Properties Of Nonlinear Randomly Switching Dynamic Systems: Mean-Field Models And Feedback Controls For Stabilization

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PROPERTIES OF NONLINEAR RANDOMLY SWITCHING
DYNAMIC SYSTEMS: MEAN-FIELD MODELS AND
FEEDBACK CONTROLS FOR STABILIZATION

by

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DISSERTATION

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DEDICATION

To my family

To my teachers
ACKNOWLEDGEMENTS

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1 Introduction

1.1 Background and Main Issues

This dissertation focuses on basic properties of nonlinear randomly switching dynamic systems. It encompasses an in-depth study of mean-field model, as well as system regularization and stabilization by using feedback controls. In this introductory chapter, we present the motivation of the study, outline of our approaches, and certain results.

Owing to the development of science and technology, many systems with simple settings are inadequate in applications. To face such challenges, more sophisticated systems have been developed and analyzed nowadays. To study systems including continuous dynamics with discrete events, one of the popular methods is to introduce regime switching into the setting. In the resulting systems, the dynamic movements are influenced by two parts: the continuous-time dynamics and random environment changes or other uncertainty of discrete nature (termed discrete events henceforth). The continuous dynamics are formulated by the usual differential equations, whereas the discrete-time events are depicted by jumps or regime switching. The switching systems may be traced back to [18]. Recently, more and more attention has been drawn to hybrid systems with regime-switching processes, and new applications have been found in various disciplines, including economic systems [15], Lotka-Volterra model [51], mean-field models for many bodies [47], replicator dynamics [17]. For systematic study, see [48] and many references therein.

Concerning such systems, one typical example is a system of ordinary differential equa-
tions (ODEs) randomly switching among themselves as follows

\[
\frac{dX(t)}{dt} = \mu(X(t), \theta(t)),
\]

(1.1)

where \( M = \{1, \ldots, m\} \) is the state space of the continuous-time regime switching process \( \theta(\cdot), x \in \mathbb{R}^r \), and \( \mu(\cdot, \cdot) : \mathbb{R}^r \times M \mapsto \mathbb{R}^r \). A popular and reasonable assumption is that \( \theta(t) \) has Markovian property:

\[
P\{\theta(t+\delta) = j | \theta(t) = i, X(s), \theta(s), s \leq t\} = q_{ij}\delta + o(\delta), \quad i \neq j.
\]

(1.2)

where \( Q = (q_{ij}) \) is the generator of \( \theta(t) \) such that \( q_{ij} \geq 0 \) for \( i \neq j \) and

\[
Q\mathbb{1} = \left( \sum_{j=1}^{m} q_{1j}, \ldots, \sum_{j=1}^{m} q_{mj} \right)' = 0.
\]

Here \( \mathbb{1} = (1, \ldots, 1)' \in \mathbb{R}^m \). Equation (1.1) represents a class of systems that are important and that have a wide range of applications. Recently, the regime-switching processes are widely used to depict the topology changes, and environmental variations. Note that the discrete events could not be represented by the continuous-time dynamic systems containing only differential equations. The model of hybrid regime-switching systems is thus natural and has main advantage in modeling complex networked systems. Since the regime switching only takes place in a finite set, one may ask that the system with switching that move back and forth between a finite number of equations may not have much difference from the system without switching. Nevertheless, practical example shows that even though each individual ODE is stable, the hybrid system with switching may be unstable.
To demonstrate, consider the linear system with regime switching

\[
d\frac{X(t)}{dt} = \mu(\theta(t))X(t)
\]  

(1.3)

where \(\mu(1) = \begin{pmatrix} -10 & 2 \\ 20 & -10 \end{pmatrix}\), \(\mu(2) = \begin{pmatrix} -10 & 20 \\ 2 & -10 \end{pmatrix}\), \(\theta(t)\) is a continuous-time Markov chain with generator \(Q = \begin{pmatrix} -100 & 100 \\ 100 & -100 \end{pmatrix}\). It is easy to see that both

\[
d\frac{X(t)}{dt} = \mu(1)X(t), \quad \text{and} \quad d\frac{X(t)}{dt} = \mu(2)X(t)
\]

are stable. However, system (1.3) is unstable; see Figure 1. From this example, it is clear that the asymptotic behavior of systems with regime switching are quite different from each individual system.

![Figure 1: Trajectory of the Euclidean norm \(|X(t)|\) as a function of \(t\) for system (1.3).](image)

To make the systems more realistic, white noise could also be considered by adding a
Brownian motion term. The system would change to

$$dX(t) = \mu(X(t), \theta(t))dt + \sigma(X(t), \theta(t))dB(t)$$

where $B(\cdot)$ is a standard $d$-dimensional Brownian motion and $B$ and the Markov chain $\theta$ are independent with each other, and $\sigma(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^{r \times d}$. This could make the systems be more applicable in financial market, communication networks, ecological and biological models.

Our contributions in this dissertation consists of two main parts. Part 1 provides a detailed study of mean-field models that take into consideration of random environment. Part 2 designs feedback controls that regularize and stabilize a given system with random switching.

In Chapter 2, the dissertation begins our study on the so-called mean-field models, which are originated from statistical mechanics. When physicists treating many body problems, they brought in the idea of the use of “mean-field” models. The main thought is: Many-body problems will be difficult to deal with because of the complicated interactions among many particles. To overcome the difficulties, one of the main ideas is to replace the interactions to each body by an average term, which changes the many particles problem to an effective mean field problem. One can however replace the many bodies by a representative, namely an arithmetic average of these bodies. Such an idea has been around for decades. In 1975, Dawson rigorously justified this idea using probabilistic argument. He first proved a law of large number type of results for measure-valued processes. Then he further revealed the phase transition properties using the mean-field models. The mathematical theory obtained gives precise description on many particles in such systems and the interactions among them. The
applications of the mean field models are not limited to the original statistical mechanics. In fact, many complex systems can be modeled by using mean field to represent the many particles interactions. For examples, mean field theory are used in many models in machine learning and artificial intelligence. Also in financial engineering, one famous model is the multivariate Ornstein-Uhlenbeck processes, which use a constant mean-reversion term as an approximation of the mean field. In such system, the constant mean reversion term acts as a force pushing the system moving to the good direction and some stationary solutions may occur [12]. The difference with mean-reversing model is that in mean field model, the mean term is not a constant, but an average of the particles, which is more complex.

The problems considered in the literature, for example, the work of Dawson and subsequent work inspired by his work mainly concentrated on diffusion systems. Such systems are modeled by stochastic differential equations. They have been in use for many years with great success. Nevertheless, such models cannot handle the situation that the environment is subject to random changes that are discrete event type. This brought us to the work to be addressed in the dissertation. As a point of departure, we replace the diffusion formulation by a formulation involving switching diffusions. Among other things, the switching is allowed to be diffusion dependent. We aim to obtain a number of properties. The main work in Chapter 2 is on developing such properties.

In control and systems theory, one often faces the problem of finding appropriate feedback controls so as to achieve a specific goal. Chapter 3 contributes to our effort in this direction. Specifically, Chapter 3 studies regularization and stabilization of regime switching dynamic systems by using feedback controls. Our motivation stems from the consideration of system stability. In greatly many systems studied in the field of communication networks,
financial markets, economic systems, control systems and optimization, biological and ecological systems, an important problem is to check the stability. The problem we work on can be described as follows. Can we find a feedback control $u(x, t)$ on the system (1.1) or (1.4) such that the system

$$\frac{dX(t)}{dt} = \mu(X(t), \theta(t)) + u(X(t), t)$$

(1.5)

or the system with noise term

$$dX(t) = \mu(X(t), \theta(t))dt + \sigma(X(t), \theta(t))dB(t) + u(X(t), t)$$

is stable? Let us point out the main difficulties here. The first problem that we encounter is that system (1.1) has “bad” nonlinearity, in that it does not have the often assumed linear growth condition in the variable $x$. For each discrete state $i$, we only know that $\mu(\cdot, i)$ is continuous. Because of the fast growth, the system has only local solution; the solution will blow up in finite time. That is, with probability one (w.p.1), the system will explode in finite time. The second difficulty is the uncertainty caused by the random switching. As shown in previous example, the hybrid system may be explode due to the regime switching, even each individual system is stable. It seems that to stabilize the system becomes hopeless. To solve the problem, we device a method based by using feedback control methods. So that we ensure that the resulting system has a global solution. In addition, the desire stabilization goal is fulfilled. This can be done in the following two steps: (i) by adding well designed feedback controls or perturbations, we show that the resulting system has a global solution; (ii) by adding another feedback control, then we ensure that the system can be stabilized.
Inspired by the recent work of Wu and Hu [41], our work extends the existing work enabling the treatment of systems involving both continuous dynamics and discrete events. Here our system is more complex that includes regime switching among a finite state space. In addition, due of the system complexity, it is usually difficult to find the closed-form solutions. Thus, we need design algorithms to find numerical solutions. Define a discrete time sequence of approximation for the original nonlinear system, we also face the problem of finite explosion. Thus, we need to study the regularity and stability of discrete sequence too.

1.2 Preliminary

Consider a continuous-time Markov chain taking values in the finite state space \( \mathcal{M} = \{1, \ldots, m\} \) such that (1.2) holds as \( \delta \to 0 \). Suppose \( B(t) \) is a standard \( d \)-dimensional Brownian motion that is independent of \( \theta(t) \). For a stochastic system hybrid with regime switching given by

\[
\begin{align*}
    dX(t) &= \mu(X(t), \theta(t))dt + \sigma(X(t), \theta(t))dB(t) \\
    X(0) &= x_0, \quad \theta(0) = \theta,
\end{align*}
\]

(1.6)

where \( x_0 \in \mathbb{R}^r, \mu(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^r, \sigma(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^{r \times d} \). For each \( i \in \mathcal{M} \) and any \( f(\cdot, i) \in C^2 \), define the operator of (1.6) by

\[
    \mathcal{L}f(x, i) = \mu'(x, i)\nabla f(x, i) + \frac{1}{2} \text{tr}(\sigma(x, i)\sigma'(x, i)\nabla^2 f(x, i)) + \mathcal{Q}f(x, \cdot)(i),
\]

(1.7)

\[
    \mathcal{Q}f(x, \cdot)(i) = \sum_{j \in \mathcal{M}} q_{ij} f(x, j) \quad \text{for each} \quad i \in \mathcal{M}.
\]
Definition 1.1. A Markov process \( X(t) \) with initial \( X(0) = x_0 \) is said to be regular if

\[
\tau_n := \inf\{ t \geq 0 : |X(t)| \geq n \} \to \infty
\]
a.s. as \( n \to \infty \)

Recall that (see [6] and [48, Section 2.3]) the two-component Markov process \((X(t), \theta(t))\) with initial data \((X(0), \theta(0)) = (x, \theta)\) is said to be regular or no finite explosion time, if for any \(0 < T < \infty\),

\[
P\{ \sup_{0 \leq t \leq T} |X^{x,\theta}(t)| = \infty \} = 0. \tag{1.8}
\]

That is, the process is regular if and only if it does not blow up in finite time w.p.1., which is however equivalent to the definition we use here.

Definition 1.2. The Markov chain \( \theta(t) \) is irreducible if the system of equations

\[
\begin{cases}
\nu Q = 0 \\
\nu \mathbb{1} = 1
\end{cases}
\]

has a unique solution such that \( \nu = (\nu_1, \ldots, \nu_m) \) satisfies \( \nu_i > 0 \).

By irreducibility, the Markov chain \( \theta(\cdot) \) is ergodic and hence has a stationary distribution. Denote its stationary distribution by \( \nu = (\nu_1, \ldots, \nu_m) \in \mathbb{R}^{1 \times m} \) (see [44]).

Let us represent the Markov switching diffusion as a Poisson jump diffusion. This is based on the fact that the discrete switching process \( \theta(\cdot) \) can be described as a stochastic integral w.r.t. a Poisson random measure \([13,39,48]\). Define \( P(\cdot, \cdot) \) on \([0, +\infty) \times \mathbb{R} \) with rate of intensity \( dt \times m(dz) \) and \( m \) being the Lebesgue measure on the real line. Let \( P(\cdot, \cdot) \) be independent of
the Brownian motions $B(\cdot)$. For $i, j \in \mathcal{M}$, with $i \neq j$, let $\Delta_{ij}$ be the consecutive left-closed, right-open intervals of the real line, with length $q_{ij}$. Let $\psi : \mathcal{M} \times \mathbb{R} \mapsto \mathbb{R}$ be

$$
\psi(i, z) = \sum_{j=1}^{m} (j - i)I_{\{z \in \Delta_{ij}\}}.
$$

Then (1.2) is equivalent to

$$
d\theta(t) = \int_{\mathbb{R}} \psi(\theta(t-), z)P(dt, dz).
$$

Denote the compensated Poisson measure by

$$
\tilde{P}(ds, dz) = P(ds, dz) - ds \times m(dz),
$$

which is a martingale measure. Then (1.2) can be represented as

$$
d\theta(t) = \int_{\mathbb{R}} \psi(\theta(t-), z)\tilde{P}(dt, dz) + \int_{\mathbb{R}} \psi(\theta(t-), z)ds \times m(dz). \quad (1.9)
$$

By representing the Markov switching process to Poisson jump process, we can apply the generalized Itô lemma (see [7, 33, 39]) to the hybrid process.

### 1.3 Outline of the Dissertation

The rest of the dissertation is arranged as follows. In Chapter 2, the mean-field models are studied. It originates from the phase transition problem in statistical physics and are formulated by nonlinear stochastic differential equations hybrid with state-dependent regime.
switching. The mean-field term is used to describe the complex interactions between multi
bodies in the system, and acts as an mean reversing effects. We study the basic properties
of such models, including regularity, non-negativity, finite moments, existence of moment
generating functions, continuity of sample path, positive recurrence, long-time behavior. We
also proved that when switching process changes much more frequently, the two-time-scale
limit exists. In Chapter 3 and Chapter 4, we consider the stabilization problem of nonlinear
dynamic systems. We work on deterministic systems with switching in Chapter 3. Many
nonlinear systems would explode in finite time. We found that Brownian motion noise can
be used as feedback control to stabilize such systems. To do so, we can use one nonlinear
feedback noise term to suppress the explosion, and then use another linear feedback noise
term to stabilize the system to the equilibrium point 0. Since it is almost impossible to
get an closed-form solutions, the discrete-time approximation algorithm is constructed. The
interpolated sequence of the discrete-time algorithm is proved to converge to the switching
diffusion process, and then the regularity and stability results of the approximating sequence
are derived. In Chapter 4, we study the stochastic systems with switching. Use the similar
methods, we can prove that well designed noise type feedback control could also regularize
and stabilize nonlinear switching diffusions. Examples are used to demonstrate the results.
Finally in Chapter 5, concluding remarks are given, and several possible directions are pro-
posed for future study.
1.4 Notation Index

Before proceeding further, we compile the following list of notation index to be used in the entire dissertation.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\mathbb{R}^{n_1 \times n_2}$</td>
<td>$n_1 \times n_2$-dimensional Euclidean space, where $n_1$ and $n_2$ are positive integers</td>
</tr>
<tr>
<td>$\mathbb{R}_+$</td>
<td>${ z \in \mathbb{R} : z &gt; 0 }$</td>
</tr>
<tr>
<td>$</td>
<td>x</td>
</tr>
<tr>
<td>$z'$</td>
<td>transpose of $z \in \mathbb{R}^{l_1 \times l_2}$</td>
</tr>
<tr>
<td>$\text{tr}(A)$</td>
<td>trace of $A \in \mathbb{R}^{n \times n}$</td>
</tr>
<tr>
<td>$\nabla f$</td>
<td>gradient of $f(x)$ w.r.t. $x$</td>
</tr>
<tr>
<td>$\nabla^2 f$</td>
<td>Hessian of $f(x)$ w.r.t. $x$</td>
</tr>
<tr>
<td>$S^C_R$</td>
<td>${ x :</td>
</tr>
<tr>
<td>$\lambda_M$</td>
<td>the largest eigenvalue of $MM'$ for $M \in \mathbb{R}^{n \times r}$</td>
</tr>
<tr>
<td>a.s.</td>
<td>almost surely</td>
</tr>
<tr>
<td>$\mathbb{P}(B)$</td>
<td>the probability of event $B$</td>
</tr>
<tr>
<td>$\mathbb{E}(B)$</td>
<td>the expectation of event $B$</td>
</tr>
<tr>
<td>$\mathcal{L}X$</td>
<td>the characteristic operator of a diffusion $X$</td>
</tr>
<tr>
<td>$\mathbb{1}$</td>
<td>$= (1, \ldots, 1)' \in \mathbb{R}^m$</td>
</tr>
</tbody>
</table>

end of proof
2 Mean-Field Models

Mean-field models could accurately depict the complex interactions of a many body system, so it is necessary to study the basic properties of such model. Dawson [9] studied the cooperative behavior of the Mean-field model, and proved the central limit theorem and law of large numbers for such models with jump in his following work [10]. Hitsuda and Mitoma [16] proved the tightness in the Kolmogorov-Prokhorov sense for a sequence of distribution valued processes, which is a generalization of mean-field models. see also related work [34, 37] and references therein. The ergodic property and exponential decay of sample path for linear interacting diffusion systems are obtained by Cox and and Green [8], Shiga and Uchiyama [38], Zeldovich, Molchanov, Ruzmaikin and Sokoloff [49], and Shiga [36]. Infinite dimensional interacting diffusion systems are studied by Rockner and Schmuland [35], Kondratiev, Lytvynov, Rochner [22]. In recent work of Xi and Yin [42], regularity, Feller continuity, strong Feller continuity, and exponential ergodicity are obtained for mean-field models. This paper is a continuous work of [42] concerning nonnegativity constraints, which is more realistic in statistical physics.

Continuing our effort in the study of regime-switching diffusions, this work focuses on investigating properties of regime-switching mean-field models. This paper is a continuation of our recent work [42], in which regularity, Feller continuity, strong Feller continuity, and exponential ergodicity are obtained. In the previous work, for example, in [9] as well as in [42], each component of the system is allowed to take values in \( \mathbb{R} \). That is, any of the \( r \) bodies is allowed to take negative values. However, in statistical physics, typically, these many bodies are only allowed to be nonnegative. Thus it will be more realistic to consider a formulation
with nonnegativity constraint. In this paper, we take nonnegativity constraint into consideration, which puts further challenges to the analysis. If $\mathbb{R}$ is used, to ensure the system is non-explosive, it suffices to verify the regularity. Under the nonnegativity constraint, it is necessary to show that each component of the system remains to be nonnegative or to be confined to the first quadrant only. This in turn, requires more careful analysis and special attention. In addition, we are interested in getting several moment bounds. With such bounds at our hands, we can proceed to obtain sample continuity as well as further asymptotic behavior. Furthermore, when the switching process is varying an order of magnitude faster than the continuous state, certain average takes place. We show that the continuous state process has a limit, which is an average with respect to the quasi-stationary measure of the fast varying switching process (more precise definition will be given in the subsequent section). This limit can be obtained by means of a martingale problem formulation.

### 2.1 Formulation

Consider a mean-field model hybrid with regime switching shown in the following system.

For $i = 1, 2, \ldots, r$, 

\[
\begin{align*}
    dX_i(t) &= \left[\gamma(\theta(t))X_i(t) - X_i^3(t) - \beta(\theta(t))(X_i(t) - \overline{X}(t))\right] dt \\
    & \quad + \sigma_{ii}(X(t), \theta(t))dB_i(t),
\end{align*}
\]

(2.1)

where $\theta(\cdot) \in \mathcal{M} := \{1, \ldots, m\}$ is a regime switching process, $\gamma(\cdot), \beta(\cdot) : \mathcal{M} \mapsto \mathbb{R}_+$, $B_i(\cdot) \in \mathbb{R}$ is a standard Brownian motion, the Mean-field term

\[
\overline{X}(t) = \frac{1}{r} \sum_{i=1}^{r} X_i(t), \quad X(t) = (X_1(t), X_2(t), \ldots, X_r(t))^\prime.
\]
For $\theta \in \mathcal{M} = \{1, \ldots, m\}$, the transition rules of $\theta(t)$ are specified by

$$\mathbf{P}\{\theta(t + \Delta) = k | \theta(t) = \theta, X(t) = x\} = q_{\theta k}(x)\Delta + o(\Delta) \text{ if } k \neq \theta, \quad (2.2)$$

where $\Delta \downarrow 0$ and $\sum_{k \in \mathcal{M}} q_{\theta k}(x) = 0$ for each $\theta \in \mathcal{M}$. Compare with (1.2), the transition probability is state dependent.

In the rest of this chapter, we need some assumption on (2.1) and (2.2).

(H2) The $Q(\cdot)$ is bounded and continuous. For each $\theta \in \mathcal{M}$ and $x \in \mathbb{R}^r$,

* $q_{\theta k}(x) > 0$ for $k \neq \theta$ and $\sigma_{ii}(x, \theta) > 0$ for each $1 \leq i \leq r$;

* $\sigma_{ii}(x, \theta)$ and $q_{\theta k}(x)$ are locally Lipschitz with respect to $x$;

* $\sigma_{ii}(x, \theta)$ is infinitely differentiable in $x$;

* there exist constants $K_0 > 0$ and $\delta > 0$ such that

$$\sum_{i=1}^{r} \sigma_{ii}^2(x, \theta) \leq K_0(|x|^{4-\delta} + 1). \quad (2.3)$$

(H3) Assume (H2) but with (2.3) replaced by

$$\sum_{i=1}^{r} \sigma_{ii}^2(x, \theta) \leq K_0|x|^{4-\delta}. \quad (2.4)$$

Note that (H2) allows the diffusion to grow at the order of $4 - \delta$ for some $\delta > 0$, whereas (H3) allows similar growth rate and requires also $\sigma_{ii}(0) = 0$. Condition (H2) is sufficient to ensure the existence and uniqueness of the solution of the switching stochastic differential equation, and condition (H3) enables us to obtain further properties such as nonnegativity.
etc. More details will be seen in the subsequent sections.

To proceed, it is convenient to use a vector notation. For \((x, \theta) \in \mathbb{R}^r \times \mathcal{M}\), set

\[
\mu(x, \theta) = \begin{pmatrix}
\mu_1(x, \theta) \\
\mu_2(x, \theta) \\
\vdots \\
\mu_r(x, \theta)
\end{pmatrix} = \begin{pmatrix}
\gamma(\theta)x_1 - x_1^3 - \beta(\theta)(x_1 - \overline{x}) \\
\gamma(\theta)x_2 - x_2^3 - \beta(\theta)(x_2 - \overline{x}) \\
\vdots \\
\gamma(\theta)x_r - x_r^3 - \beta(\theta)(x_r - \overline{x})
\end{pmatrix} \in \mathbb{R}^r, \quad \text{(2.5)}
\]

and \(\sigma(x, \theta) = \text{diag}\{\sigma_{ii}(x, \theta)\} \in \mathbb{R}^r \times \mathbb{R}^r\), where \(\overline{x} := \sum_{j=1}^r x_j / r\). Then stochastic differential equation (2.1) can be rewritten as

\[
dX(t) = \mu(X(t), \theta(t))dt + \sigma(X(t), \theta(t))dB(t). \quad \text{(2.6)}
\]

For a function \(f(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}\) such that \(f(\cdot, \theta)\) is twice continuously differentiable with respect to the variable \(x\) for each \(\theta \in \mathcal{M}\), the operator associated with the switching diffusion is given by

\[
\mathcal{L}f(x, \theta) = \frac{1}{2} \sum_{i=1}^r \sigma_{ii}^2(x, \theta) \frac{\partial^2 f(x, \theta)}{\partial x_i^2} + \sum_{i=1}^r \mu_i(x, \theta) \frac{\partial f(x, \theta)}{\partial x_i} + \sum_{k \in \mathcal{M}, k \neq \theta} q_{\theta k}(x) \left( f(x, k) - f(x, \theta) \right). \quad \text{(2.7)}
\]

### 2.2 Properties of Solutions

We begin by stating an existence and uniqueness of solution for the system of differential equations of interest. Its proof can be found in [42, Theorem 3.3].

**Lemma 2.1.** Assume condition (H2). Then for each initial condition \((X(0), \theta(0)) = (X_0, \theta)\)
with \( \theta \in \mathcal{M} = \{1, \ldots, m\} \), there exists a unique solution \((X(t), \theta(t))\) to (2.6) and (2.2) for \( t \geq 0 \).

**Remark 2.2.** To proceed, we explore the regularity and nonnegativity of solutions to (2.6) and (2.2). To get the regularity only, one can use a Lyapunov function \( V(x, \theta) = |x| \). Then it can be verified that \( \mathcal{L}V(x, \theta) \leq cV(x, \theta) \) for some \( c > 0 \). However, to show that the process will remain in the first quadrant, more complex Lyapunov function is needed as can be seen in the proof to follow.

**Theorem 2.3.** Assume (H3) and \( X_0 \in \mathbb{R}_+^r := \{(x_1, \ldots, x_r) : x_i > 0, i = 1, \ldots, r\} \). Then the solution to (2.6) will remain in \( \mathbb{R}_+^r \) almost surely. That is, \( X(t) \in \mathbb{R}_+^r \) a.s. for any \( t \geq 0 \).

**Proof.** Consider (2.1). Using an argument of [19] for diffusions, assumption (H3) indicates that the coefficients of the stochastic differential equation (2.6) are locally Lipschitz and “locally” linear growth; see [48, Chapter 2]. Therefore, there is an explosion time \( \rho_e \) such that for all \( t \in [0, \rho_e) \), there exists a local solution for (2.6). Let \( k_0 > 0 \) be sufficiently large such that \( X_i(0) \in ((1/k_0), k_0) \) for each \( i = 1, \ldots, r \). For each \( k \geq k_0 \), we define

\[
\tau_k := \inf \left\{ t \in [0, \rho_e) : X_i(t) \notin \left( \frac{1}{k}, k \right) \text{ for some } i = 1, 2, \ldots, r \right\}.
\]

The sequence \( \tau_k \) is monotonically increasing. Set \( \tau_\infty := \lim_{k \to \infty} \tau_k \). Then \( \tau_\infty \leq \rho_e \).

We are in a position to prove \( \tau_\infty = \infty \) a.s. Suppose that this were not true. Then there would exist a \( T > 0 \) and \( \varepsilon > 0 \) such that \( P\{\tau_\infty < T\} > \varepsilon \). Thus, there is a \( k_1 \) such that
\( P\{\tau_k < T\} > \varepsilon \) for all \( k \geq k_1 \). Denote

\[
S(x) = \sum_{i=1}^{r} x_i, \quad (2.9)
\]

and define a Lyapunov function

\[
V(x, \theta) = \sum_{i=1}^{r} x_i - \log S(x) \quad \text{where} \quad (x, \theta) \in \mathbb{R}_+^r \times \mathcal{M}.
\]

It is easily seen that

\[
\frac{\partial}{\partial x_i} \log S(x) = \frac{1}{S(x)}, \quad \frac{\partial^2}{\partial x_i^2} \log S(x) = -\frac{1}{S^2(x)}.
\]

Direct calculation leads to

\[
\mathcal{L}V(x, \theta) = \sum_{i=1}^{r} \left[ \gamma(\theta)x_i - x_i^3 - \beta(\theta)(x_i - \bar{x}) \right] \\
+ \frac{1}{S(x)} \sum_{i=1}^{r} \left[ -\gamma(\theta)x_i + x_i^3 + \beta(\theta)(x_i - \bar{x}) \right] \\
+ \frac{1}{2} \frac{1}{S^2(x)} \sum_{i=1}^{r} \sigma^2_{ii}(x, \theta) \quad \text{for each} \quad \theta \in \mathcal{M}. \quad (2.10)
\]

Since \( x_i > 0 \) for each \( i \), using the familiar inequality

\[
\sum_{i=1}^{r} x_i^p \leq \left( \sum_{i=1}^{r} x_i \right)^p, \quad p > 1.
\]
\[- \frac{1}{S(x)} \sum_{i=1}^{r} \gamma(\theta)x_i = -\gamma(\theta), \]
\[\frac{1}{S(x)} \sum_{i=1}^{r} \beta(\theta)(x_i - \bar{x}) = 0, \]
\[\frac{1}{S(x)} \sum_{i=1}^{r} x_i^3 \leq \frac{1}{S(x)} \left( \sum_{i=1}^{r} x_i \right)^3 = S^2(x), \]
\[\frac{1}{S(x)} \sum_{i=1}^{r} \sigma_{ii}(x, \theta) \leq \frac{1}{S^2(x)} \sum_{i=1}^{r} x_i^{4-\delta} \leq S^{2-\delta}(x). \]

Using (2.12) in (2.10), detailed estimates lead to that when \( x_i > 0 \) is large, the value of \( \mathcal{L}V(x, \theta) \) is dominated by \(-x_i^3\); when \( x_i > 0 \) is small, the value of \( \mathcal{L}V(x, \theta) \) is dominated by a constant by using the bound of \( \sigma(x, \theta) \) in assumption (2.4). Thus, in any event,

\[\mathcal{L}V(x, \theta) \leq K \text{ where } K > 0 \text{ is independent of } k. \] (2.13)

By virtue of the definitions of \( \tau_k \) and \( V(x, \theta) \),

\[V(X(\tau_k), \theta(\tau_k)) \geq (k - r \log k) \wedge \left( \frac{1}{k} + \log k \right). \]

By means of Dynkin’s formula,

\[\mathbb{E}V(X(T \wedge \tau_k), \theta(T \wedge \tau_k)) - V(X(0), \theta(0)) = \mathbb{E} \int_0^{\tau_k \wedge T} \mathcal{L}V(X(s), \theta(s))ds \leq KT. \]
By rearrangement,

\[ KT + V(X(0), \theta(0)) \geq \mathbb{E} V(X(\tau_k \wedge T), \theta(\tau_k \wedge T)) \]
\[ \geq \mathbb{E} V(X(\tau_k), \theta(\tau_k))I_{\{\tau_k < T\}} \]
\[ \geq (k - r \log k) \wedge \left( \frac{1}{k} + \log k \right) \mathbb{P}(\tau_k < T) \]
\[ \geq \left[ (k - r \log k) \wedge \left( \frac{1}{k} + \log k \right) \right] \varepsilon \]
\[ \rightarrow \infty \text{ as } k \rightarrow \infty. \]

This is a contradiction. As a result, \( \lim_{k \to \infty} \tau_k = \infty \) a.s. and hence the explosion time \( \rho_e = \infty \) a.s. \( \square \)

Next, consider moment properties of the process \( X(t) \). We show that the moment generating function

\[ M(z) = \mathbb{E} \exp(z'X(t)), \text{ for any } t \geq 0, \; z' = (z_1, \ldots, z_r) \in \mathbb{R}^r \; \text{with} \; z_i \in \mathbb{R} \]

exists. To proceed, we first obtain a finite moment result.

**Lemma 2.4.** Under the conditions of Theorem 2.3, for any \( p \geq 2 \),

\[ \sup_{t \geq 0} \mathbb{E} \left[ \sum_{i=1}^r X_i^p(t) \right] \leq K < \infty. \]

**Proof.** For any \( (x, \theta) \in \mathbb{R}_+^r \times \mathcal{M} \), consider \( V(x, \theta) = S^p(x) \) with \( S(x) \) defined in (2.9). Using
the stopping time $\tau_k$ defined in (2.8), we have

$$
\mathcal{L}V(x, \theta) = pS^{p-1}(x) \sum_{i=1}^{r} [\gamma(\theta)x_i - x_i^3 - \beta(\theta)(x_i - \overline{x})] + \frac{1}{2} p(p - 1)S^{p-2}(x) \sum_{i=1}^{r} \sigma_{ii}^2(x, \theta)
$$

$$
= p\gamma(\theta)S^p(x) - pS^{p-1}(x) \sum_{i=1}^{r} x_i^3 + \frac{1}{2} p(p - 1)S^{p-2} \sum_{i=1}^{r} \sigma_{ii}^2(x, \theta).
$$

By virtue of Dynkin’s formula,

$$
E[e^{t \wedge \tau_k} S^p(X(t \wedge \tau_k)) - S^p(X(0))]
$$

$$
= E \int_{0}^{t \wedge \tau_k} e^{s}[V(X(s), \theta(s)) + \mathcal{L}V(X(s), \theta(s))] ds
$$

$$
\leq E \int_{0}^{t \wedge \tau_k} e^s \left[ 1 + p\gamma(\theta) \right] S^p(x) + \frac{1}{2} p(p - 1)S^{p-2} \left( \sum_{i=1}^{r} x_i \right)^2 \right] ds
$$

$$
\leq E \int_{0}^{t \wedge \tau_k} e^s K ds
$$

$$
\leq K(e^t - 1).
$$

Since $\lim_{k \to \infty} \tau_k = \infty$ a.s., letting $k \to \infty$, we obtain

$$
E[e^{t \wedge \tau_k} S^p(X(t \wedge \tau_k))] - S^p(X(0)) \leq K(e^t - 1)
$$

so

$$
E S^p(X(t)) \leq e^{-t} S^p(X(0)) + K(1 - e^{-t}) \leq K < \infty.
$$

Using (2.11) and taking $\sup_{t \geq 0}$, the desired result then follows. □

By virtue of Theorem 2.3, we can show $E_{X_i}(t) \leq K < \infty$. This together with Lemma 2.4 yields that for any positive integer $l$, $E_{X_i}(t) \leq K < \infty$ for each $i = 1, \ldots, r$. Thus for any
given \( z \in \mathbb{R}^r \), we have \( \mathbb{E} z_i^l X_i^l(t) \leq K < \infty \) for each \( i = 1, \ldots, r \). As a result,

\[
\sum_{l=0}^{\infty} \sum_{i=1}^{r} \frac{z_i^l \mathbb{E} X_i^l(t)}{l!}
\]

converges absolutely and uniformly.

The existence of the moment generating function then follows. We summarize this into the following proposition.

**Proposition 2.5.** The moment generating function \( M(z) = \mathbb{E} \exp(z^t X(t)) \) exists for any \( z \in \mathbb{R}^r \) and \( t \geq 0 \).

Then we consider the sample path continuity. In fact, the desired result is obtained by means of an auxiliary bound. To proceed, we first establish the following lemma.

**Lemma 2.6.** Under the conditions of Theorem 2.3, for any positive integer \( \kappa \), \( 0 < T < \infty \), and any \( 0 \leq t, s \leq T \), there is a positive constant \( K \) such that

\[
\mathbb{E} |X(t) - X(s)|^{2\kappa} \leq K |t - s|^{\kappa}.
\]  

(2.15)

**Proof.** It suffices to examine each component \( X_i(\cdot) \). It is easily seen that for any positive integer \( \kappa \), \( 0 < T < \infty \), and any \( 0 \leq t, s \leq T \),

\[
X_i(t) - X_i(s) = \int_s^t \mu_i(X(u), \theta(u)) du + \int_s^t \sigma_{ii}(X(u), \theta(u)) dB_i(u),
\]

and as a result

\[
\left| X_i(t) - X_i(s) \right|^{2\kappa} \leq 2^{2\kappa - 1} \left[ \left| \int_s^t \mu_i(X(u), \theta(u)) du \right|^{2\kappa} + \left| \int_s^t \sigma_{ii}(X(u), \theta(u)) dB_i(u) \right|^{2\kappa} \right].
\]
An application of the Hölder inequality leads to

\[
E \left| \int_s^t \mu_i(X(u), \theta(u)) \, du \right|^{2\kappa} \\
\leq \left( \int_s^t \, du \right)^{2\kappa - 1} \int_s^t E[\mu_i(X(u), \theta(u))]^{2\kappa} \, du \\
\leq K(t-s)^{2\kappa}. \tag{2.16}
\]

The last line above follows from the moment estimate in Lemma 2.4.

Next, we estimate the diffusion term. By using \[25, \text{Lemma 4.12, p. 131}\], we have

\[
E \left| \int_s^t \sigma_{ii}(X(u), \theta(u)) \, dB_i(u) \right|^{2\kappa} \\
\leq [\kappa(2\kappa - 1)^{\kappa}(t-s)^{\kappa-1}] \int_s^t E[\sigma_{ii}(X(u), \theta(u))]^{2\kappa} \, du \\
\leq K(t-s)^{\kappa}. \tag{2.17}
\]

Combining the estimates in (2.16) and (2.17), the desired moment estimate follows.

\[\Box\]

**Theorem 2.7.** Under the conditions of Theorem 2.3, the process \(X(\cdot)\), which is the solution of (2.1), has continuous sample paths almost surely.

**Proof.** It suffices to examine each component. The assertion is a direct consequence of Lemma 2.6 and the well-known Kolmogorov continuity criterion \[40, \text{Theorem 2, p. 3}\]. \[\Box\]

### 2.3 Positive Recurrence

Recall that an \(\mathbb{R}^r\)-valued Markov process \(\xi(t)\) satisfying \(\xi(0) = x\) (denoted by \(\xi^x(t)\) when we want to emphasize the initial data \(x\)-dependence) is recurrent with respect to some nonempty bounded open set \(G \subset \mathbb{R}^r\) if \(P\{\tau^x < \infty\} = 1\) for any \(x \notin G\), where \(x = \xi(0)\) and \(\tau^x\) is the hitting time of \(G\) for \(\xi^x(t)\) (i.e., the first time that the process \(\xi^x(t)\) enters the set \(G\), or
\( \tau^x := \inf \{ t \geq 0 : \xi^x(t) \in G \} \). The process \( \xi(t) \) is said to be positive recurrent with respect to \( G \) if \( \mathbb{E}\tau^x < \infty \) for any \( x \notin G \).

**Theorem 2.8.** The solution is positive recurrent with respect to the domain

\[
G_\rho := \{ x \in \mathbb{R}_+^r : 0 < x_i < \rho, \; i = 1, 2, \ldots, r \},
\]

where \( \rho \) is a positive number to be specified.

**Proof.** By virtue of [48], it suffices for each \( \theta \in \mathcal{M} \) to find a nonnegative Lyapunov function \( V(\cdot, \theta) \) defined on \((\mathbb{R}_+^r - \overline{G}_\rho) \times \mathcal{M}\) such that \( V(\cdot, \theta) \) is twice continuously differentiable with respect to \( x \) and satisfies \( \mathcal{L}V(x, \theta) \leq -1 \) for all \((x, \theta) \in (\mathbb{R}_+^r - \overline{G}_\rho) \times \mathcal{M}\), where \( \overline{G}_\rho \) denotes the closure of \( G_\rho \). We consider the nonnegative function

\[
V(x, \theta) = \sum_{i=1}^r (\log x_i)^2, \quad (x, \theta) \in (\mathbb{R}_+^r - \overline{G}_\rho) \times \mathcal{M}.
\]

Note that for getting the positive recurrence, it suffices to work with a Lyapunov function that is defined in the exterior of the bounded set \( G_\rho \). Note also that when \( x_i > e \), \( \log x_i > 1 \).

Thus, in view of the fact \( \log x_i / x_i \leq 1 \),

\[
\sum_{i=1}^r 2\beta(\theta) \frac{\log x_i}{x_i} \sum_{j=1}^r \frac{x_j}{r} \leq 2\beta(\theta) \frac{\sum_{i=1}^r \sum_{j=1}^r x_j}{r} \leq \sum_{i=1}^r 2\beta(\theta) x_i \log x_i \tag{2.18}
\]
when $\rho$ is large enough. Note also that for $x_i > e$, \((1 - \log x_i)/x_i^2 < 0\). As a result,

$$
\sum_{i=1}^{r} \sigma_{ii}^2(x, \theta) \frac{1 - \log x_i}{x_i^2} < 0.
$$

Consequently, using (2.18) and (2.19),

$$
\mathcal{L}V(x, \theta) = \sum_{i=1}^{r} 2 \log x_i (\gamma(\theta) - x_i^2 - \beta(\theta)) + \sum_{i=1}^{r} 2 \beta(\theta) \frac{\log x_i}{x_i} \sum_{j=1}^{r} \frac{x_j}{r} \\
\quad + \sum_{i=1}^{r} \sigma_{ii}^2(x, \theta) \frac{1 - \log x_i}{x_i^2} < \sum_{i=1}^{r} 2 \log x_i (\gamma(\theta) - x_i^2 - \beta(\theta) + \beta(\theta)x_i) \\
\quad = \sum_{i=1}^{r} 2 \log x_i \left[-(x_i - \frac{\beta(\theta)}{2})^2 + \left(\frac{\beta(\theta)}{2}\right)^2 - \beta(\theta) + \gamma(\theta)\right]
$$

Thus we can find $\rho$ large enough such that for all $(x, \theta) \in (\mathbb{R}^r_+ - \overline{G}_\rho) \times \mathcal{M}$, \(\mathcal{L}V(x, \theta) < -1\). Therefore $X(t)$ is positive recurrent with respect to the domain $G_\rho$.

Since the process is positive recurrent with respect to $G_\rho$, there is an invariant measure there ([50] and also [48, Chapter 4]). Furthermore, there is an invariant density for the joint process $(X(t), \theta(t))$, denoted by \(\{\alpha(x, i) : i \in \mathcal{M}\}\) such that

$$
\mathcal{L}^* \alpha(x, i) = 0, \quad \sum_{i \in \mathcal{M}} \int_{\mathbb{R}^r} \alpha(x, i) dx = 1,
$$

where $\mathcal{L}^*$ is the adjoint of the operator $\mathcal{L}$. 
2.4 Further Asymptotic Bounds

In this section, we derive further asymptotic bounds in the sense of almost sure estimates. The result reveals long-time behavior and stability.

**Theorem 2.9.** The solution $X(t)$ satisfies

$$\limsup_{T \rightarrow \infty} \frac{\log (X(T))}{\log T} \leq K \quad \text{a.s.}$$

for some $K > 0$.

**Proof.** Choose $V(t, x, \theta) = e^t \log (\overline{x})$ for $(t, x, \theta) \in [0, \infty) \times \mathbb{R}^r_+ \times \mathcal{M}$, where $\overline{x} = \sum_{i=1}^r x_i/r$.

It is readily seen that

$$\frac{\partial V}{\partial t} = e^t \log(\overline{x}), \quad \frac{\partial V}{\partial x_i} = \frac{1}{r \overline{x}} e^t, \quad \text{and} \quad \frac{\partial^2 V}{\partial x_i^2} = -\frac{1}{r^2 \overline{x}^2} e^t.$$

Then by virtue of Itô’s formula,

$$e^t \log (\overline{X}(t)) - \log (\overline{X}(0))$$

$$= \int_0^t e^s \log (\overline{X}(s))ds$$

$$+ \int_0^t e^s \left\{ \sum_{i=1}^r \frac{1}{r \overline{X}(s)} \left[ \gamma(\theta(s))X_i(s) - X_i^3(s) - \beta(\theta(s))(X_i(s) - \overline{X}(s)) \right] + \sum_{i=1}^r \frac{1}{2} \sigma_{ii}(X(s), \theta(s)) \frac{-1}{r^2 \overline{X}^2(s)} \right\} ds$$

$$+ \int_0^t e^s \sum_{i=1}^r \frac{\sigma_{ii}(X(s), \theta(s))}{r \overline{X}(s)} dw_i(s).$$
Denote
\[ M_i(t) = \int_0^t e^{s} \frac{1}{rX(s)} \sigma_{ii}(X(s), \theta(s))dw_i(s), \]
whose quadratic variation is
\[ \langle M_i, M_i \rangle(t) = \int_0^t e^{2s} \frac{\sigma_{ii}^2(X(s), \theta(s))}{r^2X^2(s)} ds. \]

By the familiar exponential martingale inequality (e.g., [32, p. 49]), for any positive constants \( T, \delta, \) and \( \eta, \) we have
\[ P\{ \sup_{0 \leq t \leq T} [M_i(t) - \frac{\delta}{2} \langle M_i, M_i \rangle(t)] > \eta \} \leq e^{-\delta \eta}. \]

Choose
\[ T = k\omega, \ \delta = \varepsilon e^{-k\omega}, \ \text{and} \ \eta = \frac{\theta e^{k\omega} \log k}{\varepsilon}, \]
where \( k \in \mathbb{N}, \ 0 < \varepsilon < 1, \ \theta > 1, \) and \( \omega > 0. \) Then similarly to that of [51], we can show that
\[ M_i(t) \leq \frac{\varepsilon e^{-k\omega}}{2} \langle M_i, M_i \rangle(t) + \frac{\theta e^{k\omega} \log k}{\varepsilon}, \ \text{for all} \ 0 \leq t \leq k\gamma. \]

\[
e^t \log (\overline{X}(t)) - \log (\overline{X}(0))
\leq \int_0^t e^s \{ \log (\overline{X}(s)) + \sum_{i=1}^r \frac{1}{rX(s)} [\gamma(\theta(s))X_i(s) - X_i^3(s) - \beta(\theta(s))(X_i(s) - \overline{X}(s))] \\
+ \sum_{i=1}^r \frac{\sigma_{ii}^2(X(s), \theta(s))}{2} \frac{1}{r^2X^2(s)} \} ds + \frac{r\theta e^{k\omega} \log k}{\varepsilon}. \]

(2.22)
Since $\varepsilon e^{-k\omega}e^s < 1$,

$$\sum_{i=1}^{r} \frac{\sigma_i^2(X(s), \theta)}{2} \frac{1}{r^2 x^2} + \frac{\varepsilon e^{-k\omega}e^s \sigma_i^2(x, \theta)}{2} \frac{1}{r^2 x^2} = \sum_{i=1}^{r} \frac{\sigma_i^2(x, \theta)}{2r^2 x^2} (-1 + \varepsilon e^{-k\omega}e^s) < 0.$$  

In addition, note that

$$\frac{1}{r\overline{x}} \sum_{i=1}^{r} \left[ \gamma(\theta)x_i - \beta(\theta)(x_i - \overline{x}) \right] = \gamma(\theta).$$

Using the H"older inequality,

$$\left( \sum_{i=1}^{r} x_i \right)^{3} \leq \left( \sum_{i=1}^{r} 1^{3/2} \right)^{2} \sum_{i=1}^{r} x_i^{3} \leq r^2 \sum_{i=1}^{r} x_i^{3}.$$  

As a result,

$$\log \overline{x} - \frac{1}{r\overline{x}} \sum_{i=1}^{r} x_i^{3} \leq \log \overline{x} - \frac{1}{r^3 \overline{x}} \left( \sum_{i=1}^{r} x_i \right)^{3} = \log \overline{x} - \overline{x}^2 < 0.$$  

Using the above estimates, we obtain

$$\log \overline{x} + \frac{1}{r\overline{x}} \sum_{i=1}^{r} \left[ \gamma(\theta)x_i - \beta(\theta)(x_i - \overline{x}) \right] \leq \gamma(\theta). \quad (2.23)$$

Therefore, using (2.23) and (2.22), we obtain

$$e^t \log \left( \overline{X}(t) \right) - \log \left( \overline{X}(0) \right) \leq \int_{0}^{t} e^s Cds + \frac{r\theta e^{k\omega} \log k}{\varepsilon} = C(e^t - 1) + \frac{r\theta e^{k\omega} \log k}{\varepsilon}, \quad (2.24)$$

where $C$ is a positive constant. Thus for $(k - 1)\omega \leq t \leq k\omega$, similar to the development
in [51], by sending $\omega \downarrow 0$, $\varepsilon \uparrow 1$, and $\theta \downarrow 1$, we have

$$\limsup_{T \to \infty} \frac{\log X(T)}{\log T} \leq K \text{ a.s.}$$

The result is proved. \qed

As a direct consequence, we obtain the following corollary.

**Corollary 2.10.** Under the conditions of Theorem 2.9,

$$\limsup_{T \to \infty} \frac{\log X(T)}{T} \leq 0 \text{ a.s.,}$$

and

$$\limsup_{T \to \infty} \frac{\log |X(T)|}{T} \leq 0 \text{ a.s.}$$

### 2.5 A Two-Time-Scale Limit

This section is concerned with a class of Mean-field processes, in which the random switching process changes an order of magnitude faster than the continuous state (or the switching process jump change much more frequently). The basic premise is that there are inherent two-time scales. Our interest focuses on the limit behavior of the resulting process. Suppose that $\varepsilon > 0$ is a small parameter and the system of Mean-field equations is given by

$$dX_t^\varepsilon = \left[ \gamma(\theta^\varepsilon(t))X_t^\varepsilon - (X_t^\varepsilon)^3 - \beta(\theta^\varepsilon(t))(X_t^\varepsilon - \bar{X}^\varepsilon) \right]dt + \sigma_i(X_t^\varepsilon, \theta^\varepsilon(t))dB_i(t),$$

(2.25)
or using the definition of $\mu(x, \theta)$,

$$dX^\varepsilon(t) = \mu(X^\varepsilon(t), \theta^\varepsilon(t))dt + \sigma(X^\varepsilon(t), \theta^\varepsilon(t))dB(t),$$  \hspace{1cm} (2.26)

where $\theta^\varepsilon(t)$ is a fast-varying process whose generator is $Q(x)/\varepsilon$ when $X^\varepsilon(t) = x$. Compared with our previous work on two-time-scale Markov processes [44], where time-inhomogeneous Markov chains are treated, the new contribution is featured in the $x$-dependence of the switching process. In this paper, the switching process itself is non-Markov. To overcome the difficulty, we sub-divide the interval into small part, and use careful approximation techniques to resolve the $x$-dependency issue. Recall that $Q(x)$ is bounded and continuous.

**(H4)** For each $x \in \mathbb{R}^r$, $Q(x)$ is weakly irreducible. That is, the system of equations

$$\nu(x)Q(x) = 0, \quad \nu(x)\mathbb{1} = 1$$

has a unique solution where $\mathbb{1} := (1, \ldots, 1)' \in \mathbb{R}^{m \times 1}$ is a vector with all component being 1, where $\nu(x) = (\nu_1(x), \ldots, \nu_m(x))$ with $\nu_i(x) \geq 0$ for each $i \in \mathcal{M}$.

As can be seen, $\theta^\varepsilon(\cdot)$ is subject to fast variation, whereas $X^\varepsilon(\cdot)$ changes relatively slowly compared with $\theta^\varepsilon(\cdot)$. Although it is subject to rapid variations, the $\theta^\varepsilon(\cdot)$ does not go to $\infty$; it is essentially a noise process having an invariant measure. As $\varepsilon \to 0$, the noise is averaged out, and the slow component of the evolution $X^\varepsilon(\cdot)$ converges weakly to $X(\cdot)$ that is an average with respect to the invariant measure $\nu(x)$ given in (H4); see also the idea of averaging in systems with singularly perturbed diffusions in [21] and references therein. Let us consider the process $\{X^\varepsilon(\cdot)\}$, and work with $t \in [0, T]$ for some $T > 0$. 
Lemma 2.11. Assume both (H2) and (H4). Then

\[ \sup_{t \in [0,T]} E|X^\varepsilon(t)|^2 < \infty. \]

This is barely a restatement of the moment bounds in the previous sections. Using similar idea as in Lemma 2.4, we can also show that \( E|X^\varepsilon(t)|^p < \infty \).

Lemma 2.12. Assume both (H2) and (H4). Then \( \{X^\varepsilon(\cdot)\} \) is tight in \( D([0,T] : \mathbb{R}^r) \), the space of functions that are right continuous with left limits endowed with the Skorohod topology.

Proof. For any \( \Delta > 0 \), and \( t, s > 0 \) satisfying \( s \leq \Delta \), using Lemma 2.11 and the same technique as in Lemma 2.6,

\[ E_t|X^\varepsilon(t + s) - X^\varepsilon(t)|^2 \leq O(s) \leq O(\Delta), \]

where \( E_t \) denotes the expectation conditioning on the \( \sigma \)-algebra generated by \( \{X^\varepsilon(u), \theta^\varepsilon(u) : u \leq t\} \). Taking \( \limsup_\varepsilon \) followed by \( \lim_\Delta \), we obtain

\[ \lim_\Delta \limsup_\varepsilon \sup_{t \leq 0} E|X^\varepsilon(t + s) - X^\varepsilon(t)|^2 = 0. \]

Thus, by virtue of the well-known tightness criterion (for example, see [45, Lemma 14.12, p.320]), \( \{X^\varepsilon(\cdot)\} \) is tight. \( \square \)

Since \( \{X^\varepsilon(\cdot)\} \) is tight, we can extract weakly convergent subsequences by the well-known Prohorov’s theorem. Select such a subsequence and for notational simplicity, still denote the subsequence indexed by \( \varepsilon \) with limit \( X(\cdot) \). By virtue of the Skorohod representation, there is
an augmented probability space on which there is a sequence $\tilde{X}_\varepsilon(\cdot)$ defined on it having the same distribution as $X_\varepsilon(\cdot)$ such that $\tilde{X}_\varepsilon(\cdot)$ converges to $\tilde{X}(\cdot)$ in the sense of w.p.1, where $\tilde{X}(\cdot)$ have the same distribution as that of $X(\cdot)$. With a slight abuse of notation without changing notation, we still denote this sequence by $\{X_\varepsilon(\cdot)\}$ such that $X_\varepsilon(\cdot) \to X(\cdot)$ w.p.1.

Using the argument as in Theorem 2.7, $X(\cdot)$ has continuous sample paths w.p.1. We proceed to characterize the limit process.

To proceed, for an arbitrary $N$ satisfying $0 < N < \infty$, we can confine ourselves with $S_N = \{x : |x| \leq N\}$, the ball with radius $N$, and work with a truncated process $X_{\varepsilon,N}(\cdot)$, known as $N$-truncation [45, p.321]. We then obtain the limit of $X_{\varepsilon,N}(\cdot)$. Finally by letting $N \to \infty$ and using a piecing together argument, we prove the convergence of $X_\varepsilon(\cdot)$. However, for notational simplicity and without loss of generality, we can assume that $X_\varepsilon(\cdot)$ is bounded in what follows. We shall show that the limit $X(\cdot)$ is a solution of the Mean-field equation

$$dX(t) = \overline{\nu}(X(t))dt + \overline{\sigma}(X(t))dB(t), \quad (2.27)$$

where

$$\overline{\nu}(x) = \sum_{i \in M} \nu_i(x) \mu(x, i),$$
$$\overline{\sigma}(x) = \sum_{i \in M} \nu_i(x) \sigma(x, i),$$
$$\overline{a}(x) = \overline{\sigma}(x) \overline{\sigma}(x), \quad a(x, i) = \sigma(x, i) \sigma'(x, i),$$
$$\nu(x) = (\nu_1(x), \ldots, \nu_m(x)) \in \mathbb{R}^{1 \times m}. \quad (2.28)$$

Equivalently, $X(\cdot)$ is a solution of the martingale problem with operator $\overline{\mathcal{L}}$ defined by

$$\overline{\mathcal{L}}f(x) = \overline{\nu}'(x) \nabla f(x) + \text{tr}[\overline{\sigma}(x) H f(x)], \quad (2.29)$$
for any $f(\cdot) \in C^2(\mathbb{R}^r)$, where $\nabla f(x)$ and $Hf(x)$ are the usual gradient and Hessian matrix of $f(x)$, respectively.

**Theorem 2.13.** Under the conditions of Lemma 2.12, the process $X^\varepsilon(\cdot)$ converges weakly to $X(\cdot)$, which is the solution of the martingale problem with operator $\overline{L}$ given by (2.29) or $X(\cdot)$ is a solution of the limit Mean-field equation given by (2.27).

**Proof.** Since we have already established the tightness of the process $\{X^\varepsilon(\cdot)\}$, what remains to be done is to characterize the limit process $X(\cdot)$. To show that $X(\cdot)$ is a solution of the martingale problem with operator $\overline{L}$, pick out any $F(\cdot) \in C^2_0(\mathbb{R}^r)$ ($C^2$ function with compact support). We need only show that

$$F(X(t)) - F(X(0)) - \int_0^t \overline{L}F(X(u))du \text{ is a martingale.} \quad (2.30)$$

To verify (2.30), it suffices to show that for any bounded and continuous function $h(\cdot)$, any positive integer $\ell$, any $t, s > 0$, and any $t_l \leq t$ with $l \leq \ell$,

$$\mathbb{E}h(X(t_l) : l \leq \ell)\left[F(X(t + s)) - F(X(t)) - \int_t^{t+s} \overline{L}F(X(u))du\right] = 0. \quad (2.31)$$

To verify (2.31), we begin with the process indexed by $\varepsilon$, namely $\{X^\varepsilon(\cdot)\}$. Because $F(\cdot)$ is independent of $i \in \mathcal{M}$,

$$\sum_{j \in \mathcal{M}} q_{ij}^\varepsilon(x)F(x) = 0 \text{ for each } i \in \mathcal{M} \text{ and each } x. \quad (2.32)$$
Since the joint process \((X^\varepsilon(\cdot), \theta^\varepsilon(\cdot))\) is Markov,

\[
F(X^\varepsilon(t+s)) - F(X^\varepsilon(t)) - \int_t^{t+s} \mathcal{L}F(X^\varepsilon(u))du
\]

is a martingale, and as a result,

\[
\mathbb{E}h(X^\varepsilon(t_l) : l \leq l) \left[ F(X^\varepsilon(t+s)) - F(X^\varepsilon(t)) - \int_t^{t+s} \mathcal{L}F(X^\varepsilon(u))du \right] = 0,
\]

where \(\mathcal{L}\) is the operator defined in (2.7) with \(Q(x)\) replaced by \(Q(x)/\varepsilon\). That is, in view of (2.32),

\[
\mathcal{L}F(x) = \frac{1}{2} \text{tr}[a(x,i)HF(x)] + \mu'(x,i)\nabla F(x), \quad i \in \mathcal{M}.
\]

Note that since \(F(\cdot)\) is independent of \(i \in \mathcal{M}\), the term involving \(Q(x)/\varepsilon\) disappears. Note also that \(\mathcal{L}\) depends on \(\varepsilon\) and should have been written as \(\mathcal{L}^\varepsilon\), but for notational simplicity, we suppress the \(\varepsilon\)-dependence. By the weak convergence of \(X^\varepsilon(\cdot)\) to \(X(\cdot)\) and the Skorohod representation, we have

\[
\mathbb{E}h(X^\varepsilon(t_l) : l \leq l) \left[ F(X^\varepsilon(t+s)) - F(X^\varepsilon(t)) \right] \rightarrow \mathbb{E}h(X(t_l) : l \leq l) \left[ F(X(t+s)) - F(X(t)) \right] \quad \text{as} \quad \varepsilon \rightarrow 0. \tag{2.33}
\]

On the other hand,

\[
\mathbb{E}h(X^\varepsilon(t_l) : l \leq l) \left[ \int_t^{t+s} \mathcal{L}F(X^\varepsilon(u))du \right] = \mathbb{E}h(X^\varepsilon(t_l) : l \leq l) \left[ \int_t^{t+s} [\mu'(X^\varepsilon(u), \theta^\varepsilon(u))\nabla F(X^\varepsilon(u)) + \frac{1}{2} \text{tr}[a(X^\varepsilon(u), \theta^\varepsilon(u))HF(X^\varepsilon(u))]du \right]
\]
First, consider the drift term, we obtain

\[ \mathbb{E} h(X^\varepsilon(t_1) : l \leq \ell) \left[ \int_t^{t+s} \mu'(X^\varepsilon(u), \theta^\varepsilon(u)) \nabla F(X^\varepsilon(u)) du \right] \]

\[ = \mathbb{E} h(X^\varepsilon(t_1) : l \leq \ell) \left[ \int_t^{t+s} \sum_{i \in M} \mu'(X^\varepsilon(u), i) \nabla F(X^\varepsilon(u)) I_{\{\theta^\varepsilon(u) = i\}} \right] \]

\[ = \mathbb{E} h(X^\varepsilon(t_1) : l \leq \ell) \left[ \int_t^{t+s} \sum_{i \in M} \mu'(X^\varepsilon(u), i) \nabla F(X^\varepsilon(u)) [I_{\{\theta^\varepsilon(u) = i\}} - \nu_i(X^\varepsilon(u))] \right] \]

(2.34)

\[ + \mathbb{E} h(X^\varepsilon(t_1) : l \leq \ell) \left[ \int_t^{t+s} \sum_{i \in M} \mu'(X^\varepsilon(u), i) \nabla F(X^\varepsilon(u)) \nu_i(X^\varepsilon(u)) \right]. \]

By virtue of the weak convergence of \( X^\varepsilon(\cdot) \) to \( X(\cdot) \) and the Skorohod representation (without changing notation by our convention), it can be shown that for the last term in (2.34),

\[ \mathbb{E} h(X^\varepsilon(t_1) : l \leq \ell) \left[ \int_t^{t+s} \sum_{i \in M} \mu'(X^\varepsilon(u), i) \nabla F(X^\varepsilon(u)) \nu_i(X^\varepsilon(u)) \right] \]

\[ \to \mathbb{E} h(X(t_1) : l \leq \ell) \left[ \int_t^{t+s} \sum_{i \in M} \mu'(X(u), i) \nabla F(X(u)) \nu_i(X(u)) \right] \text{ as } \varepsilon \to 0 \]

(2.35)

\[ = \mathbb{E} h(X(t_1) : l \leq \ell) \left[ \int_t^{t+s} \overline{\pi}(X(u)) \nabla F(X(u)) du \right]. \]

As for the next to the last term in (2.34), we partition the interval \([t, t+s]\) as follows. For any \( 0 < \Delta < 1 \), let \( t = t_0 < t_1 < t_2 < \cdots < t_{l_\varepsilon} \leq t + s \) such that \( t_k = k\varepsilon^{1-\Delta} \). Note that \( l_\varepsilon = \lfloor s/\varepsilon^{1-\Delta} \rfloor = O(1/\varepsilon^{1-\Delta}) \). Without loss of generality and for notational simplicity, we will assume the \( t_{l_\varepsilon} \) coincides with \( t + s \). Then we can rewrite

\[ \int_t^{t+s} \sum_{i \in M} \mu'(X^\varepsilon(u), i) \nabla F(X^\varepsilon(u)) [I_{\{\theta^\varepsilon(u) = i\}} - \nu_i(X^\varepsilon(u))] \]

\[ = \sum_{k=0}^{l_\varepsilon-1} \int_{t_k}^{t_{k+1}} \sum_{i \in M} \mu'(X^\varepsilon(u), i) \nabla F(X^\varepsilon(u)) [I_{\{\theta^\varepsilon(u) = i\}} - \nu_i(X^\varepsilon(u))] du. \]
By the continuity of $\mu(\cdot, i)$ and the smoothness of $F(\cdot)$, we obtain

$$
\lim_{\varepsilon \to 0} E h(X^\varepsilon(t_\ell) : l \leq \ell) \left[ \sum_{i \in M} \varepsilon \int_{t_k}^{t_{k+1}} \mu'(X^\varepsilon(t_k), i) \nabla F(X^\varepsilon(t_k)) [I_{\{\theta^\varepsilon(u) = i\}} - \nu_i(X^\varepsilon(u))] du \right]
$$

$$
= \lim_{\varepsilon \to 0} E h(X^\varepsilon(t_\ell) : l \leq \ell) \left[ \sum_{k=0}^{l_{\ell-1}} \sum_{i \in M} \int_{t_k}^{t_{k+1}} \mu'(X^\varepsilon(t_k), i) \nabla F(X^\varepsilon(t_k)) [I_{\{\theta^\varepsilon(u) = i\}} - \nu_i(X^\varepsilon(u))] du \right].
$$

By the choice of $t_\ell$, we can rewrite the last line above as

$$
E h(X^\varepsilon(t_\ell) : l \leq \ell) \left[ \sum_{k=0}^{l_{\ell-1}} \sum_{i \in M} \int_{t_k}^{t_{k+1}} \mu'(X^\varepsilon(t_k), i) \nabla F(X^\varepsilon(t_k)) E_{t_k} \int_{t_k}^{t_{k+1}} [I_{\{\theta^\varepsilon(u) = i\}} - \nu_i(X^\varepsilon(u))] du \right]
$$

By virtue of the Cauchy-Schwarz inequality,

$$
E \left[ h(X^\varepsilon(t_\ell) : l \leq \ell) \left[ \sum_{k=0}^{l_{\ell-1}} \sum_{i \in M} \mu'(X^\varepsilon(t_k), i) \nabla F(X^\varepsilon(t_k)) E_{t_k} \int_{t_k}^{t_{k+1}} [I_{\{\theta^\varepsilon(u) = i\}} - \nu_i(X^\varepsilon(u))] du \right] \right]
$$

$$
\leq \sum_{k=0}^{l_{\ell-1}} \sum_{i \in M} E \left[ h(X^\varepsilon(t_\ell) : l \leq \ell) \mu'(X^\varepsilon(t_k), i) \nabla F(X^\varepsilon(t_k)) E_{t_k} \int_{t_k}^{t_{k+1}} [I_{\{\theta^\varepsilon(u) = i\}} - \nu_i(X^\varepsilon(u))] du \right]
$$

$$
\leq K \sum_{k=0}^{l_{\ell-1}} \sum_{i \in M} (1 + E^{1/2} |X^\varepsilon(t_k)|^2) E^{1/2} \left[ E_{t_k} \int_{t_k}^{t_{k+1}} [I_{\{\theta^\varepsilon(u) = i\}} - \nu_i(X^\varepsilon(u))] du \right]^2.
$$

(2.36)

**Lemma 2.14.** Assume the conditions of Theorem 2.13 are fulfilled. For $t \in [0, T]$, and each fixed $x$ in a bounded subset of $\mathbb{R}^r$, consider the generator $Q(x)/\varepsilon$. Then

$$
\left| \exp \left( \frac{Q(x)t}{\varepsilon} \right) - I \nu(x) \right| \leq K \exp \left( -\frac{\kappa_0 t}{\varepsilon} \right),
$$

(2.37)

for some $K > 0$ and $\kappa_0 > 0$, where $I = (1, 1, \ldots, 1)' \in \mathbb{R}^m$. 
**Proof.** For each $x$ in a bounded subset of $\mathbb{R}^r$, consider the switching process with a generator $Q(x)/\varepsilon$. Then the results in [44, Lemma A.2, p. 300] are applicable. In fact, $\exp(Q(x)t/\varepsilon)$ is the associated transition matrix. The weak irreducibility of $Q(x)$ implies that $\exp(Q(x)t/\varepsilon)$ converges to a matrix with identical rows, namely, $\mathbb{I} \nu(x)$. Moreover, the convergence takes place exponentially fast. Thus (2.37) holds. Since the set $x$ living in is bounded, $K$ and $\kappa_0$ can be chosen to be independent of $x$. \[ \Box \]

We now examine the last term in (2.36). We have by the continuity of $\nu(x)$,

$$
E_{t_k} \int_{t_k}^{t_{k+1}} [\nu_i(X^\varepsilon(t)) - \nu_i(X^\varepsilon(u))]du = o(t_{k+1} - t_k) = o(\varepsilon^{1-\Delta}).
$$

Thus

$$
\sum_{k=0}^{l_\varepsilon - 1} \sum_{i \in M} (1 + E^{1/2} |X^\varepsilon(t_k)|^2) E^{1/2} \left| E_{t_k} \int_{t_k}^{t_{k+1}} [\nu_i(X^\varepsilon(t)) - \nu_i(X^\varepsilon(u))]du \right|^2 
\leq K \sum_{k=0}^{l_\varepsilon - 1} o(\varepsilon^{1-\Delta}) \leq K' \varepsilon^{1-\Delta} = o(1) \to 0 \quad \text{as} \quad \varepsilon \to 0.
$$

(2.38)

Next, consider

$$
E_{t_k} \int_{t_k}^{t_{k+1}} [I_{\{\theta^\varepsilon(u) = i\}} - \nu_i(X^\varepsilon(t_k)))]du.
$$

For $u \in [t_k, t_{k+1}]$, to consider $E[I_{\{\theta^\varepsilon(u) = i\}} - \nu_i(X^\varepsilon(t_k))]]$, first let us examine the associated transition matrix

$$
pr^\varepsilon(u, t_k) = (p^\varepsilon_{ij}(u, t_k)) = (P(\theta^\varepsilon(u) = j|\theta^\varepsilon(t_k) = i, X^\varepsilon(t_k))).
$$
It satisfies the forward equation

\[
\frac{d}{du}pr^\varepsilon(u, t_k) = pr^\varepsilon(u, t_k) \frac{Q(X^\varepsilon(u))}{\varepsilon} = pr^\varepsilon(u, t_k) \frac{Q(X^\varepsilon(t_k))}{\varepsilon} + pr^\varepsilon(u, t_k) \frac{Q(X^\varepsilon(u)) - Q(X^\varepsilon(t_k))}{\varepsilon},
\]

\[pr^\varepsilon(t_k, t_k) = I.\]

The solution of the above matrix differential equation is given by

\[
pr^\varepsilon(u, t_k) = \exp \left( \frac{Q(X^\varepsilon(t_k))(u - t_k)}{\varepsilon} \right)
+ \int_{t_k}^{u} pr^\varepsilon(s, t_k) \frac{Q(X^\varepsilon(s)) - Q(X^\varepsilon(t_k))}{\varepsilon} \exp \left( \frac{Q(X^\varepsilon(t_k))(s - t_k)}{\varepsilon} \right) ds. \tag{2.39}
\]

Note that \(Q(X^\varepsilon(s)) I = Q(X^\varepsilon(t_k)) I = 0\) and \(\nu(x)Q(x) = 0\) for each \(x\). It then follows from (2.39) that

\[
pr^\varepsilon(u, t_k) - I \nu(X^\varepsilon(u)) = \left( \exp \left( \frac{Q(X^\varepsilon(t_k))(u - t_k)}{\varepsilon} \right) - I \nu(X^\varepsilon(t_k)) \right)
+ \int_{t_k}^{u} \left\{ pr^\varepsilon(s, t_k) - I \nu(X^\varepsilon(t_k)) \right\} \frac{Q(X^\varepsilon(s)) - Q(X^\varepsilon(t_k))}{\varepsilon}
+ I \left[ \nu(X^\varepsilon(t_k)) - \nu(X^\varepsilon(s)) \right] \frac{Q(X^\varepsilon(s))}{\varepsilon}
\exp \left( \frac{Q(X^\varepsilon(t_k))(s - t_k)}{\varepsilon} \right) - I \nu(X^\varepsilon(t_k)) \right) ds. \tag{2.40}
\]

Note that using Lemma 2.14, for some for some \(0 < \kappa_1 < \kappa_0\),

\[
\left| \int_{t_k}^{u} I \left[ \nu(X^\varepsilon(t_k)) - \nu(X^\varepsilon(s)) \right] \frac{Q(X^\varepsilon(s))}{\varepsilon} \left( \exp \left( \frac{Q(X^\varepsilon(t_k))(s - t_k)}{\varepsilon} \right) - I \nu(X^\varepsilon(t_k)) \right) ds \right|
\leq K \int_{t_k}^{u} \frac{1}{\varepsilon} \hat{g}_1(s) \exp \left( -\frac{\kappa_0(s - t_k)}{\varepsilon} \right) ds
\leq K \int_{t_k}^{u} \hat{g}_1(s) \exp \left( -\frac{\kappa_1(s - t_k)}{\varepsilon} \right) ds
\leq o(\varepsilon).
\]
In the above \( g_1^\varepsilon(s) \) is a continuous function satisfying \( g_1^\varepsilon(s) \to 0 \) as \( \varepsilon \to 0 \).

Thus, using Lemma 2.14, for some \( \kappa_0 > 0 \) and \( 0 < \kappa_2 < \kappa_0 \),

\[
|pr^\varepsilon(u, t_k) - \mathbb{1}_\nu(X^\varepsilon(u))| \\
\leq \exp\left( -\frac{\kappa_0(u - t_k)}{\varepsilon} \right) + o(\varepsilon) \\
+ K \int_{t_k}^u |pr^\varepsilon(s, t_k) - \mathbb{1}_\nu(X^\varepsilon(t_k))| g^\varepsilon(s) \frac{1}{\varepsilon} \exp\left( -\frac{\kappa_0(s - t_k)}{\varepsilon} \right) ds \\
\leq \exp\left( -\frac{\kappa_0(u - t_k)}{\varepsilon} \right) + o(\varepsilon) \\
+ K \int_{t_k}^u |pr^\varepsilon(s, t_k) - \mathbb{1}_\nu(X^\varepsilon(t_k))| g^\varepsilon(s) \exp\left( -\frac{\kappa_2(s - t_k)}{\varepsilon} \right) ds,
\]

where \( g^\varepsilon(s) \to 0 \) as \( \varepsilon \to 0 \). An application of Gronwall’s inequality [14, p. 36, Lemma 6.2] then yields that

\[
|pr^\varepsilon(u, t_k) - \mathbb{1}_\nu(X^\varepsilon(u))| \leq o(\varepsilon). \tag{2.41}
\]

It then follows that

\[
\left| E_{t_k} \int_{t_k}^{t_{k+1}} [\mathbb{1}_{\{\theta^\varepsilon(u) = i\}} - \nu_t(X^\varepsilon(u))] du \right| \\
\leq K \int_{t_k}^{t_{k+1}} o(\varepsilon) du \tag{2.42}
\]

\[
= O(\varepsilon^{2-\Delta}).
\]

Using (2.42) in the last term in (2.36), we obtain

\[
\sum_{k=0}^{l-1} \sum_{i \in M} (1 + E^{1/2} |X^\varepsilon(t_k)|^2) E^{1/2} \left| E_{t_k} \int_{t_k}^{t_{k+1}} [\mathbb{1}_{\{\theta^\varepsilon(u) = i\}} - \nu_t(X^\varepsilon(u))] du \right|^2 \\
\leq K \sum_{k=0}^{l-1} \sum_{i \in M} (1 + E^{1/2} |X^\varepsilon(t_k)|^2) E^{1/2} \left| \int_{t_k}^{t_{k+1}} o(\varepsilon) du \right|^2 \tag{2.43}
\]

\[
\leq Kl \varepsilon O(\varepsilon^{2-\Delta}) = O(\varepsilon^{1-\Delta}) \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
By virtue of the weak convergence of $X^\varepsilon(\cdot)$ to $X(\cdot)$, the Skorohod representation, (2.34), (2.35), and the estimates up to now, we obtain

$$
\mathbb{E} h(X^\varepsilon(t_l) : l \leq \ell) \left[ \int_t^{t+s} \mu'(X^\varepsilon(u), \theta^\varepsilon(u)) \nabla F(X^\varepsilon(u)) du \right] = \mathbb{E} h(X(t_l) : l \leq \ell) \left[ \int_t^{t+s} \tilde{\mu}'(X(u)) \nabla F(X(u)) du \right].
$$

(2.44)

Likewise, we can show

$$
\mathbb{E} h(X^\varepsilon(t_l) : l \leq \ell) \left[ \int_t^{t+s} \text{tr}[a(X^\varepsilon(u), \theta^\varepsilon(u)) \nabla F(X^\varepsilon(u))] du \right] = \mathbb{E} h(X(t_l) : l \leq \ell) \left[ \int_t^{t+s} \text{tr}[a(X(u)) \nabla F(X(u))] du \right].
$$

(2.45)

Combining the estimates obtained thus far, we obtain that $X(\cdot)$ is a solution of the martingale problem with operator $\mathcal{L}$. The theorem is thus proved. \qed
3 Stabilization for Switching Ordinary Differential Equations

The idea of stabilization can be traced back to the stochastic stability literature. In his pioneering work [19, Example 7.2, p.232], Khasminskii proved that a given two-dimensioned linear system can be stabilized by white noise. Subsequently, many people worked on stability problems and obtained a number of interesting results; see [1–3, 27, 28, 30, 31]. For regularization, Bahar and Mao [5] showed that the population explosion can be suppressed by environmental noise leading to the existence of global solutions almost surely. Recent work of Deng, Luo, Mao, and Pang [11] gave a general result on noise suppression for the system with one-sided linear growth rate. In Wu and Hu [41], a system with one-sided polynomial growth condition was treated and the system is regularized and stabilized by adding two Brownian noises. We considered a system of ODEs with randomly switching, nonlinear growth rates, and provided criteria for regularity and stabilization.

In this chapter, the feedback controls used are not in the classical form but rather are perturbing white noise processes. We use regularization to depict the suppression of finite time explosion, and use stabilization to mean the suppression of noise effect so as to have the resulting system stable. The rest of the chapter is arranged as follows. First feedback controls are designed to regularize continuous-time randomly switching systems. Then stability results are derived. The following topic concentrates on the approximating discrete-time systems. Finally a couple of examples for demonstration are provided.
3.1 Regularization

To suppress the finite explosion time, we add a feedback control term, which is a diffusion of the form $\sigma_1(x, \theta)dB_1$. To ensure stability, we add another feedback control $\sigma_2(x, \theta)dB_2$. Thus, we begin with the following system given by

$$
\begin{align*}
  dX(t) &= \mu(X(t), \theta(t))dt + \sigma_1(X(t), \theta(t))dB_1(t) + \sigma_2(X(t), \theta(t))dB_2(t), \\
  X(0) &= x, \quad \theta(0) = \theta,
\end{align*}
$$

(3.1)

where $\sigma_i(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^{r \times d}$, and $B_1(\cdot)$ and $B_2(\cdot)$ are independent, standard $d$-dimensional Brownian motions that are independent of the Markov chain. For each $i \in \mathcal{M}$ and each $g(\cdot, i) \in C^2$ ($C^2$ functions), the operator associated with (3.1) is given by

$$
\begin{align*}
  \mathcal{L}g(x, i) &= \mu'(x, i)\nabla g(x, i) + \frac{1}{2}\text{tr}(\sigma_1(x, i)\sigma_1'(x, i)\nabla^2 g(x, i)) \\
  &\quad + \frac{1}{2}\text{tr}(\sigma_2(x, i)\sigma_2'(x, i)\nabla^2 g(x, i)) + Qg(x, \cdot)(i), \\
  Qg(x, \cdot)(i) &= \sum_{j \in \mathcal{M}} q_{ij}g(x, j) \quad \text{for each} \ i \in \mathcal{M},
\end{align*}
$$

(3.2)

To ensure the regularization and stabilization, we need the following condition.

(H5) For each $i \in \mathcal{M}$, $\mu(\cdot, i)$, $\sigma_1(\cdot, i)$, and $\sigma_2(\cdot, i)$ are locally Lipschitz continuous such that

(a) $\mu(0, i) = 0$;

(b) $\mu'(x, i)x \leq K_0(|x|^{\beta_1+2} + |x|^2)$ for each $i \in \mathcal{M}$ and some $\beta_1 > 0$.

(c) for some $\beta > 0$ satisfying $2\beta - \beta_1 > 0$ and some $K_j > 0$ with $j = 1, \ldots, 4$ satisfying
\[ 2K_1 > K_2 \] and for each \( x \in \mathbb{R}^r \),

\[
\begin{align*}
\text{tr}(\sigma_1(x, i)\sigma_1'(x, i)xx') & \geq K_1(|x|^{4+2\beta} - |x|^4) \\
\text{tr}(\sigma_1(x, i)\sigma_1'(x, i)) & \leq K_2(|x|^{2+2\beta} + |x|^2), \\
\text{tr}(\sigma_2(x, i)\sigma_2'(x, i)xx') & \geq K_3|x|^4, \\
\text{tr}(\sigma_2(x, i)\sigma_2'(x, i)) & \leq K_4|x|^2.
\end{align*}
\] (3.3)

The conditions in (H5) are motivated by the need of treating function \( \mu(\cdot) \) with faster growth rates. Note that (H5)(c) yields that \( \sigma_1(0, i) = \sigma_2(0, i) = 0 \). This together with (H5)(a) indicates that \( (0, i) \) is an equilibrium of the system for each \( i \in \mathcal{M} \). The conditions in (H5) can often be easily verified mainly because these perturbations were added by us.

For instance, the condition \( 2K_1 > K_2 \) is not a restriction since the feedback controls-the noise perturbation \( \sigma_1(X(t), \theta(t))dB_1 \) is added by us. The motivation for the bounds in (3.3) are as follows: Suppose that for each \( i \in \mathcal{M} \), \( \text{tr}(\sigma_1(x, i)\sigma_1'(x, i)xx') \geq K_1(i)(|x|^{4+2\beta} - |x|^4) \) for some \( K_1(i) > 0 \) and each \( i \in \mathcal{M} \). Then \( K_1 = \min_{i \in \mathcal{M}} K_1(i) \). Likewise, if \( \text{tr}(\sigma_1(x, i)\sigma_1'(x, i)) \leq K_2(i)(|x|^{2+2\beta} + |x|^2) \) for some \( K_2(i) > 0 \) and each \( i \in \mathcal{M} \). Then \( K_2 = \max_{i \in \mathcal{M}} K_2(i) \).

In practical computation, it is easier to use a scalar Brownian motion and a polynomial like function in the diffusion coefficients. For example, in lieu of the general form (3.1), we may consider the following feedback controlled system

\[
dX(t) = \mu(X(t), \theta(t))dt + a_1(\theta(t))|X(t)|^\beta X(t)d\tilde{B}_1(t) + a_2(\theta(t))X(t)d\tilde{B}_2(t),
\]

where \( a_i(\cdot) : \mathcal{M} \rightarrow \mathbb{R}^+ \) and \( \tilde{B}_i(t) \) are standard independent scalar Brownian motions such that \( a_i(j) \) with \( j \in \mathcal{M} \) satisfies suitable conditions.
We proceed to obtain a result on the existence of the global unique solution of the system of interest. The idea here is that we explore regularity to obtain the existence and uniqueness. Such an idea can be traced back to the work [19]. Before proceeding further, we first recall a lemma, which is [48, Theorem 2.7]. We thus omit its proof.

**Lemma 3.1.** Suppose that \( b(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^r \) and that \( \sigma(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^{r \times d} \). Consider the two-component process \((X(\cdot), \theta(\cdot))\), satisfying

\[
    dX(t) = \mu(X(t), \theta(t))dt + \sigma(X(t), \theta(t))dB(t),
\]

\((X(0), \theta(0)) = (x, \theta)\),

and for \(i \neq j\),

\[
    \mathbb{P}\{\theta(t + \delta) = j | \theta(t) = i, X(s), \theta(s), s \leq t\} = q_{ij}(X(t))\delta + o(\delta).
\]

Suppose that for each \(i \in \mathcal{M}\), both the drift \(b(\cdot, i)\) and the diffusion coefficient \(\sigma(\cdot, i)\) satisfy the local Lipschitz condition, and that there is a nonnegative function \(V(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^+\) that is twice continuously differentiable with respect to \(x \in \mathbb{R}^r\) for each \(i \in \mathcal{M}\) such that there is an \(\gamma_0 > 0\) satisfying

\[
    \mathcal{L}V(x, i) \leq \gamma_0 V(x, i), \text{ for all } (x, i) \in \mathbb{R}^r \times \mathcal{M},
\]

\[
    V_R := \inf_{|x| \geq R, i \in \mathcal{M}} V(x, i) \to \infty \text{ as } R \to \infty.
\]

Then the process \((X(t), \theta(t))\) is regular.

Note that the above lemma can be used when the switching process is not a Markov chain, but a continuous state dependent switching process. Note also that the local Lipschitz
condition implies linear growth in every neighborhood since the system is not explicitly \( t \)
dependent (i.e., autonomous). In accordance with the lemma, we need to find an appropriate
Lyapunov function for the purpose of verifying no finite explosion time.

**Theorem 3.2.** Under condition (H5), and initial data \( x \) being bounded w.p.1, there is a
unique global solution of the system of differential equations (3.1).

**Proof.** First, by virtue of the argument in [33, pp. 90-91], the condition ensures that there
is unique maximal local solution up until the explosion time \( \tau \) defined by

\[
\tau = \inf \{ t \geq 0 : |X(t)| = \infty \}.
\]

To obtain the desired result, all needed is to show that \( \tau = \infty \) w.p.1. Define a Lyapunov
function \( V : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R} \) by

\[
V(x, i) = |x|^\gamma \text{ for } \gamma \in (0, 1), \ i \in \mathcal{M}. \tag{3.7}
\]

Since \( V(x, i) \) is independent of \( i \), we write it as \( V(x) \) in what follows. Note that

\[
\sum_{j \in \mathcal{M}} q_{ij} V(x) = 0 \text{ for each } x \text{ and each } i \in \mathcal{M}. \tag{3.8}
\]

Because the regularity is a property that concerns the solution of the system for \( |x| \) fairly
large, it suffices to work with a domain exterior to \( S_R = \{ x \in \mathbb{R}^r : |x| \leq R \} \), the ball with
radius \( R \) for sufficiently large \( R \). That is, we need only work with \( S_R^c = \{ x : |x| > R \} \), where
$R > 1$ is sufficiently large as needed. It is easily seen that

$$\nabla V(x) = \gamma |x|^\gamma - 2 x,$$

$$\nabla^2 V(x) = \gamma |x|^\gamma - 2 I + \gamma(\gamma - 2)|x|^\gamma - 4 xx'.$$

Thus, we have

$$\mathcal{L}V(x) = \mu'(x, i)\nabla V(x) + \frac{1}{2}\text{tr}[\nabla^2 V(x) \sigma_1(x, i) \sigma'_1(x, i)] + \frac{1}{2}\text{tr}[\nabla^2 V(x) \sigma_2(x, i) \sigma'_2(x, i)]$$

$$= \gamma \mu'(x, i) x |x|^\gamma - 2$$

$$+ \frac{\gamma}{2}\text{tr}[\sigma_1(x, i) \sigma'_1(x, i)] |x|^\gamma - 2 I$$

$$\leq K_0 \gamma(|x|^\beta_1 + \gamma + |x|^\gamma)$$

$$+ \frac{\gamma}{2}(\gamma - 2)\text{tr}[\sigma_1(x, i) \sigma'_1(x, i)] |x|^\gamma - 4 xx'$$

$$\leq \frac{\gamma}{2}[K_2 + K_1(\gamma - 2)] |x|^{\gamma + 2\beta}$$

$$\leq \frac{\gamma}{2}K_2 |x|\gamma + \frac{\gamma(\gamma - 2)}{2}K_3 |x|^\gamma$$

$$\leq \frac{\gamma}{2}K_4 |x|\gamma + \frac{\gamma(\gamma - 2)}{2}K_5 |x|^\gamma$$

$$:= K_5 |x|\gamma.$$
Since $2K_1 > K_2$, we can choose $\gamma$ sufficient small so that we still have $2K_1 - pK_1 > K_2$. As a result, $K_2 + K_1(\gamma - 2) < 0$. Thus (H5) implies that

$$K_0 \gamma |x|^{\beta_1 + \gamma} + \frac{\gamma}{2} [K_2 + K_1(\gamma - 2)] |x|^{\gamma + 2\beta} + (K_0 \gamma + K_5) |x|^{\gamma}$$

$$= \left[ \frac{K_0 \gamma}{|x|^{2\beta - \beta_1}} + \frac{\gamma}{2} [K_2 + K_1(\gamma - 2)] \right] |x|^{\gamma + 2\beta} + K_6 |x|^\gamma,$$

where $K_6 = K_0 \gamma + K_5$. Since we are working with $S_{R_c}$, $\frac{K_0 \gamma}{|x|^{2\beta - \beta_1}}$ can be made small enough so that

$$\frac{K_0 \gamma}{|x|^{2\beta - \beta_1}} + \frac{\gamma}{2} [K_2 + K_1(\gamma - 2)] \leq \frac{K_0 \gamma}{|R_c|^{2\beta - \beta_1}} + \frac{\gamma}{2} [K_2 + K_1(\gamma - 2)]$$

$$:= \tilde{K} < 0.$$

Using (3.12) and combining (3.8) and (3.9)–(3.11), we arrive at that

$$\mathcal{L}V(x) \leq \tilde{K} \delta V(x).$$

Note that clearly $V(x) = |x|^\gamma \to \infty$ as $|x| \to \infty$. Thus by virtue of Lemma 3.1, $\tau = \infty$ w.p.1. The global existence of a unique solution is established.

To proceed, we obtain tightness of $X(t)$ for $t$ sufficiently large. This is stated in the following theorem.

**Theorem 3.3.** Under the conditions of Theorem 3.2, for any $\delta > 0$ sufficiently small, there is a $K_\delta > 0$ such that

$$\limsup_{t \to \infty} P(|X(t)| \geq K_\delta) \leq \delta.$$
Proof. Again, take $V(x) = |x|^\gamma$ with $0 < \gamma < 1$ as in Theorem 3.2 and consider $e^{\lambda V(x)}$ for some $\lambda > 0$. By virtue of the Dynkin formula, we have

$$
\mathbb{E}V(X(t)) = e^{-\lambda t}\mathbb{E}V(X(0)) + \mathbb{E}\int_0^t e^{-\lambda(t-s)}[\mathcal{L}V(X(s)) + \lambda V(X(s))]ds.
$$

As in the proof of Theorem 3.2, it suffices to concentrate on the exterior of the ball with radius $R$, namely, $S^c_R$ for sufficiently large $R$. In view of (3.9) and (3.10), for sufficiently large $R$ and some $K > 0$, since the dominating term is the one containing $|x|^{\gamma+2\beta}$,

$$
\mathcal{L}V(x) + \lambda V(x) \leq \tilde{K}|x|^{\gamma+2\beta} + K_6|x|^\gamma + \lambda|x|^\gamma \\
\leq \left[\tilde{K} + \frac{K_6}{|R|^{2\beta}} + \frac{\lambda}{|R|^{2\beta}}\right]|x|^{\gamma+2\beta} \\
< 0 \leq K.
$$

In the above and hereafter, we use $K$ as a generic positive constant, whose value may be different from different appearances. Putting this into (3.14), we obtain

$$
\mathbb{E}V(X(t)) \leq e^{-\lambda t}\mathbb{E}V(X(0)) + K[1 - e^{-\lambda t}].
$$

Again, $K > 0$ is understood to be a generic positive constant. Therefore,

$$
\limsup_{t \to \infty} \mathbb{E}|X(t)|^\gamma \leq K < \infty.
$$

The desired result thus follows. $\square$

Remark 3.4. We may device an alternative proof. The main idea is to use Theorem 3.2.
By virtue of Theorem 3.2, $\mathcal{L}V(x) \leq c_1 V(x)$ for some $c_1 > 0$. Then we further deduce

$$\mathcal{L}V(x) + \lambda V(x) \leq c_2 V(x),$$

where $c_2 = c_1 + \lambda$. Next, substitute the above into (3.14) and use Gronwall’s inequality to obtain $EV(X(t)) \leq K < \infty$. Finally, we obtain (3.15) and therefore conclude the proof.

### 3.2 Stabilization

First represent the Markov switching diffusion as a Poisson jump diffusion as introduced in section 1.2. With the $\mathcal{L}$ defined in (3.2), noting the independence of $B_1$ and $B_2$, for each $i \in \mathcal{M}$ and $g(\cdot, i) \in C^2$, the generalized Itô lemma (see [7, 33, 39]) reads

$$g(X(t), \theta(t)) - g(X(0), \theta(0)) = \int_0^t \mathcal{L}g(X(s), \theta(s))ds + \tilde{M}_1(t) + \tilde{M}_2(t) + \tilde{M}_3(t), \quad (3.16)$$

where

$$\tilde{M}_1(t) = \int_0^t \nabla g'(X(s), \theta(s))\sigma_1(X(s), \theta(s))dB_1(s),$$

$$\tilde{M}_2(t) = \int_0^t \nabla g'(X(s), \theta(s))\sigma_2(X(s), \theta(s))dB_2(s),$$

$$\tilde{M}_3(t) = \int_0^t \int_{\mathbb{R}} [g(X(s), \theta(0) + h(X(s), \theta(s), z)) - g(X(s), \theta(s))] \alpha(ds, dz), \quad (3.17)$$

and

$$\alpha(ds, dz) = p(ds, dz) - ds \times \tilde{m}(dz)$$

is a martingale measure. First, we have the following lemma. The proof of this lemma can be found in [33] for Markov switching diffusions, and in [48] for $x$-dependent switching.
Lemma 3.5. Under the conditions of (H5), if $X(0) \neq 0$, then

$$\mathbf{P}(X(t) \neq 0 : t \geq 0) = 1. \quad (3.18)$$

Next, we show that by proper choice of the feedback control, the system is stabilizable. For stabilization, we use a modified version of (H5) in the rest of this section. In (H5), the bounding constants are independent of $i$. For the stability consideration, we let these constants be $i$ dependent.

(H5’) For each $i \in \mathcal{M}$, $\mu(\cdot, i)$, $\sigma_1(\cdot, i)$, and $\sigma_2(\cdot, i)$ are locally Lipschitz continuous such that

(a) $\mu(0, i) = 0$;

(b) $\mu'(x, i)x \leq K_0(i)(|x|^{\beta_1} + 2 + |x|^2)$ for each $i \in \mathcal{M}$ and some $\beta_1 > 0$.

(c) for some $\beta > 0$ satisfying $2\beta - \beta_1 > 0$ and some $K_j(i) > 0$ with $j = 1, \ldots, 4$ satisfying $2K_1(i) > K_2(i)$ and for each $x \in \mathbb{R}^r$,

$$\begin{align*}
K_1(i)(|x|^{4+2\beta} - |x|^4) &\leq \text{tr}(\sigma_1(x, i)\sigma'_1(x, i)xx') \leq K_5(i)|x|^{4+2\beta} \\
\text{tr}(\sigma_1(x, i)\sigma'_1(x, i)) &\leq K_2(i)(|x|^{2+2\beta} + |x|^2), \\
\text{tr}(\sigma_2(x, i)\sigma'_2(x, i)x'x) &\geq K_3(i)|x|^4, \\
\text{tr}(\sigma_2(x, i)\sigma'_2(x, i)) &\leq K_4(i)|x|^2.
\end{align*} \quad (3.19)$$

(d) The Markov chain $\theta(t)$ is irreducible.

To proceed, we compute the Lyapunov exponent by working with $\log |x|$. By virtue of
the Itô rule, we have

\[
\log |X(t)| = \log |X(0)| + \int_0^t \frac{X'(s)}{|X(s)|^2} \mu(X(s), \theta(s)) ds \\
+ \int_0^t \frac{1}{2|X(s)|^2} \left\{ \text{tr}[\sigma_1(X(s), \theta(s))\sigma_1'(X(s), \theta(s))] \\
+ \text{tr}[\sigma_2(X(s), \theta(s))\sigma_2'(X(s), \theta(s))] \right\} ds \\
- \int_0^t \frac{1}{|X(s)|^4} \left\{ \text{tr}[\sigma_1(X(s), \theta(s))\sigma_1'(X(s), \theta(s))X(s)X'(s)] \\
+ \text{tr}[\sigma_2(X(s), \theta(s))\sigma_2'(X(s), \theta(s))X(s)X'(s)] \right\} ds \\
+ M_1(t) + M_2(t),
\]

(3.20)

where \( M_1(t) \) and \( M_2(t) \) are given by

\[
M_1(t) = \int_0^t \frac{X'(s)}{|X(s)|^2} \sigma_1(X(s), \theta(s)) dB_1(s), \\
M_2(t) = \int_0^t \frac{X'(s)}{|X(s)|^2} \sigma_2(X(s), \theta(s)) dB_2(s).
\]

(3.21)

Note that since \( \log |x| \) is independent of \( \theta \), the third martingale \( M_3(t) \) in (3.16) disappears.

For the martingale term \( M_1(t) \), the quadratic variation is given by

\[
\langle M_1(t), M_1(t) \rangle = \int_0^t \frac{X'(s)\sigma_1(X(s), \theta(s))\sigma_1'(X(s), \theta(s))X(s)}{|X(s)|^4} ds.
\]

For any \( \varepsilon \in (0, 1) \), choose \( \xi > 0 \) such that \( \xi \varepsilon > 1 \). It follows from the well-known Doob’s inequality (also known as exponential martingale inequality, see, for example [30, Theorem 1.7.4, p. 44]), for each positive integer \( n \),

\[
P \left( \sup_{1 \leq t \leq n} \left[ M_1(t) - \frac{\varepsilon}{2} \langle M_1(s), M_1(s) \rangle ds \right] \geq \xi \log n \right) \leq \frac{1}{n^{\xi \varepsilon}}.
\]
Hence, we deduce from the Borel-Cantelli Lemma that for almost all \( \omega \), there is a \( \tilde{K}_1 = \tilde{K}_1(\omega) > 1 \) such that for all \( n \geq \tilde{K}_1 \) and \( n - 1 \leq t \leq n \),

\[
M_1(t) \leq \frac{\varepsilon}{2} \langle M_1(t), M_1(t) \rangle + \xi \log(t + 1) \text{ w.p.} 1.
\] (3.22)

For the second martingale, by using (H5'),

\[
\langle M_2(t), M_2(t) \rangle \leq \int_0^t \sum_i K_4(i)ds \leq Kt.
\]

Again, \( K \) is a generic positive constant. Thus, the local martingale convergence theorem in [26] implies that \( M_2(t)/t \to 0 \) w.p.1 as \( t \to \infty \).

To proceed, we rewrite all but the last two terms in (3.20) by use of the indicator functions \( I_{\{\theta(s)=i\}} \). That is, we write, for example,

\[
\int_0^t \frac{X'(s)}{|X(s)|^2} \mu(X(s), \theta(s))ds = \sum_{i=1}^m \int_0^t \frac{X'(s)}{|X(s)|^2} \mu(X(s), i) I_{\{\theta(s)=i\}} ds.
\]

Likewise, we write all the other terms using exactly the same way. Using assumption (H5')
again, we arrive at

\[
\log |X(t)| \leq \log |X(0)| + \sum_{i=1}^{m} \left\{ \int_{0}^{t} K_0(i) ||X(s)||^{\beta_1} + 1] I_{\{\theta(s)=i\}} ds \\
+ \frac{1}{2} K_2(i) \int_{0}^{t} ||X(s)||^{2\beta} + 1] I_{\{\theta(s)=i\}} ds \\
+ \frac{1}{2} K_4(i) \int_{0}^{t} I_{\{\theta(s)=i\}} ds \\
- K_1(i) \int_{0}^{t} ||X(s)||^{2\beta} - 1] I_{\{\theta(s)=i\}} ds \\
- K_3(i) \int_{0}^{t} I_{\{\theta(s)=i\}} ds \\
+ \frac{\varepsilon}{2} K_5(i) \int_{0}^{t} |X(s)|^{2\beta} I_{\{\theta(s)=i\}} ds \right\} \\
+ \xi \log(t + 1) + M_2(t).
\]  

(3.23)

Since the Markov chain is ergodic, there is a stationary distribution given by \( \nu = (\nu_1, \ldots, \nu_m) \).

Thus, as \( t \to \infty \),

\[
\frac{1}{t} \int_{0}^{t} I_{\{\theta(s)=i\}} ds \to \nu_i \text{ w.p.1.} 
\]  

(3.24)

Define

\[
\hat{K}(i) = \sup_{x \geq 0} \left[ (-K_1(i) + \frac{1}{2} K_2(i) + \frac{\varepsilon}{2} K_5(i)) |x|^{2\beta} + K_0(i) |x|^{\beta_1} + \tilde{K}(i) \right], \text{ where} \\
\tilde{K}(i) = [K_0(i) + K_1(i) + K_2(i) + \frac{1}{2} K_4(i)].
\]  

(3.25)

Note that because \(-2K_1(i) + K_2(i) < 0\) for all \( i \in \mathcal{M} \), we can choose \( \varepsilon > 0 \) sufficiently small such that

\[
-K_1(i) + \frac{1}{2} K_2(i) + \frac{\varepsilon}{2} K_5(i) < 0.
\]

This ensures the existence of sup in \( \hat{K}(i) \) in (3.25). Note that the sup above is indeed achieved.
Dividing both sides of (3.23) by $t$ and noting

$$\frac{\log |X(0)|}{t} \to 0, \quad \frac{\log(t + 1)}{t} \to 0, \quad \text{and} \quad \frac{M_2(t)}{t} \to 0 \text{ w.p.1 as } t \to \infty,$$

we have

$$\limsup_{t \to \infty} \frac{\log |X(t)|}{t} \leq -\sum_{i=1}^{m} [K_3(i) - \hat{K}(i)] \nu_i \text{ w.p.1.} \quad (3.26)$$

The above calculation is concerned about the so-called Lyapunov exponent. Recall that [48, p. 177], the equilibrium point $x = 0$ is exponential $\gamma$-stable if for some positive constants $K$ and $k$, for any $(X(0), \theta(0)) = (x, 0)$, $E|X(t)|^\gamma \leq K|x|^{\gamma} \exp(-kt)$. What we have demonstrated in fact is a stability in the almost sure sense. That is, the equilibrium point $x = 0$ is exponential stable in the almost sure sense if for some positive constants $K$ and $k$, for any $(X(0), \theta(0)) = (x, 0)$, $|X(t)|^\gamma \leq K|x|^\gamma \exp(-kt)$ a.s. Summarizing what have been derived thus far, we obtained the following result.

**Theorem 3.6.** Suppose (H5') is satisfied. In addition, assuming $\hat{K}(i)$ is defined in (3.25), if

$$\sum_{i=1}^{m} [K_3(i) - \hat{K}(i)] \nu_i > 0,$$

the system is exponentially stable.

### 3.3 Discrete-Time Approximation

Let $\theta_n$ be a discrete-time Markov chain with state space $\mathcal{M} = \{1, \ldots, m\}$ and one-step transition matrix $P = I + \varepsilon Q$, where $Q$ is a generator of a continuous-time Markov chain.

Suppose $\mu(\cdot, \cdot): \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^r$, $\sigma_i(\cdot, \cdot): \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^{r \times r}$ for $i = 1, 2$. To begin, we consider
a discretization of (1.1), namely,

$$x_{n+1} = x_n + \alpha \mu(x_n, \theta_n),$$  \hspace{1cm} (3.27)

which is a faithful reflection of the continuous-time system. Due to the faster growth in the $x$-variable, (1.1) does not possess a global solution thus is not stable resulting in that the solution of discrete counterpart (3.27) grows without bound. Corresponding to the continuous-time system, we then consider the feedback controlled system

$$x_{n+1} = x_n + \alpha \mu(x_n, \theta_n) + au_n,$$

where $u_n$ is a feedback control that regularizes and stabilizes (3.27). Similar to the continuous-time case, using appropriate feedback controls, consider the algorithm

$$x_{n+1} = x_n + \alpha \mu(x_n, \theta_n) + \sqrt{\alpha} \sigma_1(x_n, \theta_n) \eta_n + \sqrt{\alpha} \sigma_2(x_n, \theta_n) \eta_n + \sqrt{\alpha} e_1(x_n, \theta_n) \xi_n,$$

$$x_0 = x, \ \theta_0 = \theta,$$  \hspace{1cm} (3.28)

where $\alpha > 0$ is the step size. For example, let us consider a scalar system given by (1.1). The difficulty is that due to the fast growth, (3.27) does not have a global solution but only a local one as its continuous-time counterpart. One of our motivations is to treat the following scalar system:

$$x_{n+1} = x_n + \alpha \mu(x_n, \theta_n) + \sqrt{\alpha} \sigma_1(\theta_n) |x_n|^\beta x_n \eta_n + \sqrt{\alpha} e_1(\theta_n) x_n \xi_n.$$  \hspace{1cm} (3.29)
The first added feedback term $\sigma_1(x_n, \theta_n)\eta_n$ aims to suppress the fast growth of the state of the system so as to get a global solution, whereas the second feedback $\sigma_2(x_n, \theta_n)\xi_n$ is used to get the stability. To proceed, we use the following.

(H6) The noise sequences $\{\eta_n\}$ and $\{\xi_n\}$ are independent and identically distributed random variables that are independent of each other each with mean 0 and covariance matrix $I$ and that are independent of the Markov chain $\theta_n$.

**Remark 3.7.** In lieu of the i.i.d. sequence, we may treat correlated $\phi$-mixing noise, which as special cases includes for example, the most commonly used conditions that the sequences are independent and identically distributed (i.i.d.) random variables, the martingale difference sequences, moving average random variables driven by i.i.d. random variables, irreducible Markov chains with finite state spaces, and more general noise with the condition that the remote past and distant future are asymptotically independent. However, the perturbations in the current paper is added by us. Thus it is more reasonable to add simple noise perturbations. So the i.i.d. sequence is a natural candidate.

Note that the Markov chain $\theta_n$ is used to approximate the continuous Markov chain $\theta(\cdot)$.

We thus choose $\varepsilon = \alpha$ in what follows. To study the convergence of the discrete approximation (3.28), we define the piecewise constant interpolations as follows:

$$x^\alpha(t) = x_n, \quad \theta^\alpha(t) = \theta_n, \quad t \in [n\alpha, n\alpha + \alpha).$$

(3.30)

Clearly, for each $0 < T < \infty$, $x^\alpha(\cdot) \in D([0, T] : \mathbb{R}^r)$ that is the space of functions defined on $[0, T]$ that are right continuous and have left limits endowed with the Skorohod topology.
Likewise, \( \theta^\alpha(\cdot) \in D([0,T]:\mathcal{M}) \). Before proceeding to the regularity and stability study, we derive a limit result.

**Theorem 3.8.** Under assumptions (H5) and (H6), as \( \alpha \to 0 \), \((x^\alpha(\cdot), \theta^\alpha(\cdot))\) converges weakly to \((x(\cdot), \theta(\cdot))\) such that the limit is the solution to the martingale problem with operator \( \mathcal{L} \).

That is, \((x(\cdot), \theta(\cdot))\) is the solution of (3.1).

We shall prove the result by establishing a number of lemmas. The first lemma presents a uniqueness result for the martingale problem. Then we proceed with the use of a truncation argument.

**Lemma 3.9.** Under the assumptions of Theorem 3.8, the martingale problem with operator defined in (3.2) has a unique solution in the sense in distribution.

**Proof.** By virtue of Theorem 3.2, there exists a unique solution to (3.1). Thus, the lemma follows. \( \square \)

**Remark 3.10.** In fact, [41, Theorem 5.2] presents the existence and uniqueness in the pathwise sense. Here, in our result, only uniqueness in the sense in distribution is needed.

To treat the potential unboundedness, we need to use a truncation device [24, p. 284]. For a fixed but otherwise arbitrary \( N < \infty \), define \( X^N(\cdot) \) so that \( X^N(t) = X(t) \) for \( t \) up until the first exit of \( X(t) \) from \( S_N = \{ x \in \mathbb{R}^r : |x| \leq N \} \), the sphere with radius \( N \). For the
discrete-time approximation sequence, we also work with a truncated process defined by

\[ x_{n+1}^N = x_n^N + \alpha \mu^N(x_n, \theta_n) + \sqrt{\alpha} \sigma_1^N(x_n, \theta_n) \eta_n + \sqrt{\alpha} \sigma_2^N(x_n, \theta_n) \xi_n, \]

\[ \mu^N(x, \theta) = \mu(x, \theta) q_N(x), \quad \sigma_1^N(x, \theta) = \sigma_1(x, \theta) q_N(x), \quad \sigma_2^N(x, \theta) = \sigma_2(x, \theta) q_N(x), \]

\[ q_N(x) = \begin{cases} 
1, & \text{if } x \in S_N, \\
0, & \text{if } x \in \mathbb{R}^r - S_{N+1},
\end{cases} \]

where \( q_N(x) \) is termed a truncation function that is smooth, and that is equal to 1 when \( x \) is within the \( N \)-sphere, and is 0 outside the \((N+1)\)-sphere. Corresponding to the truncations, define the associated interpolations:

\[ x^{\alpha,N}(t) = x_n^N, \quad t \in [n\alpha, n\alpha + \alpha). \]

**Lemma 3.11.** Under the conditions of Theorem 3.8, \( \{x^{\alpha,N}(\cdot), \theta^\alpha(\cdot)\} \) is tight in \( D([0, T] : \mathbb{R}^r \times \mathcal{M}). \)

**Proof.** The tightness of \( \theta^\alpha(\cdot) \) can be obtained as in [45]. Thus we will concentrate on \( x^{\alpha,N}(\cdot) \).

By using the truncation, \( \{x_n^N\} \) is bounded. Thus, for any \( \delta > 0, 0 < t, \) and \( 0 < s \leq \delta, \)

\[ E |x^{\alpha,N}(t + s) - x^{\alpha,N}(t)|^2 \]

\[ \leq K \alpha^2 E \left| \sum_{k=t/\alpha}^{(t+s)/\alpha-1} \mu^N(x_k, \theta_k) \right|^2 + K \alpha E \left| \sum_{k=t/\alpha}^{(t+s)/\alpha-1} \sigma_1^N(x_k, \theta_k) \eta_k \right|^2 \]

\[ + K \alpha E \left| \sum_{k=t/\alpha}^{(t+s)/\alpha-1} \sigma_2^N(x_k, \theta_k) \xi_k \right|^2. \]
In the above and hereafter, $t/\alpha$ and $(t+s)/\alpha$ denote $\lfloor t/\alpha \rfloor$ and $\lfloor (t+s)/\alpha \rfloor$, the integer parts of these quantities. However, for notational simplicity, we will not use the floor function notation in what follows. Also, we use $K$ to denote a generic positive constant, whose value may be different for different appearances.

By use of the Hölder inequality and the $N$-truncation,

$$\alpha^2 \mathbb{E} \left| \sum_{k=t/\alpha}^{(t+s)/\alpha-1} \mu^N(x_k, \theta_k) \right|^2 \leq Ks \alpha \sum_{k=t/\alpha}^{(t+s)/\alpha-1} \mathbb{E} |\mu^N(x_k, \theta_k)|^2 \leq Ks^2 \leq K\delta^2.$$ 

Next, by virtue of the truncation and the i.i.d. assumption of $\{\eta_n\}$,

$$\alpha \mathbb{E} \left| \sum_{k=t/\alpha}^{(t+s)/\alpha-1} \sigma^N_1(x_k, \theta_k) \eta_k \right|^2 \leq K\alpha \sum_{j=t/\alpha}^{(t+s)/\alpha-1} \mathbb{E} |\eta_k|^2 \leq Ks \leq K\delta.$$ 

Likewise,

$$\alpha \mathbb{E} \left| \sum_{k=t/\alpha}^{(t+s)/\alpha-1} \sigma^N_2(x_k, \theta_k) \zeta_k \right|^2 \leq K\delta.$$ 

Combining the above estimates, we obtain that

$$\mathbb{E} |x^{\alpha,N}(t+s) - x^{\alpha,N}(t)|^2 \leq K\delta.$$
Taking $\limsup_{\alpha \to 0}$ followed by $\lim_{\delta \to 0}$, the desired tightness is obtained [23, p. 47] (see also [24, Chapter 7]). □

To proceed, define a truncated operator by

$$\mathcal{L}^Ng(x, i) = \mu^N(x, i)\nabla g(x, i) + \frac{1}{2}\text{tr}(\sigma^N(x, i)\sigma^N(x, 1)\nabla^2 g(x, i))$$

$$+ \frac{1}{2}\text{tr}(\sigma^N(x, i)\sigma^N(x, i)\nabla^2 g(x, i)) + Qg(x, \cdot)(i),$$

(3.33)

for any $g(\cdot, i) \in C^2_0$ (collection of $C^2$ functions with compact support) and each $i \in \mathcal{M}$, where $\mu^N, \sigma^N_1,$ and $\sigma^N_2$ are as defined in (3.31). We then obtain the following lemma.

**Lemma 3.12.** Under the conditions of Lemma 3.11, $(x^{\alpha N}(\cdot), \theta^\alpha(\cdot))$ converges to $(X^N(\cdot), \theta(\cdot))$ weakly, which is a solution of the martingale problem with operator $\mathcal{L}^N$.

**Proof.** By virtue of Lemma 3.11, $(x^{\alpha N}(\cdot), \theta^\alpha(\cdot))$ is tight. By Prohorov’s theorem, we can extract a convergent subsequence. For notational simplicity, still index the subsequence by $\alpha$ with the limit denoted by $(X^N(\cdot), \theta(\cdot))$. By Skorohod representation with slightly abuse of notation, we may assume that $(x^{\alpha N}(\cdot), \theta^\alpha(\cdot))$ converges to $(X^N(\cdot), \theta(\cdot))$ with probability one, and the convergence is uniform on any compact interval.

We proceed to characterize the limit process and show that $(X^N(\cdot), \theta(\cdot))$ is the solution of the martingale problem with operator defined in (3.33). To prove the martingale property, we need only show that for any bounded and continuous function $\Psi(\cdot)$, any $g(\cdot, i) \in C^2_0$ for each $i \in \mathcal{M}$, any positive integer $\kappa$, any $t, s \geq 0$, and any $t_j \leq t$ for $j \leq \kappa$,

$$E\Psi(X^N(t_j), \theta(t_j) : j \leq \kappa) \left[ g(X^N(t + s), \theta(t + s)) - g(X^N(t), \theta(t)) \right.$$

$$- \int_t^{t+s} \mathcal{L}^N g(X^N(u), \theta(u))du \bigg] = 0.$$  

(3.34)
To verify (3.34), we note that by the weak convergence and the Skorohod representation, as \( \alpha \to 0 \),

\[
\mathbb{E}\Psi(x^{\alpha,N}(t_j), \theta^{\alpha}(t_j) : j \leq \kappa)[g(x^{\alpha,N}(t+s), \theta^{\alpha}(t+s)) - g(x^{\alpha,N}(t), \theta^{\alpha}(t))] \\
\to \mathbb{E}\Psi(X(t_j), \theta(t_j) : j \leq \kappa)[g(X(t+s), \theta(t+s)) - g(X(t), \theta(t))].
\] (3.35)

Next, we choose \( m_\alpha \) such that \( m_\alpha \to \infty \) as \( \alpha \to 0 \), but \( \Delta_\alpha = \alpha m_\alpha \to 0 \). Let \( I_{l_\alpha} = \{ k : lm_\alpha \leq k \leq lm_\alpha + m_\alpha \} \). Using such a partition to \( [t/\alpha, (t+s)/\alpha) \), we obtain

\[
g(x^{\alpha,N}(t+s), \theta^{\alpha}(t+s)) - g(x^{\alpha,N}(t), \theta^{\alpha}(t)) = \sum_{l=t/\Delta_\alpha}^{(t+s)/\Delta_\alpha - 1} \{ [g(x_{l_\alpha+m_\alpha}^{N}, \theta_{l_\alpha+m_\alpha}) - g(x_{l_\alpha}^{N}, \theta_{l_\alpha})] \\
+ [g(x_{l_\alpha+m_\alpha}^{N}, \theta_{l_\alpha+m_\alpha}) - g(x_{l_\alpha+m_\alpha}, \theta_{l_\alpha})] \}
\] (3.36)

It can be shown that the limit of

\[
\mathbb{E}\Psi(x^{\alpha,N}(t), \theta^{\alpha}(t) : j \leq \kappa)[g(x_{l_\alpha+m_\alpha}^{N}, \theta_{l_\alpha+m_\alpha}) - g(x_{l_\alpha+m_\alpha}, \theta_{l_\alpha})]
\]

is the same as that of

\[
\mathbb{E}\Psi(x^{\alpha,N}(t), \theta^{\alpha}(t) : j \leq \kappa)[g(x_{l_\alpha+m_\alpha}^{N}, \theta_{l_\alpha+m_\alpha}) - g(x_{l_\alpha+m_\alpha}, \theta_{l_\alpha})] \\
\to \mathbb{E}\Psi(x(t), \theta(t) : j \leq \kappa) \int_t^{t+s} Qg(x(u), \theta(u))du \text{ as } \alpha \to 0.
\]
Using Taylor expansions up to the second order, and then taking the limits, we obtain that as $\alpha \to 0$,

\[
E\Psi(x^{\alpha,N}(t_j), \theta^{\alpha}(t_j) : j \leq \kappa) \to E\Psi(X^N(t_j), \theta(t_j) : j \leq \kappa) \\
= E\Psi(X^N(t_j), \theta(t_j) : j \leq \kappa) \left[ \int_t^{t+s} [\mu^N(X^N(u), \theta(u)) \nabla g(X^N(u), \theta(u))] du \\
+ \int_t^{t+s} \text{tr}[\sigma_1^N(X^N(u), \theta(u))\sigma_1^2(X^N(u), \theta(u)) \nabla^2 g(X^N(u), \theta(u))] du \\
+ \int_t^{t+s} \text{tr}[\sigma_2^N(X^N(u), \theta(u))\sigma_2^2(X^N(u), \theta(u)) \nabla^2 g(X^N(u), \theta(u))] du \right].
\]

Combining the estimates obtained thus far, the desired result follows. \qed

Lemma 3.13. As $N \to \infty$, the limit $X^N(\cdot)$ in Lemma 3.12 converges to $x(\cdot)$, which is a solution of the martingale problem with operator $L$.

Proof. The proof is similar to [24, Step 4, p. 285]; see also [23, p. 46]. Let $P(\cdot)$ and $P^N(\cdot)$ be measures induced by $X(\cdot)$ and $X^N(\cdot)$, respectively. The measure $P(\cdot)$ is unique by Lemma 3.9. The uniqueness further implies that $P(\cdot)$ and $P^N(\cdot)$ coincide on all Borel subsets of the set of paths with values in $S_N$ for each $t \leq T$. Using $P(\sup_{t \leq T} |X(t)| \leq N) \to 1$ as $N \to \infty$ and the weak convergence of $x^{\alpha,N}(\cdot)$ to $X^N(\cdot)$, we obtain $x^{\alpha}(\cdot)$ converges weakly to $X(\cdot)$. The uniqueness of the martingale problem yields that the result is independent of the chosen subsequence. \qed

We have shown that the continuous-time system can be regularized. Concerning the discrete-time counterpart, do we have similar results? The following theorem provides a definitive answer.

Theorem 3.14. Under assumptions of Theorem 3.8, $(x_n, \theta_n)$ is regular. That is, the process $(x_n, \theta_n)$ does not blow up in finite time.
Proof. The proof is divided into two steps. Step 1. To begin, we define \( V(x) = |x|^\gamma \) for \( 0 < \gamma < 1 \) sufficiently small as in the proof of Theorem 3.2. For some \( 0 < \lambda < 1 \), consider the sequence \( \{V(x_n)\}_{n=1}^\infty \). Use \( E_n \) to denote the conditional expectation with respect to \( F_n \), the \( \sigma \)-algebra generated by \( \{\theta_n, \eta_j, \xi_j, \theta_j : j \leq n - 1\} \). Then

\[
E_n V(x_{n+1})\lambda^{n+1} - V(x_n)\lambda^n
= [E_n V(x_{n+1}) - V(x_n)]\lambda^{n+1} + V(x_n)[\lambda^{n+1} - \lambda^n] \tag{3.37}
\leq [E_n V(x_{n+1}) - V(x_n)]\lambda^{n+1}.
\]

The last line above follows from the observation

\[
\lambda^{n+1} - \lambda^n = \lambda^n(\lambda - 1) < 0 \quad \text{and} \quad V(x_n) \geq 0.
\]

Next we claim that \( E_n V(x_{n+1}) - V(x_n) \leq 0 \). Note that

\[
E_n V(x_{n+1}) - V(x_n)
= E_n (\nabla V(x_n))[x_{n+1} - x_n] + \frac{1}{2}[x_{n+1} - x_n][\nabla^2 V(x_n)[x_{n+1} - x_n] + \frac{1}{3} \rho_n, \tag{3.38}
\]

where

\[
\rho_n = E_n \sum_{i=1}^r \sum_{j=1}^r \sum_{l=1}^r \frac{\partial^3 V(x_n^+)}{\partial x_i \partial x_j \partial x_l}[x_n^i - x_n^j][x_n^{[i+1]} - x_n^i][x_n^{[j+1]} - x_n^j], \tag{3.39}
\]

\( x_n^+ \) is on the line segment joining \( x_n \) and \( x_{n+1} \), and \( x^i \) denotes the \( i \)th component of \( x \). The independence of \( \theta_n \) and \( \eta_n \) and \( \xi_n \) yields that

\[
E_n \sigma_1(x_n, \theta_n)\eta_n = 0, \quad E_n \sigma_2(x_n, \theta_n)\xi_n = 0,
\]

\[
E_n (\nabla V(x_n))[x_{n+1} - x_n] \leq K_0[|x_n|^\beta + |x_n|\gamma].
\]
Likewise, using the independence of $\theta_n$ with $\eta_n$ and $\xi_n$ together with the assumption in (H5) and (H6), similar to the estimates in (3.10), we obtain

\[ e_n = [x_{n+1} - x_n]'\nabla^2 V(x_n)[x_{n+1} - x_n] \leq \left[ \frac{K_0\gamma}{|x|^{2\beta - \beta_1}} + \frac{\gamma}{2}[K_2 + K_1(\gamma - 2)]\right]|x|^\gamma + 2 + K_7|x|^\gamma \]

for some $K_7 > 0$. Since we are working with $S_{R_n}$, the $|x|^\gamma + 2$ is the dominating force. Using the recursion (3.28) and taking conditional expectation in (3.39), the two noise terms disappear.

It can further be shown that $\rho_n$ is bounded above by $O(\alpha)e_n$. Owing to the small stepsize $\alpha > 0$ used, we can make $e_n + \rho_n \leq 0$. Substituting the above into (3.38), we obtain

\[ \mathbb{E}_n V(x_{n+1}) - V(x_n) \leq 0. \quad (3.40) \]

Step 2. Redefine $\gamma_n = \inf\{k \geq 0 : |x_k| \geq n\}$. Assume that the statement of the theorem is not true. Then there would exist a finite integer $N > 0$ and $\varepsilon > 0$ such that $\mathbb{P}(\gamma_n \leq N) > \varepsilon$.

As a result, there is an $n_0$ such that for all $n \geq n_0$, $\mathbb{P}(\gamma_n \leq N) > \varepsilon$. Using the result in Step 1 (in particular, (3.40)), iterating on $\mathbb{E}_n V(x_{n+1}) - V(x_n)$ together with telescoping, we obtain

\[ \mathbb{E}V(x_{\gamma_n \wedge N})\lambda^N - \mathbb{E}V(x_0) = \sum_{k=0}^{\gamma_n \wedge N - 1} \mathbb{E}[\mathbb{E}_k V(x_{k+1}) - V(x_k)] \leq 0. \]

It follows that

\[ \mathbb{E}V(x_0)\lambda^{-N} \geq \mathbb{E}V(x_{\gamma_n \wedge N}) \geq \mathbb{E}V(x_{\gamma_n \wedge N})I_{\{\gamma_n \leq N\}} \geq [\inf_{|x| \geq n} V(x)]\mathbb{P}(\gamma_n \leq N) \geq \varepsilon[\inf_{|x| \geq n} V(x)] \to \infty \text{ as } n \to \infty. \quad (3.41) \]
This is a contradiction. Thus, $\gamma_n \to \infty$ w.p.1 as $n \to \infty$. □

In view of the above proof and parallel to Theorem 3.3, we can derive the following results. It is mainly based on an observation that $V(x_n)$ is a super-martingale.

**Proposition 3.15.** Under the conditions of Theorem 3.14, for any $\delta > 0$ sufficiently small, there is a $K_\delta > 0$ such that

$$
\limsup_{n \to \infty} P(|x_n| \geq K_\delta) \leq \delta. 
$$

(3.42)

**Proof.** By virtue of (3.40), $\{V(x_n)\}$ is a super-martingale. Iterating on this inequality and taking expectation, we obtain

$$
E \sum_{k=0}^{n-1} [E_k V(x_{k+1}) - V(x_k)] \leq 0.
$$

On the other hand,

$$
E \sum_{k=0}^{n-1} [E_k V(x_{k+1}) - V(x_k)] = EV(x_n) - EV(x_0).
$$

Thus, $EV(x_n) \leq EV(x_0)$. It follows that

$$
\limsup_{n \to \infty} EV(x_n) \leq K < \infty.
$$

The desired result thus follows. □

To proceed, for the purpose of stability, we need to examine $x_\alpha(\cdot + t_\alpha)$, where $t_\alpha \to \infty$ as $\alpha \to 0$. The weak convergence alone will not give us the desired result. We need in addition the tightness of the iterates.
Lemma 3.16. Under (H5) or (H5') and (H6), and assuming $E|x_0| < \infty$, the sequence $\{x_n\}$ is tight in $\mathbb{R}^r$.

Proof. To prove the result, we merely note that by virtue of (3.40), we arrive at

$$E|x_n|^\gamma \leq E|x_0|^\gamma < \infty.$$

Since the function $| \cdot |^\gamma$ is positive and strictly increasing on $(0, \infty)$, the familiar Tchebyshev inequality yields that for each $\varepsilon > 0$, there is a $K_\varepsilon \geq (E|x_0|^\gamma/\varepsilon)^{1/\gamma}$ such that

$$P(|x_n| \geq K_\varepsilon) \leq \frac{E|x_n|^\gamma}{K_\varepsilon^\gamma} \leq \frac{E|x_0|^\gamma}{K_\varepsilon^\gamma} \leq \varepsilon.$$

The desired tightness thus follows. \(\square\)

Now we are in a position to obtain the desired stability result. It is stated for the interpolated process.

Theorem 3.17. Under conditions (H5') and (H6), for any $t_\alpha \to \infty$ as $\alpha \to 0$,

$$\limsup_{\alpha \to 0} \frac{\log |x^\alpha(t_\alpha)|}{t_\alpha} < 0 \text{ w.p.1,}$$

(3.43)

where the probability measure is understood to be defined in an enlarged probability space with the use of the Skorohod representation.

Proof. Define $\tilde{x}^\alpha(\cdot) = x^\alpha(t_\alpha + \cdot)$ where $t_\alpha \to \infty$ as $\alpha \to 0$. By virtue of the argument in the proof of Theorem 3.8, $\{\tilde{x}^\alpha(\cdot)\}$ is tight and any weakly convergent subsequence has limit
that satisfies (3.1). For arbitrary $T$ satisfying $0 < T$, consider the pair $(\tilde{x}^\alpha(\cdot), \tilde{x}^\alpha(t_\alpha + \cdot - T))$.

Denote the weak limit of the pair by $(x(\cdot), x_T(\cdot))$. By virtue of the Skorohod representation (with a slight abuse of notation, without changing notation), we assume the convergence is in the sense of convergence w.p.1. It is easily seen that $x(0) = x_T(T)$. The set of possible values $\{x_T(0)\}$ is tight because of Lemma 3.16. In view of the weak convergence,

$$x(0) = x_T(T) = x_T(0) + \int_0^T \mu(x(s), \theta(s))ds + \int_0^T \sigma_1(x(s), \theta(s))dB_1(s) + \int_0^T \sigma_2(x(s), \theta(s))dB_2(s).$$

(3.44)

Using (3.44) and evaluating $\log |x_T(T)|$, we obtain (3.20) with $X(t)$ replaced by $x_T(t)$. Next evaluating $\log |x_T(T)|/T$, by the arbitrariness of $T$, the desired result follows from Theorem 3.6 and the representation (3.44).

3.4 Examples

In this section, we give several examples to demonstrate the regularization and exponential stabilization for switching ODEs.

**Example 3.18.** We begin with (1.1) together with initial condition $X(0) = 1$. Suppose that $\theta(t)$ is a Markov chain with two states $\mathcal{M} = \{1, 2\}$ and generator $Q = \begin{pmatrix} -0.1 & 0.1 \\ 1 & -1 \end{pmatrix}$, that $\mu(x, 1) = x(x + 1)$ and $\mu(x, 2) = x(2x + 1)$. Corresponding to the two states, we have two equations

$$\frac{d}{dt} X(t) = X(t)(X(t) + 1),$$

$$\frac{d}{dt} X(t) = X(t)(2X(t) + 1).$$

(3.45)
It is readily seen that neither equation has a global solution. In fact for the first equation, we have \( X(t) = e^t/(2 - e^t) \) that will blow up at time \( \ln 2 \); for the second equation, \( X(t) = e^t/(3 - 2e^t) \) that will blow up at time \( \ln(3/2) \). We plot the trajectories of the switched system as well as each individual system in Figure 2.

![Figure 2: Trajectories of switched system and the two individual systems. Solid red curve plots the trajectory of system (1.1) in Example 3.18, the dashed blue curve is the trajectory of the first equation in (3.45), and the dotted green curve is the trajectory of the second equation in 3eqex1-1. Stepsize \( \Delta t = 10^{-4} \) is used.](image)

To regularize the system, we use a feedback control with \( a_1(\theta(t))X^2(t)dB_1(t) \), where \( B_1(t) \) is a one-dimensional Brownian motion, \( a_1(i) = 2 \) for \( i = 1, 2 \). To stabilize the system, we add another feedback control \( a_2(\theta(t))X(t)dB_2(t) \), where \( B_2(t) \) is one-dimensional standard Brownian motion that is independent of \( B_1(t) \) and \( a_2(1) = 19 \) and \( a_2(2) = 24 \). That is, we have

\[
dX(t) = \mu(X(t), \theta(t))dt + a_1(\theta(t))X^2(t)dB_1(t) + a_2(\theta(t))X(t)dB_2(t). \tag{3.46}
\]

To visualize the regularization and stabilization effect, we plot the trajectories in Figure 3.

**Example 3.19.** We consider the same problem as in Example 3.18 with the modification
\[ Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \]. In Example 3.18, the switching process tends to spend more time in state 1, whereas in the current example, the switching is symmetric with equal chance of staying in each state.

The system (3.46) in Example 3.18 still meets the conditions for the new generator \( Q \), and its trajectory is shown in Figure 4.

Figure 3: Trajectory of system (3.46) in Example 3.18 with stepsize \( \Delta t = 10^{-6} \).

Figure 4: Trajectory of system (3.46) in Example 3.19 with time step \( \Delta t = 10^{-6} \).

**Example 3.20.** Consider a Lotka-Volterra model of two species competitive ecosystem given by

\[ dX(t) = X(t)(b(\theta(t)) - a(\theta(t))X(t)) \, dt \]  \hspace{1cm} (3.47)

where \( b(1) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \), \( b(2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), \( a(1) = \begin{pmatrix} 2 & 3 \\ 1 & 6 \end{pmatrix} \), \( a(2) = \begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix} \), and \( \theta(\cdot) \in \{1, 2\} \) is a continuous-time Markov chain generated by \( Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \). Again, it is shown in [51] that system (3.47) has a global solution. We plot the trajectories of (3.47) in Figure 5. To stabilize the system, We can add a feedback control with \( B_2(\cdot) \in \mathbb{R}^2 \) to change the
system into

\[ dX(t) = X(t)(b(\theta(t)) - aX(t))dt + \sigma(\theta(t))X(t)dB_2(t) \]  

(3.48)

where \( \sigma(1) = \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \sigma(2) = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \). We plot the corresponding trajectories in Figure 6.

Figure 5: Component-wise sample trajectories of system (3.47) with stepsize \( \Delta t = 10^{-4} \). The solid red curve is the first species, and the dotted blue curve is the second species.

Figure 6: Component-wise sample trajectories of system (3.48) with stepsize \( \Delta t = 10^{-4} \). The solid red curve is the first species, and the dotted blue curve is the second species.
4 Stabilization for Switching Diffusions

Now we consider an even more general dynamic system with Brownian noise added. Many applications in real life involve not only noise in the traditional setup represented by stochastic differential equations, but also discrete random events. Switching diffusion systems are the coexistence of continuous diffusive dynamics as well as discrete events represented by a switching process, which makes the systems more realistic. The switching diffusions have substantially expanded the applicability of the diffusions and the pure jump systems. The interactions of the continuous and discrete dynamics make the analysis, control, analysis of stability, and stabilization of such systems far more difficult than diffusion systems or jump systems alone. We begin with the system of stochastic differential equations involving random switching as (1.6) The motivation of studying such systems stems from that the regime switching can be used to quantify the randomness of the environment or the topology changes. Note that instead of one stochastic differential equation, we have a system of such equations switching back and forth in accordance with the dynamics of the Markov chain. In addition to the drift, the diffusion coefficient is also a nonlinear function, which makes the system more realistic, but is more difficult to handle.

Similar to the approach above, we also add two noise type feedback control to ensure the regularity and stability. This chapter is arranged as follows. First the existence of global solutions for the switching diffusion system is proved. Second the properties of $\gamma$–stable and exponentially stable by representing the Markov chain as a Poisson jump process are obtained. Finally a couple of examples are provided for demonstration.
4.1 Regularization

Consider the regime switching diffusion system (1.6), the classical existence and uniqueness conditions are not satisfied, so the system may have no global solutions. To stabilize the switching diffusion system, we need to first extend the local solutions to global solutions. To take up the challenge, we first construct a feedback control to make the resulting system have a global solution. This is accomplished by injecting a perturbation term $\sigma_1(X(t), \theta(t))dB_1(t)$ to suppress the explosion. Note that the feedback control we are using is a Brownian noise. Roughly, in real applications, the precise form of the diffusion in (1.6) may be unknown, but we have an idea how fast it grows. Consequently, we add a noise term to outperform the growth of the drift and the existing noise. For diffusions, it has been well known that one can use appropriate linear diffusions to stabilize the system. In the second step of our work, we add another feedback noise $\sigma_2(X(t), \theta(t))dB_2(t)$ (linear in $x$) to make the system stable. Both steps are essential.

Putting the aforementioned two stages together, effectively we study the system given by

\[
\begin{cases}
    dX(t) = \mu(X(t), \theta(t))dt + \sigma(X(t), \theta(t))dB(t) + \sigma_1(X(t), \theta(t))dB_1(t) \\
    \quad + \sigma_2(X(t), \theta(t))dB_2(t), \\
    X(0) = x_0, \quad \theta(0) = \theta,
\end{cases}
\]

(4.1)

where $\sigma_i(X(t), \theta(t)) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^{r \times d}$, $B_1(t)$ and $B_2(t)$ are standard $d$-dimensional Brownian motion that are independent of each other and independent of $B(t)$ and $\theta(t)$. For each
i ∈ \mathcal{M} and any \( g(\cdot, i) \in C^2 \), define the operator of (4.1) by

\[
\mathcal{L}g(x, i) = \mu'(x, i)\nabla g(x, i) + \frac{1}{2} \text{tr}(\sigma(x, i)\sigma'(x, i)\nabla^2 g(x, i)) + 1\frac{1}{2} \text{tr}(\sigma_1(x, i)\sigma_1'(x, i)\nabla^2 g(x, i)) + 1\frac{1}{2} \text{tr}(\sigma_2(x, i)\sigma_2'(x, i)\nabla^2 g(x, i)) + Qg(x, \cdot)(i),
\]

(4.2)

\[
Qg(x, \cdot)(i) = \sum_{j \in \mathcal{M}} q_{ij}g(x, j) \quad \text{for each } i \in \mathcal{M}.
\]

To study the regularity and stability of the process \( X(t) \), we assume the following conditions hold.

(H7) For any \( i \in \mathcal{M} \), \( \mu(\cdot, i) \) and \( \sigma(\cdot, i) \) are locally Lipschitz continuous and

(a) \( \mu(0, i) = 0 \);

(b) \( \mu'(x, i)x \leq K_0(i)(|x|^{\beta_1+2} + |x|^2) \) for some \( \beta_1 > 0 \);

(c) For some \( \beta_2 > 0 \), \( \lambda_x(x, i) \leq K_1(i)(|x|^{\beta_2+2} + |x|^2) \). Denote \( K_j = \max_{i \in \mathcal{M}}\{K_j(i)\} \) for \( j = 0, 1 \).

(H8) For any \( i \in \mathcal{M} \), \( \sigma_1(\cdot, i) \), \( \sigma_2(\cdot, i) \) are locally Lipschitz continuous and

(a) for some \( \beta \) satisfying \( 2\beta > \max\{\beta_1, \beta_2\} \) and for some \( K_j(i) \) > 0, \( j = 2, \ldots, 5 \) and \( i \in \mathcal{M} \), with \( 2K_2(i) > K_3(i) \),

\[
\text{tr}(\sigma_1(x, i)\sigma_1'(x, i)x^xx') \geq K_2(i)(|x|^{4+2\beta} - |x|^4),
\]

(4.3)

\[
\text{tr}(\sigma_1(x, i)\sigma_1'(x, i)) \leq K_3(i)(|x|^{2+2\beta} + |x|^2),
\]

(4.4)

\[
\text{tr}(\sigma_2(x, i)\sigma_2'(x, i)x^xx') \geq K_4(i)|x|^4,
\]

(4.5)
\[
\text{tr}(\sigma_2(x,i)\sigma'_2(x,i)) \leq K_5(i)|x|^2. \tag{4.6}
\]

(b) \( K_k = \min_{i \in \mathcal{M}} \{K_k(i)\} \) for \( k = 2, 4 \) and \( K_j = \max_{i \in \mathcal{M}} \{K_j(i)\} \) for \( j = 3, 5 \).

Let us comment on the conditions briefly. The motivation of \((H7)(c)\) is that for each matrix \( M \in \mathbb{R}^{r \times d} \) and for any \( x \in \mathbb{R}^{r \times 1} \), \(|\text{tr}(MM')| \leq r\lambda_M \) and \(|x'MM'x| \leq \lambda_M|x|^2|\). This and \((H7)(a)\) imply that system \((1.6)\) has an equilibrium point \((0,i)\) for each \( i \in \mathcal{M} \). Note that only partial information of the noise term \( \sigma(X(t),\theta(t))dB(t) \) as listed in \((H7)(c)\) is available, which makes the conditions more realistic. This however becomes more difficult compared with noise suppression and stabilization for a system of randomly switched ordinary differential equations as treated in \([46]\). The difficulty lies in that the system noise is part of the uncertainty and the diffusion matrix has growth rate much faster than linear with respect to the variable \( x \). It is readily seen that conditions \((4.4)\) and \((4.6)\) imply \( \sigma_1(0,i) = \sigma_2(0,i) = 0 \) for any \( i \in \mathcal{M} \). These and condition \((H7)\) yield that system \((4.1)\) has an equilibrium point \((0,i)\) for each \( i \in \mathcal{M} \). Condition \((H8)\) is a technical requirement, which essentially demands that the added perturbations (the controls) have desired properties so as to lead to the desired regularization and stabilization. Since the controls are designed by us, this is completely at our proposal and is not a restriction.

By virtue of Lemma 3.1 and the use of the Lyapunov function \( V(x) = |x|^\gamma \), we can get the following theorem.

**Theorem 4.1.** Assume condition \((H7)\) and \((H8)\) hold. Then system \((4.1)\) has a unique global solution for any finite initial value \( x_0 \).

**Proof of Theorem 4.1.** In view of \([33]\), under condition \((H7)\), system \((4.1)\) has a unique
maximal local strong solution $X(t)$ on $t \in [0, \tau)$. To obtain the existence of global solution, it suffices to prove that $\tau = \infty$ a.s. For the study of regularity, it is sufficient to work with a domain exterior to a neighborhood of 0 since the regularity is mainly concerned with the behavior of the system in a neighborhood of infinity. As a result, we can confine our attention to $S^C_R$, the region exterior to the ball with radius $R$, where $R$ is large enough. In this domain, for any $\gamma \in (0, 1)$, define a $C^2$ Lyapunov function by

$$V(x) = |x|^\gamma \quad \text{for each } i \in \mathcal{M}. \quad (4.7)$$

Then $V(x) \geq 0$ for any $x \in \mathbb{R}^r$, $\lim_{k \to \infty} \inf_{|x| \geq k} V(x) = \infty$ for each $i \in \mathcal{M}$, and

$$\nabla V(x) = \gamma |x|^{\gamma-2}x,$$

$$\nabla^2 V(x) = \gamma |x|^{\gamma-2}I + \gamma(\gamma - 2)|x|^{\gamma-4}xx'.$$

Using (H7) and (H8),

$$\mathcal{L}V(X) = \gamma \mu'(X,i) |X|^{\gamma-2}X$$

$$+ \frac{\gamma}{2} \text{tr}[\sigma(X,i)\sigma'(X,i)|X|^{\gamma-2}] + \frac{\gamma(\gamma - 2)}{2} \text{tr}[\sigma(X,i)\sigma'(X,i)|X|^{\gamma-4}XX']$$

$$+ \frac{\gamma}{2} \text{tr}[\sigma_1(X,i)\sigma'_1(X,i)|X|^{\gamma-2}] + \frac{\gamma(\gamma - 2)}{2} \text{tr}[\sigma_1(X,i)\sigma'_1(X,i)|X|^{\gamma-4}XX']$$

$$+ \frac{\gamma}{2} \text{tr}[\sigma_2(X,i)\sigma'_2(X,i)|X|^{\gamma-2}] + \frac{\gamma(\gamma - 2)}{2} \text{tr}[\sigma_2(X,i)\sigma'_2(X,i)|X|^{\gamma-4}XX']. \quad (4.8)$$

Thus, we obtain that

$$\mathcal{L}V(x) \leq K_0 |X|^\gamma(|X|^{\beta_1} + 1) + \frac{\gamma K_1 + \gamma(2 - \gamma)K_1}{2}|X|^\gamma(|X|^{\beta_2} + 1)$$

$$+ \frac{\gamma K_3 + \gamma(\gamma - 2)K_2}{2}|X|^\gamma \quad \text{for each } i \in \mathcal{M}. \quad (4.9)$$
Because $2\beta > \max\{\beta_1, \beta_2\}$, the dominating term is

$$\frac{\gamma K_3 + \gamma(\gamma - 2)K_2}{2} |X|^\gamma + 2\beta. \quad (4.10)$$

Given $2K_2 > K_3$, there exists some $\gamma$ small enough such that $K_3 + (\gamma - 2)K_2 < 0$. We can choose proper $R$ such that

$$\frac{\gamma K_3 + \gamma(\gamma - 2)K_2}{2} |X|^\gamma + \gamma K_0 |X|^{\gamma + \beta_1} + \frac{r\gamma K_1 + \gamma(2 - \gamma)K_1}{2} |X|^{\gamma + \beta_2} < 0.$$

As a result, for some $K > 0$, we have

$$\mathcal{L}V(X) \leq (K_0\gamma + \frac{r\gamma K_1 + \gamma(2 - \gamma)K_1}{2} + \frac{\gamma K_3 + \gamma(2 - \gamma)K_2}{2} + \frac{\gamma K_5 + \gamma(\gamma - 2)K_4}{2}) |X|^\gamma \leq KV(X).$$

By virtue of Lemma 3.1, there is a unique global solution for any initial data $x_0$. \Box

Using Theorem 4.1 and applying the Dynkin formula to $e^{\lambda t}V(x)$, we obtain tightness of $X(t)$ as $t \to \infty$ in the following theorem.

**Theorem 4.2.** Under conditions (H7) and (H8), for any $\delta > 0$ sufficiently small, there is a $K_\delta > 0$ such that

$$\limsup_{t \to \infty} \mathbf{P}(|X(t)| \geq K_\delta) \leq \delta. \quad (4.11)$$

**Proof.** Let us begin with a Lyapunov function $e^{\lambda t}V(x)$ with $V(x) = |x|^\gamma$ for $\gamma \in (0, 1)$ and $\lambda > 0$ is a constant. As in Theorem 4.1, (4.8) and (4.10) hold. Because the coefficient of the dominant term $|x|^{\gamma + 2\beta}$ in (4.8) is negative, we can choose $R$ sufficiently large and focus on
the $S^C_\kappa$ such that for some $K > 0$,

$$LV(x) + \lambda V(x) \leq K. \quad (4.12)$$

By virtue of the Dynkin formula (applied to $e^{\lambda t}V(x)$) and combining (4.12), we have

$$E\mathbb{V}(X(t)) = e^{-\lambda t}E\mathbb{V}(X(0)) + E\int_0^t e^{-\lambda(t-s)}[\mathcal{L}V(X(t)) + \lambda V(X(s))]ds$$

$$\leq e^{-\lambda t}E\mathbb{V}(X(0)) + E\int_0^t e^{-\lambda(t-s)}Kds$$

$$\leq e^{-\lambda t}E\mathbb{V}(X(0)) + K(1 - e^{-\lambda t}). \quad (4.13)$$

Here $K > 0$ is a generic positive constant. This leads to

$$\limsup_{t \to \infty} E\mathbb{V}(X(t)) \leq K < \infty.$$ 

Therefore the desired result follows. \hfill \square

### 4.2 Stabilization

To study the stability of the equilibrium point $x = 0$, first we need the following property, which can be found in [33]; see also [48, Lemma 7.1] for switching diffusions with $x$-dependent switching. This property can be phrased as if we start with $(x, i)$ for any $i \in \mathcal{M}$, $X(t)$ will not reach the state 0 for subsequent time $t > 0$ as long as the initial state $x \neq 0$. Under conditions (H7) and (H8), for $X(0) \neq 0$, we have

$$\mathbb{P}(X(t) \neq 0 \text{ on } t \in [0, \tau)) = 1,$$
where $\tau$ is the explosion time. In Section 4.1 we have shown that $\tau = +\infty$ a.s., this yields the following lemma.

**Lemma 4.3.** Under the condition of Theorem 4.1, if $X(0) \neq 0$, then

$$P(X(t) \neq 0 \text{ for all } t \geq 0) = 1.$$  \hspace{1cm} (4.14)

This lemma states that a.s. the sample path of any solution of the system (4.1) and (1.2) starting from a nonzero state will not reach the origin subsequently. With this lemma, we can choose Lyapunov functions that are smooth in a deleted neighborhood of origin. To stabilize the system, we need the stationarity of the Markov chain $\theta(t)$. The condition is given as follows.

(H9) The Markov chain $\theta(t)$ is irreducible.

To prove the equilibrium point $x = 0$ of the system (4.1) is exponentially $\gamma$-stable, we first establish certain sufficient conditions for $\gamma$-stability.

**Theorem 4.4.** Let $\gamma, K, d_1, d_2$ be any positive numbers and $d_1 \leq d_2$. Assume that there exists a function $V(X, t, i) \in C^{2,1}(\mathbb{R}_0^r \times \mathbb{R}_+ \times \mathcal{M}; \mathbb{R}_+)$ such that

$$d_1|X|^\gamma \leq V(X, t, i) \leq d_2|X|^\gamma$$ \hspace{1cm} (4.15)

and

$$\mathcal{L}V(X, t, i) \leq -K|X|^\gamma$$ \hspace{1cm} (4.16)
for all \((X,t,i) \in \mathbb{R}_0^r \times \mathbb{R}_+ \times \mathcal{M}\). Then

\[
\limsup_{t \to \infty} \frac{1}{t} \ln(\mathbb{E}|X(t;x_0,\theta)|^\gamma) \leq -\frac{K}{d_2}
\]  

(4.17)

for all \(x_0 \in \mathbb{R}^r\) and \(\theta \in \mathcal{M}\).

**Proof.** It can be seen that the result holds for \(x_0 = 0\). For \(x_0 \neq 0\), system (4.1) has a unique global solution \(X(t;x_0,\theta)\). Clearly the stopping time \(\tau_n\) satisfies

\[
\lim_{n \to \infty} \tau_n = +\infty \text{ a.s.}
\]

Then (4.17) holds, which comes directly from the result of [29, Theorem 3.1].

To prove exponential \(\gamma\)-stability, by Theorem 4.4, we need only find a Lyapunov function which satisfies (4.15) and (4.16). Consider function \(V(x,i) = (1 - \gamma c_i)|x|^\gamma \in \mathbb{R}^r \times \mathcal{M}\), where \(\gamma \in (0,1)\), \(c_i\) is positive number to be specified later, and \(1 - \gamma c_i > 0\) for each \(i \in \mathcal{M}\). It can be verified that (4.15) and (4.16) are satisfied. Then we can get the following theorem.

**Theorem 4.5.** Under condition (H7)–(H9), the equilibrium point 0 of the system (4.1) is exponential \(\gamma\)-stable if

\[
\sum_{i \in \mathcal{M}} (K_4(i) - \alpha_i)\nu_i > 0,
\]

with \(\alpha_i\) defined in (4.19).

**Proof.** It can be verified that (4.15) is satisfied. We need only verify (4.16). In view of (4.2),
we have

\[
\mathcal{L}V(x, i) = \gamma(1 - \gamma c_i) \{ \mu'(x, i) |x|^{\gamma - 2} x \\
+ \frac{1}{2} \text{tr}[\sigma(x, i)\sigma'(x, i)] |x|^{\gamma - 2} I \\
+ \frac{1}{2} \text{tr}[\sigma_1(x, i)\sigma_1'(x, i)] |x|^{\gamma - 2} I \\
+ \frac{1}{2} \text{tr}[\sigma_2(x, i)\sigma_2'(x, i)] |x|^{\gamma - 2} I \}
\]

\[+ QV(x, i) \]

\[
:= \gamma(1 - \gamma c_i) \{ T_1(i) + T_2(i) + T_3(i) + T_4(i) \} + QV(x, i).
\]

(4.18)

Using conditions (H7)–(H9), for each \( i \in \mathcal{M} \) and \( x \in \mathbb{R}^r \), we obtain

\[
T_1(i) \leq K_0(i) |x|^{\gamma}(|x|^{\beta_1} + 1),
\]

\[
T_2(i) \leq r K_1(i) |x|^{\gamma}(|x|^{\beta_2} + 1) + \frac{2 - \gamma}{2} K_1(i) |x|^{\gamma}(|x|^{\beta_2} + 1),
\]

\[
T_3(i) \leq \frac{1}{2} K_3(i) |x|^{\gamma}(|x|^{2\beta} + 1) - \frac{2 - \gamma}{2} K_2(i) |x|^{\gamma}(|x|^{2\beta} - 1),
\]

\[
T_4(i) \leq \frac{1}{2} K_5(i) |x|^{\gamma} - \frac{2 - \gamma}{2} K_4(i) |x|^{\gamma}.
\]

Set

\[
\alpha_i := \sup_x \left[ \left( -\frac{2 - \gamma}{2} K_2(i) + \frac{1}{2} K_4(i) \right) |x|^{2\beta} + K_0(i) |x|^{\beta_1} + \frac{r + 2 - \gamma}{2} K_1(i) |x|^{\beta_2} + \tilde{K}(i) \right],
\]

(4.19)

where

\[
\tilde{K}(i) = K_0(i) + \frac{1}{2} ((r + 2 - \gamma) K_1(i) + K_3(i) - (2 - \gamma) K_2(i) + K_5(i) + \gamma K_4(i)).
\]
Under condition (H8), for $\gamma$ sufficiently small, the coefficient

$$\frac{-2 - \gamma}{2} K_2(i) + \frac{1}{2} K_3(i) < 0,$$

which ensures the existence of $\alpha_i$. Thus we have

$$\sum_{j=1}^{4} T_j(i) \leq -(K_4(i) - \alpha_i)|x|\gamma.$$ (4.20)

Next, by condition (H9), when $\gamma$ is sufficiently small,

$$QV(x, \cdot)(i) = -\sum_{j \neq i} q_{ij} \gamma (c_j - c_i)|x|\gamma$$

$$= -\gamma(1 - \gamma c_i)|x|\gamma \left( \sum_{j \neq i} q_{ij} \frac{c_j - c_i}{1 - \gamma c_i} \right)$$

$$= -\gamma(1 - \gamma c_i)|x|\gamma \left( \sum_{j \in M} q_{ij} c_j + \sum_{j \neq i} q_{ij} c_i \frac{c_j - c_i}{1 - \gamma c_i} \right)$$

$$= -\gamma(1 - \gamma c_i)|x|\gamma \left( \sum_{j \in M} q_{ij} c_j + O(\gamma) \right).$$ (4.21)

Set $\alpha = (\alpha_1 - K_4(1), \ldots, \alpha_m - K_4(m))' \in \mathbb{R}^m$ and $\eta := -\nu \alpha$. Using (H9), $Qc = \alpha + \eta \mathbb{1}$ has a solution $c = (c_1, \ldots, c_m)' \in \mathbb{R}^m$. Thus for each $i \in \mathcal{M}$,

$$\alpha_i - K_4(i) - \sum_{j \in \mathcal{M}} q_{ij} c_j = -\eta.$$ (4.22)

Combining (4.18)-(4.22), and assuming $\eta > 0$, we get

$$\mathcal{L}V(x, i) \leq -(\eta + O(\gamma))\gamma(1 - \gamma c_i)|x|^\gamma \leq -K|x|^\gamma.$$
To prove $x = 0$ is almost sure exponential stability, we need a condition on $\sigma_1(x, i)$.

(H10) For some $K_6(i) > 0$ with $i \in \mathcal{M}$, $\text{tr}(\sigma_1(x, i)\sigma_1'(x, i)xx^\prime) \leq K_6(i)|x|^{4+2\beta}$.

The idea of proof can be explained as follows. First, we represent $\theta(t)$ as a Poisson process. We then consider a compensated Poisson process, we obtain the additional martingale term in addition to the martingales due to the Brownian motions. To proceed, we apply generalized Itô formula to $\ln|X(t)|$. Next, applying the ergodicity of the Markov chain $\theta(\cdot)$ using conditions (H7)–(H10), we can obtain the desired result.

**Theorem 4.6.** Suppose that (H7)–(H10) are satisfied and let $\widehat{K}(i)$ be defined by (4.33). If $\sum_{i=1}^m [K_4(i) - \widehat{K}(i)]\nu_i > 0$, then the system (4.1) is exponentially stable.

**Proof.** First represent the Markov switching diffusion as a Poisson jump diffusion as introduced in section 1.2. Applying the generalized Itô lemma (see [39]), for each $i \in \mathcal{M}$ and $g(\cdot, i) \in C^2$, we have

$$g(X(t), \theta(t)) - g(X(0), \theta(0)) = \int_0^t Lg(X(s), \theta(s))ds + \widetilde{N}_1(t) + \widetilde{N}_2(t) + \widetilde{N}_3(t) + \widetilde{N}_4(t),$$

(4.23)

where the operator $L$ is given in (4.2), and

$$\begin{align*}
\widetilde{N}_1(t) &= \int_0^t \nabla g'(X(s), \theta(s))\sigma(X(s), \theta(s))dB(s), \\
\widetilde{N}_2(t) &= \int_0^t \nabla g'(X(s), \theta(s))\sigma_1(X(s), \theta(s))dB_1(s), \\
\widetilde{N}_3(t) &= \int_0^t \nabla g'(X(s), \theta(s))\sigma_2(X(s), \theta(s))dB_2(s), \\
\widetilde{N}_4(t) &= \int_0^t \int_{\mathbb{R}} \left[ g(X(s), \theta(0) + \psi(\theta(s), z)) - g(X(s), \theta(s)) \right] \tilde{P}(ds, dz).
\end{align*}$$

(4.24)
To proceed, by choosing these feedback controls properly, we can show that the system can be stabilized. We next compute the associated Lyapunov exponent. Since 0 is an inaccessible state, we can use \( \ln |x| \) as a Lyapunov function in the following calculation. By (4.23), we have

\[
\ln |X(t)| = \ln |X(0)| + \int_0^t \frac{X'(s)}{|X(s)|^2} \mu(X(s), \theta(s)) ds \\
+ \int_0^t \frac{1}{2|X(s)|^2} \left\{ \text{tr}[\sigma(X(s), \theta(s))\sigma'(X(s), \theta(s))] \\
+ \text{tr}[\sigma_1(X(s), \theta(s))\sigma'_1(X(s), \theta(s)) + \sigma_2(X(s), \theta(s))\sigma'_2(X(s), \theta(s))] \right\} ds \\
- \int_0^t \frac{1}{|X(s)|^2} \left\{ \text{tr}[\sigma(X(s), \theta(s))\sigma'(X(s), \theta(s))X(s)X'(s)] \\
+ \text{tr}[\sigma_1(X(s), \theta(s))\sigma'_1(X(s), \theta(s))X(s)X'(s)] \\
+ \sigma_2(X(s), \theta(s))\sigma'_2(X(s), \theta(s))X(s)X'(s)] \right\} ds \\
+ N_1(t) + N_2(t) + N_3(t),
\]

where \( N_1(t), N_2(t) \) and \( N_3(t) \) are continuous local martingales given by

\[
N_1(t) = \int_0^t \frac{X'(s)}{|X(s)|^2} \sigma(X(s), \theta(s)) dB(s), \\
N_2(t) = \int_0^t \frac{X'(s)}{|X(s)|^2} \sigma_1(X(s), \theta(s)) dB_1(s), \\
N_3(t) = \int_0^t \frac{X'(s)}{|X(s)|^2} \sigma_2(X(s), \theta(s)) dB_2(s).
\]

By (H9) and using indicator function \( I_{\{\theta(s)=i\}} \), for example,

\[
\int_0^t \frac{X'(s)}{|X(s)|^2} \mu(X(s), \theta(s)) ds \\
= \sum_{i=1}^n \int_0^t \frac{X'(s)}{|X(s)|^2} \mu(X(s), i) I_{\{\theta(s)=i\}} ds.
\]

Likewise, we can represent the other integrals in (4.25) in the same way. Using (H7) and
(H8), we obtain that the quadratic variation of $N_1(t)$ satisfies
\[
\langle N_1(t), N_1(t) \rangle = \int_0^t \frac{X'(s)\sigma(X(s), \theta(s))\sigma'(X(s), \theta(s))X(s)}{|X(s)|^4} ds \\
\leq \int_0^t K_1(i)(|X(s)|^{\beta_2} + 1)I_{\{\theta(s) = i\}} ds.
\] (4.28)

The quadratic variation of $N_3(t)$ is given by
\[
\langle N_3(t), N_3(t) \rangle \leq \int_0^t rK_4(i)ds \leq Kt.
\] (4.29)

Recall that we use $K$ as a generic positive constant and thereafter. The local martingale convergence theorem in [26] yields that $\frac{N_3(t)}{t} \to 0$ a.s. as $t \to \infty$.

Consider $N_2(t)$. For any $\varepsilon \in (0, 1)$, choose $\varpi > 0$ such that $\varepsilon \varpi > 1$. Then for each integer $n$, Doob’s inequality [30] implies that
\[
P \left( \sup_{1 \leq t \leq n} \left[ N_2(t) - \frac{\varepsilon}{2} \langle N_2(s), N_2(s) \rangle ds \geq \varpi \ln n \right] \right) \leq \frac{1}{n^{\varepsilon \varpi}}.
\]

Because $\sum_{m=1}^{\infty} \frac{1}{n^{\varepsilon \varpi}} < \infty$, the Borel-Cantelli Lemma leads to that for almost all $\omega$, there exists a $\tilde{K}_2(\omega) > 1$ such that for all $n \geq \tilde{K}_2(\omega)$ and $n - 1 \leq t \leq n$,
\[
N_1(t) \leq \frac{\varepsilon}{2} \langle N_1(t), N_1(t) \rangle + \varpi \ln(t + 1).
\] (4.30)
Combining (4.25)–(4.30) and using (H7), (H8), and (H10), we have

\[
\ln |X(t)| \leq \ln |X(0)| + \sum_{i=1}^{m} \left\{ K_0(i) \int_0^t |X(s)|^{\beta_1} + 1 \right\} I_{\{\theta(s)=i\}} ds
+ \frac{r K_1(i)}{2} \int_0^t |X(s)|^{\beta_2} + 1 \right\} I_{\{\theta(s)=i\}} ds
+ \frac{1}{2} K_3(i) \int_0^t |X(s)|^{2\beta} + 1 \right\} I_{\{\theta(s)=i\}} ds
+ \frac{1}{2} K_5(i) \int_0^t I_{\{\theta(s)=i\}} ds
+ \frac{K_1(i)}{2} \int_0^t \left[ |X(s)|^{\beta_2} + 1 \right] I_{\{\theta(s)=i\}} ds
- K_2(i) \int_0^t |X(s)|^{\beta_2} - 1 \right\} I_{\{\theta(s)=i\}} ds
- K_4(i) \int_0^t I_{\{\theta(s)=i\}} ds
+ K_1(i) \int_0^t \left[ |X(s)|^{\beta_2} + 1 \right] I_{\{\theta(s)=i\}} ds
+ \frac{\varepsilon}{2} K_6(i) \int_0^t |X(s)|^{2\beta} I_{\{\theta(s)=i\}} ds \right\}
+ \omega \ln(t + 1) + N_3(t). \tag{4.31}
\]

The ergodicity of the Markov chain \(\theta(\cdot)\) implies

\[
\frac{1}{t} \int_0^t I_{\{\theta(s)=i\}} ds \to \nu_i \ \text{a.s. as } t \to \infty. \tag{4.32}
\]

Using condition (H8)(b), for sufficiently small

\[
\varepsilon \in \left( 0, \min_{i \in \mathcal{M}} \left\{ \frac{2 K_2(i) - K_3(i)}{K_6(i)} \right\} \right),
\]

we have

\[
\widehat{K}(i) = \sup_x \left[ (-K_2(i) + \frac{1}{2} K_3(i) + \frac{\varepsilon}{2} K_6(i)) |x|^{2\beta} + K_0(i) |x|^{\beta_1} + (r + 1) K_1(i) |x|^{\beta_2} + \widehat{K}(i) \right], \tag{4.33}
\]
where
\[ \hat{K}(i) = [K_0(i) + (r + 1)K_1(i) + \frac{1}{2}(K_1 + K_3 + K_5) + K_2(i)]. \]

Since
\[ -K_2(i) + \frac{1}{2}K_3(i) + \frac{\varepsilon}{2}K_6(i) < 0 \quad \text{for all} \quad i \in M, \]
\( \hat{K}(i) \) exists. Dividing both sides of (4.25) by \( t \) and noting that as \( t \to \infty \),
\[ \frac{\ln |X(0)|}{t} \to 0, \]
\[ \frac{\ln(t + 1)}{t} \to 0, \]
\[ \frac{N_3(t)}{t} \to 0 \quad \text{a.s.} \]

Thus, we have
\[ \limsup_{t \to \infty} \frac{\ln |X(t)|}{t} \leq -\sum_{i=1}^{m} [K_4(i) - \hat{K}(i)] \nu_i \quad \text{a.s.} \quad (4.34) \]

\[ \square \]

### 4.3 Examples

In this section, we give several examples to demonstrate the regularization and exponential stabilization for switching diffusions.

**Example 4.7.** To visualize why (H7) and (H8) are needed, let us consider the following example. The system can be solved explicitly when without the regularizing and stabilizing feedback controls added. The explosion in a finite time can be clearly seen, which shows the necessity of regularization and stabilization.

Consider (1.6) and assume the initial condition is \( X(0) = 1 \). Suppose that \( \mu(x, 1) = \)
\(x(4x + 1), \mu(x, 2) = x(2x + 1), \sigma(x, 1) = \sigma(x, 2) = x,\) and \(\theta(t)\) is a continuous-time Markov chain taking value in \(\mathcal{M} = \{1, 2\}\) with generator \(Q = \begin{pmatrix} -2 & 2 \\ 0.3 & -0.3 \end{pmatrix}\). Effectively, there are two diffusions given by
\[
dX_1(t) = X_1(t)(4X_1(t) + 1)dt + X_1(t)dB(t),
\]
\[
dX_2(t) = X_2(t)(2X_2(t) + 1)dt + X_2(t)dB(t),
\]
which switch back and forth in accordance with the Markov switching process \(\theta(\cdot)\). For \(X_1(0) = X_2(0) = X(0) = 1\), these two equations can be solved explicitly as
\[
X_1(t) = \frac{\exp \left( \frac{t}{2} + B(t) \right)}{1 - 4 \int_0^t \exp \left( \frac{\tau}{2} + B(s) \right) ds},
\]
\[
X_2(t) = \frac{\exp \left( \frac{t}{2} + B(t) \right)}{1 - 2 \int_0^t \exp \left( \frac{\tau}{2} + B(s) \right) ds},
\]
Thus \(X_1(t)\) and \(X_2(t)\) will explode at the random times \(\tau_1\) and \(\tau_2\) that are solutions of the following equations
\[
\int_0^{\tau_1} \exp \left( \frac{s}{2} + B(s) \right) ds = \frac{1}{4},
\]
\[
\int_0^{\tau_2} \exp \left( \frac{s}{2} + B(s) \right) ds = \frac{1}{2}.
\]
Clearly, \(P(\tau_1 > \tau_2) = 1\). In fact, we also arrive at the explicit solution of the switching system (1.6) as follows:
\[
X(t) = \frac{\exp \left( \frac{t}{2} + B(t) \right)}{1 - \int_0^t a(\theta(s)) \exp \left( \frac{\tau}{2} + B(s) \right) ds},
\]
which implies that this switching diffusion will blow up in finite time with the explosion time
\( \tau \) given by

\[
\int_0^\tau a(\theta(s)) \exp \left( \frac{s}{2} + B(s) \right) ds = 1,
\]

where \( a(1) = 4 \) and \( a(2) = 2 \). Numerical computations show that neither of these two equations has a global solution. The switching system is not regular either; Figure 7 shows the explosive behavior of switched system and the two individual equations.

![Figure 7](image1.png)

**Figure 7:** Solid curve: system (4.35) in Example 4.7, dashed curve: equation in state \( \theta(t) = 1 \), and dotted curve: equation in state \( \theta(t) = 2 \). Stepsize \( \Delta t = 10^{-4} \) is used.

![Figure 8](image2.png)

**Figure 8:** Regularized and stabilized trajectory of (4.35) with stepsize \( \Delta t = 10^{-5} \).

To ensure that the system has a global solution, we add a feedback control of the form \( \sigma_1(\theta(t))X^2(t)dB_1(t) \) with \( \sigma_1(1) = \sigma_1(2) = 2 \), and \( B_1(t) \) being a standard one-dimensional Brownian motion. Then all the conditions posed are all satisfied. Thus the resulting system has a global solution. To ensure the system asymptotically stable, we add another feedback control \( \sigma_2(\theta(t))X(t)dB_2(t) \) with \( \sigma_2(1) = 6 \), \( \sigma_2(2) = 8 \), and \( B_2(t) \) being a standard Brownian motion independent of \( B_1(t) \). We plot the trajectory of the stabilized system in Figure 8.
Example 4.8. Consider a mutual ecosystem with two species

\[ dX(t) = \text{diag}(X_1(t), X_2(t))(b(\theta(t)) + a(\theta(t))X(t))dt + \sigma(\theta(t))X(t)dB(t) \quad (4.37) \]

where \( B(t) \) is a one-dimensional standard Brownian motion,

\[
\begin{align*}
    b(1) &= b(2) = \left( \begin{array}{c}
                    1 \\
                    \frac{1}{2}
                \end{array} \right), \\
    a(1) &= \left( \begin{array}{cc}
                -1 & 1 \\
                2 & -1
            \end{array} \right), \\
    a(2) &= \left( \begin{array}{cc}
                -1 & 2 \\
                1 & -1
            \end{array} \right), \\
    \sigma(1) &= \sigma(2) = 1 \quad \text{and} \quad \theta(\cdot) \in \{1, 2\}
\end{align*}
\]

is a Markov chain with generator \( Q = \left( \begin{array}{cc}
                -1 & 1 \\
                1 & -1
            \end{array} \right) \). It can be seen that the system will explode in finite time. To regularize the system, we use a feedback control \( \sigma_1(X(t), \theta(t))dB_1(t) \), where \( \sigma_1(x, 1) = \sigma_1(x, 2) = (|x|^2, |x|^3)' \) and \( B_1(t) \) is a one-dimensional standard Brownian motion. To stabilize the system, we add \( \sigma_2(\theta(t))X(t)dB_2(t) \) with \( \sigma_2(1) = 5I, \sigma_2(2) = 4I \), and \( B_2 \) being one-dimensional standard Brownian motion independent of \( B_1(t) \). Now the system can be written as

\[
\begin{align*}
    dX(t) &= \text{diag}(X_1(t), X_2(t))(b(\theta(t)) + a(\theta(t))X(t))dt + \sigma(\theta(t))X(t)dB(t) \\
    &\quad + \sigma_1(X(t), \theta(t))dB_1(t) + \sigma_2(\theta(t))X(t)dB_2(t). \quad (4.38)
\end{align*}
\]

The trajectory of (4.38) is given in Figure 9.

The ecological meaning is that without suppression noise, the populations of each mutual species will explode in a finite time. This defies one of the ecological laws that a population should be self-limited. With the regularize feedback control added, the explosion of the system is suppressed. When the stabilize feedback control is added, the species are asymptotically
extinct.

Figure 9: System (4.38) in component form with stepsize $\Delta t = 10^{-4}$. Solid curve: the first species; dashed curve: the second species.
5 Concluding Remarks and Future Directions

In this dissertation, we have discussed some properties of nonlinear switching dynamic systems with Markov switching. In chapter 2, we established several properties for a class of mean-field models with random switching. The random switching is continuous-state dependent. One difficulty considered here is that each of the particle is required to be nonnegative. Our results include moment estimates, regularity, continuity, and certain tightness. Furthermore, we also examine the asymptotic behavior when the switching process is subject to fast variation. In the future study, it will be interesting to examine the equivalent or mean field behavior when the number of particles or bodies becomes large. In addition, the study of behavior of phase transitions will also be a worthwhile undertaking.

Chapter 3 and Chapter 4 have been devoted to ensuring the existence of global solutions and stability of systems hybrid with regime switching. We used two feedback controls. First, we added nonlinear switching diffusions that grow faster than the nonlinear drift so as to suppress the explosion. Then we added another feedback control of linear switching diffusions. Under suitable conditions, we proved that this scheme leads to the regular stable systems. For computation consideration, we constructed discrete-time approximation algorithms for deterministic system with switching. For future study, the consideration involving null recurrent diffusions (see [20]) will be of interest. Furthermore, for practical issues, one often has to deal with discrete approximation. One may use the approach from stochastic approximation tool box [24] to design approximation schemes; see also related work [43]. The question is: Can we design feedback controls for the discrete approximation for the regularization and stabilization tasks. The problems deserve to be carefully examined in the future.
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Alphen, 1980.


This dissertation concerns the properties of nonlinear dynamic systems hybrid with Markov switching. It contains two parts. The first part focuses on the mean-field models with state-dependent regime switching, and the second part focuses on the system regularization and stabilization using feedback control. Throughout this dissertation, Markov switching processes are used to describe the randomness caused by discrete events, like sudden environment change or other uncertainty.

In Chapter 2, the mean-field models we studied are formulated by nonlinear stochastic differential equations hybrid with state-dependent regime switching. It originates from the phase transition problem in statistical physics. The mean-field term is used to describe the complex interactions between multi bodies in the system, and acts as an mean reversing effects. We studied the basic properties of such models, including regularity, non-negativity, finite moments, existence of moment generating functions, continuity of sample path, positive
recurrence, long-time behavior. We also proved that when switching process changes much more frequently, the two-time-scale limit exists.

In Chapter 3 and Chapter 4, we consider the feedback control for stabilization of nonlinear dynamic systems. Chapter 3 focus on nonlinear deterministic systems with switching. Many nonlinear systems would explode in finite time. We found that Brownian motion noise can be used as feedback control to stabilize such systems. To do so, we can use one nonlinear feedback noise term to suppress the explosion, and then use another linear feedback noise term to stabilize the system to the equilibrium point 0. Since it is almost impossible to get an closed-form solutions, the discrete-time approximation algorithm is constructed. The interpolated sequence of the discrete-time algorithm is proved to converge to the switching diffusion process, and then the regularity and stability results of the approximating sequence are derived. In Chapter 4, we study the nonlinear stochastic systems with switching. Use the similar methods, we can prove that well designed noise type feedback control could also regularize and stabilize nonlinear switching diffusions. Examples are used to demonstrate the results.
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