Journal of Modern Applied Statistical Methods

Volume 15 | Issue 2 Article 29

11-1-2016

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Bhaktida S. Trivedi
H L Institute of Commerce, Ahmedabad University, Gujarat, India, bhaktida.trivedi@ahduni.edu.in

M. N. Patel
Department of Statistics, Gujarat University, Ahmedabad, mnpatel.stat@gmail.com

Recommended Citation
Trivedi, Bhaktida S. and Patel, M. N. (2016) "Estimation of Parameters of Misclassified Size Biased Borel Distribution," *Journal of Modern Applied Statistical Methods*: Vol. 15 : Iss. 2 , Article 29.
DOI: 10.22237/jmasm/1478003220
Available at: http://digitalcommons.wayne.edu/jmasm/vol15/iss2/29

Journal of Modern Applied Statistical Methods
November 2016, Vol. 15, No. 2, 475-494.
doi: 10.22237/jmasm/1478003220

ISSN 1538 − 9472

Estimation of Parameters of Misclassified Size Biased Borel Distribution

B. S. Trivedi
Ahmedabad University
Navrangpura, Ahmedabad, India

M. N. Patel
Gujarat University
Ahmedabad, India

A misclassified size-biased Borel Distribution (MSBBD), where some of the observations corresponding to $x = c + 1$ are wrongly reported as $x = c$ with probability α, is defined. Various estimation methods like the method of maximum likelihood (ML), method of moments, and the Bayes estimation for the parameters of the MSBB distribution are used. The performance of the estimators are studied using simulated bias and simulated risk. Simulation studies are carried out for different values of the parameters and sample size.

Keywords: Borel distribution, misclassification, size–biased, method of moments, maximum likelihood, Bayes estimation

Introduction

The Borel distribution is a discrete probability distribution, arising in contexts including branching processes and queueing theory. If the number of offspring that an organism has is Poisson-distributed, and if the average number of offspring of each organism is no bigger than 1, then the descendants of each individual will ultimately become extinct. The number of descendants that an individual ultimately has in that situation is a random variable distributed according to a Borel distribution.

Borel (1942) defined a one parameter Borel distribution as

$$\mathrm{P}\left(X = x\right) = \mathrm{p}\left(x;\theta\right) = \frac{\left(1+x\right)^{x-1}}{x!}\theta^{x}\mathrm{e}^{-(1+x)\theta}; \quad 0<\theta<1,\ x=1,2,3,\ldots \tag{1}$$

B. S. Trivedi is an Associate Professor in the H. L. Institute of Commerce. Email her at: bhaktida.trivedi@ahduni.edu.in. M. N. Patel is in the Department of Statistics. Email them at: mnpatel.stat@gmail.com.

This distribution describes a distribution of the number of customers served before a queue vanishes under condition of a single queue with random arrival times (at constant rate) of customers and a constant time occupied in serving each customer.

Gupta (1974) defined the Modified Power Series Distribution (MPSD) with probability function given by

$$\mathrm{P}_1\left(X=x\right)=\mathrm{a}\left(x\right)\frac{\left(g\left(\theta\right)\right)^x}{\mathrm{f}\left(\theta\right)},\quad x\in\mathrm{T} \tag{2}$$

where $\mathrm{a}(x) > 0$, T is a subset of the set of non-negative integers, $\mathrm{g}(\theta)$ and $\mathrm{f}(\theta)$ are positive, finite, and differentiable, and θ is the parameter.

Hassan and Ahmad (2009) showed the Borel distribution is a particular case of modified power series distribution (MPSD) with

$$\mathrm{a}\left(x\right)=\frac{\left(1+x\right)^{x-1}}{x!},\quad \mathrm{g}\left(\theta\right)=\theta\mathrm{e}^{-\theta},\quad \mathrm{f}\left(\theta\right)=\mathrm{e}^{\theta} \tag{3}$$

in (2).

The Borel-Tanner distribution generalizes the Borel distribution. Let k be a positive integer. If $x_1, x_2,\ldots, x_k$ are independent and each has Borel distribution with parameter θ, then their sum $w = x_1 + x_2 +\ldots+ x_k$ is said to have the Borel-Tanner distribution with parameters θ and k. This gives the distribution of the total number of individuals in a Poisson-Galton-Watson process starting with k individuals in the first generation, or of the time taken for an M/D/1 queue to empty starting with k jobs in the queue. The case $k = 1$ is simply the Borel distribution above.

Here, the M/D/1 queue represents the queue length in a system having a single server, where arrivals are determined by a Poisson process and job service times are fixed (deterministic). An extension of this model with more than one server is the M/D/c queue.

Size-Biased Borel Distribution

Size-biased distributions are a special case of the more general form known as weighted distributions. Weighted distributions have numerous applications in forestry and ecology.

Size-biased distributions were first introduced by Fisher (1934) to model ascertainment bias; weighted distributions were later formalized in a unifying theory by Rao (1965). Such distributions arise naturally in practice when observations from a sample are recorded with unequal probability, such as from probability proportional to size (PPS) designs. In short, if the random variable X has distribution $f(x; \theta)$, with unknown parameter θ, then the corresponding weighted distribution is of the form

$$f^{w}(x;\theta) = \frac{w(x) f(x;\theta)}{E\{w(x)\}} \tag{4}$$

where $w(x)$ is a non-negative weight function such that $E\{w(x)\}$ exists.

The size-biased Borel distribution is also derived from the size-biased MPSD as it is a particular case of the MPSD. A size-biased MPSD is obtained by taking the weight of MPSD (2) as x, given by

$$\begin{aligned} P(X = x) &= \frac{b_x (g(\theta))^x}{\mu(\theta) f(\theta)} \\ &= \frac{b_x (g(\theta))^x}{f^*(\theta)} \end{aligned} \tag{5}$$

where $b_x = x a(x)$ and $f^*(\theta) = \mu(\theta) f(\theta)$.

Now, by taking

$$b_x = x a(x) = \frac{(1+x)^{x-1}}{(x-1)!}, \ \mu(\theta) = \frac{\theta}{1-\theta}, \ g(\theta) = \theta e^{-\theta}, \ f(\theta) = e^{\theta}, \quad 0 < \theta < 1 \tag{6}$$

a size-biased Borel distribution is obtained with p.m.f. given by

$$P(X = x) = \frac{(1+x)^{x-1}}{(x-1)!} \theta^{x-1} (1-\theta) e^{-\theta(1+x)}, \quad x = 1, 2, 3, \ldots \tag{7}$$

Misclassified Size-Biased Borel Distribution

A dependent variable which is a discrete response causes the estimated coefficients to be inconsistent in a probit or logit model when misclassification is present. By 'misclassification' we mean that the response is reported or recorded in the wrong category; for example, a variable is recorded as a one when it should have the value zero. This mistake might easily happen in an interview setting where the respondent misunderstands the question or the interviewer simply checks the wrong box. Other data sources where the researcher suspects measurement error, such as historical data, certainly exist as well. It will be shown that, when a dependent variable is misclassified in a probit or logit setting, the resulting coefficients are biased and inconsistent.

Assume that some of the values (c + 1) are erroneously reported as c, and let the probabilities of these observation be α. Then the resulting distribution of the size-biased random variable X is called the misclassified size-biased distribution. Trivedi and Patel (2013) have considered misclassified size-biased generalized negative binomial distributions and parameter estimation. The misclassified size-biased Borel distribution can be obtained as

$$p_x = \mathrm{P}(X = x) = \begin{cases} \left(\theta e^{-\theta}\right)^c \left\{ \dfrac{(1+c)^{c-1}}{(c-1)!} + \alpha \dfrac{(2+c)^c \left(\theta e^{-\theta}\right)}{c!} \right\} \left\{ \dfrac{\theta e^{\theta}}{(1-\theta)} \right\}^{-1}, & x = c \\ (1-\alpha)(c+2)^c \left(\theta e^{-\theta}\right)^{c+1} \left\{ c! \dfrac{\theta e^{\theta}}{(1-\theta)} \right\}^{-1}, & x = c+1 \\ \dfrac{(1+i)^{i-1} \left(\theta e^{-\theta}\right)^i}{(i-1)!} \left\{ \dfrac{\theta e^{\theta}}{(1-\theta)} \right\}^{-1}, & x \in \mathrm{S} \end{cases} \tag{8}$$

where S is the set of non-negative integers excluding integers c and c + 1, $0 \le \alpha \le 1$, $0 < \theta < 1$, and $x = 1, 2, 3, \ldots$. The mean and variance of this distribution are obtained from the moments of misclassified size-biased MPSD given by Hassan and Ahmad (2009) as

$$\text{Mean} = \mu_1' = \frac{1}{(1-\theta)^2} + \frac{\theta}{(1-\theta)} - \alpha (1-\theta) b_{c+1}\, g^c\, e^{-2\theta} \tag{9}$$

$$\text{Variance} = \mu_2$$
$$= \frac{\left(3\theta-\theta^2\right)}{\left(1-\theta\right)^4} + \left[\begin{array}{l} \alpha \left\{\begin{array}{c} 2+2\theta(1-\theta)-(2c+1)(1-\theta)^2 \\ -\alpha b_{c+1}\,\mathrm{g}^c\,(1-\theta)^3\,\mathrm{e}^{-2\theta} \end{array}\right\} \\ \times\left\{b_{c+1}\,\mathrm{g}^c\,(1-\theta)^3\,\mathrm{e}^{-2\theta}\right\} \end{array}\right] \qquad (10)$$

Method of Maximum Likelihood Estimation

Let $x_1, x_2,\ldots, x_k$ be the probable values of the random variable X in a random sample of misclassified size-biased Borel distribution and n_k denote the number of observations corresponding to the value x_k in the sample (where $k > 0$). Thus the likelihood function L is given by

$$\begin{aligned} \mathrm{L} &\propto \prod_{i=1}^{k} P_i^{n_i} \\ &= P_c^{n_c} P_{c+1}^{n_{c+1}} \prod_{i\neq c,c+1} P_i^{n_i} \\ &= \left[\left(\theta\mathrm{e}^{-\theta}\right)^c \left\{\frac{(1+c)^{c-1}}{(c-1)!} + \alpha\frac{(2+c)^c\left(\theta\mathrm{e}^{-\theta}\right)}{c!}\right\}\left\{\frac{\theta\mathrm{e}^{\theta}}{(1-\theta)}\right\}^{-1}\right]^{n_c} \\ &\quad\times\left[(1-\alpha)(c+2)^c\left(\theta\mathrm{e}^{-\theta}\right)^{c+1}\left\{c!\frac{\theta\mathrm{e}^{\theta}}{(1-\theta)}\right\}^{-1}\right]^{n_{c+1}} \\ &\quad\times\prod_{i\neq c,c+1}\left[\frac{(1+i)^{i-1}\left(\theta\mathrm{e}^{-\theta}\right)^i}{(i-1)!}\left\{\frac{\theta\mathrm{e}^{\theta}}{(1-\theta)}\right\}^{-1}\right]^{n_i} \end{aligned} \qquad (11)$$

where

$$\sum_{i=1}^{k} n_i = N$$

$$
\begin{aligned}
\ln \mathrm{L} &= n_c \ln\left[\left(\theta \mathrm{e}^{-\theta}\right)^c \left\{\frac{(1+c)^{c-1}}{(c-1)!} + \alpha \frac{(2+c)^c \left(\theta \mathrm{e}^{-\theta}\right)}{c!}\right\}\left\{\frac{\theta \mathrm{e}^{\theta}}{(1-\theta)}\right\}^{-1}\right] \\
&\quad + n_{c+1} \ln\left[(1-\alpha)(c+2)^c \left(\theta \mathrm{e}^{-\theta}\right)^{c+1}\left\{c!\frac{\theta \mathrm{e}^{\theta}}{(1-\theta)}\right\}^{-1}\right] \\
&\quad + \sum_{i \neq c, c+1} n_i \ln\left[\frac{(1+i)^{i-1}\left(\theta \mathrm{e}^{-\theta}\right)^i}{(i-1)!}\left\{\frac{\theta \mathrm{e}^{\theta}}{(1-\theta)}\right\}^{-1}\right] \\
&= n_c c \ln\theta - n_c c\theta + n_c \ln\left\{\frac{(1+c)^{c-1}}{(c-1)!} + \alpha \frac{(2+c)^c \left(\theta \mathrm{e}^{-\theta}\right)}{c!}\right\} - n_c \ln\theta \\
&\quad - n_c\theta + n_c \ln(1-\theta) + n_{c+1} \ln(1-\alpha) + n_{c+1} c \ln(c+2) \\
&\quad + n_{c+1}(c+1)\ln\theta - n_{c+1}(c+1)\theta - n_{c+1}\ln c! - n_{c+1}\ln\theta - n_{c+1}\theta \\
&\quad + n_{c+1}\ln(1-\theta) + \sum_{i \neq c, c+1} n_i \left\{\begin{array}{l}(i-1)\ln(i+1) + (i-1)\ln\theta \\ \quad -i\theta - \ln(i-1)! - \theta + \ln(1-\theta)\end{array}\right\}
\end{aligned}
\tag{12}
$$

Let the derivative of ln L with respect to α and θ be zero. The solutions of $\frac{\partial \ln \mathrm{l}}{\partial \alpha} = 0$ and $\frac{\partial \ln \mathrm{l}}{\partial \theta} = 0$ gives us the ML estimators of α and θ:

$$
\frac{\partial \ln \mathrm{l}}{\partial \alpha} = n_c \frac{(2+c)^c \left(\theta \mathrm{e}^{-\theta}\right)}{c!}\left\{\frac{(1+c)^{c-1}}{(c-1)!} + \alpha \frac{(2+c)^c \left(\theta \mathrm{e}^{-\theta}\right)}{c!}\right\}^{-1} + n_{c+1}\frac{(-1)}{(1-\alpha)} \tag{13}
$$

$$
\begin{aligned}
\frac{\partial \ln \mathrm{l}}{\partial \theta} &= \left(\frac{1}{\theta} - 1\right)\sum_{i=1}^{k} i n_i + n_c \frac{\alpha \frac{(2+c)^c}{c!}\left(\mathrm{e}^{-\theta} - \theta \mathrm{e}^{-\theta}\right)}{\left\{\frac{(1+c)^{c-1}}{(c-1)!} + \alpha \frac{(2+c)^c \left(\theta \mathrm{e}^{-\theta}\right)}{c!}\right\}} - \left(\frac{1}{\theta} + 1\right)\sum_{i=1}^{k} n_i \\
&\quad - \frac{1}{1-\theta}\sum_{i=1}^{k} n_i
\end{aligned}
\tag{14}
$$

Equating $\frac{\partial \ln l}{\partial \alpha}$ and $\frac{\partial \ln l}{\partial \theta}$ to zero, we get

$$\alpha = \frac{n_c\left(2+c\right)^c\left(\theta \mathrm{e}^{-\theta}\right) - n_{c+1}c\left(1+c\right)^{c-1}}{\left(2+c\right)^c\left(\theta \mathrm{e}^{-\theta}\right)\left(n_c+n_{c+1}\right)} \tag{15}$$

$$\left(\frac{1-\theta}{\theta}\right)\sum_{i=1}^{k} in_i + n_c\left\{\frac{c\left(1+c\right)^{c-1}}{\alpha\left(2+c\right)^c \mathrm{e}^{-\theta}\left(1-\theta\right)} + \frac{\theta}{\left(1-\theta\right)}\right\}^{-1} - \left(\frac{1+\theta}{\theta}\right)\sum_{i=1}^{k} n_i - \left(\frac{1}{1-\theta}\right)\sum_{i=1}^{k} n_i = 0 \tag{16}$$

In the equation (16), substituting α from the equation (15), we get an equation consisting only parameter θ, say $g(\theta) = 0$. By solving this equation for θ using any iterative method, we get the solution, known as the MLE of θ. Using this MLE of θ in (15), we get the MLE of α.

Asymptotic Variance–Covariance Matrix of ML Estimators

The second order derivatives with respect to α and θ of the likelihood function L are obtained as below:

$$\frac{\partial^2 \ln \mathrm{L}}{\partial \alpha^2} = -\frac{n_c\left(2+c\right)^{2c}\left(\theta \mathrm{e}^{-\theta}\right)^2}{\left\{c\left(1+c\right)^{c-1} + \alpha\left(2+c\right)^c\left(\theta \mathrm{e}^{-\theta}\right)\right\}^2} - \frac{n_{c+1}}{\left(1-\alpha\right)^2} \tag{17}$$

$$\frac{\partial^2 \ln \mathrm{L}}{\partial \theta \partial \alpha} = \frac{n_c\left(2+c\right)^c \mathrm{e}^{-\theta}\left(1-\theta\right)\left\{c\left(1+c\right)^{c-1}\right\}}{\left\{c\left(1+c\right)^{c-1} + \alpha\left(2+c\right)^c\left(\theta \mathrm{e}^{-\theta}\right)\right\}^2} \tag{18}$$

$$\frac{\partial^2 \ln \mathrm{L}}{\partial \theta^2} = \left(\frac{-1}{\theta^2}\right)\sum_{i=1}^{k} i n_i + \left(\frac{1-2\theta}{\theta^2(1-\theta)^2}\right)\sum_{i=1}^{k} n_i$$
$$- \frac{n_c \alpha (2+c)^c \mathrm{e}^{-\theta}\left\{(2-\theta)\left[c(1+c)^{c-1} + \alpha(2+c)^c\left(\theta \mathrm{e}^{-\theta}\right)\right]\right\}}{\left\{c(1+c)^{c-1} + \alpha(2+c)^c\left(\theta \mathrm{e}^{-\theta}\right)\right\}^2} \quad (19)$$
$$+ \frac{n_c \alpha (2+c)^c \mathrm{e}^{-\theta}\left\{\mathrm{e}^{-\theta}(1-\theta)^2 \alpha (2+c)^c\right\}}{\left\{c(1+c)^{c-1} + \alpha(2+c)^c\left(\theta \mathrm{e}^{-\theta}\right)\right\}^2}$$

Using the above equations, the asymptotic variance covariance matrix $\boldsymbol{\Sigma}$ of MLE is obtained from the inverse of the Fisher information matrix

$$\mathrm{J}(\theta,\alpha) = \begin{bmatrix} -\mathrm{E}\left(\frac{\partial^2 \ln \mathrm{L}}{\partial \theta^2}\right) & -\mathrm{E}\left(\frac{\partial^2 \ln \mathrm{L}}{\partial \theta \partial \alpha}\right) \\ -\mathrm{E}\left(\frac{\partial^2 \ln \mathrm{L}}{\partial \alpha \partial \alpha}\right) & -\mathrm{E}\left(\frac{\partial^2 \ln \mathrm{L}}{\partial \alpha^2}\right) \end{bmatrix} \quad (20)$$

That is

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathrm{v}(\theta) & \mathrm{cov}(\theta,\alpha) \\ \mathrm{cov}(\theta,\alpha) & \mathrm{v}(\alpha) \end{bmatrix}, \quad \mathrm{SE}\left(\hat{\theta}\right) = \sqrt{\mathrm{v}(\theta)}, \mathrm{SE}(\hat{\alpha}) = \sqrt{\mathrm{v}(\hat{\alpha})} \quad (21)$$

Method of Moments

The mean and variance of the misclassified size-biased Borel distribution are

$$\mathrm{Mean} = \mu_1' = \frac{1}{(1-\theta)^2} + \frac{\theta}{(1-\theta)} - \alpha(1-\theta) b_{c+1} \mathrm{g}^c \mathrm{e}^{-2\theta} \quad (22)$$

$$\text{Variance} = \mu_2$$
$$= \frac{\left(3\theta - \theta^2\right)}{\left(1-\theta\right)^4} + \left\{ \begin{array}{l} \alpha \left[\begin{array}{c} 2 + 2\theta\left(1-\theta\right) - \left(2c+1\right)\left(1-\theta\right)^2 \\ -\alpha b_{c+1}\, \mathrm{g}^c \left(1-\theta\right)^3 \mathrm{e}^{-2\theta} \end{array} \right] \\ \times \left[b_{c+1}\, \mathrm{g}^c \left(1-\theta\right)^{-1} \mathrm{e}^{-2\theta} \right] \end{array} \right\} \tag{23}$$

The recurrence relation of row moments of the misclassified size-biased Borel distribution is

$$\mu'_{r+1} = \frac{\mathrm{g}}{\mathrm{g}'} \frac{\partial \mu'_r}{\partial \theta} + \alpha \frac{\left(\mu'_r - c^r\right) \mathrm{g}^{c+1} b_{c+1}}{\mathrm{f}\,\mu} + \mu'_1 \mu'_r \tag{24}$$

where g(θ), f(θ), μ(θ), and b_x are as per (6). By taking different values of r, different row moments are obtained. Taking $r = 1$ will obtain the second row moments of the misclassified size-biased Borel distribution.

$$\mu'_2 = \frac{\mathrm{g}}{\mathrm{g}'} \frac{\partial \mu'_1}{\partial \theta} + \alpha \frac{\left(\mu'_1 - c^1\right) \mathrm{g}^{c+1} b_{c+1}}{\mathrm{f}\,\mu} + \left(\mu'_1\right)^2 \tag{25}$$

Solving (22) and (25) for α and θ yields moment estimators of α and θ.

The explicit form cannot be obtained for the moment estimators but, by the method of iteration, the solution for the equations may be obtained.

Asymptotic Variance–Covariance Matrix of Moment Estimators

Denote μ'_1 by $H_1(\theta, \alpha)$ and μ'_2 by $H_2(\theta, \alpha)$, i.e.

$$H_1\left(\theta, \alpha\right) = \frac{1}{\left(1-\theta\right)^2} + \frac{\theta}{\left(1-\theta\right)} - \alpha\left(1-\theta\right) b_{c+1}\, \mathrm{g}^c\, \mathrm{e}^{-2\theta} \tag{26}$$

and

$$H_2\left(\theta, \alpha\right) = \frac{\mathrm{g}}{\mathrm{g}'} \frac{\partial \mu'_1}{\partial \theta} + \alpha \frac{\left(\mu'_1 - c^1\right) \mathrm{g}^{c+1} b_{c+1}}{\mathrm{f}\,\mu} + \left(\mu'_1\right)^2 \tag{27}$$

Then, the asymptotic variance–covariance matrix of moment estimators $\tilde{\theta}$ and $\tilde{\alpha}$ are given by

$$\mathbf{V} = \mathbf{A}^{-1}\mathbf{\Sigma}\left(\mathbf{A}^{-1}\right)' = \begin{bmatrix} \mathrm{v}\left(\tilde{\theta}\right) & \mathrm{cov}\left(\tilde{\theta},\tilde{\alpha}\right) \\ \mathrm{cov}\left(\tilde{\theta},\tilde{\alpha}\right) & \mathrm{v}\left(\tilde{\alpha}\right) \end{bmatrix} \tag{28}$$

where the matrix **A** is

$$\mathbf{A} = \begin{bmatrix} \dfrac{\partial \mathrm{H}_1}{\partial\theta} & \dfrac{\partial \mathrm{H}_1}{\partial\alpha} \\ \dfrac{\partial \mathrm{H}_2}{\partial\theta} & \dfrac{\partial \mathrm{H}_2}{\partial\alpha} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{29}$$

and

$$\mathbf{\Sigma} = \begin{bmatrix} \mathrm{v}\left(\mathrm{m}_1'\right) & \mathrm{cov}\left(\mathrm{m}_1',\mathrm{m}_2'\right) \\ \mathrm{cov}\left(\mathrm{m}_1',\mathrm{m}_2'\right) & \mathrm{v}\left(\mathrm{m}_2'\right) \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \tag{30}$$

where

$$\mathrm{V}\left(\mathrm{m}_r'\right) = \frac{\mu_{2r}' - \left(\mu_r'\right)^2}{n}, \quad r = 1,2$$

$$\mathrm{COV}\left(\mathrm{m}_r',\mathrm{m}_s'\right) = \frac{\mu_{r+s}' - \mu_r'\mu_s'}{n}, \quad r \neq s = 1,2$$

and m_r' is the r^{th} sample raw moment of the MSBPL distribution, i.e.

$$\mathrm{m}_r' = \frac{1}{n}\sum_{i=1}^{n} x_i^r$$

Bayes Estimation

The ML method, as well as other classical approaches, is based only on the empirical information provided by the data. However, when there is some technical knowledge on the parameters of the distribution available, a Bayes procedure seems to be an attractive inferential method. The Bayes procedure is based on a posterior density, say $\pi(\alpha, \theta \mid x)$, which is proportional to the product of the likelihood function $L(\alpha, \theta \mid x)$ with a prior joint density, say $g(\alpha, \theta)$, representing the uncertainty on the parameters values. Assume before the observations were made knowledge about the parameters α and θ was vague. Consequently, the non-informative vague prior $\pi_1(\alpha) = g_1(\alpha) = 1$ is applicable to a good approximation.

The non-informative priors of α and θ are

$$\pi_1(\alpha) = g_1(\alpha) = 1 \tag{31}$$

$$\pi_2(\theta) = g_2(\theta) = 1 \tag{32}$$

Hence, the joint prior of θ and α is given by

$$\begin{aligned} g(\theta, \alpha) &= g_1(\alpha) g_2(\theta) \\ g(\alpha, \theta) &= \pi_1(\alpha) \pi_2(\theta) = 1 \end{aligned} \tag{33}$$

If L is the likelihood function indexed by a continuous parameter $\mathbf{\Theta} = (\theta, \alpha)$ with prior density $g(\theta, \alpha)$, then the posterior density for $\mathbf{\Theta}$ is given by

$$\pi(\boldsymbol{\Theta} \mid x) = \frac{\mathrm{L}(\boldsymbol{\Theta})\,\mathrm{g}(\alpha,\theta)}{\int_0^1\int_0^1 \mathrm{L}(\boldsymbol{\Theta})\,\mathrm{g}(\alpha,\theta)\,d\alpha\, d\theta}$$

$$= \frac{\left[\prod_{i=1}^{k}\left[\frac{\left(\theta \mathrm{e}^{-\theta}\right)^i}{\theta \mathrm{e}^{\theta}/1-\theta}\right]^{n_i}\right]\left[\frac{(1+c)^{c-1}}{(c-1)!}+\alpha\frac{(2+c)^c\left(\theta\mathrm{e}^{-\theta}\right)}{c!}\right]^{n_c}(1-\alpha)^{n_{c+1}}}{\int_0^1\left[\prod_{i=1}^{k}\left[\frac{\left(\theta \mathrm{e}^{-\theta}\right)^i}{\theta \mathrm{e}^{\theta}/1-\theta}\right]^{n_i}\right]\sum_{j=0}^{n_c}\binom{n_c}{j}\frac{(2+c)^{cj}\left(\theta\mathrm{e}^{-\theta}\right)^j}{(c!)^j}\left[\frac{(1+c)^{c-1}}{(c-1)!}\right]^{n_c-j}\int_0^1\alpha^j(1-\alpha)^{n_{c+1}}\,d\alpha\, d\theta}$$

$$= \frac{\left[\prod_{i=1}^{k}\left[\frac{\left(\theta \mathrm{e}^{-\theta}\right)^i}{\theta \mathrm{e}^{\theta}/1-\theta}\right]^{n_i}\right]\left[\frac{(1+c)^{c-1}}{(c-1)!}+\alpha\frac{(2+c)^c\left(\theta\mathrm{e}^{-\theta}\right)}{c!}\right]^{n_c}(1-\alpha)^{n_{c+1}}}{\int_0^1\left[\prod_{i=1}^{k}\left[\frac{\left(\theta \mathrm{e}^{-\theta}\right)^i}{\theta \mathrm{e}^{\theta}/1-\theta}\right]^{n_i}\right]\sum_{j=0}^{n_c}\binom{n_c}{j}\frac{(2+c)^{cj}\left(\theta\mathrm{e}^{-\theta}\right)^j}{(c!)^j}\left[\frac{(1+c)^{c-1}}{(c-1)!}\right]^{n_c-j}\beta_{(n_{c+1}+1,\, j+1)}\,d\theta}$$

$$= \frac{\left[\prod_{i=1}^{k}\left[\frac{\left(\theta \mathrm{e}^{-\theta}\right)^i}{\theta \mathrm{e}^{\theta}/1-\theta}\right]^{n_i}\right]\left[\frac{(1+c)^{c-1}}{(c-1)!}+\alpha\frac{(2+c)^c\left(\theta\mathrm{e}^{-\theta}\right)}{c!}\right]^{n_c}(1-\alpha)^{n_{c+1}}}{\sum_{j=0}^{n_c}\binom{n_c}{j}\frac{(2+c)^{cj}}{(c!)^j}\left[\frac{(1+c)^{c-1}}{(c-1)!}\right]^{n_c-j}\beta_{(n_{c+1}+1,\, j+1)}\int_0^1\theta\sum_{i=1}^{k}\left[n_i(i-1)+j\right](1-\theta)\sum_{i=1}^{k}n_i e^{-\theta}\left[\sum_{i=1}^{k}n_i(i+1)+j\right]d\theta} \quad (34)$$

Using the result given by Gradshteĭn and Ryzhik (2007, p. 347),

$$\pi(\boldsymbol{\Theta} \mid x) = \frac{\prod_{i=1}^{k}\left[\frac{\left(\theta e^{-\theta}\right)^{i}}{\theta e^{\theta} / 1-\theta}\right]^{n_i}\left[\frac{(1+c)^{c-1}}{(c-1)!}+\alpha\frac{(2+c)^{c}\left(\theta e^{-\theta}\right)}{c!}\right]^{n_c}(1-\alpha)^{n_{c+1}}}{\sum_{j=0}^{n_c}\binom{n_c}{j}\frac{(2+c)^{cj}}{(c!)^{j}}\left[\frac{(1+c)^{c-1}}{(c-1)!}\right]^{n_c-j}\beta_{(n_{c+1}+1,\, j+1)}\beta_{(\mu+1,\gamma+1)}\Phi(\gamma+1,\mu+\gamma+2;-\eta)} \tag{35}$$

where

$$\Phi(\gamma+1;\mu+\gamma+2;-\eta)=1+\frac{(\gamma+1)}{(\mu+\gamma+2)}\left(\frac{-\eta}{1!}\right)+\frac{(\gamma+1)(\gamma+2)}{(\mu+\gamma+2)(m+\gamma+3)}\left(\frac{-\eta^{2}}{2!}\right)+\frac{(\gamma+1)(\gamma+2)(\gamma+3)}{(\mu+\gamma+2)(m+\gamma+3)(m+\gamma+4)}\left(\frac{-\eta^{3}}{3!}\right)$$
$$+\frac{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)}{(\mu+\gamma+2)(m+\gamma+3)(m+\gamma+4)(m+\gamma+5)}\left(\frac{-\eta^{4}}{4!}\right)+\ldots$$

where

$$\gamma=\sum_{i=1}^{k}n_i(i-1)+j=\mathrm{N}(\bar{x}-1)+j, \quad \mu=\sum_{i=1}^{k}n_i=N, \quad \eta=-\left[\left(\sum_{i=1}^{k}n_i(i+1)\right)+j\right]=\mathrm{N}(\bar{x}+1)+j \tag{36}$$

From (35), the marginal posterior of α will be

$$
\begin{aligned}
\pi(\alpha \mid x) &= \int_0^1 \pi(\alpha,\theta \mid x)\, d\theta \\
&= \int_0^1 \frac{\prod_{i=1}^{k}\left[\left(\theta e^{-\theta}\right)^i \Big/ \dfrac{\theta e^{\theta}}{(1-\theta)}\right]^{n_i}\left[\dfrac{(1+c)^{c-1}}{(c-1)!}+\alpha\dfrac{(2+c)^c\left(\theta e^{-\theta}\right)}{c!}\right]^{n_c}(1-\alpha)^{n_{c+1}}}{\sum_{j=0}^{n_c}\binom{n_c}{j}\dfrac{(2+c)^{cj}}{(c!)^j}\left[\dfrac{(1+c)^{c-1}}{(c-1)!}\right]^{n_c-j}\beta_{(n_{c+1}+1,\,j+1)}\beta_{(\mu+1,\gamma+1)}\Phi(\gamma+1;\mu+\gamma+2;-\eta)}\, d\theta \\
&= \frac{(1-\alpha)^{n_{c+1}}\sum_{j=0}^{n_c}\binom{n_c}{j}\dfrac{\alpha^j (2+c)^{cj}}{(c!)^j}\left[\dfrac{(1+c)^{c-1}}{(c-1)!}\right]^{n_c-j}\beta_{(\mu+1,\gamma+1)}\Phi(\gamma+1;\mu+\gamma+2;-\eta)}{\sum_{j=0}^{n_c}\binom{n_c}{j}\dfrac{(2+c)^{cj}}{(c!)^j}\left[\dfrac{(1+c)^{c-1}}{(c-1)!}\right]^{n_c-j}\beta_{(n_{c+1}+1,\,j+1)}\beta_{(\mu+1,\gamma+1)}\Phi(\gamma+1;\mu+\gamma+2;-\eta)}
\end{aligned}
\tag{37}
$$

From (37), the Bayes estimate of α is given by

$$
\begin{aligned}
\hat{\alpha}_{BS} &= \int_0^1 \alpha\,\pi(\alpha \mid x)\, d\alpha \\
&= \int_0^1 \alpha \frac{(1-\alpha)^{n_{c+1}}\sum_{j=0}^{n_c}\binom{n_c}{j}\dfrac{\alpha^j (2+c)^{cj}}{(c!)^j}\left[\dfrac{(1+c)^{c-1}}{(c-1)!}\right]^{n_c-j}\beta_{(\mu+1,\gamma+1)}\Phi(\gamma+1;\mu+\gamma+2;-\eta)}{\sum_{j=0}^{n_c}\binom{n_c}{j}\dfrac{(2+c)^{cj}}{(c!)^j}\left[\dfrac{(1+c)^{c-1}}{(c-1)!}\right]^{n_c-j}\beta_{(n_{c+1}+1,\,j+1)}\beta_{(\mu+1,\gamma+1)}\Phi(\gamma+1;\mu+\gamma+2;-\eta)}\, d\alpha
\end{aligned}
$$

$$= \frac{\sum_{j=0}^{n_c} \binom{n_c}{j} \frac{(2+c)^{cj}}{(c!)^j} \left[\frac{(1+c)^{c-1}}{(c-1)!} \right]^{n_c - j} \beta_{(n_{c+1}+1,\, j+2)} \beta_{(\mu+1,\gamma+1)} \Phi(\gamma+1; \mu+\gamma+2; -\eta)}{\sum_{j=0}^{n_c} \binom{n_c}{j} \frac{(2+c)^{cj}}{(c!)^j} \left[\frac{(1+c)^{c-1}}{(c-1)!} \right]^{n_c - j} \beta_{(n_{c+1}+1,\, j+1)} \beta_{(\mu+1,\gamma+1)} \Phi(\gamma+1; \mu+\gamma+2; -\eta)} \tag{38}$$

Similarly, from (35), the marginal posterior of θ will be

$$\pi(\theta \mid x) = \int_0^1 \pi(\alpha, \theta \mid x)\, d\alpha$$

$$= \int_0^1 \frac{\prod_{i=1}^{k} \left[\left(\theta e^{-\theta}\right)^i \Big/ \frac{\theta e^{\theta}}{(1-\theta)} \right]^{n_i} \left[\frac{(1+c)^{c-1}}{(c-1)!} + \alpha \frac{(2+c)^c \left(\theta e^{-\theta}\right)}{c!} \right]^{n_c} (1-\alpha)^{n_{c+1}}}{\sum_{j=0}^{n_c} \binom{n_c}{j} \frac{(2+c)^{cj}}{(c!)^j} \left[\frac{(1+c)^{c-1}}{(c-1)!} \right]^{n_c - j} \beta_{(n_{c+1}+1,\, j+1)} \beta_{(\mu+1,\gamma+1)} \Phi(\gamma+1; \mu+\gamma+2; -\eta)} d\alpha$$

$$= \frac{\theta^{\sum_{i=1}^{k} n_i (i-1) + j} (1-\theta)^{\sum_{i=1}^{k} e^{-\theta\left[\left(\sum_{i=1}^{k} n_i (i+1)\right) + j\right]}} \sum_{j=0}^{n_c} \binom{n_c}{j} \frac{(2+c)^{cj}}{(c!)^j} \left[\frac{(1+c)^{c-1}}{(c-1)!} \right]^{n_c - j} \beta_{(n_{c+1}+1,\, j+1)}}{\sum_{j=0}^{n_c} \binom{n_c}{j} \frac{(2+c)^{cj}}{(c!)^j} \left[\frac{(1+c)^{c-1}}{(c-1)!} \right]^{n_c - j} \beta_{(n_{c+1}+1,\, j+1)} \beta_{(\mu+1,\gamma+1)} \Phi(\gamma+1; \mu+\gamma+2; -\eta)} \tag{39}$$

From (39), the Bayes estimate of θ is given by

$$\hat{\theta}_{BS} = \int_0^1 \theta \pi(\theta \mid x) d\theta$$

$$= \int_0^1 \theta \frac{\theta^{\sum_{i=1}^{k} n_i (i-1)+j} (1-\theta)^{\sum_{i=1}^{k} n_i} \mathrm{e}^{-\theta\left[\left(\sum_{i=1}^{k} n_i (i+1)\right)+j\right]} \sum_{j=0}^{n_c} \binom{n_c}{j} \frac{(2+c)^{cj}}{(c!)^j} \left[\frac{(1+c)^{c-1}}{(c-1)!}\right]^{n_c-j} \beta_{(n_{c+1}+1,\, j+1)}}{\sum_{j=0}^{n_c} \binom{n_c}{j} \frac{(2+c)^{cj}}{(c!)^j} \left[\frac{(1+c)^{c-1}}{(c-1)!}\right]^{n_c-j} \beta_{(n_{c+1}+1,\, j+1)} \beta_{(\mu+1,\gamma+1)} \Phi(\gamma+1; \mu+\gamma+2; -\eta)} d\theta$$

$$= \frac{\sum_{j=0}^{n_c} \binom{n_c}{j} \frac{(2+c)^{cj}}{(c!)^j} \left[\frac{(1+c)^{c-1}}{(c-1)!}\right]^{n_c-j} \beta_{(n_{c+1}+1,\, j+1)} \beta_{(\mu+1,\gamma+2)} \Phi(\gamma+2; \mu+\gamma+3; -\eta)}{\sum_{j=0}^{n_c} \binom{n_c}{j} \frac{(2+c)^{cj}}{(c!)^j} \left[\frac{(1+c)^{c-1}}{(c-1)!}\right]^{n_c-j} \beta_{(n_{c+1}+1,\, j+1)} \beta_{(\mu+1,\gamma+1)} \Phi(\gamma+1; \mu+\gamma+2; -\eta)} \quad (40)$$

where γ, μ, and η are as given in (36) above.

Simulation Study

One thousand random samples, each of size n, were generated by using Monte Carlo simulation with different choices of sample size n, θ, α, and value of $c = 1$ from the misclassified size-biased Borel distribution defined in equation (8). Using these different values of sample size n, θ, and α, we calculated the simulated risk (SR) and simulated bias of estimators α and θ by the method of MLE, method of moments, and Bayes estimation. The simulated results are shown in Tables 1 and 2. The SR is defined as

$$\text{SR}=\sqrt{\frac{\sum_{i=1}^{1000}\left(\hat{\theta}_1-\theta\right)^2}{1000}}$$

Conclusion

A comparison was made between different methods of estimation for the parameters of the misclassified size-biased Borel distribution. From Table 1 and 2, it was found that the method of maximum likelihood estimator works better compared to the moment estimator and the Bayes estimator on the basis of SR. As sample size increases, SR of both parameters of all three methods decreases. For fixed misclassification error α, as θ increases, the SR of α and θ decreases in the case of maximum likelihood estimation, moment estimation method, and Bayes estimation. For fixed values of θ and sample size n, as α increases, there is not much difference in the SR of α as well as θ. At the same time, if these values were compared in context of sample size, observe that, for a fixed value of θ and as α increases, the SR of α and θ decreases in most of the cases with the increase in sample size. As sample size increases, the bias in α and θ decreases in the case of all the three methods.

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Appendix A

Table 1. Simulated risk of ML, moment, and Bayes estimators for different values of α, θ, and sample size n

			ML		Moment		Bayes	
θ	α	n	SR(θ)	SR(α)	SR(θ)	SR(α)	SR(θ)	SR(α)
0.03	0.12	20	0.070621	0.687617	0.070009	0.730376	0.428511	0.731791
		50	0.017214	0.662598	0.070000	0.716240	0.366045	0.722317
		90	0.028486	0.576466	0.070000	0.661494	0.342946	0.729741
	0.15	20	0.086849	0.637623	0.090139	0.712136	0.428910	0.774369
		50	0.018903	0.615088	0.070000	0.695215	0.366122	0.695103
		90	0.016796	0.386507	0.070000	0.649579	0.343211	0.675911
	0.20	20	0.072757	0.600000	0.075005	0.681954	0.428803	0.683406
		50	0.022814	0.489319	0.070000	0.668798	0.365955	0.653836
		90	0.022814	0.489319	0.070000	0.668798	0.365955	0.653836
0.06	0.12	20	0.040157	0.409082	0.042393	0.606705	0.408054	0.659958
		50	0.012628	0.391981	0.040017	0.591911	0.349603	0.628596
		90	0.015325	0.280505	0.040019	0.524187	0.327791	0.610602
	0.15	20	0.034921	0.525708	0.042064	0.564374	0.407451	0.595577
		50	0.032482	0.237705	0.040083	0.559160	0.348794	0.565870
		90	0.030247	0.194459	0.040000	0.508564	0.327905	0.567689
	0.20	20	0.041125	0.453903	0.041515	0.533885	0.408379	0.554755
		50	0.031410	0.319943	0.040203	0.521217	0.350368	0.546684
		90	0.029152	0.212999	0.040016	0.476619	0.328233	0.531593
0.09	0.12	20	0.031714	0.386623	0.034880	0.413639	0.392743	0.556575
		50	0.028941	0.338622	0.029383	0.376251	0.338557	0.558982
		90	0.003557	0.010874	0.012466	0.336422	0.320139	0.556492
	0.15	20	0.040798	0.301392	0.043413	0.409123	0.392115	0.556858
		50	0.023699	0.105444	0.025690	0.347586	0.339796	0.539688
		90	0.020707	0.086850	0.021824	0.321968	0.319821	0.520310
	0.20	20	0.032107	0.361397	0.032177	0.415253	0.391115	0.504882
		50	0.021050	0.240808	0.024129	0.348901	0.339039	0.499720
		90	0.014885	0.214971	0.021792	0.326637	0.319959	0.454457

Table 2. Simulated Bias of ML, Moment and Bayes estimators for different values of α, θ, and sample size n

			ML		Moment		Bayes	
θ	α	n	Bias(θ)	Bias(α)	Bias(θ)	Bias(α)	Bias(θ)	Bias(α)
0.03	0.12	20	0.070474	0.691185	0.070010	0.699012	0.428432	0.696161
		50	0.026334	0.571479	0.070000	0.637222	0.365951	0.619558
		90	0.016826	0.555057	0.070000	0.558062	0.342872	0.528030
	0.15	20	0.084899	0.613982	0.070120	0.694661	0.428820	0.668716
		50	0.017858	0.506054	0.070000	0.684802	0.366023	0.641862
		90	0.012158	0.374220	0.070000	0.647191	0.343131	0.604162
	0.20	20	0.072757	0.357243	0.070005	0.688818	0.428718	0.667923
		50	0.020000	0.348958	0.070000	0.662306	0.365868	0.659111
		90	0.002144	0.292983	0.070000	0.622817	0.343073	0.657824
0.06	0.12	20	0.046324	0.193095	0.041704	0.575521	0.407688	0.649596
		50	0.042542	0.146035	0.040018	0.550282	0.349271	0.623053
		90	0.035325	0.080505	0.040017	0.545392	0.327534	0.622236
	0.15	20	0.059598	0.334418	0.041511	0.557115	0.407108	0.685204
		50	0.051860	0.290584	0.040073	0.502591	0.348482	0.600330
		90	0.015826	0.263941	0.039999	0.482067	0.327645	0.600231
	0.20	20	0.058381	0.366643	0.041210	0.422684	0.408050	0.583953
		50	0.043795	0.205674	0.040177	0.377713	0.349991	0.569713
		90	0.039152	0.202999	0.040012	0.351268	0.327979	0.568386
0.09	0.12	20	0.024845	0.190314	0.018532	0.223233	0.391976	0.542166
		50	0.005821	0.282052	0.013171	0.233392	0.337880	0.551094
		90	0.003557	0.010874	0.011603	0.210933	0.319659	0.552079
	0.15	20	0.040859	0.167709	0.017899	0.196935	0.391373	0.552278
		50	0.021317	0.008981	0.013323	0.191992	0.339088	0.538764
		90	0.020707	0.008685	0.011186	0.191486	0.319373	0.535741
	0.20	20	0.025665	0.115874	0.016674	0.193710	0.390411	0.499345
		50	0.019843	0.071383	0.012421	0.183469	0.338378	0.491407
		90	0.015508	0.021350	0.011075	0.175713	0.319515	0.490788