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Saima Naqash
University of Kashmir, Jammu and Kashmir, India, naqashsaima@gmail.com

S. P. Ahmad
University of Kashmir, Jammu and Kashmir, India, sprvz@yahoo.com

Aquil Ahmed
Aligarh Muslim University, Aligarh, UP, Inida, aqlstat@yahoo.co.in

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Bayesian Analysis of Generalized Exponential Distribution

Saima Naqash
University of Kashmir
Jammu and Kashmir, India

S. P. Ahmad
University of Kashmir
Jammu and Kashmir, India

Aquil Ahmed
Aligarh Muslim University
Aligarh, UP, Inida

Bayesian estimators of unknown parameters of a two parameter generalized exponential distribution are obtained based on non-informative priors using different loss functions.

Keywords: Generalized exponential distribution, Bayesian estimators, loss function, R-software

Introduction

One of the simplest and most commonly used distributions (and often erroneously overused due to its simplicity) is the exponential distribution. The two-parameter exponential distribution, which is an extension of the exponential distribution, was first introduced by Gupta and Kundu (1999), and is very popular in analyzing lifetime or survival data. Like Weibull and gamma distributions, the generalized exponential distribution can have an increasing, constant, or decreasing hazard function depending on the shape parameter.

It was observed by Gupta and Kundu (2001) that the generalized exponential (GE) distribution and the gamma distribution have very similar properties in many respects, and in some situations the generalized exponential distribution provides a better fit than Gamma and Weibull distributions in terms of maximum likelihood (ML) or minimum chi-square. Sanku Dey (2010) obtained Bayes estimators of the parameters of GE and its associated risk using different loss functions. Raqab (2002), Raqab and Ahsanullah (2001), Raqab and Madi (2005), Jaheen (2004), Kundu and Gupta (2008) extensively studied this distribution. Singh, Singh, Singh, and Singh (2008) studied the estimation problem of the parameters of this

Saima Naqash is in the Department of Statistics. Email them at: naqashsaima@gmail.com. S. P. Ahmad is in the Department of Statistics. Email them at: sprvz@yahoo.com. Aquil Ahmed is in the Department of Statistics and O. R. Email them at: aqlstat@yahoo.co.in.

distribution under some symmetric and asymmetric loss functions using Lindley's method.

Let x_1, x_2,…, x_n be independently and identically distributed GE random variables with shape parameter α and scale parameter λ (= 1). Then the C.D.F. of x will become

$$\mathrm{F}\left(x,\alpha\right)=\left[1-\exp\left(-x\right)\right]^{\alpha}, \quad x>0,\alpha>0 \tag{1}$$

and the corresponding P.D.F. is

$$\mathrm{f}\left(x,\alpha\right)=\alpha\left[1-\exp\left(-x\right)\right]^{\alpha-1}\exp\left(-x\right), \quad x>0,\alpha>0 \tag{2}$$

For $\alpha = 1$, the GE distribution reduces to the one parameter (standard) exponential distribution. The GE distribution is unimodal with mode at $z = \log\alpha$, $\alpha > 1$, and its median is $M=-\log\left[1-\left(0.5\right)^{\frac{1}{\alpha}}\right]$.

Maximum Likelihood Estimation

Assume that $X = (x_1, x_2,\ldots, x_n)$ is a random sample from GE distribution. The likelihood function of α for the given sample observation is:

$$\mathrm{L}\left(\alpha,x\right)=\alpha^{n}\prod_{i=1}^{n}\left(1-\exp\left(-x_i\right)\right)^{\alpha-1}\exp\left(-\sum_{i=i}^{n}x_i\right) \tag{3}$$

$$\Rightarrow \log\mathrm{L}\left(\alpha,x\right)=n\log\alpha+\left(\alpha-1\right)\sum_{i=1}^{n}\log\left(1-\exp\left(-x_i\right)\right)-\sum_{i=1}^{n}x_i$$

the maximum likelihood estimation (MLE) of α is given by

$$\hat{\alpha}=-\frac{n}{T} \tag{4}$$

where $T=\sum_{i=1}^{n}\log\left(1-\exp\left(-x_i\right)\right)^{-1}$.

Prior and Posterior Distributions

Consider that the parameter α has the non-informative Jeffrey's prior and is given by $g(\alpha) \propto \sqrt{\det(\mathbf{I}(\alpha))}$, where $\mathbf{I}(\alpha)$ is the Fisher Information Matrix given by

$$\mathbf{I}(\alpha) = -n\,\mathrm{E}\left[\frac{\partial^2}{\partial\alpha^2}\log f(x,\alpha)\right] = \frac{n}{\alpha^2}$$

and Jeffery's prior distribution becomes

$$g(\alpha) \propto \frac{1}{\alpha} \tag{5}$$

The posterior distribution is given by

$$\begin{aligned}
&\pi(\alpha \mid x) \propto g(\alpha)\,L(\alpha)\\
&\Rightarrow \pi(\alpha \mid x) \propto \frac{1}{\alpha}\left\{\alpha^n \prod_{i=1}^{n}\left(1-\exp(-x_i)\right)^{(\alpha-1)} \exp\left(-\sum_{i=1}^{n} x_i\right)\right\}\\
&= k\alpha^{n-1}\prod_{i=1}^{n}\left(1-\exp(-x_i)^{\alpha-1}\exp\left(-\sum_{i=1}^{n} x_i\right)\right)\\
&= k\alpha^{n-1}\exp\left[-(\alpha-1)\sum_{i=1}^{n}\log\left(1-\exp(-x_i)\right)^{-1}\right]\exp\left(-\sum_{i=1}^{n} x_i\right)\\
&\Rightarrow \pi(\alpha \mid x) = k\alpha^{n-1}\exp\left[-(\alpha-1)T\right]\exp\left(-\sum_{i=1}^{n} x_i\right)\\
&\qquad\qquad\quad = k\alpha^{n-1}\exp(\alpha T)\exp(T)\exp\left(-\sum_{i=1}^{n} x_i\right)
\end{aligned}$$

The constant k is determined such that

$$\int_0^\infty \pi(\alpha \mid x)\,d\alpha = 1 \quad \Rightarrow k = \frac{T^n}{\Gamma(n)\exp\left(-\sum_{i=1}^{n} x_i\right)\exp(T)}$$

With this value of k, the posterior distribution of α becomes

$$\pi(\alpha \mid x) = \frac{T^n}{\Gamma(n)} \exp(-\alpha T)\alpha^{n-1} \tag{6}$$

which is a Gamma distribution with parameters n and T, where $T = \sum_{i=1}^{n} \log\left(1 - \exp(-x_i)\right)^{-1}$, i.e., $\alpha \sim \mathrm{G}\left(n, \sum_{i=1}^{n} \log\left(1 - \exp(-x_i)\right)^{-1}\right)$.

The expected value (mean) and variance of the distribution is given by

$$\mathrm{E}\left(\pi(\alpha \mid x)\right) = \frac{n}{T} \tag{7}$$

and

$$\mathrm{V}\left(\pi(\alpha \mid x)\right) = \frac{n}{T^2}$$

where T is given as above.

Bayes Estimator under Jeffery's Prior Using Different Loss Functions

Squared Error Loss Function (SELF)

Consider the following SELF: $\mathrm{l}(\hat{\alpha}, \alpha) = c(\hat{\alpha} - \alpha)^2$ and obtain the Risk function as:

$$\begin{aligned}
\mathrm{R}(\hat{\alpha}, \alpha) &= \int_0^\infty \mathrm{l}(\hat{\alpha}, \alpha)\, \pi(\alpha \mid x)\, d\alpha \\
&= c\frac{T^n}{\Gamma(n)} \int_0^\infty \left(\hat{\alpha}^2 - 2\hat{\alpha}\alpha + \alpha^2\right) \exp(-\alpha T)\alpha^{n-1} d\alpha \\
&= c\hat{\alpha}^2 - 2\hat{\alpha}c\frac{n}{T} + c\frac{n(n+1)}{T^2}
\end{aligned}$$

Solving the equation $\frac{\partial}{\partial \hat{\alpha}} \mathrm{R}(\hat{\alpha}, \alpha) = 0$ will give the Bayes estimator:

$$\hat{\alpha}_{B_1} = \frac{n}{T} \tag{8}$$

which is the same as the MLE of α given in (4).

Quadratic Loss Function (QLF)

Consider the following QLF:

$$\mathrm{l}(\hat{\alpha}, \alpha) = \left(\frac{\alpha - \hat{\alpha}}{\alpha} \right)^2$$

and obtain the Risk function as

$$\begin{aligned} \mathrm{R}(\hat{\alpha}, \alpha) &= \int_0^\infty \mathrm{l}(\hat{\alpha}, \alpha) \pi(\alpha \mid x) d\alpha \\ &= \int_0^\infty \left(\frac{\alpha - \hat{\alpha}}{\alpha} \right)^2 \frac{T^n}{\Gamma(n)} \exp(-\alpha T) \alpha^{n-1} d\alpha \\ &= 1 - 2\hat{\alpha} \frac{T}{n-1} + \hat{\alpha}^2 \frac{T^2}{(n-1)(n-2)} \end{aligned}$$

Solving the equation $\frac{\partial}{\partial \hat{\alpha}} \mathrm{R}(\hat{\alpha}, \alpha) = 0$, we get the Bayes estimator of α as:

$$\hat{\alpha}_{B_2} = \frac{n-2}{T} \tag{9}$$

Al-Bayyati's Loss Function

Al-Bayyati's loss function is of the form $\mathrm{l}(\hat{\alpha}, \alpha) = \alpha^{c_2} (\hat{\alpha} - \alpha)^2, c_2 \in \mathbb{R}^+$. This loss function is used to obtain the estimator of the parameter of GE distribution. The risk function is obtained as:

$$\mathrm{R}\left(\hat{\alpha},\alpha\right)=\int_{0}^{\infty}\mathrm{l}\left(\hat{\alpha},\alpha\right)\pi\left(\alpha\mid x\right)d\alpha=\int_{0}^{\infty}\alpha^{c_2}\left(\hat{\alpha}-\alpha\right)^{2}\frac{T^{n}}{\Gamma\left(n\right)}\exp\left(-\alpha T\right)d\alpha$$
$$=\frac{1}{T^{c_2}\Gamma\left(n\right)}\left[\hat{\alpha}^{2}\Gamma\left(n+c_2\right)-2\hat{\alpha}\frac{\Gamma\left(n+c_2+1\right)}{T}+\frac{\Gamma\left(n+c_2+2\right)}{T^{2}}\right]$$

Solving the equation $\frac{\partial}{\partial\hat{\alpha}}\mathrm{R}\left(\hat{\alpha},\alpha\right)=0$, we get the Bayes estimator of α as:

$$\hat{\alpha}_{B_3}=\frac{n+c_2}{T} \tag{10}$$

Remark 1:

1. For c_2 = -2 in (10), we get $\hat{\alpha}=\frac{n-2}{T}$, which gives the Bayes estimator under QLF using Jeffery's prior.
2. For c_2 = 0 in (10), we get $\hat{\alpha}=\frac{n}{T}$, which gives the Bayes estimator under SELF using Jeffery's prior.

Precautionary Loss Function (PLF)

Consider the following PLF:

$$\mathrm{l}\left(\hat{\alpha},\alpha\right)=\frac{\left(\alpha-\hat{\alpha}\right)^{2}}{\alpha}$$

and obtain the Risk function as:

$$\mathrm{R}\left(\hat{\alpha},\alpha\right)=\int_{0}^{\infty}\mathrm{l}\left(\hat{\alpha},\alpha\right)\pi\left(\alpha\mid x\right)d\alpha$$
$$=\int_{0}^{\infty}\frac{\left(\hat{\alpha}-\alpha\right)^{2}}{\hat{\alpha}}\frac{T^{n}}{\Gamma\left(n\right)}\exp\left(-\alpha T\right)\alpha^{n-1}d\alpha$$
$$=\hat{\alpha}-2\frac{n}{T}+\frac{\left(n+1\right)n}{\hat{\alpha}T^{2}}$$

Solving the equation $\frac{\partial}{\partial\hat{\alpha}}\mathrm{R}\left(\hat{\alpha},\alpha\right)=0$, the Bayes estimator of α is

$$\hat{\alpha}_{B_4}=\frac{\sqrt{n\left(n+1\right)}}{T} \tag{11}$$

New Extension of Jeffery's Prior Information

The new extension of Jeffrey's prior information is given by:

$$\begin{aligned}
&\mathrm{g}\left(\alpha\right)\propto\left[\mathbf{I}\left(\alpha\right)\right]^{c_1},\ c_1\in\mathbb{R}^{+}\\
&\Rightarrow\mathrm{g}\left(\alpha\right)\propto\left[\frac{n}{\alpha^2}\right]^{c_1}=\frac{n^{c_1}}{\alpha^{2c_1}}\\
&\Rightarrow\mathrm{g}\left(\alpha\right)\propto\frac{1}{\alpha^{2c_1}}
\end{aligned} \tag{12}$$

The posterior distribution is obtained in a similar way as in the case of Jeffrey's prior information and is given by

$$\begin{aligned}
&\int_0^\infty \pi_1\left(\alpha\mid x\right)d\alpha=1\\
&\Rightarrow k=\frac{T^{n-2c_1+1}}{\Gamma\left(n-2c_1+1\right)\exp\left(-\sum_{i=1}^{n}x_i\right)\exp\left(T\right)}
\end{aligned}$$

Hence the posterior distribution of α becomes

$$\pi_1\left(\alpha\mid x\right)=\frac{T^{n-c_1+1}}{\Gamma\left(n-2c_1+1\right)}\alpha^{n-2c_1}\exp\left(-\alpha T\right) \tag{13}$$

which is the Gamma distribution with parameters $(n-2c_1+1)$ and T, i.e.

$$\alpha \sim \mathrm{G}\left(n-2c_1+1, \sum_{i=1}^{n}\log\left(1-\exp\left(-x_i\right)\right)^{-1}\right)$$

The expected value (mean) and variance of the distribution is given by

$$\mathrm{E}\left(\pi_1\left(\alpha \mid x\right)\right)=\frac{n-2c_1+1}{T} \tag{14}$$

and

$$\mathrm{V}\left(\pi_1\left(\alpha \mid x\right)\right)=\frac{n-2c_1+1}{T^2}$$

Remark 2:

1. For $c_1 = 1/2$ in (14), the posterior distribution under the extension of Jeffery's prior reduces to the posterior distribution under the Jeffery's prior.
2. For $c_1 = 3/2$ in (14), the posterior distribution under the extension of Jeffery's prior reduces to the posterior distribution under the Hartigan's prior.

Bayes Estimation under the Extension of Jeffery's Prior using Different Loss Functions

Squared Error Loss Function

The risk function under SELF is obtained as

$$\begin{aligned}
\mathrm{R}\left(\hat{\alpha},\alpha\right) &= \int_0^\infty 1\left(\hat{\alpha},\alpha\right)\pi_1\left(\alpha \mid x\right)d\alpha \\
&= c\frac{T^{n-c_1+1}}{\Gamma\left(n-2c_1+1\right)}\int_0^\infty\left(\hat{\alpha}^2-2\hat{\alpha}\alpha+\alpha^2\right)\exp\left(-\alpha T\right)\alpha^{n-2c_1}d\alpha \\
&= c\hat{\alpha}^2-2\hat{\alpha}c\frac{n-2c_1+1}{T}+c\frac{\left(n-2c_1+2\right)\left(n-2c_1+1\right)}{T^2}
\end{aligned}$$

Solving the equation $\frac{\partial}{\partial\hat{\alpha}}\mathrm{R}\left(\hat{\alpha},\alpha\right)=0$, the Bayes estimator of α is

$$\hat{\alpha}_{B_5}=\frac{n-2c_1+1}{T} \tag{15}$$

Remark 3: For $c_1 = 1/2$ in (15), $\hat{\alpha}=n/T$, which gives the Jeffery's estimator under SELF.

Quadratic Loss Function

Using the QLF,

$$\mathrm{l}\left(\hat{\alpha},\alpha\right)=\left(\frac{\alpha-\hat{\alpha}}{\alpha}\right)^2$$

The risk function under QLF is obtained as

$$\begin{aligned}\mathrm{R}\left(\hat{\alpha},\alpha\right)&=\int_0^\infty \mathrm{l}\left(\hat{\alpha},\alpha\right)\pi_1\left(\alpha\,|\,x\right)d\alpha\\&=\frac{T^{n-2c_1+1}}{\Gamma\left(n-2c_1+1\right)}\int_0^\infty\left(\hat{\alpha}^2-2\hat{\alpha}\alpha+\alpha^2\right)\exp\left(-\alpha T\right)\alpha^{n-2c_1-2}d\alpha\\&=1-2\hat{\alpha}\frac{T}{n-2c_1}+\hat{\alpha}^2\frac{T^2}{\left(n-2c_1\right)\left(n-2c_1-1\right)}\end{aligned}$$

Solving the equation $\frac{\partial}{\partial\hat{\alpha}}\mathrm{R}\left(\hat{\alpha},\alpha\right)=0$, the Bayes estimator of α is

$$\hat{\alpha}_{B_6}=\frac{n-2c_1-1}{T} \tag{16}$$

Remark 4: For $c_1 = 1/2$ in (16), $\hat{\alpha}=\frac{n-2}{T}$, which gives the Bayes estimator under QLF using Jeffery's prior.

Al-Bayyati's Loss Function

Al-Bayyati's loss function is of the form $l(\hat{\alpha},\alpha)=\alpha^{c_2}(\hat{\alpha}-\alpha)^2$, $c_2 \in \mathbb{R}^+$. The risk function is given by

$$\begin{aligned}
\mathrm{R}(\hat{\alpha},\alpha) &= \int_0^\infty l(\hat{\alpha},\alpha)\pi_1(\alpha \mid x)\,d\alpha \\
&= \frac{T^{n-2c_1+1}}{\Gamma(n-2c_1+1)}\int_0^\infty (\hat{\alpha}-\alpha)^2 \exp(-\alpha T)\alpha^{n+c_2-2c_1}\,d\alpha \\
&= \frac{1}{T^{c_2}\Gamma(n-2c_1+1)}\left[\hat{\alpha}^2\Gamma(n+c_2-2c_1+1)-2\hat{\alpha}\frac{\Gamma(n+c_2-2c_1+1)}{T}\right. \\
&\qquad\left. +\frac{\Gamma(n+c_2-2c_1+3)}{T^2}\right]
\end{aligned}$$

Solving the equation $\frac{\partial}{\partial\hat{\alpha}}\mathrm{R}(\hat{\alpha},\alpha)=0$, the Bayes estimator of α is

$$\hat{\alpha}_{B_7} = \frac{n+c_2-2c_1+1}{T} \tag{17}$$

Remark 5:

1. For $c_1 = 1/2$ and $c_2 = 0$ in (17), $\hat{\alpha} = n/T$, which gives the Bayes' estimator under SELF using Jeffery's prior.
2. For $c_1 = 1/2$ and $c_2 = -2$ in (17), $\hat{\alpha} = \frac{n-2}{T}$, which gives the Bayes' estimator under QLF using Jeffery's prior.

Precautionary Loss Function

Using the PLF

$$l(\hat{\alpha},\alpha) = \frac{(\alpha-\hat{\alpha})^2}{\alpha}$$

obtain the Risk function under PLF as

$$\begin{aligned}
\mathrm{R}\left(\hat{\alpha},\alpha\right) &= \int_0^\infty \mathrm{l}\left(\hat{\alpha},\alpha\right)\pi_1\left(\alpha \mid x\right)d\alpha \\
&= \int_0^\infty \frac{\left(\hat{\alpha}-\alpha\right)^2}{\hat{\alpha}} \frac{T^{n-2c_1+1}}{\Gamma\left(n-2c_1+1\right)} \exp\left(-\alpha T\right)\alpha^{n-2c_1} d\alpha \\
&= \hat{\alpha} - 2\frac{\left(n-2c_1+1\right)}{T} + \frac{\left(n-2c_1+2\right)\left(n-2c_1+1\right)}{\hat{\alpha}T^2}
\end{aligned}$$

Solving the equation $\frac{\partial}{\partial\hat{\alpha}}\mathrm{R}\left(\hat{\alpha},\alpha\right)=0$, the Bayes' estimator of α is

$$\hat{\alpha}_{B_8} = \frac{\sqrt{\left(n-2c_1+2\right)\left(n-2c_1+1\right)}}{T} \tag{18}$$

Remark 6: For c_1 = 1/2 in (18), $\hat{\alpha} = \frac{\sqrt{n\left(n+1\right)}}{T}$, which gives the Bayes' estimator under PLF using Jeffery's prior.

Simulation Study of Generalized Exponential Distribution

In the simulation study, sample sizes were chosen at n = 25, 50, and 100 to represent small, medium, and large data sets. The scale parameter is estimated for Generalized Exponential distribution with Maximum Likelihood and Bayesian using Jeffrey's & extension of Jeffrey's prior methods. For the scale parameter, α = 0.5, 1.0, and 1.5. The values of Jeffrey's extension are chosen as c_1 = 1.0, 1.5, and 2. The value for the loss parameter c_2 = ±1.0 and ±2.0. This was iterated 5000 times and the scale parameter for each method was calculated. A simulation study was conducted in R-software to examine and compare the performance of the estimates for different sample sizes with different values for Jeffrey's prior and the extension of Jeffrey's prior under different loss functions. The results are presented in tables for different selections of the parameters and c extension of Jeffrey's prior.

In Table 2, Bayes' estimation with Al-Bayyati's Loss function under Jeffrey's prior provides the smallest values in most cases especially when loss parameter c_2 is ±2.0. Similarly, in Table 4, Bayes' estimation with Al-Bayyati's Loss function under extension of Jeffrey's prior provides the smallest values in most cases, especially when loss parameter c_2 is ±2.0 whether the extension of Jeffrey's prior is 0.5, 1.0, or 1.5. Moreover, when the sample size increases from 25 to 100, the Mean Squared Error decreases quite significantly.

Table 1. Posterior mean for $\hat{\alpha}$ under Jeffery's prior

							α_{AL}			
n	**λ**	**α**	**α_{ML}**	**α_{SL}**	**α_{QL}**	**α_{PL}**	**c_2=1.0**	**c_2=-1.0**	**c_2=2.0**	**c_2=-2.0**
		0.5	0.3815	0.3815	0.3510	0.3890	0.3967	0.3662	0.4120	0.3510
25	1.0	1.0	0.9899	0.9899	0.9107	1.0095	1.0295	0.9503	1.0691	0.9107
		1.5	2.3398	2.3398	2.1526	2.3861	2.4334	2.2462	2.5270	2.1526
		0.5	0.4521	0.4521	0.4340	0.4566	0.4512	0.4431	0.4702	0.4340
50	1.0	1.0	0.8861	0.8861	0.8507	0.8950	0.9038	0.8684	0.9215	0.8507
		1.5	1.5164	1.5164	1.4557	1.5315	1.5467	1.4861	1.5770	1.4557
		0.5	0.4845	0.4845	0.4748	0.4869	0.4893	0.4797	0.4941	0.4748
100	1.0	1.0	0.8911	0.8911	0.8733	0.8956	0.9000	0.8822	0.9089	0.8733
		1.5	1.4541	1.4545	1.4250	1.4613	1.4686	1.4395	1.4831	1.4250

Note: ML=Maximum Likelihood, SL=Squared Error Loss Function, QL=Quadratic Loss Function, PL=Precautionary Loss Function, AL=Al-Bayyati's Loss Function

Table 2. Mean squared error for $\hat{\alpha}$ under Jeffery's prior

							α_{AL}			
n	**λ**	**α**	**α_{ML}**	**α_{SL}**	**α_{QL}**	**α_{PL}**	**c_2=1.0**	**c_2=-1.0**	**c_2=2.0**	**c_2=-2.0**
		0.5	0.0258	0.0258	0.2722	0.0246	0.0235	0.0288	**0.0215**	0.0322
25	1.0	1.0	0.0473	0.0473	1.0080	0.0492	0.0519	**0.0460**	0.0600	0.0479
		1.5	0.8114	0.8114	2.6759	0.8956	0.9860	0.6546	1.1785	**0.5157**
		0.5	0.0077	0.0077	0.2544	0.0074	0.0080	0.0084	**0.0068**	0.0094
50	1.0	1.0	0.0347	0.0347	1.0223	0.0331	0.0319	0.0381	**0.0297**	0.0423
		1.5	0.0491	0.0491	2.2520	0.0508	0.0530	0.0471	0.0582	**0.0470**
		0.5	0.0028	0.0028	0.2506	0.0028	0.0028	0.0030	**0.0027**	0.0031
100	1.0	1.0	0.0223	0.0223	1.0161	0.0214	0.0206	0.0241	**0.0191**	0.0261
		1.5	0.0255	0.0255	2.2556	0.0252	0.0249	0.0267	**0.0247**	0.0281

Note: ML=Maximum Likelihood, SL=Squared Error Loss Function, QL=Quadratic Loss Function, PL=Precautionary Loss Function, AL=Al-Bayyati's Loss Function

Table 3. Posterior mean for $\hat{\alpha}$ under extension of Jeffery's prior

n	λ	α	α_{ML}	α_{SL}	α_{QL}	α_{PL}	α_{AL}			
							c_2=1.0	c_2=-1.0	c_2=2.0	c_2=-2.0
		0.5	0.3815	0.3662	0.3357	0.3738	0.3815	0.3510	0.3967	0.3357
25	1.0	1.0	0.9899	0.9107	0.8315	0.9303	0.9503	0.8711	0.9899	0.8315
		1.5	2.3398	2.0590	1.8718	2.1053	2.1526	1.9654	2.2462	1.8718
		0.5	0.4521	0.4431	0.4250	0.4476	0.4521	0.4340	0.4612	0.4250
50	1.0	1.0	0.8861	0.8107	0.8152	0.8595	0.8684	0.8330	0.8861	0.8152
		1.5	1.5164	1.4254	1.3648	1.4557	1.4405	1.3951	1.4861	1.3648
		0.5	0.4845	0.4797	0.4700	0.4821	0.4845	0.4748	0.4893	0.4700
100	1.0	1.0	0.8911	0.8733	0.8555	0.8777	0.8822	0.8644	0.8911	0.8555
		1.5	1.4541	1.4104	1.3814	1.4177	1.4250	1.3959	1.4395	1.3814

Note: ML=Maximum Likelihood, SL=Squared Error Loss Function, QL=Quadratic Loss Function, PL=Precautionary Loss Function, AL=Al-Bayyati's Loss Function

Table 4. Mean squared error for $\hat{\alpha}$ under extension of Jeffery's prior

n	λ	α	α_{ML}	α_{SL}	α_{QL}	α_{PL}	α_{AL}			
							c_2=1.0	c_2=-1.0	c_2=2.0	c_2=-2.0
		0.5	0.0258	0.0288	0.0361	0.0272	0.0258	0.0322	**0.0235**	0.0361
25	1.0	1.0	0.0473	0.0479	0.0617	0.0466	**0.0460**	0.0531	0.0473	0.0617
		1.5	0.8114	0.3947	0.2062	0.4253	0.5157	0.2915	0.6546	**0.2061**
		0.5	0.0077	0.0552	0.0104	0.0080	0.0077	0.0094	**0.0071**	0.0104
50	1.0	1.0	**0.0347**	0.0558	0.0526	0.0401	0.0381	0.0471	**0.0347**	0.0526
		1.5	0.0491	0.0487	0.0578	**0.0460**	0.0485	0.0523	0.0471	0.0578
		0.5	**0.0028**	0.0032	0.0033	0.0029	**0.0028**	0.0031	**0.0028**	0.0033
100	1.0	1.0	**0.0223**	0.0261	0.0305	0.0251	0.0241	0.0282	**0.0223**	0.0305
		1.5	**0.0255**	0.0300	0.0352	0.0291	0.0281	0.0324	0.0267	0.0352

Note: ML=Maximum Likelihood, SL=Squared Error Loss Function, QL=Quadratic Loss Function, PL=Precautionary Loss Function, AL=Al-Bayyati's Loss Function

Conclusion

The Bayes' estimator of the parameter of the Generalized Exponential distribution was studied under Jeffrey's prior and the extended Jeffrey's prior assuming different loss functions. The extended Jeffrey's prior gives the opportunity of covering wide spectrum of priors to get Bayes' estimates of the parameter –

particular cases of which are Jeffrey's prior and Hartigan's prior. We have also addressed the problem of Bayesian estimation for the Generalized Exponential distribution, under symmetric loss functions and that of Maximum Likelihood Estimation. In most cases, the Bayesian Estimator under Al-Bayyati's Loss function has the smallest Mean Squared Error values for both prior's i.e, Jeffrey's and an extension of Jeffrey's prior information. Moreover, when the sample size increases from 25 to 100, the MSE decreases quite significantly.

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