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The multistage balanced groups ranked set samples (MBGRSS) method is considered for estimating the population mean for samples of size $m = 3k$ where k is a positive real integer. It is compared with the simple random sampling (SRS) and ranked set sampling (RSS) schemes. For the symmetric distributions considered in this study, the MBGRSS estimator is an unbiased estimator of the population mean and it is more efficient than SRS and RSS methods based on the same number of measured units. Its efficiency is increasing in s for fixed value of the sample size, where s is the number of stages. For non symmetric distributions considered in this paper, the MBGRSS estimator is biased. The method is applied in a study of bioleaching.

Keywords: Ranked set sampling, simple random sampling, multistage balanced groups, ranked set samples, symmetric and asymmetric distribution

Introduction

Ranked set sampling is a sampling procedure, which is a less costly as compared to the widely used simple random sampling in cases where visual ranking of a set of observations can be easily done, while the exact measurement of observations is not easy and cost. The RSS mean was considered by McIntyre (1952) as an estimator of the population mean. The RSS mean estimator was considered more efficient than the SRS counterpart.

Takahasi and Wakimoto (1968) introduced the mathematical theory of ranked set sampling. Al-Saleh and Al-Kadiri (2000) suggested double RSS method in order to estimate the population mean. Al-Saleh and Al-Omari (2002) suggested multistage RSS method to increase the efficiency of estimating the

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mean for fixed value of the sample size. Jemain and Al-Omari (2006a, 2006b) considered double percentile RSS and multistage median RSS methods, respectively, for the mean estimation. They found that both methods are more efficient than the SRS based on the same sample size.

Jemain, Al-Omari, and Ibrahim (2008) investigated balanced groups RSS method for estimating the population mean. Jemain, Al-Omari, and Ibrahim (2007) suggested multistage extreme ranked set sampling method for estimating the population mean. Al-Hadhrami and Al-Omari (2009) considered the Bayesian inference of the variance of the normal distribution using moving extreme ranked set sampling. Ozturk (2011) used the RSS for parametric inference about the parameters of the location-scale family of distributions. Dong and Cui (2011) investigated the optimal sign test for quantiles in ranked set samples. Al-Omari, Ibrahim, Jemain, and Al-Hadhrami (2009) proposed multistage balanced groups ranked set samples for estimating the population median. For more details about RSS see Herrera and Al-Omari (2011), Al-Omari (2011), Vock and Balakrishnan (2011), and Drikvandi, Modarres, and Jalilian (2011).

Let $X_1, X_2, \ldots, X_m$ be a SRS of size m from cdf $F(x)$. The i^{th} order statistic $X_{(i:m)}$ has the probability density function (pdf) and the cumulative distribution function (cdf), $f_{(i:m)}(x)$ and $F_{(i:m)}(x)$, respectively, given by

$$f_{(i:m)}(x) = \frac{1}{B[i, m-i+1]}\left[F(x)\right]^{i-1}\left[1-F(x)\right]^{m-i} f(x) \tag{1}$$

$$F_{(i:m)}(x) = \int_{-\infty}^{x} \frac{1}{B[i, m-i+1]}\left[F(t)\right]^{i-1}\left[1-f(t)\right]^{m-i} f(t)\,dt, \quad -\infty < x < \infty \tag{2}$$

where $B[\alpha, \beta] = \int_0^1 t^{\alpha-1}[1-t]^{\beta-1}dt$ is the complete beta function. The mean and the variance of $X_{(i:m)}$ are given by $\mu_{(i:m)} = \int_{-\infty}^{\infty} x f_{(i:m)}(x)\,dx$ and $\sigma^2_{(i:m)} = \int_{-\infty}^{\infty}\left[x-\mu_{(i:m)}\right]^2 f_{(i:m)}(x)\,dx$, respectively.

Multistage Balanced Groups Ranked Set Samples

The RSS can be described as: randomly select m^2 units from the target population. Allocate these units into m sets, each of size m. Rank the m units within each set visually or by any cheap method with respect to the characteristic of interest. From the i^{th} set select the i^{th} ranked unit for $i = 1, 2, \ldots, m$. The process can be repeated n cycles to obtain a set of size mn from the initial m^2n units.

The MBGRSS as suggested by Al-Omari et al. (2009) consists from the following steps:

Step 1: Randomly select $(3k)^{s+1}$ for $k = 1, 2, 3, \ldots$ units from the target population, and then allocate them into $(3k)^s$ sets, each of size $3k$.

Step 2: The $3k$ units of each set are ranked based on professional judgment or by any cheap method in terms of the variable of interest. Then the $(3k)^s$ sets are divided into three groups, each of $3^{s-1}k^s$ sets.

Step 3: From each set in the first group, the smallest ranked unit is selected; from each set in the second group; the median ranked unit is selected, and from each set in the third group, the largest ranked unit is selected. This step yields $(3k)^{s-1}$ sets, $3^{s-2}k^{s-1}$ sets in each group.

Step 4: Without doing any actual measurement, from the $3^{s-2}k^{s-1}$ sets in the first group the smallest ranked unit is selected, from the $3^{s-2}k^{s-1}$ sets in the second group the median ranked unit is selected, and from the $3^{s-2}k^{s-1}$ sets in the third group the largest ranked unit is selected. This step yields $(3k)^{s-2}$ sets, each group of $3^{s-3}k^{s-2}$ sets of size $3k$.

Step 5: The process is continued using Steps (3) and (4) until we end up with one s^{th} stage balanced groups RSS of size $3k$.

The procedure can be repeated n times if needed to obtain a sample of size $3kn$ from the initial $(3k)^{s+1}n$ units.

Al-Omari et al. (2009) introduced an example to illustrate the MBGRSS when $m = 3$. In this paper we will illustrate the MBGRSS in estimating the population mean using $m = 9$.

Example

Let $s = 3$ and $k = 3$, then $m = 9$. Therefore, we have to select 6561 units, say $X_1, X_2, \ldots, X_{6561}$. Allocate the 6561 selected units into 729 sets each of size 9. The 9 observations of each set are ranked with respect to the study variable as follows: $\{X_{i(1:9)}, X_{i(2:9)}, \ldots, X_{i(9:9)}\}$, for $i = 1, 2, \ldots, 729$. Now, allocate the 729 sets into 3 groups, each of 243 sets as:

1st Group: $\{X_{i(1:9)}, X_{i(2:9)}, \ldots, X_{i(9:9)}\}$, for $i = 1, 2, \ldots, 243$,
2nd Group: $\{X_{i(1:9)}, X_{i(2:9)}, \ldots, X_{i(9:9)}\}$, for $i = 244, 245, \ldots, 486$,
3rd Group: $\{X_{i(1:9)}, X_{i(2:9)}, \ldots, X_{i(9:9)}\}$, for $i = 487, 488, \ldots, 729$.

For $s = 1$, select the smallest ranked unit, $X^{(1)}_{i(1:9)}$ for $i = 1, 2, \ldots, 243$ from each set in the first group, and the median ranked unit, $X^{(1)}_{i(5:9)}$ for $i = 244, 245, \ldots, 486$ from each set in the second group, and finally, the largest ranked unit, $X^{(1)}_{i(9:9)}$ for $i = 487, 488, \ldots, 729$ from each set in the third group. This step yields 729 units, which are $X^{(1)}_{1(1:9)}, X^{(1)}_{2(1:9)}, \ldots, X^{(1)}_{243(1:9)}, X^{(1)}_{244(5:9)}, X^{(1)}_{245(5:9)}, \ldots, X^{(1)}_{486(5:9)}, X^{(1)}_{487(9:9)}, X^{(1)}_{488(9:9)}, \ldots, X^{(1)}_{729(9:9)}$. Allocate these units into 81 sets, 27 sets in each group as follows:

1st Group: $\left\{X^{(1)}_{9(i-1)+1(1:9)}, X^{(1)}_{9(i-1)+2(1:9)}, \ldots, X^{(1)}_{9(i-1)+9(1:9)}\right\}$, for $i = 1, 2, \ldots 27$,
2nd Group: $\left\{X^{(1)}_{9(i-1)+1(5:9)}, X^{(1)}_{9(i-1)+2(5:9)}, \ldots, X^{(1)}_{9(i-1)+9(5:9)}\right\}$, for $i = 28, 29, \ldots, 54$,
3rd Group: $\left\{X^{(1)}_{9(i-1)+1(9:9)}, X^{(1)}_{9(i-1)+2(9:9)}, \ldots, X^{(1)}_{9(i-1)+9(9:9)}\right\}$, for $i = 55, 56, \ldots, 81$.

Now, for $s = 2$, rank the units within each set in all the three groups and then select the smallest ranked unit, $X^{(2)}_{i(1:9)}$ for $i = 1, 2, \ldots 27$ from each set in the 1st group, and the median ranked unit, $X^{(2)}_{i(5:9)}$ for $i = 28, 29, \ldots, 54$ from each set in the 2nd group, and the largest ranked unit, $X^{(2)}_{i(9:9)}$ for $i = 55, 56, \ldots, 81$ from each set in the 3rd group. This step yields 81 units, which are $X^{(2)}_{1(1:9)}, X^{(2)}_{2(1:9)}, \ldots, X^{(2)}_{27(1:9)}, X^{(2)}_{28(5:9)}, X^{(2)}_{29(5:9)}, \ldots, X^{(2)}_{54(5:9)}, X^{(2)}_{55(9:9)}, X^{(2)}_{56(9:9)}, \ldots, X^{(2)}_{81(9:9)}$. Allocate them into 9 sets, 3 sets in each group as follows:

1st Group: $\left\{X^{(2)}_{9(i-1)+1(1:9)}, X^{(2)}_{9(i-1)+2(1:9)}, \ldots, X^{(2)}_{9(i-1)+9(1:9)}\right\}$, for $i = 1, 2, 3,$

2nd Group: $\left\{X^{(2)}_{9(i-1)+1(5:9)}, X^{(2)}_{9(i-1)+2(5:9)}, \ldots, X^{(2)}_{9(i-1)+9(5:9)}\right\}$, for $i = 4, 5, 6,$

3rd Group: $\left\{X^{(2)}_{9(i-1)+1(9:9)}, X^{(2)}_{9(i-1)+2(9:9)}, \ldots, X^{(2)}_{9(i-1)+9(9:9)}\right\}$, for $i = 7, 8, 9.$

Next, for $s = 3$ rank the units within each set in each group, then select the smallest ranked unit, $X^{(3)}_{i(1:9)}$ for $i = 1, 2, 3$ from each set in the 1st group, the median ranked unit, $X^{(3)}_{i(5:9)}$ for $i = 4, 5, 6$ from each set in the 2nd group, and the largest ranked unit, $X^{(3)}_{i(9:9)}$ for $i = 7, 8, 9$ from each set in the 3rd group. This step yields 9 units, which are $X^{(3)}_{1(1:9)}, X^{(3)}_{2(1:9)}, X^{(3)}_{3(1:9)}, X^{(3)}_{4(5:9)}, X^{(3)}_{5(5:9)}, X^{(3)}_{6(5:9)}, X^{(3)}_{7(9:9)}, X^{(3)}_{8(9:9)},$ $X^{(3)}_{9(9:9)}$ to be a MBGRSS of size 9. The mean of these units is considered as an estimator of the population mean.

It is of interest to note here that the RSS and the MBGRSS are equivalent when $m = 3$ for $s = 1$.

Estimation of the Population Mean

Assume that $X_1, X_2, \ldots, X_m$ is a random sample from the cdf $F(x)$ with a finite mean μ and variance σ^2 Also, assume that $X_{11h}, X_{12h}, \ldots, X_{1mh}$; $X_{21h}, X_{22h}, \ldots, X_{2mh}$; $X_{m1h}, X_{m2h}, \ldots, X_{mmh}$ are m independent SRS of size m each in the h^{th} cycle for $h = 1, 2, \ldots, n$. If $X_{i(1:m)h}, X_{i(2:m)h}, \ldots, X_{i(m:m)h}$ are the order statistics of the i^{th} sample $X_{i1h}, X_{i2h}, \ldots, X_{imh}$, for $i = 1, 2, \ldots, m$. Then, the measured RSS units are $X_{1(1:m)h}$, $X_{2(2:m)h}, \ldots, X_{m(m:m)h}$.

The SRS estimator of the population mean based on a sample of size m is defined as

$$\bar{X}_{SRS} = \frac{1}{mn}\sum_{h=1}^{n}\sum_{i=1}^{m} X_{ih} \ , \tag{3}$$

with variance

$$\operatorname{Var}\left(\bar{X}_{SRS}\right) = \frac{\sigma^2}{mn} \ . \tag{4}$$

The RSS estimator of the population mean (see McIntyre (1952)) is given by

$$\bar{X}_{RSS} = \frac{1}{mn}\sum_{h=1}^{n}\sum_{i=1}^{m} X_{i(i:m)h} \ , \tag{5}$$

with variance

$$\mathrm{Var}\left(\bar{X}_{RSS}\right) = \frac{1}{m^2 n}\sum_{h=1}^{n}\sum_{i=1}^{m}\mathrm{Var}\left(X_{i(i:m)h}\right) = \frac{\sigma^2}{mn} - \frac{1}{m^2 n}\sum_{i=1}^{m}\left[\mu_{(i:m)} - \mu\right]^2 \tag{6}$$

If m is odd, in the h^{th} cycle ($h = 1, 2,\ldots, n$), let $X^{(s)}_{i(1:m)h}$ be the smallest ranked observation of the i^{th} sample for $i = 1, 2,\ldots, k$, $X^{(s)}_{i\left(\frac{m+1}{2}:m\right)h}$ be the median ranked observation of the i^{th} sample for $i = k + 1, k + 2,\ldots, 2k$, and $X^{(s)}_{i(m:m)h}$ be the largest ranked observation of the i^{th} sample when $i = 2k + 1, 2k + 2,\ldots, 3k$. Therefore, when m is odd, the measured units $X^{(s)}_{1(1:m)h}, X^{(s)}_{2(1:m)h}, \ldots X^{(s)}_{k(1:m)h}, X^{(s)}_{k+1\left(\frac{m+1}{2}:m\right)h}, \ldots, X^{(s)}_{2k\left(\frac{m+1}{2}:m\right)h}, X^{(s)}_{2k+1(m:m)h}, \ldots, X^{(s)}_{3k(m:m)h}$ will be denoted by MBGRSSO. It is of interest to mention here that the measured units within each group are identically independent (iid) but all units are independent but not identically distributed.

The suggested estimator of the population mean based on MBGRSSO is given by

$$\bar{X}^{(s)}_{MBGRSSO} = \frac{1}{3kn}\sum_{h=1}^{n}\left(\sum_{i=1}^{k} X^{(s)}_{i(1:m)h} + \sum_{i=k+1}^{2k} X^{(s)}_{i\left(\frac{m+1}{2}:m\right)h} + \sum_{i=2k+1}^{3k} X^{(s)}_{i(m:m)h}\right) \tag{7}$$

with variance

$$\mathrm{Var}\left(\bar{X}^{(s)}_{MBGRSSO}\right) = \frac{1}{9k^2 n}\left[\sum_{i=1}^{k}\mathrm{Var}\left(X^{(s)}_{i(1:m)}\right) + \sum_{i=k+1}^{2k}\mathrm{Var}\left(X^{(s)}_{i\left(\frac{m+1}{2}:m\right)}\right) + \sum_{i=2k+1}^{3k}\mathrm{Var}\left(X^{(s)}_{i(m:m)}\right)\right] \tag{8}$$

For even sample size, let $X^{(s)}_{i(1:m)h}$ be the smallest ranked observation of the i^{th} sample for $i = 1, 2,\ldots, k$,, $\frac{1}{2}\left(X^{(s)}_{i\left(\frac{m}{2}:m\right)h} + X^{(s)}_{i\left(\frac{m+2}{2}:m\right)h}\right)$ be the median ranked observation of the i^{th} sample for $i = k+1,\ k+2,\ldots, 2k$, and $X^{(s)}_{i(m:m)h}$ be the largest ranked observation of the i^{th} sample for $i = 2k+1,\ 2k+2,\ldots, 3k$. However, the measured observations $X^{(s)}_{1(1:m)h}$, $X^{(s)}_{2(1:m)h}$,…, $X^{(s)}_{k(1:m)h}$, $\frac{1}{2}\left(X^{(s)}_{k+1\left(\frac{m}{2}:m\right)h} + X^{(s)}_{k+1\left(\frac{m+2}{2}:m\right)h}\right)$,…, $\frac{1}{2}\left(X^{(s)}_{2k\left(\frac{m}{2}:m\right)h} + X^{(s)}_{2k\left(\frac{m+2}{2}:m\right)h}\right)$, $X^{(s)}_{2k+1(m:m)h}$,…, $X^{(s)}_{3k(m:m)h}$ will be denoted as MBGRSSE. The suggested MBGRSSE estimator of the population mean is defined as

$$\bar{X}^{(s)}_{MBGRSSE} = \frac{1}{3kn}\sum_{h=1}^{n}\left\{\sum_{i=1}^{k} X^{(s)}_{i(1:m)h} + \sum_{i=k+1}^{2k}\left[\frac{1}{2}\left(X^{(s)}_{i\left(\frac{m}{2}:m\right)h} + X^{(s)}_{i\left(\frac{m+2}{2}:m\right)h}\right)\right] + \sum_{i=2k+1}^{3k} X^{(s)}_{i(m:m)h}\right\} \quad (9)$$

with variance

$$\operatorname{Var}\left(\bar{X}^{(s)}_{MBGRSSE}\right) = \frac{1}{9k^2n}\left\{\begin{array}{l}\frac{1}{4}\sum_{i=k+1}^{2k}\left[\begin{array}{l}\operatorname{Var}\left(X^{(s)}_{i\left(\frac{m}{2}:m\right)}\right) + \operatorname{Var}\left(X^{(s)}_{i\left(\frac{m+2}{2}:m\right)}\right) \\ +2\operatorname{Cov}\left(X^{(s)}_{i\left(\frac{m}{2}:m\right)}, X^{(s)}_{i\left(\frac{m+2}{2}:m\right)}\right)\end{array}\right] \\ +\sum_{i=1}^{k}\operatorname{Var}\left(X^{(s)}_{i(1:m)}\right) + \sum_{i=2k+1}^{3k}\operatorname{Var}\left(X^{(s)}_{i(m:m)}\right)\end{array}\right\} \quad (10)$$

Define the following notations. For $i = 1, 2,\ldots,m$ in the h^{th} cycle, $h = 1, 2,\ldots, n$, let $\mu^{(s)}_{(j:m)} = E\left(X^{(s)}_{i(j:m)h}\right)$, $\sigma^{2(s)}_{(j:m)} = \operatorname{Var}\left(X^{(s)}_{i(j:m)h}\right)$, where $j = 1,\ \frac{m}{2}, \frac{m+2}{2}, \frac{m+1}{2}, m$. Whether the sample size is even or odd the measured units $X^{(s)}_{1(1:m)h}$, $X^{(s)}_{2(1:m)h}$,…, $X^{(s)}_{k(1:m)h}$ are iid, and also $X^{(s)}_{2k+1(m:m)h}$, $X^{(s)}_{2k+2(m:m)h}$,…, $X^{(s)}_{3k(m:m)h}$ are iid. Also, when the sample size is odd, $X^{(s)}_{k+1\left(\frac{m+1}{2}:m\right)h}$, $X^{(s)}_{k+2\left(\frac{m+1}{2}:m\right)h}$,…, $X^{(s)}_{2k\left(\frac{m+1}{2}:m\right)h}$ are iid. Hence, Equations (8) and (10), respectively, can be written as

$$\operatorname{Var}\left(\bar{X}_{MBGRSSO}^{(s)}\right)=\frac{1}{9kn}\left(\sigma_{(1:m)}^{2(s)}+\sigma_{\left(\frac{m+1}{2}:m\right)}^{2(s)}+\sigma_{(m:m)}^{2(s)}\right) \tag{11}$$

$$\operatorname{Var}\left(\bar{X}_{MBGRSSE}^{(s)}\right)=\frac{1}{9kn}\left[\sigma_{(1:m)}^{2(s)}+\frac{1}{4}\left(\sigma_{\left(\frac{m}{2}:m\right)}^{2(s)}+\sigma_{\left(\frac{m+2}{2}:m\right)}^{2(s)}+2\sigma_{\left(\frac{m}{2}:m\right),\left(\frac{m+2}{2}:m\right)}^{(s)}\right)+\sigma_{(m:m)}^{2(s)}\right] \tag{12}$$

If the parent distribution is symmetric about its mean μ, then $X_{(i:m)}^{(s)}=X_{(m-i+1:m)}^{(s)}$ in distribution and then $\operatorname{Var}\left(X_{(i:m)}^{(s)}\right)=\operatorname{Var}\left(X_{(m-i+1:m)}^{(s)}\right)$ for $i = 1, 2,\ldots,m$ (David & Nagaraja, 2003). Therefore, we have

$$\operatorname{Var}\left(\bar{X}_{MBGRSSO}^{(s)}\right)=\frac{1}{9kn}\left(2\sigma_{(1:m)}^{2(s)}+\sigma_{\left(\frac{m+1}{2}:m\right)}^{2(s)}\right) \tag{13}$$

and

$$\operatorname{Var}\left(\bar{X}_{MBGRSSE}^{(s)}\right)=\frac{1}{9kn}\left[2\sigma_{(1:m)}^{2(s)}+\frac{1}{2}\left(\sigma_{\left(\frac{m}{2}:m\right)}^{2(s)}+\sigma_{\left(\frac{m}{2}:m\right),\left(\frac{m+2}{2}:m\right)}^{(s)}\right)\right] \tag{14}$$

Lemma 3.1. If the population of study is symmetric about its mean μ, then $\bar{X}_{MBGRSSO}^{(s)}$ and $\bar{X}_{MBGRSSE}^{(s)}$ are unbiased estimators of the population mean.

Proof: When the sample size is odd, the expectation of (7) is

$$E\left(\bar{X}_{MBGRSSO}^{(s)}\right)=\frac{1}{3kn}\sum_{h=1}^{n}\left[\sum_{i=1}^{k}E\left(X_{i(1:m)h}^{(s)}\right)+\sum_{i=k+1}^{2k}E\left(X_{i\left(\frac{m+1}{2}:m\right)h}^{(s)}\right)+\sum_{i=2k+1}^{3k}E\left(X_{i(m:m)h}^{(s)}\right)\right]$$

$$=\frac{1}{3kn}\sum_{h=1}^{n}\left(\sum_{i=1}^{k}\mu_{(1:m)h}^{(s)}+\sum_{i=k+1}^{2k}\mu_{\left(\frac{m+1}{2}:m\right)h}^{(s)}+\sum_{i=2k+1}^{3k}\mu_{(m:m)h}^{(s)}\right)$$

$$=\frac{1}{3}\left(\mu_{(1:m)}^{(s)}+\mu_{\left(\frac{m+1}{2}:m\right)}^{(s)}+\mu_{(m:m)}^{(s)}\right)$$

Because the distribution is symmetric about μ, then we have $\mu_{(1:m)}^{(s)} + \mu_{(m:m)}^{(s)} = 2\mu$ and $\mu_{\left(\frac{m+1}{2}:m\right)}^{(s)} = \mu$. Therefore, $E\left(\bar{X}_{MBGRSSO}^{(s)}\right) = \frac{1}{3}\left(2\mu + \mu\right) = \mu$.
Also, the expectation of (9) is

$$E\left(\bar{X}_{MBGRSSE}^{(s)}\right) = \frac{1}{3kn}\sum_{h=1}^{n}\left\{\sum_{i=1}^{k} E\left(X_{i(1:m)h}^{(s)}\right) + \frac{1}{2}\sum_{i=k+1}^{2k}\left[E\left(X_{i\left(\frac{m}{2}:m\right)h}^{(s)}\right) + E\left(X_{i\left(\frac{m+2}{2}:m\right)h}^{(s)}\right)\right] + \sum_{i=2k+1}^{3k} E\left(X_{i(m:m)h}^{(s)}\right)\right\}$$

$$= \frac{1}{3kn}\sum_{h=1}^{n}\left[\sum_{i=1}^{k}\mu_{(1:m)h}^{(s)} + \sum_{i=k+1}^{2k}\left[\frac{1}{2}\left(\mu_{\left(\frac{m}{2}:m\right)h}^{(s)} + \mu_{\left(\frac{m+2}{2}:m\right)h}^{(s)}\right)\right] + \sum_{i=2k+1}^{3k}\mu_{(m:m)h}^{(s)}\right]$$

$$= \frac{1}{3}\left[\mu_{(1:m)}^{(s)} + \frac{1}{2}\left(\mu_{\left(\frac{m}{2}:m\right)}^{(s)} + \mu_{\left(\frac{m+2}{2}:m\right)}^{(s)}\right) + \mu_{(m:m)}^{(s)}\right]$$

$$= \frac{1}{3}\left[\left(\mu_{(1:m)}^{(s)} + \mu_{(m:m)}^{(s)}\right) + \frac{1}{2}\left(\mu_{\left(\frac{m}{2}:m\right)}^{(s)} + \mu_{\left(\frac{m+2}{2}:m\right)}^{(s)}\right)\right]$$

Because the distribution is symmetric about μ, then we have $\mu_{(1:m)}^{(s)} + \mu_{(m:m)}^{(s)} = 2\mu$ and $\mu_{\left(\frac{m}{2}:m\right)}^{(s)} + \mu_{\left(\frac{m+2}{2}:m\right)}^{(s)} = 2\mu$. Therefore, $E\left(\bar{X}_{MBGRSSE}^{(s)}\right) = \frac{1}{3}\left[2\mu + \frac{1}{2}\left(2\mu\right)\right] = \mu.$

Theorem 3.2:

1) $\bar{X}_{MBGRSSO}^{(s)}$ is more efficient than $\bar{X}_{SRS}$ if $2\sigma_{(1:m)}^{2(s)} + \sigma_{\left(\frac{m+1}{2}:m\right)}^{2(s)} < 3\sigma^2$.
2) $\bar{X}_{MBGRSSE}^{(s)}$ is more efficient than $\bar{X}_{SRS}$ if $4\sigma_{(1:m)}^{2(s)} + \sigma_{\left(\frac{m}{2}:m\right)}^{2(s)} + \sigma_{\left(\frac{m}{2}:m\right)\left(\frac{m+2}{2}:m\right)}^{(s)} < 6\sigma^2$.

Proof: The proof is directly using the MSE equations of the MBGRSS estimators with that of SRS method.

Simulation Study

The suggested MBGRSS estimators of the population mean will be compared with their competitors using RSS and SRS schemes. Six probability distribution functions are investigated for the populations: uniform, normal, beta, exponential, gamma and Weibull. The averages of 60,000 samples estimates using $k = 1, 2, 3$ corresponding to the sample sizes $m = 3, 6, 9$ are compared. Assume that the cycle is repeated once. The efficiency of RSS relative to SRS is defined as

$$eff\left(\bar{X}_{RSS}, \bar{X}_{SRS}\right) = \frac{\mathrm{Var}\left(\bar{X}_{SRS}\right)}{\mathrm{Var}\left(\bar{X}_{RSS}\right)} = 1 - \frac{\sum_{i=1}^{m}\left[\mu_{(i:m)} - \mu\right]^2}{m\sigma^2}. \quad (15)$$

If the distribution is symmetric, the efficiency of MBGRSSO and MBGRSSE relative to SRS are defined as:

$$eff\left(\bar{X}_{MBGRSSO}^{(s)}, \bar{X}_{SRS}\right) = \frac{\mathrm{Var}\left(\bar{X}_{SRS}\right)}{\mathrm{Var}\left(\bar{X}_{MBGRSSO}^{(s)}\right)} = \frac{3\sigma^2}{\sigma_{(1:m)}^{2(s)} + \sigma_{\left(\frac{m+1}{2}:m\right)}^{2(s)} + \sigma_{(m:m)}^{2(s)}}$$

$$eff\left(\bar{X}_{MBGRSSE}^{(s)}, \bar{X}_{SRS}\right) = \frac{\mathrm{Var}\left(\bar{X}_{SRS}\right)}{\mathrm{Var}\left(\bar{X}_{MBGRSSE}^{(s)}\right)} = \frac{3\sigma^2}{\sigma_{(1:m)}^{2(s)} + \frac{1}{4}\left[\sigma_{\left(\frac{m}{2}:m\right)}^{2(s)} + \sigma_{\left(\frac{m+2}{2}:m\right)}^{2(s)} + 2\sigma_{\left(\frac{m}{2}:m\right)\left(\frac{m+2}{2}:m\right)}^{(s)}\right] + \sigma_{(m:m)}^{2(s)}}. \quad (16)$$

The mean square errors of $\bar{X}_{MBGRSSO}^{(s)}$ and $\bar{X}_{MBGRSSE}^{(s)}$ are defined as

$$\mathrm{MSE}\left(\bar{X}_{MBGRSSO}^{(s)}\right) = \frac{1}{9kn}\left(\sigma_{(1:m)}^{2(s)} + \sigma_{\left(\frac{m+1}{2}:m\right)}^{2(s)} + \sigma_{(m:m)}^{2(s)}\right) + \left[E\left(\bar{X}_{MBGRSSO}^{(s)}\right) - \mu\right]^2 \quad (17)$$

and

$$\text{MSE}\left(\bar{X}^{(s)}_{MBGRSSE}\right)=\frac{1}{9kn}\left[\sigma^{2(s)}_{(1:m)}+\frac{1}{4}\left(\sigma^{2(r)}_{\left(\frac{m}{2}:m\right)}+\sigma^{2(s)}_{\left(\frac{m+2}{2}:m\right)}+2\sigma^{(s)}_{\left(\frac{m}{2}:m\right)\left(\frac{m+2}{2}:m\right)}\right)+\sigma^{2(s)}_{(m:m)}\right] +\left[E\left(\bar{X}^{(s)}_{MBGRSSE}\right)-\mu\right]^2. \tag{18}$$

If the distribution is asymmetric, the efficiency is defined as:

$$\begin{aligned} eff\left(\bar{X}^{(s)}_{MBGRSSO},\bar{X}_{SRS}\right)&=\frac{\text{Var}\left(\bar{X}_{SRS}\right)}{\text{MSE}\left(\bar{X}^{(s)}_{MBGRSSO}\right)} \\ &=\frac{3\sigma^2}{\sigma^{2(s)}_{(1:m)}+\sigma^{2(s)}_{\left(\frac{m+1}{2}:m\right)}+\sigma^{2(s)}_{(m:m)}+9kn\left[E\left(\bar{X}^{(s)}_{MBGRSSO}\right)-\mu\right]^2}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} eff\left(\bar{X}^{(s)}_{MBGRSSE},\bar{X}_{SRS}\right)&=\frac{\text{Var}\left(\bar{X}_{SRS}\right)}{\text{MSE}\left(\bar{X}^{(s)}_{MBGRSSE}\right)} \\ &=\frac{3\sigma^2}{\sigma^{2(s)}_{(1:m)}+\frac{1}{4}\left[\begin{array}{l}\sigma^{2(s)}_{\left(\frac{m}{2}:m\right)}+\sigma^{2(s)}_{\left(\frac{m+2}{2}:m\right)} \\ +2\sigma^{(s)}_{\left(\frac{m}{2}:m\right)\left(\frac{m+2}{2}:m\right)}\end{array}\right]+\sigma^{2(s)}_{(m:m)}+9kn\left[E\left(\bar{X}^{(s)}_{MBGRSSE}\right)-\mu\right]^2}. \end{aligned} \tag{20}$$

In terms of the efficiency and bias values, the results are summarized in Tables 1-3 with m = 3, 6, 9, respectively for several values of s.

Table 1. The efficiency of RSS and MBGRSSO for estimating the population mean with $m = 3$ and $1 \leq s \leq 5$

Distribution	RSS		MBGRSSO				
			$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$
Uniform (0,1)	2.000	*Eff*	2.000	5.746	16.148	38.257	89.134
Normal (0,1)	1.914	*Eff*	1.914	3.288	5.010	6.937	9.046
Beta (4,4)	1.989	*Eff*	1.989	4.249	8.763	17.514	35.017
Exponential (1)	1.636	*Eff*	1.636	1.355	0.683	0.331	0.185
		Bias	0.000	0.232	0.546	0.900	1.263
Gamma (2,1)	1.767	*Eff*	1.767	1.764	1.100	0.565	0.321
		Bias	0.000	0.243	0.587	0.966	1.355
Weibull (1,3)	1.802	*Eff*	1.802	1.402	0.683	0.336	0.190
		Bias	0.000	0.688	1.650	2.697	3.765

Table 2. The efficiency of RSS and MBGRSSE for estimating the population mean with $m = 6$ and $1 \leq s \leq 3$

Distribution	RSS		MBGRSSE		
			$s = 1$	$s = 2$	$s = 3$
Uniform (0,1)	3.500	*Eff*	4.258	32.067	89.541
Normal (0,1)	3.226	*Eff*	2.880	5.906	8.472
Beta (4,4)	3.319	*Eff*	3.287	11.647	28.294
Exponential (1)	2.460	*Eff*	1.497	0.335	0.154
		Bias	0.135	0.647	0.996
Gamma (2,1)	2.725	*Eff*	1.916	0.557	0.268
		Bias	0.141	0.688	1.057
Weibull (1,3)	2.424	*Eff*	1.472	0.324	0.154
		Bias	0.408	1.943	2.984

Table 3. The efficiency of RSS and MBGRSSO for estimating the population mean with $m = 9$ and $1 \leq s \leq 3$

Distribution	RSS		MBGRSSO		
			$s = 1$	$s = 2$	$s = 3$
Uniform (0,1)	5.000	*Eff*	6.395	53.838	342.878
Normal (0,1)	4.442	*Eff*	3.467	7.283	11.386
Beta (4,4)	4.726	*Eff*	4.278	18.740	71.316
Exponential (1)	3.251	*Eff*	0.967	0.130	0.041
		Bias	0.230	0.896	1.622
Gamma (2,1)	3.610	*Eff*	1.449	0.222	0.071
		Bias	0.242	0.960	1.747
Weibull (1,3)	3.162	*Eff*	0.971	0.128	0.041
		Bias	0.685	2.696	4.867

Based on the results in Tables 1-3, we can conclude the following:

(1) When the parent distribution is symmetric about its mean we have:

a. MBGRSS method is more efficient than the usual SRS. For example, for $m = 9$ and $s = 2$, the efficiency of MBGRSSO is 18.740 for estimating the mean of beta (4, 4).

b. MBGRSS estimators are unbiased of the population mean.

c. The efficiency of MBGRSS is increasing in s for specific value of the sample size. For example, for $m = 6$, the efficiency values for $s = 1, 2, 3$ are 4.258, 32.067, and 89.541 respectively for estimating the mean of the uniform distribution.

d. The efficiency of MBGRSS estimators is increasing as the sample size increasing. As an example, for the standard normal distribution, for $s = 2$ and $m = 3, 6, 9$ the efficiency values are 3.288, 5.906, and 7.283, respectively.

(2) When the underlying distribution is asymmetric about the population mean we have:

a. MBGRSS estimators are biased of the population mean. For example, with $m = 9$ and $s = 1$, the efficiency of MBGRSSO is 0.971 with bias 0.685 when estimating the mean of the Weibull distribution with parameters 1 and 3.

b. The efficiency is decreasing in s for specific value of the sample size. For example, for $m = 6$ and $s = 1, 2, 3$, the efficiency values of MBGRSSE are 1.497, 0.335 and 0.154, respectively for estimating the mean of exponential distribution with parameter 1.

c. The bias of MBGRSS estimators is increasing in s. For example, if the parent distribution is gamma with parameters 2 and 1, then for $m = 3$ and $s = 1, 2, 3, 4$, the bias values are 0, 0.243, 0.587 and 0.966 respectively.

(3) For $m = 3$ and $s = 1$, MBGRSSO is the same as RSS. Otherwise, when $s > 1$ and for any m the MERSSO is found to be more efficient.

Application to Bioleaching Studies

Generally, RSS is more efficient than SRSWR. In practice, the interest is in estimating confidence intervals (CI). When the distribution is not known using resampling seems to be a good approach for evaluating the efficiency of RSS and for performing inferences. Bootstrap has proven to be good general resampling method for deriving the sampling distribution of statistics in SRS. Chen et al. (2004) considered a bootstrapping procedure for RSS re-sampling row-wise. Hui et al. (2005) proposed bootstrapping as a method to obtain confidence interval for estimation. We are going to use their proposals for deriving estimations of the sampling errors and CI`s.

A Bootstrap procedure for RSS, BRSSR, is instrumented by the following algorithm.

BRSSR algorithm:

1. Assign to each element of the r^{th} row a probability the same probability of being selected and select m units randomly from $F_{(r),m}$ with replacement to get

$$X^{*(s)}_{1(1:m)h}, X^{*(s)}_{2(1:m)h}, \dots, X^{*(s)}_{k(1:m)h,} X^{*(s)}_{k+1\left(\frac{m+1}{2}:m\right)h},$$
$$\dots, X^{*(s)}_{2k\left(\frac{m+1}{2}:m\right)h}, X^{*(s)}_{2k+1(m:m)h}, \dots, X^{*(s)}_{3k\left(\frac{m+1}{2}:m\right)h}$$

2. Perform Step 1 for $r = 1, 2, \dots, k$ to get a bootstrap ranked set samples
3. Define the Bootstrap distributions

$$F^{*}_{(1)m}(t) = \frac{1}{k}\sum_{h=1}^{k} I\left(X^{*(s)}_{1(1:m)h} \le t\right), F^{*}_{\left(\frac{m+1}{2}\right)m}(t) = \frac{1}{k}\sum_{h=1}^{k} I\left(X^{*(s)}_{k+1\left(\frac{m+1}{2}:m\right)h} \le t\right)$$

and

$$F^{*}_{(m)m}(t) = \frac{1}{k}\sum_{h=1}^{k} I\left(X^{*(s)}_{k+1(m:m)h} \le t\right)$$

The BRSSR scheme of Bootstrap resamples from each $F_{(r),m}(t)$ independently and then combine to have a Bootstrap sample.

Denote by $F(2)$ a collection of distribution functions having finite second moments, $H_{n,F}$ as the sampling distribution of $T_n = \frac{\sum_{r=1}^{k}\left[\bar{X}_{(r)} - \mu_{(r:m)}\right]\sqrt{m_r}}{\sqrt{k}}$, $H_{n,Fn}$ as the sampling distribution of the corresponding BRSSR replica T_n^*, and $\rho(2)(G,H) = inf_{\tau_{X,Y}} \sqrt{E|X-Y|^2}$, where $\tau_{X,Y}$ is the collection of all possible joint distributions of the pairs (X,Y) whose marginal distributions are G and H, respectively. An important result is the following proposition:

Proposition 5.1: (Modarres et al. 2006).

If $F \in F(2)$ the statistics $T_n = \frac{\sum_{r=1}^{k}\left[\bar{X}_{(r)} - \mu_{(r:m)}\right]\sqrt{m_r}}{\sqrt{k}}$, T_n^* as the corresponding BRSSR replica. Then, $(2)\left(H_{n,F_n}, H_n\right) \xrightarrow{a.s} 0$.

Once a number of Bootstrap samples B is fixed the Monte Carlo approximation of $H_{n,F_n}(t)$ is defined as

$$\hat{H}_{n,F_n}(t) = \frac{\sum_{b=1}^{B} I\left(\bar{X}_{MBGRSSOb}^{*(s)} \le t\right)}{B}.$$

Because F_n is completely specified, we can make $\hat{H}_{n,F_n}(t)$ arbitrarily close to $H_{n,F_n}(t)$ by taking a sufficiently large B. Now, we can estimate the moments of MBGRSSA, $A = O,E$ using

$$\hat{\mu}_n^q = \frac{\sum_{b=1}^{B}\left(\bar{X}_{MBGRSSAb}^{*(s)}\right)^q}{B}.$$

These estimators allow estimating the variance of the estimator using

$$\hat{V}_B\left(\bar{X}_{MBGRSSA}^{*(s)}\right) = \hat{\mu}_n^2 - \left(\hat{\mu}_n\right)^2$$

Approximate Bootstrap confidence intervals can also be determined computing the needed quantiles

$$IC_{NP}\left(\mu\right)=\left(\bar{X}^{*(s)}_{MBGRSSAL},X^{*(s)}_{MBGRSSAU}\right)$$

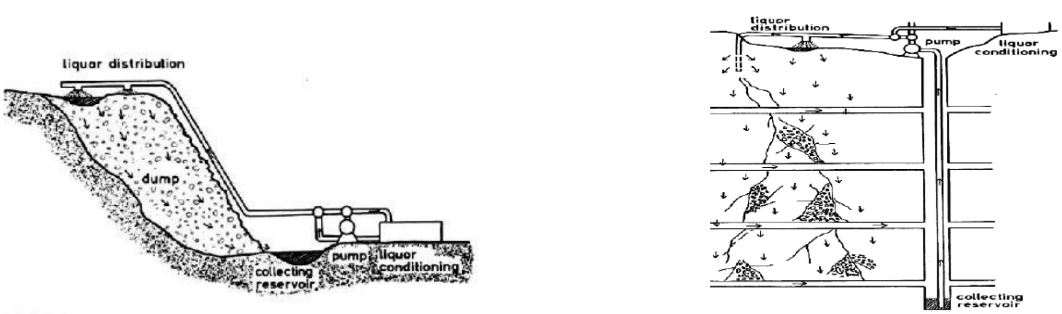
such that

$$P\left(\bar{X}^{*(s)}_{MBGRSSAL}<\mu\right)\leq\frac{\alpha}{2},P\left(X^{*(s)}_{MBGRSSAU}<\mu\right)\leq 1-\frac{\alpha}{2}$$

or the *T*-Student approximation:

$$IC_{P}\left(\mu\right)=\left(\bar{X}^{*(s)}_{MBGRSSA}-t_{\left(B-1,1-\frac{\alpha}{2}\right)}\sqrt{\hat{V}_{B}\left(\bar{X}^{*(s)}_{MBGRSSA}\right)},\bar{X}^{*(s)}_{MBGRSSA}+t_{\left(B-1,1-\frac{\alpha}{2}\right)}\sqrt{\hat{V}_{B}\left(\bar{X}^{*(s)}_{MBGRSSA}\right)}\right).$$

Bioleaching is increasingly being used because of its economical and environmental advantages. A bioleaching is the most acceptable manner of processing of ores since it does not require elaboration of mining complexes and allows increasing the source of raw materials along with providing integrated approach to metals extraction. In terms of economy and environmental protection, biotechnological methods are more sufficient than chemical methods used for processing of ores. It consists of the acid leaching of the mineral enhanced by bacteria.



Dump ore leaching on a slope *In situ leaching in a mine*

Figure 1. Two leaching procedures

Figure 1 is a sketch of engineering procedures and Figure 2 of the mechanism. Direct molecular analysis of DNA has greatly enhanced the ability to assess the diversity of microorganisms growing in an ecosystem. The samples were collected in the agglomeration of mineral ores of a combination of nickel and cobalt. It is of interest to grant that the observations cover small, medium and large concentrations after bioleaching.

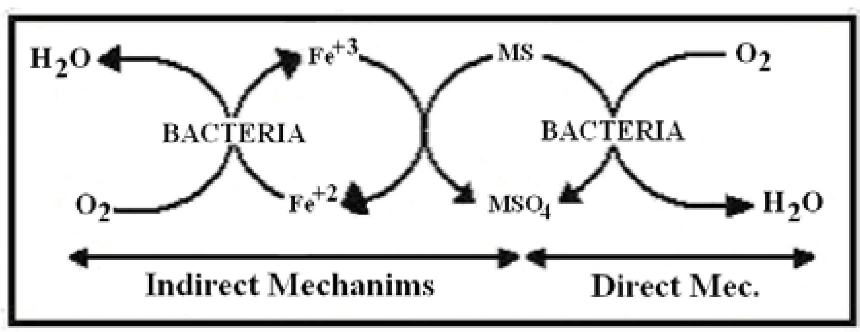


Figure 2 Bioleaching mechanism

The geophysics evaluate the contents of the mineral samples by a cheap method periodically. They are interested in evaluating the mean contents of cobalt in the ore. We considered the use of MBGRSS. The parameters of the example developed previously in which $s = 1, 2,\ldots, 5$, $k = 1, 2, 3$, $m = 3, 6, 9$. The sample units were taken from the existent data base compiled in the last 5 years. We computed the estimation of the variances as well as the estimation of $\operatorname{Var}\left(\bar{X}_{RSS}\right)$ using the Bootstrap estimator

$$\hat{V}_B\left(\bar{X}^*_{RSS}\right) = \frac{\sum_{b=1}^{B}\left(\bar{X}^*_{RSSb}\right)^2}{B} - \left(\frac{\sum_{b=1}^{B}\bar{X}^*_{RSSb}}{B}\right)^2$$

The variance of the SRSWR mean, $\operatorname{Var}\left(\bar{X}_{RSS}\right)$, was estimated computing the usual estimator of σ^2 as

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}\sum_{h=1}^{m} X_{ih}}{mn-1}$$

The estimated efficiencies were computed as follows

$$eff\left(\bar{X}^{*(s)}_{B_{MBGRSSA}}, \bar{X}_{SRS}\right) = \frac{\hat{V}_B\left(\bar{X}^{*(s)}_{MBGRSSA}\right)}{Var\left(\bar{X}_{SRS}\right)}, A = O, E$$

$$eff\left(\bar{X}^{*(s)}_{B_{MBGRSSA}}, \bar{X}_{RSS}\right) = \frac{\hat{V}_B\left(\bar{X}^{*(s)}_{MBGRSSA}\right)}{Var\left(\bar{X}_{RSS}\right)}, A = O, E$$

The behavior of the proposed model was studied in the variable "Estimated lengths in nucleotides of the 16-23S intergenic spacer region in strains." The data were collected on:

X(1)=*T. ferrooxidans* 530 545
X(2)=*T. thiooxidans* 480 555
X(3) =*L. ferrooxidans* 495 505

The ranking variable was a consideration on the concentrations reported by the engineers associated with each sample send to the laboratory. An R-code was developed for selecting the multistage RSS sample and performing the Bootstrap samples selections and the needed calculations. The results are presented in next tables.

Table 4. Estimated efficiency of RSS and MBGRSSO estimators for estimating the population mean m = 3 and $1 \leq s \leq 5$

	RSS	*s* = 1	*s* = 2	*s* = 3	*s* = 4	*s* = 5
X(1)	2.391	2.391	5.019	5.158	7.541	8.400
X(2)	1.634	1.634	4.402	8.460	10.020	13.066
X(3)	2.703	2.703	5.969	8.709	9.522	12.893

Table 5. Estimated efficiency of RSS and MBGRSSE estimators for estimating the population mean for m = 6 and $1 \leq s \leq 3$

	RSS	*s* = 1	*s* = 2	*s* = 3
X(1)	2.013	2.013	5.096	5.222
X(2)	1.900	1.900	5.158	8.467
X(3)	2.198	2.198	5.202	9.741

Table 6. Estimated **e**fficiency of RSS and MBGRSSO estimators for estimating the population mean for m = 9 and $1 \le s \le 3$

	RSS	s = 1	s = 2	s = 3
$X(1)$	2.223	2.223	7.771	8.138
$X(2)$	2.650	8.467	10.332	10.910
$X(3)$	2.073	9.741	10.779	12.189

Based on Tables 4-6, we conclude the following:

a) MBGRSS method is more efficient than the SRS and RSS methods.

b) MBGRSSO estimators with m = 3 and s = 5 obtains the best results in terms of efficiency for all the variables.

c) The efficiency of MBGRSS estimators is increasing as s increasing.

Conclusion

Based on MBGRSS, it can be conclude that

1) If the underlying distribution is symmetric about the population mean μ, then
    - The MBGRSS estimators are unbiased of the population mean.
    - $\text{Var}\left(\bar{X}^{(s)}_{MBGRSS}\right) < \text{Var}\left(\bar{X}_{SRS}\right)$,
    - $\text{Var}\left(\bar{X}^{(s)}_{MBGRSS}\right) < \text{Var}\left(\bar{X}_{RSS}\right)$ for $s > 1$, and $s \ge 1$ for the uniform distribution.
    - The efficiency of MBGRSS estimators is increasing in s.

2) If the parent distribution is asymmetric about μ, then
    - $\bar{X}^{(s)}_{MBGRSS}$ is biased.
    - For m = 3, 6 and s = 1, the MSE of $\bar{X}^{(s)}_{MBGRSS}$ is less than the variance of $\bar{X}_{SRS}$,

3) It seems that MBGRSS should be preferred in bioleaching studies to RSS and SRS.

It is recommended that the MBGRSS be used to estimate the population mean of symmetric distribution.

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