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Well-Posedness Properties In Variational Analysis And Its Applications

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WELL-POSEDNESS PROPERTIES IN VARIATIONAL ANALYSIS AND ITS APPLICATIONS

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DEDICATION

To Dad and Mom.

Thank you for your endless support and encouragement.

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CHAPTER 1

Introduction

This dissertation mainly concerntrates on the study and applications of some significant properties in well-posedness and sensitivity analysis, which includes the notions of *uniform* metric regularity, *higher-order* metric *sub*regularity and its *strong* subregularity counterpart. For definiteness, we use the number $q > 0$ to indicate the *order/rate* of the corresponding regularity under consideration. Recall first that a set-valued mapping $F : X \rightrightarrows Y$ between Banach spaces is *metrically q-regular* at (better *around*) $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there exist a number $\eta > 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d\big(x; F^{-1}(y)\big) \le \eta\, d^q\big(y; F(x)\big) \ \text{ for all } \ x \in U \ \text{ and } \ y \in V, \tag{1.1}$$

where $d(\cdot; \Omega)$ is the distance function associated with Ω. It has been well recognized in nonlinear and variational analysis that metric regularity ($q = 1$) and the equivalent notions of linear openness and Lipschitzian stability play an important role in optimization, control, equilibria, and various applications as documented, e.g., in the books [5, 10, 21, 25] with many references therein. On the other hand, metric regularity often fails for broad classes of parametric variational systems given by the generalized equations in the sense of Robinson [24]:

$$0 \in f(x, y) + Q(y), \tag{1.2}$$

where f is single-valued while $Q(y) = \partial\varphi(y)$ is a set-valued mapping of the subdifferential/normal cone type generated by nonsmooth functions; see [22] and also [2, 3, 4, 15, 17, 28] for more details and further results in this direction. However, this phenomenon does not appear if metric regularity is replaced by a weaker property of *metric subregularity* of F at $(\bar{x}, \bar{y})$ defined by

$$d\big(x; F^{-1}(\bar{y})\big) \le \eta\, d\big(\bar{y}; F(x)\big) \text{ for all } x \in U. \tag{1.3}$$

Considering $\|x - \bar{x}\|$ instead of $d(x; F^{-1}(\bar{y}))$ in (1.3), we get the notion of *strong metric subregularity.* In the aforementioned books and in an increasing number of papers, the reader can find more information about these subregularity properties, their *calmness* (resp. *isolated calmness*) equivalents for inverse mappings, as well as their various applications to optimization.

In [6, 13, 27, 29], the authors studied the notion of *Hölder metric regularity*, which corresponds to (1.1) with the replacement of $d(y; F(x))$ by $d^q(y; F(x))$ as $0 < q < 1$. Replacing $d(\bar{y}; F(x))$ by $d^q(\bar{y}; F(x))$ in (1.3) as $0 < q < 1$ gives us the notion of *Hölder metric subregularity* considered recently in [14, 18, 19] from different viewpoints while without its *strong* counterpart.

It is essential to mention that there is no sense to study metric q-regularity of single-valued or set-valued mappings for $q > 1$, since only constant mappings satisfy this property. However, it is not the case for q-subregularity that is equally important whenever $q > 0$ as demonstrated in this paper, where—to the best of our knowledge—the notion of q-subregularity for $q > 1$ is studied and applied for the first time in the literature.

In what follows we investigate both notions of metric q-subregularity and strong metric q-subregularity for any positive q concentrating mainly on the higher-order case of $q > 1$. In this way we derive verifiable sufficient conditions and necessary conditions for these notions of q-subregularity in terms of appropriate generalized differential constructions of variational analysis, study their behavior with respect to perturbations, and obtain their applications to the rate of convergence of Newton's and quasi-Newton methods for solving generalized equations.

Accordingly, we organize the rest of the paper. Chapter 2 contains some preliminaries from variational analysis and generalized differentiation widely used in the formulations and proofs of the main results given below. Chapter 3 is devoted to a detailed study of q-subregularity of set-valued mappings between general Banach and Asplund spaces concentrating mainly on subdifferential mappings. In addition to deriving verifiable conditions that imply and are implied by these notions, we compare them (when appropriate) with the corresponding notions of metric regularity and provide several numerical examples illustrating the new phenomena.

Chapter 4 studies behavior of strong metric q-subregularity as $q \geq 1$ under parameter perturbations. The obtained results, being of their own interest, allow us to establish the convergence rate for Newton's and quasi-Newton methods of solving generalized equations depending on the order of the strong metric subregularity for the underlying set-valued mapping in the generalized equation under consideration.

Chapter 5 is devoted to investigating metric q-subregularity for constraint system of the form $0 \in f(x) - C$, where f is a Lipchitzian mapping and C is a closed set. Verifiable sufficient conditions as well as modulus estimation of q-subregularity are obtained through the approach of projection points. In addition, we established a relationship between metric q-subregularity of the constraint system and the $\frac{1}{q}$-order growth condition on f for any positive order q. Concrete examples are also given to illustrate the phenomena. The second part of this chapter concentrates on *uniform* metric regularity for a collection of multifunctions $\{G_i\}$ of the aforementioned constraint form. Through the approach of Lyusternik-Graves iterative process, we established a sufficient condition for this uniform regularity to be true.

Chapter 6 studies the optimality conditions of parametric variational systems (PVS) under equilibrium constraints, see [20, 31, 33] for more details in this direction. The first part of this chapter involves coderivative estimation for the solution mapping of (PVS) as well as canonically perturbed variational systems, which allows us to apply the results obtained to the study of constrained optimization and equilibrium problems with possibly nonsmooth data. We derived *necessary optimality* and *suboptimality* conditions for various problems of constrained optimization and equilibria such as MPECs with amenable/full rank potentials and EPECs with closed preferences in finite-dimensional spaces.

CHAPTER 2

Preliminary

2.1 Basic Notation

Throughout the paper we use standard notation of variational analysis and generalized differentiation. Recall that, given a set-valued mapping $F\colon X \rightrightarrows X^*$ from the Banach space X into its topological dual X^* endowed with the weak* topology w^*, the symbol

$$\begin{aligned} \operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \Big|\ \ & \exists\,\text{seqs. } x_k \to \bar{x},\ x_k^* \stackrel{w^*}{\to} x^* \text{ such that} \\ & x_k^* \in F(x_k) \text{ for all } k \in I\!N\}\Big\} \end{aligned} \tag{2.1}$$

signifies the *sequential Painlevé-Kuratowski outer limit* of F as $x \to \bar{x}$, where $I\!N := \{1, 2, \ldots\}$. Given a set $\Omega \subset I\!R^n$ and an extended-real-valued function $\varphi\colon I\!R^n \to \overline{I\!R} := (-\infty, \infty]$ finite at $\bar{x}$, the symbols $x \stackrel{\Omega}{\to} \bar{x}$ and $x \stackrel{\varphi}{\to} \bar{x}$ stand for $x \to \bar{x}$ with $x \in \Omega$ and for $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$, respectively. As usual, $I\!B(x, r) = I\!B_r(x)$ denotes the closed ball of the space in question centered at x with radius $r > 0$, while the symbols $I\!B$ and $I\!B^*$ signify the corresponding closed unit ball in the primal and dual spaces, respectively. Dealing with empty sets, we let $\inf \phi := \infty$, $\sup \phi := -\infty$, and $\|\phi\| := \infty$.

Definition 2.1 *[21, (1.22)] A set-valued mapping $F : X \rightrightarrows Y$ is said to be* POSITIVELY HOMOGENEOUS *if $0 \in F(0)$ and $F(\alpha x) \supset \alpha F(x)$ for all $x \in X$ and $\alpha > 0$, or equivalently, when the graph of F is a cone in $X \times Y$. The* NORM *of a positively homogeneous mapping F is defined by*

$$\|F\| := \sup\{\|y\| \ \big|\ y \in F(x) \text{ and } \|x\| \le 1\}.$$

The following lemma will be helpful in our later proof.

Lemma 2.2 *[21, Theorem 1.54] Let H be a positively homogeneous mapping from X to Y. then the following equality holds:*

$$\|H^{-1}\| = \frac{1}{\inf\{\|v\| \mid v \in H(u), \|u\| = 1\}}$$

Recall that for a set-valued mapping $F : X \rightrightarrows Y$, the DOMAIN of F is defined as $\operatorname{dom} F := \{x \in X \mid F(x) \neq \phi\}$.

Definition 2.3 (inner semicontinuous and inner semicompact multifunctions) *[21, Definition 1.63] Let $S : X \rightrightarrows Y$ with $\bar{x} \in \operatorname{dom} S$.*

(i) Given $\bar{y} \in S(\bar{x})$, we say that the mapping S is INNER SEMICONTINUOUS *at $(\bar{x}, \bar{y})$ if for every sequence $x_k \to \bar{x}$ there is a sequence $y_k \in S(x_k)$ converging to $\bar{y}$ as $k \to \infty$.*

(ii) S is INNER SEMICOMPACT *at $\bar{x}$ if for every sequence $x_k \to \bar{x}$ there is a sequence $y_k \in S(x_k)$ that contains a convergent subsequence as $k \to \infty$.*

Finally, given a mapping $g\colon X \to Y$ between Banach spaces that is locally Lipschitzian around $\bar{x}$, we denote

$$\operatorname{lip} g(\bar{x}) := \limsup_{x,u \to \bar{x}} \frac{\|g(x) - g(u)\|}{\|x - u\|}.$$

2.2 Variational Geometry

Throughout this section, Let A be a closed subset of a Banach space X and $a \in A$. The CLARKE TANGENT CONE to A at a is defined as

$$T_c(a, A) := \liminf_{x \xrightarrow{A} a, t \to 0^+} \frac{A - x}{t}$$

where $x \xrightarrow{A} a$ means that $x \to a$ with $x \in A$. Thus, $v \in T_c(a, A)$ if and only if, for each sequence $\{a_n\}$ in A converging to a and each sequence $\{t_n\}$ in $(0, +\infty)$ decreasing to 0, there exists a sequence $\{v_n\}$ in X converging to v such that $a_n + t_n v_n \in A$ for all $n \in I\!N$.

The Bouligand (or Contingent) tangent cone to A at a is defined as

$$T_B(a, A) := \left\{ \lim_{k\to\infty} \frac{x_k - a}{t_k} : x_k \xrightarrow{A} a, t_k \to 0^+ \right\}$$

The Contingent tangent cone can be characterized by virtue of the distant function and therefore we have the equivalence relationship:

$$T_C(a, A) = T_B(a, A)$$

when A is convex due to the fact that the distant function of a convex set is convex as well.

Definition 2.4 (weak contigent derivative) *For a set-valued function $F : X \rightrightarrows Y$, the* weak contigent derivative *of F at $\bar{x}$ for $\bar{y}$, where $\bar{y} \in F(\bar{x})$, is the set-valued mapping $C_w F(\bar{x}, \bar{y}) : X \rightrightarrows Y$ where for each $u \in X$, the set $C_w F(\bar{x}, \bar{y})(u)$ is given by the set of all $v \in Y$ for which there are sequences $\{t_k\} \downarrow 0$ and $(u_k, v_k) \xrightarrow{w} (u, v)$ with $(\bar{x} + t_k u_k, \bar{y} + t_k v_k) \in \operatorname{gph} F$.*

Remark 2.5 We claim that $C_w F(\bar{x}, \bar{y}) : X \rightrightarrows Y$ is a positive homogeneous mapping. To prove this is true, it suffices to show that (i) $0 \in C_w F(\bar{x}, \bar{y})(0)$ which is obvious; and (ii) for any positive λ, $\lambda C_w F(\bar{x}, \bar{y}) \subset C_w F(\bar{x}, \bar{y})(\lambda u)$ holds. Indeed, if we select any $v \in C_w F(\bar{x}, \bar{y})(u)$, then there exists $t_k \downarrow 0$, $(u_k, v_k) \xrightarrow{w} (u, v)$ such that $(\bar{x} + t_k u_k, \bar{y} + t_k v_k) \in gphF$. For any $\lambda > 0$, by rescaling $\widetilde{t_k} = \frac{t_k}{\lambda}$, one has $\widetilde{t_k} \downarrow 0$ and $(\bar{x} + \widetilde{t_k} u_k, \bar{y} + \widetilde{t_k} v_k) \in gphF$, which means that $\lambda v \in C_w F(\bar{x}, \bar{y})(\lambda u)$. Hence (ii) is true.

A natural counterpart to "tangent cone" is "normal cone", which we develop next. The

Clarke normal cone to A at a is defined as:

$$N_c(a,A) := \{x^* \in X^* : \langle x^*, h\rangle \leq 0 \quad \forall h \in T_c(a,A)\}$$

For $\varepsilon \geq 0$, we define the Fréchet ε-normals of A at a by

$$\hat{N}_\varepsilon(a,A) := \left\{ x^* \in X^* : \limsup_{x \xrightarrow{A} a} \langle x^*, x-a\rangle / \parallel x-a \parallel \leq \varepsilon \right\}.$$

When $\varepsilon = 0$, it is simply denoted by $\hat{N}(A,a)$ and called as REGULAR NORMAL CONE (also known as FRECHET NORMAL CONE) to A at a. Then the LIMITING (MORDUKHOVICH/BASIC) NORMAL CONE to A at a is defined by

$$N(a,A) := \limsup_{x \xrightarrow{A} a, \varepsilon \to 0^+} \hat{N}_\varepsilon(x,A).$$

Thus, $x^* \in N(a,A)$ if and only if there exists a sequence $\{(x_n, \varepsilon_n, x_n^*)\}$ in $A \times R_+ \times X^*$ such that $(x_n, \varepsilon_n) \to (a,0)$, $x_n^* \xrightarrow{w^*} x^*$ and $x_n^* \in \hat{N}_{\varepsilon_n}(x_n, A)$ for each $n \in I\!N$. It is well-known that we have the following relationship regarding the aforementioned normal cones:

$$\hat{N}(a,A) \subset N(a,A) \subset N_c(a,A).$$

When A is convex, the above constructions all reduce to the classical case of convex analysis:

$$N_c(a,A) = \hat{N}(a,A) = N(a,A) = \{x^* \in X^* : \langle x^*, x-a\rangle \leq 0 \ \ \forall x \in A\}.$$

Recall that a Banach space X is called an ASPLUND space if every continuous convex function on X is Fréchet differentiable at each point of a dense subset of X. It is well known that X is an Asplund space if and only if every separable subspace of X has a separable dual. In particular, every reflexive Banach space is an Asplund space. When X is an Asplund space, Mordukhovich and Shao [21, Theorem 3.57] proved that

$$N_c(a, A) = \overline{\text{co}}^{w^*}(N(a, A)) \quad \text{and} \quad N(a, A) = \limsup_{x \xrightarrow{A} a} \hat{N}(x, A).$$

The following type of limiting normal cone to moving sets is useful in both finite and infinite dimensions.

Definition 2.6 (limiting normals to moving sets.) *[21, Definition 5.69] Let $S : Z \rightrightarrows X$ be a set-valued mapping from a metric space Z into a Banach space X, and let $(\bar{z}, \bar{x}) \in \operatorname{gph} S$. Then*

$$N_+(\bar{x}, S(\bar{z})) := \limsup_{(z,x) \overset{\operatorname{gph} s}{\to} (\bar{z},\bar{x}), \varepsilon \to 0^+} \hat{N}_\varepsilon(x; S(z)).$$

is the extended normal cone to $S(\bar{z})$ at $\bar{x}$. One can equivalently put $\epsilon = 0$ in the above definition if X is Asplund and S is closed-valued around $\bar{x}$.

The following Theorem is helpful in the proof of our later result.

Theorem 2.7 (basic normals to inverse images) *[21, Theorem 3.8] Let $\bar{x} \in F^{-1}(\Theta)$, where $F : X \rightrightarrows Y$ is a closed-graph mapping and where $\Theta \subset Y$ is a closed set. Assume that the set-valued mapping $x \to F(x) \cap \Theta$ is inner semicompact at $\bar{x}$ and that for every $\bar{y} \in F(\bar{x}) \cap \Theta$ the following hold:*

(a) Either F^{-1} is PSNC at $(\bar{y}, \bar{x})$ or Θ is SNC at $\bar{y}$.

(b) $\{F,\Theta\}$ *satisfies the qualification condition*

$$N(\bar{y};\Theta)\cap\ker\widetilde{D}^*_M F(\bar{x},\bar{y})=\{0\}.$$

Then one has

$$N(\bar{x};F^{-1}(\Theta))\subset\bigcup[D^*_N F(\bar{x},\bar{y})(y^*)|y^*\in N(\bar{y};\Theta),\bar{y}\in F(\bar{x})\cap\Theta].$$

2.3 Coderivatives of Set-Valued Mappings

In this section, we let $F:X\rightrightarrows Y$ be a multifunction and denote by $\operatorname{gph}F$ the graph of F, that is ,

$$\operatorname{gph}F:=\{(x,y)\in X\times Y: y\in F(x)\}.$$

As usual, F is said to be closed (resp., convex) if $\operatorname{gph}F$ is a closed (resp., convex) subset of $X\times Y$. It is known that F is convex if and only if

$$tF(x_1)+(1-t)F(x_2)\subset F(tx_1+(1-t)x_2)\quad\forall x_1,x_2\in X\,\text{and}\,\forall t\in[0,1]$$

Following the scheme in [20], we define its REGULAR CODERIVATIVE and (BASIC, NORMAL) CODERIVATIVE at the graphical point $(\bar{x},\bar{y})\in\operatorname{gph}F$ by

$$\begin{aligned}\widehat{D}^*F(\bar{x},\bar{y})(y^*)&:=\Big\{x^*\in X^*\big|\ (x^*,-y^*)\in\widehat{N}\big((\bar{x},\bar{y});\operatorname{gph}F\big)\Big\},\quad y^*\in Y^*\\ D^*F(\bar{x},\bar{y})(y^*)&:=\Big\{x^*\in X^*\big|\ (x^*,-y^*)\in N\big((\bar{x},\bar{y});\operatorname{gph}F\big)\Big\},\quad y^*\in Y^*,\end{aligned}\tag{2.2}$$

respectively. The coderivative mappings from Y^* into X^* in (2.2) are clearly positive-homogeneous. Furthermore, if $F\colon I\!R^n\to I\!R^m$ is single-valued (then we omit $\bar{y}=F(\bar{x})$ in the coderivative notation) and *strictly differentiable* at $\bar{x}$ (which is automatic when it is $\mathcal{C}^1$ around this point), then both coderivatives in (2.2) are also single-valued and reduces to the *adjoint* derivative operator

$$\widehat{D}^*F(\bar{x})(y^*) = D^*F(\bar{x})(y^*) = \big\{\nabla F(\bar{x})^*y^*\big\}, \quad y^* \in Y^*.$$

Let's provide next some calculus results for coderivatives of set-valued mapppings between arbitrary Banach spaces.

Theorem 2.8 (Coderivative sum rules with equalities) *[21, Theorem 1.62] Let $f : X \to Y$ be Fréchet differentiable at $\bar{x}$, and let $F : X \rightrightarrows Y$ be an arbitrary set-valued mapping such that $\bar{y} - f(\bar{x}) \in F(\bar{x})$ for some $\bar{y} \in Y$. The following hold:*

(i) For all $y^* \in Y^*$ one has

$$\hat{D}^*(f+F)(\bar{x},\bar{y})(y^*) = \triangledown f(\bar{x})^*y^* + \hat{D}^*F(\bar{x},\bar{y}-f(\bar{x}))(y^*).$$

(ii) If f is strictly differentiable at $\bar{x}$, then

$$D^*(f+F)(\bar{x},\bar{y})(y^*) = \triangledown f(\bar{x})^*y^* + D^*F(\bar{x},\bar{y}-f(\bar{x}))(y^*)$$

for all $y^* \in Y^*$, where D^* stands either for the normal coderivative or mixed coderivative. Moreover, the mapping $f+F$ is N-regular(resp. M-regular) at $(\bar{x},\bar{y})$ if and only if F is N-regular (resp. M-regular) at the point $(\bar{x},\bar{y}-f(\bar{x}))$.

Theorem 2.9 (scalarization of the mixed coderivative) *. Let $f : X \to Y$ be continuous around $\bar{x}$. Then*

$$\partial\langle y^*, f\rangle(\bar{x}) \subset D^*_M f(\bar{x})(y^*) \text{ for all } y^* \in Y^*.$$

If in addition f is Lipschitz continuous around $\bar{x}$, then

$$D^*_M f(\bar{x})(y^*) = \partial\langle y^*, f\rangle(\bar{x}) \text{ for all } y^* \in Y^*.$$

2.4 Subdifferentials of Nonsmooth Functions

In this section we present for the reader's convenience some basic tools of generalized differentiation widely employed in what follows. We refer to the books [5, 21, 25, 26] for more details in both finite and infinite dimensions.

Given $\varphi\colon X \to \overline{I\!R}$ with $\bar{x} \in \operatorname{dom}\varphi$, the *regular subdifferential* (known also as the presubdifferential and as the Fréchet or viscosity subdifferential) of φ at $\bar{x}$ is defined by

$$\widehat{\partial}\varphi(\bar{x}) := \Big\{ v \in I\!R^n \Big| \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle v, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq 0 \Big\}. \tag{2.3}$$

It reduces to $\{\nabla\varphi(\bar{x})\}$ if φ is Fréchet differentiable at $\bar{x}$ and to the subdifferential of convex analysis if φ is convex, while the set $\widehat{\partial}\varphi(\bar{x})$ may often be empty for nonconvex and nonsmooth functions as, e.g., for $\varphi(x) = -|x|$ at $\bar{x} = 0 \in I\!R$. A serious disadvantage of (2.3) is the failure of standard calculus rules required in variational analysis and its applications to optimization.

We come to the different picture while performing a limiting procedure/robust regularization over the mapping $x \mapsto \widehat{\partial}\varphi(x)$ as $x \xrightarrow{\varphi} \bar{x}$ vis the sequential outer limit (2.1), which gives us the (basic first-order) *subdifferential* of φ at $\bar{x}$ defined by

$$\partial\varphi(\bar{x}) := \operatorname*{Lim\,sup}_{x \xrightarrow{\varphi} \bar{x}} \widehat{\partial}\varphi(x) \tag{2.4}$$

and known also as the general, or limiting, or Mordukhovich subdifferential. The construction

$$\partial^{\infty}\varphi(\bar{x}) := \operatorname*{Lim\,sup}_{x \xrightarrow{\varphi} \bar{x},\, \lambda \downarrow 0} \lambda \widehat{\partial}\varphi(x) \tag{2.5}$$

is known as the *singular* (or horizon) *subdifferential* of φ at $\bar{x}$; it reduces to $\{0\}$ if and only if φ is locally Lipschitzian around $\bar{x}$. In contrast to (2.3), the set (2.4) is often nonconvex (e.g., $\partial\varphi(0) = \{-1, 1\}$ for $\varphi(x) = -|x|$) enjoying nevertheless comprehensive calculus based on variational/extremal principles of variational analysis.

Recall that φ is said to be PROPER if $\varphi(x) > -\infty$ for all $x \in X$ and its DOMAIN

$$\mathrm{dom}(\varphi) := \{x \in X :\ \varphi(x) < +\infty\}$$

is not empty. With such φ, we write its EPIGRAPH as

$$\mathrm{epi}(\varphi) := \{(x, t) \in X \times \mathbb{R} :\ \varphi(x) \le t\}.$$

It is well known that

$$\hat{\partial}\phi(x) \subset \partial\phi(x).$$

When φ is convex, the Mordukhovich and Fréchet subdifferentials reduce to the one in the sense of convex analysis, i.e.

$$\partial\varphi(x) = \hat{\partial}\varphi(x) = \{x^* \in X^* :\ \langle x^*, y - x\rangle \le \varphi(y) - \varphi(x) \quad \forall y \in X\} \quad \forall x \in \mathrm{dom}(\varphi).$$

For a closed set A in X, let δ_A denote the indicator function of A, then we have

$$N(a, A) = \partial\delta_A(a) \quad \forall a \in A$$

Besides the above analytical definition of subgradient, we may as well describe the generalized differentiation through a geometric approach as follows:

$$\partial\varphi(x) = \{x^* \in X^* :\ (x^*, -1) \in N((x, \varphi(x)), \mathrm{epi}(\varphi))\} \quad \forall x \in \mathrm{dom}(\varphi).$$

Recall that the singular subdiffernetial $\partial^\infty\varphi(x)$ of φ at x is defined by

$$\partial^\infty\varphi(x) := \{x^* \in X^* :\ (x^*, 0) \in N((x, \varphi(x)), \mathrm{epi}(\varphi))\}.$$

It is clear that if φ is convex then

$$\partial^\infty\varphi(x) = N(x, \mathrm{dom}(\varphi)).$$

Now let us present some *second-order* subdifferential constructions for extended-real-valued functions, which are at the heart of the variational techniques developed in this paper. Given $\varphi\colon I\!\!R^n \to \overline{I\!\!R}$ finite at $\bar{x}$, pick a subgradient $\bar{y} \in \partial\varphi(\bar{x})$ and, following Mordukhovich [30], we introduce the *second-order subdifferential* (or generalized Hessian) of φ at $\bar{x}$ relative to $\bar{y}$ by

$$\partial^2\varphi(\bar{x}, \bar{y})(u) := (D^*\partial\varphi)(\bar{x}, \bar{y})(u), \quad u \in I\!\!R^n, \tag{2.6}$$

via the coderivative (2.2) of the first-order subdifferential mapping (2.4). Observe that for

$\varphi \in \mathcal{C}^2$ with the (symmetric) Hessian matrix $\nabla^2\varphi(\bar{x})$ we have

$$\partial^2\varphi(\bar{x})(u) = \big\{\nabla^2\varphi(\bar{x})u\big\} \text{ for all } u \in I\!R^n.$$

Referring the reader to the book [21] and the recent paper [32] (as well as the bibliographies therein) for the theory and applications of the second-order subdifferential (2.6). From now on we focus on an appropriate partial counterpart of (2.6) for functions $\varphi\colon I\!R^n \times I\!R^d \to \overline{I\!R}$ of two variables $(x,w) \in I\!R^n \times I\!R^d$. Consider the partial first-order subgradient mapping

$$\partial_x\varphi(x,w) := \Big\{\text{set of subgradients } v \text{ of } \varphi_w := \varphi(\cdot,w) \text{ at } x\Big\} = \partial\varphi_w(x), \tag{2.7}$$

take $(\bar{x},\bar{w})$ with $\varphi(\bar{x},\bar{w}) < \infty$, and define the *extended partial second-order subdifferential* of φ with respect to x at $(\bar{x},\bar{w})$ relative to some $\bar{y} \in \partial_x\varphi(\bar{x},\bar{w})$ by

$$\widetilde{\partial}^2_x\varphi(\bar{x},\bar{w},\bar{y})(u) := (D^*\partial_x\varphi)(\bar{x},\bar{w},\bar{y})(u), \quad u \in I\!R^n. \tag{2.8}$$

This second-order construction was first employed by Levy, Poliquin and Rockafellar [40] for characterizing full stability of extended-real-valued functions in the unconstrained format of optimization. Note that the second-order construction (2.8) is different from the standard partial second-order subdifferential

$$\partial^2_x\varphi(\bar{x},\bar{w},\bar{y})(u) := (D^*\partial\varphi_{\bar{w}})(\bar{x},\bar{y})(u) = \partial^2\varphi_{\bar{w}}(\bar{x},\bar{y})(u), \quad u \in I\!R^n,$$

of $\varphi = \varphi(x,w)$ with respect to x at $(\bar{x},\bar{w})$ relative to $\bar{y} \in \partial_x\varphi(\bar{x},\bar{w})$, even in the classical $\mathcal{C}^2$

setting. Indeed, for such functions φ with $\bar{y} = \nabla_x \varphi(\bar{x}, \bar{w})$ we have

$$\partial_x^2 \varphi(\bar{x}, \bar{w})(u) = \left\{ \nabla_{xx}^2 \varphi(\bar{x}, \bar{w})u \right\} \text{ while}$$

$$\widetilde{\partial}_x^2 \varphi(\bar{x}, \bar{w})(u) = \{(\nabla_{xx}^2 \varphi(\bar{x}, \bar{w})u, \nabla_{xw}^2 \varphi(\bar{x}, \bar{w})u)\} \text{ for all } u \in I\!R^n. \tag{2.9}$$

This is due to the fact that the definition of the extended partial second-order subdifferential set $\widetilde{\partial}_x^2 \varphi(\bar{x}, \bar{w})(u)$ involves the limiting procedure of $w \to \bar{w}$.

In this paper, we only consider *extended partial second-order subdifferential* of φ and for convenience, we write it as $\partial_x^2 \varphi(\bar{x}, \bar{w}, \bar{y})(u)$ unless otherwise specified. $\square$

CHAPTER 3

Characterization of Metric q-Subregularity

3.1 Overview

This chapter is devoted to investigating the notion of metric q-subregularity for set-valued mappings in the general framework of Banach and Asplund space settings. It is introduced as Hölder metric subregularity in the case of $q \leq 1$, which has been extensively studied in terms of coderivative by Li and Mordukhovich in [19]. Different from that, we consider in this paper mainly the notion of q-subregularity for the case of $q > 1$.

Let us introduce the basic definition of positive-order metric subregularity for arbitrary set-valued mappings between Banach spaces.

Definition 3.1 (metric q-subregularity and strong q-subregularity) *Let $F : X \rightrightarrows Y$ with $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, and let $q > 0$. We say that:*

(i) *F is* METRICALLY q-SUBREGULAR *at $(\bar{x}, \bar{y})$ if there are constants $\eta, \gamma > 0$ such that*

$$d\big(x; F^{-1}(\bar{y})\big) \leq \eta d^q\big(\bar{y}; F(x)\big) \ \textit{ for all } \ x \in I\!B(\bar{x}; \gamma) \tag{3.1}$$

The infimum over all constants/moduli $\eta > 0$ for which (3.1) *holds with some $\gamma > 0$ is called the* EXACT Q-SUBREGULARITY BOUND *of F at $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{subreg}^q F(\bar{x}, \bar{y})$.

(ii) *F is* STRONGLY METRICALLY q-SUBREGULAR *at $(\bar{x}, \bar{y})$ if there are $\eta, \gamma > 0$ such that*

$$\|x - \bar{x}\| \leq \eta d^q\big(\bar{y}; F(x)\big) \ \textit{ for all } \ x \in I\!B(\bar{x}; \gamma). \tag{3.2}$$

The infimum over all $\eta > 0$ for which (3.2) *holds with some $\gamma > 0$ is called the* EXACT STRONG Q-SUBREGULARITY BOUND *of F at $(\bar{x}, \bar{y})$ and is denoted by* $\text{ssubreg}^{\,q} F(\bar{x}, \bar{y})$.

For brevity, in what follows we omit the adjective "metric" for q-subregularity. It is easy to see from the definitions that the strong q-subregularity of F at $(\bar{x}, \bar{y})$ implies the corresponding q-subregularity of F. Furthermore, the validity of $\bar{q}$-subregularity (resp. strong $\bar{q}$-subregularity) of F ar $(\bar{x}, \bar{y})$ for the fixed number $\bar{q} > 0$ ensures this property for any $0 < q \leq \bar{q}$.

Clearly, the larger q in the above subregularity properties the better the corresponding estimate (error bound) in (3.1) and (3.2) is. The following simple one-dimensional example shows that it makes sense to consider the q-subregularity property of order $q > 1$, in contrast to its metric regularity counterpart of such (higher) orders, even in the case of real functions.

Example 3.2 (q-subregularity of higher order) Consider the continuous function $f(x) := |x|^{\frac{1}{2}}$, $x \in I\!\!R$, which is not Lipschitz continuous around $\bar{x} = 0$. We have

$$|x| \leq |x^{\frac{1}{2}}|^q \text{ for any } q \in (0, 2] \text{ and all } x \in I\!\!B(0, 1).$$

This shows that f is strongly q-subregular at $(0, 0)$ whenever $q \in (0, 2]$.

The next example is more involved, being still one-dimensional, and reveals an interesting phenomenon: a set-valued mapping may not be metrically regular around the given point while it is metrically subregular at this point with some $q > 1$. This example concerns in fact solution maps of the parametric generalized equations of type (1.2), which fails to have the metric regularity property in common situations; see Chapter 1.

Example 3.3 (q-subregular but not metrically regular solution maps to parametric

generalized equations) Consider the solution map

$$S(x) = \left\{ y \in R^n \middle|\, 0 \in f(x,y) + Q(y) \right\} \tag{3.3}$$

of the parametric generalized equation (1.2) with $f(x,y) := x$ and $Q\colon I\!R \rightrightarrows I\!R$ given by

$$Q(y) := \begin{cases} [\frac{1}{2^{k+1}}, \frac{1}{2^k}] & \text{for} \quad y = \frac{1}{(\sqrt[3]{2})^k}, \\ \frac{1}{2^{k+1}} & \text{for} \quad y \in \left(\frac{1}{(\sqrt[3]{2})^{k+1}}, \frac{1}{(\sqrt[3]{2})^k}\right), \\ 0 & \text{for} \quad y = 0, \\ [-\frac{1}{2^k}, -\frac{1}{2^{k+1}}] & \text{for} \quad y = -\frac{1}{(\sqrt[3]{2})^k}, \\ -\frac{1}{2^{k+1}} & \text{for} \quad y \in \left(-\frac{1}{(\sqrt[3]{2})^k}, -\frac{1}{(\sqrt[3]{2})^{k+1}}\right) \end{cases}$$

as depicted in Figure 1. Then S is not metrically regular around $(0,0)$ while it is strongly q-subregular of any order $q \in (0,2]$ at this point.

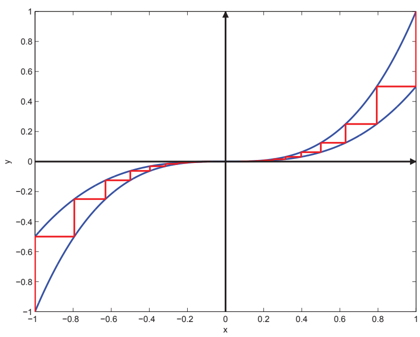

Figure 3.1: Q(y)

Indeed, the failure of metric regularity of S in (3.3) around $(0,0)$ follows from more general results of [22]. Let us verify this directly for the mapping S under consideration. Due to the form of S in (3.3) and the well-known equivalence between metric regularity of the given mapping and the Lipschitz-like/Aubin property of its inverse (see, e.g., [21, Theorem 1.49]), it suffices to show that Q in (3.3) is not Lipschitz-like around $(0,0)$. By [21, Theorem 1.41] this is equivalent to the fact that the scalar function

$$\rho(x,y) := \mathrm{dist}\big(x; Q(y)\big) = \inf\big\{\|x-v\|\,\big|\, v \in Q(y)\big\}$$

is not locally Lipschitzian around $(0,0)$. To check the latter, we construct two sequences $\{(x_{1k}, y_{1k})\}$ and $\{(x_{2k}, y_{2k})\}$, which converge to $(0,0)$ when $k \to \infty$ as follows:

$$x_{1k} := \frac{1}{2^{k-1}},\ y_{1k} := \frac{1}{(\sqrt[3]{2})^{k-1}} - \alpha_k, \quad \text{where} \quad 0 < \alpha_k < \min\Big\{\frac{1}{k2^k}, \frac{1}{(\sqrt[3]{2})^{k-1}} - \frac{1}{(\sqrt[3]{2})^k}\Big\};$$

$$x_{2k} := \frac{1}{2^{k-1}}, \quad y_{2k} := \frac{1}{(\sqrt[3]{2})^{k-1}}, \quad k \in \mathcal{N}.$$

Then we have the equalities

$$\mathrm{dist}(x_{2k}; Q(y_{2k})) = \mathrm{dist}\Big(x_{2k}; \Big[\frac{1}{2^k}, \frac{1}{2^{k-1}}\Big]\Big) = \mathrm{dist}\Big(\frac{1}{2^{k-1}}, \Big[\frac{1}{2^k}, \frac{1}{2^{k-1}}\Big]\Big) = 0,$$

$$\mathrm{dist}\big(x_{1k}; Q(y_{1k})\big) = \mathrm{dist}\Big(\frac{1}{2^{k-1}}; \frac{1}{2^k}\Big) = \frac{1}{2^k},$$

where the last one holds due to the estimates

$$\frac{1}{(\sqrt[3]{2})^k} < y_{1k} < \frac{1}{(\sqrt[3]{2})^{k-1}}, \quad k \in \mathcal{N}.$$

Since $\|y_{1k} - y_{2k}\| = \alpha_k \le (k2^k)^{-1}$, we have

$$\big\|d\big(x_{1k}; Q(y_{1k})\big) - d\big(x_{2k}; Q(y_{2k})\big)\big\| = \frac{1}{2^k} \ge k\alpha_k = k\|(x_{1k} - x_{2k}, y_{1k} - y_{2k})\|,$$

which indicates that $\rho(x,y)$ is not locally Lipschitzian around $(0,0)$ and yields therefore that the solution map S from (3.3) is not metrically regular around $(0,0)$.

Now we show that S is q-subregular for $q=2$ and thus for any $q\in(0,2]$ at $(0,0)$. To proceed, take $\eta=\gamma=1$ and, given $x\in B(0,\gamma)$, find k_0 so that $|x|\in[\frac{1}{2^{k_0+1}},\frac{1}{2^{k_0}}]$ and the corresponding value $|Q^{-1}(x)|$ belongs to $\Big[\frac{1}{(\sqrt[3]{2})^{k+1}},\frac{1}{(\sqrt[3]{2})^{k-1}}\Big]$. Notice that $S^{-1}(0)=\{0\}$, we have

$$d\big(x;S^{-1}(0)\big)=|x|\quad\leq\quad\eta d^2\big(0;S(x)\big)=\eta\big(\inf\big\{\|y\|\,\big|\;y\in Q^{-1}(x)\big\}\big)^2,\quad x\in I\!\!B(0,\gamma),$$

due to $\frac{1}{2^k}\leq\frac{1}{(\sqrt[3]{2})^{2(k+1)}}$ for any $k\geq 2$, which verifies the 2-subregularity of S at $(0,0)$ and thus completes our justification in this example.

It is worth mentioning that the solution map (3.3) in Example 3.3 happens to be even *strongly* 2-subregular at $(0,0)$. It follows from the arguments above since $S^{-1}(0)=\{0\}$ is a singleton.

3.2 Necessary and Sufficient Conditions for the Subdifferential Mapping

In this section, we derive *characterizations* of q-subregularity of any rate $q>0$ for the *subdifferential* mappings (2.4) generated by extended-real-valued lower semicontinuous (l.s.c.) functions on Banach (sufficient conditions) and Asplund (necessary conditions) spaces. For the case of subregularity ($q=1$) the obtained characterization reduces to [11, Theorem 3.1]. For convex functions on Banach spaces this case while concerning only local minimizers x of f has been independently characterized in [1, Theorem 2.1] with a weaker modulus estimate; see more discussions in [11] presented around Corollary 3.2 and also in [1, Remark 2.2] for convex and nonconvex functions with $q=1$. In the general case of q-subregularity the formulation and proof of the theorem below are essentially more involved following the lines of the approach in [23, Theorem 3.2] (for strong metric regularity) and of [11, Theorem 3.1] (for subregularity).

Theorem 3.4 (characterization of q-subregularity of the basic subdifferential) *Let $f\colon X \to \overline{R}$ be l.s.c. around $\bar{x} \in \mathrm{dom}\, f$ on a Banach space X, let $\bar{x}^* \in \partial f(\bar{x})$, and let q be an arbitrary positive number. Consider the following two statements:*

(i) *∂f is q-subregular at $(\bar{x}, \bar{x}^*)$ with modulus $\widetilde{\kappa}$ and there exist numbers $\gamma > 0$ and $r \in (0, q/\kappa)$ with $\kappa := q\widetilde{\kappa}^{\frac{1}{q}}$ such that*

$$f(x) \geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x} \rangle - \frac{qr}{1+q} d^{\frac{1+q}{q}}\big(x; (\partial f)^{-1}(\bar{x}^*)\big) \ \text{ for all } \ x \in I\!B(\bar{x}, \gamma). \tag{3.4}$$

(ii) *There are two positive numbers α and η such that*

$$f(x) \geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x} \rangle + \frac{q\alpha}{1+q} d^{\frac{1+q}{q}}\big(x; (\partial f)^{-1}(\bar{x}^*)\big) \ \text{ for all } \ x \in I\!B(\bar{x}, \eta). \tag{3.5}$$

Then we have **(ii)$\Longrightarrow$(i)** *provided that there is $\beta \in (0, \alpha)$ with*

$$f(u) \geq f(x) + \langle x^*, u - x \rangle - \frac{q\beta}{1+q} d^{\frac{1+q}{q}}\big(x, (\partial f)^{-1}(\bar{x}^*)\big) \tag{3.6}$$

whenever $(u, \bar{x}^*), (x, x^*) \in (\mathrm{gph}\, \partial f) \cap I\!B\big((\bar{x}, \bar{x}^*), \eta + (\frac{q\eta}{1+q})^{\frac{1}{q}}\big)$. Conversely, we have **(i)$\Longrightarrow$(ii)** *for any fixed $\alpha \in (0, q/\kappa)$ provided that the space X is Asplund.*

Proof. Let us first justify implication **(ii)$\Longrightarrow$(i)** in the case of the general Banach space X assuming condition (3.6) with some $\beta \in (0, \alpha)$. Since (3.5) clearly yields (3.4), we will arrive at

(i) by showing that there exists a number $\widetilde{\kappa} > 0$ such that

$$d\big(x; (\partial f)^{-1}(\bar{x}^*)\big) \le \widetilde{\kappa} d^q\big(\bar{x}^*; \partial f(x)\big) \ \text{ for all } \ x \in I\!B\big(\bar{x}, \eta q/1+q\big). \tag{3.7}$$

To proceed, fix $x \in I\!B(\bar{x}, \eta q/1+q)$. Pick any $u \in (\partial f)^{-1}(\bar{x}^*)$ with $\|x-u\| \le q^{-1}\|x-\bar{x}\|$ and get

$$\|u-\bar{x}\| \le \|u-x\| + \|x-\bar{x}\| \le \eta.$$

Then it follows from (3.6) the estimates

$$f(\bar{x}) \ge f(u) + \langle \bar{x}^*, \bar{x}-u\rangle - \frac{\beta q}{1+q} d^{\frac{1+q}{q}}\big(u; (\partial f)^{-1}(\bar{x}^*)\big) \ge f(u) + \langle \bar{x}^*, \bar{x}-u\rangle, \tag{3.8}$$

which ensure in turn that for any such u and $x^* \in \partial f(x) \cap I\!B(\bar{x}^*, [\eta q/(1+q)]^{\frac{1}{q}})$ we have

$$\begin{aligned}
\langle x^*-\bar{x}^*, x-u\rangle &= \langle x^*, x-u\rangle + \langle \bar{x}^*, u-\bar{x}\rangle - \langle \bar{x}^*, x-\bar{x}\rangle \\
&\ge f(x) - f(u) - \frac{\beta q}{1+q} d^{\frac{q+1}{q}}\big(x; (\partial f)^{-1}(\bar{x}^*)\big) + f(u) - f(\bar{x}) - \langle \bar{x}^*, x-\bar{x}\rangle \\
&= f(x) - f(\bar{x}) - \langle \bar{x}^*, x-\bar{x}\rangle - \frac{\beta q}{1+q} d^{\frac{q+1}{q}}\big(x; (\partial f)^{-1}(\bar{x}^*)\big) \\
&\ge \frac{q(\alpha-\beta)}{q+1} d^{\frac{q+1}{q}}\big(x; (\partial f)^{-1}(\bar{x}^*)\big),
\end{aligned}$$

where the first inequality follows from (3.6) and (3.8) while the second one from (3.5). Thus

$$\|x^*-\bar{x}^*\| \cdot \|x-u\| \ge \frac{q(\alpha-\beta)}{q+1} d^{\frac{1+q}{q}}\big(x; (\partial f)^{-1}(\bar{x}^*)\big),$$

which gives us the estimate

$$\|x^*-\bar{x}^*\|d\big(x;(\partial f)^{-1}(\bar{x}^*)\big)\ge\frac{q(\alpha-\beta)}{q+1}d^{\frac{q+1}{q}}\big(x;(\partial f)^{-1}(\bar{x}^*)\big)$$

due to the arbitrary choice of $u\in(\partial f)^{-1}(\bar{x}^*)$ with $\|x-u\|\le\frac{1}{q}\|x-\bar{x}\|\le\frac{\eta}{1+q}$. Hence

$$\|x^*-\bar{x}^*\|\ge\frac{q(\alpha-\beta)}{q+1}d^{\frac{1}{q}}\big(x;(\partial f)^{-1}(\bar{x}^*)\big)\ \text{ for all }\ x^*\in\partial f(x)\cap I\!B\big(\bar{x}^*,[\eta q/(1+q)]^{\frac{1}{q}}\big).\tag{3.9}$$

If $d(\bar{x}^*;\partial f(x))\le(\eta q/1+q)^{\frac{1}{q}}$, we deduce from (3.9) that

$$d\big(\bar{x}^*;\partial f(x)\big)\ge\frac{q(\alpha-\beta)}{q+1}d^{\frac{1}{q}}\big(x;(\partial f)^{-1}(\bar{x}^*)\big)\ \text{ for all }\ x\in I\!B\big(\bar{x},\eta q/(1+q)\big),$$

which justifies (3.7) with $\widetilde{\kappa}:=[(1+q)/(q(\alpha-\beta)]^q$. In the remaining case of $d(\bar{x}^*;\partial f(x))>(\eta q/1+q)^{\frac{1}{q}}$ we obviously have the estimates

$$d\big(x;(\partial f)^{-1}(\bar{x}^*)\big)\le\|x-\bar{x}\|\le\frac{\eta q}{1+q}<d^q\big(\bar{x}^*;\partial f(x)\big),$$

which also justify (3.7) is and thus completes the proof of implication **(ii)**$\Longrightarrow$**(i)**.

Next we verify the converse **(i)**$\Longrightarrow$**(ii)** assuming that X is Asplund. Arguing by contradiction, suppose that **(i)** holds while property (3.5) is not satisfied whenever $\alpha,\eta>0$. Choose

$$0<\frac{2^{\frac{1+q}{q}}}{1+q}<a<\infty$$

and pick θ from the interval $\big(\frac{2^{-q}}{q(1+q)},\frac{1}{2}\big)$, which ensures that $\theta+\frac{1}{\theta^q a^q(1+q)^q}<1$. Now we claim that there exists a positive number ν satisfying

$$f(x) \geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x} \rangle + \frac{q - (aq+1)\kappa r}{a(1+q)\kappa} d^{\frac{1+q}{q}} \big(x; (\partial f)^{-1}(\bar{x}^*)\big) \text{ for all } x \in I\!B(\bar{x}, \nu). \quad (3.10)$$

Indeed, otherwise there is a sequence $x_k \to \bar{x}$ such that

$$f(x_k) < f(\bar{x}) + \langle \bar{x}^*, x_k - \bar{x} \rangle + \frac{q - (aq+1)\kappa r}{a(1+q)\kappa} d^{\frac{1+q}{q}} \big(x_k; (\partial f)^{-1}(\bar{x}^*)\big), \quad k \in \mathcal{N}.$$

This together with (3.4) implies that all x_k lie outside of $(\partial f)^{-1}(\bar{x}^*)$. Consequently we have

$$\begin{aligned} &\inf_{x \in I\!B(\bar{x},\gamma)} \Big\{ f(x) - \langle \bar{x}^*, x - \bar{x} \rangle + \frac{qr}{1+q} d^{\frac{1+q}{q}} \big(x; (\partial f)^{-1}(\bar{x}^*)\big) \Big\} \geq f(\bar{x}) \\ &> f(x_k) - \langle \bar{x}^*, x_k - \bar{x} \rangle + \frac{qr}{1+q} d^{\frac{1+q}{q}} \big(x_k; (\partial f)^{-1}(\bar{x}^*)\big) - \frac{q - \kappa r}{a(1+q)\kappa} d^{\frac{1+q}{q}} \big(x_k; (\partial f)^{-1}(\bar{x}^*)\big). \end{aligned} \quad (3.11)$$

Denote $\epsilon_k := \big(\frac{q-\kappa r}{a(1+q)\kappa}\big) d^{\frac{1+q}{q}} \big(x_k; (\partial f)^{-1}(\bar{x}^*)\big) \downarrow 0$ as $k \to \infty$ and define the function

$$g(x) := f(x) - \langle \bar{x}^*, x - \bar{x} \rangle + \frac{qr}{1+q} d^{\frac{1+q}{q}} \big(x; (\partial f)^{-1}(\bar{x}^*)\big), \quad x \in X.$$

It follows from (3.11) that $g(x_k) < \inf_{x \in I\!B(\bar{x},\gamma)} g(x) + \epsilon_k$. Applying Ekeland's variational principle (see, e.g., [21, Theorem 2.26]) to the function $g + \delta_{I\!B(\bar{x},\gamma)}$ with $\lambda_k := \theta d\big(x_k; (\partial f)^{-1}(\bar{x}^*)\big)$ ensures the existence of a new sequence $\widehat{x}_k$ satisfying $\|\hat{x}_k - x_k\| \leq \lambda_k$ and such that for each $k \in \mathcal{N}$ we have $\widehat{x}_k \in \text{int} I\!B(\bar{x}, \gamma)$ (due to $x_k \to \bar{x}$ and $\lambda_k \downarrow 0$ as $k \to \infty$) and that

$$g(\hat{x}_k) < g(x) + \frac{\epsilon_k}{\lambda_k} \|x - \hat{x}_k\| \text{ for all } x \in I\!B(\bar{x}, \gamma).$$

Employing the Fermat stationary rule in the above optimization problem and then using the subdifferential sum rule held in Asplund spaces by [21, Theorem 3.41], we arrive at

$$\begin{array}{rcl} 0 & \in & \partial\Big(g(\cdot)+\dfrac{\epsilon_k}{\lambda_k}\|\cdot-\hat{x}\|\Big)(\hat{x}_k) \\ & \subset & -\bar{x}^*+\partial f(\hat{x}_k)+\dfrac{\epsilon_k}{\lambda_k}I\!B^*+\dfrac{qr}{1+q}\partial d^{\frac{1+q}{q}}\big(\cdot;(\partial f)^{-1}(\bar{x}^*)\big)(\hat{x}_k) \\ & \subset & -\bar{x}^*+\partial f(\hat{x}_k)+\Big(rd^{\frac{1}{q}}\big(\hat{x}_k;(\partial f)^{-1}(\bar{x}^*)\big)+\dfrac{\epsilon_k}{\lambda_k}\Big)I\!B^*. \end{array}$$

Combining this with the metric q-subregularity property of ∂f at $(\bar{x},\bar{x}^*)$ ensures the estimates

$$d^{\frac{1}{q}}\big(\hat{x}_k;(\partial f)^{-1}(\bar{x}^*)\big)\le\frac{\kappa}{q}d\big(\bar{x}^*;\partial f(\hat{x}_k)\big)\le\frac{\kappa}{q}\Big(rd^{\frac{1}{q}}\big(\hat{x}_k;(\partial f)^{-1}(\bar{x}^*)\big)+\frac{\epsilon_k}{\lambda_k}\Big)$$

for all $k\in\mathcal{N}$ sufficiently large. Hence for such numbers k we get the inequality

$$\Big(1-\frac{\kappa r}{q}\Big)d^{\frac{1}{q}}\big(\hat{x}_k;(\partial f)^{-1}(\bar{x}^*)\big)\le\frac{\kappa}{q}\frac{\epsilon_k}{\lambda_k}.$$

This allows us to successively deduce that

$$\begin{array}{rcl} \Big(1-\dfrac{\kappa r}{q}\Big)^q d\big(x_k;(\partial f)^{-1}(\bar{x}^*)\big) & \le & \Big(1-\dfrac{\kappa r}{q}\Big)^q\|\hat{x}_k-x_k\|+\Big(1-\dfrac{\kappa r}{q}\Big)^q d\big(\hat{x}_k;(\partial f)^{-1}(\bar{x}^*)\big) \\ & \le & \Big(1-\dfrac{\kappa r}{q}\Big)^q\lambda_k+\Big(\dfrac{\kappa\epsilon_k}{q\lambda_k}\Big)^q \\ & \le & \Big(1-\dfrac{\kappa r}{q}\Big)^q d\big(x_k;(\partial f)^{-1}(\bar{x}^*)\big)\Big(\theta+\dfrac{1}{a^q\theta^q(q+1)^q}\Big) \\ & < & \Big(1-\dfrac{\kappa r}{q}\Big)^q d\big(x_k;(\partial f)^{-1}(\bar{x}^*)\big), \end{array}$$

where the last strict inequality follows from our choices of a and θ. Thus we arrive at the obvious contradiction, which justifiers our claim in (3.10). We conclude therefore that $q-(aq+1)\kappa r\le 0$.

Define now the real number

$$r_1 := \frac{\frac{aq+1}{q}\kappa r - 1}{a\kappa} \in \big[0, q/\kappa\big)$$

and observe that inequality (3.10) can be transformed into (3.4) with replacing r by r_1 and γ by ν, respectively. Consequently there is some real number ν_1 such that

$$f(x) \geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x} \rangle + \frac{q - (aq+1)\kappa r_1}{a(1+q)\kappa} d^{\frac{1+q}{q}}\big(x; (\partial f)^{-1}(\bar{x}^*)\big) \text{ for all } x \in I\!B(\bar{x}, \nu_1).$$

As before, we get the inequality $\kappa r_1 > \frac{q}{aq+1}$, or equivalently $\kappa r > \frac{q}{aq+1} + \frac{aq^2}{(aq+1)^2}$. Defining

$$r_2 := \frac{\frac{aq+1}{q}\kappa r_1 - 1}{a\kappa} \in \big[0, q/\kappa\big)$$

and proceeding in the same way as above lead us to the inequality $\kappa r > \frac{q}{aq+1} + \frac{aq^2}{(aq+1)^2} + \frac{a^2q^3}{(aq+1)^3}$. Then we get by induction the progressively stronger bounds

$$\kappa r > \frac{q}{aq+1} + \frac{aq^2}{(aq+1)^2} + \ldots + \frac{a^{k-1}q^k}{(aq+1)^k} = q\Big(1 - \Big(\frac{aq}{aq+1}\Big)^k\Big) \text{ for all } k \in \mathcal{N}.$$

Letting $k \to \infty$ gives us $\kappa r > q$, which is a contradiction. Therefore there exist real numbers $\alpha, \eta > 0$ such that inequality (3.5) is satisfied.

To justify **(ii)**, we verify now that α may be chosen arbitrarily close to q/κ while being smaller than this number. It suffices to consider the case of $\alpha < q/\kappa$. Take $\widetilde{a}, \widetilde{\theta} > 0$ such that

$$\frac{q2^{\frac{1}{q}}}{\widetilde{a}(1+q)} < \frac{1}{2}, \quad \widetilde{\theta} \in \Big(\frac{2^{\frac{1}{q}}}{\widetilde{a}(1+q)}, \frac{1}{2}\Big), \text{ and so } \widetilde{\theta} + \Big(\frac{q}{\widetilde{\theta}\widetilde{a}(1+q)}\Big)^q < 1.$$

Given $\alpha, \eta > 0$ for which (3.5) holds, let us prove the existence $\mu > 0$ such that

$$f(x) \geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x} \rangle + \frac{q(q + (\widetilde{a} - 1)\alpha\kappa)}{\widetilde{a}(1+q)\kappa} d^{\frac{1+q}{q}}\big(x; (\partial f)^{-1}(\bar{x}^*)\big) \ \text{ for all } \ x \in I\!B(\bar{x}, \mu). \quad (3.12)$$

We just sketch the proof of (3.12) observing that it is similar to the proof of (3.10) given above. Arguing by contradiction, find a sequence of $u_k \to \bar{x}$ so that

$$f(u_k) < f(\bar{x}) + \langle \bar{x}^*, u_k - \bar{x} \rangle + \frac{q(q + (\widetilde{a} - 1)\alpha\kappa)}{\widetilde{a}(1+q)\kappa} d^{\frac{1+q}{q}}\big(u_k; (\partial f)^{-1}(\bar{x}^*)\big), \quad k \in I\!N.$$

This gives us together with (3.5) that

$$\begin{aligned} &\inf_{x \in I\!B(\bar{x},\eta)} \Big\{ f(x) - \langle \bar{x}^*, x - \bar{x} \rangle - \frac{q\alpha}{1+q} d^{\frac{1+q}{q}}\big(x; (\partial f)^{-1}(\bar{x}^*)\big) \Big\} \geq f(\bar{x}) \\ &> f(u_k) - \langle \bar{x}^*, u_k - \bar{x} \rangle - \frac{q\alpha}{1+q} d^{\frac{1+q}{q}}\big(u_k; (\partial f)^{-1}(\bar{x}^*)\big) - \frac{q(q - \kappa\alpha)}{\widetilde{a}(1+q)\kappa} d^{\frac{1+q}{q}}\big(u_k; (\partial f)^{-1}(\bar{x}^*)\big). \end{aligned}$$

Denote further $\nu_k := \frac{q(q-\kappa\alpha)}{\widetilde{a}(1+q)\kappa} d^{\frac{1+q}{q}}\big(u_k; (\partial f)^{-1}(\bar{x}^*)\big) \downarrow 0$ as $k \to \infty$ and consider the function

$$h(x) := f(x) - \langle \bar{x}^*, x - \bar{x} \rangle - \frac{q\alpha}{1+q} d^{\frac{1+q}{q}}\big(x; (\partial f)^{-1}(\bar{x}^*)\big), \quad x \in X,$$

for which we have $h(u_k) < \inf_{x \in I\!B(\bar{x},\eta)} h(x) + \nu_k$ whenever $k \in I\!N$. Applying Ekeland's variational principle to the function $h + \delta_{I\!B(\bar{x},\eta)}$ with $\rho_k := \widetilde{\theta} d(u_k; (\partial f)^{-1}(\bar{x}^*))$ ensures the existence of a new sequence $\{\hat{u}_k\}$ satisfying $\|\hat{u}_k - u_k\| \leq \rho_k$ and such that $\hat{u}_k \in \operatorname{int} I\!B(\bar{x}, \eta)$ with

$$h(\hat{u}_k) < h(x) + \frac{\nu_k}{\rho_k} \|x - \hat{u}_k\| \ \text{ for all } \ x \in I\!B(\bar{x}, \gamma), \ k \in I\!N.$$

By the calculus rules as above we get the inclusions

$$0\in\partial\Big(h(\cdot)+\frac{\nu_k}{\rho_k}\|\cdot-\hat{x}\|\Big)(\hat{u}_k)\subset-\bar{x}^*+\partial f(\hat{u}_k)+\Big(\alpha d^{\frac{1}{q}}\big(\hat{u}_k;(\partial f)^{-1}(\bar{x}^*)\big)+\frac{\nu_k}{\rho_k}\Big)I\!B^*,$$

which ensure together with the q-subregularity of ∂f at $(\bar{x},\bar{x}^*)$ that

$$d^{\frac{1}{q}}\big(\hat{x}_k;(\partial f)^{-1}(\bar{x}^*)\big)\le\frac{\kappa}{q}d\big(\bar{x}^*,\partial f(\hat{x}_k)\big)\le\frac{\kappa}{q}\Big(\alpha d^{\frac{1}{q}}\big(\hat{u}_k;(\partial f)^{-1}(\bar{x}^*)\big)+\frac{\nu_k}{\rho_k}\Big)$$

for all $k\in\mathcal{N}$ sufficiently large. Thus for such k we arrive at the estimate

$$\Big(1-\frac{\kappa\alpha}{q}\Big)d^{\frac{1}{q}}\big(\hat{u}_k;(\partial f)^{-1}(\bar{x}^*)\big)\le\frac{\kappa}{q}\frac{\nu_k}{\rho_k}.$$

This allows us to successively deduce that

$$\begin{aligned}\Big(1-\frac{\kappa\alpha}{q}\Big)^q d\big(u_k;(\partial f)^{-1}(\bar{x}^*)\big) &\le \Big(1-\frac{\kappa\alpha}{q}\Big)^q\|\hat{u}_k-u_k\|+\Big(1-\frac{\kappa\alpha}{q}\Big)^q d\big(\hat{u}_k;(\partial f)^{-1}(\bar{x}^*)\big)\\ &\le \Big(1-\frac{\kappa\alpha}{q}\Big)^q\rho_k+\Big(\frac{\kappa\nu_k}{q\rho_k}\Big)^q\\ &\le \Big(1-\frac{\kappa\alpha}{q}\Big)^q d\big(u_k;(\partial f)^{-1}(\bar{x}^*)\big)\Big(\widetilde{\theta}+\Big(\frac{q}{\widetilde{a}\widetilde{\theta}(q+1)}\Big)^q\Big)\\ &< \Big(1-\frac{\kappa\alpha}{q}\Big)^q d\big(u_k;(\partial f)^{-1}(\bar{x}^*)\big),\end{aligned}$$

which is a contradiction justifying the existence of $\mu>0$ such that (3.12) holds.

In the last part of the proof we define the number

$$\alpha_1:=\frac{(q+(\widetilde{a}-1)\alpha\kappa)}{\widetilde{a}\kappa}\in\big(0,q/\kappa\big)$$

and observe that (3.12) yields (3.5) with this number α_1 and $\eta=\mu$ from (3.12). Then proceeding as above allows us to find $\mu_1 > 0$ such that

$$f(x) \geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x} \rangle + \frac{q(q + (\widetilde{a} - 1)\alpha_1 \kappa)}{\widetilde{a}(1+q)\kappa} d^{\frac{1+q}{q}}\big(x; (\partial f)^{-1}(\bar{x}^*)\big) \text{ for all } x \in I\!B(\bar{x}, \mu_1).$$

Define further $\alpha_2 := \frac{(q+(\widetilde{a}-1)\alpha_1\kappa)}{\widetilde{a}\kappa} \in (0, q/\kappa)$ and deduce from the above inequality that (3.5) holds for α_2 and $\eta = \mu_1$. By induction we find sequences $\{\alpha_k\}$ and $\{\mu_k\}$ satisfying

$$f(x) \geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x} \rangle + \frac{q}{1+q} \alpha_k d^{\frac{1+q}{q}}\big(x; (\partial f)^{-1}(\bar{x}^*)\big) \text{ for all } x \in B(\bar{x}, \mu_k)$$

with $\alpha_k := \frac{(q+(\widetilde{a}-1)\alpha_{k-1}\kappa)}{\widetilde{a}\kappa} \in (0, q/\kappa)$ for $k \in \mathcal{N}$ and $\alpha_0 = \alpha$. Letting finally $k \to \infty$ gives us $\alpha_k \to q/\kappa$ and thus completes proof of the theorem. $\square$

Now we present two examples illustrating the results of Theorem 3.4 and the assumptions made therein. The first example shows that the conditions of the theorem ensures the validity of 2-subregularity of the subdifferential mapping while metric regularity fails.

Example 3.5 (2-subregularity versus metric regularity of the subdifferential) Consider the convex and continuous function $f\colon I\!R \to I\!R_+$ defined by

$$f(x) := \begin{cases} -x & \text{for} \quad x < -1, \\ 1 & \text{for} \quad -1 \leq x \leq 1, \\ x & \text{for} \quad x > 1. \end{cases}$$

It is not hard to calculate its subdifferential mapping as follows:

$$\partial f(x) = \begin{cases} \{-1\} & \text{for} \quad x < -1, \\ [-1, 0] & \text{for} \quad x = -1, \\ \{0\} & \text{for} \quad -1 < x < 1, \\ [0, 1] & \text{for} \quad x = 1, \\ \{1\} & \text{for} \quad x > 1. \end{cases}$$

Take $\bar{x} = 0$, $\bar{x}^* = 0 \in \partial f(0)$ and show that the mapping ∂f is 2-subregular (and hence q-subregular for any $0 < q \le 2$) at $(0,0)$ while not metrically regular around this point. To verify 2-subregularity, it suffices to check by Theorem 3.4 that both conditions (3.5) and (3.6) of this theorem hold with $q = 2$. To proceed, take $\alpha = 1$ and $\eta = \frac{1}{2}$ in Theorem 3.4(ii) and observe that $f(x) = f(0) = 1$ for any $x \in I\!B(0, \eta)$. Then for any $q \in (0, 2]$ we have

$$f(x) \ge f(0) + \frac{1}{2} d^{1+\frac{1}{q}}\big(x; (\partial f)^{-1}(0)\big)$$

due to $(\partial f)^{-1}(0) = (-1, 1)$, and so condition (3.5) holds. To verify (3.6), take $\beta = 0.5$ and get

$$f(u) = f(x) = 1 \ \text{ for any } \ (u, \bar{x}^*), (x, x^*) \in \operatorname{gph}(\partial f) \cap I\!B\big((\bar{x}, \bar{x}^*), \eta + (\eta/2)^{\frac{1}{q}}\big),$$

which justifies (3.6) and thus shows that ∂f is 2-subregular at $(0,0)$ by Theorem 3.4.

However, the validity of conditions (3.5) and (3.6) do not guarantee that the mapping ∂f is metrically regular around $(0,0)$. Indeed, we have for any $x, y \neq 0$ sufficiently close to 0 that $\partial f(x) = \{0\}$ while the values of $(\partial f)^{-1}(y)$ are either $\{-1\}$ or $\{1\}$. Then it is easy to see by considering two sequences $\{x_k\} = \{k^{-1}\}$ and $\{y_k\} = \{(2k)^{-1}\}$ that there is no positive number

κ, which ensures the distance estimate

$$d\big(x_k;(\partial f)^{-1}(y_k)\big)\le\kappa d\big(y_k;\partial f(x_k)\big),\quad k\in\mathcal{N},$$

i.e., the subdifferential mapping ∂f fails to be metrically regular around $(0,0)$.

The next example shows that condition (3.6) is *not necessary* for q-subregularity of ∂f whenever $q>0$. In particular, this illustrates that (i) implies (ii) but does not imply (3.6).

Example 3.6 (on assumptions and conclusions of Theorem 3.4) Define $f\colon I\!R\to I\!R_+$ by

$$f(x):=\begin{cases} x^{\frac{1}{2}} & \text{for}\quad x>0,\\ 0 & \text{for}\quad x=0,\\ (-x)^{\frac{1}{2}} & \text{for}\quad x<0.\end{cases}$$

It follows immediately that $\partial f(x)=\{f'(x)\}$ at $x\ne 0$ with

$$f'(x)=\begin{cases} \frac{1}{2\sqrt{x}} & \text{for}\quad x>0,\\ -\frac{1}{2\sqrt{-x}} & \text{for}\quad x<0\end{cases}$$

and that $[-1,1]\subset\partial f(0)$. The latter implies that $\partial f^{-1}(0)=\{0\}$. Considering now $\bar{x}=0$ and $\bar{x}^*=0$, we claim that the mapping ∂f is q-subregular at $(0,0)$ with any positive order q, which we fix in what follows. To proceed, take $\gamma:=2^{-\frac{2q}{q+2}}$ and get

$$|x|\le\big(1/2\big)^{\frac{2q}{2+q}}\Longleftrightarrow|x|\le\big(1/(2\sqrt{|x|})\big)^q\;\text{ for }\;x\in I\!B(0,\gamma),$$

which verifies the validity of the q-subregularity condition (3.1) for the mapping ∂f.

However, condition (3.6) fails here whenever $\beta,\eta>0$. To justify it, we argue by contradiction and suppose that (3.6) holds with some β_0, η_0. It is easy to see that the inclusion

$$(u,0),(x,x^*) \in \operatorname{gph}(\partial f) \cap I\!B\Big((0,0), \eta_0 + (\frac{q\eta_0}{1+q})^{\frac{1}{q}}\Big)$$

yields $u = 0$, which implies in turn that condition (3.6) reduces to

$$0 \geq f(x) + \langle x^*, -x\rangle - \frac{q\beta_0}{1+q}|x|^{\frac{q+1}{q}} \tag{3.13}$$

Considering $x > 0$ in (3.13) gives us the inequality

$$0 \geq \frac{\sqrt{x}}{2} - \frac{q\beta_0}{1+q}|x|^{\frac{q+1}{q}}, \text{ i.e., } \frac{2q\beta_0}{1+q}|x|^{\frac{q+1}{q}} \geq x^{\frac{1}{2}},$$

which is a contradiction, since the latter inequality is obviously violated for x sufficiently small.

□

CHAPTER 4

Strong Q-Subregularity and Its Applications

4.1 Strong q-Subregularity and Its Perturbations

The main goal of this section is to investigate characterization of strong metric q-subregularity as well as its robustness property under perturbation. Similarly to [11, Corollaries 3.2, 3.3, 3.5] given in the case of $q = 1$, we can easily deduce from the obtained Theorem 3.4 its consequences for *local minimizers* as well as the characterization of *strong q-subregularity* of the basic subdifferential in the case of arbitrary $q > 0$.

Theorem 4.1 (characterization of strong q-subregularity of the basic subdifferential) *Let $f : X \to \overline{R}$ be l.s.c. around $\bar{x} \in \operatorname{dom} f$ on a Banach space X, let $(\bar{x}, \bar{x}^*) \in \operatorname{gph}(\partial f)$ and q be an arbitrary positive number. Consider the following two statements:*

(i) ∂f is strongly metrically q-subregular at $(\bar{x}, \bar{x}^*)$ with modulus $\widetilde{\kappa}$ and there exist numbers $\gamma > 0$ and $r \in (0, \frac{q}{\kappa})$ with $\kappa := q\widetilde{\kappa}^{\frac{1}{q}}$ such that

$$f(x) \geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x} \rangle - \frac{qr}{1+q}\|x - \bar{x}\|^{(\frac{1+q}{q})} \quad \textit{for all } x \in B(\bar{x}, \gamma) \tag{4.1}$$

(ii) There exists $\alpha,\ \eta > 0$ such that

$$f(x) \geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x} \rangle + \frac{q\alpha}{1+q}\|x - \bar{x}\|^{(\frac{1+q}{q})} \quad \textit{for all } x \in B(\bar{x}, \eta) \tag{4.2}$$

Then (i)$\Longrightarrow$(ii), where α may be chosen arbitrarily in $(0, \frac{q}{\kappa})$. Conversely if in addition there exists some $\beta \in (0, \alpha)$ such that

$$f(\bar{x}) \geq f(x) + \langle x^*, \bar{x} - x \rangle - \frac{q\beta}{1+q} \|x - \bar{x}\|^{(\frac{1+q}{q})} \tag{4.3}$$

holds for all $(x, x^*) \in \mathrm{gph}\,(\partial f) \cap B((\bar{x}, \bar{x}^*), \eta + (\frac{q\eta}{1+q})^{\frac{1}{q}})$, then (ii) $\Longrightarrow$(i) as well.

Proof. (i) $\Longrightarrow$ (ii): Assume that (i) is true. Then by Theorem 3.4, it suffices to show that $d(x; (\partial f)^{-1}(\bar{x}^*)) = \|x - \bar{x}\|$ for all x around $\bar{x}$ which follows from the assumed strong metric q-subregularity. Hence (ii) is true.

(ii)$\Longrightarrow$(i): Suppose that both (4.2) and (4.3) are satisfied for some constant $\alpha > \beta > 0$ and $\eta > 0$. Then according to Theorem 3.4 it suffices to show that $(\partial f)^{-1}(\bar{x}^*) \cap I\!B(\bar{x}, \eta) = \{\bar{x}\}$. Indeed, take any $u \in (\partial f)^{-1}(\bar{x}^*) \cap I\!B(\bar{x}, \eta)$, it follows from (4.2) and (4.3) that

$$f(u) - f(\bar{x}) \geq \langle \bar{x}^*, u - \bar{x} \rangle + \frac{q\alpha}{1+q} \|u - \bar{x}\|^{\frac{q+1}{q}}$$

and

$$f(u) - f(\bar{x}) \leq \langle \bar{x}^*, u - \bar{x} \rangle + \frac{q\beta}{1+q} \|u - \bar{x}\|^{\frac{q+1}{q}}$$

Combining the above two inequalities gives us only one possibility that $u = \bar{x}$ due to the fact of $\alpha > \beta$. Hence ∂f is metrically q-subregular at $(\bar{x}, \bar{x}^*)$ and our proof is completed. $\square$

Next we provide an example showing that $\frac{q+1}{q}$-order growth alone may not imply strong q-subregularity.

Example 4.2 Consider the following function:

$$f(x) = \begin{cases} \frac{1}{2}x - 2x^{\frac{3}{2}}\sin(\frac{1}{\sqrt{x}}) & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0 \\ \infty & \text{for} \quad x < 0 \end{cases}$$

Take $\bar{x} = 0,\ \bar{x}^* = 0$ and $\alpha = 1$. It is easy to see that f satisfies condition (4.2) when η is small enough and $q = 2$. However ∂f is not strongly metrically 2-subregular at $(0, 0)$ since $(\partial f)^{-1}(0)$ is not single-valued. This shows that in Theorem 4.1, condition (4.3) is also necessary.

The following result provides a necessary condition in terms of weak contigent coderivative for strong q-subregularity of a general multifunction in the case of $q > 1$ between Banach spaces.

Theorem 4.3 (Necessary condition for strong q-subregularity) *Let $F : X \rightrightarrows Y$ be a closed set-valued function between Banach spaces, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and $q \geq 1$. If F is strongly metrically q-subregular at $(\bar{x}, \bar{y})$, then we have $\|C_w F(\bar{x}, \bar{y})\| = \infty$ when $q > 1$, and* $\ker\ C_w F(\bar{x}, \bar{y}) = \{0\}$ *when $q = 1$.*

Proof. Let $\eta > 0$ be the modulus with which F is strongly metrically q-subregular at $(\bar{x}, \bar{y})$. Take any $v \in I\!B_Y, u \in C_w F^{-1}(\bar{y}, \bar{x})(v)$ and let them be fixed. Then there exists $\{t_k\} \downarrow 0$ and $(u_k, v_k) \xrightarrow{w} (u, v)$ such that $(\bar{x} + t_k u_k, \bar{y} + t_k v_k) \in gphF$. Hence it follows from (3.2) that for all k large enough,

$$\|u_k\| \leq \eta |t_k|^{q-1} \|v_k\|^q \quad \text{for all } k \text{ large enough} \tag{4.4}$$

For the case of $q > 1$, since $\{t_k\} \downarrow 0$ and $\{\|v_k\|\}$ converges weakly hence it's bounded, passing (4.4) to the limit gives us $\|u\| = 0$ due to the weak lower semicontinuity property of norm. Since

u is arbitrarily chosen, one has

$$\|C_w F^{-1}(\bar{y},\bar{x})\| = \sup\{\|u\| \mid u \in C_w F^{-1}(\bar{y},\bar{x})(v), \|v\| \leq 1\} = 0$$

Since the weak contigent derivative function is positively homogeneous, it follows from Lemma 2.2 that

$$\begin{aligned} \|C_w F(\bar{x},\bar{y})\| &= \|(C_w F^{-1}(\bar{y},\bar{x}))^{-1}\| \\ &= \frac{1}{\inf\{\|u\| \mid u \in C_w F^{-1}(\bar{y},\bar{x})(v), \|v\| = 1\}} \\ &\geq \frac{1}{\sup\{\|u\| \mid u \in C_w F^{-1}(\bar{y},\bar{x})(v), \|v\| = 1\}} \\ &\geq \frac{1}{\sup\{\|u\| \mid u \in C_w F^{-1}(\bar{y},\bar{x})(v), \|v\| \leq 1\}} \\ &= \frac{1}{\|C_w F^{-1}(\bar{y},\bar{x})\|} = \infty \end{aligned}$$

For the case of $q = 1$, the inequality (4.4) implies that ker $C_w F(\bar{x},\bar{y}) = \{0\}$, which completes our proof. $\square$

However, the above necessary condition is not sufficient for strong q-subregularity, even not for metric q-subregularity. Here is an example in finite dimensional space:

Example 4.4 Let $F : R \rightrightarrows R$ be a set-valued function defined as $F(x) = [\max\{x, 0\}, \infty)$ and $(\bar{x},\bar{y}) = (0,0)$. It is easy to see that $\operatorname{gph} F$ is a convex set. Therefore, it follows from [7, Exe.2.7.2 and Prop.2.5.5] that $\operatorname{gph} C_w F(0,0) = T((0,0), \operatorname{gph} F)$, where

$$T((0,0), \operatorname{gph} F) = \operatorname{cl}\{\lambda(x,y) \ : \ (x,y) \in \operatorname{gph} F \text{ and } \lambda \geq 0\}$$

Hence,

$$\|C_w F(0,0)\| = \sup\{\|v\| \mid (u,v) \in T((0,0), \operatorname{gph} F) \text{ with } \|u\| \leq 1\} = \infty.$$

However, F is not metrically subregular at $(0,0)$ of order $q > 1$. To see this, we argue by contradiction. Assume to the contrary that there exist $\eta, \delta \in (0, \infty)$ such that

$$d(x, F^{-1}(0)) \leq \eta d^q(0, F(x)) \quad \text{for any } x \in (-\delta, \delta)$$

which indicates that

$$d(x, (-\infty, 0]) \leq \eta d^q(0, [\max\{x, 0\}, \infty))$$

holds for any $0 < x < \delta$. i.e $x \leq \eta x^q$ holds for any positive small x, which is obviously impossible.

It has been well recognized in the literature that metric subregularity, in contrast to metric regularity, is not robust/stable with respect to perturbations of the initial data; see, e.g., [10]. In this section we show that the situation is different for *strong* subregularity and higher-order q-subregularity ($q \geq 1$). Namely, it is proved below that such strong q-subregularity is *stable* with respect to appropriate perturbations of the initial set-valued mapping by Lipschitzian single-valued ones. Furthermore, we estimate the exact bound of strong q-subregularity moduli together with the radius of perturbations that keep strong q-subregularity of the perturbed mappings. These results are used in Section 4.2 for establishing the convergence rate for a class of quasi-Newton methods depending on the order q of the assumed strong q-subregularity of the initial mapping. Unless otherwise stated, we have $q \geq 1$ in the rest of this section.

Theorem 4.5 (strong q-subregularity under Lipschitzian perturbations) *Let $F\colon X \rightrightarrows Y$ be a set-valued mapping between Banach spaces with $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, and let $g : X \to Y$ be a* *single-valued perturbation locally Lipschitzian around $\bar{x}$. If there exist $\kappa, \lambda \in (0, \infty)$ such that*

$$\operatorname{ssubreg}{}^q F(\bar{x}, \bar{y}) < \kappa \;\; \textit{and} \;\; \operatorname{lip} g(\bar{x}) < \lambda < \big(\kappa^{\frac{1}{q}}\big)^{-1},$$

then we have the modulus upper estimate

$$\operatorname{ssubreg}{}^q(\widetilde{F} + g)(\bar{x}, \bar{y}) \le \frac{\kappa}{(1 - \lambda\kappa^{\frac{1}{q}})^q} \;\; \textit{with} \;\; \widetilde{F}(x) := F(x) - g(\bar{x}). \tag{4.5}$$

Furthermore, if $\operatorname{ssubreg}{}^q F(\bar{x}, \bar{y}) > 0$*, we have the modulus relationship*

$$\operatorname{ssubreg}{}^q(\widetilde{F} + g)(\bar{x}, \bar{y}) \le \frac{\operatorname{ssubreg}{}^q F(\bar{x}, \bar{y})}{\big(1 - (\operatorname{ssubreg}{}^q F(\bar{x}, \bar{y})\big)^{\frac{1}{q}} \operatorname{lip} g(\bar{x})^q} \tag{4.6}$$

whenever $(\operatorname{ssubreg} F(\bar{x}, \bar{y}))^{\frac{1}{q}} \operatorname{lip} g(\bar{x}) < 1$.

Proof. Since $\operatorname{ssubreg}{}^q F(\bar{x}, \bar{y}) < \kappa$ and $\operatorname{lip} g(\bar{x}) < \lambda$, there exists $\gamma \in (0, 1)$ such that

$$\|x - \bar{x}\| \le \kappa d^q\big(\bar{y}; F(x)\big) \;\; \text{and} \;\; \|g(x) - g(\bar{x})\| \le \lambda\|x - \bar{x}\|, \quad x \in I\!B(\bar{x}, \gamma).$$

For such x, take $z \in \widetilde{F}(x) + g(x)$ and find $y \in F(x)$ such that $z - y = g(x) - g(\bar{x})$. It yields

$$\|x - \bar{x}\|^{\frac{1}{q}} \le \kappa^{\frac{1}{q}}\|y - \bar{y}\| \le \kappa^{\frac{1}{q}}\big(\|\bar{y} - z\| + \|y - z\|\big)$$

$$\le \kappa^{\frac{1}{q}}\|\bar{y} - z\| + \kappa^{\frac{1}{q}}\lambda\|x - \bar{x}\| \le \kappa^{\frac{1}{q}}\|\bar{y} - z\| + \kappa^{\frac{1}{q}}\lambda\|x - \bar{x}\|^{\frac{1}{q}},$$

where the last inequality holds due to $q \geq 1$ and $\gamma < 1$. It gives us the estimate

$$\|x - \bar{x}\| \leq \frac{\kappa}{(1 - \lambda \kappa^{\frac{1}{q}})^q} \|\bar{y} - z\|^q \ \text{ for all } \ x \in I\!B(\bar{x}, \gamma), \ z \in \widetilde{F}(x) + g(x),$$

which implies both inequalities (4.5), (4.6) and so completes the proof of the theorem. $\square$

Next we derive two useful consequences of Theorem 4.5 of their own interest. The first one concerns strictly differentiable (in particular, C^1-smooth) perturbations and is employed to establish convergence rates of the quasi-Newton methods considered in Section 4.2.

Corollary 4.6 (strong q-subregularity under smooth perturbations) *Let in the setting of Theorem* 4.5 *the perturbation g is strictly differentiable at $\bar{x}$, and let $\bar{y} \in F(\bar{x}) + g(\bar{x})$. Then the mapping $x \mapsto F(x) + g(x)$ is strongly q-subregular at $(\bar{x}, \bar{y})$ if and only if the mapping $G\colon x \mapsto g(\bar{x}) + \nabla g(\bar{x})(x - \bar{x}) + F(x)$ is strongly q-subregular at $(\bar{x}, \bar{y})$ with the exact strong q-subregularity bound* $\operatorname{ssubreg}^q(F + g)(\bar{x}, \bar{y})$.

Proof. Pick $\kappa \in (0, \infty)$ such that $\operatorname{ssubreg}^q(F + g)(\bar{x}) < \kappa$. Define the mapping

$$\widetilde{g}(x) := \nabla g(\bar{x})(x - \bar{x}) + g(\bar{x}) - g(x)$$

and observe that $\widetilde{g}(\bar{x}) = 0$ and $\operatorname{lip} \widetilde{g}(\bar{x}) = 0$ since g is strictly differentiable at $\bar{x}$. Thus it follows from Theorem 4.5 that the mapping $x \mapsto (\widetilde{F + g})(x) + \widetilde{g}(x) = F(x) + g(x) - \widetilde{g}(\bar{x}) + \widetilde{g}(x) = G(x)$ is also strongly q-subregular at $(\bar{x}, \bar{y})$ with the exact bound not exceeding κ. The proof of the converse implication is similar with replacing $\widetilde{g}$ by $-\widetilde{g}$. $\square$

The second consequence of Theorem 4.5 provides a lower estimate of moduli of Lipschitzian perturbations, which fails strong q-subregularity of the original mapping.

Corollary 4.7 (perturbation radius for failure of strong q-subregularity) *In the setting of Theorem* 4.5 *we have the following estimate:*

$$\inf_{g\colon X\to Y}\left\{\operatorname{lip} g(\bar{x})\middle|\ \widetilde{F}+g\ \ \textit{is not}\ \ \textit{strongly metrically}\ \ q\textit{-subregular}\ \ \textit{at}\ (\bar{x},\bar{y})\ \right\}\tag{4.7}$$
$$\geq\frac{1}{\big(\operatorname{ssubreg}{}^{q}F(\bar{x},\bar{y})\big)^{\frac{1}{q}}}.$$

Proof. We split the proof into considering the three possible cases in the theorem.

(i) If $\operatorname{ssubreg}{}^{q}F(\bar{x},\bar{y})=\infty$, then the right-hand side of (4.7) is zero. Observing that for $g\equiv 0$ the mapping $\widetilde{F}$ is not strong q-subregular, we conclude that the infimum in (4.7) the is also zero, and thus the inequality therein holds.

(ii) If $\operatorname{ssubreg}{}^{q}F(\bar{x},\bar{y})=0$, then the right-hand side of (4.7) becomes ∞. For any $g:X\to Y$ with $\operatorname{lip} g(\bar{x})<\infty$ we deduce from Theorem 4.5 that the mapping $\widetilde{F}+g$ is strongly q-subregular as well. Hence the infimum in (4.7) is also ∞, and thus the inequality holds therein.

(iii) Consider the major case of $0<\operatorname{ssubreg}{}^{q}F(\bar{x},\bar{y})<\infty$ and suppose that (4.7) is violated. Then we find a mapping $g:X\to Y$ locally Lipschitzian around $\bar{x}$ such that

$$\operatorname{lip} g(\bar{x})\big(\operatorname{ssubreg}{}^{q}F(\bar{x},\bar{y})\big)^{\frac{1}{q}}<1$$

and the mapping $\widetilde{F}+g$ is not strongly q-subregular at $(\bar{x},\bar{y})$. This clearly contradicts Theorem 4.5 and thus completes the proof. $\square$

The concluding result of this section concerns strong q-subregularity of parameterized mappings being important, in particular, in the framework of Section 4.2.

Theorem 4.8 (strong q-subregularity of parameterized mappings) *Let $F:X\rightrightarrows Y$ be as above, and let $g\colon X\to Y$ be C^1-smooth around $\bar{x}$. Assume that the mapping $G\colon x\mapsto$*

$g(\bar{x}) + \nabla g(\bar{x})(x - \bar{x}) + F(x)$ is strongly q-subregular at $(\bar{x}, \bar{y})$ with $\bar{y} \in G(\bar{x})$. Then for any $\lambda > \text{ssubreg}^{\,q} G(\bar{x}, \bar{y})$ there exists $\gamma > 0$ such that the parameterized form of G defined by

$$x \mapsto G(u, x) := g(\bar{x}) + \nabla g(u)(x - \bar{x}) + F(x) \quad \textit{with} \quad u \in I\!B(\bar{x}, \gamma)$$

is strongly q-subregular at $(\bar{x}, \bar{y})$ with modulus λ, i.e., there is $\eta > 0$ for which

$$\|x - \bar{x}\| \leq \lambda d^q\big(\bar{y}; G(u, x)\big) \quad \textit{whenever} \quad x \in I\!B(\bar{x}, \eta).$$

Proof. Take $\lambda > \kappa > \text{ssubreg}^{\,q}(G, \bar{x}, \bar{y})$ and select $\mu > 0$ so that

$$\lambda > \frac{\kappa}{(1 - \mu\kappa^{\frac{1}{q}})^q} \quad \text{and} \quad \mu\kappa^{\frac{1}{q}} < 1.$$

By the assumed C^1 property of g, find $\gamma > 0$ such that

$$\|\nabla g(u) - \nabla g(\bar{x})\| \leq \mu \quad \text{for all} \quad u \in I\!B(\bar{x}, \gamma).$$

Fix further u as above and define a new parameterized mapping $\widetilde{g} : X \to Y$ by

$$\widetilde{g}(x) := g(u) + \nabla g(u)(x - u) - g(\bar{x}) - \nabla g(\bar{x})(x - \bar{x}).$$

Then we have $\widetilde{g}(\bar{x}) = g(u) + \nabla g(u)(\bar{x} - u) - g(\bar{x})$, and hence

$$\begin{aligned} \|\widetilde{g}(x) - \widetilde{g}(x')\| &= \|\nabla g(u)(x - x') - \nabla g(\bar{x})(x - x')\| \\ &\leq \|\nabla g(u) - \nabla g(\bar{x})\| \cdot \|x - x'\| \leq \mu\|x - x'\| \end{aligned}$$

for any $x, x' \in I\!B(\bar{x}, \gamma)$. Thus $\operatorname{lip} \widetilde{g}(\bar{x}) \le \mu$. Applying Theorem 4.5 to the mappings G and $\widetilde{g}$ ensures that the mapping

$$
\begin{aligned}
& x \mapsto \widetilde{G}(x) + \widetilde{g}(x) = G(x) - \widetilde{g}(\bar{x}) + \widetilde{g}(x) \\
& g(\bar{x}) + \nabla g(\bar{x})(x - \bar{x}) + F(x) + (\nabla g(u) - \nabla g(\bar{x}))(x - \bar{x}) \\
& = g(\bar{x}) + \nabla g(u)(x - \bar{x}) + F(x) = G(u, x)
\end{aligned}
$$

is strongly q-subregular at $(\bar{x}, \bar{y})$ with th exact bound not exceeded λ. This tells us that for any $u \in I\!B(\bar{x}, \gamma)$ there is $\eta > 0$ such that

$$
\|x - \bar{x}\| \le \lambda d^q\big(\bar{y}; G(u, x)\big) \text{ for all } x \in I\!B(\bar{x}, \eta),
$$

which thus completes the proof of the theorem. $\square$

4.2 Application to Quasi-Newton Methods

In this section we discuss some applications of the results on strong q-subregularity under perturbations obtained in Section 4.1 to the convergence rate for a class of *quasi-Newton methods* to solve generalized equations given in the form

$$
0 \in g(x) + F(x), \tag{4.8}
$$

where $g : X \to Y$ is a single-valued mapping while $F \colon X \rightrightarrows Y$ is a set-valued mapping between Banach spaces. As in [9], we consider the following class of quasi-Newton methods to solve (4.8):

$$
0 \in g(x_k) + B_k(x_{k+1} - x_k) + F(x_{k+1}), \quad k = 0, 1, \ldots, \tag{4.9}
$$

where B_k signify a sequence of linear and bounded operators acting from X to Y. In the case of $B_k = \nabla g(x_k)$ algorithm (4.9) corresponds to *Newton's method* while particular choices of the operator sequence $\{B_k\}$ make it possible to include in this scheme various versions of quasi-Newton methods; see more discussions in [8, 9].

Let $\bar{x}$ be a solution to (4.8), and let $\{x_k\}$ be a sequence generated by (4.9) that converges to $\bar{x}$. The classical Dennis-Moré theorem [8, Theorem 2.2] for $F \equiv 0$ establishes a certain characterization of the superlinear convergence (called in [8] the Q-superlinear convergence, where Q stands for "quotient") of the quasi-Newton iterations

$$\lim_{k\to\infty} \frac{\|x_{k+1}-\bar{x}\|}{\|x_k-\bar{x}\|} = 0 \tag{4.10}$$

under the smoothness of $g\colon I\!\!R^n \to I\!\!R^n$ and nonsingularity of its Jacobian $\nabla g(\bar{x})$. Recently [9, Theorem 3], Dontchev extended this result to the case of generalized equations (4.8) assuming that the mapping $x \mapsto g(\bar{x}) + \nabla g(\bar{x})(x-\bar{x}) + F(x)$ is strongly subregular at $(\bar{x}, 0)$, which reduces to the nonsingularity of $\nabla g(\bar{x})$ in the setting of [8].

The following theorem imposes the q-subregularity of F at $(\bar{x}, -g(\bar{x}))$ as $q \geq 1$ and shows, by using the approach somewhat different from both papers [8, 9] and based on the stability results of Section 4.1, that we have the *higher convergence rate*

$$\lim_{k\to\infty} \frac{\|x_{k+1}-\bar{x}\|}{\|x_k-\bar{x}\|^q} = 0, \quad q \geq 1, \tag{4.11}$$

which reduces to the superlinear one in (4.10) for $q = 1$.

Theorem 4.9 (convergence rate for quasi-Newton iterations) *Let $\bar{x}$ be a solution of the generalized equation* (4.8)*, where $g\colon X \to Y$ be a mapping between Banach spaces that is C^1-*

smooth on some convex neighborhood U of $\bar{x}$. Given a starting point $x_0 \in U$ and a sequence of linear and bounded operators $B_k\colon X \to Y$, consider the corresponding sequence $\{x_k\}$ generated by (4.9) such that $\{x_k\} \subset U$ and $x_k \to \bar{x}$ as $k \to \infty$. Assume that the set-valued mapping $F\colon X \rightrightarrows Y$ in (4.8) is strongly q-subregular at $(\bar{x}, -g(\bar{x}))$ with some $q \geq 1$ and that there exist positive numbers κ, λ for which

$$\operatorname{ssubreg}^{q} F\big(\bar{x}, -g(\bar{x})\big) < \kappa \quad \textit{and} \quad \operatorname{lip} g(\bar{x}) < \lambda < \big(\kappa^{\frac{1}{q}}\big)^{-1}. \tag{4.12}$$

Suppose also that $x_k \neq \bar{x}$ for all $k \in \mathcal{N}$. Then we have the implication

$$\Big[\lim_{k\to\infty} \frac{\big\|\big(B_k - \nabla g(\bar{x})\big)(x_{k+1} - x_k)\big\|}{\|x_{k+1} - x_k\|} = 0\Big] \Longrightarrow (4.11). \tag{4.13}$$

Proof. By (4.12) it follows from Theorem 4.5 that

$$\operatorname{ssubreg}^{q}(\widetilde{F} + g)\big(\bar{x}, -g(\bar{x})\big) \leq \frac{\kappa}{(1 - \lambda\kappa^{\frac{1}{q}})^{q}} \quad \text{with} \quad \widetilde{F}(x) = F(x) - g(\bar{x}).$$

Then Corollary 4.6 tells us that the mapping $\widetilde{G}\colon x \mapsto g(\bar{x}) + \nabla g(\bar{x})(x - \bar{x}) + \widetilde{F}(x) = \nabla g(\bar{x})(x - \bar{x}) + F(x)$ is strongly q-subregular at $(\bar{x}, -g(\bar{x})$ with modulus $\mu \leq \frac{\kappa}{(1-\lambda\kappa^{\frac{1}{q}})^{q}}$. To prove implication (4.13) with $q \geq 1$, take any small $\epsilon > 0$ and find a natural number k_0 sufficiently large so that for all $k \geq k_0$ we have the relationships

$$\begin{aligned}
\|x_{k+1}-\bar{x}\|^{\frac{1}{q}} &\leq \mu^{\frac{1}{q}} d\big(-g(\bar{x}); \widetilde{G}(x_{k+1})\big) = \mu^{\frac{1}{q}} d\big(-g(\bar{x}); \nabla g(\bar{x})(x_{k+1}-\bar{x}) + F(x_{k+1})\big) \\
&\leq \mu^{\frac{1}{q}} \| -g(\bar{x}) + g(x_k) + B_k(x_{k+1}-x_k) - \nabla g(\bar{x})(x_{k+1}-\bar{x})\| \\
&= \mu^{\frac{1}{q}} \|g(x_k) - g(\bar{x}) - \nabla g(\bar{x})(x_k-\bar{x}) + \big(B_k - \nabla g(\bar{x})\big)(x_{k+1}-x_k)\| \\
&\leq \mu^{\frac{1}{q}} \Big\| \int_0^1 \nabla g\big(\bar{x}+t(x_k-\bar{x})\big)(x_k-\bar{x})\,dt - \nabla g(\bar{x})(x_k-\bar{x})\Big\| \\
&+ \mu^{\frac{1}{q}} \|B_k - \nabla g(\bar{x})\|\|x_{k+1}-x_k\| \\
&\leq \mu^{\frac{1}{q}} \Big\| \int_0^1 \big(\nabla g\big(\bar{x}+t(x_k-\bar{x})\big) - \nabla g(\bar{x})\big)(x_k-\bar{x})dt\Big\| + \mu^{\frac{1}{q}}\epsilon\|x_{k+1}-x_k\| \\
&\leq \mu^{\frac{1}{q}}\epsilon\|x_k-\bar{x}\| + \mu^{\frac{1}{q}}\epsilon\|x_{k+1}-\bar{x}\| + \mu^{\frac{1}{q}}\epsilon\|x_k-\bar{x}\| \\
&\leq 2\mu^{\frac{1}{q}}\epsilon\|x_k-\bar{x}\| + \mu^{\frac{1}{q}}\epsilon\|x_{k+1}-\bar{x}\|^{\frac{1}{q}}.
\end{aligned}$$

This gives us the upper estimate

$$\|x_{k+1}-\bar{x}\| \leq \frac{2^q \mu \epsilon^q}{(1-\mu^{\frac{1}{q}}\epsilon)^q}\|x_k-\bar{x}\|^q,$$

which ensures the validity of (4.11) and thus completes the proof of the theorem. □

It worth mentioning that the inverse implication also holds in (4.13), which in fact follows from the proof of [9, Theorem 3] for $q=1$; cf. also [8].

As shown by the examples of Chapter 3, strong higher-order subregularity ($q>1$) holds in natural situations when metric regularity fails. The following simple example, where $F(x)$ is a non-Lipschitzian function, illustrates the application of Theorem 4.9 in such settings.

Example 4.10 (quasi-Newton method for non-Lipschitzian 2-subregular equations) Let $g(x) := x^2$ and $F(x) := |x|^{\frac{1}{2}}$, $x \in I\!R$. Then $\bar{x} = 0$ is a solution to the generalized equation (4.8), which in this case reduces to a nonsmooth equation defined by a non-Lipschitzian function. As has been well recognized in the literature (see, e.g., [12, 16] and the references therein), the vast majority of the results on Newton-type methods for nonsmooth equations concerns Lipschitzian ones, while non-Lipschitzian settings are highly challenging. Based on Example 3.2, we conclude that F is strongly 2-subregular at $(0,0)$. It is easy to check that the other conditions of Theorem 4.9 are also satisfied. Consider now the quasi-Newton method (4.9) with

$$B_k := \frac{\left(2^{\frac{(k+1)!}{2}}\right)^{-1} + \left(2^{2k!}\right)^{-1}}{\left(2^{k!}\right)^{-1} - \left(2^{(k+1)!}\right)^{-1}}.$$

It is easy to verify that $\lim_{k\to\infty} |B_k - \nabla g(\bar{x})| = 0$. Then for any starting point x_0 close to $\bar{x}$, it follows that algorithm (4.9) generates a sequence $\{x_k\} = \{(2^{k!})^{-1}\}$ converging to $\bar{x} = 0$ with the convergence rate that exceeds $q = 2$.

$\square$

CHAPTER 5

Q-Subregularity for Constraint Systems

For many optimization problems the constraints can be formulated as an inclusion of the abstract form

$$0 \in F(x),$$

where $F : X \rightrightarrows Y$ is a multifunction between Banach spaces X and Y. An important special case, appearing in many applications, is given by constraint systems of the form

$$0 \in f(x) - C, \tag{5.1}$$

where $f : X \to Y$ is a mapping and $C \subset Y$ is a set, i.e., we have $F(x) = f(x) - C$. A prominent example of a constraint system of the form (5.1) is given by the constraints of a possibly infinite dimensional mathematical programming problem with $Y := Y_1 \times \mathbb{R}^m, C = \{0\} \times \mathbb{R}^m_+$ and $f = (P, f_1, \cdots, f_m)$, i.e.,

$$\begin{aligned} &P(x) = 0, \\ &f_i(x) \leq 0, i = 1, \cdots, m. \end{aligned} \tag{5.2}$$

Our main task in this chapter is to investigate the metric q-subregularity as well as uniform metric regularity of the constraint system (5.1). We intend to characterize the aforementioned properties in terms of its initial data. Let's provide the following result, which is a direct consequence of [34, Theorem 4.10] and [21, Theorem 3.41 and Corollary 1.81] and will be helpful in the later proofs.

Theorem 5.1 (chain rule for Lipschitz functions) *Assume that X and Y are Asplund spaces. Let $f : X \to Y$ be Lipschitz continuous around $\bar{x}$, and let $g : Y \to \mathbb{R}$ be Lipschitz continuous around $\bar{y}$ with $\bar{y} := f(\bar{x})$. Then one has*

(i) *for any $\varepsilon, \sigma > 0$ and each fixed $x^* \in \hat{\partial}(g \circ f)(\bar{x})$, there exist $x \in B(\bar{x}, \varepsilon), y \in B(\bar{y}, \varepsilon)$ with $|g(y) - g(\bar{y})| \leq \varepsilon$ as well as $y^* \in \hat{\partial} g(y) + \sigma B_{Y^*}$ and $\widetilde{x}^* \in \hat{\partial}\langle y^*, f\rangle(x)$ such that*

$$x^* \in \widetilde{x}^* + \sigma B_{X^*}; \tag{5.3}$$

(ii)

$$\partial(g \circ f)(\bar{x}) \subset \bigcup_{y^* \in \partial g(\bar{y})} D^* f(\bar{x})(y^*). \tag{5.4}$$

5.1 Characterizations on q-Subregularity

Let us start this section with some basic notations. Let $C \subset Y, y \in Y \setminus C$, recall that the Euclidean projector of y to C is defined as follows:

$$\Pi(y; C) := \{w \in C | \|y - w\| = d(y, C)\},$$

For convenience, we set

$$\mathfrak{N}(y; C) := \bigcap_{w \in \Pi(y;C)} \hat{N}(w; C) \cap S_{Y^*}$$

and

$$\mathfrak{N}_\varepsilon(y; C) := \left\{ \frac{y^* + \varepsilon b^*}{\|y^* + \varepsilon b^*\|} \middle| y^* \in \mathfrak{N}(y), b^* \in B_{Y^*} \right\}. \tag{5.5}$$

We first provide the following sufficient condition as well as modulus estimation for metric q-subregularity of the constraint system (5.1) under Lipschitz condition.

Theorem 5.2 (Sufficient condition of q-subregularity in terms of projection points) *Let X and Y be Asplund spaces, $\beta, q \in (0, +\infty)$, C be a closed set of Y, $f : X \to Y$ and $F(x) = f(x) - C$. Suppose $\bar{x} \in f^{-1}(C)$ and f is local Lipschitz around $\bar{x}$. Also assume that $\Pi(f(x); C)$ is not empty for all x close to $\bar{x}$ with $f(x) \notin C$. Then the following condition*

$$\begin{aligned}\kappa := \sup_{\varepsilon>0} \inf \Bigg\{ \frac{\|x^*\|}{d(x, f^{-1}(C))^{\frac{1-q}{q}}} \Bigg| \, & x, x', x'' \in B(\bar{x}, \varepsilon) \setminus f^{-1}(C), x' \in B(x, d(f(x), C)), \\ & x'' \in B(x', d(f(x'), C)), y \in B(f(x'), \varepsilon) \setminus C, d(f(x), C)^q < \beta d(x, f^{-1}(C)), \\ & y^* \in \mathfrak{N}_\varepsilon(y; C), x^* \in \hat{\partial}\langle y^*, f\rangle(x'') \Bigg\} > 0.\end{aligned} \tag{5.6}$$

implies that F is metrically q-subregular at $(\bar{x}, 0)$. Moreover, we have the following modulus estimate:

$$subreg^q F(\bar{x}, 0) \leq \begin{cases} \max\left\{\frac{2}{\kappa^q}, \frac{2}{\beta^q}\right\}, & q \in (0, 1], \\ \max\left\{\frac{4^q}{(3\kappa)^q}, \frac{2}{\beta^q}\right\}, & q \in (1, +\infty). \end{cases} \tag{5.7}$$

Proof. It is clear that estimate (5.7) with $\kappa > 0$ ensures all the statements of the theorem. To justify this, assume the contrary and thus find $\tau > 0$ with

$$subreg^q F(\bar{x}, 0) > \tau > \begin{cases} \max\left\{\frac{2}{\kappa^q}, \frac{2}{\beta^q}\right\}, & q \in (0, 1], \\ \max\left\{\frac{4^q}{(3\kappa)^q}, \frac{2}{\beta^q}\right\}, & q \in (1, +\infty). \end{cases} \tag{5.8}$$

Then, there are sequences x_k with $\|x_k - \bar{x}\| \to 0$ such that

$$\tau d(f(x_k), C)^q < d(x_k, f^{-1}(C)). \tag{5.9}$$

And hence, we have that

$$d(f(x_k),C) < \inf_{x\in X} d(f(x),C) + (\frac{d(x_k, f^{-1}(C))}{\tau})^{\frac{1}{q}}. \tag{5.10}$$

Now we apply Ekeland's variational principle [21, Theorem 2.26] to the function $d(\cdot, C)\circ f$ on X with $\epsilon_k = (\frac{d(x_k,f^{-1}(C))}{\tau})^{\frac{1}{q}}$ and $\lambda_k := \frac{d(x_k,f^{-1}(C))}{2}$, which ensures us optimal solutions $\widetilde{x}_k \in X$ satisfying $\|\widetilde{x}_k - x_k\| \le \lambda_k$, $d(f(\widetilde{x}_k),C) \le d(f(x_k),C)$ and

$$d(f(\widetilde{x}_k),C) \le d(f(x),C) + \frac{d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\alpha\tau^{\frac{1}{q}}}\|x-\widetilde{x}_k\| \ \forall x \in X. \tag{5.11}$$

It follows from the triangle inequality that

$$d(\widetilde{x}_k, f^{-1}(C)) \le d(x_k, f^{-1}(C)) + \|\widetilde{x}_k - x_k\| \le \frac{3d(x_k, f^{-1}(C))}{2}$$

and

$$d(\widetilde{x}_k, f^{-1}(C)) \ge d(x_k, f^{-1}(C)) - \|\widetilde{x}_k - x_k\| \ge \frac{d(x_k, f^{-1}(C))}{2}.$$

which guarantee that

$$\frac{3}{2}d(\widetilde{x}_k, f^{-1}(C)) \le d(x_k, f^{-1}(C)) \le 2d(\widetilde{x}_k, f^{-1}(C)) \tag{5.12}$$

and $d(\widetilde{x}_k, f^{-1}(C)) > 0$. Then $\widetilde{x}_k \notin f^{-1}(C)$, and hence $f(\widetilde{x}_k) \notin C$. Note that $d(f(\widetilde{x}_k),C) \le d(f(x_k),C)$, combining (5.8), (5.9) and (5.11) gives us

$$d(\widetilde{x}_k, f^{-1}(C)) \le (\frac{d(x_k, f^{-1}(C))}{\tau})^{\frac{1}{q}} < \frac{2^{\frac{1}{q}}}{\tau^{\frac{1}{q}}}d(\widetilde{x}_k, f^{-1}(C))^{\frac{1}{q}} \le \beta d(\widetilde{x}_k, f^{-1}(C))^{\frac{1}{q}}. \tag{5.13}$$

Observe that $0 \in \hat{\partial}(d(f(\cdot), C) + \frac{2d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\tau^{\frac{1}{q}}}\|\cdot - \widetilde{x}_k\|)(\widetilde{x}_k)$ due to the validity of (5.11). Then by applying the semi-Lipschitzian sum rule [21, Theorem 2.33 (b)], we find $\widetilde{x}'_k \in B(\widetilde{x}_k, d(\widetilde{x}_k, f^{-1}(C)))$ and $u^*_k \in B_{X^*}$ such that

$$\frac{2d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\tau^{\frac{1}{q}}} u^* \in \hat{\partial}(d(f(\cdot), C))(\widetilde{x}'_k). \tag{5.14}$$

Clearly, $d(\widetilde{x}'_k, f^{-1}(C)) > 0$, then $\widetilde{x}'_k \notin f^{-1}(C)$, and hence $f(\widetilde{x}'_k) \notin C$. Note that $d(\cdot, C)$ and f are Lipschitz continuous, using the fuzzy chain rule in Theorem 5.1 (i), we can find $\widetilde{x}''_k \in B(\widetilde{x}'_k, d(\widetilde{x}'_k, f^{-1}(C))), y_k \in B(f(\widetilde{x}'_k), d(f(\widetilde{x}'_k), C)),\ y^*_k \in \hat{\partial}d(\cdot, C)(y_k), y'^*_k \in B(y^*_k, d(f(\widetilde{x}'_k), C))$ and $v^*_k \in \hat{\partial}\langle y'^*_k, f\rangle(\widetilde{x}''_k)$ satisfying

$$\|v^*_k - \frac{2d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\tau^{\frac{1}{q}}} u^*\| \leq d(f(\widetilde{x}'_k), C) \downarrow 0. \tag{5.15}$$

It is easy to see that $y_k \notin C$ and $\Pi(y_k; C) \neq \emptyset$. Since $y^*_k \in \hat{\partial}d(\cdot, C)(y_k)$, it follows from [21, Theorem 1.102] that $y^*_k \in \mathfrak{N}(y_k; C)$. Now let

$$\widetilde{y}^*_k := \frac{y'^*_k}{\|y'^*_k\|} \text{ and } \widetilde{v}^*_k := \frac{v^*_k}{\|y'^*_k\|},$$

then $\widetilde{y}^*_k \in \mathfrak{N}_\varepsilon(y_k; C)$ and $\widetilde{v}^*_k \in \hat{\partial}\langle \widetilde{y}^*_k, f\rangle(\widetilde{x}''_k)$, when k is sufficient large. By (5.12), one has

$$\frac{2d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\tau^{\frac{1}{q}}} \leq \begin{cases} (\frac{2}{\tau})^{\frac{1}{q}} d(\widetilde{x}_k, f^{-1}(C))^{\frac{1-q}{q}}, & q \in (0, 1], \\ 3(\frac{2}{3\tau})^{\frac{1}{q}} d(\widetilde{x}_k, f^{-1}(C))^{\frac{1-q}{q}}, & q \in (1, +\infty). \end{cases}$$

This and (5.8) show that

$$\frac{2d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\tau^{\frac{1}{q}}} < \kappa d(\widetilde{x}_k, f^{-1}(C))^{\frac{1-q}{q}} \ \forall q \in (0, +\infty).$$

Therefore by taking into account of $d(f(\widetilde{x}'_k), C) \downarrow 0$ as $k \to \infty$, we have

$$\|\widetilde{v}^*_k\| \leq \frac{\frac{2d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\tau^{\frac{1}{q}}} + d(f(\widetilde{x}'_k), C)}{1 - d(f(\widetilde{x}'_k), C)} < \kappa d(\widetilde{x}_k, f^{-1}(C))^{\frac{1-q}{q}} \ \forall q \in (0, +\infty).$$

when k is large enough. This is a contradiction to the definition of κ by taking into account of the construction of $\widetilde{x}_k, \widetilde{x}'_k, \widetilde{x}''_k, y_k, \widetilde{y}^*_k, \widetilde{v}^*_k$ and (5.13). Our proof is completed. □

The following example illustrates that our sufficient condition in Theorem 5.2 is only valid for metric q-subregularity, not for q-regularity.

Example 5.3 Let $X = Y = \mathbb{R}$, $\bar{x} = 0$, $f(x) = x^2$ and $C = (-\infty, 0]$. It is easy to see that

$$f^{-1}(C) = \{0\} \ \text{ and } \ d(f(x), C) = x^2$$

for all $x \in \mathbb{R}$. Pick $q = \frac{1}{2}$, $\beta = 2$ and let them be fixed. Observe that for any $\varepsilon > 0$ and selections of $x, x', x'' \in (-\varepsilon, \varepsilon) \setminus \{0\}$ satisfying

$$x' \in (x - x^2, x + x^2), x'' \in (x' - x'^2, x' + x'^2), y \in (0, x'^2 + \varepsilon)$$

and

$$d(f(x), C)^{\frac{1}{2}} < 2d(x, f^{-1}(C))$$

, it's not difficult to calculate that

$$\mathfrak{N}_\varepsilon(y;C) = \{1\} \text{ and } \hat{D}^* f(x'')(y^*) = \nabla f(x'') * y^* = 2x''y^*$$

for all $y^* \in \mathbb{R}$. Therefore for any $y^* \in \mathfrak{N}_\varepsilon(y;C)$ and $x^* \in \hat{D}^* f(x'')(y^*) = \hat{D}^* f(x'')(1) = 2x''$, one has

$$\|x^*\| = 2|x''| \geq 2|x| - \varepsilon$$

, which implies that $\kappa = 2$ according to the definition of κ in (5.6). It follows from Theorem 5.2 that $F(x) = x^2 - (-\infty, 0]$ is $\frac{1}{2}$-metrically subregular at $(0,0)$ with modulus no more than $\sqrt{2}$. However, $F(x) = x^2 - (-\infty, 0]$ is not metrically $\frac{1}{2}$-regular around $(0,0)$ due to the fact that $F^{-1}(y) = \phi$ when $y < 0$.

Remark 5.4 It is easy to observe that the formulation of the constant κ in (5.6) involves triple selections regarding candidates x's. This is a result of the fuzzy sum rule which was applied to the Fréchet subdifferentials. Indeed, by employing the generalized Fermat Rule and the semi-Lipschitzian sum rule [21, Theorem 2.33 (c)] to Mordukhovich subdifferential instead of the Fréchet case on (5.11), we are then able to simplify the calculations of κ with help of the following result [35, Example 2.130]:

$$\partial d(\cdot, C)(y_0) = N(C, y') \cap \partial \| \cdot - y' \|(y_0) \tag{5.16}$$

where $y' \in \Pi(y_0; C)$ for a closed convex subset C of Y and $y_0 \in Y$. Note that y' need not to be unique and the right hand side of (5.16) is the same for all possible $y' \in \Pi(y_0; C)$. It is clear that if $y_0 \in Y \setminus C$, then $\partial d(\cdot, C)(y_0) \subset S_{X^*}$.

Theorem 5.5 *Let X be an Asplund space and Y be a reflexive Banach space. Let $\beta, q \in (0, +\infty)$ and C be a closed convex subset of Y. Suppose that $F(x) = f(x) - C$, $\bar{x} \in f^{-1}(C)$, where the mapping $f : X \to Y$ is locally Lipschitzian around $\bar{x}$. Then the condition*

$$\kappa := \sup_{\varepsilon>0} \inf \left\{ \frac{\|x^*\|}{d(x, f^{-1}(C))^{\frac{1-q}{q}}} \middle| x \in B(\bar{x}, \varepsilon) \setminus f^{-1}(C), d(f(x), C)^q < \beta d(x, f^{-1}(C)), \right. \tag{5.17}$$
$$\left. \exists y' \in \Pi(f(x); C), x^* \in D^* f(x) \left(N(C, y') \cap \partial \| \cdot - y'\|(f(x))\right) \right\} > 0.$$

is sufficient for F to be metrically q-subregular at $(\bar{x}, 0)$. More precisely, we have the following estimates of modulus

$$subreg^q F(\bar{x}, 0) \leq \begin{cases} \max\left\{\frac{2}{\kappa^q}, \frac{2}{\beta^q}\right\}, & q \in (0, 1], \\ \max\left\{\frac{4^q}{(3\kappa)^q}, \frac{2}{\beta^q}\right\}, & q \in (1, +\infty). \end{cases} \tag{5.18}$$

Proof. It suffices to show that (5.18) is true. Taking the similar approach as in Theorem 5.2, we assume to the contrary that there exists a $\tau > 0$ such that (5.8) holds. Then there are sequences x_k with $\|x_k - \bar{x}\| \to 0$ such that (5.9) and (5.10) holds. From that we are ready to apply Ekeland's Variational Principle to the function $d(\cdot, C) \circ f$ and get $\widetilde{x}_k \in X$ such that $\|\widetilde{x}_k - x_k\| \leq \lambda_k := \frac{d(x_k, f^{-1}(C))}{2}$, $d(f(\widetilde{x}_k), C) \leq d(f(x_k), C)$ and satisfy (5.11), (5.12), (5.13) as well as $f(\widetilde{x}_k) \notin C$. It then follows from (5.11) and then the semi-Lipschitzian Sum Rule [21, Theorem 2.33] that

$$\begin{aligned} 0 &\in \partial \left(d(f(\cdot), C) + \frac{2d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\tau^{\frac{1}{q}}} \| \cdot - \widetilde{x}_k\| \right) (\widetilde{x}_k) \\ &\subset \partial d(f(\cdot), C)(\widetilde{x}_k) + \frac{2d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\tau^{\frac{1}{q}}} B_{X^*}. \end{aligned}$$

Applying the chain rule in Theorem 5.1 (ii) ensures the existence of $u_k^* \in B_{X^*}$ such that

$$\begin{aligned}\frac{2d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\tau^{\frac{1}{q}}} u_k^* &\in \partial(d(f(\cdot), C))(\widetilde{x}_k)\\ &\subset \bigcup_{y^* \in \partial d(\cdot, C)(f(\widetilde{x}_k))} D^* f(\widetilde{x}_k)(y^*).\end{aligned}$$

Note that $f(\widetilde{x}_k) \notin C$ and $d(\cdot, C)$ and f are Lipschitz continuous, it follows from the above inclusion and (5.16) that, there exist $y' \in \Pi(f(\widetilde{x}_k); C)$ and $y^* \in N(C, y') \cap \partial \| \cdot - y' \| (f(\widetilde{x}_k))$ satisfying

$$\frac{2d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\tau^{\frac{1}{q}}} u_k^* \in D^* f(\widetilde{x}_k)(y^*). \tag{5.19}$$

Recall that by (5.12), one has

$$\frac{2d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\tau^{\frac{1}{q}}} \leq \begin{cases} (\frac{2}{\tau})^{\frac{1}{q}} d(\widetilde{x}_k, f^{-1}(C))^{\frac{1-q}{q}}, & q \in (0, 1],\\ 3(\frac{2}{3\tau})^{\frac{1}{q}} d(\widetilde{x}_k, f^{-1}(C))^{\frac{1-q}{q}}, & q \in (1, +\infty). \end{cases}$$

This and (5.8) imply that

$$\frac{2d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\tau^{\frac{1}{q}}} < \kappa d(\widetilde{x}_k, f^{-1}(C))^{\frac{1-q}{q}} \ \forall q \in (0, +\infty).$$

Therefore we conclude that

$$\|\frac{2d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\tau^{\frac{1}{q}}} u_k^*\| \leq \frac{2d(x_k, f^{-1}(C))^{\frac{1-q}{q}}}{\tau^{\frac{1}{q}}} < \kappa d(\widetilde{x}_k, f^{-1}(C))^{\frac{1-q}{q}} \ \forall q \in (0, +\infty).$$

which is a contradiction to our definition of κ by taking into account of the constructions of (5.13) and (5.19). The proof is completed. $\square$

The next example shows that Theorem 5.5 does not hold when C is nonconvex.

Example 5.6 Let $X = \mathbb{R}, Y = \mathbb{R}^2$, $f(x) = (x, x^2)$ and $C = \{(x, y) \in \mathbb{R}^2 | y \geq x \geq 0 \text{ or } y \leq x \leq 0\}$. Apparently, C is not convex. Let $\bar{x} = 0$, then $\bar{x} \in f^{-1}(C) = \{0\} \cup [1, +\infty)$, hence $d(x, f^{-1}(C)) = |x|$ and $d(f(x), C) = \frac{|x^2 - x|}{\sqrt{2}}$ when x is sufficiently close to 0. It is clear that

$$\lim_{x \to 0} \frac{d(x, f^{-1}(C)}{d(f(x), C)} = \lim_{x \to 0} \frac{\sqrt{2}|x|}{|x^2 - x|} = \infty.$$

which implies that $F(x) := f(x) - C$ is not metrically subregular at $(0, 0)$. However, it is not difficult to calculate that for any nonzero element x close enough to 0, we have $\Pi(f(x); C) = \{(\frac{x+x^2}{2}, \frac{x+x^2}{2})\}$ and

$$N\left(C, \left(\frac{x+x^2}{2}, \frac{x+x^2}{2}\right)\right) \bigcap \partial \left\| \cdot - \left(\frac{x+x^2}{2}, \frac{x+x^2}{2}\right) \right\| (x, x^2) = \begin{cases} \left\{(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\right\}, & x > 0, \\ \left\{(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\right\}, & x < 0. \end{cases}$$

Since

$$D^* f(x)(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = \nabla f(x)^* (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = (1, 2x)^* (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2} - \sqrt{2}x$$

and

$$D^* f(x)(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = \sqrt{2}x - \frac{\sqrt{2}}{2},$$

we can claim that the κ in (5.17) is no less then $\frac{\sqrt{2}}{2}$ with $\beta = 1$ and $q = 1$.

The rest of this section investigate the application of our previous results on the constraint system (5.1) to q-order weak sharp minimizer [36, Definition 1.1]. For convenience, we reformulate the definition here. Consider the problem

$$(P) \qquad \min\{f(x)|x \in X\}$$

where $f : X \to R$ is a continuous function. Denote the solution set of (P) by S. Recall that

Definition 5.7 (q-order growth condition) *Let $p > 0$. Then the objective function f in (P) is said to satisfy a* P-ORDER GROWTH CONDITION *if there exist a constant $c > 0$ and a neighborhood $\mathcal{N}$ of S such that*

$$f(x) \geq f_* + cd(x, S)^p \quad \forall x \in \mathcal{N}. \tag{5.20}$$

where $f_* = \inf\{f(x)|x \in X\}$.

The following result provides a relationship between metric q-subregularity of the constraint system (5.1) and $\frac{1}{q}$-order growth condition on f for any positive q.

Theorem 5.8 *Let the solution set S of (P) be nonempty and compact, and suppose that the multifunction $F(x) := f(x) - (-\infty, f_*]$ is metrically q-subregular at $(S \times \{0\})$ (i.e., metrically q-subregular at $(x, 0)$ for all $x \in S$). Then f satisfies a $\frac{1}{q}$-order growth condition in (P).*

Proof. Pick an arbitrary $x_0 \in S$ and let it be fixed, i.e. $f(x_0) = f_*$. Therefore the metric q-subregularity of F at $(x_0, 0)$ implies the existence of $\delta, \tau > 0$ such that

$$d(x, F^{-1}(0)) \leq \tau d(0, f(x) - (-\infty, f_*])^q = |f(x_0) - f(x)|^q$$

for all $x \in B(x_0, \delta)$. Since $F^{-1}(0) = S$, it follows that

$$\frac{1}{\tau} d(x, S)^{\frac{1}{q}} \leq |f(x_0) - f(x)|^q \qquad \forall\, x \in B(x_0, \delta).$$

Note that $f(x) \geq f(x_0)$ for all x around x_0, we have

$$f(x) \geq f_* + \frac{1}{\tau} d(x, S)^{\frac{1}{q}} \quad \forall x \in B(x_0, \delta).$$

Since x_0 is arbitrarily chosen and S is compact, there exist a finite number of x_i's from S, $\delta_i > 0$, and $\tau_i > 0$, such that $S \subset \cup_{i=1}^n B(x_i, \delta_i)$ and for each i from $\{1, \ldots, n\}$,

$$f(x) \geq f_* + \frac{1}{\tau_i} d(x, S)^{\frac{1}{q}} \qquad \forall x \in B(x_i, \delta_i),$$

which implies that f satisfies a $\frac{1}{q}$-order growth condition with $c = \max_{i=1}^n \tau_i$ and $\mathcal{N} = \cup_{i=1}^n B(x_i, \delta_i)$. The proof is completed. $\square$

By Theorem 5.5 and Theorem 5.8, we have the following result:

Corollary 5.9 *Let X be an Asplund space. Let $\beta, q \in (0, +\infty)$, f_* and S be defined as in (5.20). Suppose that for all $x \in S$, the mapping $f : X \to \mathbb{R}$ is local Lipschitz around x and*

$$\kappa_x := \sup_{\varepsilon > 0} \inf \left\{ \frac{\|x^*\|}{d(x', S)^{\frac{1-q}{q}}} \middle| x' \in B(x, \varepsilon) \setminus S, (f(x') - f_*)^q < \beta d(x', S), x^* \in \hat{\partial} f(x') \right\} > 0. \tag{5.21}$$

Then, f satisfies a $\frac{1}{q}$-order growth condition in (P).

Proof. Let $C := (-\infty, f_*]$ and $F(x) := f(x) - C$, then $f^{-1}(C) = S$. Fix any $\varepsilon > 0, x \in S$ and $x' \in B(x, \varepsilon) \setminus S$, one has $d(f(x'), C) = f(x') - f_*$, $\Pi(f(x'); C) = \{f_*\}, N(C, f_*) = [0, +\infty)$

and $\partial|\cdot - f_*| = \{1\}$. Note that $\hat{D}^*f(x')(1) = \hat{\partial}f(x')$, then (5.21) implies (5.17). Hence it follows from Theorem 5.5 that $F(x)$ is metrically q-subregular at $(x, 0)$. Now we are ready to apply Theorem 5.8 and get that f satisfies a $\frac{1}{q}$-order growth condition in (P). The proof is completed.

□

5.2 Uniform Metric Regularity for Constraint Systems

This section studies the notion of uniform metric regularity of a collection of multifunctions $G_i : X \rightrightarrows Y$ defined by

$$G_i(x) := g(x) - \Theta_i \;\; \forall x \in X, \tag{5.22}$$

where g is a mapping between X and Y and $\{\Theta_i \subset Y : i \in I\}$ is a collection of closed sets. Let's start by recalling the following definition:

Definition 5.10 *Let I be any index set, $G_i : X \rightrightarrows Y, i \in I$ be a collection of multifunctions with $(\bar{x}, \bar{y}) \in \cap_{i\in I}\mathrm{gph}G_i$. Then the multifunction collection $\{G_i : i \in I\}$ is said to be uniformly metrically regular at $(\bar{x}, \bar{y})$ if there exist $\tau, \delta \in (0, +\infty)$ such that*

$$d(x, G_i^{-1}(y)) \le \tau d(y, G_i(x)) \;\forall (i, x, y) \in I \times B(\bar{x}, \delta) \times B(\bar{y}, \delta). \tag{5.23}$$

Many authors have studied the uniform metric regularity, see [37, 38]. In [37], the authors provided the following Proposition under the assumptions of surjectivity on $g'(\bar{x})$ and uniform subsmoothness of $\{G_i : i \in I\}$.

Proposition 5.11 *[37, Proposition 3.5] Let X and Y be Banach spaces and $g : X \to Y$ be a smooth mapping. Let I be an arbitrary index set, $\{G_i : i \in I\}$ be a collection of closed sets in Y and let each G_i be defined by (5.22). Suppose that $\bar{x}$ is an element in $\cap_{i\in I}g^{-1}(\Theta_i)$ such that*

$g'(\bar{x})$ is surjective and that $\{\Theta_i : i \in I\}$ is uniformly subsmooth at $g(\bar{x})$. Then $\{G_i : i \in I\}$ is uniformly metrically regular at $(\bar{x}, 0)$.

In what follows, we relaxed the surjectivity assumption on $g'(\bar{x})$ and dropped the uniform subsmoothness assumption on $\{\Theta_i : i \in I\}$ from Proposition 5.11 with a different approach. The proof of the following Theorem is a modification of the *Lyusternik-Graves iterative process [21, Theorem 1.57].*

Theorem 5.12 *Let X and Y be Banach spaces,I be an arbitrary index set, $\{\Theta_i \subset Y : i \in I\}$ be a collection of closed sets and $\{G_i\}$ be defined as in (5.22). Suppose that $\bar{x} \in \cap_{i\in I} g^{-1}(\Theta_i)$ and $g : X \to Y$ is a mapping such that there exists a surjective linear operator $A \in \mathcal{B}(X, Y)$, some neighborhood U of $\bar{x}$, $\mu > 0$ and $\gamma > regA$ satisfying the relationships $\mu\gamma < 1$ and*

$$\|g(x) - g(x') - A(x - x')\| \leq \mu\|x - x'\| \quad \forall x, x' \in U. \tag{5.24}$$

Then $\{G_i : i \in I\}$ is uniformly metrically regular at $(\bar{x}, 0)$ and more over

$$reg(G_i) \leq \frac{\gamma}{1 - \mu\gamma} \quad \forall i \in I. \tag{5.25}$$

Proof. It suffices to justify (5.25). To this end, we fix any $i \in I$ and pick an $a > 0$ such that (5.24) holds for every $x, x' \in B(\bar{x}, a)$. Choose further $\alpha \in (0, \min\{2(\mu + \|A\|)a, a\})$ sufficiently small satisfying the following estimates

$$(\gamma + 1)\alpha \leq a \text{ and } \frac{(\gamma + 1 - \mu\gamma)\alpha}{1 - \mu\gamma} \leq a. \tag{5.26}$$

It is clear that g is locally Lipschitz around $\bar{x}$. Indeed by (5.24), we have that

$$\|g(x) - g(x')\| \leq \|g(x) - g(x') - A(x - x')\| + \|A(x - x')\| \leq (\mu + \|A\|)\|x - x'\|. \tag{5.27}$$

holds for all $x, x' \in B(\bar{x}, a)$. Take any arbitrary pair (x, y) from $B(\bar{x}, \frac{\alpha}{2(\mu+\|A\|)}) \times B(0, \frac{\alpha}{2})$ and let it be fixed. It then follows from (5.27) that $g(x) \in B(g(\bar{x}), \frac{\alpha}{2})$ and

$$d(g(x), y + \Theta_i) \leq \|g(x) - y - g(\bar{x})\| < \alpha. \tag{5.28}$$

Thus $\Theta_i \cap B(g(x) - y, \alpha) \neq \emptyset$. Now pick an arbitrary $\theta \in \Theta_i \cap B(g(x) - y, \alpha)$ and start with $x_0 := x$ to construct a sequence $x_k \in B(\bar{x}, a)$ by using the following iterates of the Lyusternik-Graves type as $k \in \mathbb{N}$:

$$A(x_k - x_{k-1}) = y + \theta - g(x_{k-1}) \text{ with } \|x_k - x_{k-1}\| \leq \gamma(\mu\gamma)^{k-1}\|g(x) - y - \theta\|. \tag{5.29}$$

Indeed, since A is surjective, it follows from [21, Lemma 1.18] and [39, Theorem 4.13] or [21, (1.43), Theorem 1.57] that there exists $x_1 \in X$ such that

$$A(x_1 - x_0) = y + \theta - g(x_0) \text{ and } \|x_1 - x_0\| \leq \gamma\|g(x_0) - y - \theta\|.$$

The above inequality together with (5.26) and (5.27) ensure the estimates

$$\begin{aligned} \|x_1 - \bar{x}\| &\leq \|x_1 - x_0\| + \|x_0 - \bar{x}\| \\ &\leq \gamma\|g(x_0) - y - \theta\| + \alpha \\ &\leq (\gamma + 1)\alpha \leq a, \end{aligned}$$

which implies that $x_1 \in B(\bar{x}, a)$ and thus (5.29) is true for $k = 1$. Suppose now that we have constructed the iterates $x_1, \ldots, x_n$ in $B(\bar{x}, a)$ satisfying (5.29) for some $n \in \mathbb{N}$. Using again the surjectivity we get $x_{n+1} \in X$ such that $A(x_{n+1} - x_n) = y + \theta - g(x_n)$ and

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \gamma\|g(x_n) - y - \theta\| = \gamma\|g(x_n) - g(x_{n-1}) - A(x_n - x_{n+1})\| \\ &\leq \gamma\mu\|x_n - x_{n-1}\| \leq \gamma(\mu\gamma)^n(\|g(x_0) - y - \theta\|). \end{aligned}$$

The latter yields furthermore that

$$\begin{aligned} \|x_{n+1} - x_0\| \leq \sum_{i=1}^{n+1} \|x_i - x_{i-1}\| &\leq \gamma(\|g(x_0) - y - \theta\|) \sum_{i=1}^{n+1} (\mu\gamma)^i \\ \leq \frac{\gamma}{1 - \mu\gamma}\|g(x_0) - y - \theta\| &\leq \frac{\gamma\alpha}{1 - \mu\gamma}. \end{aligned} \tag{5.30}$$

Then by this and (5.26), we have that

$$\|x_{n+1} - \bar{x}\| \leq \|x_{n+1} - x_0\| + \|x_0 - \bar{x}\| \leq \frac{\gamma\alpha}{1 - \mu\gamma} + \alpha \leq a.$$

which verifies by induction the validity of choosing a sequence $x_k \in B(\bar{x}, a)$ satisfying (5.29). Therefore, for any $m > n \geq 1$, we have

$$\begin{aligned} \|x_m - x_n\| &\leq \sum_{i=n+1}^{m} \|x_i - x_{i-1}\| \leq \gamma(\|g(x_0) - y - \theta\|) \sum_{i=n+1}^{m} (\mu\gamma)^i \\ &\leq \gamma(\|g(x_0) - y - \theta\|) \sum_{i=n+1}^{\infty} (\mu\gamma)^i. \end{aligned}$$

Since $\sum_{i=n+1}^{\infty}(\mu\gamma)^i \to 0$ as $n \to \infty$, $\{x_k\}$ is a Cauchy sequence and it converges therefore to some $\hat{x} \in B(\bar{x}, a)$. Passing to the limit in (5.29) and (5.30) as $k \to \infty$, we obtain that

$$\|x_0 - \hat{x}\| \leq \frac{\gamma}{1 - \mu\gamma}\|g(x_0) - y - \theta\| \text{ and } g(\hat{x}) = y + \theta.$$

which implies that

$$d(x, g^{-1}(y + \Theta_i)) \leq \|x_0 - \hat{x}\| \leq \frac{\gamma}{1 - \mu\gamma}\|g(x_0) - y - \theta\|.$$

Note that θ is arbitrarily chosen in $\Theta_i \cap B(g(x) - y, \alpha)$, it then follows from (5.28) that

$$d(x, g^{-1}(y + \Theta_i)) \leq \frac{\gamma}{1 - \mu\gamma} d(g(x), y + \Theta_i).$$

for all $(i, x, y) \in I \times B(\bar{x}, \frac{\alpha}{2(\mu + \|A\|)}) \times B(0, \frac{\alpha}{2})$, which implies that (5.25) is true by taking into account of the definition of G_i. Our proof is completed. $\square$

Next we provide two examples illustrating the assumptions made in Theorem 5.12. The first example shows that condition (5.24) itself is not enough without the surjectivity property on A even for the case when the index set I and $\{\Theta_i\}$ are all singleton.

Example 5.13 Consider a function $g : \mathbb{R} \to \mathbb{R}^2$ defined as follows:

$$g(x) = \begin{cases} (0, x^2) & \text{for} \quad x \geq 0 \\ (0, x) & \text{for} \quad x < 0 \end{cases}$$

Let $\bar{x} = 0$ and $\Theta := \{(0, 0)\}$. Then $G(x) = g(x) - \Theta = g(x)$. It is easy to see that condition (5.24) holds with $A = 0$ and $\mu = 1$ for sufficiently small x and x'. Now select $y_k = (0, \frac{1}{k})$ and

$x_k = \frac{1}{k}$, then we have

$$\lim_{k\to\infty} \frac{d(x_k, G^{-1}(y_k)}{d(y_k, G(x_k))} = \lim_{k\to\infty} \frac{|\frac{1}{k} - \frac{1}{\sqrt{k}}|}{|\frac{1}{k^2} - \frac{1}{k}|} = \infty.$$

which implies that G is not metrically regular around $(0,(0,0))$. The reason is of course that the linear operator A is not surjective.

The next example shows that Theorem 5.12 does not require the constraint sets $\{\Theta_i\}$ to be uniformly subsmooth as in Theorem 5.11 even in the setting of a single Θ and G.

Example 5.14 Consider the multifunction $G : \mathbb{R} \rightrightarrows \mathbb{R}^2$ defined as $G(x) = g(x) - \Theta$, where

$$g(x) = (x,x) \text{ and } \Theta = \{(x,y) \in \mathbb{R}^2 | xy \geq 0\}.$$

Take $A = g'(0) = (1,1)$, then it is easy to verify that A is surjective and (5.24) holds for any sufficient small positive real number μ. Therefore the assumptions in Theorem 5.12 is fulfilled and hence G is metrically regular around $(0,(0,0))$. Indeed, take any x and $y := (y_1, y_2)$ close to 0. Without loss of generality, we may assume that $y_2 \geq y_1$. Then it is easy to calculate that

$$\begin{aligned} d(y, G(x)) &= d(g(x) - y, \Theta) \\ &= d((x - y_1, x - y_2), \Theta) \\ &= \begin{cases} 0 & \text{if } \; x \leq y_1 \text{ or } x \geq y_2 \\ \min\{|x - y_1|, |x - y_2|\} & \text{if } \; y_1 < x < y_2 \end{cases} \end{aligned}$$

and that

$$\begin{aligned} d(x, G^{-1}(y)) &= d(x, g^{-1}(y+\Theta)) \\ &= d(x, (-\infty, y_1] \cup [y_2, +\infty)) \\ &= \begin{cases} 0 & \text{if} \quad x \le y_1 \text{ or} x \ge y_2 \\ \min\{|x-y_1|, |x-y_2|\} & \text{if} \quad y_1 < x < y_2 \end{cases} \end{aligned}$$

Therefore we have that $d(x, G^{-1}(y)) = d(y, G(x))$ for small x and y around 0. However, it is obvious that Θ is not subsmooth at $g(0) = (0, 0)$.

CHAPTER 6

Sensitivity Analysis of PVS

6.1 Coderivatives of Variational Systems

In this section we consider a special class of parametric generalized equations of type (1.2), where the set-valued (field) mapping Q depends not only on the decision variable but also on the parameter, which is a more general form than (1.2). On the other hand, our attention below is paid to the most useful specifications of such generalized equations given in the form of finite-dimensional *parametric variational systems* (PVS)

$$0 \in f(x,w) + \partial_w(\phi \circ g)(x,w), \tag{6.1}$$

where $w \in I\!R^d$ is the decision variable (we use this notation here for further convenience), $x \in I\!R^n$ is the parameter, $f\colon I\!R^n \times I\!R^d \to I\!R^n$, $g\colon I\!R^n \times I\!R^d \to I\!R^m$, and $\phi\colon I\!R^m \to \overline{I\!R}$ with $m \le d$. The subdifferential operator of the composition in (6.1) is understood in the sense of the partial first-order subdifferential (2.4) with respect to the decision variable. Our main object is

$$S(x) := \big\{ w \in I\!R^d \big|\, 0 \in f(x,w) + \partial_w(\phi \circ g)(x,w) \big\}, \tag{6.2}$$

which is the parameter-dependent *solution set* for (6.1).

The main goal we pursue in what follows is evaluating the coderivative (2.2) of the solution map (6.2). The results obtained in this direction, which are of their own interest, extend those presented in [21, Section 4.4] in the case of the full versus partial subdifferential in (6.1). We also study these issues for solution maps of canonically perturbed variational systems. The obtained conditions are rather involved, which reflects significant difficulties of the problems under consideration.

Let us start with the coderivative estimation. Recall that the composition $\phi \circ g$ as above is *strongly amenable* in x at $\bar{x}$ with *compatible parametrization* by w at $\bar{w}$ if g is C^2-smooth around $(\bar{x}, \bar{w})$ while ϕ is convex, proper, l.s.c., and finite at $g(\bar{x}, \bar{w})$ under the qualification condition

$$\partial^\infty \phi\big(g(\bar{x}, \bar{w})\big) \cap \ker \nabla_x g(\bar{x}, \bar{w})^* = \{0\} \tag{6.3}$$

expressed via the singular subdifferential (2.5). This broad class of functions has been well recognized in variational analysis and parametric optimization being important in many applications; see, e.g., the book [25] and more recent paper [32] with the references and discussions therein.

The first coderivative estimate concerns the case of strongly amenable compositions under the partial subdifferential in (6.2) known as potentials.

Theorem 6.1 (coderivative estimate for PVS with strongly amenable potentials) *Let S be given in* (6.2)*, where the potential $\phi \circ g$ is strongly amenable in w at $\bar{w}$ with compatible parametrization by x at $\bar{x}$, and where f is continuous around $(\bar{x}, \bar{w}) \in \operatorname{gph} S$. Denote $\bar{z} := g(\bar{x}, \bar{w})$, $\bar{y} := -f(\bar{x}, \bar{w}) \in \partial_w(\phi \circ g)(\bar{x}, \bar{w})$,*

$$M(\bar{x}, \bar{w}, \bar{y}) := \big\{ v \in I\!R^m \big|\ v \in \partial \phi(\bar{z}) \ \textit{with}\ \nabla_w g(\bar{x}, \bar{w})^* v = \bar{y} \big\} \tag{6.4}$$

and impose the following two conditions:

(A1) *$\partial^2 \phi(\bar{z}, v)(0) \cap \ker \nabla_w g(\bar{x}, \bar{w})^* = \{0\}$ for all $v \in M(\bar{x}, \bar{w}, \bar{y})$;*

(A2) *The relationship*

$$
\begin{aligned}
(x^*,w^*) \quad \in & \bigcup_{v\in M(\bar{x},\bar{w},\bar{y})}\Big[\big(\nabla^2_{wx}\langle v,g\rangle(\bar{x},\bar{w})u,\nabla^2_{ww}\langle v,g\rangle(\bar{x},\bar{w})u\big)\\
+ & \quad \big(\nabla_x g(\bar{x},\bar{w}),\nabla_w g(\bar{x},\bar{w})\big)^*\partial^2\phi(\bar{z},v)(\nabla_w g(\bar{x},\bar{w})u)\Big]\cap\big(-D^*f(\bar{x},\bar{w})(u)\big)
\end{aligned}
$$

holds only for the trivial triple $(x^*,w^*,u)=(0,0,0)$.

Then we have the coderivative upper estimate

$$
\begin{aligned}
D^*S(\bar{x},\bar{w})(w^*) \quad \subset & \Big\{x^*\in I\!\!R^n\Big|\,\exists u\in I\!\!R^n \;\; \textit{with} \;\; (x^*,-w^*)\in D^*f(\bar{x},\bar{w})(u)\\
& + \bigcup_{v\in M(\bar{x},\bar{w},\bar{y})}\Big[\big(\nabla^2_{wx}\langle v,g\rangle(\bar{x},\bar{w})u,\triangledown^2_{ww}\langle v,g\rangle(\bar{x},\bar{w})u\big)\\
& +\big(\triangledown_x g(\bar{x},\bar{w}),\nabla_w g(\bar{x},\bar{w})\big)^*\partial^2\phi(\bar{z},v)(\nabla_w g(\bar{x},\bar{w})u)\Big]\Big\}.
\end{aligned}
\tag{6.5}
$$

Proof. Picking any $x^*\in D^*S(\bar{x},\bar{w})(w^*)$, we have

$$
(x^*,-w^*)\in N\big((\bar{x},\bar{w});\operatorname{gph}S\big)=N\big((\bar{x},\bar{w});h^{-1}(\operatorname{gph}\partial_w(\phi\circ g))\big),
$$

where $h\colon I\!\!R^n\times I\!\!R^d\to I\!\!R^n\times I\!\!R^d\times I\!\!R^n$ is defined by $h(x,w):=(x,w,-f(x,w))$. This follows from

$$
\operatorname{gph}S=\big\{(x,w)\in I\!\!R^n\times I\!\!R^d\big|\;h(x,w)\in\operatorname{gph}\partial_w(\phi\circ g)\big\}=h^{-1}\big(\operatorname{gph}\partial_w(\phi\circ g)\big).
$$

Next we employ the obvious representation

$$
h(x,w)=h_1(x,w)+h_2(x,w) \;\;\text{with}\;\; h_1(x,w):=(x,w,0),\;h_2(x,w):=(0,0,-f(x,w)).
$$

Since h_1 is C^1-smooth, it follows from the coderivative sum rule of Theorem 2.8 that

$$\begin{aligned} D^*h(\bar{x},\bar{w})(x^*,w^*,y^*) &= \nabla h_1(\bar{x},\bar{w})^*(x^*,w^*,y^*) + D^*h_2(\bar{x},\bar{w})(x^*,w^*,y^*) \\ &= (x^*,w^*) + D^*f(\bar{x},\bar{w})(-y^*), \quad (x^*,w^*,y^*) \in I\!\!R^n \times I\!\!R^d \times I\!\!R^n. \end{aligned} \tag{6.6}$$

Applying the second-order chain rule from [32, Theorem 3.3] valid under **(A1)**, we get

$$\begin{aligned} \partial^2_w(\phi\circ g)(\bar{x},\bar{w},\bar{y})(u) \;\subset & \bigcup_{v\in M(\bar{x},\bar{w},\bar{y})} \Big[\big(\nabla^2_{wx}\langle v,g\rangle(\bar{x},\bar{w})u, \nabla^2_{ww}\langle v,g\rangle(\bar{x},\bar{w})u\big) \\ & +\big(\nabla_x g(\bar{x},\bar{w}),\nabla_w g(\bar{x},\bar{w})\big)^*\partial^2\phi(\bar{z},v)\big(\nabla_w g(\bar{x},\bar{w})u\big)\Big]. \end{aligned} \tag{6.7}$$

It follows from the definitions and constructions above that for any triple $(x^*,w^*,y^*) \in \ker D^*h(\bar{x},\bar{w}) \cap N(h(\bar{x},\bar{w}); \operatorname{gph}\partial_w(\phi\circ g))$ we have the inclusion

$$(x^*,w^*) \in \big(-D^*f(\bar{x},\bar{w})(-y^*)\big) \cap \partial^2_w(\phi\circ g)(\bar{x},\bar{w},\bar{y})(-y^*),$$

which implies by assumption **(A2)** and inclusion (6.7) that

$$N\big(h(\bar{x},\bar{w}); \operatorname{gph}\partial_w(\phi\circ g)\big) \cap \ker D^*h(\bar{x},\bar{w}) = \{0\}.$$

Since ϕ is l.s.c. and g is C^2-smooth around the points in question, it follows that the sets $\operatorname{epi}(\phi\circ g)$ and $\operatorname{gph}\partial_w(\phi\circ g)$ are locally closed around the corresponding points. Furthermore the mapping $(x,w)\mapsto h(x,w)\cap\operatorname{gph}\partial_w(\phi\circ g)$ is inner semicompact at $(\bar{x},\bar{w})$ in the sense of Definition 2.3. Applying thus Theorem 2.7 gives us the inclusion

$$N\big((\bar{x},\bar{w}); h^{-1}(\operatorname{gph}\partial_w(\phi\circ g))\big) \subset D^*h(\bar{x},\bar{w}) \circ N\big((\bar{x},\bar{w},\bar{y}); \operatorname{gph}\partial_w(\phi\circ g)\big)$$

This yields in turn together with (6.6) that there exists a triple

$$(u_1^*, u_2^*, -v^*) \in N\big((\bar{x}, \bar{w}, \bar{y}); \operatorname{gph} \partial_w(\phi \circ g)\big) \Longleftrightarrow (u_1^*, u_2^*) \in \partial_w^2(\phi \circ g)(\bar{x}, \bar{w}, \bar{y})(v^*)$$

such that we have the equalities

$$(x^*, -w^*) = D^*h(\bar{x}, \bar{w})(u_1^*, u_2^*, -v^*) = (u_1^*, u_2^*) + D^*f(\bar{x}, \bar{w})(v^*).$$

Finally, using (6.7) leads us to (6.5) and thus completes the proof of the theorem. $\square$

The next result estimating the coderivative of (6.2) is independent from Theorem 6.1 in both assumptions and conclusions made. The imposed full rank condition (6.8) ensures the validity of the constraint qualification (6.3) and in **(A1)** of Theorem 6.1 as well as that the set $M(\bar{x}, \bar{w}, \bar{y})$ is a singleton—hence the corresponding simplifications in conditions **(ii)** and (6.5) of the previous theorem, but the composition $\phi \circ g$ may not be strongly amenable under (6.8), i.e., the outer function ϕ is not assumed to be l.s.c. and convex.

Theorem 6.2 (coderivative estimate for PVS under full rank condition) *In the setting of Theorem* 6.1 *with a C^2-smooth mapping g, let us impose the full rank condition*

$$\operatorname{rank} \nabla_w g(\bar{x}, \bar{w}) = m \tag{6.8}$$

instead of the assumption that the composition $\phi \circ g$ is strongly amenable with compatible parametrization. Denoting by $\bar{v}$ the unique solution to the system

$$\bar{v} \in \partial\phi(\bar{z}), \quad \nabla_w g(\bar{x}, \bar{w})^* \bar{v} = \bar{y}$$

and assuming the validity of condition **(A2)** *from Theorem* 6.1 *with* $M(\bar{x}, \bar{w}, \bar{y}) = \{\bar{v}\}$*, we have the coderivative estimate* (6.5) *with the replacement of* $M(\bar{x}, \bar{w}, \bar{y})$ *by* $\bar{v}$ *therein.*

Proof. We proceed similarly to the proof of Theorem 6.1 with replacing the usage of [32, Theorem 3.3] by the second-order subdifferential chain rule

$$\begin{aligned} \partial_w^2(\phi \circ g)(\bar{x}, \bar{w}, \bar{y})(u) \; &= \big(\nabla_{wx}^2 \langle \bar{v}, g\rangle(\bar{x}, \bar{w})u, \nabla_{ww}^2 \langle \bar{v}, g\rangle(\bar{x}, \bar{w})u\big) \\ &\quad + \big(\nabla_x g(\bar{x}, \bar{w}), \nabla_w g(\bar{x}, \bar{w})\big)^* \partial^2 \phi(\bar{z}, \bar{v})\big(\nabla_w g(\bar{x}, \bar{w})u\big) \end{aligned}$$

from [32, Theorem 3.1] valid for all $u \in I\!R^n$ under the full rank condition (6.8). The remaining part of the proof of the theorem follows the lines in the proof of Theorem 6.1 with taking into account that $M(\bar{x}, \bar{w}, \bar{y}) = \{\bar{v}\}$ when (6.8) is imposed. $\square$

As in [21, Section 4.4.1], we can derive various consequences and specifications of the obtained Theorems 6.1 and 6.2 ensuring, in particular, the validity of the assumptions made therein. It is worth also mentioning that, as follows from the proof, the coderivative estimate of Theorem 6.2 can be extended to the infinite-dimensional setting of Asplund spaces.

Theorem 6.3 (metric regularity of subdifferential PVS with composite potentials) *Let* $(\bar{x}, \bar{w}) \in \operatorname{gph} S$ *for the parametric variational system* S *given in* (6.1) *under notation and assumptions of Theorem 6.1. Let* $f : I\!R^n \times I\!R^d \to I\!R^n$ *be strictly differentiable around* $(\bar{x}, \bar{w})$ *and assume that for any* $v \in M(\bar{x}, \bar{w}, \bar{y})$,

$$[0 = \nabla_x f(\bar{x}, \bar{w})u + \nabla_{wx}^2 \langle v, g\rangle(\bar{x}, \bar{w})u + \nabla_x g(\bar{x}, \bar{w})^* \partial^2 \phi(\bar{z}, v)(\triangle_w g(\bar{x}, \bar{w})u)] \Rightarrow u = 0 \qquad (6.9)$$

Then we have the following inclusion

$$\ker D^*S(\bar{x},\bar{w}) \subset -\nabla_w g(\bar{x},\bar{w})^* \partial^2\phi(\bar{z},v)(0) \tag{6.10}$$

for some $v \in M(\bar{x},\bar{w},\bar{y})$. Hence the Lipschitz-like property of $\partial\phi$ around $(\bar{z},v)$ for all $v \in M(\bar{x},\bar{w},\bar{y})$ implies the metric regularity of the property of S.

Proof. Take any $w^* \in \ker D^*S(\bar{x},\bar{w})$, it follows from (6.5) that there exists $u \in I\!R^n$ with

$$\begin{aligned}(0,-w^*) \ &\in\ (\nabla_x f(\bar{x},\bar{w}),\nabla_w f(\bar{x},\bar{w}))(u) + \bigcup_{v\in M(\bar{x},\bar{w},\bar{y})}\Big[\big(\nabla^2_{wx}\langle v,g\rangle(\bar{x},\bar{w})u,\nabla^2_{ww}\langle v,g\rangle(\bar{x},\bar{w})u\big)\\ &+\ \big(\nabla_x g(\bar{x},\bar{w}),\nabla_w g(\bar{x},\bar{w})\big)^*\partial^2\phi(\bar{z},v)(\nabla_w g(\bar{x},\bar{w})u\big)\end{aligned}$$

Then our assumption (6.9) implies that $u=0$. Hence

$$-w^* = \nabla_w g(\bar{x},\bar{w})^*\partial^2\phi(\bar{z},v)(0)$$

for some $v \in M(\bar{x},\bar{w},\bar{y})$. i.e. (6.10) is true. Therefore, if $\partial\phi$ is Lipshcitz-like, by [21, Theorem4.10], one has $w^*=0$, i.e. $\ker D^*S(\bar{x},\bar{w})=\{0\}$, which implies that S is metrically regular around $(\bar{x},\bar{w})$ from [21, Theorem4.18]. □

Finally in this section, we consider the *canonically* perturbed version of PVS (6.2) given by

$$\Sigma(x,p)=\big\{w\in I\!R^d\big|\ p\in f(x,w)+\partial_w(\phi\circ g)(x,w)\big\}, \tag{6.11}$$

with the same data as in (6.1) and the pair of parameters (p,x) one of which enters in a special/canonical way; see, e.g., [10, 21] for more details on this type of perturbations.

The next theorem unifies the coderivative estimate for Σ in the case of strongly amenable potentials. Observe that the special (while fairly general) structure of (6.11) allows us, in particular, to avoid somewhat involved assumptions of type **(A2)** as imposed in Theorems 6.1 and 6.2. Note also that out treatment of the canonically perturbed variational systems is different from more conventional approaches based on preliminary first-order approximations as in, e.g., [10, 21].

Theorem 6.4 (coderivative estimate of canonically perturbed systems) *Consider the triple* $(\bar{x},\bar{p},\bar{w})\in\operatorname{gph}\Sigma$ *for system* (6.11)*, denote* $\bar{z}:=g(\bar{x},\bar{w})$ *and* $\bar{y}:=\bar{p}-f(\bar{x},\bar{w})\in\partial_w(\phi\circ g)(\bar{x},\bar{w})$*, and form the set* $M(\bar{x},\bar{w},\bar{y})$ *by* (6.4)*. Assume that* f *is locally Lipschitzian around* $(\bar{x},\bar{w})$*, the composition* $\phi\circ g$ *is strongly amenable in* w *at* $\bar{w}$ *with compatible parameterization by* x *at* $\bar{x}$*, and the second-order qualification condition* **(A1)** *of Theorem* 6.1 *is satisfied. Then for all* $w^*\in I\!R^d$ *we have the coderivative estimate*

$$
\begin{aligned}
&D^*\Sigma(\bar{x},\bar{p},\bar{w})(w^*)\subset\Big\{(x_1^*,x_2^*)\in I\!R^n\times I\!R^n\Big|\ \exists(y^*,z^*)\in\big(\partial_x\langle x_2^*,f\rangle(\bar{x},\bar{w}),\partial_w\langle x_2^*,f\rangle(\bar{x},\bar{w})\big)\\
&\text{with }\ (x_1^*,-w^*+y^*+z^*)\in\bigcup_{v\in M(\bar{x},\bar{w},\bar{y})}\Big[\big(-\nabla^2_{wx}\langle v,g\rangle(\bar{x},\bar{w})x_2^*,-\nabla^2_{ww}\langle v,g\rangle(\bar{x},\bar{w})x_2^*\big)\\
&+\big(\nabla_x g(\bar{x},\bar{w}),\nabla_w g(\bar{x},\bar{w})\big)^*\partial^2\phi(\bar{z},v)\big(-\nabla_w g(\bar{x},\bar{w})x_2^*\big)\Big]\Big\}.
\end{aligned}
\tag{6.12}
$$

Proof. Let us justify inclusion (6.12). Picking any $(x_1^*,x_2^*)\in D^*\Sigma(\bar{x},\bar{p},\bar{w})(w^*)$, we get

$$
(x_1^*,x_2^*,-w^*)\in N\big((\bar{x},\bar{p},\bar{w});\operatorname{gph}\Sigma\big)=N\big((\bar{x},\bar{p},\bar{w});h^{-1}(\operatorname{gph}\partial_w(\phi\circ g))\big)
$$

for the mapping $h\colon I\!R^n\times I\!R^n\times I\!R^d\to I\!R^n\times I\!R^d\times I\!R^n$ defined by

$$
h(x,p,w):=\big(x,w,p-f(x,w)\big)=h_1(x,p,w)+h_2(x,p,w)
$$

with $h_1(x,p,w) := (x,w,p)$ and $h_2(x,p,w) := (0,0,-f(x,w))$. Then we claim the equalities

$$
\begin{array}{rcl}
D^*h(\bar{x},\bar{p},\bar{w})(x^*,w^*,z^*) & = & \nabla h_1(\bar{x},\bar{p},\bar{w})^*(x^*,w^*,z^*) + D^*h_2(\bar{x},\bar{p},\bar{w})(x^*,w^*,z^*) \\
& = & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x^* \\ w^* \\ z^* \end{pmatrix} + D^*h_2(\bar{x},\bar{p},\bar{w})(x^*,w^*,z^*) \\
& = & (x^*,z^*,w^*) + \partial\langle (x^*,w^*,z^*), h_2\rangle(\bar{x},\bar{p},\bar{w}) \\
& = & (x^*,z^*,w^*) + \big(0,0,(\partial_x\langle x^*,-f\rangle + \partial_w\langle z^*,-f\rangle)(\bar{x},\bar{p},\bar{w})\big) \\
& = & (x^*,z^*,w^* + \partial_x\langle x^*,-f\rangle(\bar{x},\bar{p},\bar{w}) + \partial_w\langle z^*,-f\rangle(\bar{x},\bar{p},\bar{w})).
\end{array}
$$

Indeed, the first equality follows from Theorem 2.8 due to the smoothness of h_1. The third equality holds by Theorem 2.9 since h_2 is locally Lipschitzian around $(\bar{x},\bar{w})$.

This allows us to conclude that the first-order qualification condition

$$
N\big(h(\bar{x},\bar{p},\bar{w}); \operatorname{gph}\partial_x(\phi\circ g)\big) \cap \ker D^*h(\bar{x},\bar{p},\bar{w}) = \{0\}
$$

is satisfied. To verify this, take any $(x^*,w^*,z^*) \in \ker D^*h(\bar{x},\bar{p},\bar{w})$ and deduced from the above coderivative calculation that $x^* = z^* = 0$ and so $w^* = 0$, i.e., $\ker D^*h(\bar{x},\bar{p},\bar{w}) = \{0\}$.

Since h is continuous and the subgradient set $\operatorname{gph}\partial_w(\phi\circ g)$ is locally closed due to the amenability of $\psi\circ g$, we can apply Theorem 2.7 and get from it that

$$
N\big((\bar{x},\bar{p},\bar{w}); h^{-1}(\operatorname{gph}\partial_w(\phi\circ g))\big) \subset D^*h(\bar{x},\bar{p},\bar{w}) \circ N\big((\bar{x},\bar{w},\bar{z}),; \operatorname{gph}\partial_w(\phi\circ g)\big).
$$

Hence there exists a triple of normal vectors

$$(u_1^*, u_2^*, -v^*) \in N\big((\bar{x}, \bar{w}, \bar{y}); \operatorname{gph} \partial_w(\phi \circ g)\big) \Longleftrightarrow (u_1^*, u_2^*) \in D^*\big(\partial_w(\phi \circ g)\big)(\bar{x}, \bar{w}, \bar{y})(v^*)$$

for which we have the inclusion $(x_1^*, x_2^*, -w^*) \in D^*h(\bar{x}, \bar{p}, \bar{w})(u_1^*, u_2^*, -v^*)$. It follows then from the above calculation of $D^*h(\bar{x}, \bar{p}, \bar{w})$ that the relationships

$$x_1^* = u_1^*, \quad x_2^* = -v^*, \quad -w^* \in u_2^* + \partial_x\langle u_1^*, -f\rangle(\bar{x}, \bar{p}, \bar{w}) + \partial_w\langle v^*, f\rangle(\bar{x}, \bar{p}, \bar{w})$$

are satisfied and imply therefore the inclusion

$$u_2^* \in -w^* + \partial_x\langle u_1^*, f\rangle(\bar{x}, \bar{p}, \bar{w}) + \partial_w\langle x_2^*, f\rangle(\bar{x}, \bar{p}, \bar{w}).$$

The latter gives us by the partial second-order subdifferential construction (2.8) that there are

$$(y^*, z^*) \in \big(\partial_x\langle x_2^*, f\rangle(\bar{x}, \bar{w}), \partial_w\langle x_2^*, f\rangle(\bar{x}, \bar{w})\big) \ \text{ with } \ (x_1^*, -w^* + y^* + z^*) \in \partial_w^2(\phi \circ g)(\bar{x}, \bar{w}, \bar{y})(-x_2^*),$$

Then it follows from [32, Theorem 3.3] that (6.12) is true and our proof is completed. □

The results of Theorem 6.4 admit various simplifications under additional assumptions on the initial data, e.g., when the base mapping f is C^1-smooth. We can also derive their counterpart in the case when the amenability assumption on $\phi \circ g$ is replaced by the full rank condition (6.8).

6.2 Optimality Conditions of PVS

This section is devoted to applications of coderivative result obtained above to the study of constrained optimization and equilibrium problems with possibly nonsmooth data. The primary objective of this chapter is to derive *necessary optimality* and *suboptimality* conditions for various problems of constrained optimization and equilibria in finite-dimensional spaces.

First, we consider MPECs with equilibrium constraints governed by parametric variational systems in the form of (6.1):

$$\textit{minimize} \quad \psi(x,w) \quad \textit{subject to } 0 \in f(x,w) + \partial_w(\phi \circ g)(x,w), \ (x,w) \in \Omega \tag{6.13}$$

where $\psi\colon I\!\!R^n \times I\!\!R^d \to \bar{I\!\!R}$ is an extended real valued function. In what follows, we are going to derive both lower and upper subdifferential necessary optimality conditions of (6.13) in terms of its initial data.

Theorem 6.5 (upper, lower subdifferential conditions for MPECs (6.13) **with strongly amenable potentials)** *Let $(\bar{x},\bar{w})$ be a local optimal solution to* (6.13)*, f be continuous around $(\bar{x},\bar{w})$. Let $\phi \circ g$ be strongly amenable in w at $\bar{w}$ with compatible parametrization by x at $\bar{x}$, Ω be locally closed at $(\bar{x},\bar{w})$ and $\bar{z},\bar{y},M(\bar{x},\bar{w},\bar{y})$ be the same as in Theorem 6.1. Assume that conditions* **(A1)***,* **(A2)** *of Theorem 6.1 hold and in addition*

$$\begin{aligned} \Big[(x^*,w^*) \quad &\in \quad D^*f(\bar{x},\bar{w})u + \bigcup_{v\in M(\bar{x},\bar{w},\bar{y})} \Big[\big(\nabla^2_{wx}\langle v,g\rangle(\bar{x},\bar{w})u, \nabla^2_{ww}\langle v,g\rangle(\bar{x},\bar{w})u\big) \\ &+ \quad \big(\nabla_x g(\bar{x},\bar{w}), \nabla_w g(\bar{x},\bar{w})\big)^*\partial^2\phi(\bar{z},v)(\nabla_w g(\bar{x},\bar{w})u)\Big], \ \textit{and} \\ &(\quad -x^*,-w^*) \in N((\bar{x},\bar{w}),\Omega)\Big] \Rightarrow x^* = w^* = 0 \end{aligned} \tag{6.14}$$

Then for every $(x^*, w^*) \in \hat{\partial}^+\psi(\bar{x}, \bar{w})$*, there is* $u \in I\!R^n \times I\!R^d$*, such that*

$$\begin{aligned} 0 \in (x^*, w^*) \quad &+ \quad D^*f(\bar{x}, \bar{w})u + \bigcup_{v \in M(\bar{x}, \bar{w}, \bar{y})} \Big[\big(\nabla^2_{wx} \langle v, g \rangle (\bar{x}, \bar{w}) u, \nabla^2_{ww} \langle v, g \rangle (\bar{x}, \bar{w}) u \big) \\ &+ \quad \big(\nabla_x g(\bar{x}, \bar{w}), \nabla_w g(\bar{x}, \bar{w}) \big)^* \partial^2 \phi(\bar{z}, v) (\nabla_w g(\bar{x}, \bar{w}) u) \Big] + N((\bar{x}, \bar{w}), \Omega) \end{aligned} \tag{6.15}$$

If furthermore, ψ *is Lipschitzian around* $(\bar{x}, \bar{w})$*, there are elements* $(x^*, w^*) \in \partial\psi(\bar{x}, \bar{w})$ *and* $u \in I\!R^n \times I\!R^d$ *satisfying* (6.15)*.*

Proof. Observe that $(\bar{x}, \bar{w})$ provides a local minimum to the function ψ subject to the constraints $(x, w) \in \Omega_1 := \operatorname{gph} S$ and $(x, w) \in \Omega_2 := \Omega$, where S is defined in (6.2). First we show that the set system $\{\Omega_1, \Omega_2\}$ satisfies the limitting qualification condition at $(\bar{x}, \bar{w})$ in the sense of [21, Definition 3.2]. To this end, take $(x^*, w^*) \in (-N((\bar{x}, \bar{w}), \Omega)) \cap N((\bar{x}, \bar{w}), \operatorname{gph} S)$, i.e. $x^* \in D^*S(\bar{x}, \bar{w})(-w^*)$. Since all the assumptions of Theorem 6.1 are fullfilled, we can conclude from (6.5) and condition (6.14) that $x^* = w^* = 0$. Hence $(N((\bar{x}, \bar{w}), \Omega_1)) \cap (-N((\bar{x}, \bar{w}), \Omega_2)) = \{0\}$. Note that Ω_1 is locally closed at $(\bar{x}, \bar{w})$ as it has been discussed in the proof of Theorem 6.1, we are now ready to apply [21, Theorem 5.5 (i)] to get

$$-\hat{\partial}^+\psi(\bar{x}, \bar{w}) \subset N((\bar{x}, \bar{w}), \operatorname{gph} S) + N((\bar{x}, \bar{w}), \Omega)$$

The above inclusion together with (6.5) indicates that for every $(x^*, w^*) \in \hat{\partial}^+\psi(\bar{x}, \bar{w})$, there is $u \in I\!R^n \times I\!R^d$, such that

$$\begin{aligned} (-x^*, -w^*) \quad &\in \quad D^*f(\bar{x}, \bar{w})u + \bigcup_{v \in M(\bar{x}, \bar{w}, \bar{y})} \Big[\big(\nabla^2_{wx} \langle v, g \rangle (\bar{x}, \bar{w}) u, \nabla^2_{ww} \langle v, g \rangle (\bar{x}, \bar{w}) u \big) \\ &+ \quad \big(\nabla_x g(\bar{x}, \bar{w}), \nabla_w g(\bar{x}, \bar{w}) \big)^* \partial^2 \phi(\bar{z}, v) (\nabla_w g(\bar{x}, \bar{w}) u) \Big] + N((\bar{x}, \bar{w}), \Omega) \end{aligned}$$

Hence (6.15) is true. If in addition, ψ is Lipschitzian around $(\bar{x}, \bar{w})$, then we apply [21, Theorem 5.5 (ii)] to get $(x^*, w^*) \in \partial\psi(\bar{x}, \bar{w})$ and $u \in I\!R^n \times I\!R^d$ such that (6.15) holds. $\square$

Remark 6.6 In the setting of MPECs (6.13), there is a special form when the geometric constraint only involves x but no w:

$$\textit{minimize} \quad \psi(x,w) \quad \textit{subject to } 0 \in f(x,w) + \partial_w(\phi \circ g)(x,w), \ x \in \Omega$$

which indicates that $\Omega_2 = \Omega \times I\!R^d$ in the proof of Theorem 6.5. In this case, to ensure the limitting qualification condition $N((\bar{x}, \bar{w}), \operatorname{gph} S) \cap (-N((\bar{x}, \bar{w}), \Omega \times I\!R^d)) = \{0\}$, assumption (6.14) can be reduced to the following:

$$\begin{aligned}\Big[(x^*, 0) \;\; &\in \;\; D^*f(\bar{x}, \bar{w})u + \bigcup_{v \in M(\bar{x}, \bar{w}, \bar{y})} \Big[\big(\nabla^2_{wx}\langle v, g\rangle(\bar{x}, \bar{w})u, \nabla^2_{ww}\langle v, g\rangle(\bar{x}, \bar{w})u\big) \\ &+ \;\; \big(\nabla_x g(\bar{x}, \bar{w}), \nabla_w g(\bar{x}, \bar{w})\big)^* \partial^2\phi(\bar{z}, v)(\nabla_w g(\bar{x}, \bar{w})u)\Big], \ \textit{and} \ -x^* \in N(\bar{x}, \Omega)\Big] \Rightarrow x^* = 0\end{aligned}$$

Also, observe that in (6.15), $N((\bar{x}, \bar{w}), \Omega_2) = N(\bar{x}, \Omega) \times \{0\}$.

Theorem 6.7 (upper, lower subdifferential conditions for MPECs (6.13) **with full rank conditions)** *Let* $(\bar{x}, \bar{w})$ *be a local optimal solution to* (6.13) *where* $\phi, g, \bar{z}, \bar{y}, \bar{v}$ *are the same as in the setting of Theorem 6.2. Assume that condition* **(A2)** *and* (6.14) *holds with* $M(\bar{x}, \bar{w}, \bar{y}) = \{\bar{v}\}$*, then we have both lower and upper subdifferential necessary condition of Theorem 6.5 holds with the replacement of* $M(\bar{x}, \bar{w}, \bar{y})$ *by* $\bar{v}$ *provided that* ψ *is Lipschitz continuous around* $(\bar{x}, \bar{w})$ *in the lower subdifferential case.*

Proof. The proof is similar to the one of Theorem 6.5 but apply Theorem 6.2 instead of

Theorem 6.1. $\square$

The next part of our application is devoted to discuss necessary conditions for generalized order optimality with constraint understood in the sense of [21, Definition 5.53]. For convenience, we recall that for a given single valued mapping $\psi : X \to Z$ between Banach spaces and a set $0 \in \Lambda \subset Z, \Omega \subset X$, we say that a point $\bar{x} \in \Omega$ is *locally* $(\psi, \Lambda) -$ *optimal* relative to Ω, if there are a neighborhood U of $\bar{x}$ and a sequence $\{z_k\} \subset Z$ with $\|z_k\| \to 0$ as $k \to \infty$ such that

$$\psi(x) - \psi(\bar{x}) \notin \Lambda - z_k \text{ for all } x \in U \cap \Omega \text{ and } k \in \mathcal{N}.$$

Theorem 6.8 (Generalized order optimality with equilibrium constraints of type (6.1) **for strongly amenable potentials)** *Let $\psi : I\!R^n \times I\!R^d \to I\!R^s$ be continuous, let $\Lambda \subset I\!R^s$ be a closed set with $0 \in \Lambda$ and let $(\bar{x}, \bar{w})$ be* locally *$(\psi, \Lambda) -$ optimal subject to the parameter-dependent equilibrium constraints* (6.2)*, where $\Theta = \phi \circ g$ is strongly amenable in x at $\bar{x}$ with compatible parameterizations in w at $\bar{w}$. Let all the assumptions of Theorem 6.1 be true, then there are $0 \neq (x^*, w^*, z^*)$ with $z^* \in N(0, \Lambda)$ satisfy the relations*

$$\begin{aligned}
(-x^*, -w^*) \quad &\in \quad D^*\psi(\bar{x}, \bar{w})(z^*) \ \textit{and} \qquad (6.16)\\
&(\quad x^*, w^*) \in D^*f(\bar{x}, \bar{w})u + \bigcup_{v \in M(\bar{x}, \bar{w}, \bar{y})} \Big[\big(\nabla^2_{wx}\langle v, g\rangle(\bar{x}, \bar{w})u, \nabla^2_{ww}\langle v, g\rangle(\bar{x}, \bar{w})u\big)\\
&+ \quad \big(\nabla_x g(\bar{x}, \bar{w}), \nabla_w g(\bar{x}, \bar{w})\big)^* \partial^2\phi(\bar{z}, v)(\nabla_w g(\bar{x}, \bar{w})u)\Big]
\end{aligned}$$

with some $u \in I\!R^n \times I\!R^d$. Furthermore, these optimality conditions are equivalent to the existence *of* $z^* \in N(0,\Lambda)\diagdown\{0\}$ *and* $u \in I\!R^n \times I\!R^d$ *satisfying*

$$
\begin{aligned}
0 \in \partial\langle z^*,\psi\rangle(\bar{x},\bar{w}) + \partial\langle u,f\rangle(\bar{x},\bar{w}) \quad &+ \bigcup_{v\in M(\bar{x},\bar{w},\bar{y})} \Big[\big(\nabla^2_{wx}\langle v,g\rangle(\bar{x},\bar{w})u, \nabla^2_{ww}\langle v,g\rangle(\bar{x},\bar{w})u\big) \\
&+ \quad \big(\nabla_x g(\bar{x},\bar{w}),\nabla_w g(\bar{x},\bar{w})\big)^*\partial^2\phi(\bar{z},v)(\nabla_w g(\bar{x},\bar{w})u)\Big]
\end{aligned}
$$

when f *is strictly Lipschitzian at* $(\bar{x},\bar{w})$.

Proof. The proof follows from [21, Corollary 5.80] with S in the form of (6.1) by applying the coderivative estimate with strongly amenable partial potentials derived in Theorem 6.1. □

Theorem 6.9 (Generalized order optimality with equilibrium constraints of type (6.1) **for full rank conditions)** *Let* $(\bar{x},\bar{w})$ *be locally* $(\psi,\Lambda)-$ *optimal in the setting of Theorem 6.8 with a* C^2- *smooth mapping* g *with the full rank condition*

$$
rank\nabla_w g(\bar{x},\bar{w}) = m
$$

instead of the assumption that the composition $\phi\circ g$ *is strongly amenable with compatible parameterization. Let us impose the validity of condition* **(A2)** *from Theorem 6.1 with* $M(\bar{x},\bar{w},\bar{y}) = \{\bar{v}\}$. *Then there are* $0 \neq (x^*,w^*,z^*)$ *with* $z^* \in N(0,\Lambda)$ *satisfy the relations* (6.16) *with* $M(\bar{x},\bar{w},\bar{y}) = \{\bar{v}\}$ *for some* $u \in I\!R^n \times I\!R^d$.

Proof. Omitted. □

The next part of our application goes to optimality conditions for EPECs with closed preferences and variational constraints:

$$\textit{minimize} \qquad \psi(x,w) \quad \textit{with respect to } \prec \tag{6.17}$$
$$\textit{subject to } 0 \in f(x,w) + \partial_w(\phi \circ g)(x,w)$$

where the preference $\prec$ is closed , in two different settings: **(B1)**:$\Theta = \phi \circ g$ is amenable with compatible parameterization; **(B2)**: g is C^2 with full rank condition: $rank \nabla_w g(\bar{x},\bar{w}) = m$ and $\mathcal{L}(z)$ is the level set at $z \in Z$ with respect to the given preference $\prec$ defined as follows:

$$\mathcal{L}(z) := \{u \in Z | u \prec z\}$$

For more information about closed preference relations, see [21, Definition 5.55].

Theorem 6.10 (Optimality conditions for EPECs with closed preferences and variational constraints) *Let $(\bar{x},\bar{w})$ be a local optimal solution to the multiobjective optimization problem* (6.17). *Let ψ, f be continuous around $(\bar{x},\bar{w})$. Then,*

(i)In the setting of **(B1)**, *assume that conditions* **(A1)** *and* **(A2)** *of Theorem 6.1 holds, then there are $0 \neq (x^*,w^*,z^*) \in I\!R^n \times I\!R^d \times I\!R^s$ with $z^* \in N_+(\bar{z}, cl\mathcal{L}(\bar{z}))$ such that*

$$\begin{aligned}(-x^*,-w^*) \in D^*\psi(\bar{x},\bar{w})(z^*) \textit{ and } (x^*,w^*) \in D^*f(\bar{x},\bar{w})u + \\ \bigcup_{v \in M(\bar{x},\bar{w},\bar{y})} \Big[\big(\nabla^2_{wx}\langle v,g\rangle(\bar{x},\bar{w})u, \nabla^2_{ww}\langle v,g\rangle(\bar{x},\bar{w})u\big) \\ +\big(\nabla_x g(\bar{x},\bar{w}), \nabla_w g(\bar{x},\bar{w})\big)^* \partial^2\phi(\bar{z},v)(\nabla_w g(\bar{x},\bar{w})u)\Big]\end{aligned} \tag{6.18}$$

(ii)In the setting of **(B2)**, *the assumptions of Theorem 6.2 guarantees that* (6.18) *holds for* $M(\bar{x},\bar{w},\bar{y}) = \{\bar{v}\}$

Proof. The proof follows from Theorem 6.1, Theorem 6.2 and [21, Theorem 5.86]. □

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ABSTRACT

WELL-POSEDNESS PROPERTIES IN VARIATIONAL ANALYSIS AND ITS APPLICATIONS

by

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Advisor: Prof. Boris S. Mordukhovich

Major: Mathematics (Applied)

Degree: Doctor of Philosophy

This dissertation focuses on the study and applications of some significant properties in well-posedness and sensitivity analysis, among which the notions of *uniform* metric regularity, *higher-order* metric *sub*regularity and its *strong* subregularity counterpart play an essential role in modern variational analysis. We derived verifiable sufficient conditions and necessary conditions for those notions in terms of appropriate generalized differential as well as geometric constructions of variational analysis. Concrete examples are provided to illustrate the behavior and compare the results. Optimality conditions of parametric variational systems (PVS) under equilibrium constraints are also investigated via the terms of coderivatives. We derived *necessary optimality* and *suboptimality* conditions for various problems of constrained optimization and equilibria such as MPECs with amenable/full rank potentials and EPECs with closed preferences in finite-dimensional spaces.

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1. B. S. Mordukhovich and Wei Ouyang, Higher-order metric subregularity and its applications, to appear in J. Global Optim. (2015), DOI:10.1007/s10898-015-0271-x
2. B.S. Mordukhovich and Wei Ouyang, Metric q-subregularity for constraint systems, in preparation.
3. B.S. Mordukhovich and Wei Ouyang, Second-order optimality conditions of PVS under equilibrium constraints, in preparation.