le\varepsilon\\ t & \varepsilon<t\le 1-\varepsilon\\ 1-\varepsilon\psi\left(\frac{1-t}{\varepsilon}\right) & 1-\varepsilon<t\le 1\\ 0 & 1<t\end{cases}$$

and consider Adams' test functions

$$\psi_r(|x|)=H\left(\frac{\log\frac{1}{|x|}}{\log\frac{1}{r}}\right).$$

By construction, $\psi_r\in W^{2,\frac{N}{2}}\left(\mathbb{R}^N\right)$ and $\psi_r(|x|)=1$ for $x\in B_r$. Moreover, by [2]:

$$\begin{aligned}\|\Delta\psi_r\|_{\frac{N}{2}}^{\frac{N}{2}}&\le\omega_{N-1}a(N,2)^{\frac{N}{2}}\log\left(\frac{1}{r}\right)^{1-\frac{N}{2}}A_r;\\ \|\psi_r\|_{\frac{N}{2}}^{\frac{N}{2}}&=o\left(\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)^{\frac{N-2}{2}}\right)\\ a(N,2)&=\frac{\beta(N,2)^{\frac{N-2}{N}}}{N\sigma_N^{\frac{2}{N}}};\\ A_r=A_r(N,2)&=\left[1+2\varepsilon\left(\|\psi'\|_\infty+O\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right)^{\frac{N}{2}}\right];\end{aligned}$$

Now, we set

$$u_r(|x|)=\left(\log\left(\frac{1}{r}\right)\right)^{\frac{N-2}{N}}\psi_r(|x|).$$

Then

$$u_r(|x|)=\left(\log\left(\frac{1}{r}\right)\right)^{\frac{N-2}{N}}\quad\text{for }x\in B_r$$

$$\|\Delta u_r\|_{\frac{N}{2}}^{\frac{N}{2}} \leq \omega_{N-1} a\,(N,2)^{\frac{N}{2}}\, A_r \text{ and}$$
$$\|\Delta u_r\|_{\frac{N}{2}}^{\frac{N}{N-2}} \leq \frac{\beta\,(N,2)}{N} A_r^{\frac{2}{N-2}}.$$

Now,

$$\begin{aligned}
AT\,(\alpha_N,\beta) &\geq \lim_{r\to 0^+} \frac{1}{\left\| \frac{u_r}{\|\Delta u_r\|_{\frac{N}{2}}} \right\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}} \int_{B_r} \phi_{N,2}\left(\beta\,(N,2)\,(1-\frac{\beta}{N}) \left| \frac{u_r}{\|\Delta u_r\|_{\frac{N}{2}}} \right|^{N/(N-2)} \right) \frac{dx}{|x|^\beta} \\
&\geq \lim_{r\to 0^+} \frac{\|\Delta u_r\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}}{\|u_r\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}} \int_{B_r} \phi_{N,2}\left(\frac{\beta\,(N,2)\,(1-\frac{\beta}{N}) \log\left(\frac{1}{r}\right)}{\|\Delta u_r\|_{\frac{N}{2}}^{\frac{N}{N-2}}} \right) \frac{dx}{|x|^\beta} \\
&\geq \lim_{r\to 0^+} \frac{\|\Delta u_r\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}}{\|u_r\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}} \omega_{N-1} \frac{r^{N-\beta}}{N-\beta} \phi_{N,2}\left(\frac{(N-\beta)\log\left(\frac{1}{r}\right)}{\left[1+2\varepsilon\left(\|\psi'\|_\infty + O\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right)^{\frac{N}{2}}\right]^{\frac{2}{N-2}}} \right) \\
&\to \infty \text{ as } r\to 0^+.
\end{aligned}$$

Now, consider the following sequence

$$u_k\,(x) = \begin{cases} \left[\frac{1}{\beta(N,2)} \ln k\right]^{1-\frac{2}{N}} - \frac{|x|^2}{\left(\frac{\ln k}{k}\right)^{\frac{2}{N}}} + \frac{1}{(\ln k)^{\frac{2}{N}}} & \text{if } 0 \leq |x| \leq \left(\frac{1}{k}\right)^{\frac{1}{N}} \\ N\beta\,(N,2)^{\frac{2}{N}-1} (\ln k)^{-\frac{2}{N}} \ln \frac{1}{|x|} & \text{if } \left(\frac{1}{k}\right)^{\frac{1}{N}} \leq |x| \leq 1. \\ 0 & \text{if } |x| > 1. \end{cases}$$

Then, we can check that

$$1 \leq \|\Delta u_k\|_{\frac{N}{2}}^{\frac{N}{2}} \leq 1 + O\left(\frac{1}{\ln k}\right).$$

Also,

$$\begin{aligned}\|u_k\|_{\frac{N}{2}}^{\frac{N}{2}} &\le \omega_{N-1}\left(N\beta\left(N,2\right)^{\frac{2}{N}-1}\left(\ln k\right)^{-\frac{2}{N}}\right)^{\frac{N}{2}}\int_0^1 r^{N-1}\ln\frac{1}{r}dr \\ &+\frac{\omega_{N-1}}{N}\left(\left[\frac{1}{\beta\left(N,2\right)}\ln k\right]^{1-\frac{2}{N}}+\frac{1}{\left(\frac{\ln k}{k}\right)^{\frac{2}{N}}}\right)^{\frac{N}{2}}\frac{1}{k} \\ &\le A\left(\ln k\right)^{-1}+B\left(\ln k\right)^{\frac{N-2}{2}}\frac{1}{k}\end{aligned}$$

for some constants $A, B > 0$.

Let

$$v_k = \frac{u_k}{\|\Delta u_k\|_{\frac{N}{2}}}$$

then

$$\|\Delta v_k\|_{\frac{N}{2}} = 1$$

and

$$\|v_k\|_{\frac{N}{2}}^{\frac{N}{2}} \le \|u_k\|_{\frac{N}{2}}^{\frac{N}{2}} \le A\left(\ln k\right)^{-1}+B\left(\ln k\right)^{\frac{N-2}{2}}\frac{1}{k}.$$

By the definition of $ATA\left(\alpha,\beta\right)$, we get

$$\begin{aligned}ATA\left(\alpha,\beta\right) &\ge \frac{1}{\|v_k\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}}\int_{\mathbb{R}^N}\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|v_k|^{\frac{N}{N-2}}\right)\frac{dx}{|x|^{\beta}} \\ &\ge \frac{1}{\|v_k\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}}\int_{|x|\le\left(\frac{1}{k}\right)^{\frac{1}{N}}}\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|v_k|^{\frac{N}{N-2}}\right)\frac{dx}{|x|^{\beta}}\end{aligned}$$

$$\geq C \frac{\exp\left(\frac{\alpha}{\beta(N,2)}\left(1-\frac{\beta}{N}\right)\left(\frac{1}{\|\Delta u_k\|_{\frac{N}{2}}^{\frac{2}{N-2}}} - \frac{\beta(N,2)}{\alpha}\right)\ln k\right)}{\left(A(\ln k)^{-1} + B(\ln k)^{\frac{N-2}{2}}\frac{1}{k}\right)^{1-\frac{\beta}{N}}}$$

Note that when k (independent of α) is large

$$\frac{1}{\|\Delta u_k\|_{\frac{N}{2}}^{\frac{2}{N-2}}} - \frac{\beta(N,2)}{\alpha} \approx 1 - \frac{\beta(N,2)}{\alpha}.$$

So we have

$$ATA(\alpha,\beta) \gtrsim \exp\left\{\left(1-\frac{\beta}{N}\right)\left(\frac{\alpha}{\beta(N,2)}-1\right)\ln k\right\} \cdot (\ln k)^{1-\frac{\beta}{N}}$$

When α is close enough to $\beta(N,2)$, we are able to choose k large enough as required before such that

$$\ln k \approx \frac{1}{1-\frac{\alpha}{\beta(N,2)}}$$

or

$$\left(1-\frac{\beta}{N}\right)\left(\frac{\alpha}{\beta(N,2)}-1\right)\ln k \approx 1.$$

Then

$$ATA(\alpha,\beta) \gtrsim C \cdot \left(\frac{1}{1-\frac{\alpha}{\beta(N,2)}}\right)^{1-\frac{\beta}{N}} \approx \left(\frac{1}{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}}\right)^{1-\frac{\beta}{N}}$$

when α is close enough to $\beta(N,2)$. □

We now offer another proof to Theorem 4.4 using the improved sharp subcritical Adams inequality (4.8).

Proof of Theorem 4.4. Assume $0 < b \leq \frac{N}{2}$. Let $u \in W^{2,\frac{N}{2}}\left(\mathbb{R}^N\right) \setminus \{0\} : \|\Delta u\|_{\frac{N}{2}}^{a} + \|u\|_{\frac{N}{2}}^{b} \leq 1$.

Assume that

$$\|\Delta u\|_{\frac{N}{2}} = \theta \in (0,1)\,;\ \ \|u\|_{\frac{N}{2}}^{b} \leq 1 - \theta^{a}.$$

If $\frac{1}{4} < \theta < 1$, then we set

$$\begin{aligned} v(x) &= \frac{u(\lambda x)}{\theta} \\ \lambda &= \frac{(1-\theta^{a})^{\frac{1}{2b}}}{\theta^{\frac{1}{2}}} > 0. \end{aligned}$$

Hence

$$\begin{aligned} \|\Delta v\|_{\frac{N}{2}} &= \frac{\|\Delta u\|_{\frac{N}{2}}}{\theta} = 1; \\ \|v\|_{\frac{N}{2}}^{\frac{N}{2}} &= \int_{\mathbb{R}^N} |v|^{\frac{N}{2}}\, dx = \frac{1}{\theta^{\frac{N}{2}}} \int_{\mathbb{R}^N} |u(\lambda x)|^{\frac{N}{2}}\, dx = \frac{1}{\theta^{\frac{N}{2}} \lambda^N} \|u\|_{\frac{N}{2}}^{\frac{N}{2}} \leq \frac{(1-\theta^{a})^{\frac{N}{2b}}}{\theta^{\frac{N}{2}} \lambda^N} = 1. \end{aligned}$$

By Theorem 4.3, we get

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{\phi_{N,2}\left(\beta(N,2)\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-2}}\right)}{|x|^{\beta}} dx = \int_{\mathbb{R}^N} \frac{\phi_{N,2}\left(\beta(N,2)\left(1-\frac{\beta}{N}\right)|u(\lambda x)|^{\frac{N}{N-2}}\right)}{|\lambda x|^{\beta}} d(\lambda x) \\ &\leq \lambda^{N-\beta} \int_{\mathbb{R}^N} \frac{\phi_{N,2}\left(\theta^{\frac{N}{N-2}}\beta(N,2)\left(1-\frac{\beta}{N}\right)|v|^{N/(N-2)}\right)}{|x|^{\beta}} dx \\ &\leq \lambda^{N-\beta} ATA\left(\theta^{\frac{N}{N-2}}\beta(N,2), \beta\right) \leq \left(\frac{(1-\theta^{a})^{\frac{1}{2b}}}{\theta^{\frac{1}{2}}}\right)^{N-\beta} \frac{C(N,\beta)}{\left[1-\left(\frac{\theta^{\frac{N}{N-2}}\beta(N,2)}{\beta(N,2)}\right)^{\frac{N-2}{2}}\right]^{1-\frac{\beta}{N}}} \\ &\leq \frac{\left((1-\theta^{a})^{\frac{N}{2b}}\right)^{1-\frac{\beta}{N}}}{\left(1-\theta^{\frac{N}{2}}\right)^{1-\frac{\beta}{N}}} C(N,\beta) \leq C(N,\beta,a,b) \ \text{ since } b \leq \frac{N}{2}. \end{aligned}$$

If $0<\theta\leq\frac{1}{4}$, then with

$$v\left(x\right)=2^{2}u\left(2x\right),$$

we have

$$\left\Vert \Delta v\right\Vert _{\frac{N}{2}}=4\left\Vert \Delta u\right\Vert _{\frac{N}{2}}\leq1$$

$$\left\Vert v\right\Vert _{\frac{N}{2}}\leq1.$$

By Theorem 4.3:

$$\int_{\mathbb{R}^{N}}\frac{\phi_{N,2}\left(\beta\left(N,2\right)\left(1-\frac{\beta}{N}\right)\left|u\right|^{\frac{N}{N-2}}\right)}{\left|x\right|^{\beta}}dx\leq4^{N}\int_{\mathbb{R}^{N}}\frac{\phi_{N}\left(\frac{\beta(N,2)(1-\frac{\beta}{N})}{4^{\frac{N}{N-2}}}\left|v\right|^{N/(N-2)}\right)}{\left|x\right|^{\beta}}dx$$

$$\leq C\left(N,\beta\right).$$

We now also consider the Adams' test functions as in the proof of Theorem 4.3. Let $\beta>\beta\left(N,2\right)$. Set

$$w_{r}(|x|)=\lambda_{r}\frac{u_{r}\left(|x|\right)}{\left\Vert \Delta u_{r}\right\Vert _{\frac{N}{2}}}\text{ where }\lambda_{r}\in\left(0,1\right)\text{ is a solution of }\lambda_{r}^{a}+\frac{\lambda_{r}^{b}\left\Vert u_{r}\right\Vert _{\frac{N}{2}}^{b}}{\left\Vert \Delta u_{r}\right\Vert _{\frac{N}{2}}^{b}}=1.$$

$$\lambda_{r}\rightarrow_{r\rightarrow0^{+}}1.$$

Then

$$\left\Vert \Delta w_{r}\right\Vert _{\frac{N}{2}}^{a}+\left\Vert w_{r}\right\Vert _{\frac{N}{2}}^{b}=1$$

and

$$\lim_{r\to 0^+}\int_{\mathbb{R}^N}\frac{\phi_{N,2}\left(\beta\left(1-\frac{\beta}{N}\right)|w_r|^{\frac{N}{N-2}}\right)}{|x|^\beta}dx \geq \lim_{r\to 0^+}\int_{B_r}\phi_{N,2}\left(\frac{\beta\left(1-\frac{\beta}{N}\right)\lambda_r^{\frac{N}{N-2}}|u_r|^{\frac{N}{N-2}}}{\|\Delta u_r\|_{\frac{N}{2}}^{\frac{N}{N-2}}}\right)\frac{dx}{|x|^\beta}$$

$$\geq \lim_{r\to 0^+}\omega_{N-1}\frac{r^{N-\beta}}{N-\beta}\phi_{N,2}\left(\frac{\beta}{\beta(N,2)}\frac{(N-\beta)\log\left(\frac{1}{r}\right)}{\left[1+2\varepsilon\left(\|\psi'\|_\infty+O\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right)^{\frac{N}{2}}\right]^{\frac{2}{N-2}}}\right)\to\infty$$

as $r\to 0^+$ if we choose ϵ small enough.

It now remains to show that

$$A_{a,b}(\beta)=\sup_{\alpha\in(0,\beta(N,2))}\left(\frac{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}}{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}\right)^{\frac{N-\beta}{2b}} ATA(\alpha,\beta).$$

By (4.12):

$$\sup_{\alpha\in(0,\beta(N,2))}\left(\frac{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}}{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}\right)^{\frac{N-\beta}{2b}} ATA(\alpha,\beta)\leq A_{a,b}(\beta).$$

Now, let (u_n) be the maximizing sequence of $A_{a,b}(\beta)$, i.e., $u_n\in W^{2,\frac{N}{2}}\left(\mathbb{R}^N\right)\setminus\{0\}:\|\Delta u_n\|_{\frac{N}{2}}^a+\|u_n\|_{\frac{N}{2}}^b\leq 1$ and

$$\int_{\mathbb{R}^N}\phi_{N,2}\left(\beta(N,2)\left(1-\frac{\beta}{N}\right)|u_n|^{\frac{N}{N-2}}\right)\frac{dx}{|x|^\beta}\to_{n\to\infty}A_{a,b}(\beta).$$

We define a new sequence:

$$v_n(x)=\frac{u(\lambda_n x)}{\|\Delta u_n\|_{\frac{N}{2}}}$$

$$\lambda_n = \left(\frac{1 - \|\Delta u_n\|_{\frac{N}{2}}^{a}}{\|\Delta u_n\|_{\frac{N}{2}}^{b}} \right)^{\frac{1}{2b}} > 0.$$

Hence

$$\|\Delta v_n\|_{\frac{N}{2}} = 1 \text{ and } \|v_n\|_{\frac{N}{2}} \leq 1.$$

Also,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \phi_{N,2} \left(\beta(N,2) \left(1 - \frac{\beta}{N} \right) |u_n|^{\frac{N}{N-2}} \right) \frac{dx}{|x|^{\beta}} \\
&= \lambda_n^{N-\beta} \int_{\mathbb{R}^N} \frac{\phi_{N,2} \left(\|\Delta u_n\|_{\frac{N}{2}}^{N/(N-2)} \beta(N,2) (1 - \frac{\beta}{N}) |v_n|^{N/(N-2)} \right)}{|x|^{\beta}} dx \\
&\leq \lambda_n^{N-\beta} ATA \left(\|\Delta u_n\|_{\frac{N}{2}}^{N/(N-2)} \beta(N,2), \beta \right) \leq \sup_{\alpha \in (0, \beta(N,2))} \left(\frac{1 - \left(\frac{\alpha}{\beta(N,2)} \right)^{\frac{N-2}{N} a}}{\left(\frac{\alpha}{\beta(N,2)} \right)^{\frac{N-2}{N} b}} \right)^{\frac{N-\beta}{2b}} ATA(\alpha, \beta).
\end{aligned}$$

Now, we assume that there is some $b > \frac{N}{2}$ such that $A_{a,b}(\beta) < \infty$. Then

$$A_{a,b}(\beta) = \sup_{\alpha \in (0, \beta(N,2))} \left(\frac{1 - \left(\frac{\alpha}{\beta(N,2)} \right)^{\frac{N-2}{N} a}}{\left(\frac{\alpha}{\beta(N,2)} \right)^{\frac{N-2}{N} b}} \right)^{\frac{N-\beta}{2b}} ATA(\alpha, \beta)$$

and so

$$\overline{\lim}_{\alpha \uparrow \beta(N,2)} \left(\frac{1 - \left(\frac{\alpha}{\beta(N,2)} \right)^{\frac{N-2}{N} a}}{\left(\frac{\alpha}{\beta(N,2)} \right)^{\frac{N-2}{N} b}} \right)^{\frac{N-\beta}{2b}} ATA(\alpha, \beta) < \infty.$$

Also, by Theorem 4.3:

$$\overline{\lim}_{\alpha \uparrow \beta(N,2)} \left(\frac{1 - \left(\frac{\alpha}{\beta(N,2)} \right)^{\frac{N-2}{N} a}}{\left(\frac{\alpha}{\beta(N,2)} \right)^{\frac{N-2}{N} b}} \right)^{\frac{N-\beta}{N}} ATA(\alpha, \beta) > 0, \tag{4.16}$$

Hence:

$$\underline{\lim}_{\alpha\uparrow\beta(N,2)}\frac{\left(1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}\right)^{\frac{N-\beta}{2b}}}{\left(1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}\right)^{\frac{N-\beta}{N}}}>0$$

which is impossible since $b>\frac{N}{2}$. The proof is now completed. □

4.4.2 Adams Inequalities on $W^{\gamma,\frac{N}{\gamma}}\left(\mathbb{R}^N\right)$-Proof of Theorem 4.5

Let $u\in W^{\gamma,p}\left(\mathbb{R}^N\right)\setminus\{0\}:\left\|(-\Delta)^{\frac{\gamma}{2}}u\right\|_p^a+\|u\|_p^b\leq 1$. We set

$$\left\|(-\Delta)^{\frac{\gamma}{2}}u\right\|_p=\theta\in(0,1)\,;\;\;\|u\|_p^b\leq 1-\theta^a.$$

If $\frac{1}{2^\gamma}<\theta<1$, then by define a new function

$$\begin{aligned}v(x)&=\frac{u(\lambda x)}{\theta}\\ \lambda&=\frac{(1-\theta^a)^{\frac{1}{\gamma b}}}{\theta^{\frac{1}{\gamma}}}>0.\end{aligned}$$

we get

$$(-\Delta)^{\frac{\gamma}{2}}v(x)=\frac{\lambda^\gamma}{\theta}\left((-\Delta)^{\frac{\gamma}{2}}u\right)(\lambda x)\,.$$

Hence

$$\begin{aligned}\left\|(-\Delta)^{\frac{\gamma}{2}}v\right\|_p&=\frac{\left\|(-\Delta)^{\frac{\gamma}{2}}u\right\|_p}{\theta}=1;\\ \|v\|_p^p&=\int_{\mathbb{R}^N}|v|^p\,dx=\frac{1}{\theta^p}\int_{\mathbb{R}^N}|u(\lambda x)|^p\,dx=\frac{1}{\theta^p\lambda^N}\|u\|_p^p\leq\frac{(1-\theta^a)^{\frac{p}{b}}}{\theta^p\lambda^N}=1.\end{aligned}$$

By the definition of $GATA\left(\alpha,\beta\right)$, we get

$$\int_{\mathbb{R}^N}\frac{\phi_{N,\gamma}\left(\beta_0\left(N,\gamma\right)\left(1-\frac{\beta}{N}\right)|u|^{\frac{p}{p-1}}\right)}{|x|^{\beta}}dx=\int_{\mathbb{R}^N}\frac{\phi_{N,\gamma}\left(\beta_0\left(N,\gamma\right)\left(1-\frac{\beta}{N}\right)|u\left(\lambda x\right)|^{\frac{p}{p-1}}\right)}{|\lambda x|^{\beta}}d\left(\lambda x\right)$$
$$\leq\lambda^{N-\beta}\int_{\mathbb{R}^N}\frac{\phi_{N,\gamma}\left(\theta^{\frac{p}{p-1}}\beta_0\left(N,\gamma\right)\left(1-\frac{\beta}{N}\right)|v|^{\frac{p}{p-1}}\right)}{|x|^{\beta}}dx$$
$$\leq\lambda^{N-\beta}GATA\left(\theta^{\frac{p}{p-1}}\beta_0\left(N,\gamma\right),\beta\right)\leq\left(\frac{\left(1-\theta^{a}\right)^{\frac{1}{\gamma b}}}{\theta^{\frac{1}{\gamma}}}\right)^{N-\beta}\frac{C\left(N,\beta\right)}{\left[1-\left(\frac{\theta^{\frac{p}{p-1}}\beta_0(N,\gamma)}{\beta_0(N,\gamma)}\right)^{\frac{p-1}{p}}\right]^{1-\frac{\beta}{N}}}$$
$$\leq\frac{\left(\left(1-\theta^{a}\right)^{\frac{N}{\gamma b}}\right)^{1-\frac{\beta}{N}}}{\left(1-\theta\right)^{1-\frac{\beta}{N}}}C\left(N,\beta\right)\leq C\left(N,\beta,a,b\right)\ \text{ since } b\leq p.$$

If $0<\theta\leq\frac{1}{2^{\gamma}}$, then with

$$v\left(x\right)=2^{\gamma}u\left(2x\right),$$

we have

$$\left\|\left(-\Delta\right)^{\frac{\gamma}{2}}v\right\|_p=2^{\gamma}\left\|\left(-\Delta\right)^{\frac{\gamma}{2}}u\right\|_p\leq 1$$
$$\|v\|_p\leq 1.$$

By the definition of $GATA\left(\alpha,\beta\right)$:

$$\int_{\mathbb{R}^N}\frac{\phi_{N,\gamma}\left(\beta_0\left(N,\gamma\right)\left(1-\frac{\beta}{N}\right)|u|^{\frac{p}{p-1}}\right)}{|x|^{\beta}}dx\leq 2^{N}\int_{\mathbb{R}^N}\frac{\phi_{N,\gamma}\left(\frac{\beta_0(N,\gamma)}{2^{\gamma\frac{p}{p-1}}}\left(1-\frac{\beta}{N}\right)|v|^{\frac{p}{p-1}}\right)}{|x|^{\beta}}dx$$
$$\leq C\left(N,\beta\right).$$

CHAPTER 5 TRUDINGER-MOSER INEQUALITIES WITH EXACT GROWTH AND THEIR EXTREMALS

5.1 Introduction

In this section, we consider the Trudinger-Moser inequalities with exact growth, which allows the critical inequalities under the restriction of the semi-norm.

to get the critical case $\alpha = \alpha_N$ while still using the seminorm $\left(\int_{\mathbb{R}^N} |\nabla u|^N \, dx\right)^{1/N}$, in dimension two, Ibrahim, Masmoudi and Nakanishi [39] used $\int\limits_{\mathbb{R}^2} \frac{e^{4\pi u^2}-1}{1+u^2} dx$ instead of $\int\limits_{\mathbb{R}^2} e^{4\pi u^2} - 1dx$.

Recently, Lam and Lu studied some sharp versions of the Trudinger-Moser inequalities for functions in $D^{1,N}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right)$. More precisely, they proved that

Theorem B (Lam–Lu-2014). *Let $N \geq 2$, $\alpha_N = N\left(\frac{N\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}\right)^{\frac{1}{N-1}}$ and $0 \leq \beta < N$. Then for all $0 \leq \alpha < \alpha_N\left(1 - \frac{\beta}{N}\right)$, $q \geq 1$ and $p > q\left(1 - \frac{\beta}{N}\right)$ ($p \geq q$ if $\beta = 0$), there exists a positive constant $C_{p,N,\alpha,\beta} > 0$ such that*

$$\int_{\mathbb{R}^N} \frac{\exp\left(\alpha |u|^{\frac{N}{N-1}}\right) |u|^p}{|x|^\beta} dx \leq C_{N,p,q,\alpha,\beta} \|u\|_q^{q\left(1-\frac{\beta}{N}\right)}, \ \forall u \in D^{1,N}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right), \ \|\nabla u\|_N \leq 1.$$

Moreover, this constant $\alpha_N\left(1 - \frac{\beta}{N}\right)$ is sharp.

A consequence of the above theorem is the following inequality:

Theorem C (Lam–Lu-2014). *Let $0 \leq \beta < N$* and *$q \geq 1$. Then for all $0 \leq \alpha < \alpha_N\left(1 - \frac{\beta}{N}\right)$,* there exists a positive constant $C_{q,N,\alpha,\beta}$ such that

$$\int_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha |u|^{\frac{N}{N-1}}\right)}{|x|^\beta} dx \leq C_{q,N,\alpha,\beta} \|u\|_q^{q\left(1-\frac{\beta}{N}\right)}, \ \forall u \in D^{1,N}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right) : \ \|\nabla u\|_N \leq 1.$$

The constant $\alpha_N\left(1-\frac{\beta}{N}\right)$ *is sharp. Here*

$$\Phi_{N,q,\beta}(t)=\begin{cases}\sum\limits_{j\in\mathbb{N},\ j>\frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)}\frac{t^j}{j!}\text{ if }\beta>0.\\ \sum\limits_{j\in\mathbb{N},\ j\geq\frac{q(N-1)}{N}}\frac{t^j}{j!}\text{ if }\beta=0.\end{cases}.$$

Here we study a version of the Trudinger-Moser inequality with exact growth on $D^{1,N}\left(\mathbb{R}^N\right)\cap L^q\left(\mathbb{R}^N\right)$:

Theorem 5.1. *Let* $\lambda>0,\ 0\leq\beta<N,\ p\geq q\geq 1$ *and* $0<\alpha\leq\alpha_N\left(1-\frac{\beta}{N}\right)$*. Then there exists a constant* $C=C(N,p,q,\lambda,\beta)>0$ *such that for all* $u\in D^{1,N}\left(\mathbb{R}^N\right)\cap L^q\left(\mathbb{R}^N\right)$: $\|\nabla u\|_N\leq 1$, *there holds*

$$\int\limits_{\mathbb{R}^N}\frac{\Phi_{N,q,\beta}\left(\alpha u^{\frac{N}{N-1}}\right)}{\left(1+\lambda|u|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx\leq C\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}. \tag{5.1}$$

Moreover, the inequality does not hold when $p<q$.

We also studied the maximizers of the above Trudinger-Moser inequalities in the subcritical case $p>q$. We actually proved

Theorem 5.2. *Let* $\lambda>0,\ 0\leq\beta<N,\ q>1,\ 0<\alpha\leq\alpha_N$ *and* $p>q$*. Denote*

$$TME_{p,q,N,\lambda,\alpha,\beta}=\sup_{u\in D^{1,N}(\mathbb{R}^N)\cap L^q(\mathbb{R}^N):\ \|\nabla u\|_N\leq 1}\frac{1}{\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}}\int\limits_{\mathbb{R}^N}\frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)u^{\frac{N}{N-1}}\right)}{\left(1+\lambda|u|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx.$$

Then $TME_{p,q,N,\lambda,\alpha,\beta}$ *can be attained in any of the following cases*

(a) $\beta>0$ *and all* $0<\alpha\leq\alpha_N$,

(b) $\beta=0$, $\frac{q(N-1)}{N}\notin\mathbb{N}$ *and all* $0<\alpha\leq\alpha_N$,

(c) $\beta = 0$, $\frac{q(N-1)}{N} \in \mathbb{N}$, $p > N$ *and all* $0 < \alpha \leq \alpha_N$,

(d) $\beta = 0$, $\frac{q(N-1)}{N} \in \mathbb{N}$, $p \leq N$, $p < \frac{N-1}{N-2}q$ *and* $\alpha = \alpha_N$.

5.2 Some Useful Results

In this section, we introduce some useful results that will be used in our proofs. We first recall the definition of rearrangement and some useful inequalities. Let $\Omega \subset \mathbb{R}^N,\ N \geq 2$, be a measurable set. We denote by $\Omega^{\#}$ the open ball $B_R \subset \mathbb{R}^N$ centered at 0 of radius $R > 0$ such that $|B_R| = |\Omega|$.

Let $u : \Omega \to \mathbb{R}$ be a real-valued measurable function that vanishes at infinity, that is $|\{x : |u(x)| > t\}|$ is finite for all $t > 0$. The distribution function of u is the function

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}|$$

and the decreasing rearrangement of u is the right-continuous, nonincreasing function u^* that is equimeasurable with u :

$$u^*(s) = \sup\{t \geq 0 : \mu_u(t) > s\}.$$

It is clear that $suppu^* \subseteq [0, |\Omega|]$. We also define

$$u^{**}(s) = \frac{1}{s}\int_0^s u^*(t)dt \geq u^*(s).$$

Moreover, we define the spherically symmetric decreasing rearrangement of u :

$$u^{\#} : \Omega^{\#} \to [0, \infty]$$
$$u^{\#}(x) = u^{*}\left(\sigma_N |x|^N\right).$$

Then we have the following important result that could be found in [55]:

Lemma 5.3 (Pólya-Szegö inequality)**.** *Let $u \in W^{1,p}(\mathbb{R}^n)$, $p \geq 1$. Then $f^{\#} \in W^{1,p}(\mathbb{R}^n)$ and*

$$\left\|\nabla f^{\#}\right\|_p = \|\nabla f\|_p .$$

Lemma 5.4. *Let f and g be nonnegative functions on $\mathbb{R}^N$, vanishing at infinity. Then*

$$\int_{\mathbb{R}^N} f(x)\, g(x)\, dx \leq \int_{\mathbb{R}^N} f^{\#}(x)\, g^{\#}(x)\, dx$$

in the sense that when the ldft side is infinite so is the right. Moreover, if f is strictly symmetric-decreasing, then there is equality if and only if $g = g^{\#}$.

We will next prove a lemma that will be used several times in our work.

Lemma 5.5. *Let $\Omega \subset \mathbb{R}^N$, $|\Omega| < \infty$. Suppose that*

$$f_n \to f \ \text{a.e. in } \Omega$$

and there exists $q > 1$ such that f_n is uniformly bounded in $L^q(\Omega)$ and $f \in L^q(\Omega)$. Then

$$f_n \to f \ \text{in } L^1(\Omega).$$

Proof. For arbitrary $\varepsilon > 0$, by Egorov's theorem, we can find a measurable $D \subset \Omega$ such that

$$f_n \to f \text{ uniformly in } D,$$

$$|\Omega \setminus D| < \varepsilon.$$

Thus

$$\int\limits_D |f_n - f|\, dx \to 0.$$

Also, by Holder's inequality

$$\int\limits_{\Omega\setminus D} |f_n - f|\, dx \le \left(\int\limits_{\Omega\setminus D} |f_n - f|^q\, dx\right)^{1/q} \left(\int\limits_{\Omega\setminus D} 1^{q^*}\, dx\right)^{1/q^*}$$

$$\le C\varepsilon^{1/q^*}.$$

Hence $f_n \to f$ in $L^1(\Omega)$. □

Now, we recall a compactness lemma of Strauss [7, 84].

Lemma 5.6. *Let P and $Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying*

$$\frac{P(s)}{Q(s)} \to 0 \text{ as } |s| \to \infty \text{ and } \frac{P(s)}{Q(s)} \to 0 \text{ as } s \to 0$$

Let (u_n) be a sequence of measurable functions: $\mathbb{R}^N \to \mathbb{R}$ such that

$$\sup_n \int\limits_{\mathbb{R}^N} |Q(u_n(x))|\, dx < \infty$$

and

$$P\left(u_n\left(x\right)\right) \overset{n\to\infty}{\to} v(x) \text{ a.e., and } \lim_{|x|\to\infty}|u_n(x)| = 0 \text{ uniformly with respect to } n.$$

Then $P\left(u_n\right) \to v$ *in* $L^1\left(\mathbb{R}^N\right)$.

Using Lemma 5.6, we will study the continuity and compactness of the embeddings from $D^{1,N}_{rad}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right)$ into $L^a\left(\mathbb{R}^N\right)$ and $L^a\left(\mathbb{R}^N; \frac{dx}{|x|^\beta}\right)$. More precisely, we have the following lemma:

Lemma 5.7. *Let* $N \geq 2,\ 0 < t < N$. *Then the embedding* $D^{1,N}_{rad}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right) \hookrightarrow L^r\left(\mathbb{R}^N\right)$ *is continuous when* $r \geq q$ *and compact for all* $r > q$. *Also, the embedding* $D^{1,N}_{rad}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right) \hookrightarrow L^r\left(\mathbb{R}^N; \frac{dx}{|x|^t}\right)$ *is continuous when* $r > q\left(1 - \frac{t}{N}\right)$ *and compact for all* $r \geq q$.

Proof. By the Caffarelli-Kohn-Nirenberg inequality [9]: There exists a positive constant C such that for all $u \in C_0^\infty\left(\mathbb{R}^N\right)$:

$$\left\||x|^\gamma u\right\|_r \leq C \left\||x|^\alpha |\nabla u|\right\|_p^a \left\||x|^\beta u\right\|_q^{1-a}$$

where

$$p, q \geq 1,\ r > 0,\ 0 \leq a \leq 1$$

$$\frac{1}{p} + \frac{\alpha}{N},\ \frac{1}{q} + \frac{\beta}{N},\ \frac{1}{r} + \frac{\gamma}{N} > 0 \text{ where}$$

$$\gamma = a\sigma + (1-a)\beta$$

$$\frac{1}{r} + \frac{\gamma}{N} = a\left(\frac{1}{p} + \frac{\alpha - 1}{N}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{N}\right),$$

and

$$0 \le \alpha - \sigma \text{ if } a > 0 \text{ and}$$
$$\alpha - \sigma \le 1 \text{ if } a > 0 \text{ and } \frac{1}{p} + \frac{\alpha - 1}{N} = \frac{1}{r} + \frac{\gamma}{N}.$$

we could obtain the continuity of the embedding $D^{1,N}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right) \hookrightarrow L^r\left(\mathbb{R}^N; \frac{dx}{|x|^t}\right)$ with $r > q\left(1 - \frac{t}{N}\right)$ ($r \ge q$ if $t = 0$). Indeed, we choose $p = N$; $\alpha = \beta = 0$; $\gamma = -\frac{t}{r}$; $a = 1 - \frac{q}{r}\left(1 - \frac{t}{N}\right)$.

Now, let $r > q$, we now will prove that the embedding $D^{1,N}_{rad}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right) \hookrightarrow L^r\left(\mathbb{R}^N\right)$ is compact.

Indeed, let $\{u_n\} \in D^{1,N}_{rad}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right)$ be bounded. Then we can assume that

$$u_n \rightharpoonup u \text{ weakly in } D^{1,N}_{rad}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right).$$

Set

$$v_n = u_n - u.$$

By Radial lemma, we get that

$$\lim_{|x|\to\infty} |v_n(x)| = 0 \text{ uniformly with respect to } n.$$

Also, using the Lemma 5.6 with

$$P(s) = s^r;\ Q(s) = s^q + s^{r+1},$$

then we can conclude that v_n converges to 0 in L^1. It means that u_n converges to u in L^r.

Now, let $r \geq q$, we will prove that the embedding $D^{1,N}_{rad}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right) \hookrightarrow L^r\left(\mathbb{R}^N; \frac{dx}{|x|^t}\right)$ is compact.

First, let $\{u_n\} \in D^{1,N}_{rad}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right)$ be bounded. Again, we can assume that

$$u_n \rightharpoonup u \text{ weakly in } D^{1,N}_{rad}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right).$$

Choose p such that $1 < p < \frac{N}{t}$, then for R arbitrary, we get

$$\begin{aligned}\int\limits_{|x|<R} |u_n - u|^r \frac{dx}{|x|^t} &\leq \left(\int\limits_{|x|<R} |u_n - u|^{rp'} dx\right)^{1/p'} \left(\int\limits_{|x|<R} \frac{1}{|x|^{tp}} dx\right)^{1/p} \\ &\leq CR^{\frac{N}{p}-t} \left(\int\limits_{|x|<R} |u_n - u|^{rp'} dx\right)^{1/p'}.\end{aligned}$$

Also,

$$\int\limits_{|x|\geq R} |u_n - u|^r \frac{dx}{|x|^t} \leq \frac{1}{R^t} \int\limits_{|x|\geq R} |u_n - u|^r dx \leq \frac{C}{R^t}.$$

Using the compactness of the embedding $D^{1,N}_{rad}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right) \hookrightarrow L^{rp'}\left(\mathbb{R}^N\right)$, choose R sufficiently large, we get that u_n converges to u in $L^r\left(\mathbb{R}^N; \frac{dx}{|x|^t}\right)$. □

Now, we will prove a variant of Lemma 2.2 in [64]:

Lemma 5.8. *Given any sequence* $s = \{s_k\}_{k\geq 0}$, *let* $\|s\|_1 = \sum_{k=0}^{\infty} |s_k|$, $\|s\|_N = \left(\sum_{k=0}^{\infty} |s_k|^N\right)^{1/N}$, $\|s\|_{(e)} = \left(\sum_{k=0}^{\infty} |s_k|^q e^k\right)^{1/q}$ *and*

$$\mu(h) = \inf\left\{\|s\|_{(e)} : \|s\|_1 = h,\ \|s\|_N \leq 1\right\}.$$

Then for $h>1$, we have

$$\mu(h)\sim\frac{\exp\left(\frac{h^{\frac{N}{N-1}}}{q}\right)}{h^{\frac{1}{N-1}}}.$$

Proof. Since $\mu(h)$ is increasing in h, we just need to show that

$$\mu\left(n^{1-\frac{1}{N}}\right)\sim\frac{e^{\frac{n}{q}}}{n^{\frac{1}{N}}}\text{ for all natural number } n\in\mathbb{N}.$$

Choose

$$s_k=\begin{cases}\frac{1}{n^{\frac{1}{N}}}\text{ if } k\le n-1\\ 0\text{ if } k>n-1\end{cases}.$$

It's clear that

$$\|s\|_N=1;\ \|s\|_1=n^{1-\frac{1}{N}};$$
$$\|s\|_{(e)}\sim\frac{e^{\frac{n}{q}}}{n^{\frac{1}{N}}}$$

so

$$\mu\left(n^{1-\frac{1}{N}}\right)\lesssim\frac{e^{\frac{n}{q}}}{n^{\frac{1}{N}}}.$$

Now, assume that for some $\varepsilon\ll 1,\ n\gg 1$ and sequence s :

$$\|s\|_N=1;\ \|s\|_1=\sqrt{n};\ \|s\|_{(e)}\le\varepsilon\frac{e^{\frac{n}{q}}}{n^{\frac{1}{N}}}$$

It means that for $k\ge n$:

$$|s_k|\lesssim\varepsilon\frac{e^{\frac{n-k}{q}}}{n^{\frac{1}{N}}}.$$

Consider the new sequence $b_k = s_k : k \le n$ and $b_k = 0 : k > n$, we get

$$\|b\|_1 = \|s\|_1 - \sum_{j>n} |s_j| \ge n^{1-\frac{1}{N}} - C\frac{\varepsilon}{n^{\frac{1}{N}}}.$$

Hence

$$\|b\|_1^{\frac{N}{N-1}} \ge \left(n^{1-\frac{1}{N}} - C\frac{\varepsilon}{n^{\frac{1}{N}}}\right)^{\frac{N}{N-1}} = n\left(1 - C\frac{\varepsilon}{n}\right)^{\frac{N}{N-1}} \ge n - C\varepsilon.$$

On the other hand,

$$\|b\|_1^{\frac{N}{N-1}} = \left(\|b\|_1^2\right)^{\frac{N}{2(N-1)}} \le n - \frac{1}{2}\frac{N}{N-1}\frac{\sum_{j,k\le n} \frac{(s_j - s_k)^2}{2}}{n^{1-\frac{2}{N}}}$$

Hence

$$\sum_{j,k\le n} (s_j - s_k)^2 \lesssim \varepsilon n^{1-\frac{2}{N}}$$

Choose $m \le n$ such that

$$\min_{j\le n} |s_j| = |s_m|.$$

Then

$$\|b\|_1 - n|s_m| \lesssim \sqrt{\varepsilon} n^{1-\frac{1}{N}}.$$

Hence

$$|s_m| \gtrsim \frac{1}{n^{\frac{1}{N}}}$$

and we get

$$\|s\|_{(e)} \gtrsim \frac{e^{\frac{n}{q}}}{n^{\frac{1}{N}}}$$

which is a contradiction. □

Using the above lemma, we can now prove a Radial Sobolev inequality in the spirit of Ibrahim-Masmoudi-Nakanishi [39]:

Theorem 5.9 (RadialSobolev)**.** *There exists a constant $C > 0$ such that for any radially nonnegative nonincreasing function $\varphi \in D^{1,N}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right)$ satisfying $u(R) > 1$ and*

$$\omega_{N-1}\int_R^\infty \left|\varphi'(t)\right|^N t^{N-1}dt \le K$$

for some $R,\ K > 0$, then we have

$$\frac{\exp\left[\frac{\alpha_N}{K}\varphi^{\frac{N}{N-1}}(R)\right]}{\varphi^{\frac{q}{N-1}}(R)}R^N \le C\frac{\int_R^\infty \left|\varphi(t)\right|^q t^{N-1}dt}{K^{\frac{q}{N-1}}}$$

Proof. By scaling, we can assume that $R = 1;\ K = 1$, i.e., $\omega_{N-1}\int_1^\infty \left|\varphi'(t)\right|^N t^{N-1}dt \le 1$. Set

$$h_k = \alpha_N^{\frac{N}{N-1}}\varphi\left(e^{k/N}\right);\ s_k = h_k - h_{k+1} \ge 0$$

then

$$\|s\|_1 = h_0 = \alpha_N^{\frac{N}{N-1}}\varphi(1).$$

Also

$$s_k = h_k - h_{k+1} = \alpha_N^{\frac{N-1}{N}}\left[\varphi\left(e^{k/N}\right) - \varphi\left(e^{(k+1)/N}\right)\right]$$

$$= \alpha_N^{\frac{N-1}{N}} \int_{e^{(k+1)/N}}^{e^{k/N}} u'(t)\,dt$$

$$\leq \alpha_N^{\frac{N-1}{N}} \left(\int_{e^{k/N}}^{e^{(k+1)/N}} |u'(t)|^N t^{N-1} dt \right)^{1/N} \left(\int_{e^{k/N}}^{e^{(k+1)/N}} \frac{1}{t} dt \right)^{(N-1)/N}$$

$$\leq \left(\omega_{N-1} \int_{e^{k/N}}^{e^{(k+1)/N}} |u'(t)|^N t^{N-1} dt \right)^{1/N} .$$

Hence

$$\|s\|_N \leq 1.$$

Now

$$\int_1^\infty |\varphi(t)|^q t^{N-1} dt = \sum_{k\geq 0} \int_{e^{k/N}}^{e^{(k+1)/N}} |\varphi(t)|^q t^{N-1} dt \geq \sum_{k\geq 0} \left|\varphi(e^{(k+1)/N})\right|^q \int_{e^{k/N}}^{e^{(k+1)/N}} t^{N-1} dt$$

$$\gtrsim \sum_{k\geq 0} \left|\varphi(e^{(k+1)/N})\right|^q e^{k+1} \gtrsim \sum_{k\geq 0} |h_{k+1}|^q e^{k+1}$$

$$= \sum_{k\geq 1} |h_k|^q e^k \geq \sum_{k\geq 1} |s_k|^q e^k$$

Thus

$$\|s\|_{(e)}^q = \sum_{k=0}^{\infty} |s_k|^q e^k = s_0^q + \sum_{k\geq 1} |s_k|^q e^k \lesssim h_0^q + \int_1^\infty |\varphi(t)|^q t^{N-1} dt.$$

Also, for $1 < r < \exp\left(\frac{1}{2^{\frac{N}{N-1}} \cdot N}\right)$:

$$h_0 - \alpha_N^{\frac{N-1}{N}} \varphi(r) = \alpha_N^{\frac{N-1}{N}} \varphi(1) - \alpha_N^{\frac{N-1}{N}} \varphi(r)$$

$$= \alpha_N^{\frac{N-1}{N}} \int_r^1 u'(t)\,dt$$

$$\leq \alpha_N^{\frac{N-1}{N}} \left(\int\limits_1^r |u'(t)|^N t^{N-1} dt \right)^{1/N} \left(\int\limits_1^r \frac{1}{t} dt \right)^{(N-1)/N}$$

$$< \frac{1}{2} \leq \frac{h_0}{2}$$

Hence

$$h_0 \lesssim \varphi(r).$$

So

$$\int\limits_1^\infty |\varphi(t)|^q t^{N-1} dt \geq \int\limits_1^{e^{1/2n}} |\varphi(t)|^q t^{N-1} dt \gtrsim h_0^q.$$

Now, we can conclude that

$$\int\limits_1^\infty |\varphi(t)|^q t^{N-1} dt \gtrsim \|s\|_{(e)}^q \gtrsim \frac{\exp\left(h_0^{\frac{N}{N-1}} \right)}{h_0^{\frac{q}{N-1}}} = C \frac{e^{\alpha_N \varphi^{\frac{N}{N-1}}(1)}}{(\varphi(1))^{\frac{q}{N-1}}}.$$

□

5.3 Trudinger-Moser Inequalities with Exact Growth-Proof of Theorem 5.1

Proof. It is enough to prove the inequality (5.1) when $\lambda = 1$ and $p = q$. By the symmetrization arguments: the Pólya-Szegö inequality, the Hardy-Littlewood inequality and the density arguments, we may assume that u is a smooth, nonnegative and radially nonincreasing function (we just need to make sure that the function $\frac{\Phi_{N,q,\beta}\left(\alpha t^{\frac{N}{N-1}}\right)}{\left(1+t^{q\left(1-\frac{\beta}{N}\right)}\right)}$ is nondecreasing on $\mathbb{R}^+$ but it is easy since $\frac{\Phi_{N,q,\beta}\left(\alpha t^{\frac{N}{N-1}}\right)}{t^{q\left(1-\frac{\beta}{N}\right)}}$ and $\Phi_{N,q,\beta}\left(\alpha t^{\frac{N}{N-1}}\right)$ are both nondecreasing on $\mathbb{R}^+$). Let

$R_1 = R_1(u)$ be such that

$$\int\limits_{B_{R_1}} |\nabla u|^N \, dx = \omega_{N-1} \int\limits_0^{R_1} |u_r|^N \, r^{N-1} dr \leq 1 - \varepsilon_0,$$

$$\int\limits_{\mathbb{R}^N \setminus B_{R_1}} |\nabla u|^N \, dx = \omega_{N-1} \int\limits_{R_1}^{\infty} |u_r|^N \, r^{N-1} dr \leq \varepsilon_0.$$

Here $\varepsilon_0 \in (0,1)$ is fixed and does not depend on u.

By the Holder's inequality, we have

$$\begin{aligned} u\left(r_1\right) - u\left(r_2\right) &\leq \int\limits_{r_1}^{r_2} - u_r dr \qquad (5.2) \\ &\leq \left(\int\limits_{r_1}^{r_2} |u_r|^N \, r^{N-1} dr \right)^{1/N} \left(\ln \frac{r_2}{r_1} \right)^{\frac{N-1}{N}} \\ &\leq \left(\frac{1 - \varepsilon_0}{\omega_{N-1}} \right)^{1/N} \left(\ln \frac{r_2}{r_1} \right)^{\frac{N-1}{N}} \text{ for } 0 < r_1 \leq r_2 \leq R_1, \end{aligned}$$

and

$$u\left(r_1\right) - u\left(r_2\right) \leq \left(\frac{\varepsilon_0}{\omega_{N-1}} \right)^{1/N} \left(\ln \frac{r_2}{r_1} \right)^{\frac{N-1}{N}} \text{ for } R_1 \leq r_1 \leq r_2. \qquad (5.3)$$

We define $R_0 := \inf \{ r > 0 : u(r) \leq 1 \} \in [0, \infty)$. Hence $u(s) \leq 1$ when $s \geq R_0$. WLOG, we assume $R_0 > 0$.

Now, we split the integral as follows:

$$\begin{aligned} \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha \, |u|^{\frac{N}{N-1}} \right)}{\left(1 + \lambda u^{\frac{q}{N-1}\left(1 - \frac{\beta}{N}\right)} \right) |x|^\beta} dx &= \int\limits_{B_{R_0}} \frac{\Phi_{N,q,\beta}\left(\alpha \, |u|^{\frac{N}{N-1}} \right)}{\left(1 + \lambda u^{\frac{q}{N-1}\left(1 - \frac{\beta}{N}\right)} \right) |x|^\beta} dx + \int\limits_{\mathbb{R}^N \setminus B_{R_0}} \frac{\Phi_{N,q,\beta}\left(\alpha \, |u|^{\frac{N}{N-1}} \right)}{\left(1 + \lambda u^{\frac{q}{N-1}\left(1 - \frac{\beta}{N}\right)} \right) |x|^\beta} dx \\ &= I + J. \end{aligned}$$

First, we will estimate J. Since $u \leq 1$ on $\mathbb{R}^N \setminus B_{R_0}$, we have if $\beta > 0$:

$$J = \int\limits_{\mathbb{R}^N \setminus B_{R_0}} \frac{\Phi_{N,q,\beta}\left(\alpha |u|^{\frac{N}{N-1}}\right)}{\left(1 + \lambda u^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}} dx \tag{5.4}$$

$$\leq C \int_{\{u \leq 1\}} \frac{|u|^{\left(\left\lfloor q\frac{N-1}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor + 1\right)\frac{N}{N-1}}}{|x|^{\beta}} dx$$

$$\leq C \left\|u\right\|_q^{q\left(1-\frac{\beta}{N}\right)} \text{ by Lemma 5.7.}$$

Similarly for the case $\beta = 0$, we also have

$$J \leq C \left\|u\right\|_q^{q\left(1-\frac{\beta}{N}\right)}.$$

Hence, now, we just need to deal with the integral I.

Case 1: $0 < R_0 \leq R_1$.

In this case, using (5.2), we have for $0 < r \leq R_0$:

$$u(r) \leq 1 + \left(\frac{1-\varepsilon_0}{\omega_{N-1}}\right)^{1/N} \left(\ln \frac{R_0}{r}\right)^{\frac{N-1}{N}}.$$

By using

$$(a+b)^{\frac{N}{N-1}} \leq (1+\varepsilon)a^{\frac{N}{N-1}} + A(\varepsilon)\, b^{\frac{N}{N-1}},$$

where

$$A(\varepsilon) = \left(1 - \frac{1}{(1+\varepsilon)^{N-1}}\right)^{\frac{1}{1-N}},$$

we get

$$u^{\frac{N}{N-1}}(r) \leq (1+\varepsilon)\left(\frac{1-\varepsilon_0}{\omega_{N-1}}\right)^{1/(N-1)} \ln \frac{R_0}{r} + C(\varepsilon).$$

Thus, we can estimate the integral I as follows:

$$\begin{aligned} I &= \int_{B_{R_0}} \frac{\Phi_{N,q,\beta}\left(\alpha |u|^{\frac{N}{N-1}}\right)}{\left(1+\lambda u^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}} dx \\ &\leq \int_{B_{R_0}} \frac{\exp\left(\alpha(1+\varepsilon)\left(\frac{1-\varepsilon_0}{\omega_{N-1}}\right)^{1/(N-1)} \ln\frac{R_0}{r} + \alpha A(\varepsilon)\right)}{|x|^{\beta}} dx \\ &\leq C R_0^{\alpha(1+\varepsilon)\left(\frac{1-\varepsilon_0}{\omega_{N-1}}\right)^{1/(N-1)}} \int_0^{R_0} r^{N-1-\alpha(1+\varepsilon)\left(\frac{1-\varepsilon_0}{\omega_{N-1}}\right)^{1/(N-1)}-\beta} dr \\ &\leq C R_0^{N-\beta} \\ &\leq C\left(\int_{B_{R_0}} 1 dx\right)^{1-\frac{\beta}{N}} \\ &\leq C \|u\|_q^{q\left(1-\frac{\beta}{N}\right)}. \end{aligned} \tag{5.5}$$

Case 2: $0 < R_1 < R_0$.

We have

$$\begin{aligned} I &= \int_{B_{R_0}} \frac{\Phi_{N,q,\beta}\left(\alpha |u|^{\frac{N}{N-1}}\right)}{\left(1+\lambda u^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}} dx \\ &= \int_{B_{R_1}} \frac{\Phi_{N,q,\beta}\left(\alpha |u|^{\frac{N}{N-1}}\right)}{\left(1+\lambda u^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}} dx + \int_{B_{R_0}\setminus B_{R_1}} \frac{\Phi_{N,q,\beta}\left(\alpha |u|^{\frac{N}{N-1}}\right)}{\left(1+\lambda u^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}} dx \\ &= I_1 + I_2. \end{aligned}$$

Using (5.3), we get

$$u(r) - u(R_0) \leq \left(\frac{\varepsilon_0}{\omega_{N-1}}\right)^{1/N} \left(\ln \frac{R_0}{r}\right)^{\frac{N-1}{N}} \text{ for } r \geq R_1.$$

Hence

$$u(r) \leq 1 + \left(\frac{\varepsilon_0}{\omega_{N-1}}\right)^{1/N} \left(\ln \frac{R_0}{r}\right)^{\frac{N-1}{N}}.$$

Then, we have

$$u^{\frac{N}{N-1}}(r) \leq (1+\varepsilon)\left(\frac{\varepsilon_0}{\omega_{N-1}}\right)^{\frac{1}{N-1}} \ln \frac{R_0}{r} + A(\varepsilon), \ \forall \varepsilon > 0.$$

So

$$\begin{aligned}
I_2 &= \int_{B_{R_0} \setminus B_{R_1}} \frac{\Phi_{N,q,\beta}\left(\alpha |u|^{\frac{N}{N-1}}\right)}{\left(1 + \lambda u^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta} dx \\
&\leq C \int_{R_1}^{R_0} \exp\left(\alpha(1+\varepsilon)\left(\frac{\varepsilon_0}{\omega_{N-1}}\right)^{\frac{1}{N-1}} \ln \frac{R_0}{r} + \alpha A(\varepsilon)\right) r^{N-1-\beta} dr \\
&\leq C R_0^{\alpha(1+\varepsilon)\left(\frac{\varepsilon_0}{\omega_{N-1}}\right)^{\frac{1}{N-1}}} \frac{R_0^{N-\beta-\alpha(1+\varepsilon)\left(\frac{\varepsilon_0}{\omega_{N-1}}\right)^{\frac{1}{N-1}}} - R_1^{N-\beta-\alpha(1+\varepsilon)\left(\frac{\varepsilon_0}{\omega_{N-1}}\right)^{\frac{1}{N-1}}}}{N-\beta-\alpha(1+\varepsilon)\left(\frac{\varepsilon_0}{\omega_{N-1}}\right)^{\frac{1}{N-1}}} \\
&\leq \frac{C}{N-\beta-\alpha(1+\varepsilon)\left(\frac{\varepsilon_0}{\omega_{N-1}}\right)^{\frac{1}{N-1}}} \left(R_0^{N-\beta} - R_1^{N-\beta}\right) \\
&\leq C\left(R_0^N - R_1^N\right)^{1-\frac{\beta}{N}} \\
&\leq C\left(\int_{B_{R_0} \setminus B_{R_1}} 1 dx\right)^{1-\frac{\beta}{N}}
\end{aligned}$$

$$\leq C \left\| u \right\|_q^{q\left(1-\frac{\beta}{N}\right)},$$

(since $\alpha \leq \alpha_N \left(1 - \frac{\beta}{N}\right)$, we can choose $\varepsilon > 0$ such that $N - \beta - \alpha \left(1+\varepsilon\right) \left(\frac{\varepsilon_0}{\omega_{N-1}}\right)^{\frac{1}{N-1}} > 0$).

So, we need to estimate $I_1 = \int\limits_{B_{R_1}} \frac{\Phi_{N,q,\beta}\left(\alpha |u|^{\frac{N}{N-1}}\right)}{\left(1+\lambda u^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}} dx$ with $u\left(R_1\right) > 1$.

First, we define

$$v(r) = u(r) - u\left(R_1\right) \text{ on } 0 \leq r \leq R_1.$$

It's clear that $v \in W_0^{1,N}\left(B_{R_1}\right)$ and that $\int\limits_{B_{R_1}} \left|\nabla v\right|^N dx = \int\limits_{B_{R_1}} \left|\nabla u\right|^N dx \leq 1 - \varepsilon_0$.

Moreover, for $0 \leq r \leq R_1$:

$$u^{\frac{N}{N-1}}(r) \leq (1+\varepsilon) v^{\frac{N}{N-1}}(r) + A(\varepsilon) u^{\frac{N}{N-1}}\left(R_1\right).$$

Hence

$$\begin{aligned} I_1 &= \int\limits_{B_{R_1}} \frac{\Phi_{N,q,\beta}\left(\alpha u^{\frac{N}{N-1}}\right)}{\left(1+\lambda u^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}} dx \\ &\leq \frac{1}{\lambda} \frac{e^{\alpha A(\varepsilon) u^{\frac{N}{N-1}}\left(R_1\right)}}{u^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}\left(R_1\right)} \int\limits_{B_{R_1}} \frac{e^{(1+\varepsilon)\alpha v^{\frac{N}{N-1}}(r)}}{|x|^{\beta}} dx \\ &= \frac{1}{\lambda} \frac{e^{\alpha A(\varepsilon) u^{\frac{N}{N-1}}\left(R_1\right)}}{u^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}\left(R_1\right)} \int\limits_{B_{R_1}} \frac{e^{\alpha w^{\frac{N}{N-1}}(r)}}{|x|^{\beta}} dx \end{aligned} \tag{5.6}$$

where $w = (1+\varepsilon)^{\frac{N-1}{N}} v$.

It's clear that $w \in W_0^{1,N}\left(B_{R_1}\right)$ and $\int\limits_{B_{R_1}} \left|\nabla w\right|^N dx = (1+\varepsilon)^{N-1} \int\limits_{B_{R_1}} \left|\nabla v\right|^N dx \leq (1+\varepsilon)^{N-1}\left(1-\varepsilon_0\right) \leq 1$ if we choose $0 < \varepsilon \leq \left(\frac{1}{1-\varepsilon_0}\right)^{\frac{1}{N-1}} - 1$. Hence, using the singular Trudinger-

Moser inequality, we have

$$\int\limits_{B_{R_1}} \frac{e^{\alpha w^{\frac{N}{N-1}}(r)}}{|x|^{\beta}} dx \le C\left|B_{R_1}\right|^{1-\frac{\beta}{N}} \le C R_1^{N-\beta}. \tag{5.7}$$

Also, using Theorem 5.9, we have

$$\begin{aligned} \frac{e^{\alpha A(\varepsilon) u^{\frac{N}{N-1}}(R_1)}}{u^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}(R_1)} R_1^{N-\beta} &\le \left[\frac{\exp\left(\frac{N\alpha A(\varepsilon)}{N-\beta} u^{\frac{N}{N-1}}(R_1)\right)}{u^{\frac{q}{N-1}}(R_1)} R_1^{N}\right]^{1-\frac{\beta}{N}} \\ &\le \left(C A(\varepsilon)^{\frac{q}{N-1}} \int\limits_{\mathbb{R}^N \setminus B_{R_1}} |u|^q \, dx\right)^{1-\frac{\beta}{N}} \\ &\le \left(C \|u\|_q^q\right)^{1-\frac{\beta}{N}} \end{aligned} \tag{5.8}$$

if we choose

$$\varepsilon = \left(\frac{1}{1-\varepsilon_0}\right)^{\frac{1}{N-1}} - 1.$$

By (5.6), (5.7) and (5.8), the proof is now completed. □

Remark 5.10. When $\beta = 0$, we note that the inequality (5.1) still holds when we replace $\Phi_{N,q,0}$ by a function Φ such that there exists $C_{N,q} > 0$:

$$\Phi\left(\alpha_N |u|^{\frac{N}{N-1}}\right) \le C_{N,q} \exp\left(\alpha_N |u|^{\frac{N}{N-1}}\right) \ \forall u;$$

$$\Phi\left(\alpha_N |u|^{\frac{N}{N-1}}\right) \le C_{N,q} |u|^q \text{ for every } |u| \le 1.$$

5.4 Sharpness

We define the sequence

$$u_n(x) = \begin{cases} (\frac{1}{\omega_{N-1}})^{1/N}(\frac{n}{N-\beta})^{\frac{N-1}{N}}, & 0 \le |x| \le e^{-\frac{n}{N-\beta}}, \\ (\frac{N-\beta}{\omega_{N-1}n})^{1/N}\log(\frac{1}{|x|}), & e^{-\frac{n}{N-\beta}} < |x| < 1, \\ 0, & |x| \ge 1. \end{cases} \tag{5.9}$$

Note that

$$\|\nabla u_n\|_N = 1$$

and for sufficiently large n,

$$\begin{aligned} \|u_n\|_q^q &= \int_0^{e^{-\frac{n}{N-\beta}}} (\frac{1}{\omega_{N-1}})^{q/N}(\frac{n}{N-\beta})^{\frac{q(N-1)}{N}} r^{N-1}dr + \int_{e^{-\frac{n}{N-\beta}}}^1 (\frac{N-\beta}{\omega_{N-1}n})^{q/N}(\log(\frac{1}{r}))^q r^{N-1}dr \\ &\approx n^{\frac{q(N-1)}{N}} \int_0^{e^{-\frac{n}{N-\beta}}} r^{N-1}dr + \frac{1}{n^{q/N}}\int_0^{\frac{n}{N-\beta}} y^q e^{-Ny}dy \\ &\approx n^{\frac{q(N-1)}{N}} e^{-\frac{nN}{N-\beta}} + \frac{1}{n^{q/N}} \approx \frac{1}{n^{q/N}} \end{aligned}$$

Now we consider the LHS of (5.1),

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}(\alpha_N(1-\beta/N)|u|^{\frac{N}{N-1}})}{(1+\lambda|u|^l)|x|^\beta}dx \\ &\gtrsim \int_0^{e^{-\frac{n}{N-\beta}}} \frac{\Phi_{N,q,\beta}(\alpha_N(1-\beta/N)(\frac{1}{\omega_{N-1}})^{\frac{1}{N-1}}(\frac{n}{N-\beta}))}{(1+\lambda|(\frac{1}{\omega_{N-1}})^{1/N}(\frac{n}{N-\beta})^{\frac{N-1}{N}}|^l)} r^{N-1-\beta}dr \\ &\gtrsim \int_0^{e^{-\frac{n}{N-\beta}}} \frac{\Phi_{N,q,\beta}(n)}{n^{\frac{l(N-1)}{N}}} r^{N-1-\beta}dr \gtrsim \frac{\Phi_{N,q,\beta}(n)e^{-n}}{n^{\frac{l(N-1)}{N}}} \gtrsim \frac{1}{n^{\frac{l(N-1)}{N}}} \end{aligned}$$

Note to make (5.1) true providing n sufficiently large, we need

$$\frac{1}{n^{\frac{l(N-1)}{N}}} \gtrsim \|u_n\|_q^{q(1-\beta/N)} \approx \frac{1}{n^{\frac{q(1-\beta/N)}{N}}} \Rightarrow l \geq \frac{q}{N-1}(1-\beta/N)$$

5.5 Proof of Theorem 5.2

Before proving Theorem 5.2, we will study a lower estimate for $TME_{p,q,N,\lambda,\alpha,\beta}$ when $\beta = 0$; $\frac{q(N-1)}{N} \in \mathbb{N}$.

Lemma 5.11. *Let $\beta = 0$, $\frac{q(N-1)}{N} \in \mathbb{N}$ and $0 < \alpha \leq \alpha_N$, the following estimates hold*

(a) if $p > N$, then $TME_{p,q,N,\lambda,\alpha} > \frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}$.

(b) if $p \leq N$ and $p < \frac{N-1}{N-2}q$, then $TME_{p,q,N,\lambda,\alpha_N} > \frac{\alpha_N^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}$.

Proof. Define

$$u_n(x) = \begin{cases} (\frac{1}{\omega_{N-1}})^{1/N}(\frac{n}{N})^{\frac{N-1}{N}}, & 0 \leq |x| \leq e^{-\frac{n}{N}}, \\ (\frac{N}{\omega_{N-1}n})^{1/N}\log(\frac{1}{|x|}), & e^{-\frac{n}{N}} < |x| < 1, \\ 0, & |x| \geq 1. \end{cases} \tag{5.10}$$

Note

$$\begin{aligned} \|u_n\|_q^q &= \omega_{N-1}\int_0^{e^{-\frac{n}{N}}} \left(\frac{n}{\alpha_N}\right)^{\frac{q(N-1)}{N}} r^{N-1}dr + \omega_{N-1}\int_{e^{-\frac{n}{N}}}^1 (\frac{N}{\omega_{N-1}n})^{q/N}\left(\log\frac{1}{r}\right)^q r^{N-1}dr \\ &= \omega_{N-1}(\frac{n}{\alpha_N})^{\frac{q(N-1)}{N}}\frac{e^{-n}}{N} + \omega_{N-1}\left(\frac{N}{\omega_{N-1}n}\right)^{\frac{q}{N}}\int_0^{\frac{n}{N}} y^q e^{-Ny}dy \\ &:= A + B \end{aligned}$$

In this case $\phi_{N,q,0}(x)=\sum_{j=q\frac{N-1}{N}}^{\infty}\frac{x^j}{j!}$, and

$$\begin{aligned}
&\int_{\mathbb{R}^n}\frac{\phi_{N,q,0}(\alpha u_n^{\frac{N}{N-1}})}{1+\lambda u^{\frac{p}{N-1}}}dx\\
=&\ \omega_{N-1}\int_0^{e^{-\frac{n}{N}}}\frac{\phi_{N,q,0}\left(\alpha\frac{n}{\alpha_N}\right)}{1+\lambda\left(\frac{n}{\alpha_N}\right)^{\frac{p}{N}}}r^{N-1}dr+\omega_{N-1}\int_{e^{-\frac{n}{N}}}^1\frac{\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}(\log\frac{1}{r})^{\frac{N}{N-1}}\right)}{1+\lambda(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}(\log\frac{1}{r})^{\frac{p}{N-1}}}r^{N-1}dr\\
=&\ \omega_{N-1}\frac{\phi_{N,q,0}\left(\alpha\frac{n}{\alpha_N}\right)}{1+\lambda\left(\frac{n}{\alpha_N}\right)^{\frac{p}{N}}}\frac{e^{-n}}{N}+\omega_{N-1}\int_0^{\frac{n}{N}}\frac{\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}y^{\frac{N}{N-1}}\right)}{1+\lambda(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}y^{\frac{p}{N-1}}}r^{N-1}dr\\
:=&\ I+II.
\end{aligned}$$

Note that for sufficiently large n,

(a) $I>\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}A$ for all $0<\alpha\le N\omega_{N-1}^{\frac{1}{N-1}}$,

(b) In particular, $I-\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}A\approx\frac{1}{n^{\frac{p}{N}}}$ when $\alpha=\alpha_N$.

and

$$\begin{aligned}
II =&\ \omega_{N-1}\int_{e^{-\frac{n}{N}}}^1\frac{\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}(\log\frac{1}{r})^{\frac{N}{N-1}}\right)}{1+\lambda(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}(\log\frac{1}{r})^{\frac{p}{N-1}}}r^{N-1}dr\\
=&\ \omega_{N-1}\int_0^{\frac{n}{N}}\left(\frac{\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}y^{\frac{N}{N-1}}\right)}{1+\lambda(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}y^{\frac{p}{N-1}}}-\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}}y^q\right)e^{-Ny}dy\\
&+\omega_{N-1}\int_0^{\frac{n}{N}}\left(\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}}y^q\right)e^{-Ny}dy\\
=&\ \omega_{N-1}\int_0^{\frac{n}{N}}\left(\frac{\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}y^{\frac{N}{N-1}}\right)}{1+\lambda(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}y^{\frac{p}{N-1}}}-\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}}y^q\right)e^{-Ny}dy\\
&+\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}\omega_{N-1}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}}\int_0^{\frac{n}{N}}y^qe^{-Ny}dy\\
=&\ III+\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}B
\end{aligned}$$

where

$$III :=$$

$$\omega_{N-1}\int_0^{\frac{n}{N}}\left(\frac{\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}y^{\frac{N}{N-1}}\right)-\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}}y^q-\lambda\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}+\frac{p}{N(N-1)}}y^{q+\frac{p}{N-1}}}{1+\lambda(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}y^{\frac{p}{N-1}}}\right)$$

$$\cdot e^{-Ny}dy$$

$$:=\omega_{N-1}\int_0^{\frac{n}{N}}S(y)e^{-Ny}dy.$$

If $\alpha=\alpha_N$ and $p\le N$, note

$$\begin{aligned}
III &\gtrsim -\lambda\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}+\frac{p}{N(N-1)}}\cdot\omega_{N-1}\int_0^{\frac{n}{N}}\left(\frac{y^{q+\frac{p}{N-1}}}{1+\lambda(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}y^{\frac{p}{N-1}}}\right)e^{-Ny}dy\\
&\approx -\left(\frac{1}{n}\right)^{\frac{q}{N}+\frac{p}{N(N-1)}}.
\end{aligned}$$

Taking advantage of the assumption $\frac{p}{q}<\frac{N-1}{N-2}$, we have $\frac{p}{N}<\frac{q}{N}+\frac{p}{N(N-1)}$,then

$$\begin{aligned}
&\frac{1}{\|u_n\|_q^q}\int_{\mathbb{R}^n}\frac{\phi_{N,q,0}(\alpha u_n^{\frac{N}{N-1}})}{1+u^{\frac{p}{N-1}}}dx=\frac{I+II}{A+B}\\
\approx\;&\frac{\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}A+\left(\frac{1}{n}\right)^{\frac{p}{N}}-\left(\frac{1}{n}\right)^{\frac{q}{N}+\frac{p}{N(N-1)}}+\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}B}{A+B}>\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}.
\end{aligned}$$

Note that if $p>N$, we can conclude that when

$$y<\left(\frac{\alpha}{\lambda(\frac{q(N-1)}{N}+1)}\right)^{\frac{N-1}{p-N}}\left(\frac{\omega_{N-1}n}{N}\right)^{\frac{1}{N}}:=c(n),$$

we have

$$
\begin{aligned}
&\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}y^{\frac{N}{N-1}}\right)-\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}}y^{q}-\lambda\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}+\frac{p}{N(N-1)}}y^{q+\frac{p}{N-1}}\\
= &\left(\frac{\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}y^{\frac{N}{N-1}}\right)^{\frac{q(N-1)}{N}+1}}{\left(\frac{q(N-1)}{N}+1\right)!}-\lambda\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}+\frac{p}{N(N-1)}}y^{q+\frac{p}{N-1}}\right)\\
&\quad+\sum_{j=\frac{q(N-1)}{N}+2}^{\infty}\frac{\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}y^{\frac{N}{N-1}}\right)^{j}}{j!}\\
\geq &\sum_{j=\frac{q(N-1)}{N}+2}^{\infty}\frac{\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}y^{\frac{N}{N-1}}\right)^{j}}{j!}.
\end{aligned}
$$

Then we get the following estimates

$$
\begin{aligned}
&\int_0^{c(n)}S(y)e^{-Ny}dy\\
&\geq\int_0^{c(n)}\left(\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}y^{\frac{N}{N-1}}\right)-\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}}y^{q}-\lambda\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}+\frac{p}{N(N-1)}}y^{q+\frac{p}{N-1}}\right)\\
&\qquad\cdot e^{-Ny}dy\cdot\frac{\omega_{N-1}}{1+(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}(\frac{n}{N})^{\frac{p}{N-1}}}\\
&\gtrsim\left(\frac{1}{n}\right)^{\frac{q}{N}+\frac{2}{N-1}}\int_0^1 y^{q+\frac{2N}{N-1}}e^{-Ny}dy\cdot\frac{1}{n^{\frac{p}{N}}}\gtrsim\left(\frac{1}{n}\right)^{\frac{p}{N}+\frac{q}{N}+\frac{2}{N-1}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{c(n)}^{\frac{n}{N}}S(y)e^{-Ny}dy\\
&\geq\int_{c(n)}^{\frac{n}{N}}\left(\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}y^{\frac{N}{N-1}}\right)-\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}}y^{q}-\lambda\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}+\frac{p}{N(N-1)}}y^{q+\frac{p}{N-1}}\right)\\
&\qquad\cdot e^{-Ny}dy
\end{aligned}
$$

$$\gtrsim -\left(\frac{1}{n}\right)^{\frac{q}{N}+\frac{p}{N(N-1)}}\int_{c(n)}^{\frac{n}{N}} y^{q+\frac{p}{N-1}}e^{-Ny}dy \geq -n^{\frac{q(N-1)}{N}+\frac{p}{N}}\int_{c(n)}^{\frac{n}{N}} e^{-Ny}dy$$
$$\gtrsim -n^{\frac{q(N-1)}{N}+\frac{p}{N}}e^{-Nc(n)}.$$

Since $c(n) \sim n^{\frac{1}{N}}$ and n is sufficiently large, we conclude

$$III \gtrsim \left(\frac{1}{n}\right)^{\frac{p}{N}+\frac{q}{N}+\frac{2}{N-1}} - n^{\frac{q(N-1)}{N}+\frac{p}{N}}e^{-Nc(n)} \geq 0.$$

Now we have

$$\frac{1}{\|u_n\|_q^q}\int_{\mathbb{R}^n}\frac{\phi_{N,q,0}(\alpha u_n^{\frac{N}{N-1}})}{1+u^{\frac{p}{N-1}}}dx = \frac{I+II}{A+B} > \frac{\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}A + III + \frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}B}{A+B} \geq \frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}.$$

□

Proof of Theorem 5.2: We recall that

$$TME_{p,q,N,\lambda,\alpha,\beta} = \sup_{u\in D^{1,N}(\mathbb{R}^N)\cap L^q(\mathbb{R}^N):\ \|\nabla u\|_N\leq 1}\frac{1}{\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}}\int_{\mathbb{R}^N}\frac{\Phi_{N,q,\beta}\left(\alpha u^{\frac{N}{N-1}}\right)}{\left(1+\lambda|u|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx$$

where $0 \leq \beta < N$, $0 < \lambda$, $q \geq 1$, $p > q$, and

$$\Phi_{N,q,\beta}(t) = \begin{cases} \sum_{j\in\mathbb{N},\ j>\frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)} \frac{t^j}{j!} \text{ if } \beta > 0. \\ \sum_{j\in\mathbb{N},\ j\geq\frac{q(N-1)}{N}} \frac{t^j}{j!} \text{ if } \beta = 0. \end{cases}.$$

Let (u_k) be a maximizing sequence of $TME_{p,q,N,\lambda,\alpha,\beta}$ in $D^{1,N}\left(\mathbb{R}^N\right)\cap L^q\left(\mathbb{R}^N\right)$ such that

$\|\nabla u_k\|_N \leq 1$, i.e.,

$$\frac{1}{\|u_k\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha u_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\,|u_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta} dx \to TME_{p,q,N,\lambda,\alpha,\beta}.$$

By symmetrization arguments, we can also assume that each u_k is radially nonnegative nonincreasing function. Now, setting

$$v_k\left(x\right) = u_k\left(\lambda_k x\right) \text{ where } \lambda_k = \|u_k\|_q^{q/N},$$

then we have that v_k is radially nonnegative nonincreasing function in $D^{1,N}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right)$; $\|\nabla v_k\|_N \leq 1$; $\|v_k\|_q = 1$ and

$$\begin{aligned}\frac{1}{\|u_k\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha u_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\,|u_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta} dx &= \frac{1}{\|v_k\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\,|v_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta} dx \\ &= \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\,|v_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta} dx \\ &\to TME_{p,q,N,\lambda,\alpha,\beta}.\end{aligned}$$

Hence, we an assume without loss of generality that

$$v_k \rightharpoonup v \text{ weakly in } D^{1,N}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right);$$

$$v_k \to v \text{ a.e. in } \mathbb{R}^N; \;\; \|\nabla v\|_N \leq 1; \|v\|_q \leq 1.$$

Case 1: $\beta > 0$

We note here that for all $R>0:$

$$1=\|v_k\|_q^q=\int\limits_{\mathbb{R}^N}|v_k|^q\,dx\geq\left|S^{N-1}\right|\int\limits_0^R|v_k(r)|^q\,r^{N-1}dr$$
$$\geq\left|S^{N-1}\right||v_k(R)|^q\int\limits_0^R r^{N-1}dr=\left|S^{N-1}\right||v_k(R)|^q\,\frac{R^N}{N}.$$

Hence

$$v_k(R)\leq\left(\frac{N}{|S^{N-1}|}\right)^{1/q}\frac{1}{R^{N/q}}.$$

We now fix $\varepsilon>0$ and set $R_\varepsilon=\left(\frac{N}{|S^{N-1}|\varepsilon}\right)^{1/N}$. Then for every $R\geq R_\varepsilon: v_k(R)\leq\varepsilon$.

We denote $j_{N,q,\beta}\geq 1$ to be the smallest natural number such that $j_{N,q,\beta}>\frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)$.

Then

$$\int\limits_{|x|\geq R_\varepsilon}\frac{\Phi_{N,q,\beta}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx\leq\frac{1}{R_\varepsilon^\beta}\int\limits_{|x|\geq R_\varepsilon}\sum_{j\geq j_{N,q,\beta}}\frac{\left(\alpha v_k^{\frac{N}{N-1}}\right)^j}{j!}$$
$$\leq\frac{1}{R_\varepsilon^\beta}\alpha_N^{j_{N,q,\beta}}\int\limits_{\mathbb{R}^N}\exp\left(\alpha v_k^{\frac{N}{N-1}}\right)v_k^{\frac{N}{N-1}j_{N,q,\beta}}$$
$$\leq\frac{1}{R_\varepsilon^\beta}\alpha_N^{j_{N,q,\beta}}C_{N,q,\beta}\|v_k\|_q^q\ \text{(by Theorem B)}$$
$$\leq\frac{1}{R_\varepsilon^\beta}\alpha_N^{j_{N,q,\beta}}C_{N,q,\beta}\ (\to 0\text{ as }\varepsilon\to 0).$$

Now, consider

$$\int\limits_{|x|<R_\varepsilon}\frac{\Phi_{N,q,\beta}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx.$$

Since $\frac{1+\lambda|s|^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}}{1+\lambda|s|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}}\to 0$ as $|s|\to\infty$, we can find L_ε (that goes to ∞ as $\varepsilon\to 0$) such that

$\frac{1+\lambda|s|^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}}{1+\lambda|s|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}} \leq \varepsilon$ for every $|s| \geq L_\varepsilon$. Then

$$\int\limits_{|x|<R_\varepsilon;\ |v_k|\geq L_\varepsilon} \frac{\Phi_{N,q,\beta}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx \leq \varepsilon \int\limits_{|x|<R_\varepsilon;\ |v_k|\geq L_\varepsilon} \frac{\Phi_{N,q,\beta}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v_k\right|^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx$$

$$\leq \varepsilon C_{p,q,N,\lambda,\beta}\ (\to 0 \text{ as } \varepsilon \to 0).$$

It remains to consider

$$I_{k,\varepsilon} = \int\limits_{|x|<R_\varepsilon;\ |v_k|<L_\varepsilon} \frac{\Phi_{N,q,\beta}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx.$$

But by Dominated Convergence Theorem, it is easy to deduce that (since $\frac{\Phi_{N,q,\beta}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta} \to \frac{\Phi_{N,q,\beta}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}$ a.e. and $\frac{\Phi_{N,q,\beta}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta} \leq \frac{\Phi_{N,q,\beta}\left(\alpha L_\varepsilon^{\frac{N}{N-1}}\right)}{|x|^\beta} \in L^1\left(B_{R_\varepsilon}\right)$)

$$\overline{\lim}_{k\to\infty} I_{k,\varepsilon} \leq \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v\right|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx$$

Hence, when we let $\varepsilon \to 0$, we have

$$TME_{p,q,N,\lambda,\alpha,\beta} \leq \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v\right|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx.$$

Thus $v \neq 0$ and then

$$TME_{p,q,N,\lambda,\alpha,\beta} \leq \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+|v|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}}dx$$
$$\leq \frac{1}{\|v\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+\lambda\,|v|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}}dx.$$

As a consequence, v is a maximizer for $TME_{p,q,N,\lambda,\alpha,\beta}$.

Case 2: $\beta = 0$; $\frac{q(N-1)}{N} \notin \mathbb{N}$

We can denote $j_{N,q} \geq 1$ to be the smallest natural number such that $j_{N,q} \geq \frac{q(N-1)}{N}$ (actually, $j_{N,q} > \frac{q(N-1)}{N}$ since $\frac{q(N-1)}{N} \notin \mathbb{N}$). In this case, we have

$$\int\limits_{|x|\geq R_\varepsilon} \frac{\Phi_{N,q,0}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\,|v_k|^{\frac{p}{N-1}}\right)}dx \leq \int\limits_{|x|\geq R_\varepsilon} \frac{\sum\limits_{j=0}^{\infty} \frac{\left(\alpha v_k^{\frac{N}{N-1}}\right)^{j_{N,q}+j}}{(j_{N,q}+j)!}}{\left(1+\lambda\,|v_k|^{\frac{p}{N-1}}\right)}dx$$
$$\leq \int\limits_{|x|\geq R_\varepsilon} v_k^{\frac{N}{N-1}j_{N,q}-q} \frac{\sum\limits_{j=0}^{\infty} \frac{\alpha_N^{j_{N,q}+j} v_k^{q+\frac{N}{N-1}j}}{(j_{N,q}+j)!}}{\left(1+\lambda\,|v_k|^{\frac{p}{N-1}}\right)}dx$$
$$\leq \varepsilon^{\frac{N}{N-1}j_{N,q}-q} \int\limits_{|x|\geq R_\varepsilon} \frac{\sum\limits_{j=0}^{\infty} \frac{\alpha_N^{j_{N,q}+j} v_k^{q+\frac{N}{N-1}j}}{(j_{N,q}+j)!}}{\left(1+\lambda\,|v_k|^{\frac{p}{N-1}}\right)}dx$$
$$\leq \varepsilon^{\frac{N}{N-1}j_{N,q}-q} C_{p,q,N,\lambda}\,\|v_k\|_q^q$$
$$= \varepsilon^{\frac{N}{N-1}j_{N,q}-q} C_{p,q,N,\lambda} \to 0 \text{ as } \varepsilon \to 0.$$

Here, the last inequality comes from Remark 5.10, since the function

$$\Phi\left(\alpha\left|u\right|^{\frac{N}{N-1}}\right)=\sum_{j=0}^{\infty}\frac{\alpha^{j_{N,q}+j}u^{q+\frac{N}{N-1}j}}{(j_{N,q}+j)!}$$

satisfying

$$\begin{aligned}&\Phi\left(\alpha\left|u\right|^{\frac{N}{N-1}}\right)\leq C_{N,q}\exp\left(\alpha\left|u\right|^{\frac{N}{N-1}}\right);\\&\Phi\left(\alpha\left|u\right|^{\frac{N}{N-1}}\right)\leq C_{N,q}\left|u\right|^{q}\text{ for every }\left|u\right|\leq 1.\end{aligned}$$

(the second one is clear since the smallest power in $\Phi\left(\alpha\left|u\right|^{\frac{N}{N-1}}\right)$ is $|u|^q$; To explain the first one, we note that $j_{N,q}>\frac{q(N-1)}{N}>j_{N,q}-1\geq 0:$

$$\Phi\left(\alpha\left|u\right|^{\frac{N}{N-1}}\right)=\sum_{j=0}^{\infty}\frac{\alpha^{j_{N,q}+j}u^{q+\frac{N}{N-1}j}}{(j_{N,q}+j)!}=\sum_{j=0}^{\infty}\frac{\alpha^{j_{N,q}+j}u^{\frac{N}{N-1}\left(j_{N,q}-1+j\right)}u^{\frac{q(N-1)}{N}+1-j_{N,q}}}{(j_{N,q}+j)!}.$$

Since $0<\frac{q(N-1)}{N}+1-j_{N,q}<1$, we can find two positive numbers A and B such that $u^{\frac{q(N-1)}{N}+1-j_{N,q}}\leq Au+B$. Hence

$$\begin{aligned}\Phi\left(\alpha\left|u\right|^{\frac{N}{N-1}}\right)&\leq\sum_{j=0}^{\infty}\frac{\alpha^{j_{N,q}+j}u^{\frac{N}{N-1}\left(j_{N,q}-1+j\right)}\left(Au+B\right)}{(j_{N,q}+j)!}\\&\leq A\sum_{j=0}^{\infty}\frac{\alpha^{j_{N,q}+j}u^{\frac{N}{N-1}\left(j_{N,q}+j\right)}}{(j_{N,q}+j)!}+B\alpha_N\sum_{j=0}^{\infty}\frac{\alpha^{j_{N,q}+j-1}u^{\frac{N}{N-1}\left(j_{N,q}+j-1\right)}}{(j_{N,q}+j-1)!}\\&\leq C_{N,q}\exp\left(\alpha\left|u\right|^{\frac{N}{N-1}}\right)).\end{aligned}$$

The integral $\int\limits_{|x|<R_\varepsilon}\frac{\Phi_{N,q,0}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v_k|^{\frac{p}{N-1}}\right)}dx$ can be dealed similarly as in Case 1. Hence, again we have that v is a maximizer for $TME_{p,q,N,\lambda,\alpha,\beta}$.

Case 3: $\beta = 0$; $\frac{q(N-1)}{N} \in \mathbb{N}$

We assume that in this case, the supremum cannot be attained. Then, we set

$$F(u) = \frac{\Phi_{N,q,0}\left(\alpha u^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|u\right|^{\frac{p}{N-1}}\right)} - \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}u^q.$$

Hence

$$\begin{aligned} TME_{p,q,N,\lambda,\alpha} &= \lim_{k\to\infty}\int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,0}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)}dx \\ &= \lim_{k\to\infty}\int\limits_{\mathbb{R}^N} F(v_k) + \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}. \end{aligned}$$

Again, we will first consider here $\int\limits_{|x|\geq R_\varepsilon} F(v_k)dx$. We have

$$\begin{aligned} \int\limits_{|x|\geq R_\varepsilon} F(v_k)dx &\leq \int\limits_{|x|\geq R_\varepsilon} \frac{\Phi_{N,q,0}\left(\alpha v_k^{\frac{N}{N-1}}\right) - \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}v_k^q}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)} \\ &= \int\limits_{|x|\geq R_\varepsilon} \frac{\sum\limits_{j=0}^{\infty}\frac{\left(\alpha v_k^{\frac{N}{N-1}}\right)^{\frac{q(N-1)}{N}+j}}{\left(\frac{q(N-1)}{N}+j\right)!} - \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}v_k^q}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)} \\ &= \int\limits_{|x|\geq R_\varepsilon} \frac{\sum\limits_{j=1}^{\infty}\frac{\left(\alpha v_k^{\frac{N}{N-1}}\right)^{\frac{q(N-1)}{N}+j}}{\left(\frac{q(N-1)}{N}+j\right)!}}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)} \\ &\leq \int\limits_{|x|\geq R_\varepsilon} \alpha v_k^{\frac{N}{N-1}} \frac{\sum\limits_{j=0}^{\infty}\frac{\left(\alpha v_k^{\frac{N}{N-1}}\right)^{\frac{q(N-1)}{N}+j}}{\left(\frac{q(N-1)}{N}+j\right)!}}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)} \end{aligned}$$

$$\leq \alpha \varepsilon^{\frac{N}{N-1}} \int\limits_{|x|\geq R_\varepsilon} \frac{\Phi_{N,q,0}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)}$$

$$\leq \alpha \varepsilon^{\frac{N}{N-1}} C_{N,q,p,\lambda} \ (\rightarrow 0 \text{ as } \varepsilon \rightarrow 0).$$

Also,

$$\int\limits_{|x|<R_\varepsilon;\ |v_k|\geq L_\varepsilon} F(v_k) \leq \varepsilon \int\limits_{|x|<R_\varepsilon;\ |v_k|\geq L_\varepsilon} \frac{\Phi_{N,q,0}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v_k\right|^{\frac{q}{N-1}}\right)} \leq \varepsilon C_{N,q,p,\lambda} \ (\rightarrow 0 \text{ as } \varepsilon \rightarrow 0).$$

Considering $\int\limits_{|x|<R_\varepsilon;\ |v_k|<L_\varepsilon} F(v_k)$, again we can use the Dominated Convergence Theorem to conclude that

$$\overline{\lim}_{k\rightarrow\infty} \int\limits_{|x|<R_\varepsilon;\ |v_k|<L_\varepsilon} F(v_k) \leq \int\limits_{|x|<R_\varepsilon;\ |v|\leq L_\varepsilon} F(v).$$

Hence, we have

$$TME_{p,q,N,\lambda,\alpha} \leq \alpha \varepsilon^{\frac{N}{N-1}} C_{N,q,p,\lambda} + \varepsilon C_{N,q,p,\lambda} + \int\limits_{|x|<R_\varepsilon;\ |v|\leq L_\varepsilon} \left(\frac{\Phi_{N,q,0}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v\right|^{\frac{p}{N-1}}\right)} - \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} v^q \right) dx + \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}.$$

Letting $\varepsilon \rightarrow 0$, and noting that $R_\varepsilon \rightarrow \infty;\ L_\varepsilon \rightarrow \infty$, we get

$$TME_{p,q,N,\lambda,\alpha} \leq \int\limits_{\mathbb{R}^N} \left(\frac{\Phi_{N,q,0}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v\right|^{\frac{p}{N-1}}\right)} - \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} v^q \right) dx + \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}.$$

If $v \neq 0$, then

$$TME_{p,q,N,\lambda,\alpha} \leq \int\limits_{\mathbb{R}^N} \left(\frac{\Phi_{N,q,0}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v\right|^{\frac{p}{N-1}}\right)} - \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} v^q \right) dx + \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}$$

$$\leq \frac{1}{\|v\|_q^q} \int\limits_{\mathbb{R}^N} \left(\frac{\Phi_{N,q,0}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v\right|^{\frac{p}{N-1}}\right)} - \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} v^q \right) dx + \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}$$
$$= \frac{1}{\|v\|_q^q} \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,0}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v\right|^{\frac{p}{N-1}}\right)} dx.$$

In other words, v is a maximizer for $TME_{p,q,N,\lambda,\alpha}$.

Hence $v = 0$, then $TME_{p,q,N,\lambda,\alpha} \leq \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}$. This is impossible either when $p > N$ or $q < p < \frac{N-1}{N-2}q$ with $\alpha = \alpha_N$ by Lemma 5.11.

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ABSTRACT

MULTPARAMETER AND MULTILINEAR PSEUDO-DIFFERENTIAL OPERATORS AND TRUDINGER-MOSER INEQUALITIES

by

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August 2016

Advisor: Dr. Guozhen Lu

Major: Mathematics

Degree: Doctor of Philosophy

Pseudo-differential operators play important roles in harmonic analysis, several complex variables, partial differential equations and other branches of modern mathematics. We s-tudied some types of multilinear and multiparameter Pseudo-differential operators. They include a class of trilinear Pseudo-differential operators, where the symbols are in the for-m of products of Hörmader symbols defined on lower dimensions, and we established the Hölder type L^p estimates for such operators. They derive from the trilinear Coifman-Meyer type operators with flag singularities. And we also studied a class of bilinear bi-parameter Pseudo-differential operators, where the symbols are taken from the general Hörmander class, and we studied the restriction for the order of the symbols which could imply the Hölder type L^p estimates. Such types of operators are motivated by the Calderón-Vaillancourt theorem in single parameter setting.

Trudinger-Moser inequalities can be treated as the limiting case of the Sobolev embed-dings. Sharp Trudinger-Moser inequalities on the first order Sobolev spaces and their anal-

ogous Adams inequalities on high order Sobolev spaces play an important role in geometric analysis, partial differential equations and other branches of modern mathematics. There are two types of such optimal inequalities: critical and subcritical sharp inequalities, both are with best constants. Critical sharp inequalities are under the restriction of the full Sobolev norms for the functions under consideration, while the subcritical inequalities are under the restriction of the partial Sobolev norms for the functions under consideration. There are subtle differences between these two type of inequalities. Surprisingly, we proved that these critical and subcritical Trudinger-Moser and Adams inequalities are actually equivalent.

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List of Publications and Preprints

1. *L^p estimate for multi-parameter and multilinear Fourier multipliers and Pseudo-differential operators,* Some Topics in Harmonic Analysis and Applications, ALM34 (2015), 113-144, (with W.Dai and G. Lu).
2. *L^p estimate for a tri-linear Pseudo-differential operator,* accepted by Indiana Univ. Math. J. (2015), (with G. Lu).
3. *Bi-Parameter and bilinear Calderón-vaillancourt theorem with critical order,* accepted by Forum Mathematicum, (with G. Lu, 2015).
4. *Equivalence of critical and subcritical sharp Trudinger-Moser-Adams inequalities*, arXiv:1504.04858, accepted by Rev. Math. Iberoam, (with N. Lam and G. Lu).
5. *L^p boundedness of bi-parameter Fourier integral operators*, arXiv:1510.00986, (with Q. Hong and G. Lu).