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Moser-Trudinger And Adams Type Inequalities And Their Applications

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MOSER-TRUDINGER AND ADAMS TYPE INEQUALITIES AND THEIR APPLICATIONS

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DEDICATION

To my family

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Chapter 1

Introduction

Sobolev spaces and geometric inequalities can be considered as one of the central tools in many areas such as analysis, differential geometry, mathematical physics, partial differential equations, calculus of variations, etc. The main aim of this dissertation is to study such inequalities in several settings such as Euclidean spaces, Heisenberg groups, etc and their applications. More precisely, we will prove many versions of singular Moser-Trudinger and Adams type inequalities, which are the borderline cases of the Sobolev embeddings. Basically, the Sobolev embeddings assert that $W_0^{k,p}(\Omega) \subset L^q(\Omega)$ for $1 \leq q \leq \frac{np}{n-kp}$, $kp < n$, $n \geq 2$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. However, in the limiting case: $n = kp$, we can show by many examples that $W_0^{k,\frac{n}{k}}(\Omega) \not\subseteq L^\infty(\Omega)$. Thus, it is a very good question to ask what the best possible target is of the Sobolev inequalities in this borderline situation. The Moser-Trudinger and Adams inequalities answer this question and hence, can be considered as the perfect replacements for the Sobolev embeddings when $n = kp$. For the convenience of the reader, we will first introduce some versions of Moser-Trudinger and Adams inequalities in the literature.

1.1 Moser-Trudinger inequalities on Eucledian spaces

In the 1960s, Yudovich [96], Pohozaev [84] and Trudinger [93] worked independently and proved that $W_0^{1,n}(\Omega) \subset L_{\varphi_n}(\Omega)$ where $L_{\varphi_n}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_n(t) = \exp\left(\beta |t|^{n/(n-1)}\right) - 1$ for some $\beta > 0$. More precisely, they proved that there exist constants $\beta > 0$ and $C_n > 0$ depending only on n such that

$$\sup_{u \in W_0^{1,n}(\Omega),\ \int_\Omega |\nabla u|^n dx \leq 1} \int_\Omega \exp\left(\beta |u|^{\frac{n}{n-1}}\right) dx \leq C_n |\Omega|$$

However, the best possible constant β is much more interesting and was not exhibited until the 1971 paper [80] of J. Moser. In fact, using the symmetrization argument to reduce to the one dimensional case, J. Moser established the following result which is now called the Moser-Trudinger inequality:

Theorem 1.1 (Moser [80]) *Let Ω be a domain with finite measure in Euclidean $n-$ space $\mathbb{R}^n$, $n \geq 2$. Then there exist a sharp constant $\alpha_n = n\left(\frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\right)^{\frac{1}{n-1}}$ and a positive constant $C_0 = C_0(n)$ such that*

$$\frac{1}{|\Omega|} \int_\Omega \exp\left(\alpha |u|^{\frac{n}{n-1}}\right) dx \leq C_0$$

for any $\alpha \leq \alpha_n$, any $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n dx \leq 1$. This constant α_n is sharp in the sense that if $\alpha > \alpha_n$, then the above inequality can no longer hold with some c_0 independent of u.

The Moser-Trudinger inequalities are refined and extended to many different settings. For instance, a singular Moser-Trudinger inequality which is an interpolation of Hardy inequality and Moser-Trudinger inequality was studied by Adimurthi and Sandeep in [5]: if $\Omega \subset \mathbb{R}^n$, $n \geq 2$, $|\Omega| < \infty$, then there exists a constant $C_0 = C_0(n) > 0$ such that

$$\frac{1}{|\Omega|^{1-\frac{\alpha}{n}}} \int_\Omega \frac{\exp\left(\alpha |u|^{\frac{n}{n-1}}\right)}{|x|^\beta} dx \leq C_0$$

for any $\beta \in [0, n)$, $0 \leq \alpha \leq \left(1 - \frac{\beta}{n}\right)\alpha_n$, any $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n dx \leq 1$. Moreover, this constant $\left(1 - \frac{\beta}{n}\right)\alpha_n$ is sharp in the sense that if $\alpha > \left(1 - \frac{\beta}{n}\right)\alpha_n$, then the above inequality can no longer hold with some C_0 independent of u.

Recently, using the L^p affine energy $\mathcal{E}_p(f)$ of f instead of the standard L^p energy of gradient $\|\nabla f\|_p$, where

$$\mathcal{E}_p(f) = c_{n,p}\left(\int\limits_{S^{n-1}} \|D_v f\|_p^{-n}\, dv\right)^{-1/n},$$
$$c_{n,p} = \left(\frac{n\omega_n\omega_{p-1}}{2\omega_{n+p-2}}\right)^{1/p} (n\omega_n)^{1/n},$$
$$\|D_v f\|_p = \left(\int\limits_{\mathbb{R}^n} |v\cdot\nabla f(x)|^p\, dx\right)^{1/p},$$

the authors in [19] proved a sharp version of affine Moser-Trudinger inequality, namely,

Theorem 1.2 (Cianchi-Lutwak-Yang-Zhang [19]) *Let Ω be a domain with finite measure in Euclidean $n-$space $\mathbb{R}^n$, $n \geq 2$. Then there exists a constant $m_n > 0$ such that*

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\alpha |u|^{\frac{n}{n-1}}\right) dx \leq m_n$$

for any $\alpha \leq \alpha_n$, any $u \in W_0^{1,n}(\Omega)$ with $\mathcal{E}_n(u) \leq 1$. The constant α_n is sharp in the sense that if $\alpha > \alpha_n$, then the above inequality can no longer hold with some m_n independent of u.

It is worthy to note that by the Holder inequality and Fubini's theorem, we have that

$$\mathcal{E}_p(f) \leq \|\nabla f\|_p$$

for every $f \in W^{1,p}(\mathbb{R}^n)$ and $p \geq 1$. Moreover, since the ratio $\frac{\|\nabla f\|_p}{\mathcal{E}_p(f)}$ is not uniformly bounded from above by any constant, this affine Moser-Trudinger inequality is actually stronger than the standard Moser-Trudinger inequality.

When Ω has infinite volume, the above results become meaningless. In this case, the subcritical Moser-Trudinger type inequalities for unbounded domains were first proposed by D.M. Cao [17] when $n = 2$ and Do Ó [26] for the general case $n \geq 2$. More precisely, they proved that for any $u \in W^{1,n}(\mathbb{R}^n)$ with $\|\nabla u\|_n \leq m < 1$ and $\|u\|_n \leq M < \infty$, there exist a constant $C(m, M) > 0$ and $\alpha > 0$ independent of u such that

$$\int_{\mathbb{R}^n} \phi_{n,1}\left(\alpha |u|^{\frac{n}{n-1}}\right) dx \leq C(m, M)$$

where

$$\phi_{n,1}(t) = e^t - \sum_{i=0}^{n-2} \frac{t^i}{i!}.$$

These results were extended later by Adachi and Tanaka [1] in order to determine the best constant α. In fact, they proved that

Theorem 1.3 (Adachi-Tanaka [1]) *For any $\alpha \in (0, \alpha_n)$, there exists a constant $C_\alpha > 0$ such that*

$$\int_{\mathbb{R}^n} \phi_{n,1}\left(\alpha |u|^{\frac{n}{n-1}}\right) dx \leq C_\alpha \|u\|_n^n, \ \forall u \in W^{1,n}(\mathbb{R}^n), \ \ \|\nabla u\|_n \leq 1,$$

This inequality is false for $\alpha \geq \alpha_n$.

It can be noted that unlike the case of the bounded domains, the best constant α_n cannot be achieved. Thus, the result of Adachi-Tanaka can be considered as the sharp subcritical Moser-Trudinger type inequality on unbounded domains.

We notice that if we replace the norm $\|\nabla u\|_n$ by the full norm $\|\nabla u\|_n + \|u\|_n$ in the Sobolev space $W^{1,n}(\mathbb{R}^n)$, the best constants in the Moser-Trudinger inequalities in unbounded domains can be attained. Thus, they can be considered as the critical Moser-Trudinger inequalities on unbounded domains. In fact, these results are studied in the last 10 years by the work of B. Ruf [85] for the Moser-Trudinger type inequality when $n = 2$. This result was extended later to the general dimension n by Y. X. Li and B. Ruf [65] and more recently, by Adimurthi and Yang [6] for the singular case. Indeed, using the full norm of the Sobolev space $W^{1,n}(\mathbb{R}^n)$ instead of $\|\nabla u\|_n$, they can proved that for all $\alpha \leq \left(1 - \frac{\beta}{n}\right)\alpha_n$ and $\tau > 0$,

$$\sup_{\|u\|_{1,\tau} \leq 1} \int_{\mathbb{R}^n} \frac{\phi_{n,1}\left(\alpha |u|^{\frac{n}{n-1}}\right)}{|x|^\beta} dx < \infty$$

where

$$\|u\|_{1,\tau} = \left(\int_{\mathbb{R}^n} \left(|\nabla u|^n + \tau |u|^n\right) dx\right)^{1/n}.$$

Moreover, this constant $\left(1 - \frac{\beta}{n}\right)\alpha_n$ is sharp in the sense that if $\alpha > \left(1 - \frac{\beta}{n}\right)\alpha_n$, then the supremum is infinity.

We recall that in the paper [4], Adimurthi and Druet used the blow-up technique to study an improvement of the Trudinger-Moser inequality in the spirit of Lions [69]. In fact, they proved that

$$C_\alpha(\Omega) := \sup_{u \in W_0^{1,2}(\Omega),\ \|\nabla u\|_2 \leq 1} \int_\Omega e^{4\pi u^2\left(1+\alpha\|u\|_2^2\right)} dx < \infty \text{ iff } 0 \leq \alpha < \lambda_1(\Omega),$$
$$\lambda_1(\Omega) = \inf_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}.$$

We note that $\lambda_1(\Omega)$ is the first eigenvalue for the Dirichlet problem of the Laplace operator on $\Omega \subset \mathbb{R}^2$. It is easy to see that this inequality is stronger than the original one of Moser where 4π is the best constant, while this inequality of [4] has the constant $4\pi\left(1 + \alpha\|u\|_2^2\right)$ which is larger than 4π. This result is extended to n-dimensional case by Lu and Yang [75] and Zhu [98].

1.2 Moser-Trudinger inequalities on Heisenberg groups

Analysis and study of partial differential equations on the Heisenberg group has received great attention in the past decades. Heisenberg group is the simplest example of noncommutative nilpotent Lie groups which has a close connection with several complex variables and CR geometry. Sharp geometric inequalities on the Heisenberg group have particularly played an important role in harmonic analysis, partial differential equations and differential geometry. A good example of this role is the identification of the sharp constant and extremal functions for the L^2 Sobolev inequality on the Heisenberg group. This was achieved in a series of celebrated works of Jerison and Lee in conjunction with the solution of the CR Yamabe problem [43, 44, 45].

The following Sobolev inequality on the Heisenberg group is well known: for $f \in C_0^\infty(\mathbb{H})$

$$\left(\int_{\mathbb{H}} |f(z,t)|^q dzdt\right)^{\frac{1}{q}} \leq C_{p,q}\left(\int_{\mathbb{H}} |\bigtriangledown_{\mathbb{H}} f(z,t)|^p dzdt\right)^{\frac{1}{p}} \tag{1.1}$$

provided that $1 \leq p < Q = 2n+2$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{Q}$. This inequality was first proved by Folland-Stein [31], see also [34], [70]. In the above inequality, we have used $|\bigtriangledown_{\mathbb{H}} f|$ to express the (Euclidean) norm of the subelliptic gradient of f:

$$|\bigtriangledown_{\mathbb{H}} f| = \sum_{i=1}^{n}\left((X_i f)^2 + (Y_i f)^2\right)^{\frac{1}{2}}.$$

It is then clear that the above inequality is also true for functions in the anisotropic Sobolev space $W_0^{1,p}(\mathbb{H})$ $(p \geq 1)$, where $W_0^{1,p}(\Omega)$ for open set $\Omega \subset \mathbb{H}$ is the completion of $C_0^\infty(\Omega)$ under the norm

$$||f||_{L^p(\Omega)} + ||\bigtriangledown_{\mathbb{H}} f||_{L^p(\Omega)}.$$

Nevertheless, much less is known about sharp constants for Sobolev inequality (1.1) for the

Heisenberg group than for Euclidean space. In fact, the first major breakthrough came after the works by D. Jerison and J. Lee [45] on the sharp constants for the Sobolev inequality and extremal functions on the Heisenberg group in conjunction with the solution to the CR Yamabe problem (we should note the well-known results of Talenti [90] and Aubin [7] for sharp constants and extremal functions in the isotropic case). More precisely, in a series of papers [43, 44, 45], the Yamabe problem on CR manifolds was first studied. In particular, Jerison and Lee study the problem of conformally changing the contact form to one with constant Webster curvature in the compact setting.

In particular, the best constant $C_{p,q}$ for the Sobolev inequality (1.1) on $\mathbb{H}$ for $p = 2$ was found and the extremal functions were identified in [45].

Theorem 1.4 (Jerison and Lee [45]) *The best constant for the inequality (1.1) on* $\mathbb{H}$ *is*

$$C_{2,\frac{2n+2}{n}} = (4\pi)^{-1} n^{-2} \left[\Gamma(n+1)\right]^{\frac{1}{n+1}}$$

and all the extremals of (1.1) are obtained by dilations and left translations of the function

$$K|\left(t + i(|z|^2 + 1)\right)|^{-n}.$$

Furthermore, the extremals in (1.1) are constant multiples of images under the Cayley transform of extremals for the Yamabe functional on the sphere $\mathbb{S}^{2n+1}$ *in* $\mathbb{C}^{n+1}$*.*

The sharp Sobolev inequality on the Heisenberg group for $p = 2$ is closely related to the sharp Hardy-Littlewood-Sobolev inequality, also known as the HLS inequality:

$$\left| \int\int_{\mathbb{H}\times\mathbb{H}} \frac{\overline{f(u)} g(v)}{|u^{-1}v|^{\lambda}} du dv \right| \leq C_{r,\lambda,n} \|f\|_r \|g\|_s. \tag{1.2}$$

In fact, the result of Jerison and Lee is equivalent to the sharp version of HLS inequality (1.2) when $\lambda = Q-2$ and $r = s = 2Q/(2Q-\lambda) = 2Q/(Q+2)$.

Very recently, in a remarkable paper of Frank and Lieb [35], they have succeeded to establish the sharp constants and extremal functions of the HLS inequality on the Heisenberg group for all $0 < \lambda < Q$ and $r = s = \frac{2Q}{2Q-\lambda}$, an analogue to Lieb's celebrated result in Euclidean spaces [66]. We can state the result in [35] as the following theorem.

Theorem 1.5 (Frank & Lieb, Theorem 2.1 in [35]) *Let $0 < \lambda < Q$ and $r = 2Q/(2Q-\lambda)$. Then for any $f, g \in L^r(\mathbb{H})$,*

$$\left| \int\int_{\mathbb{H}\times\mathbb{H}} \frac{\overline{f(u)}g(v)}{|u^{-1}v|^{\lambda}} dudv \right| \leq \left(\frac{\pi^{n+1}}{2^{n-1}n!} \right)^{\frac{\lambda}{Q}} \frac{n!\Gamma((Q-\lambda)/2)}{\Gamma^2((2Q-\lambda)/4)} \|f\|_r \|g\|_r, \tag{1.3}$$

with equality if and only if

$$f(u) = cH(\delta(a^{-1}u)),\ g(v) = c'H(\delta(a^{-1}v))$$

for some $c, c' \in \mathbb{C}$, $\delta > 0$, $a \in \mathbb{H}$ (unless $f \equiv 0$ or $g \equiv 0$), and

$$H = \left[(1+|z|^2)^2 + t^2 \right]^{-\frac{2Q-\lambda}{4}}.$$

Their results also justified Branson, Fontana and Morpurgo's guess in [15] about the optimizer H.

The work of Jerison and Lee [45] raised two natural questions. What is the best constant $C_{p,q}$ for the L^p to L^q Sobolev inequality (1.1) for all $1 \leq p < Q$ and $q = \frac{Qp}{Q-p}$ when $p \neq 2$? What is the sharp constant for the borderline case $p = Q$? While the first question still seems to be open, the second question was answered in the work of Cohn and Lu in [21] on domains of finite measure in the Heisenberg group. Namely, they proved [21] the sharp Moser-Trudinger inequality on any domain Ω with $|\Omega| < \infty$ on the Heisenberg group.

As has been the case in most proofs of sharp constants in Euclidean spaces, one often attempts to use the radial non-increasing rearrangement u^* of functions u (in terms of a certain norm) on the Heisenberg group. However, it is not known whether or not the L^p norm of the subelliptic gradient of the rearrangement of a function is dominated by the L^p norm of the subelliptic gradient of the function. In other words, an inequality like

$$|| \nabla_{\mathbb{H}} u^*||_{L^p} \leq || \nabla_{\mathbb{H}} u||_{L^p} \tag{1.4}$$

is not available on the Heisenberg group. In fact, the work of Jerison-Lee on the best constant and extremals [45] indicates that this inequality fails to hold for the case $p = 2$. Thus, in the works of Jerison and Lee [45] and Frank and Lieb [35], substantially new ideas are needed in deriving sharp Sobolev and Hardy-Littlewood-Sobolev inequalities on the Heisenberg group.

As for the Moser-Trudinger inequality on bounded domains on the Heisenberg group, the borderline case of the Sobolev inequality when $p = Q$, we also have to avoid the rearrangement argument due to the unavailability of the symmetrization inequality (1.4) when $p = Q$. This was carried out in the work of Cohn and Lu [21]. In fact, we can adapt D. Adams' idea in deriving the Moser-Trudinger inequality for higher order derivatives in Euclidean space [2], which requires, roughly speaking, an optimal bound on the size of a function in terms of the potential of its gradient, namely a sharp representation formula. By using this one parameter representation formula on the Heisenberg group, we are able to avoid considering the subelliptic gradient of the rearrangement function. Instead, we will consider the rearrangement of the convolution of the subelliptic gradient with an optimal kernel (see [21] for more details).

The sharp constant for the Moser-Trudinger inequality on domains of finite measure in the Heisenberg group is stated as follows. Throughout the remaining of this dissertation, we use $\xi = (z,t)$ to denote any point $(z,t) \in \mathbb{H}$ and $\rho(\xi) = (|z|^4 + t^2)^{\frac{1}{4}}$ to denote the homogeneous norm of $\xi \in \mathbb{H}$.

Theorem 1.6 *Let* $\alpha_Q = Q\left(2\pi^n\Gamma(\frac{1}{2})\Gamma(\frac{Q-1}{2})\Gamma(\frac{Q}{2})^{-1}\Gamma(n)^{-1}\right)^{Q'-1}$. *Then there exists a uniform constant* C_0 *depending only on* Q *such that for all* $\Omega \subset \mathbb{H}$, $|\Omega| < \infty$ *and* $\alpha \leq \alpha_Q$

$$\sup_{u \in W_0^{1,Q}(\Omega),\ \|\nabla_{\mathbb{H}} u\|_{L^Q} \leq 1} \frac{1}{|\Omega|} \int_\Omega \exp(\alpha \left|u(\xi)\right|^{Q'}) d\xi \leq C_0 < \infty. \tag{1.5}$$

The constant α_Q *is the best possible in the sense that if* $\alpha > \alpha_Q$, *then the supremum in the inequality (1.5) is infinite.*

It is clear that when $|\Omega| = \infty$, the above inequality (1.5) in Theorem 1.6 is not meaningful. We also remark that using similar ideas of representation formulas and rearrangement of convolutions as done on the Heisenberg group in [21], Theorem 1.6 was extended to the groups of Heisenberg type in [21] and to general stratified groups in [9]. We refer to [8] for more introduction of stratified groups.

Using a similar argument, the authors established the following version of singular Moser-Trudinger inequality on bounded domains in [57]:

Theorem 1.7 *Let* $\Omega \subset \mathbb{H}$, $|\Omega| < \infty$ *and* $0 \leq \beta < Q$. *Then there exists a uniform constant* $C_0 < \infty$ *depending only on* Q, β *such that*

$$\sup_{u \in W_0^{1,Q}(\Omega),\ \|\nabla_{\mathbb{H}} u\|_{L^Q} \leq 1} \frac{1}{|\Omega|^{1-\frac{\beta}{Q}}} \int_\Omega \frac{\exp(\alpha_Q \left(1 - \frac{\beta}{Q}\right) \left|u(\xi)\right|^{Q'}) d\xi}{\rho\left(\xi\right)^{\beta}} \leq C_0.$$

The constant $\alpha_Q\left(1-\frac{\beta}{Q}\right)$ *is sharp in the sense that if* $\alpha_Q\left(1-\frac{\beta}{Q}\right)$ *is replaced by any larger number, then the supremum is infinite.*

The situation is more complicated when dealing with unbounded domains on the Heisenberg group. Before we state the Moser-Trudinger inequality on the entire Heisenberg group, we need to recall some preliminaries.

Let $u : \mathbb{H} \to \mathbb{R}$ be a nonnegative function in $W^{1,Q}(\mathbb{H})$, and u^* be the decreasing rearrangement of u, namely

$$u^*(\xi) := \sup\{s \geq 0 : \xi \in \{u > s\}^*\}$$

where

$$\{u > s\}^* = B_r = \{\xi : \rho(\xi) \leq r\}$$

such that $|\{u > s\}| = |B_r|$. It is known from a result of Manfredi and V. Vera De Serio [79] that there exists a constant $c \geq 1$ depending only on Q such that

$$\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u^*|^Q \, d\xi \leq c \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q \, d\xi \tag{1.6}$$

for all $u \in W^{1,Q}(\mathbb{H})$. Thus we can define

$$c^* = \inf\left\{c^{1/(Q-1)} : \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u^*|^Q \, d\xi \leq c \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q \, d\xi,\, u \in W^{1,Q}(\mathbb{H})\right\} \geq 1.$$

We can now state the following version of the Moser-Trudinger type inequality (see [20]):

Theorem 1.8 *Let* $\alpha^* = \alpha_Q/c^*$. *Then for any pair* β, α *satisfying* $0 \leq \beta < Q$ *and* $\alpha \leq$

$\alpha^*(1-\frac{\beta}{Q})$, there holds

$$\sup_{\|u\|_{W^{1,Q}(\mathbb{H})}\leq 1}\int_{\mathbb{H}}\frac{1}{\rho\left(\xi\right)^{\beta}}\left\{\exp\left(\alpha\left|u\right|^{Q/(Q-1)}\right)-S_{Q-2}\left(\alpha,u\right)\right\}<\infty \tag{1.7}$$

where

$$S_{Q-2}\left(\alpha,u\right)=\sum_{k=0}^{Q-2}\frac{\alpha^{k}}{k!}\left|u\right|^{kQ/(Q-1)}.$$

Moreover, the supremum is infinite if $\alpha>\alpha_Q(1-\frac{\beta}{Q})$.

We menton in passing that inequality (1.7) in Euclidean spaces when $\beta=0$ was established in two dimensional case $\mathbb{R}^2$ in [85] and high dimensional case $\mathbb{R}^N$ in [65], while the singular case $0\leq\beta<N$ was treated in [6]. A subcritical case was studied first in two dimension $\mathbb{R}^2$ in [17].

We briefly outline the proof of Theorem 1.8 given in [20]. By using the rearrangement inequality (2.6), we can reduce the inequality to the case where the functions are radial in terms of the homogeneous norm on the Heisenberg group. Then we break the integral over the space $\mathbb{H}$ into two parts, the interior of a large ball and the exterior of the ball. Over the finite ball, we can use the sharp Moser-Trudinger inequality on finite domains proved in [21]. On the exterior of the ball, we will then use the radial lemma for radial functions on the Heisenberg group. However, we should note that, in the above Theorem 1.8, we cannot exhibit the best constant $\alpha^*(1-\frac{\beta}{Q})$ due to the loss of the non-optimal rearrangement argument in the Heisenberg group. In fact, in the inequality controlling the norm of the subelliptic gradient of the rearranged function u^*, the constant c^* is not known to be 1. Therefore, the constant $\frac{\alpha_Q}{c^*}(1-\frac{\beta}{Q})$ is not known to be equal to $\alpha_Q(1-\frac{\beta}{Q})$. We note that using a cut-off function argument and thus avoiding the rearrangement inequality (2.6), the above inequality (1.7) has also shown to be true for α strictly smaller than $\alpha_Q(1-\frac{\beta}{Q})$. Nevertheless, the more difficult critical case $\alpha=\alpha_Q(1-\frac{\beta}{Q})$ is still left open from

[20].

1.3 Adams inequalities

We now turn to the discussion of high order Adams inequalities. Regarding the case of high order derivatives, since the symmetrization is not available, D. Adams [2] proposed a new idea to find the sharp constants for higher order Moser's type inequality, namely, to express u as the Riesz potential of its gradient of order m, and then apply O'Neil's result on the rearrangement of convolution functions and use techniques of symmetric decreasing rearrangements. To state Adams' result, we use the symbol $\nabla^m u$, m is a positive integer, to denote the $m-$th order gradient for $u \in C^m$, the class of $m-$th order differentiable functions:

$$\nabla^m u = \begin{cases} \triangle^{\frac{m}{2}} u & \text{for } m \text{ even} \\ \nabla \triangle^{\frac{m-1}{2}} u & \text{for } m \text{ odd} \end{cases}.$$

where ∇ is the usual gradient operator and $\triangle$ is the Laplacian. We use $||\nabla^m u||_p$ to denote the L^p norm ($1 \le p \le \infty$) of the function $|\nabla^m u|$, the usual Euclidean length of the vector $\nabla^m u$. We also use $W_0^{k,p}(\Omega)$ to denote the Sobolev space which is a completion of $C_0^\infty(\Omega)$ under the norm of $\left[||u||^p_{L^p(\Omega)} + \sum_{j=1}^{k} ||\nabla^j u||^p_{L^p(\Omega)} \right]^{1/p}$. Then Adams proved the following:

Theorem 1.9 (Adams [2]) *Let Ω be an open and bounded set in $\mathbb{R}^n$. If m is a positive integer less than n, then there exists a constant $C_0 = C(n,m) > 0$ such that for any $u \in W_0^{m,\frac{n}{m}}(\Omega)$ and $||\nabla^m u||_{L^{\frac{n}{m}}(\Omega)} \le 1$, then*

$$\frac{1}{|\Omega|} \int_\Omega \exp(\beta |u(x)|^{\frac{n}{n-m}}) dx \le C_0$$

for all $\beta \leq \beta(n,m)$ *where*

$$\beta(n,\ m)\ =\begin{cases} \frac{n}{w_{n-1}}\left[\frac{\pi^{n/2}2^m\Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})}\right]^{\frac{n}{n-m}} & \text{when } m \text{ is odd} \\ \frac{n}{w_{n-1}}\left[\frac{\pi^{n/2}2^m\Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})}\right]^{\frac{n}{n-m}} & \text{when } m \text{ is even}\end{cases}.$$

Furthermore, the constant $\beta(n,m)$ *is best possible in the sense that for any* $\beta > \beta(n,m)$*, the integral can be made as large as possible.*

It's easy to check that $\beta(n,1)$ coincides with Moser's value of α_n and $\beta(2m,m) = 2^{2m}\pi^m\Gamma(m+1)$ for both odd and even m. In fact, Adams' result was extended recently by Tarsi [91] to a larger space, namely, the Sobolev space with homogeneous Navier boundary conditions $W_N^{m,\frac{n}{m}}(\Omega)$:

$$W_N^{m,\frac{n}{m}}(\Omega) := \left\{u \in W^{m,\frac{n}{m}} : \Delta^j u = 0 \text{ on } \partial\Omega \text{ for } 0 \leq j \leq \left[\frac{m-1}{2}\right]\right\}.$$

We note that the Adams inequality was extended to compact Riemannian manifolds without boundary by Fontana [32] and to measure spaces by Fontana and Morpurgo [33].

Concerning the Adams inequality for unbounded domains, in the spirit of Adachi-Tanaka [1], T. Ogawa and T. Ozawa [83] in the case $\frac{n}{m} = 2$ and T. Ozawa [82] in the general case proved that there exist positive constants α and C_α such that

$$\int\limits_{\mathbb{R}^n} \phi_{n,m}\left(\alpha\,|u|^{\frac{n}{n-m}}\right) dx \leq C_\alpha \,\|u\|_{\frac{n}{m}}^{\frac{n}{m}},\ \forall u \in W^{m,\frac{n}{m}}(\mathbb{R}^n),\ \ \|\nabla^m u\|_{\frac{n}{m}} \leq 1,$$

where

$$\phi_{n,m}(t) = e^t - \sum_{j=0}^{j_{\frac{n}{m}}-2} \frac{t^j}{j!}$$

$$j_{\frac{n}{m}} = \min\left\{j \in \mathbb{N} : j \geq \frac{n}{m}\right\} \geq \frac{n}{m}.$$

Their approach of proving the above result is similar to the idea of Yudovich [96], Pohozaev [84] and Trudinger [93]. However, their constant is not best possible. Therefore, the problem of determining the best constant cannot be investigated by this way. In fact, as pointed out in [87], it is still left as **an open problem** to identify the best constant. Thus, it is very interesting to determine the best constants in such inequalities.

Similar to the Moser-Trudinger inequality, the critical Adams type inequality was also studied using the full norm in order to get the best constant. Indeed, it was investigated by Ruf-Sani [86] when m is even, Lam-Lu [47] when m is odd and Lam-Lu [49] for the fractional derivative case in Sobolev spaces of fractional orders. Moreover, using a similar idea to that of Adams **[2],** but using the Bessel (type) potential instead of Riesz potential, and using a new idea of rearrangement-free argument on domains of infinite volume, the sharp fractional singular Adams type inequalities were proved by Lam and Lu in [53].

Theorem 1.10 (Lam-Lu [49]) *Let $0 < \alpha < n$ be an arbitrary real positive number, $p = \frac{n}{\alpha}$ and $\tau > 0$. There holds*

$$\sup_{u \in W^{\alpha,p}(\mathbb{R}^n), \left\|(\tau I - \Delta)^{\frac{\alpha}{2}} u\right\|_p \leq 1} \int_{\mathbb{R}^n} \phi_{n,\alpha}\left(\beta_0\left(n,\alpha\right)|u|^{p'}\right) dx < \infty$$

where

$$\beta_0(n,\alpha) = \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^{\alpha} \Gamma(\alpha/2)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \right]^{p'}.$$

Furthermore this inequality is sharp, i.e., if $\beta_0(n,\alpha)$ is replaced by any $\gamma > \beta_0(n,\alpha)$, then the supremum is infinite.

There is also an improved version of the Adams type inequality in the Sobolev space $W^{2,\frac{n}{2}}(\mathbb{R}^n)$, $n \geq 3$. In this special case, it has been proved in [49] that: Let $0 \leq \alpha < n$, $n \geq 3$ and $\tau > 0$. Then for all $0 \leq \beta \leq \left(1 - \frac{\alpha}{n}\right) \beta(n,2)$, we have

$$\sup_{u \in W^{2,\frac{n}{2}}(\mathbb{R}^n),\ \int_{\mathbb{R}^n} |\Delta u|^{\frac{n}{2}} + \tau |u|^{\frac{n}{2}} \leq 1} \int_{\mathbb{R}^n} \frac{\phi_{n,2}\left(\beta |u|^{\frac{n}{n-2}}\right)}{|x|^{\alpha}} dx < \infty.$$

Moreover, the constant $\left(1 - \frac{\alpha}{n}\right) \beta(n,2)$ is sharp in the sense that if $\beta > \left(1 - \frac{\alpha}{n}\right) \beta(n,2)$, then the supremun is infinite. We should note this result does not require the restriction on the full standard norm and hence, it extends the results in [53]. Indeed, the results there are for the special case $n = 4$ and they require that the full standard norm $\int_{\mathbb{R}^4} \left(|\Delta u|^2 + \sigma |\nabla u|^2 + \tau |u|^2\right) dx$ is not greater than 1. Hen our result is an extension of the previous Adams type inequalities in the spirit of [69], [4], [75].

Chapter 2

Preliminary

2.4 Rearrangement

Let $\Omega \subset \mathbb{R}^N,\ N \geq 2$, be a measurable set. We denote by $\Omega^{\#}$ the open ball $B_R \subset \mathbb{R}^N$ centered at 0 of radius $R > 0$ such that $|B_R| = |\Omega|$.

Let $u : \Omega \to \mathbb{R}$ be a real-valued measurable function. The distribution function of u is the function

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}|$$

and the decreasing rearrangement of u is the right-continuous, nonincreasing function u^* that is equimeasurable with u :

$$u^*(s) = \sup\{t \geq 0 : \mu_u(t) > s\}.$$

It is clear that $supp u^* \subseteq [0, |\Omega|]$. We also define

$$u^{**}(s) = \frac{1}{s}\int_0^s u^*(t)dt \geq u^*(s).$$

Moreover, we define the spherically symmetric decreasing rearrangement of u :

$$u^{\#} : \Omega^{\#} \to [0, \infty]$$

$$u^{\#}(x) = u^*\left(\sigma_N |x|^N\right).$$

Then we have the following important result that could be found in [19, 36, 67]:

Lemma 2.11 (Pólya-Szegö inequality) *Let $u \in W^{1,p}(\mathbb{R}^n)$, $p \geq 1$. Then $f^{\#} \in W^{1,p}(\mathbb{R}^n)$,*

$$\mathcal{E}_p^+\left(f^{\#}\right) = \mathcal{E}_p\left(f^{\#}\right) = \left\|\nabla f^{\#}\right\|_p$$

and

$$\mathcal{E}_p^+\left(f^{\#}\right) \leq \mathcal{E}_p^+(f)\,;\ \mathcal{E}_p\left(f^{\#}\right) = \mathcal{E}_p(f)\,;\ \left\|\nabla f^{\#}\right\|_p = \|\nabla f\|_p\,.$$

We now recall here following result that is a modified version of the key lemma used to prove the Adams inequality in [2]. The proof of this Lemma can be carried out with a slight modification of that in [2] and can be found in [53].

Lemma 2.12 *Let $0 < \alpha \leq 1$, $1 < p < \infty$ and $a(s,t)$ be a non-negative measurable function on $(-\infty,\infty) \times [0,\infty)$ such that (a.e.)*

$$a(s,t) \leq 1, \ \textit{when } 0 < s < t, \tag{2.1}$$

$$\sup_{t>0}\left(\int_{-\infty}^{0} + \int_{t}^{\infty} a(s,t)^{p'}ds\right)^{1/p'} = b < \infty. \tag{2.2}$$

Then there is a constant $c_0 = c_0(p,b)$ such that if for $\phi \geq 0$,

$$\int_{-\infty}^{\infty} \phi(s)^p ds \leq 1, \tag{2.3}$$

then

$$\int_{0}^{\infty} e^{-F_\alpha(t)}dt \leq c_0 \tag{2.4}$$

where

$$F_\alpha(t) = \alpha t - \alpha\left(\int_{-\infty}^{\infty} a(s,t)\phi(s)ds\right)^{p'}. \tag{2.5}$$

2.5 Heisenberg group

We first introduce some preliminaries on the Heisenberg group. Let $\mathbb{H}$ be the n-dimensional Heisenberg group

$$\mathbb{H} = \mathbb{C}^n \times \mathbb{R}$$

whose group structure is given by

$$(z,t)\cdot(z',t') = (z+z', t+t'+2Im(z\cdot\overline{z'})),$$

for any two points (z,t) and (z',t') in $\mathbb{H}$.

The Lie algebra of $\mathbb{H}$ is generated by the left invariant vector fields

$$T = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial t}$$

for $i = 1, \cdots, n$. These generators satisfy the non-commutative relationship

$$[X_i, Y_j] = -4\delta_{ij}T.$$

Moreover, all the commutators of length greater than two vanish, and thus this is a nilpotent, graded, and stratified group of step two.

For each real number $r \in \mathbb{R}$, there is a dilation naturally associated with the Heisenberg group struture which is usually denoted as

$$\delta_r u = \delta_r(z,t) = (rz, r^2t).$$

However, for simplicity we will write ru to denote $\delta_r u$. The Jacobian determinant of δ_r is r^Q, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}$.

The anisotropic dilation structure on $\mathbb{H}$ introduces a homogeneous norm

$$|u| = |(z,t)| = (|z|^4 + t^2)^{\frac{1}{4}}.$$

With this norm, we can define the Heisenberg ball centered at $u = (z,t)$ with radius R

$$B(u,R) = \{v \in \mathbb{H} : |u^{-1} \cdot v| < R\}.$$

The volume of such a ball is $C_Q R^Q$ for some constant depending on Q.

The subelliptic gradient on the Heisenberg group is denoted by

$$\triangledown_{\mathbb{H}} f(z,t) = \sum_{j=1}^{n} \left((X_j f(z,t))X_j + (Y_j f(z,t))Y_j\right).$$

Let $u : \mathbb{H} \to \mathbb{R}$ be a nonnegative function in $W^{1,Q}(\mathbb{H})$, and u^* be the decreasing rearrangement of u, namely

$$u^*(\xi) := \sup\{s \geq 0 : \xi \in \{u > s\}^*\}$$

where

$$\{u > s\}^* = B_r = \{\xi : \rho(\xi) \leq r\}$$

such that $|\{u > s\}| = |B_r|$. It is known from a result of Manfredi and V. Vera De Serio [79] that there exists a constant $c \geq 1$ depending only on Q such that

$$\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u^*|^Q \, d\xi \leq c \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q \, d\xi \tag{2.6}$$

for all $u \in W^{1,Q}(\mathbb{H})$. Thus we can define

$$c^{*} = \inf\left\{c^{1/(Q-1)} : \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u^{*}|^{Q} \, d\xi \leq c \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^{Q} \, d\xi, \, u \in W^{1,Q}(\mathbb{H})\right\} \geq 1.$$

2.6 Bessel Potential

In this section, we provide some preliminaries. For $u \in W^{m,\frac{n}{m}}(\mathbb{R}^{n})$, we will denote by $\nabla^{j} u$, $j \in \{1, 2, ..., m\}$, the $j-th$ order gradient of u, namely

$$\nabla^{j} u = \begin{cases} \triangle^{\frac{j}{2}} u & \text{for } j \text{ even} \\ \nabla \triangle^{\frac{j-1}{2}} u & \text{for } j \text{ odd} \end{cases}.$$

We now introduce the Sobolev space of functions with homogeneous Navier boundary conditions:

$$W_{N}^{m,\frac{n}{m}}(\Omega) := \left\{u \in W^{m,\frac{n}{m}}(\Omega) : \Delta^{j} u = 0 \text{ on } \partial\Omega \text{ for } 0 \leq j \leq \left[\frac{m-1}{2}\right]\right\}.$$

It is easy to see that $W_{N}^{m,\frac{n}{m}}(\Omega)$ contains $W_{0}^{m,\frac{n}{m}}(\Omega)$ as a closed subspace.

Now, for $\tau > 0$, $\alpha \geq 0$, we define the operator $L_{\tau,\alpha}(x)$ by

$$L_{\tau,\alpha}(x) = \tau^{\frac{n-\alpha}{2}} \frac{1}{(4\pi)^{\alpha/2}} \frac{1}{\Gamma(\alpha/2)} \int_{0}^{\infty} e^{-\frac{\pi\tau|x|^{2}}{\delta}} e^{-\delta/4\pi} \delta^{(-n+\alpha)/2} \frac{d\delta}{\delta}.$$

We notice that $L_{1,\alpha}$ is the famous Bessel potential. Now, by Fourier transform, we can prove the following lemma:

Lemma 2.13 *(1)* $L_{\tau,\alpha} \in L^{1}(\mathbb{R}^{n})$.

(2) $\widehat{L_{\tau,\alpha}}(x) = \left(\tau + 4\pi^{2} |x|^{2}\right)^{-\frac{\alpha}{2}}$.

(3) Let $1 < p < \infty$, and k is a positive integer. Then $u \in W^{k,p}(\mathbb{R}^n)$ if and only if $u = L_{\tau,k} * f$ for some $f \in L^p(\mathbb{R}^n)$.

In fact, the properties of the potential $L_{\tau,\alpha}$ are pretty much the same with the properties of the Bessel potential. Also, noticing that from the following identity (see [89]):

$$\frac{|x|^{-n+\alpha}}{\gamma(\alpha)} = \frac{1}{(4\pi)^{\alpha/2}} \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-\frac{\pi|x|^2}{\delta}} \delta^{(-n+\alpha)/2} \frac{d\delta}{\delta}$$

where

$$\gamma(\alpha) = \pi^{n/2} 2^\alpha \Gamma(\alpha/2) / \Gamma(n/2 - \alpha/2)$$

we have that

$$L_{\tau,\alpha}(x) \leq \frac{|x|^{-n+\alpha}}{\gamma(\alpha)}. \tag{2.7}$$

Now, for $\tau > 0$, we have the following observations by Fourier transform:

$$\left\| (\tau I - \Delta)^{\frac{m}{2}} u \right\|_2^2 = \sum_{i=0}^m \binom{m}{i} \tau^{m-i} \left\| \nabla^i u \right\|_2^2 \tag{2.8}$$

where

$$\binom{m}{i} = \frac{m!}{i!(m-i)!}.$$

From (2.8), we have

Lemma 2.14 *Assume $m \in \mathbb{N}$. Let $a_0 = 1, a_1, ..., a_m > 0$. There exists a real number $\tau > 0$ such that for all $u \in W^{m,2}(\mathbb{R}^{2m})$:*

$$\left\| (\tau I - \Delta)^{\frac{m}{2}} u \right\|_2^2 \leq \sum_{j=0}^m a_{m-j} \left\| \nabla^j u \right\|_2^2$$

Proof. We just need to choose $\tau > 0$ such that

$$\binom{m}{j}\tau^{m-j} \leq a_{m-j},\ \ j = 0, 1, ..., m.$$

□

Chapter 3

Sharp Moser-Trudinger inequality in the entire Heisenberg group

3.1 Introduction

Our main purpose in this chapter is to establish the Moser-Trudinger type inequalities in the critical case $\alpha = \alpha_Q(1 - \frac{\beta}{Q})$ using a new method. Our new argument is completely different from and much simpler than those used in [20]. Most importantly, our method allows us to derive the best constant $\alpha = \alpha_Q(1 - \frac{\beta}{Q})$.

Indeed, our main result concerning the best constant for the Moser-Trudinger inequality on the entire Heisenberg group $\mathbb{H}$ can be read as follows:

Theorem 3.1 *Let τ be any positive real number. Then for any pair $\beta,\ \alpha$ satisfying $0 \leq \beta < Q$ and $0 < \alpha \leq \alpha_Q(1 - \frac{\beta}{Q})$, there holds*

$$\sup_{\|u\|_{1,\tau} \leq 1} \int_{\mathbb{H}} \frac{1}{\rho(\xi)^{\beta}} \left\{ \exp\left(\alpha |u|^{Q/(Q-1)} \right) - S_{Q-2}(\alpha, u) \right\} < \infty \tag{3.1}$$

When $\alpha > \alpha_Q(1-\frac{\beta}{Q})$, the integral in (3.1) is still finite for any $u \in W^{1,Q}(\mathbb{H})$, but the supremum is infinite. Here

$$\|u\|_{1,\tau} = \left[\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q + \tau \int_{\mathbb{H}} |u|^Q \right]^{1/Q}.$$

3.2 Proof of Theorem 3.1: The sharp Moser-Trudinger inequality

The primary purpose of this section is to offer a completely different and much simpler proof of the best constant $\alpha_Q(1-\frac{\beta}{Q})$ for the Moser-Trudinger inequality on unbounded domains in the Heisenberg group $\mathbb{H}$. All existing proofs on the Heisenberg group only give the subcritical case for $\alpha < \alpha_Q(1-\frac{\beta}{Q})$. Our proof does not rely on the special structure of the Heisenberg group and applies to much more general cases including the stratified groups ([8]), Euclidean spaces and complete Riemannian manifolds, etc. However, for its simplicity and clarity, we only present it on the Heisenberg group.

Proof. It suffices to prove that for any $\beta,\ \tau$ satisfying $0 \leq \beta < Q$ and $\tau > 0$, there exists a constant $C = C\left(\beta, \tau, Q\right)$ such that for all $u \in C_0^{\infty}\left(\mathbb{H}\right)\backslash\{0\}\,,\ u \geq 0$ and $\int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} u\right|^Q + \tau \int_{\mathbb{H}} \left|u\right|^Q \leq 1$, there holds

$$\int_{\mathbb{H}} \frac{1}{\rho\left(\xi\right)^{\beta}} \left\{\exp\left(\alpha_Q\left(1-\frac{\beta}{Q}\right)\left|u\right|^{Q/(Q-1)}\right) - S_{Q-2}\left(\alpha_Q\left(1-\frac{\beta}{Q}\right), u\right)\right\} \leq C\left(\beta, \tau, Q\right). \quad (3.2)$$

We fix some notations here:

$$A(u) = 2^{-\frac{1}{Q(Q-1)}} \tau^{\frac{1}{Q}} \left\|u\right\|_Q$$

$$\Omega(u) = \left\{\xi \in \mathbb{H} : u(\xi) > A(u)\right\}.$$

Then, it is clear that

$$A(u) < 1. \quad (3.3)$$

Moreover, since

$$\begin{aligned}\int_{\mathbb{H}} |u|^Q &\geq \int_{\Omega(u)} |u|^Q \\ &\geq \int_{\Omega(u)} |A(u)|^Q \\ &= 2^{-\frac{1}{(Q-1)}} \tau \, \|u\|_Q^Q \, |\Omega(u)|\end{aligned}$$

we have

$$|\Omega(u)| \leq 2^{\frac{1}{(Q-1)}} \frac{1}{\tau}. \tag{3.4}$$

Now, we write

$$\int_{\mathbb{H}} \frac{1}{\rho(\xi)^\beta} \left\{ \exp\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right) |u|^{Q/(Q-1)} \right) - S_{Q-2}\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right), u \right) \right\} = I_1 + I_2$$

where

$$I_1 = \int_{\Omega(u)} \frac{1}{\rho(\xi)^\beta} \left\{ \exp\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right) |u|^{Q/(Q-1)} \right) - S_{Q-2}\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right), u \right) \right\}$$

and

$$I_2 = \int_{\mathbb{H}\setminus\Omega(u)} \frac{1}{\rho(\xi)^\beta} \left\{ \exp\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right) |u|^{Q/(Q-1)} \right) - S_{Q-2}\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right), u \right) \right\}.$$

We will prove that both I_1 and I_2 are bounded by a constant $C = C(\beta, \tau, Q)$.

Indeed, from (5.1), we see

$$\begin{aligned} I_2 &\leq \int_{\{u(\xi)<1\}} \frac{1}{\rho(\xi)^{\beta}} \sum_{k=Q-1}^{\infty} \frac{\left[\alpha_Q\left(1-\frac{\beta}{Q}\right)\right]^k}{k!} |u|^{kQ/(Q-1)} \\ &\leq \int_{\{u(\xi)<1\}} \frac{1}{\rho(\xi)^{\beta}} \sum_{k=Q-1}^{\infty} \frac{\left[\alpha_Q\left(1-\frac{\beta}{Q}\right)\right]^k}{k!} |u|^{Q} \\ &\leq \int_{\{\rho(\xi)\geq 1\}} \sum_{k=Q-1}^{\infty} \frac{\left[\alpha_Q\left(1-\frac{\beta}{Q}\right)\right]^k}{k!} |u|^{Q} \\ &+ \int_{\{\rho(\xi)<1\}} \frac{1}{\rho(\xi)^{\beta}} \sum_{k=Q-1}^{\infty} \frac{\left[\alpha_Q\left(1-\frac{\beta}{Q}\right)\right]^k}{k!} \\ &\leq C(\beta,\tau,Q). \end{aligned}$$

Now, to estimate I_1, we first notice that if we set

$$v(\xi) = u(\xi) - A(u) \text{ in } \Omega(u),$$

then $v \in W_0^{1,Q}(\Omega(u))$. Moreover, in $\Omega(u)$:

$$\begin{aligned} |u|^{Q'} &= (|v| + A(u))^{Q'} \\ &\leq |v|^{Q'} + Q'2^{Q'-1}\left(|v|^{Q'-1} A(u) + |A(u)|^{Q'}\right) \\ &\leq |v|^{Q'} + Q'2^{Q'-1}\frac{|v|^{Q'}|A(u)|^{Q}}{Q} + Q'2^{Q'-1}\left(\frac{1}{Q'} + |A(u)|^{Q'}\right) \\ &\leq |v|^{Q'}\left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}|A(u)|^{Q}\right) + C(Q) \end{aligned}$$

where we did use Young's inequality and the following elementary inequality:

$$(a+b)^q \leq a^q + q2^{q-1}\left(a^{q-1}b + b^q\right) \text{ for all } q \geq 1 \text{ and } a,\ b \geq 0.$$

Let

$$w(\xi) = \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}\,|A(u)|^Q\right)^{\frac{Q-1}{Q}} v(\xi) \text{ in } \Omega(u),$$

then it's clear that

$$w \in W_0^{1,Q}(\Omega) \text{ and } |u|^{Q'} \le |w|^{Q'} + C(Q). \tag{3.5}$$

Moreover, we have

$$\nabla_{\mathbb{H}} w = \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}\,|A(u)|^Q\right)^{\frac{Q-1}{Q}} \nabla_{\mathbb{H}} v.$$

Thus

$$\begin{aligned}
\int_{\Omega(u)} |\nabla_{\mathbb{H}} w|^Q &= \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}\,|A(u)|^Q\right)^{Q-1} \int_{\Omega(u)} |\nabla_{\mathbb{H}} v|^Q \\
&= \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}\,|A(u)|^Q\right)^{Q-1} \int_{\Omega(u)} |\nabla_{\mathbb{H}} u|^Q \\
&\le \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}\,|A(u)|^Q\right)^{Q-1} \left[1 - \tau \int_{\mathbb{H}} |u|^Q\right].
\end{aligned}$$

Then

$$\begin{aligned}
\left(\int_{\Omega(u)} |\nabla_{\mathbb{H}} w|^Q\right)^{\frac{1}{Q-1}} &= \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}\,|A(u)|^Q\right) \left[1 - \tau \int_{\mathbb{H}} |u|^Q\right]^{\frac{1}{Q-1}} \\
&\le \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}\,|A(u)|^Q\right) \left(1 - \frac{\tau}{Q-1} \int_{\mathbb{H}} |u|^Q\right) \\
&= \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1} 2^{-\frac{1}{(Q-1)}} \tau \, \|u\|_Q^Q\right) \left(1 - \frac{\tau}{Q-1} \int_{\mathbb{H}} |u|^Q\right) \\
&= \left(1 + \frac{\tau}{Q-1} \int_{\mathbb{H}} |u|^Q\right) \left(1 - \frac{\tau}{Q-1} \int_{\mathbb{H}} |u|^Q\right) \\
&\le 1.
\end{aligned} \tag{3.6}$$

Here, we used the inequality

$$(1-x)^q \leq 1 - qx \text{ for all } 0 \leq x \leq 1,\ 0 < q \leq 1.$$

From (5.3) and (5.4), using Theorem 1.4 and (5.2), we get

$$\begin{aligned}
I_1 &\leq \int_{\Omega(u)} \frac{\exp\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right) |u|^{Q/(Q-1)}\right)}{\rho\left(\xi\right)^{\beta}} \\
&\leq e^{\alpha_Q \left(1-\frac{\beta}{Q}\right) C(Q)} \int_{\Omega(u)} \frac{\exp\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right) |w|^{Q/(Q-1)}\right)}{\rho\left(\xi\right)^{\beta}} \\
&\leq e^{\alpha_Q \left(1-\frac{\beta}{Q}\right) C(Q)} C_0 \left|\Omega\left(u\right)\right|^{1-\frac{\beta}{Q}} \\
&\leq C\left(\beta, \tau, Q\right).
\end{aligned}$$

The proof is then completed.

$\square$

Chapter 4

Q-sub-Laplacian type equation with critical exponential growth

4.1 Introduction

As applications of our critical Moser-Trudinger type inequality, we study a class of partial differential equations of exponential growth on the Heisenberg group. More precisely, we consider the existence of nontrivial weak solutions for the nonuniforly subelliptic equations of $Q-$sub-Laplacian type of the form:

$$-\operatorname{div}_{\mathbb{H}}\left(a\left(\xi,\nabla_{\mathbb{H}}u\right)\right)+V(\xi)\left|u\right|^{Q-2}u=\frac{f(\xi,u)}{\rho\left(\xi\right)^{\beta}}+\varepsilon h(\xi) \qquad (NU)$$

where

$$\left|a\left(\xi,\tau\right)\right|\leq c_0\left(h_0\left(\xi\right)+h_1\left(\xi\right)\left|\tau\right|^{Q-1}\right)$$

for any τ in $\mathbb{R}^{Q-2}$ and a.e. ξ in $\mathbb{H}$, $h_0\in L^{Q'}\left(\mathbb{H}\right)$ and $h_0\in L^{\infty}_{loc}\left(\mathbb{H}\right)$, $0\leq\beta<Q$, $V:\mathbb{H}\to\mathbb{R}$ is a continuous potential, $f:\mathbb{H}\times\mathbb{R}\to\mathbb{R}$ behaves like $\exp\left(\alpha\left|u\right|^{Q'}\right)$ when $|u|\to\infty$ and satisfy those assumptions $(V1)$, $(V2)$, $(V3)$ and $(f1)$, $(f2)$, $(f3)$ in Section 3, and $h\in\left(W^{1,Q}\left(\mathbb{H}\right)\right)^*$, $h\neq 0$ and ε is a positive parameter. The main features of this class of problems are that it is defined in the whole space $\mathbb{H}$ and involves critical growth and the nonlinear operator $Q-$sub-Laplacian type. In spite of a possible failure of the Palais-Smale (PS) compactness condition, in this article we apply the mountain-pass theorem to obtain the weak solution of (NU) in the suitable

subspace E of $W^{1,Q}(\mathbb{H})$. Moreover, in the case of $Q-$sub-Laplacian, i.e.,

$$a(\xi, \nabla_{\mathbb{H}} u) = |\nabla_{\mathbb{H}} u|^{Q-2} \nabla_{\mathbb{H}} u,$$

we will apply minimax methods, more precisely, the mountain-pass theorem combined with minimization and the Ekeland variational principle to obtain multiplicity of weak solutions to the nonhomogeneous problem

$$-\operatorname{div}_{\mathbb{H}}\left(|\nabla_{\mathbb{H}} u|^{Q-2} \nabla_{\mathbb{H}} u\right) + V(\xi)|u|^{Q-2} u = \frac{f(\xi, u)}{\rho(\xi)^{\beta}} + \varepsilon h(\xi). \qquad (NH)$$

We mention that the existence of nontrivial nonnegative solutions to the equation (NH) was established in [20]. However, the multiplicity of solutions was not treated in [20]. It is also worthy to note that the Moser-Trudinger type inequalities in Euclidean spaces play an essential role in the study of elliptic partial differential equations the exponential growth. Here we mention [18], [29], [68], [3], [25], [63], [47, 48] and the references therein.

We next state our main results concerning the existence and multiplicity of nontrivial nonnegative solutions to the $N-$sub-Laplacian equations (NH) on the Heisenberg group.

Theorem 4.1 *Suppose that (V1) and V(2) (or (V3)) and (f1)-(f3) are as stated in Section 3 and $\lambda_1(Q)$ is as defined in Section 3. Furthermore, assume that*

$$\text{(f4)} \quad \limsup_{s \to 0+} \frac{F(\xi, s)}{k_0 |s|^Q} < \lambda_1(Q) \text{ uniformly in } \xi \in \mathbb{H}.$$

Then there exists $\varepsilon_1 > 0$ such that for each $0 < \varepsilon < \varepsilon_1$, (NU) has a weak solution of mountain-pass type.

Theorem 4.2 *In addition to the hypotheses in Theorem 4.1, assume that*

$$\text{(f5)}\quad \lim_{s\to\infty} sf(\xi,s)\exp\left(-\alpha_0 |s|^{Q/(Q-1)}\right) = +\infty$$

uniformly on compact subsets of $\mathbb{H}$*. Then, there exists* $\varepsilon_2 > 0$*, such that for each* $0 < \varepsilon < \varepsilon_2$*, problem (NH) has at least two weak solutions and one of them has a negative energy.*

In the case where the function h does not change sign, we have

Theorem 4.3 *Under the assumptions in Theorems 4.1 and 4.2, if* $h(\xi) \geq 0$ $(h(\xi) \leq 0)$ *a.e., then the solutions of problem (NH) are nonnegative (nonpositive).*

Our last result is

Theorem 4.4 *Under the same hypotheses in Theorems 4.1 and 4.2, the problem (NH) with* $\varepsilon = 0$ *has a nontrivial weak solution.*

4.2 Assumptions on the nonlinearity and the potential and variational framework

In this section, we will provide conditions on the nonlinearity and potential of Eq. (NU) and (NH). Motivated by the Moser-Trudinger inequality (Theorem 3.1), we consider here the maximal growth on the nonlinear term $f(\xi,u)$ which allows us to treat Eq.(NU) and (NH) variationally in a subspace of $W^{1,Q}(\mathbb{H})$. We assume that $f:\mathbb{H}\times\mathbb{R}\to\mathbb{R}$ is continuous, $f(\xi,0)=0$ and f behaves like $\exp\left(\alpha|u|^{Q/(Q-1)}\right)$ as $|u|\to\infty$. More precisely, we assume the following growth conditions on the nonlinearity $f(\xi,u)$:

$(f1)$ There exist constants $\alpha_0,\ b_1,\ b_2>0$ such that for all $(\xi,u)\in\mathbb{H}\times\mathbb{R}$,

$$|f(\xi,u)| \leq b_1 |u|^{Q-1} + b_2\left[\exp\left(\alpha_0 |u|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha_0,u)\right],$$

$(f2)$ There exists $p > Q$ such that for all $\xi \in \mathbb{H}$ and $s > 0$,

$$0 < pF(\xi, s) = p\int_{0}^{s} f(\xi, \tau)d\tau \leq sf(\xi, s)$$

$(f3)$ There exist constant $R_0,\ M_0 > 0$ such that for all $\xi \in \mathbb{H}$ and $s \geq R_0$,

$$F(\xi, s) \leq M_0 f(\xi, s).$$

Since we are interested in nonnegative weak solutions, we will assume

$$f(\xi, u) = 0 \text{ for all } (\xi, u) \in \mathbb{H} \times (-\infty, 0].$$

Let A be a measurable function on $\mathbb{H} \times \mathbb{R}^{Q-2}$ such that $A(\xi, 0) = 0$ and $a(\xi, \tau) = \nabla_\tau A(\xi, \tau)$ is a Caratheodory function on $\mathbb{H} \times \mathbb{R}^{Q-2}$. Assume that there are positive real numbers c_0, c_1, k_0, k_1 and two nonnegative measurable functions $h_0,\ h_1$ on $\mathbb{H}$ such that $h_1 \in L^{\infty}_{loc}(\mathbb{H})$, $h_0 \in L^{Q/(Q-1)}(\mathbb{H})$, $h_1(\xi) \geq 1$ for a.e. ξ in $\mathbb{H}$ and the following conditions hold:

$(A1)$ $|a(\xi, \tau)| \leq c_0 \left(h_0(\xi) + h_1(\xi)|\tau|^{Q-1}\right) \quad \forall \tau \in \mathbb{R}^{Q-2},\ a.e.\ \xi \in \mathbb{H}$

$(A2)$ $c_1 |\tau - \tau_1|^Q \leq \langle a(\xi, \tau) - a(\xi, \tau_1), \tau - \tau_1 \rangle \quad \forall \tau, \tau_1 \in \mathbb{R}^{Q-2},\ a.e.\ \xi \in \mathbb{H}$

$(A3)$ $0 \leq a(\xi, \tau).\tau \leq QA(\xi, \tau) \quad \forall \tau \in \mathbb{R}^{Q-2},\ a.e.\ \xi \in \mathbb{H}$

$(A4)$ $A(\xi, \tau) \geq k_0 h_1(\xi)|\tau|^Q \quad \forall \tau \in \mathbb{R}^{Q-2},\ a.e.\ \xi \in \mathbb{H}.$

Then A verifies the growth condition:

$$|A(\xi, \tau)| \leq c_0 \left(h_0(\xi)|\tau| + h_1(\xi)|\tau|^Q\right) \ \forall \tau \in \mathbb{R}^{Q-2},\ a.e. \xi \in \mathbb{H} \tag{4.1}$$

For examples of A, we can consider $A(\xi,\tau)=h(\xi)\frac{|\tau|^Q}{Q}$ where $h\in L^\infty_{loc}(\mathbb{H})$.

We also propose the following conditions on the potential:

$(V1)$ V is a continuous function such that $V(\xi)\geq V_0>0$ for all $\xi\in\mathbb{H}$,

and one of the following two conditions:

$(V2)$ $V(\xi)\to\infty$ as $\rho(\xi)\to\infty$; or more generally, for every $M>0$,

$$\mu(\{\xi\in\mathbb{H}:V(\xi)\leq M\})<\infty.$$

or

$(V3)$ The function $[V(\xi)]^{-1}$ belongs to $L^1(\mathbb{H})$.

We introduce some notations:

$$\begin{aligned}
&E=\left\{u\in W_0^{1,Q}(\mathbb{H}):\textstyle\int_{\mathbb{H}}h_1(\xi)\,|\nabla_{\mathbb{H}}u|^Q\,d\xi+\int_{\mathbb{H}}V(\xi)\,|u|^Q<\infty\right\}\\
&\|u\|=\left(\textstyle\int_{\mathbb{H}}\left(h_1(\xi)\,|\nabla_{\mathbb{H}}u|^Q+\frac{1}{k_0Q}V(\xi)\,|u|^Q\right)d\xi\right)^{1/Q},\ u\in E\\
&\lambda_1(Q)=\inf\left\{\frac{\|u\|^Q}{\int_{\mathbb{H}}\frac{|u|^Q}{\rho(\xi)^\beta}d\xi}:u\in E\setminus\{0\}\right\}
\end{aligned}$$

Under the condition on the potential $(V1)$, we can see that E is a reflexive Banach space when endowed with the norm

$$\|u\|=\left(\int_{\mathbb{H}}\left(h_1(\xi)\,|\nabla_{\mathbb{H}}u|^Q+\frac{1}{k_0Q}V(\xi)\,|u|^Q\right)d\xi\right)^{1/Q}$$

and for all $Q\leq q<\infty$,

$$E\hookrightarrow W^{1,Q}(\mathbb{H})\hookrightarrow L^q(\mathbb{H})$$

with continuous embedding. Furthermore,

$$\lambda_1(Q) = \inf\left\{ \frac{\|u\|^Q}{\int_{\mathbb{H}} \frac{|u|^Q}{\rho(\xi)^\beta} d\xi} : u \in E \setminus \{0\} \right\} > 0 \text{ for any } 0 \le \beta < Q. \tag{4.2}$$

By the assumptions $(V2)$ or $(V3)$, we can get the compactness of the embedding

$$E \hookrightarrow L^p(\mathbb{H}) \text{ for all } p \ge Q.$$

Following from $(f1)$, we can conclude for all $(\xi, u) \in \mathbb{H} \times \mathbb{R}$,

$$|F(\xi, u)| \le b_3 \left[\exp\left(\alpha_1 |u|^{Q/(Q-1)} \right) - S_{Q-2}(\alpha_1, u) \right]$$

for some constants $\alpha_1,\ b_3 > 0$. Thus, by the Moser-Trudinger type inequalities, we have $F(\xi, u) \in L^1(\mathbb{H})$ for all $u \in W^{1,Q}(\mathbb{H})$. Define the functional $E,\ T,\ J_\varepsilon : E \to \mathbb{R}$ by

$$\begin{aligned} E(u) &= \textstyle\int_{\mathbb{H}} A(\xi, \nabla_H u) d\xi + \frac{1}{Q} \int_{\mathbb{H}} V(\xi) |u|^Q d\xi \\ T(u) &= \textstyle\int_{\mathbb{H}} \frac{F(\xi,u)}{\rho(\xi)^\beta} d\xi \\ J_\varepsilon(u) &= E(u) - T(u) - \varepsilon \textstyle\int_{\mathbb{H}} hu d\xi \end{aligned}$$

then the functional J_ε is well-defined. Moreover, J_ε is a C^1 functional on E with

$$DJ_\varepsilon(u)v = \int_{\mathbb{H}} a(\xi, \nabla_H u) \nabla_{\mathbb{H}} v d\xi + \int_{\mathbb{H}} V(\xi) |u|^{Q-2} v d\xi - \int_{\mathbb{H}} \frac{f(\xi, u)v}{\rho(\xi)^\beta} d\xi - \varepsilon \int_{\mathbb{H}} hv d\xi,\ \forall u, v \in E$$

4.3 Some basic lemmas

First, we recall what we call the Radial Lemma (see [20]) which asserts:

$$|u^*(\xi)|^Q \leq \frac{Q}{\omega_{Q-1}} \frac{\|u^*\|_Q^Q}{\rho(\xi)^Q}, \forall \xi \in \mathbb{H} \setminus \{0\}$$

where u^* is the decreasing rearrangement of $|u|$ and $\omega_{Q-1} = \int_{\rho(\xi)=1} d\xi$. Use this Radial Lemma, we can prove the following two lemmas (see [27] and [20]):

Lemma 4.5 *For $\kappa > 0$ and $\|u\|_E \leq M$ with M sufficiently small and $q > Q$, we have*

$$\int_{\mathbb{H}} \frac{\left[\exp\left(\kappa |u|^{Q/(Q-1)}\right) - S_{Q-2}(\kappa, u)\right] |u|^q}{\rho(\xi)^\beta} d\xi \leq C(Q, \kappa) \|u\|^q .$$

Proof. The proof is analogous to the proof of Theorem 1.1 in [20]. For the completeness, we give the details here.

Setting

$$R(\alpha, u) = \exp\left(\alpha |u|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha, u) .$$

Assume that u^* is the decreasing rearrangement of $|u|$. We have by the Hardy-Littlewood inequality that

$$\int_{\mathbb{H}} \frac{R(\kappa, u) |u|^q}{\rho(\xi)^\beta} d\xi \leq \int_{\mathbb{H}} \frac{R(\kappa, u^*) |u^*|^q}{\rho(\xi)^\beta} d\xi \tag{4.3}$$

Let γ be a positive number to be chosen later, we estimate

$$\begin{aligned}
&\int_{\rho(\xi)\le\gamma}\frac{R(\kappa,u^*)\,|u^*|^q}{\rho(\xi)^\beta}d\xi\\
&\le\left(\int_{\rho(\xi)\le\gamma}(R(\kappa,u^*))^p\,d\xi\right)^{1/p}\left(\int_{\rho(\xi)\le\gamma}\frac{1}{\rho(\xi)^{\beta s}}d\xi\right)^{1/p's}\left(\int_{\rho(\xi)\le\gamma}|u^*|^{qp's'}\,d\xi\right)^{1/p's'}\\
&\le C\left(\int_{\rho(\xi)\le\gamma}R(p\kappa,u^*)\,d\xi\right)^{1/p}\left(\int_{\rho(\xi)\le\gamma}|u^*|^{qp's'}\,d\xi\right)^{1/p's'}
\end{aligned}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{p'}=1,\ 1<s<\frac{Q}{\beta},$ and $\frac{1}{s}+\frac{1}{s'}=1.$ This together with Moser-Trudinger type inequalities and the continuous embedding of $E\hookrightarrow L^t(\mathbb{H}),\ t\ge Q$ implies

$$\int_{\rho(\xi)\le\gamma}\frac{R(\kappa,u^*)\,|u^*|^q}{\rho(\xi)^\beta}d\xi\le C\,\|u\|^q \tag{4.4}$$

for some constant $C=C(Q,\kappa,\gamma)$, provided that $\|u\|_E$ is sufficiently small such that $p\kappa\,\|u\|_E^{Q/(Q-1)}\le\alpha^*$.

On the other hand, choosing γ sufficiently large such that $(Q/\omega_{Q-1})^{1/Q}\,\gamma^{-1}\,\|u\|_E<1/2$, we obtain by the Radial lemma and the continuous embedding of $E\hookrightarrow L^q(\mathbb{H})$,

$$\begin{aligned}
\int_{\rho(\xi)\ge\gamma}\frac{R(\kappa,u^*)\,|u^*|^q}{\rho(\xi)^\beta}d\xi&\le\frac{R(\kappa,u^*(\gamma))}{\gamma^\beta}\int_{\rho(\xi)\ge\gamma}|u^*|^q\,d\xi\\
&\le\frac{R(\kappa,1/2)}{\gamma^\beta}\,\|u^*\|_q^q\le C\,\|u\|_E^q
\end{aligned} \tag{4.5}$$

for some constant C. By (4.3), (4.6) and (4.7), we then complete the proof of the lemma. □

Lemma 4.6 *If $\kappa>0$, $0\le\beta<Q$, $u\in E$ and $\|u\|_E\le M$ with $\kappa M^{Q/(Q-1)}<\left(1-\frac{\beta}{Q}\right)\alpha_Q$, then*

$$\int_{\mathbb{H}}\frac{\left[\exp\left(\kappa\,|u|^{Q/(Q-1)}\right)-S_{Q-2}(\kappa,u)\right]|u|}{\rho(\xi)^\beta}d\xi\le C(Q,M,\kappa)\,\|u\|_s$$

for some $s > Q$.

Proof. First, recall the following inequality: For $\alpha \geq 0$, $r \geq 1$, we have

$$\left(e^{\alpha} - \sum_{k=0}^{Q-2} \frac{\alpha^k}{k!}\right)^r \leq e^{r\alpha} - \sum_{k=0}^{Q-2} \frac{(r\alpha)^k}{k!} \tag{4.6}$$

Now, using Holder inequality, (4.6) and Theorem 1.6, we have

$$\begin{aligned}
&\int_{\mathbb{H}} \frac{\left[\exp\left(\kappa\left|u\right|^{Q/(Q-1)}\right) - S_{Q-2}\left(\kappa, u\right)\right]\left|u\right|}{\rho(\xi)^{\beta}} d\xi \\
&\leq \left[\int_{\mathbb{H}} \frac{\left[\exp\left(\kappa\left|u\right|^{Q/(Q-1)}\right) - S_{Q-2}\left(\kappa, u\right)\right]^r}{\rho(\xi)^{r\beta}} d\xi\right]^{1/r} \left[\int_{\mathbb{H}} \left|u\right|^s\right]^{1/s} \\
&\leq \left[\int_{\mathbb{H}} \frac{\left[\exp\left(\kappa r\left|u\right|^{Q/(Q-1)}\right) - S_{Q-2}\left(\kappa r, u\right)\right]}{\rho(\xi)^{r\beta}} d\xi\right]^{1/r} \left\|u\right\|_s \\
&\leq C\left(Q, M, \kappa\right)\left\|u\right\|_s.
\end{aligned}$$

where $r, s \geq 1$, $\frac{1}{r} + \frac{1}{s} = 1$ and r is sufficiently close to 1. □

We also have the following lemma (for Euclidean case, see [47]):

Lemma 4.7 *Let $\{w_k\} \subset E$, $\left\|w_k\right\|_E = 1$. If $w_k \to w \neq 0$ weakly and almost everywhere, $\nabla_{\mathbb{H}} w_k \to \nabla_{\mathbb{H}} w$ almost everywhere, then $\frac{R(\alpha, w_k)}{\rho(\xi)^{\beta}}$ is bounded in $L^1(\mathbb{H})$ for*

$$0 < \alpha < \alpha_Q\left(1 - \frac{\beta}{Q}\right)\left(1 - \left\|w\right\|_E^Q\right)^{-1/(Q-1)}.$$

Proof. Using Brezis-Lieb lemma in [16], we deduce that

$$\left\|w_k\right\|_E^Q - \left\|w_k - w\right\|_E^Q \to \left\|w\right\|_E^Q.$$

Thus for k large enough and $\delta > 0$ small enough:

$$0 < \alpha\left(1+\delta\right)\left\|w_k - w\right\|_E^{Q/(Q-1)} < \alpha_Q\left(1 - \frac{\beta}{Q}\right).$$

Now, by noticing that the function $e^x - \sum_{k=0}^{Q-2} \frac{x^k}{k!}$ is increasing and convex in $x \geq 0$ and the fact that for all $\varepsilon > 0$ sufficiently small, there exists $C\left(\varepsilon\right) > 0$ such that for all real numbers a, b :

$$\left|a+b\right|^{Q'} \leq \left(1+\varepsilon\right)\left|a\right|^{Q'} + C(\varepsilon)\left|b\right|^{Q'},$$

we have

$$\int_{\mathbb{H}} \frac{R\left(\alpha, w_k\right)}{\rho(\xi)^{\beta}} d\xi \leq \frac{1}{p}\int_{\mathbb{H}} \frac{R\left(\left(1+\varepsilon\right)p\alpha, w_k - w\right)}{\rho(\xi)^{\beta}} d\xi + \frac{1}{q}\int_{\mathbb{H}} \frac{R\left(qC(\varepsilon)\alpha, w\right)}{\rho(\xi)^{\beta}} d\xi$$

where $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Now, by choosing p sufficiently close to 1 and ε small enough such that $\left(1+\varepsilon\right)p < \left(1+\delta\right)$ and using Theorem 3.1, we get the conclusion. □

4.4 The existence of solution to the problem (NU)

The existence of nontrivial solution to Eq. (NU) will be proved by a mountain-pass theorem without a compactness condition such as the one of the Palais-Smale (PS) type. This version of the mountain-pass theorem is a consequence of the Ekeland's variational principle. First of all, we will check that the functional J_ε satisfies the geometric conditions of the mountain-pass theorem.

Lemma 4.8 *Suppose that* $(V1)$, $(f1)$ *and* $(f4)$ *hold. Then there exists* $\varepsilon_1 > 0$ *such that for* $0 < \varepsilon < \varepsilon_1$, *there exists* $\rho_\varepsilon > 0$ *such that* $J_\varepsilon(u) > 0$ *if* $\left\|u\right\|_E = \rho_\varepsilon$. *Furthermore,* ρ_ε *can be chosen such that* $\rho_\varepsilon \to 0$ *as* $\varepsilon \to 0$.

Proof. From $(f4)$, there exist $\tau,\ \delta > 0$ such that $|u| \le \delta$ implies

$$F(\xi,u) \le k_0 \left(\lambda_1\left(Q\right) - \tau\right) |u|^Q \tag{4.7}$$

for all $\xi \in \mathbb{H}$. Moreover, using $(f1)$ for each $q > Q$, we can find a constant $C = C(q,\delta)$ such that

$$F(\xi,u) \le C\,|u|^q \left[\exp\left(\kappa\,|u|^{Q/(Q-1)}\right) - S_{Q-2}\left(\kappa,u\right)\right] \tag{4.8}$$

for $|u| \ge \delta$ and $\xi \in \mathbb{H}$. From (4.7) and (4.8) we have

$$F(\xi,u) \le k_0 \left(\lambda_1\left(Q\right) - \tau\right) |u|^Q + C\,|u|^q \left[\exp\left(\kappa\,|u|^{Q/(Q-1)}\right) - S_{Q-2}\left(\kappa,u\right)\right]$$

for all $(\xi,u) \in \mathbb{H}\times\mathbb{R}$. Now, by $(A4)$, Lemma 4.3, (4.2) and the continuous embedding $E \hookrightarrow L^Q\left(\mathbb{H}\right)$, we obtain

$$\begin{aligned} J_\varepsilon(u) &\ge k_0\,\|u\|_E^Q - k_0\left(\lambda_1\left(Q\right) - \tau\right)\int_{\mathbb{H}} \frac{|u|^Q}{\rho\left(\xi\right)^\beta}d\xi - C\,\|u\|_E^q - \varepsilon\,\|h\|_*\,\|u\|_E \\ &\ge k_0\left(1 - \frac{\left(\lambda_1\left(Q\right) - \tau\right)}{\lambda_1\left(Q\right)}\right)\|u\|_E^Q - C\,\|u\|_E^q - \varepsilon\,\|h\|_*\,\|u\|_E \end{aligned}$$

Thus

$$J_\varepsilon(u) \ge \|u\|_E\left[k_0\left(1 - \frac{\left(\lambda_1\left(Q\right) - \tau\right)}{\lambda_1\left(Q\right)}\right)\|u\|_E^{Q-1} - C\,\|u\|_E^{q-1} - \varepsilon\,\|h\|_*\right] \tag{4.9}$$

Since $\tau > 0$ and $q > Q$, we may choose $\rho > 0$ such that $k_0\left(1 - \frac{(\lambda_1(Q)-\tau)}{\lambda_1(Q)}\right)\rho^{Q-1} - C\rho^{q-1} > 0$. Thus, if ε is sufficiently small then we can find some $\rho_\varepsilon > 0$ such that $J_\varepsilon(u) > 0$ if $\|u\| = \rho_\varepsilon$ and even $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$. □

Lemma 4.9 *There exists $e \in E$ with $\|e\|_E > \rho_\varepsilon$ such that $J_\varepsilon(e) < \inf_{\|u\|=\rho_\varepsilon} J_\varepsilon(u)$.*

Proof. Let $u \in E \setminus \{0\}$, $u \geq 0$ with compact support $\Omega = supp(u)$. By $(f2)$ and $(f3)$, we have that for $p > Q$, there exists a positive constant $C > 0$ such that

$$\forall s \geq 0,\ \forall \xi \in \Omega :\ F(\xi, s) \geq cs^p - d. \tag{4.10}$$

Then by (4.1), we get

$$J_\varepsilon(tu) \leq Ct \int_\Omega h_0(\xi) |\nabla_H u| \, d\xi + Ct^Q \|u\|_E^Q - Ct^p \int_\Omega \frac{|u|^p}{\rho(\xi)^\beta} d\xi + C + \varepsilon t \left| \int_\Omega hud\xi \right|$$

Since $p > Q$, we have $J_\varepsilon(tu) \to -\infty$ as $t \to \infty$. Setting $e = tu$ with t sufficiently large, we get the conclusion. □

In studying this class of sub-elliptic problems involving critical growth and unbounded domains, the loss of the (PS) compactness condition raises many difficulties. In the following lemmas, we will analyze the compactness of (PS) sequences of J_ε.

Lemma 4.10 *Let $(u_k) \subset E$ be an arbitrary (PS) sequence of J_ε, i.e.,*

$$J_\varepsilon(u_k) \to c,\ DJ_\varepsilon(u_k) \to 0 \text{ in } E' \text{ as } k \to \infty.$$

Then there exists a subsequence of (u_k) (still denoted by (u_k)) and $u \in E$ such that

$$\begin{cases} \frac{f(\xi,u_k)}{\rho(\xi)^\beta} \to \frac{f(\xi,u)}{\rho(\xi)^\beta} & \text{strongly in } L^1_{loc}(\mathbb{H}) \\ \nabla_{\mathbb{H}} u_k(\xi) \to \nabla_{\mathbb{H}} u(\xi) & \text{almost everywhere in } \mathbb{H} \\ a(\xi, \nabla_{\mathbb{H}} u_k) \rightharpoonup a(\xi, \nabla_{\mathbb{H}} u) & \text{weakly in } \left(L^{Q/(Q-1)}_{loc}(\mathbb{H})\right)^{Q-2} \\ u_k \rightharpoonup u & \text{weakly in } E \end{cases}$$

Furthermore u is a weak solution of (NU).

In order to prove this lemma, we need the following two lemmas that can be found in [74], [20].

Lemma 4.11 *Let $B_r(\xi^*)$ be a Heisenberg ball centered at $(\xi^*) \in \mathbb{H}$ with radius r. Then there exists a positive ε_0 depending only on Q such that*

$$\sup_{\int_{B_r(\xi^*)} |\nabla_H u|^Q d\xi \leq 1,\ \int_{B_r(\xi^*)} u d\xi = 0} \frac{1}{|B_r(\xi^*)|} \int_{B_r(\xi^*)} \exp\left(\varepsilon_0 |u|^{Q/(Q-1)}\right) d\xi \leq C_0$$

for some constant C_0 depending only on Q.

Lemma 4.12 *Let (u_n) be in $L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ and let f be a continuous function. Then $\frac{f(\xi,u_n)}{\rho(\xi)^\beta} \to \frac{f(\xi,u)}{\rho(\xi)^\beta}$ in $L^1(\Omega)$, provided that $\frac{f(\xi,u_n(\xi))}{\rho(\xi)^\beta} \in L^1(\Omega)$ $\forall n$ and $\int_\Omega \frac{|f(\xi,u_n(\xi))u_n(\xi)|}{\rho(\xi)^\beta} d\xi \leq C_1$.*

Now we are ready to prove Lemma 4.10.

Proof. By the assumption, we have

$$\int_{\mathbb{H}} A(\xi, \nabla_H u_k) d\xi + \frac{1}{Q} \int_{\mathbb{H}} V(\xi) |u_k|^Q d\xi - \int_{\mathbb{H}} \frac{F(\xi, u_k)}{\rho(\xi)^\beta} d\xi - \varepsilon \int_{\mathbb{H}} h u_k d\xi \overset{k\to\infty}{\to} c \tag{4.11}$$

and

$$\left| \int_{\mathbb{H}} a(\xi, \nabla_H u_k) \nabla_{\mathbb{H}} v d\xi + \int_{\mathbb{H}} V(\xi) |u_k|^{Q-2} u_k v d\xi - \int_{\mathbb{H}} \frac{f(\xi, u_k) v}{\rho(\xi)^\beta} d\xi - \varepsilon \int_{\mathbb{H}} h v d\xi \right| \leq \tau_k \|v\|_E \tag{4.12}$$

for all $v \in E$, where $\tau_k \to 0$ as $k \to \infty$.

Choosing $v = u_k$ in (4.12) and by $(A3)$, we get

$$\int_{\mathbb{H}} \frac{f(\xi,u_k)u_k}{\rho(\xi)^{\beta}} d\xi + \varepsilon \int_{\mathbb{H}} h u_k d\xi - Q \int_{\mathbb{H}} A(\xi, \nabla_H u_k) - \int_{\mathbb{H}} V(\xi) |u_k|^{Q-2} u_k d\xi \leq \tau_k \|u_k\|_E$$

This together with (4.11), $(f2)$ and $(A4)$ leads to

$$\left(\frac{p}{Q} - 1\right) \|u_k\|_E^Q \leq C(1 + \|u_k\|_E)$$

and hence $\|u_k\|_E$ is bounded and thus

$$\int_{\mathbb{H}} \frac{f(\xi,u_k)u_k}{\rho(\xi)^{\beta}} d\xi \leq C, \ \int_{\mathbb{H}} \frac{F(\xi,u_k)}{\rho(\xi)^{\beta}} d\xi \leq C. \tag{4.13}$$

Note that the embedding $E \hookrightarrow L^q(\mathbb{H})$ is compact for all $q \geq Q$, by extracting a subsequence, we can assume that

$$u_k \to u \text{ weakly in } E \text{ and for almost all } \xi \in \mathbb{H}.$$

Thanks to Lemma 4.12, we have

$$\frac{f(\xi,u_n)}{\rho(\xi)^{\beta}} \to \frac{f(\xi,u)}{\rho(\xi)^{\beta}} \text{ in } L^1_{loc}(\mathbb{H}). \tag{4.14}$$

Now, similarly as in [20], up to a subsequence, we define an energy concentration set for any fixed $\delta > 0$,

$$\Sigma_\delta = \left\{ \xi \in \mathbb{H} : \lim_{r\to 0} \lim_{k\to\infty} \int_{B_r(\xi)} \left(|u_k|^Q + |\nabla_{\mathbb{H}} u_k|^Q \right) d\xi' \geq \delta \right\}$$

Since (u_k) is bounded, Σ_δ must be a finite set. For any $\xi^* \in \mathbb{H} \setminus \Sigma_\delta$, there exist $r : 0 < r <$

$dist\left(\xi^*, \Sigma_\delta\right)$ such that

$$\lim_{k\to\infty}\int_{B_r(\xi^*)}\left(|u_k|^Q+|\nabla_{\mathbb{H}}u_k|^Q\right)d\xi<\delta$$

so for large k :

$$\int_{B_r(\xi^*)}\left(|u_k|^Q+|\nabla_{\mathbb{H}}u_k|^Q\right)d\xi<\delta \tag{4.15}$$

By results in [20], we have:

$$\begin{aligned}&\int_{B_r(\xi^*)}\frac{|f(\xi,u_k)|\,|u_k-u|}{\rho(\xi)^\beta}d\xi \qquad (4.16)\\ &\leq\left\|\frac{f(\xi,u_k)}{\rho(\xi)^{\beta/q}}\right\|_{L^q}\left\|\frac{1}{\rho(\xi)^\beta}\right\|_{L^s}^{1/q'}\|u_k-u\|_{L^{q's'}}\leq C\,\|u_k-u\|_{L^{q's'}}\to 0\end{aligned}$$

and for any compact set $K\subset\subset\mathbb{H}\setminus\Sigma_\delta$,

$$\lim_{k\to\infty}\int_K\frac{|f(\xi,u_k)\,u_k-f(\xi,u)\,u|}{\rho(\xi)^\beta}d\xi=0 \tag{4.17}$$

So now, we will prove that for any compact set $K\subset\subset\mathbb{H}\setminus\Sigma_\delta$,

$$\lim_{k\to\infty}\int_K|\nabla_{\mathbb{H}}u_k-\nabla_{\mathbb{H}}u|^Q\,d\xi=0 \tag{4.18}$$

It is enough to prove for any $\xi^*\in\mathbb{H}\setminus\Sigma_\delta$, and r given by (4.15), there holds

$$\lim_{k\to\infty}\int_{B_{r/2}(\xi^*)}|\nabla_{\mathbb{H}}u_k-\nabla_{\mathbb{H}}u|^Q\,d\xi=0 \tag{4.19}$$

For this purpose, we take $\phi\in C_0^\infty\left(B_r\left(\xi^*\right)\right)$ with $0\leq\phi\leq1$ and $\phi=1$ on $B_{r/2}\left(\xi^*\right)$. Obviously

ϕu_k is a bounded sequence. Choose $v = \phi u_k$ and $v = \phi u$ in (4.12), we have:

$$\int_{B_r(\xi^*)} \phi \left(a\left(\xi, \nabla_{\mathbb{H}} u_k\right) - a\left(\xi, \nabla_{\mathbb{H}} u\right)\right)\left(\nabla_{\mathbb{H}} u_k - \nabla_{\mathbb{H}} u\right) d\xi \leq \int_{B_r(\xi^*)} a\left(\xi, \nabla_{\mathbb{H}} u_k\right) \nabla_{\mathbb{H}} \phi \left(u - u_k\right) d\xi$$
$$+ \int_{B_r(\xi^*)} \phi a\left(\xi, \nabla_{\mathbb{H}} u\right)\left(\nabla_{\mathbb{H}} u - \nabla_{\mathbb{H}} u_k\right) d\xi + \int_{B_r(\xi^*)} \phi \left(u_k - u\right) \frac{f\left(\xi, u_k\right)}{\rho\left(\xi\right)^{\beta}} d\xi$$
$$+ \tau_k \left\|\phi u_k\right\|_E + \tau_k \left\|\phi u\right\|_E - \varepsilon \int_{B_r(\xi^*)} \phi h \left(u_k - u\right) d\xi$$

Note that by Holder inequality and the compact embedding of $E \hookrightarrow L^Q\left(\Omega\right)$, we get

$$\lim_{k \to \infty} \int_{B_r(\xi^*)} a\left(\xi, \nabla_{\mathbb{H}} u_k\right) \nabla_{\mathbb{H}} \phi \left(u - u_k\right) d\xi = 0 \tag{4.20}$$

Since $\nabla_{\mathbb{H}} u_k \rightharpoonup \nabla_{\mathbb{H}} u$ and $u_k \rightharpoonup u$, there holds

$$\lim_{k \to \infty} \int_{B_r(\xi^*)} \phi a\left(\xi, \nabla_{\mathbb{H}} u\right)\left(\nabla_{\mathbb{H}} u - \nabla_{\mathbb{H}} u_k\right) d\xi = 0 \text{ and } \lim_{k \to \infty} \int_{B_r(\xi^*)} \phi h \left(u_k - u\right) d\xi = 0 \tag{4.21}$$

The Holder inequality and (4.16) implies that

$$\lim_{k \to \infty} \int_{B_r(\xi^*)} \phi \left(u_k - u\right) f\left(\xi, u_k\right) d\xi = 0$$

So we can conclude that

$$\lim_{k \to \infty} \int_{B_r(\xi^*)} \phi \left(a\left(\xi, \nabla_{\mathbb{H}} u_k\right) - a\left(\xi, \nabla_{\mathbb{H}} u\right)\right)\left(\nabla_{\mathbb{H}} u_k - \nabla_{\mathbb{H}} u\right) d\xi = 0$$

and hence we get (4.19) by $(A2)$. So we have (4.18) by a covering argument. Since Σ_δ is finite, it follows that $\nabla_{\mathbb{H}} u_k$ converges to $\nabla_{\mathbb{H}} u$ almost everywhere. This immediately implies, up to a subsequence, $a\left(\xi, \nabla_{\mathbb{H}} u_k\right) \rightharpoonup a\left(\xi, \nabla_{\mathbb{H}} u\right)$ weakly in $\left(L_{loc}^{Q/(Q-1)}\left(\Omega\right)\right)^{Q-2}$. Let k tend to infinity in

(4.12) and combine with (4.14), we obtain

$$\langle DJ_{\varepsilon}(u), h\rangle = 0 \ \forall h \in C_0^{\infty}(\Omega).$$

This completes the proof of the Lemma. □

4.4.1 The proof of Theorem 4.1

Proposition 4.13 *Under the assumptions (V1) and (V2) (or V(3)), and (f1)-(f4), Then there exists $\varepsilon_1 > 0$ such that for each $0 < \varepsilon < \varepsilon_1$, the problem (NU) has a solution u_M via mountain-pass theorem.*

Proof. For ε sufficiently small, by Lemmas 4.3 and 4.6, J_{ε} satisfies the hypotheses of the mountain-pass theorem except possibly for the (PS) condition. Thus, using the mountain-pass theorem without the (PS) condition, we can find a sequence (u_k) in E such that

$$J_{\varepsilon}(u_k) \to c_M > 0 \text{ and } \|DJ_{\varepsilon}(u_k)\| \to 0$$

where c_M is the mountain-pass level of J_{ε}. Now, by Lemma 4.10, the sequence (u_k) converges weakly to a weak solution u_M of (NU) in E. Moreover, $u_M \neq 0$ since $h \neq 0$. □

4.5 The multiplicity results to the problem (NH): Theorem 4.2

In this section, we study the problem (NH). Note that Eq. (NH) is a special case of the problem (NU) where $A(\xi, \tau) = \frac{|\tau|^Q}{Q}$. As a consequence, there exists a nontrivial solution of standard "mountain-pass" type as in Theorem 4.1. Now, we will prove the existence of the second solution.

Lemma 4.14 *There exists $\eta > 0$ and $v \in E$ with $\|v\|_E = 1$ such that $J_\varepsilon(tv) < 0$ for all $0 < t < \eta$. In particular,* $\inf_{\|u\|_E \le \eta} J_\varepsilon(u) < 0$.

Proof. Let $v \in E$ be a solution of the problem

$$-\operatorname{div}_{\mathbb{H}}\left(|\nabla_{\mathbb{H}} v|^{Q-2} \nabla_{\mathbb{H}} v\right) + V(\xi)\,|v|^{Q-2}\, v = h \text{ in } \mathbb{H}.$$

Then, for $h \neq 0$, we have $\int_{\mathbb{H}} hv = \|v\|_E^Q > 0$. Moreover,

$$\frac{d}{dt} J_\varepsilon(tv) = t^{Q-1}\,\|v\|_E^Q - \int_{\mathbb{H}} \frac{f\,(\xi, tv)\, v}{\rho(\xi)^\beta} d\xi - \varepsilon \int_{\mathbb{H}} hv d\xi$$

for $t > 0$. Since $f(\xi, 0) = 0$, by continuity, it follows that there exists $\eta > 0$ such that $\frac{d}{dt} J_\varepsilon(tv) < 0$ for all $0 < t < \eta$ and thus $J_\varepsilon(tv) < 0$ for all $0 < t < \eta$ since $J_\varepsilon(0) = 0$. □

Next, we define the Moser Functions (see [20, 47]):

$$\widetilde{m}_l(\xi, r) = \frac{1}{\sigma_Q^{1/Q}} \begin{cases} (\log l)^{(Q-1)/Q} & \text{if } \rho(\xi) \le \frac{r}{l} \\ \frac{\log \frac{r}{\rho(\xi)}}{(\log l)^{1/Q}} & \text{if } \frac{r}{l} \le \rho(\xi) \le r \\ 0 & \text{if } \rho(\xi) \ge r \end{cases}$$

Using the fact that $|\nabla_{\mathbb{H}} \rho(\xi)| = \frac{|z|}{\rho(\xi)}$ where $\xi = (z, t) \in \mathbb{H}$, we can conclude that $\widetilde{m}_l\,(., r) \in W^{1,Q}(\mathbb{H})$, the support of $\widetilde{m}_l(\xi, r)$ is the ball B_r,

$$\int_{\mathbb{H}} |\nabla_{\mathbb{H}} \widetilde{m}_l(\xi, r)|^Q \, d\xi = 1, \text{ and } \|\widetilde{m}_l\|_{W^{1,Q}(\mathbb{H})} = 1 + O(1/\log l). \tag{4.22}$$

Let $m_l(\xi,r)=\widetilde{m}_l(\xi,r)/\left\|\widetilde{m}_l\right\|_E$. Then by straightforward calculation, we have

$$m_l^{Q/(Q-1)}(\xi,r)=\sigma_Q^{-1/(Q-1)}\log l+d_l \text{ for } \rho(\xi)\leq r/l, \tag{4.23}$$

where $d_l=\sigma_Q^{-1/(Q-1)}\log l\left(\|\widetilde{m}_l\|^{-1/(Q-1)}-1\right)$. Moreover, we have

$$\|\widetilde{m}_l\|\to 1 \text{ as } l\to\infty$$
$$\frac{d_l}{\log l}\to 0 \text{ as } l\to\infty.$$

It's now standard to check the following lemma (for the Euclidean case, see [27, 47]):

Lemma 4.15 *Suppose that (V1) and (f1)-(f5) hold. Then there exists $k\in\mathbb{N}$ such that*

$$\max_{t\geq 0}\left\{\frac{t^Q}{Q}-\int_{\mathbb{H}}\frac{F(\xi,tm_k)}{\rho(\xi)^\beta}d\xi\right\}<\frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}$$

Corollary 4.16 *Under the hypotheses (V1) and (f1)-(f5), if ε is sufficiently small then*

$$\max_{t\geq 0}J_\varepsilon(tm_k)=\max_{t\geq 0}\left\{\frac{t^Q}{Q}-\int_{\mathbb{H}}\frac{F(\xi,tm_k)}{\rho(\xi)^\beta}d\xi-t\int_{\mathbb{H}}\varepsilon hm_kd\xi\right\}<\frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}$$

Proof. Since $\left|\int_H \varepsilon hm_kd\xi\right|\leq\varepsilon\|h\|_*$, taking ε sufficiently small and using Moser-Trudinger type inequalities, the result follows. □

Note that we can conclude by inequality (4.9) and Lemma 4.14 that

$$-\infty<c_0=\inf_{\|u\|_E\leq\rho_\varepsilon}J_\varepsilon(u)<0. \tag{4.24}$$

Next, we will prove that this infimum is achieved and generate a solution. In order to obtain convergence results, we need to improve the estimate of Lemma 4.15.

Corollary 4.17 *Under the hypotheses (V1) and (f1)-(f5), there exist* $\varepsilon_2 \in (0, \varepsilon_1]$ *and* $u \in W^{1,Q}(\mathbb{H})$ *with compact support such that for all* $0 < \varepsilon < \varepsilon_2$,

$$J_\varepsilon(tu) < c_0 + \frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1} \quad \textit{for all } t \geq 0$$

Proof. It is possible to raise the infimum c_0 by reducing ε. By Lemma 4.22, $\rho_\varepsilon \overset{\varepsilon\to 0}{\to} 0$. Consequently, $c_0 \overset{\varepsilon\to 0}{\to} 0$. Thus there exists $\varepsilon_2 > 0$ such that if $0 < \varepsilon < \varepsilon_2$ then, by Corollary 6.1, we have

$$\max_{t\geq 0} J_\varepsilon(tm_k) < c_0 + \frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}$$

Taking $u = m_k \in W^{1,Q}(\mathbb{H})$, the result follows. □

Now, similarly as in the Euclidean case (see [6, 27, 47]), we have the following lemma:

Lemma 4.18 *If* (u_k) *is a (PS) sequence for* J_ε *at any level with*

$$\liminf_{k\to\infty} \|u_k\|_E < \left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{(Q-1)/Q} \tag{4.25}$$

then (u_k) *possesses a subsequence which converges strongly to a solution* u_0 *of (NH).*

4.5.1 Proof of Theorem 4.2

The proof of the existence of the second solution of (NH) follows by a minimization argument and Ekeland's variational principle.

Proposition 4.19 *There exists* $\varepsilon_2 > 0$ *such that for each* ε *with* $0 < \varepsilon < \varepsilon_2$, *Eq. (NH) has a minimum type solution* u_0 *with* $J_\varepsilon(u_0) = c_0 < 0$, *where* c_0 *is defined in (4.24).*

Proof. Let ρ_ε be as in Lemma 4.22. We can choose $\varepsilon_2 > 0$ sufficiently small such that

$$\rho_\varepsilon < \left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{(Q-1)/Q}$$

Since $\overline{B}_{\rho_\varepsilon}$ is a complete metric space with the metric given by the norm of E, convex and the functional J_ε is of class C^1 and bounded below on $\overline{B}_{\rho_\varepsilon}$, by the Ekeland's variational principle there exists a sequence (u_k) in $\overline{B}_{\rho_\varepsilon}$ such that

$$J_\varepsilon(u_k) \to c_0 = \inf_{\|u\|_E \le \rho_\varepsilon} J_\varepsilon(u) \text{ and } \|DJ_\varepsilon(u_k)\| \to 0$$

Observing that

$$\|u_k\|_E \le \rho_\varepsilon < \left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{(Q-1)/Q}$$

by Lemma 6.3, it follows that there exists a subsequence of (u_k) which converges to a solution u_0 of (NH). Therefore, $J_\varepsilon(u_0) = c_0 < 0$. □

Remark 4.20 By Corollary 6.2, we can conclude that

$$0 < c_M < c_0 + \frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}$$

Proposition 4.21 *If $\varepsilon_2 > 0$ is enough small, then the solutions of (NH) obtained in Propositions 5.1 and 6.1 are distinct.*

Proof. By Proposition 5.1 and 6.1, there exist sequences (u_k), (v_k) in E such that

$$u_k \to u_0,\ J_\varepsilon(u_k) \to c_0 < 0,\ DJ_\varepsilon(u_k)\, u_k \to 0$$

and

$$v_k \rightharpoonup u_M,\ J_\varepsilon(v_k) \to c_M > 0,\ DJ_\varepsilon(v_k)\,v_k \to 0,\ \nabla_{\mathbb{H}} v_k(\xi) \to \nabla_{\mathbb{H}} u_M(\xi) \text{ almost everywhere in } \mathbb{H}$$

Now, suppose by contradiction that $u_0 = u_M$. As in the proof of Lemma 4.10 we obtain

$$\frac{f(\xi, v_k)}{\rho(\xi)^\beta} \to \frac{f(\xi, u_0)}{\rho(\xi)^\beta} \text{ in } L^1(B_R) \text{ for all } R > 0. \tag{4.26}$$

From this, we have by $(f2)$, $(f3)$ and the generalized Lebesgue dominated convergence theorem:

$$\frac{F(\xi, v_k)}{\rho(\xi)^\beta} \to \frac{F(\xi, u_0)}{\rho(\xi)^\beta} \text{ in } L^1(B_R) \text{ for all } R > 0.$$

Now, recall the following inequalities: there exists $c > 0$ such that for all $(\xi, s) \in \mathbb{H} \times \mathbb{R}^+$:

$$F(\xi, s) \le c\,|s|^Q + cf(\xi, s) \tag{4.27}$$

$$F(\xi, s) \le c\,|s|^Q + cR(\alpha_0, s)\,s$$

$$\int_{\mathbb{H}} \frac{f(\xi, v_k)v_k}{\rho(\xi)^\beta} d\xi \le C, \quad \int_{\mathbb{H}} \frac{F(\xi, v_k)}{\rho(\xi)^\beta} d\xi \le C.$$

We will prove that for arbitrary $\delta > 0$, we can find $R > 0$ such that

$$\int\limits_{\rho(\xi) > R} \frac{F(\xi, v_k)}{\rho(\xi)^\beta} d\xi \le 3\delta \text{ and } \int\limits_{\rho(\xi) > R} \frac{F(\xi, u_0)}{\rho(\xi)^\beta} d\xi \le \delta.$$

As a consequence, we get

$$\frac{F(\xi, v_k)}{\rho(\xi)^\beta} \to \frac{F(\xi, u_0)}{\rho(\xi)^\beta} \text{ in } L^1(\mathbb{H}). \tag{4.28}$$

First, we have

$$\int\limits_{\substack{\rho(\xi)>R\\|v_k|>A}} \frac{F(\xi,v_k)}{\rho(\xi)^\beta}d\xi \le c\int\limits_{\substack{\rho(\xi)>R\\|v_k|>A}} \frac{|v_k|^Q}{\rho(\xi)^\beta}d\xi + c\int\limits_{\substack{\rho(\xi)>R\\|v_k|>A}} \frac{f(\xi,v_k)}{\rho(\xi)^\beta}d\xi$$
$$\le \frac{c}{R^\beta A}\int\limits_{\rho(\xi)>R} |v_k|^{Q+1}\,d\xi + c\frac{1}{A}\int_{\mathbb{H}} \frac{f(\xi,v_k)v_k}{\rho(\xi)^\beta}d\xi$$
$$\le \frac{c}{R^\beta A}\left\|v_k\right\|_E^{Q+1} + c\frac{1}{A}\int_{\mathbb{H}} \frac{f(\xi,v_k)v_k}{\rho(\xi)^\beta}d\xi.$$

Hence, since $\left\|v_k\right\|_E$ is bounded and using (4.27), we can choose A and R such that

$$\int\limits_{\substack{\rho(\xi)>R\\|v_k|>A}} \frac{F(\xi,v_k)}{\rho(\xi)^\beta}d\xi \le 2\delta.$$

Next, we have

$$\int\limits_{\substack{\rho(\xi)>R\\|v_k|\le A}} \frac{F(\xi,v_k)}{\rho(\xi)^\beta}d\xi \le \frac{C(\alpha_0,A)}{R^\beta}\int\limits_{\substack{\rho(\xi)>R\\|v_k|\le A}} |v_k|^Q\,d\xi$$
$$\le \frac{2^{Q-1}C(\alpha_0,A)}{R^\beta}\left(\int\limits_{\substack{\rho(\xi)>R\\|v_k|\le A}} |v_k-u_0|^Q\,d\xi + \int\limits_{\substack{\rho(\xi)>R\\|v_k|\le A}} |u_0|^Q\,d\xi\right).$$

Now, using the compactness of embedding $E \hookrightarrow L^q(\mathbb{H})$, $q \ge Q$ and noticing that $v_k \rightharpoonup u_0$, again we can choose R such that

$$\int\limits_{\substack{\rho(\xi)>R\\|v_k|\le A}} \frac{F(\xi,v_k)}{\rho(\xi)^\beta}d\xi \le \delta.$$

Thus we have

$$\int\limits_{\rho(\xi)>R} \frac{F(\xi, v_k)}{\rho(\xi)^{\beta}} d\xi \leq 3\delta.$$

Similarly, we also have

$$\int\limits_{\rho(\xi)>R} \frac{F(\xi, u_0)}{\rho(\xi)^{\beta}} d\xi \leq 3\delta.$$

Thus, we can get (4.28).

Now, by standard arguments (see [27, 47]), we can deduce a contradiction. □

4.5.2 Proof of Theorem 4.3

Corollary 4.22 *There exists $\varepsilon_3 > 0$ such that if $0 < \varepsilon < \varepsilon_3$ and $h(\xi) \geq 0$ for all $\xi \in \mathbb{H}$, then the weak solutions of (NH) are nonnegative.*

Proof. Let u be a weak solution of (NH), that is,

$$\int_H \left(|\nabla_H u|^{Q-2} \nabla_H u \nabla_H v + V(\xi) |u|^{Q-2} uv \right) d\xi - \int_H \frac{f(\xi, u) v}{\rho(\xi)^{\beta}} d\xi - \int_H \varepsilon h v d\xi = 0$$

for all $v \in E$. Taking $v = u^- \in E$ and observing that $f(\xi, u(\xi)) u^-(\xi) = 0$ a.e., we have

$$\|u^-\|_E^Q = - \int_H \varepsilon h u^- d\xi \leq 0$$

Consequently, $u = u^+ \geq 0$. □

4.5.3 Proof of Theorem 4.4

It's similar to the proof of Theorem 4.1 and 4.2. First, we can find a sequence (v_k) in E such that

$$J_0(v_k) \to c_M > 0 \text{ and } DJ_0(v_k) \to 0$$

where c_M is the Mountain-pass level of J_0. Moreover, we have that the sequence (v_k) converges weakly to a weak solution v of (NH) with $\varepsilon = 0$. It's now enough to show that $v \neq 0$. Indeed, suppose that $v = 0$. Similarly as in the previous part, we get

$$\frac{F(\xi, v_k)}{\rho(\xi)^\beta} \to 0 \text{ in } L^1(\mathbb{H}). \tag{4.29}$$

Thus

$$\|v_k\|_E^Q \to Qc_M > 0. \tag{4.30}$$

Also, we have from the previous sections that $c_M \in \left(0, \frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}\right)$. Hence, we can find $\delta > 0$ and $K \in \mathbb{N}$ such that

$$\|v_k\|_E^Q \leq \left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0} - \delta\right)^{Q-1} \text{ for all } k \geq K. \tag{4.31}$$

Now, if we choose $\tau > 1$ sufficiently close to 1, then by $(f1)$ we have

$$|f(\xi, v_k)v_k| \leq b_1 |v_k|^Q + b_2 \left[\exp\left(\alpha_0 |v_k|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha_0, v_k)\right] |v_k|.$$

Hence

$$\int_H \frac{|f(\xi, v_k)v_k|}{\rho(\xi)^\beta} \leq b_1 \int_H \frac{|v_k|^Q}{\rho(\xi)^\beta} + b_2 \int_H \frac{\left[\exp\left(\alpha_0 |v_k|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha_0, v_k)\right] |v_k|}{\rho(\xi)^\beta}.$$

Using Holder inequality, Theorem 3.1, Lemma 4.6 and (4.31), we can conclude that

$$\int_H \frac{|f(\xi, v_k)v_k|}{\rho(\xi)^\beta} \to 0.$$

Since $DJ_0(v_k) \to 0$, we get $\|v_k\|_E \to 0$ and it's a contradiction.

Chapter 5

Adams type inequalities

5.1 Introduction

It is well-known that sharp geometric inequalities such as Sobolev inequalities, Hardy-Littlewood-Sobolev inequalities, Moser-Trudinger inequalities, Adams inequalities, etc and their extremals play an important role in the study of partial differential equations and geometric analysis. Proofs of such sharp inequalities often require rearrangement argument to reduce the underlying problems to the radial case.

Our main purpose in this chapter is to develop a new approach to prove a general version of Adams type inequality. The method developed here does not use the rearrangement argument and therefore applies to settings where symmetrization is not available. Indeed, as our first main result in this chapter, we will prove the following singular Adams type inequality in the high order Sobolev spaces $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ for arbitrary positive integer m:

Theorem 5.1 *Let m be a positive integer less than n, $\tau > 0$ and $0 \leq \alpha < n$. There holds*

$$\sup_{u \in W^{m,\frac{n}{m}}(\mathbb{R}^n), \left\|(\tau I - \Delta)^{\frac{m}{2}} u\right\|_{\frac{n}{m}} \leq 1} \int_{\mathbb{R}^n} \frac{\phi\left(\beta_{\alpha,n,m} |u|^{\frac{n}{n-m}}\right)}{|x|^{\alpha}} dx < \infty$$

where

$$\phi(t) = e^t - \sum_{j=0}^{j_{\frac{n}{m}}-2} \frac{t^j}{j!},$$

$$j_{\frac{n}{m}} = \min\left\{j \in \mathbb{N} : j \geq \frac{n}{m}\right\} \geq \frac{n}{m},$$

$$\beta_{\alpha,n,m} = \left(1 - \frac{\alpha}{n}\right)\beta_0(n,m),$$

$$\beta_0\left(n,m\right) = \frac{n}{\omega_{n-1}}\left[\frac{\pi^{\frac{n}{2}} 2^m \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)}\right]^{\frac{n}{n-m}}.$$

Moreover, the constant $\beta_{\alpha,n,m}$ is sharp in the sense that if we replace $\beta_{\alpha,n,m}$ by any $\beta > \beta_{\alpha,n,m}$, then the supremum is infinity.

We notice that for arbitrary positive integer number m and any $a_0 = 1, a_2 > 0, \cdots, a_m > 0$, there is some $\tau > 0$ such that (see Lemma 2.2):

$$\left\|(\tau I - \Delta)^{\frac{m}{2}} u\right\|_2^2 \leq \sum_{j=0}^{m} a_{m-j} \int_{\mathbb{R}^n} \left|\nabla^j u\right|^2 dx.$$

Consequently, in the special case: $n = 2m$ and m is an arbitrary positive integer, using our Theorem 5.1, we can prove the following stronger result which is another main theorem of this chapter. Namely, we will replace the norm $\left\|(\tau I - \Delta)^{\frac{m}{2}} u\right\|_{\frac{n}{m}}$ by the standard Sobolev norm in the above Theorem 5.1 in the case $n = 2m$ for all positive integer m.

Theorem 5.2 *Let m be a positive integer number and $0 \leq \alpha < 2m$. For all constants $a_0 = 1, a_1, ..., a_m > 0$, there holds*

$$\sup_{u \in W^{m,2}(\mathbb{R}^{2m}), \int_{\mathbb{R}^{2m}} \left(\sum_{j=0}^{m} a_{m-j} |\nabla^j u|^2\right) dx \leq 1} \int_{\mathbb{R}^{2m}} \frac{\left[\exp\left(\left(1 - \frac{\alpha}{2m}\right)\beta_0\left(2m,m\right)|u|^2\right) - 1\right]}{|x|^{\alpha}} dx < \infty.$$

Furthermore this inequality is sharp, i.e., if $\left(1-\frac{\alpha}{2m}\right)\beta_0(2m,m)$ *is replaced by any* $\beta > \left(1-\frac{\alpha}{2m}\right)\beta_0(2m,m)$, *then the supremum is infinite.*

As a consequence of Theorem 5.2, we can conclude the following sharp Adams inequality with the standard Sobolev norms.

Theorem 5.3 *Let* $m \geq 1$ *be a positive integer number. There holds*

$$\sup_{u\in W^{m,2}(\mathbb{R}^{2m}),\|u\|_{W^{m,2}}\leq 1}\int_{\mathbb{R}^{2m}}\frac{\left[\exp\left(\left(1-\frac{\alpha}{2m}\right)\beta_0\left(2m,m\right)|u|^2\right)-1\right]}{|x|^\alpha}dx<\infty.$$

Furthermore this inequality is sharp, i.e., if $\beta_0(2m,m)$ *is replaced by any* $\beta > \beta_0(2m,m)$, *then the supremum is infinite.*

We are ready to make some comments about the Adams type inequalities. First, let us outline the existing proof of sharp Moser-Trudinger type inequalities in the entire Euclidean space $\mathbb{R}^n$:

Step 1: Using standard symmetrization arguments, we can reduce the problem to radial case.

Step 2: We now break the integral into two parts:

$$\int_{\mathbb{R}^n}\frac{\phi\left(\beta\,|u|^{\frac{n}{n-1}}\right)}{|x|^\alpha}dx=\int_{B_{R_0}}\frac{\phi\left(\beta\,|u|^{\frac{n}{n-1}}\right)}{|x|^\alpha}dx+\int_{\mathbb{R}^n\setminus B_{R_0}}\frac{\phi\left(\beta\,|u|^{\frac{n}{n-1}}\right)}{|x|^\alpha}dx$$

where B_{R_0} is a ball centered at the origin and with radius R_0 and R_0 can be chosen sufficiently large such that the second term $\int_{\mathbb{R}^n\setminus B_{R_0}}\frac{\phi\left(\beta|u|^{\frac{n}{n-1}}\right)}{|x|^\alpha}dx$ can be handled easily by Radial Lemmas. With the integral on the finite volume ball B_{R_0}, we will use the Moser-Trudinger inequality on the finite domain and symmetrization. Our concern here is that u is not in the Sobolev space $W_0^{1,n}(B_{R_0})$ and thus, we need to modify our function in order to use the classical Moser-Trudinger inequality. In fact, if we set $v(|x|) = u(|x|) - u(R_0)$, then v is now in $W_0^{1,n}(B_{R_0})$ that enables us to use the Moser-Trudinger inequality. Indeed, thanks to the perturbation term $\tau \int_{\mathbb{R}^n} |u|^n \, dx$ in the norm we choose, we now are able to use the Moser-Trudinger inequality on finite domains to estimate the first integral. It can be noted that symmetrization argument plays an important role in this approach.

As far as the Adams inequality in high order Sobolev space $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ is concerned, when Ω has infinite volume, Kozono et al. [46] could find a constant $\beta^*_{n,m} \leq \beta(n,\ m)$, with $\beta^*_{2m,m} = \beta(2m,\ m)$, such that if $\beta < \beta^*_{n,m}$ then

$$\sup_{u \in W^{m,\frac{n}{m}}(\mathbb{R}^n), \|u\|_{m,n} \leq 1} \int_\Omega \phi\left(\beta |u|^{\frac{n}{n-m}}\right) dx \leq C_{n,m,\beta}$$

where $C_{n,m,\beta} > 0$ is a constant depending on $\beta,\ n$ and m, while if $\beta > \beta(n,\ m)$, the supremum is infinite. Here they used the norm

$$\|u\|_{m,n} = \left\|(I - \Delta)^{\frac{m}{2}} u\right\|_{\frac{n}{m}}$$

which is equivalent to the Sobolev norm

$$\|u\|_{W^{m,\frac{n}{m}}} = \left(\|u\|_{\frac{n}{m}}^{\frac{n}{m}} + \sum_{j=1}^{m} \left\|\nabla^j u\right\|_{\frac{n}{m}}^{\frac{n}{m}}\right)^{\frac{m}{n}}.$$

In particular, if $u \in W_0^{m,\frac{n}{m}}(\Omega)$ or $u \in W^{m,\frac{n}{m}}(\mathbb{R}^n)$, then $\|u\|_{W^{m,\frac{n}{m}}} \leq \|u\|_{m,n}$.

To do this, they followed main steps similar to those of Adams. Namely, using the Bessel potentials instead of Riesz potentials, they apply O'Neil's result on the rearrangement of convolution functions and use techniques of symmetric decreasing rearrangements. However, with this approach, the critical case $\beta = \beta(n,\ m)$ cannot be established. Recently, Ruf and Sani studied the Adams type inequality for **higher derivatives of even orders** when Ω has infinite volume in this critical case. Indeed, they proved the following Adams type inequality (see [86]):

Theorem 5.4 *Let m be an even integer less than n. There exists a constant $C_{m,n} > 0$ such that for any domain $\Omega \subseteq \mathbb{R}^n$*

$$\sup_{u \in W_0^{m,\frac{n}{m}}(\Omega), \|u\|_{m,n} \leq 1} \int_\Omega \phi\left(\beta_0\left(n,m\right)|u|^{\frac{n}{n-m}}\right) dx \leq C_{m,n}$$

where

$$\beta_0\left(n,m\right) = \frac{n}{\omega_{n-1}}\left[\frac{\pi^{\frac{n}{2}} 2^m \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)}\right]^{\frac{n}{n-m}}.$$

This inequality is sharp in the sense that if we replace $\beta_0\left(n,m\right)$ by any $\beta > \beta_0\left(n,m\right)$, then the supremum is infinite.

The method of Ruf and Sani in [86] to prove the above Theorem is similar to the way we prove the first order Moser-Trudinger type inequalities in the whole space as we just outlined above. Namely, they also break the whole space into a ball and its exterior. Hence, again, radial function plays a crucial role in this approach. However, it's well-known that symmetrization does not hold in general for the higher order operators. To overcome this difficulty, they carry out the following steps:

Step 1: They must establish a comparison principle for the polyharmonic operator. More precisely, they prove that if u is a weak solution of

$$\begin{cases} (-\Delta + I)^k u = f \text{ in } B_R \\ u \in W_N^{2k,2}(B_R) \end{cases}$$

where $f \in L^{\frac{2n}{n+2}}(B_R)$ and v is a weak solution of

$$\begin{cases} (-\Delta + I)^k v = f^{\#} \text{ in } B_R \\ v \in W_N^{2k,2}(B_R) \end{cases}$$

where $f^{\#}$ is the Schwarz rearrangement of f, then for every $r \in (0, R)$ we have

$$\int_{B_r} u^{\#} dx \leq \int_{B_r} v dx.$$

As a consequence, for every convex nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ we have

$$\int_{B_r} \phi(|u|)\, dx \leq \int_{B_r} \phi(|v|)\, dx.$$

In order to do this, they use an induction argument and apply some comparison principles of the second order elliptic equation in [90] and [92]. Therefore, we can reduce the problem to the radial case.

Step 2: This step is very similar to the way as the one we did for the Moser-Trudinger inequality.

However, it can be noted that the argument in Step 1 just hold for the case when m is an even number in [86]. Thus, the work of Ruf and Sani raised a good open question: **Does**

Theorem 5.4 hold when m is odd? In fact, Lam and Lu answered and extended partly this question in the recent papers [52, 53]. More precisely, it was proved that

Theorem 5.5 *Let m be an odd integer less than n: $m = 2k+1,\ k \in \mathbb{N}$. There holds*

$$\sup_{u\in W^{m,\frac{n}{m}}(\mathbb{R}^n),\left\|\nabla(-\Delta+I)^k u\right\|_{\frac{n}{m}}^{\frac{n}{m}}+\left\|(-\Delta+I)^k u\right\|_{\frac{n}{m}}^{\frac{n}{m}}\leq 1}\int_{\mathbb{R}^n}\phi\left(\beta\left(n,m\right)|u|^{\frac{n}{n-m}}\right)dx<\infty.$$

Moreover, the constant $\beta(n,m)$ is sharp in the sense that if we replace $\beta(n,m)$ by any $\beta > \beta(n,m)$, then the supremun is infinite.

Theorem 5.6 *Let $0 \leq \alpha < n$, $m > 0$ be an even integer less than n. Then for all $0 \leq \beta \leq \beta_{\alpha,n,m} = \left(1-\frac{\alpha}{n}\right)\beta_0(n,m)$, we have*

$$\sup_{u\in W^{m,\frac{n}{m}}(\mathbb{R}^n),\left\|(-\Delta+I)^{\frac{m}{2}}u\right\|_{\frac{n}{m}}\leq 1}\int_{\mathbb{R}^n}\frac{\phi\left(\beta|u|^{\frac{n}{n-m}}\right)dx}{|x|^{\alpha}}dx<\infty$$

where $\phi(t) = e^t - \sum_{j=0}^{j\frac{n}{m}-2}\frac{t^j}{j!}$. Moreover, $\beta(n,m)$ is sharp in the sense that if $\beta > \beta_{\alpha,n,m}$, the supremum is infinite.

In order to prove Theorems 5.5 and 5.6, we used the same techniques as of Ruf and Sani. Namely, we also established the comparison principle for the polyharmonic operator

$$\begin{cases}(-\Delta+I)^k u = f \text{ in } B_R\\ u\in W_N^{2k,2}(B_R)\end{cases}$$

and

$$\begin{cases}(-\Delta+I)^k v = f^{\#} \text{ in } B_R\\ v\in W_N^{2k,2}(B_R)\end{cases}$$

and get that for every $r \in (0, R)$ we have *

$$\int_{B_r} u^{\#} dx \leq \int_{B_r} v dx.$$

Now, noting that

$$\left\| \nabla \left(-\Delta + I\right)^k u \right\|_{\frac{n}{m}} = \left\| \nabla f \right\|_{\frac{n}{m}} \geq \left\| \nabla f^{\#} \right\|_{\frac{n}{m}} = \left\| \nabla \left(-\Delta + I\right)^k v \right\|_{\frac{n}{m}},$$

again we can reduce the problem to the radial case which is much easier to deal with and then Theorem 5.5 follows. This has been carried out in [53]. However, this method only works for the very restricted norm

$$\left\| \nabla \left(-\Delta + I\right)^k u \right\|_{\frac{n}{m}}^{\frac{n}{m}} + \left\| \left(-\Delta + I\right)^k u \right\|_{\frac{n}{m}}^{\frac{n}{m}}$$

but does not work for other norms such as $\left\| (\tau I - \Delta)^{\frac{m}{2}} u \right\|_{\frac{n}{m}}$ with arbitrary positive integer number m. Therefore, this motivates us to discover a new method of proving Theorem E when we have a standard norm $\left\| (\tau I - \Delta)^{\frac{m}{2}} u \right\|_{\frac{n}{m}}$.

To prove Theorem 5.6, again we used the comparison principle to reduce our problem to radial case. Then using the ideas of Adams, we can set up a singular Adams inequality for bounded domains. Finally, following the ideas in proving Moser-Trudinger type inequalities, we split the whole space to the interior and the exterior of a ball and thanks to Radial Lemmas, we can derive our Theorem 5.6. However, this method only works when m is even. Again, this motivates us to develop a new method to establish Theorem 5.6 for arbitrary integer m.

*This comparison inequality for solutions to polyharmonic operators $(-\Delta + I)^{\alpha} u = f$ in B_R for any positive α has also been established by a different and much simpler way using the Bessel potentials and Riesz rearrangement inequality in [52].

It can be noted that when m is even, our main result Theorem 5.1 recovers Theorem 5.6. Moreover, Theorem 5.1 is still available in the case that m is odd. It is also very interesting to note that when $\alpha = 0$, our best constant $\beta_0(n,m)$ is different than the best constant $\beta(n,m)$ in Theorem 5.5. This marks a substantial difference in sharp constants for the Adams inequality when we use a different norm here.

Now, let us describe our method in the proof of Theorem 5.1. First, using an idea of Adams [2], we write our functions in terms of the Bessel potentials and then apply O'Neil's result on the rearrangement of convolution functions to set up the following singular Adams type inequality for arbitrary bounded domain:

Theorem 5.7 *Let m be an integer less than n, $\tau > 0$, $0 \leq \alpha < n$. There holds*

$$\sup_{\Omega\subset\mathbb{R}^n;\ |\Omega|<\infty}\ \sup_{u\in W^{m,\frac{n}{m}}(\mathbb{R}^n),\left\|(\tau I-\Delta)^{\frac{m}{2}}u\right\|_{\frac{n}{m}}\leq 1}\frac{1}{\max\left(1,|\Omega|^{1-\frac{\alpha}{n}}\right)}\int_{\Omega}\frac{\exp\left(\beta_{\alpha,n,m}|u|^{\frac{n}{n-m}}\right)}{|x|^{\alpha}}dx<\infty.$$

Moreover, the constant $\beta_{\alpha,n,m}$ is sharp in the sense that if we replace $\beta_{\alpha,n,m}$ by any $\beta > \beta_{\alpha,n,m}$, then the supremum is infinity.

With the help of Theorem 5.7, we will find a "good" way to split our domain: instead of breaking the whole Euclidean space into the interior and exterior of a ball, we will divide the whole space into 2 domains: a domain on which our function is small enough and the remainder. Our key observation is that the volume of this remainder must be finite. Thus, we can use the result of Theorem 5.7. It is crucial that our approach does not require us to reduce to the radial case. Hence, our approach can be applied to other non-Euclidean settings where the symmetrization does not hold.

Next, still using this new method, we can prove a strengthened version of sharp Adams inequality in the second order Sobolev spaces $W^{2,m}(\mathbb{R}^{2m})$.

Theorem 5.8 *Let $0 \leq \alpha < 2m$ and $\tau > 0$. Then for all $0 \leq \beta \leq \left(1 - \frac{\alpha}{2m}\right)\beta(2m, 2)$, we have*

$$\sup_{u \in W^{2,m}(\mathbb{R}^{2m}),\ \int_{\mathbb{R}^{2m}} |\Delta u|^m + \tau |u|^m \leq 1} \int_{\mathbb{R}^{2m}} \frac{\phi\left(\beta |u|^{\frac{m}{m-1}}\right)}{|x|^{\alpha}} dx < \infty$$

where

$$\phi(t) = e^t - \sum_{j=0}^{m-2} \frac{t^j}{j!}.$$

Moreover, the constant $\left(1 - \frac{\alpha}{2m}\right)\beta(2m, 2)$ is sharp in the sense that if $\beta > \left(1 - \frac{\alpha}{2m}\right)\beta(2m, 2)$, then the supremun is infinite.

We should note that our Theorem 5.8 does not require the restriction on the full standard norm and hence, our Theorem 5.8 extends the results in [53, 95]. Indeed, the results there are for the special case $m = 2$ and they require that the full standard norm $\int_{\mathbb{R}^4} \left(|\Delta u|^2 + \sigma |\nabla u|^2 + \tau |u|^2\right) dx$ is less than 1.

Finally, as our last result in this chapter, we will set up some Adams type inequalities on Sobolev spaces $W^{\gamma, \frac{n}{\gamma}}(\mathbb{R}^n)$ of arbitrary positive fractional order $\gamma < n$. More precisely, we will prove that:

Theorem 5.9 *Let $0 < \gamma < n$ be an arbitrary real positive number, $p = \frac{n}{\gamma}$ and $\tau > 0$. Then for every domain Ω with finite $n-$measure in Euclidean $n-$space $\mathbb{R}^n$, there exists $C = C(\gamma, n, \tau) > 0$ such that*

$$\sup_{u \in W^{\gamma,p}(\mathbb{R}^n), \left\|(\tau I - \Delta)^{\frac{\gamma}{2}} u\right\|_p \leq 1} \frac{1}{|\Omega|} \int_{\Omega} \exp\left(\beta_0(n,\gamma) |u|^{p'}\right) dx < C.$$

Here

$$p' = \frac{p}{p-1},$$
$$\beta_0\left(n,\gamma\right) = \frac{n}{\omega_{n-1}}\left[\frac{\pi^{n/2}2^{\gamma}\Gamma\left(\gamma/2\right)}{\Gamma\left(\frac{n-\gamma}{2}\right)}\right]^{p'}.$$

Furthermore this inequality is sharp, i.e., if $\beta_0\left(n,\gamma\right)$ *is replaced by any* $\beta > \beta_0\left(n,\gamma\right)$*, then the supremum is infinite.*

Theorem 5.10 *Let* $0 < \gamma < n$ *be an arbitrary real positive number,* $p = \frac{n}{\gamma}$ *and* $\tau > 0$*. There holds*

$$\sup_{u\in W^{\gamma,p}(\mathbb{R}^n),\left\|(\tau I-\Delta)^{\frac{\gamma}{2}}u\right\|_p\leq 1}\int_{\mathbb{R}^n}\phi\left(\beta_0\left(n,\gamma\right)|u|^{p'}\right)dx<\infty$$

where

$$\phi(t) = e^t - \sum_{j=0}^{j_p-2}\frac{t^j}{j!},$$
$$j_p = \min\left\{j\in\mathbb{N} : j\geq p\right\}\geq p.$$

Furthermore this inequality is sharp, i.e., if $\beta_0\left(n,\gamma\right)$ *is replaced by any* $\beta > \beta_0\left(n,\gamma\right)$*, then the supremum is infinite.*

Theorems 5.9 and 5.10 are extensions of Theorems 5.7 and 5.1 to fractional Sobolev spaces respectively. Their proofs are very similar to those of Theorems 5.7 and 5.1 and will only be sketched.

5.2 Proof of Theorem 5.7

The main purpose of this section is to establish the sharp local singular Adams inequality on domains Ω in $\mathbb{R}^n$ of finite measure. We take a perspective that any function in the high order Sobolev spaces $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ can be represented as a Bessel potential. Thus, we can fully use the tools from harmonic analysis and the kernel properties of the polyharmonic operators $(\tau I-\Delta)^{\frac{m}{2}}$. As a result, we will avoid using the deep comparison principle of Talenti [90] and Trombett-Vasquez [92] to establish the Adams inequality on finite domains by as done in [86] and [53, 54].

Once we have established this sharp local Adams inequality, then we can adapt the splitting method we will develop in this chapter to derive a global sharp Adams inequality from a local one.

Proof. Since $u\in W^{m,\frac{n}{m}}(\mathbb{R}^n)$, we can write u as a potential $L_{\tau,m}*f$, $f\in L^p(\mathbb{R}^n)$, $p=\frac{n}{m}$, $p'=\frac{n}{n-m}$. Then, we have $\|f\|_{L^p(\mathbb{R}^n)}\leq 1$ since $\left\|(\tau I-\Delta)^{\frac{m}{2}}u\right\|_p\leq 1$.

By O'Neil's Lemma, we have for all $t>0$:

$$u^*(t)\leq \frac{1}{t}\int\limits_0^t f^*(s)ds\int\limits_0^t L^*_{\tau,m}(s)ds+\int\limits_t^\infty f^*(s)L^*_{\tau,m}(s)ds.$$

Here, we notice that $L^*_{\tau,m}(s)=L_{\tau,m}(\sigma_n^{-1/n}s^{1/n})$.

Next, we change the variables

$$\begin{aligned}\phi(t)&=|\Omega|^{1/p}e^{-\frac{t}{p}}f^*(|\Omega|e^{-t})\\ \psi(t)&=\frac{1}{M}|\Omega|^{1/p'}e^{-\frac{t}{p'}}L^*_{\tau,m}(|\Omega|e^{-t})\\ M&=\left(\frac{1}{\beta_0(n,m)}\right)^{p'}.\end{aligned}$$

After some calculation, the Hardy-Littlewood inequality and the fact that with $h(x) = \frac{1}{|x|^\alpha}$, then $h^*(t) = \left(\frac{\sigma_n}{t}\right)^{\frac{\alpha}{n}}$, we have:

$$\int_\Omega \frac{\exp\left(\beta_{\alpha,n,m}\,|u|^{p'}\right)}{|x|^\alpha}dx \leq \sigma_n^{\frac{\alpha}{n}} \int_0^{|\Omega|} \frac{\exp\left(\beta_{\alpha,n,m}\,|u^*(t)|^{p'}\right)}{|x|^\alpha}dt$$
$$\leq C(\alpha)\,\frac{1}{|\Omega|^{1-\alpha}} \int_0^\infty e^{-F_{\left(1-\frac{\alpha}{n}\right)}(t)}dt$$

where

$$F_{\left(1-\frac{\alpha}{n}\right)}(t) = \left(1-\frac{\alpha}{n}\right)t - \left(1-\frac{\alpha}{n}\right)\left[e^t \int_t^\infty e^{-\frac{s}{p'}}\phi(s)ds \int_t^\infty e^{-\frac{s}{p}}\psi(s)ds + \int_{-\infty}^t \phi(s)\psi(s)ds\right]^{p'}.$$

Using Lemma 2.1, it is easy to check that

$$\int_{-\infty}^\infty \phi^p(s)ds \leq 1$$

$$\sup_{s>0}\psi(s) \leq 1$$

$$\int_{-\infty}^0 \psi^{p'}(s)ds < \infty.$$

We notice that the second inequality comes from (2.7).

Thus, Theorem 1.4 is just a consequence of Lemma 2.12 with

$$a(s,t) = \begin{cases} \psi(s), \;\; s < t \\ e^t\left(\int_t^\infty e^{-\frac{s}{p}}\psi(s)ds\right)e^{-\frac{s}{p'}}, \;\; s > t \end{cases}.$$

To prove that the constant $\beta_{\alpha,n,m}$ is sharp, we use the approach as in [2]. We set

$$L_{\tau,m,p}(E) = \inf \|f\|_p^p$$

where the infimum is taken over all $f \geq 0$ in L^p such that $L_{\tau,m} * f(x) \geq 1$, for all $x \in E$. Similarly as in [2], it's enough to check that

$$L_{\tau,m,\frac{n}{m}}(B(0,r)) \leq \omega_{n-1}^{-1}(2\pi)^n (\log 1/r)^{-1} \text{ as } r \to 0.$$

Again, this follows from the inequality (2.7). □

5.3 Proof of Theorem 5.1, Theorem 5.2 and Theorem 5.3

In this section, we will develop a new approach to prove the sharp Adams inequality in high order Sobolev spaces $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ for arbitrary integer order m. The main novelty is to carry out an argument to derive the global sharp Adams inequality in the entire space $\mathbb{R}^n$ from the local one on domains of finite measure. This method will allow us to avoid using the symmetrization which is not available in the high order Sobolev spaces $W^{m,\frac{n}{m}}(\mathbb{R}^n)$. In [86], the authors use the Talenti [90] and Trombetti-Vázquez [92] comparison principle for the polyharmonic operators $(I-\Delta)^{\frac{m}{2}}$ to reduce the functions to radial ones. Then they use rather involved construction of auxiliary radial functions to conclude the sharp Adams inequality in $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ when m is even. In [53, 54], we extend the ideas of [86] to include all odd integers. However, when $m = 2k+1$ is odd this method only works for the very restricted norm

$$\left\|\nabla(-\Delta+I)^k u\right\|_{\frac{n}{m}}^{\frac{n}{m}} + \left\|(-\Delta+I)^k u\right\|_{\frac{n}{m}}^{\frac{n}{m}}$$

but does not work for other norms such as $\left\|(\tau I-\Delta)^{\frac{m}{2}}u\right\|_{\frac{n}{m}}$ with arbitrary positive integer number m.

In this chapter, we will employ a completely different and surprisingly simple method to establish the sharp Adams inequality in $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ for arbitrary integer m under the norm $\left\|(\tau I-\Delta)^{\frac{m}{2}}u\right\|_{\frac{n}{m}}$.

Now, we are ready to prove Theorem 1.1

Proof. For any $u\in W^{m,p}(\mathbb{R}^n)\setminus\{0\}$, $\left\|(\tau I-\Delta)^{\frac{m}{2}}u\right\|_p\le 1$, we write

$$\int_{\mathbb{R}^n}\frac{\phi\left(\beta_{\alpha,n,m}|u|^{p'}\right)}{|x|^\alpha}dx=\int_{\Omega(u)}\frac{\phi\left(\beta_{\alpha,n,m}|u|^{p'}\right)}{|x|^\alpha}dx+\int_{\mathbb{R}^n\setminus\Omega(u)}\frac{\phi\left(\beta_{\alpha,n,m}|u|^{p'}\right)}{|x|^\alpha}dx$$

where

$$\Omega(u)=\{x:|u(x)|\ge 1\}.$$

We have

$$\begin{aligned}|\Omega(u)|&=\int_{\Omega(u)}1dx\\&\le\int_{\mathbb{R}^n}|u(x)|^p\\&\le A^{-p}\end{aligned}$$

where

$$A=\inf_{u\in W^{m,p}(\mathbb{R}^n)\setminus\{0\}}\frac{\left\|(\tau I-\Delta)^{\frac{m}{2}}u\right\|_p}{\|u\|_p}>0.$$

Hence, by Theorem 1.4, we get

$$\int_{\Omega(u)} \frac{\phi\left(\beta_{\alpha,n,m}|u|^{p'}\right)}{|x|^{\alpha}}dx \leq C\,|\Omega(u)|^{1-\frac{\alpha}{n}}$$
$$\leq CA^{-p\left(1-\frac{\alpha}{n}\right)}.$$

for some universal constant $C > 0$.

Now, noting that on the domain $\mathbb{R}^n \setminus \Omega(u)$, we have $|u(x)| < 1$. Thus

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus\Omega(u)} \frac{\phi\left(\beta_{\alpha,n,m}|u|^{p'}\right)}{|x|^{\alpha}}dx \\
=&\int_{\{|u(x)|<1;\ |x|<1\}} \frac{\phi\left(\beta_{\alpha,n,m}|u|^{p'}\right)}{|x|^{\alpha}}dx + \int_{\{|u(x)|<1;\ |x|>1\}} \frac{\phi\left(\beta_{\alpha,n,m}|u|^{p'}\right)}{|x|^{\alpha}}dx \\
\leq& C_1 \int_{\{|x|<1\}} \frac{1}{|x|^{\alpha}}dx + \int_{\{|u(x)|<1\}} \phi\left(\beta_{\alpha,n,m}|u|^{p'}\right)dx \\
\leq& C_2
\end{aligned}$$

for some universal constants C_1 and C_2.

When $\beta > \beta_{\alpha,n,m}$, $\alpha = 0$ and $\tau = 1$, using Bessel potential, Kozono, Sato and Wadade in [46] showed that the supremum is infinite. Moreover, when m is even, Ruf and Sani can exhibit a sequence of test functions that made the integral arbitrarily large. See Proposition 6.2 in [86]. In our case, we can use the similar method as in [46]. Indeed, using $L_{\tau,m,p}$ instead of Bessel potential as in the proof of Theorem 1.4, we can again show that our integral can be made arbitrarily large when $\beta > \beta_{\alpha,n,m}$. □

Proof of Theorem 5.2 and Theorem 5.3. Choosing $\tau > 0$ as in Lemma 2.2, we have

$$\begin{aligned}
&\sup_{u\in W^{m,2}(\mathbb{R}^{2m}),\int_{\mathbb{R}^{2m}}\left(\sum_{j=0}^{m}a_{m-j}|\nabla^j u|^2\right)dx\leq 1}\int_{\mathbb{R}^{2m}}\frac{\left[\exp\left(\left(1-\frac{\alpha}{2m}\right)\beta_0\left(2m,m\right)|u|^2\right)-1\right]}{|x|^{\alpha}}dx\\
&\leq \sup_{u\in W^{m,2}(\mathbb{R}^{2m}),\left\|(\tau I-\Delta)^{\frac{m}{2}}u\right\|_2\leq 1}\int_{\mathbb{R}^{2m}}\frac{\left[\exp\left(\left(1-\frac{\alpha}{2m}\right)\beta_0\left(2m,m\right)|u|^2\right)-1\right]}{|x|^{\alpha}}dx\\
&< \infty.
\end{aligned}$$

Now, if we choose $a_0 = a_1 = ... = a_m = 1$, we can derive Theorem 1.3.

5.4 Proof of Theorem 5.8

In this section, we deal with the sharp Adams inequality in the special case of second order Sobolev spaces $W^{2,m}(\mathbb{R}^{2m})$. The main novelty of this result is that we use a much less restricted norm $\int_{\mathbb{R}^{2m}} |\Delta u|^m + \tau |u|^m$. This norm is smaller than the standard norm $\|(\tau I - \Delta) u\|_m$.

Proof of Theorem 5.8: For any $u \in W^{2,m}\left(\mathbb{R}^{2m}\right)$, $\int_{\mathbb{R}^{2m}} |\Delta u|^m + \tau |u|^m \leq 1$, we set

$$A(u) = 2^{-\frac{1}{m(m-1)}}\tau^{\frac{1}{m}}\|u\|_m$$

$$\Omega(u) = \left\{x \in \mathbb{R}^{2m} : u(x) > A(u)\right\}.$$

Then, we have

$$A(u) < 1 \tag{5.1}$$

and

$$|\Omega(u)| \leq 2^{\frac{1}{(m-1)}}\frac{1}{\tau} \tag{5.2}$$

since

$$\begin{aligned}\int_{\mathbb{R}^{2m}} |u|^m &\geq \int_{\Omega(u)} |u|^m \\ &\geq \int_{\Omega(u)} |A(u)|^m \\ &= 2^{-\frac{1}{(m-1)}} \tau \left\|u\right\|_m^m |\Omega(u)| \,.\end{aligned}$$

Now, we write

$$\int_{\mathbb{R}^{2m}} \frac{\phi\left(\beta |u|^{\frac{m}{m-1}}\right)}{|x|^\alpha} = I_1 + I_2$$

where

$$I_1 = \int_{\Omega(u)} \frac{\phi\left(\beta |u|^{\frac{m}{m-1}}\right)}{|x|^\alpha}$$

and

$$I_2 = \int_{\mathbb{R}^{2m}\setminus\Omega(u)} \frac{\phi\left(\beta |u|^{\frac{m}{m-1}}\right)}{|x|^\alpha}.$$

We will prove that both I_1 and I_2 are bounded by a constant $C = C(\alpha, \tau, m)$.

Indeed, from (5.1), it easy to see that

$$\begin{aligned}I_2 &\leq \int_{\{u(x)<1\}} \frac{1}{|x|^\alpha} \sum_{k=m-1}^{\infty} \frac{\beta^k}{k!} |u|^{km/(m-1)} \\ &\leq \int_{\{u(x)<1\}} \frac{1}{|x|^\alpha} \sum_{k=m-1}^{\infty} \frac{\beta^k}{k!} |u|^m \\ &\leq \int_{\{|x|\geq 1\}} \sum_{k=m-1}^{\infty} \frac{\beta^k}{k!} |u|^m \\ &+ \int_{\{|x|<1\}} \frac{1}{|x|^\alpha} \sum_{k=m-1}^{\infty} \frac{\beta^k}{k!} \\ &\leq C(\alpha, \tau, m) \,.\end{aligned}$$

Now, to estimate I_1, we first notice that if we set

$$v(x) = u(x) - A(u) \text{ in } \Omega(u),$$

then $v \in W_N^{2,m}(\Omega\,(u))$. Moreover, in $\Omega(u)$:

$$\begin{aligned}
|u|^{\frac{m}{m-1}} &= (|v| + A(u))^{\frac{m}{m-1}} \\
&\leq |v|^{\frac{m}{m-1}} + \frac{m}{m-1} 2^{\frac{1}{m-1}} \left(|v|^{\frac{1}{m-1}} A(u) + |A(u)|^{\frac{m}{m-1}} \right) \\
&\leq |v|^{\frac{m}{m-1}} + \frac{m}{m-1} 2^{\frac{1}{m-1}} \frac{|v|^{\frac{m}{m-1}} |A(u)|^m}{m} + \frac{m}{m-1} 2^{\frac{1}{m-1}} \left(\frac{m-1}{m} + |A(u)|^{\frac{m}{m-1}} \right) \\
&\leq |v|^{\frac{m}{m-1}} \left(1 + \frac{2^{\frac{1}{m-1}}}{m-1} |A(u)|^m \right) + C(m)
\end{aligned}$$

where we did use Young's inequality and the following elementary inequality:

$$(a+b)^q \leq a^q + q2^{q-1} \left(a^{q-1} b + b^q \right) \text{ for all } q \geq 1 \text{ and } a,\ b \geq 0.$$

Let

$$w(x) = \left(1 + \frac{2^{\frac{1}{m-1}}}{m-1} |A(u)|^m \right)^{\frac{m-1}{m}} v(x) \text{ in } \Omega(u),$$

then it's clear that

$$w \in W_N^{2,m}(\Omega) \text{ and } |u|^{\frac{m}{m-1}} \leq |w|^{\frac{m}{m-1}} + C(m). \tag{5.3}$$

Moreover, we have

$$\Delta w = \left(1 + \frac{2^{\frac{1}{m-1}}}{m-1} |A(u)|^m \right)^{\frac{m-1}{m}} \Delta v.$$

Thus

$$\begin{aligned}\int_{\Omega(u)} |\Delta w|^m &= \left(1+\frac{2^{\frac{1}{m-1}}}{m-1}|A(u)|^m\right)^{m-1}\int_{\Omega(u)}|\Delta v|^m \\ &= \left(1+\frac{2^{\frac{1}{m-1}}}{m-1}|A(u)|^m\right)^{m-1}\int_{\Omega(u)}|\Delta u|^m \\ &\leq \left(1+\frac{2^{\frac{1}{m-1}}}{m-1}|A(u)|^m\right)^{m-1}\left[1-\tau\int_{\mathbb{R}^{2m}}|u|^m\right]\end{aligned}$$

Then

$$\begin{aligned}\left(\int_{\Omega(u)} |\Delta w|^m\right)^{\frac{1}{m-1}} &= \left(1+\frac{2^{\frac{1}{m-1}}}{m-1}|A(u)|^m\right)\left[1-\tau\int_{\mathbb{R}^{2m}}|u|^m\right]^{\frac{1}{m-1}} \\ &\leq \left(1+\frac{2^{\frac{1}{m-1}}}{m-1}|A(u)|^m\right)\left(1-\frac{\tau}{m-1}\int_{\mathbb{R}^{2m}}|u|^m\right) \\ &= \left(1+\frac{2^{\frac{1}{m-1}}}{m-1}2^{-\frac{1}{(m-1)}}\tau\|u\|_m^m\right)\left(1-\frac{\tau}{m-1}\int_{\mathbb{R}^{2m}}|u|^m\right) \\ &= \left(1+\frac{\tau}{m-1}\int_{\mathbb{R}^{2m}}|u|^m\right)\left(1-\frac{\tau}{m-1}\int_{\mathbb{R}^{2m}}|u|^m\right) \\ &\leq 1 \qquad (5.4)\end{aligned}$$

Here, we used the inequality

$$(1-x)^q \leq 1-qx \text{ for all } 0 \leq x \leq 1,\ 0 < q \leq 1.$$

From (5.3) and (5.4), using Theorem 1.6 and (5.2), we get

$$\begin{aligned} I_1 &\leq \int_{\Omega(u)} \frac{\exp\left(\beta\,|u|^{m/(m-1)}\right)}{|x|^{\alpha}} \\ &\leq e^{\beta C(m)} \int_{\Omega(u)} \frac{\exp\left(\beta\,|w|^{m/(m-1)}\right)}{|x|^{\alpha}} \\ &\leq e^{\beta C(m)} C\left(\alpha, \tau, m\right) \left|\Omega\left(u\right)\right|^{1-\frac{\alpha}{2m}} \\ &\leq C\left(\alpha, \tau, m\right). \end{aligned}$$

The proof now is completed.

5.5 Sharp Adams inequalities for fractional order Sobolev spaces $W^{\gamma,\frac{n}{\gamma}}(\mathbb{R}^n))$

In this section, we will give proofs of Theorems 5.9 and 5.10. These are results concerning sharp Adams inequalities on Sobolev spaces $W^{\gamma,\frac{n}{\gamma}(\mathbb{R}^n)}$ of arbitrary fractional order $0 < \gamma < n$.

5.5.1 Proof of Theorem 5.9

Since $u \in W^{\gamma,p}\left(\mathbb{R}^n\right)$, we first write u as a convolution $L_{\tau,\gamma} * f$, $f \in L^p\left(\mathbb{R}^n\right)$. Since $\left\|(\tau I - \Delta)^{\frac{\gamma}{2}}\, u\right\|_p \leq 1$, we have $\|f\|_{L^p(\mathbb{R}^n)} \leq 1$.

Now, applying O'Neil's lemma, we have for all $t > 0$:

$$u^*(t) \leq \frac{1}{t}\int_0^t f^*(s)ds \int_0^t L^*_{\tau,\gamma}(s)ds + \int_t^{\infty} f^*(s)L^*_{\tau,\gamma}(s)ds.$$

Here $L^*_{\tau,\gamma}(s) = L_{\tau,\gamma}(\sigma_n^{-1/n} s^{1/n})$.

Now, we will change the variables (without loss of generality, we may assume that $|\Omega| = 1$):

$$\phi(t) = e^{-\frac{t}{p}} f^*(e^{-t})$$

$$\psi(t) = \frac{1}{M} e^{-\frac{t}{p'}} L^*_{\tau,\gamma}(e^{-t})$$

$$M = \frac{\sigma_n^{1/p'} \Gamma\left(\frac{n-\gamma}{2}\right)}{\pi^{n/2} 2^{\gamma} \Gamma\left(\frac{\gamma}{2}\right)}.$$

By direct calculation, we have

$$\frac{1}{|\Omega|} \int_{\Omega} \exp\left(\beta_0\left(n,\gamma\right)|u|^{p'}\right) dx = \int_0^1 \exp\left(\beta_0\left(n,\gamma\right)|u^*(t)|^{p'}\right) dt \leq \int_0^{\infty} e^{-F(t)} dt$$

where

$$F(t) = t - \left[e^t \int_t^{\infty} e^{-\frac{s}{p'}} \phi(s) ds \int_t^{\infty} e^{-\frac{s}{p}} \psi(s) ds + \int_{-\infty}^{t} \phi(s)\psi(s) ds \right]^{p'}.$$

We can also check that

$$\int_{-\infty}^{\infty} \phi^p(s) ds \leq 1$$

$$\sup_{s>0} \psi(s) \leq 1$$

$$\int_{-\infty}^{0} \phi^{p'}(s) ds < \infty.$$

Thus, Theorem 1.1 is just a consequence of Lemma 2.12 with $\alpha = 1$, where

$$a(s,t) = \begin{cases} \psi(s), \ s < t \\ e^{t}\left(\int\limits_{t}^{\infty} e^{-\frac{s}{p}}\psi(s)ds\right)e^{-\frac{s}{p'}}, \ s > t \end{cases}.$$

To show the sharpness of $\beta_0\left(n,\gamma\right)$, we proceed as in [2, 52, 46, 86]. Indeed, we just need to use the $L_{\tau,\gamma}$ potential instead of Bessel potential and the result follows by (2.7).

5.5.2 Proof of Theorem 5.10

We need to prove that

$$\sup_{u\in W^{\gamma,p}(\mathbb{R}^n)\setminus\{0\},\left\|(I-\Delta)^{\frac{\gamma}{2}}u\right\|_p\leq 1}\int_{\mathbb{R}^n}\phi\left(\beta_0\left|u\right|^{p'}\right)dx<\infty$$

where $\beta_0 = \beta_0\left(n,\gamma\right)$.

Indeed, for any $u \in W^{\gamma,p}\left(\mathbb{R}^n\right)\setminus\left\{0\right\}, \ \left\|(\tau I-\Delta)^{\frac{\gamma}{2}}u\right\|_p \leq 1$, we can write

$$\begin{aligned}\int_{\mathbb{R}^n}\phi\left(\beta_0\left|u\right|^{p'}\right)dx &= \int_{\Omega(u)}\phi\left(\beta_0\left|u\right|^{p'}\right)dx+\int_{\mathbb{R}^n\setminus\Omega(u)}\phi\left(\beta_0\left|u\right|^{p'}\right)dx \\ &= I_1+I_2.\end{aligned} \tag{5.5}$$

Here

$$\Omega(u) = \left\{x\in\mathbb{R}^n : \left|u(x)\right|\geq 1\right\}.$$

We notice that since

$$\int_{\mathbb{R}^n}\left|u\right|^p \geq \int_{\Omega(u)}\left|u\right|^p \geq \left|\Omega(u)\right|,$$

we can conclude that there exists a constant $C_1(\gamma, n, \tau)$ such that

$$|\Omega(u)| \leq C_1(\gamma, n, \tau). \tag{5.6}$$

Now, we have

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^n \setminus \Omega(u)} \phi\left(\beta_0 |u|^{p'}\right) dx \\ &\leq \sum_{k=j_p-1}^{\infty} \frac{(\beta_0)^k}{k!} \int_{\{|u(x)|<1\}} |u|^{p'k} \, dx \\ &\leq \sum_{k=j_p-1}^{\infty} \frac{(\beta_0)^k}{k!} \int_{\Omega(u)} |u|^{p} \\ &\leq C_2(\gamma, n, \tau). \end{aligned} \tag{5.7}$$

Moreover, by (5.6) and Theorem 5.9, we get

$$\begin{aligned} I_1 &= \int_{\Omega(u)} \phi\left(\beta_0 |u|^{p'}\right) dx \\ &\leq C_1(\gamma, n, \tau) \frac{1}{|\Omega(u)|} \int_{\Omega(u)} \exp\left(\beta_0 |u|^{p'}\right) dx \\ &\leq C_3(\gamma, n, \tau). \end{aligned} \tag{5.8}$$

From (5.5), (5.7) and (5.8), we get our desired result.

The sharpness of the constant $\beta_0\,(n, \gamma)$ can be verified by the process similar to that in the proof of Theorem 5.9.

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ABSTRACT

MOSER-TRUDINGER AND ADAMS TYPE INEQUALITIES AND THEIR APPLICATIONS

by

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Major: Mathematics

Degree: Doctor of Philosophy

In this dissertation, we study some variants of the Moser-Trudinger inequalities and Adams inequalities. The proofs of these inequalities relied crucially on the symmetrization arguments in the literature. By proposing new arguments and approaches, we develop successfully the critical versions of these well-known inequalities in many different settings where the rearrangement arguments may not be existed. As applications of our results, we also study in this dissertation the elliptic equations that contain the exponential nonlinearities.

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1. Cohn, W. S.; Lam, N.; Lu, G.; Yang, Y.: The Moser-Trudinger inequality in unbounded domains of Heisenberg group and sub-elliptic equations. Nonlinear Anal. **75** (2012), no. 12, 4483-4495.
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