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A compound of Geeta distribution with Generalized Beta distribution (GBD) is obtained and the compound is specialized for different values of β. The first order factorial moments of some special compound distributions are also obtained. A chronological overview of recent developments in the compounding of distributions is provided in the introduction.

Keywords: Compound distribution, Geeta distribution, Generalized Beta Distribution (GBD), factorial moments

Introduction

Regarding the problem of the compounding of the probability distributions, work has been conducted in this area since 1920. It is well known that the parameter in a Poisson distribution is considered to be a gamma variate in the famous article by Greenwood and Yule (1920). Skellam (1948) derived a probability distribution from the binomial distribution by regarding the probability of success as a beta variable between sets of trials. The interrelationships among compound and generalized distributions were first explored by Gurland (1957) after which, Molenaar (1965) discussed some important remarks on mixtures of distributions.

Dubey (1970) derived compound gamma, beta and F distributions by compounding a gamma distribution with another gamma distribution and reducing it to the beta 1st and 2nd kind and to the F distribution via suitable transformations. The application of compounding of distributions to calculate moments was explored by Dyczka (1973). The problem of compounding of distributions was further addressed by Gerstenkorn (1993, 1996) who proposed several compound distributions; Gerstenkorn obtained a compound of gamma

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distribution with exponential distribution by treating the parameter of a gamma distribution as an exponential variate and obtained a compound of polya with beta. Gerstenkorn (2004) also found a compound of a generalized negative binomial distribution with generalized beta distribution by treating the parameter of generalized negative binomial distribution as a generalized beta distribution. Ali, Aslam and Kazmi (2011) improved the informative prior for the mixture of a Laplace distribution under different loss functions. Rashid and Jan (2013) recently obtained a compound of zero truncated generalized negative binomial distribution with that of generalized beta distribution. A broad range of relevant references can be found in studies by Johnson, Kotz and Kemp (1992).

The compounding of probability distributions

The following definition and relations are needed for compounding probability distributions. A certain compound distribution arises when all (or some) parameters of a distribution vary according to some probability distribution, called the compounding distribution. Suppose *Xy* is a random variable with a distribution function $F(x|y)$ that depends on parameter y. If parameter y is considered to be a random variable Y with distribution function $G(y)$, then the distribution that has the distribution function of X is defined by

$$H(x) = \int_{-\infty}^{\infty} F(x \mid cy)\, dG(y) \tag{1}$$

which is called compound, where c is an arbitrary constant or a constant bounded on some interval (Gurland, 1957).

The occurrence of the constant c in (1) has a practical justification inasmuch as the distribution of a random variable, in describing a phenomenon, often depends on a parameter that is itself a realization of another random variable multiplied by a certain constant. A variable that has distribution function (1) will be symbolized by $X \wedge Y$ and will be called a compound of the variable X with respect to the compounding Y.

Relation (1) is symbolized as follows:

$$H(x) \equiv F(x \mid cy) \underset{Y}{\bigwedge} G(y) \tag{2}$$

Consider the case when one variable is discrete with probability function $P\left(X = x_i \middle| cy\right)$, if parameter y is a random variable Y with density $g(y)$, then (1) is expressed by

$$h\left(x_i\right) = P(X = x_i) = \int_{-\infty}^{\infty} g(y)P(X = x_i \mid cy)dy \tag{3}$$

Compounding the Geeta Distribution with the Generalized Beta distribution

Suppose X is a discrete random variable defined over positive integers. The random variable X is said to have a Geeta distribution with parameters θ and β if

$$P_\beta\left(x;\theta\right) = \begin{cases} \dfrac{1}{\beta x - 1}\dbinom{\beta x - 1}{x}\theta^{x-1}\left(1-\theta\right)^{\beta x - x} & ; x = 1, 2, \ldots \\ 0 & ;\ otherwise \end{cases} \tag{4}$$

where $0 < \theta < 1$ and $1 < \beta < \dfrac{1}{\theta}$. The upper limit on β has been imposed for the existence of the mean. When $\beta \to 1$, the Geeta distribution degenerates and its probability mass is concentrated at point $x = 1$ (Consul, 1990).

The Generalized Beta Distribution (GBD) is a distribution given by the density function

$$GB\left(y;a,b,w,r\right) = \begin{cases} \dfrac{ay^{r-1}}{\left(bw\right)^{r/a} B\left(r/a, w\right)}\left(1 - {y^a}/{bw}\right)^{w-1} & ; 0 < y < \left(bw\right)^{1/a} \\ 0;\ y \le 0\ or\ y \ge \left(bw\right)^{1/a} & \end{cases} \tag{5}$$

where $a, b, w, r > 0$ and $B(r/a, w)$ is a beta function. Distribution (5) is a special limit case of the Bessel distribution (Srodka, 1973; Seweryn, 1986) that has been applied in reliability theory (Oginski, 1979).

Consider a Geeta distribution (4) that depends on cy :

$$P_\beta\left(x;cy\right)=\frac{1}{\beta x-1}\binom{\beta x-1}{x}\left(cy\right)^{x-1}\left(1-cy\right)^{\beta x-x},\; x=1,2,3\ldots \tag{6}$$

where $0<cy<1$, $1<\beta<\frac{1}{\theta}$ and Y is a random variable with GBD (5).

Theorem 1

The probability function of the compound of Geeta distribution with GBD is

$$P_\beta GB\left(x\right)=D_1\sum_{K=0}^{\infty}\binom{\beta x-x}{k}\left(-c\right)^k\left(bw\right)^{k/a}B\left(\frac{x+r+k-1}{a},w\right) \tag{7}$$

where $D_1=\dfrac{\frac{1}{\beta x-1}\binom{\beta x-1}{x}c^{x-1}\left(bw\right)^{\frac{x-1}{a}}}{B\left(r/a,w\right)}$

and $x=1,2,3,\ldots\ldots,a,b,w,r>0,\quad 0<cy<1$ and $\beta cy<1$

***Proof*:** From (3), (5) and (6)

$$\begin{aligned}P_\beta GB\left(x\right)&=aD_1^\star\int_0^{(bw)^{1/a}}y^{x+r-2}\left(1-\frac{y^a}{bw}\right)^{w-1}\left(1-cy\right)^{\beta x-x}dy\\&=aD_1^\star\sum_{K=0}^{\infty}\left(-c\right)^K\binom{\beta x-x}{k}\int_0^{(bw)^{1/a}}y^{x+r+k-2}\left(1-\frac{y^a}{bw}\right)^{w-1}dy\end{aligned}$$

where $D_1^\star=\dfrac{\frac{1}{\beta x-1}\binom{\beta x-1}{x}c^{x-1}}{\left(bw\right)^{r/a}B\left(r/a,w\right)}$.

Substituting, $\frac{y^a}{bw}=t$, results in

$$P_{\beta}GB(x) = D_1 \sum_{K=0}^{\infty} (-c)^k \binom{\beta x - x}{k} (bw)^{k/a} \int_0^1 t^{\frac{x+r+k-1}{a}-1} (1-t)^{w-1}\, dt \tag{8}$$

where , $x = 1, 2, 3, \ldots\ldots, a, b,\ w, r > 0,\ \beta \geq 1$ and $0 < c \leq \dfrac{1}{\beta (bw)^{\frac{1}{a}}}$.

Using the definition of beta function, (7) is obtained.

Special Cases

***Case I*:** When $\beta = 2$ in (4), Haight's distribution results and a compound of the Haight distribution with generalized beta follows from (8):

$$P_2 GB(x) = D_2^* \sum_{k=0}^{\infty} (-c)^k \binom{x}{k} (bw)^{k/a}\, B\left(\frac{x+r+k-1}{a}, w\right) \tag{9}$$

where $D_2^* = \dfrac{\frac{1}{2x-1}\binom{2x-1}{x} c^{x-1} (bw)^{\frac{x-1}{a}}}{B(r/a, w)}$; $x = 1, 2, 3 \ldots, a, b, w, r > 0,\ 0 < cy < 1$.

***Case II*:** If $b = 1/w$ and $a = 1$, in (5), the a beta distribution and a compound of Geeta distribution with beta distribution follow from (8):

$$P_{\beta} B(x) = D_3^* \sum_{k=0}^{\infty} (-c)^k \binom{\beta x - x}{k} B(x+r+k-1, w) \tag{10}$$

where $D_3^{\star} = \dfrac{\frac{1}{\beta x - 1}\binom{\beta x - 1}{x} c^{x-1}}{B(r, w)}$.

***Case III*:** When $\beta = 2$ and $b = 1/w$, $a = 1$ in (4) and (5), respectively obtained are the Haight and beta distributions and a compound of the Haight distribution with beta distribution follows from (8):

$$P_2B(x) = D_4^* \sum_{k=0}^{x} (-c)^k \binom{x}{k} B(x+r+k-1, w) \tag{11}$$

where $D_4^* = \dfrac{\frac{1}{2x-1}\binom{2x-1}{x} c^{x-1}}{B(r,w)}$

Factorial moments of the Compound of Geeta distribution with Generalized Beta distribution and some special cases

Let X_y and X be a random variable with distribution function $F(x|y)$ and $H(X)$, respectively (see (1)), and let parameter y have distribution $G(y)$. Keeping in mind the formula for the so-called factorial polynomial

$$x^{[l]} = x(x-1)(x-1)\dots(x-(l-1))$$

$$m_{[l]} = E\left(X^{[l]}\right) = \int_{-\infty}^{+\infty} E\left(X_y^{[l]}\right) dG(y) \tag{12}$$

is called a factorial moment of order l of the variable X with compound distribution (1).

Relation (12) is symbolized as

$$E\left(X_y^{[l]}\right) \underset{Y}{\wedge} G(y). \tag{13}$$

Theorem 2

The first order factorial moments of the compound of Geeta distribution with GBD is given by

$$m_{[1]}(\beta; cy) \underset{Y}{\wedge} GB(y; a, b, r, w) = \frac{\sum_{k=0}^{\infty} (\beta c)^k (bw)^{k/a}}{B(r/a, w)} \left(B\left(\frac{r+k}{a}, w\right) - c(bw)^{\frac{1}{a}} B\left(\frac{r+k+1}{a}, w\right) \right) \tag{14}$$

***Proof*:** The first order factorial moments of the Geeta distribution is given by

$$m_{[1]}(\beta,\theta)=\frac{1-\theta}{1-\beta\theta}.$$

Thus, from (13), the 1st order factorial moment of the compound of the Geeta distribution with a Generalized beta distribution if $\theta=cy$ is

$$\begin{aligned}m_{[1]}(\beta;cy)\underset{Y}{\wedge}GB(y,a,b,r,w)&=\frac{a}{(bw)^{r/a}B\left(r/a,w\right)}\int_0^{(bw)^{\frac{1}{a}}}\left(\frac{1-cy}{1-\beta cy}\right)y^{r-1}\left(1-\frac{y^a}{bw}\right)^{w-1}dy\\&=\frac{a\sum_{k=0}^{\infty}(\beta c)^k}{(bw)^{r/a}B\left(r/a,w\right)}\left(\int_0^{(bw)^{\frac{1}{a}}}y^{r+k-1}\left(1-\frac{y^a}{bw}\right)^{w-1}dy-c\int_0^{(bw)^{\frac{1}{a}}}y^{r+k}\left(1-\frac{y^a}{bw}\right)^{w-1}dy\right).\end{aligned}$$

Substituting, $\frac{y^a}{bw}=t$ results in

$$\begin{aligned}&=\frac{\sum_{k=0}^{\infty}(\beta c)^k}{B\left(r/a,w\right)}\left((bw)^{k/a}\int_0^1 t^{\frac{r+k}{a}-1}(1-t)^{w-1}dt-c(bw)^{\frac{k+1}{a}}\int_0^1 t^{\frac{r+k+1}{a}-1}(1-t)^{w-1}dt\right)\\&=\frac{\sum_{k=0}^{\infty}(\beta c)^k}{B\left(r/a,w\right)}(bw)^{k/a}\left(\int_0^1 t^{\frac{r+k}{a}-1}(1-t)^{w-1}dt-c(bw)^{\frac{1}{a}}\int_0^1 t^{\frac{r+k+1}{a}-1}(1-t)^{w-1}dt\right)\end{aligned}\tag{15}$$

Using the definition of beta function, (14) is obtained.

Special Case

When $b = 1/w$ and $a = 1$ in (15), the 1st order factorial moment of the compound of Geeta distribution with beta distribution is obtained as

$$m_{[1]}(\beta;cy)\underset{Y}{\wedge}B(y;1,1/w,r,w)=\frac{\sum_{k=0}^{\infty}(\beta c)^k}{B(r,w)}\big(B(r+k,w)-cB(r+k+1,w)\big).$$

Theorem 3

First order factorial moments of the compound of Haight with generalized beta distribution.

$$m_{[1]}\left(2;cy\right)\underset{Y}{\wedge}GB\left(y;a,b,r,w\right)=$$
$$\frac{\sum_{k=0}^{\infty}\left(2c\right)^{k}\left(bw\right)^{k/a}}{B\left(r/a,w\right)}\left(B\left(\frac{r+k}{a},w\right)-c\left(bw\right)^{\frac{1}{a}}B\left(\frac{r+k+1}{a},w\right)\right). \tag{16}$$

***Proof*:** The result follows directly from (15) for $\beta=2$,

$$m_{[1]}\left(2;cy\right)\underset{Y}{\wedge}GB\left(y,a,b,r,w\right)=$$
$$\frac{\sum_{k=0}^{\infty}\left(2c\right)^{k}\left(bw\right)^{k/a}}{B\left(r/a,w\right)}\left(\int_{0}^{1}t^{\frac{r+k}{a}-1}\left(1-t\right)^{w-1}dt-c\left(bw\right)^{\frac{1}{a}}\int_{0}^{1}t^{\frac{r+k+1}{a}-1}\left(1-t\right)^{w-1}dt\right) \tag{17}$$

which yields (16).

Special case

When b=1/w, a=1 in (17) the following result is obtained:

$$m_{[1]}\left(2;cy\right)\underset{Y}{\wedge}B\left(y;1,1/w,r,w\right)=\frac{\sum_{k=0}^{\infty}\left(2c\right)^{k}}{B\left(r,w\right)}\left(B\left(r+k,w\right)-cB\left(r+k+1,w\right)\right)$$

which gives the 1st order factorial moment of the compound of the Haight distribution with beta distribution.

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