Median Based Modified Ratio Estimators with Known Quartiles of an Auxiliary Variable

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New median based modified ratio estimators for estimating a finite population mean using quartiles and functions of an auxiliary variable are proposed. The bias and mean squared error of the proposed estimators are obtained and the mean squared error of the proposed estimators are compared with the usual simple random sampling without replacement (SRSWOR) sample mean, ratio estimator, a few existing modified ratio estimators, the linear regression estimator and median based ratio estimator for certain natural populations. A numerical study shows that the proposed estimators perform better than existing estimators; in addition, it is shown that the proposed median based modified ratio estimators outperform the ratio and modified ratio estimators as well as the linear regression estimator.

Keywords: Bias, inter-quartile range, linear regression estimator, mean squared error, natural population, simple random sampling

Introduction

Consider a finite population \( U = \{U_1, U_2, \ldots, U_N\} \) of \( N \) distinct and identifiable units. Let \( Y \) be a study variable with value \( Y_i \) measured on \( U_i, i = 1, 2, 3, \ldots, N \) giving a vector \( Y = \{Y_1, Y_2, \ldots, Y_N\} \). The goal is to estimate the population mean, \( \bar{Y} = \frac{1}{N} \sum_{i=1}^{N} Y_i \), with some desirable properties on the basis of a random sample of size \( n \) selected from the population \( U \). The simplest estimator of population mean is the sample mean, obtained by using simple random sampling without replacement.
replacement (SRSWOR), when there is no information on the auxiliary variable available. Let $X$ be an auxiliary variable that is positively correlated with the study variable $Y$: Sometimes the information on auxiliary variable $X$, positively correlated with $Y$, may be utilized to obtain a more efficient estimator of the population mean (for further details on ratio estimators see Cochran, 1977 and Murthy, 1967.) When the population parameters of an auxiliary variable $X$, such as, population mean, coefficient of variation, coefficient of kurtosis, coefficient of skewness and median are known, ratio, product and linear regression estimators (and their modifications) have been proposed in the literature – many of which perform better than the SRSWOR sample mean for estimating the population mean of a study variable.

Subramani (2013a) proposed a median based ratio estimator by using the median of a study variable as auxiliary information, and it has been shown that this median based ratio estimator outperforms the usual SRSWOR sample mean, ratio estimator, modified ratio estimator and linear regression estimator. Based on Subramani’s (2013a) median based ratio estimator, some new median based modified ratio estimators with known quartiles of the auxiliary variable are proposed.

The first quartile, also called lower quartile, is denoted by $Q_1$; the third quartile, also called the upper quartile, is denoted by $Q_3$. The lower quartile is a point where 25% of the observations are less than $Q_1$ and 75% are above $Q_1$. The upper quartile is a point where 75% observations are less than $Q_3$ and 25% are above $Q_3$. Quartiles are unaffected by extreme values unlike the population mean, variance, correlation coefficient, etc.

The inter-quartile range used as a measure of spread in a data set. The inter-quartile range of a distribution is the difference between the upper and lower quartiles. The formula for computing the inter-quartile range is

$$Q_r = Q_3 - Q_1.$$  \hspace{1cm} (1)

The semi-quartile range of a distribution is half the difference between the upper and lower quartiles, or half the inter-quartile range. The formula for computing the semi-quartile range is

$$Q_{sd} = \frac{Q_3 - Q_1}{2}.$$  \hspace{1cm} (2)
Another measure, the quartile average, noted by $Q_a$, was suggested by Subramani and Kumarapandiyan (2012a) and is defined as

$$Q_a = \frac{Q_3 + Q_1}{2} \quad (3)$$

The notations and formulae used in this article are:

- $N$: Population size
- $n$: Sample size
- $Y$: Study variable
- $M$: Median of the study variable
- $X$: Auxiliary variable
- $Q_i$: $i^{th}$ Quartile of auxiliary variable, $i=1,3$
- $\rho$: Correlation coefficient between $X$ and $Y$
- $\bar{X}, \bar{Y}$: Population means
- $x, y$: Sample means
- $\bar{M}$: Average of sample medians of $Y$
- $m$: Sample median of $Y$
- $\beta$: Regression coefficient of $Y$ on $X$
- $B(.)$: Bias of the estimator
- $V(.)$: Variance of the estimator
- $MSE(.)$: Mean squared error of the estimator
- $PRE(e, p) = \frac{MSE(e)}{MSE(p)} \times 100$ Percent relative efficiency of the proposed estimator $p$ with respect to the existing estimator $e$

The formulae for computing various measures including the variance and the covariance of the SRSWOR sample mean and sample median are:
In the case of SRSWOR, the sample mean, \( \bar{y} \), is used to estimate the population mean, \( \bar{Y} \). That is, the estimator of \( \bar{Y} = \hat{Y}_r = \bar{y} \) with variance

\[
V(\hat{Y}_r) = \frac{1-f}{n} S_y^2, \tag{4}
\]

The classical ratio estimator for estimating the population mean \( \bar{Y} \) of a study variable \( Y \) is defined as \( \hat{Y}_r = \frac{\bar{y}}{\bar{x}} = \bar{y} \hat{X} \). The bias and mean squared error of \( \hat{Y}_r \) are:

\[
B(\hat{Y}_r) = \bar{Y} \left( C'_{xx} - C'_{yx} \right) \tag{5}
\]

and

\[
MSE(\hat{Y}_r) = V(\bar{y}) + R^2 V(\bar{x}) - 2R\text{Cov}(\bar{y},\bar{x}) \tag{6}
\]
MEDIAN BASED MODIFIED RATIO ESTIMATORS

The other commonly used estimator using the auxiliary variable $X$ is the linear regression estimator. The linear regression estimator and its variance with known regression coefficient are:

$$\hat{\bar{Y}}_{lr} = \bar{y} + \beta(\bar{X} - \bar{x})$$  \hspace{1cm} (7)

$$V(\hat{\bar{Y}}_{lr}) = V(\bar{y})(1 - \rho^2) \text{ where } \rho = \frac{Cov(\bar{y}, \bar{x})}{\sqrt{V(\bar{x})} * V(\bar{y})}$$  \hspace{1cm} (8)

Subramani & Kumarapandiyan (2012a) suggested some modified ratio estimators using known quartiles and their functions of an auxiliary variable, these are:

$$\hat{\bar{Y}}_{RM1} = \bar{y} \left( \frac{\bar{X} + Q_1}{\bar{x} + Q_1} \right)$$  \hspace{1cm} (9)

$$\hat{\bar{Y}}_{RM2} = \bar{y} \left( \frac{\bar{X} + Q_3}{\bar{x} + Q_3} \right)$$  \hspace{1cm} (10)

$$\hat{\bar{Y}}_{RM3} = \bar{y} \left( \frac{\bar{X} + Q_r}{\bar{x} + Q_r} \right)$$  \hspace{1cm} (11)

$$\hat{\bar{Y}}_{RM4} = \bar{y} \left( \frac{\bar{X} + Q_d}{\bar{x} + Q_d} \right)$$  \hspace{1cm} (12)

$$\hat{\bar{Y}}_{RM5} = \bar{y} \left( \frac{\bar{X} + Q_a}{\bar{x} + Q_a} \right)$$  \hspace{1cm} (13)

The bias and the mean squared error of the modified ratio estimators in (9) to (13) are:

$$B\left(\hat{\bar{Y}}_{RMi}\right) = \bar{y} \left\{ \theta_i^rC_{xx}^r - \theta_i^C_{xx} \right\}$$  \hspace{1cm} (14)
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\[ \text{MSE}\left(\hat{Y}_{RMi}\right) = V(\overline{Y}) + R^2 \theta_i^2 V(\overline{x}) - 2R\theta_i \text{Cov}(\overline{Y}, \overline{x}) \] (15)

where \( R = \frac{\overline{Y}}{X} \) and

\[ \theta_1 = \frac{\overline{X}}{\overline{X} + Q_1}, \theta_2 = \frac{\overline{X}}{\overline{X} + Q_3}, \theta_3 = \frac{\overline{X}}{\overline{X} + Q_r}, \theta_4 = \frac{\overline{X}}{\overline{X} + Q_d}, \theta_5 = \frac{\overline{X}}{\overline{X} + Q_a}; i = 1, 2, 3, 4, 5 \]

Recently Subramani (2013a) suggested a median based ratio estimator for estimating \( \overline{Y} \) when the median of the study variable \( Y \) is known. The estimator with its bias and mean squared error are:

\[ \hat{Y}_M = \frac{\overline{Y}}{m} \] (16)

\[ B\left(\hat{Y}_M\right) = \overline{Y}\left\{C'_{mm} - C'_{ym} - \frac{\text{Bias}(m)}{M}\right\} \] (17)

\[ \text{MSE}\left(\hat{Y}_M\right) = V(\overline{Y}) + R^2 V(m) - 2R' \text{Cov}(\overline{Y}, m) \text{ where } R' = \frac{\overline{Y}}{M}. \] (18)

For further details on modified ratio estimators with known population parameters of an auxiliary variable, such as coefficient of variation, skewness, kurtosis, correlation coefficient, quartiles and their linear combinations, readers are referred to Kadilar and Cingi (2004, 2006a, b, 2009) Koyuncu and Kadilar (2009), Singh and Kakran (1993), Singh and Tailor (2003, 2005), Singh (2003), Sisodia and Dwivedi (1981), Subramani (2013a, b), Subramani and Kumarapandiyan (2012a, b, c, 2013), Tailor and Sharma (2009), Tin (1965), and Yan and Tian (2010).

The median based ratio estimator proposed by Subramani (2013a) is extended and, as a result, some new median based modified ratio estimators \( \hat{Y}_{SP1}, \hat{Y}_{SP2}, \hat{Y}_{SP3}, \hat{Y}_{SP4} \) and \( \hat{Y}_{SP5} \) with known quartiles and their functions of auxiliary variables are proposed.
Proposed Median Based Modified Ratio Estimators

The proposed median based modified ratio estimators for estimating a population mean $\bar{Y}$ based on Subramani’s (2013a) ratio estimator are:

$$\hat{Y}_{sp1} = \bar{Y} \left( \frac{M + Q_1}{m + Q_1} \right)$$ (19)

$$\hat{Y}_{sp2} = \bar{Y} \left( \frac{M + Q_2}{m + Q_2} \right)$$ (20)

$$\hat{Y}_{sp3} = \bar{Y} \left( \frac{M + Q_3}{m + Q_3} \right)$$ (21)

$$\hat{Y}_{sp4} = \bar{Y} \left( \frac{M + Q_d}{m + Q_d} \right)$$ (22)

and

$$\hat{Y}_{sp5} = \bar{Y} \left( \frac{M + Q_a}{m + Q_a} \right).$$ (23)

To the first degree of approximation, the bias and mean squared error of $\hat{Y}_{spj}$ are derived as:

$$B\left( \hat{Y}_{spj} \right) = \bar{Y} \left\{ \theta_j^2 C_{mm} - \theta_j C_{ym} - \frac{\text{Bias}(m)}{M} \right\}, \ j = 1, 2, 3, 4, 5,$$ (24)

$$MSE\left( \hat{Y}_{spj} \right) = V(\bar{Y}) + R^2 \theta_j^2 V(m) - 2R\theta_j \text{Cov}(\bar{Y}, m), \ j = 1, 2, 3, 4, 5$$ (25)

where

$$R = \frac{\bar{Y}}{M}, \ \theta_1 = \frac{M}{M + Q_1}, \ \theta_2 = \frac{M}{M + Q_3}, \ \theta_3 = \frac{M}{M + Q_r}, \ \theta_4 = \frac{M}{M + Q_d}, \ \theta_5 = \frac{M}{M + Q_a}.$$
See Appendix A for detailed derivation of the bias and the mean squared error of \( \hat{Y}_{SPj} \).

**Efficiency Comparisons**

**Comparison with SRSWOR Sample Mean**
The conditions (see Appendix B) for which the proposed estimators \( \hat{Y}_{SPj}, j=1,2,3,4,5 \) are more efficient than the SRSWOR sample mean \( \hat{Y}_r \) were derived from expressions (25) and (4) and are:

\[
MSE\left(\hat{Y}_{SPj}\right) \leq V\left(\hat{Y}_r\right) \text{ if } 2C_{ymj} \geq \theta_j \theta_{mm}^j; \, j=1,2,3,4,5. \tag{26}
\]

**Comparison with Ratio Estimators**
The conditions (see Appendix B) for which the proposed estimators \( \hat{Y}_{SPj}, j=1,2,3,4,5 \) are more efficient than the usual ratio estimator \( \hat{Y}_R \) were derived from expressions (25) and (6) and are:

\[
MSE\left(\hat{Y}_{SPj}\right) \leq MSE\left(\hat{Y}_R\right) \text{ if } \theta_j^2C_{mm} - \theta_i^2C_{xx} \leq 2\left(\theta_j C_{ymj} - \theta_i C_{yij}\right); \, i, j=1,2,3,4,5. \tag{27}
\]

**Comparison with Modified Ratio Estimators**
From expressions (25) and (15), the conditions (see Appendix B) for which the proposed estimators \( \hat{Y}_{SPj}, j=1,2,3,4,5 \) are more efficient than the existing modified ratio estimator \( \hat{Y}_{Rmi}, i=1,2,3,4,5 \) were derived and are:

\[
MSE\left(\hat{Y}_{SPj}\right) \leq MSE\left(\hat{Y}_{Rmi}\right)
\]

if

\[
MSE\left(\hat{Y}_{SPj}\right) \leq MSE\left(\hat{Y}_{Rmi}\right) \text{ if } \theta_j^2C_{mm} - \theta_i^2C_{xx} \leq 2\left(\theta_j C_{ymj} - \theta_i C_{yij}\right); \, i, j=1,2,3,4,5 \tag{28}
\]
Comparison with Linear Regression Estimator

From expressions (25) and (8), the conditions (see Appendix B) for which the proposed estimators \( \hat{Y}_{SPj} \), \( j = 1, 2, 3, 4, 5 \) are more efficient than the usual linear regression estimator \( \hat{Y}_n \) were derived and are:

\[
MSE\left( \hat{Y}_{SPj} \right) \leq V\left( \hat{Y}_n \right) \text{ if } 2\theta_j C'_{ym} - \theta_j^2 C'_{mm} \geq \frac{C'_{sy}}{C'_{xx}} ; j = 1, 2 . \tag{29}
\]

Comparison with Median Based Ratio Estimator

From expressions (25) and (18), the conditions (see Appendix B) for which the proposed estimators \( \hat{Y}_{SPj} \), \( j = 1, 2, 3, 4, 5 \) are more efficient than the existing modified ratio type estimator \( \hat{Y}_M \) were derived and are:

\[
MSE\left( \hat{Y}_{SPj} \right) \leq MSE\left( \hat{Y}_M \right) \text{ if } 2C'_{ym} \leq (\theta_j + 1) C'_{mm} ; j = 1, 2, 3, 4, 5 . \tag{30}
\]

Numerical Comparison

The conditions for which the proposed median based modified ratio estimators performed better than the other usual estimators considered in this study have been obtained. In order to show that the proposed estimators perform better than the other estimators, numerical comparisons were made to determine the efficiencies of the proposed estimators. Two populations were used to assess the efficiencies of the proposed median based modified ratio estimators with that of the existing estimators. Populations 1 and 2 are from Singh and Chaudhary (1986, p. 177). The parameter values and constants computed for the populations are given in Table 1, the bias for the proposed and existing estimators computed for the two populations are given in Table 2 and the mean squared errors are given in Table 3.
Table 1. Parameter values and constants for 2 different populations

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( n = 3 )</th>
<th>( n = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pop 1</td>
<td>Pop 2</td>
</tr>
<tr>
<td>( N )</td>
<td>34.0000</td>
<td>34.0000</td>
</tr>
<tr>
<td>( n )</td>
<td>3.0000</td>
<td>3.0000</td>
</tr>
<tr>
<td>( \bar{N}_c )</td>
<td>5,984.0000</td>
<td>5,984.0000</td>
</tr>
<tr>
<td>( \bar{Y} )</td>
<td>856.4118</td>
<td>856.4118</td>
</tr>
<tr>
<td>( \bar{M} )</td>
<td>747.7223</td>
<td>747.7223</td>
</tr>
<tr>
<td>( \bar{X} )</td>
<td>208.8824</td>
<td>199.4412</td>
</tr>
<tr>
<td>( Q_1 )</td>
<td>94.2500</td>
<td>99.2500</td>
</tr>
<tr>
<td>( Q_2 )</td>
<td>254.7500</td>
<td>278.0000</td>
</tr>
<tr>
<td>( Q_3 )</td>
<td>160.5000</td>
<td>178.7500</td>
</tr>
<tr>
<td>( Q_4 )</td>
<td>80.2500</td>
<td>89.3750</td>
</tr>
<tr>
<td>( Q_5 )</td>
<td>174.5000</td>
<td>188.6250</td>
</tr>
<tr>
<td>( R )</td>
<td>4.0999</td>
<td>4.2941</td>
</tr>
<tr>
<td>( R' )</td>
<td>1.1158</td>
<td>1.1158</td>
</tr>
<tr>
<td>( \theta )</td>
<td>0.6891</td>
<td>0.6677</td>
</tr>
<tr>
<td>( \phi_1 )</td>
<td>0.4505</td>
<td>0.4177</td>
</tr>
<tr>
<td>( \phi_2 )</td>
<td>0.5655</td>
<td>0.5274</td>
</tr>
<tr>
<td>( \phi_3 )</td>
<td>0.7224</td>
<td>0.6905</td>
</tr>
<tr>
<td>( \phi_4 )</td>
<td>0.5448</td>
<td>0.5139</td>
</tr>
<tr>
<td>( \phi_5 )</td>
<td>0.8906</td>
<td>0.8855</td>
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<tr>
<td>( \phi_6 )</td>
<td>0.7508</td>
<td>0.7341</td>
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<td>( \phi_7 )</td>
<td>0.8270</td>
<td>0.8111</td>
</tr>
<tr>
<td>( \phi_8 )</td>
<td>0.9053</td>
<td>0.8957</td>
</tr>
<tr>
<td>( \phi_9 )</td>
<td>0.8148</td>
<td>0.8027</td>
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\[
\text{var}(\bar{Y}) = 163,356.4086 \quad 163,356.4086 \quad 91,690.3713 \quad 91,690.3713
\]
\[
\text{var}(\bar{X}) = 6,884.4455 \quad 6,857.8555 \quad 3,864.1726 \quad 3,849.2480
\]
\[
\text{var}(m) = 101,518.7738 \quad 101,518.7738 \quad 59,396.2836 \quad 59,396.2836
\]
\[
\text{cov}(\bar{Y}, m) = 90,236.2939 \quad 90,236.2939 \quad 48,074.9542 \quad 48,074.9542
\]
\[
\text{cov}(\bar{Y}, \bar{X}) = 15,061.4011 \quad 14,905.0488 \quad 8,453.8187 \quad 8,366.0597
\]
\[
\rho = 0.4491 \quad 0.4453 \quad 0.4491 \quad 0.4453
\]
Table 2. Bias of existing and proposed estimators

<table>
<thead>
<tr>
<th>Estimators</th>
<th>$n = 3$</th>
<th></th>
<th>$n = 5$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pop 1</td>
<td>Pop 2</td>
<td>Pop 1</td>
<td>Pop 2</td>
</tr>
<tr>
<td>Existing</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{r}_{\text{Y}}$</td>
<td>63.0241</td>
<td>72.9186</td>
<td>35.3748</td>
<td>40.9285</td>
</tr>
<tr>
<td>$\hat{r}_{\text{RM}}$</td>
<td>14.4774</td>
<td>15.9291</td>
<td>8.1261</td>
<td>8.9409</td>
</tr>
<tr>
<td>$\hat{r}_{\text{MY}}$</td>
<td>-5.0570</td>
<td>-5.4535</td>
<td>-2.8385</td>
<td>-3.0610</td>
</tr>
<tr>
<td>Proposed</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{r}_{\text{SP}}$</td>
<td>52.0924</td>
<td>52.0924</td>
<td>57.7705</td>
<td>57.7705</td>
</tr>
</tbody>
</table>

Table 3. Variance/mean squared error of existing and proposed estimators

<table>
<thead>
<tr>
<th>Estimators</th>
<th>$n = 3$</th>
<th></th>
<th>$n = 5$</th>
<th></th>
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</thead>
<tbody>
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<td></td>
<td>Pop 1</td>
<td>Pop 2</td>
<td>Pop 1</td>
<td>Pop 2</td>
</tr>
<tr>
<td>Existing</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{r}$</td>
<td>163356.4086</td>
<td>163356.4086</td>
<td>91690.3713</td>
<td>91690.3713</td>
</tr>
<tr>
<td>$\hat{r}_{\text{Y}}$</td>
<td>155577.8155</td>
<td>161802.8878</td>
<td>87324.3215</td>
<td>90818.3961</td>
</tr>
<tr>
<td>$\hat{r}_{\text{RM}}$</td>
<td>133203.7861</td>
<td>134261.9210</td>
<td>74765.9957</td>
<td>75359.9173</td>
</tr>
<tr>
<td>$\hat{r}_{\text{MY}}$</td>
<td>131205.2291</td>
<td>131950.5079</td>
<td>73644.2252</td>
<td>74062.5432</td>
</tr>
<tr>
<td>Proposed</td>
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<tr>
<td>$\hat{r}_{\text{SP}}$</td>
<td>84266.7092</td>
<td>84147.8927</td>
<td>54798.7634</td>
<td>54675.1252</td>
</tr>
</tbody>
</table>

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The percentage relative efficiencies of the proposed estimators with respect to the existing estimators were also obtained and are shown in Tables 4-5.

**Table 4. Percentage Relative Efficiency of $\hat{Y}_{SPj}$ for Population 1**

<table>
<thead>
<tr>
<th>Existing Estimators</th>
<th>For sample size $n=3$</th>
<th>For sample size $n=5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Proposed Estimators</td>
<td>Proposed Estimators</td>
</tr>
<tr>
<td>$\hat{Y}_{SP1}$</td>
<td>193.86</td>
<td>167.32</td>
</tr>
<tr>
<td>$\hat{Y}_{SP2}$</td>
<td>184.63</td>
<td>159.35</td>
</tr>
<tr>
<td>$\hat{Y}_{SP3}$</td>
<td>158.07</td>
<td>157.72</td>
</tr>
<tr>
<td>$\hat{Y}_{SP4}$</td>
<td>155.70</td>
<td>157.29</td>
</tr>
<tr>
<td>$\hat{Y}_{SP5}$</td>
<td>154.89</td>
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<tr>
<td>$\hat{r}_{Y}$</td>
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<td>167.32</td>
</tr>
<tr>
<td>$\hat{R}_{Y}$</td>
<td>184.63</td>
<td>159.35</td>
</tr>
<tr>
<td>$\hat{r}_{RM1}$</td>
<td>158.07</td>
<td>157.72</td>
</tr>
<tr>
<td>$\hat{r}_{RM2}$</td>
<td>155.70</td>
<td>157.29</td>
</tr>
<tr>
<td>$\hat{r}_{RM3}$</td>
<td>154.89</td>
<td>156.90</td>
</tr>
<tr>
<td>$\hat{r}_{RM4}$</td>
<td>159.65</td>
<td>161.71</td>
</tr>
<tr>
<td>$\hat{r}_{RM5}$</td>
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<td>156.77</td>
</tr>
<tr>
<td>$\hat{r}_{s}$</td>
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<td>106.24</td>
</tr>
<tr>
<td>$\hat{r}_{u}$</td>
<td>105.95</td>
<td>108.99</td>
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</tbody>
</table>

**Table 5: Percentage Relative Efficiency of $\hat{Y}_{SPj}$ for Population 2**

<table>
<thead>
<tr>
<th>Existing Estimators</th>
<th>For sample size $n=3$</th>
<th>For sample size $n=5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Proposed Estimators</td>
<td>Proposed Estimators</td>
</tr>
<tr>
<td>$\hat{Y}_{SP1}$</td>
<td>194.13</td>
<td>167.70</td>
</tr>
<tr>
<td>$\hat{Y}_{SP2}$</td>
<td>192.28</td>
<td>166.11</td>
</tr>
<tr>
<td>$\hat{Y}_{SP3}$</td>
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<td>161.46</td>
</tr>
<tr>
<td>$\hat{Y}_{SP4}$</td>
<td>156.81</td>
<td>158.68</td>
</tr>
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<td>$\hat{Y}_{SP5}$</td>
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<td>157.56</td>
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<tr>
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<tr>
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</tr>
<tr>
<td>$\hat{r}_{u}$</td>
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<td>106.73</td>
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</table>
Tables 4 and 5 show that the percent relative efficiencies of the proposed estimators, with respect to existing estimators, range in general from 104.41 to 196.45. In particular, the PRE ranges from 166.18 to 196.45 for comparing with the SRSWOR sample mean; ranging from 158.27 to 194.58 for comparing with the ratio estimator; ranging from 132.68 to 162.66 for comparing with the modified ratio estimators; ranging from 132.67 to 157.50 for comparing with the linear regression estimator and ranging from 104.41 to 110.56 for comparing with the median based ratio estimator. This demonstrates that the proposed estimators perform better than the existing SRSWOR sample mean, ratio, modified ratio and linear regression estimators for the two populations considered. Further it is observed from the numerical comparisons that the following inequalities hold:

\[
MSE\left(\hat{Y}_{\text{SPE}}\right) \leq MSE\left(\hat{Y}_{\text{M}}\right) \leq V\left(\hat{Y}_{\nu}\right) \leq MSE\left(\hat{Y}_{\text{RM}}\right) \leq MSE\left(\hat{Y}_{R}\right) \leq V\left(\hat{Y}_{E}\right)
\]

**Conclusion**

This article proposed some new median based modified ratio estimators using known quartiles and their functions of the auxiliary variable. The conditions for which the proposed estimators are more efficient than the existing estimators were derived. Further the percentage relative efficiencies of the proposed estimators with respect to existing estimators were shown to range in general from 104.41 to 196.45 for certain natural populations available in the literature. It is usually believed that the linear regression estimator is the optimum estimator for estimating the population mean whenever an auxiliary variable exists that is positively correlated with that of a study variable. However, it was shown that the proposed median based modified ratio estimators outperform not only the ratio and modified ratio estimators but also the linear regression estimator. Based on results of this study, the proposed median based modified ratio estimators are recommended for estimating finite population means.

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References


Subramani, J. (2013a). A new median based ratio estimator for estimation of the finite population mean. (Submitted for publication).


Appendix A

The derivation of the bias and the mean squared error of $\overline{Y}_{SP1}$ are given below:

Consider

$$\hat{Y}_{SP1} = \overline{Y} \left( \frac{M + Q_1}{m + Q_1} \right)$$  \hspace{1cm} (A1)

Let $e_0 = \frac{\overline{Y} - \overline{Y}}{\overline{Y}}$ and $e_1 = \frac{m - M}{M}$

$$\Rightarrow E(e_0) = 0; \ E(e_1) = \frac{\overline{M} - M}{M} = \frac{Bias(m)}{M}$$  \hspace{1cm} (A2)

$$\Rightarrow E(e_0^2) = \frac{V(\overline{Y})}{\overline{Y}^2}; \ E(e_1^2) = \frac{V(m)}{M^2}; \ E(e_0e_1) = \frac{Cov(\overline{Y},m)}{\overline{Y}M}$$  \hspace{1cm} (A3)

The estimator $\overline{Y}_{SP1}$ can be written in terms of $e_0$ and $e_1$ as

$$\hat{Y}_{SP1} = \overline{Y} (1 + e_0) \left( \frac{M + Q_1}{M (1 + e_1) + Q_1} \right)$$

$$\Rightarrow \hat{Y}_{SP1} = \overline{Y} (1 + e_0) \left( \frac{M + Q_1}{(M + Q_1) + Me_1} \right)$$

$$\Rightarrow \hat{Y}_{SP1} = \overline{Y} (1 + e_0) \left( \frac{1}{1 + \left( \frac{M}{M + Q_1} \right)e_1} \right)$$

$$\Rightarrow \hat{Y}_{SP1} = \overline{Y} (1 + e_0) \left( \frac{1}{1 + \theta_ie_1} \right); \ \text{where} \ \theta_i = \frac{M}{M + Q_1}$$
\[ \hat{Y}_{SP1} = \bar{Y} \left( 1 + e_0 \right) \left( 1 + \theta_1^2 \right)^{-1} \]

Neglecting the terms of higher order, we have

\[ \hat{Y}_{SP1} = \bar{Y} \left( 1 + e_0 \right) \left( 1 - \theta_1 e_1 + \theta_1^2 e_1^2 \right) \]

\[ \Rightarrow \hat{Y}_{SP1} = \bar{Y} + \bar{Y} e_0 - \bar{Y} \theta_1 e_1 - \bar{Y} \theta_1^2 e_1^2 + \bar{Y} \theta_1 e_1 + \bar{Y} \theta_1^2 e_1^2 \]

\[ \Rightarrow \hat{Y}_{SP1} - \bar{Y} = \bar{Y} e_0 - \bar{Y} \theta_1 e_1 - \bar{Y} \theta_1^2 e_1^2 + \bar{Y} \theta_1 e_1 + \bar{Y} \theta_1^2 e_1^2 \quad (A4) \]

Taking expectations on both sides of (A4) we have,

\[ E\left( \hat{Y}_{SP1} - \bar{Y} \right) = \bar{Y} E\left( e_0 \right) - \bar{Y} \theta_1 E\left( e_1 \right) - \bar{Y} \theta_1^2 E\left( e_1^2 \right) + \bar{Y} \theta_1 e_1 + \bar{Y} \theta_1^2 e_1^2 \]

\[ \Rightarrow E\left( \hat{Y}_{SP1} - \bar{Y} \right) = \bar{Y} \left\{ \theta_1^2 C_{mm} - \theta_1^2 C_{ym} - \theta_1 \frac{Bias(m)}{M} \right\} \] from (A2) and (A3)

\[ \Rightarrow Bias\left( \hat{Y}_{SP1} \right) = \bar{Y} \left\{ \theta_1^2 C_{mm} - \theta_1^2 C_{ym} - \theta_1 \frac{Bias(m)}{M} \right\} \quad (A5) \]

The derivation of mean squared error of \( \bar{Y}_{SP1} \) is given below:

\[ MSE\left( \hat{Y}_{SP1} \right) = E\left( \hat{Y}_{SP1} - \bar{Y} \right)^2 = E\left( \bar{Y} e_0 - \bar{Y} \theta_1 e_1 \right)^2 \]

\[ \Rightarrow MSE\left( \hat{Y}_{SP1} \right) = \bar{Y} \left\{ E\left( e_0^2 \right) + \theta_1^2 E\left( e_1^2 \right) - 2 \theta_1 E\left( e_0 e_1 \right) \right\} \]

\[ \Rightarrow MSE\left( \hat{Y}_{SP1} \right) = \bar{Y} \left\{ \frac{V\left( \bar{Y} \right)}{\bar{Y}^2} + \theta_1^2 \frac{V\left( m \right)}{M^2} - 2 \theta_1 \frac{Cov\left( \bar{Y}, m \right)}{\bar{Y} M} \right\} \]

\[ \Rightarrow MSE\left( \hat{Y}_{SP1} \right) = V\left( \bar{Y} \right) + \frac{\bar{Y}^2}{M^2} \theta_1^2 V\left( m \right) - 2 \frac{\bar{Y}}{M} \theta_1 Cov\left( \bar{Y}, m \right) \]
\[ \Rightarrow \text{MSE}\left(\hat{Y}_{sp1}\right) = V\left(\bar{y}\right) + R^2 \theta_i^2 V\left(m\right) - 2R' \hat{\theta}_i \text{Cov}\left(\bar{y}, m\right) + R' = \frac{\bar{Y}}{M} \] 

(A6)

In the similar manner, the bias and mean squared error of \(\hat{Y}_{sp2}, \hat{Y}_{sp3}, \hat{Y}_{sp4}\) and \(\hat{Y}_{sp5}\) can be obtained.

**Appendix B**

The conditions for which the proposed estimators perform better than the existing estimators are derived here and are given below:

**Comparison with that of SRSWOR sample mean**

Consider \(\text{MSE}\left(\hat{Y}_{sp}\right) \leq V\left(\hat{\bar{y}}\right)\)

\[ \Rightarrow V\left(\bar{y}\right) + R^2 \theta_j^2 V\left(m\right) - 2R' \hat{\theta}_j \text{Cov}\left(\bar{y}, m\right) \leq V\left(\bar{y}\right) \]

\[ \Rightarrow R^2 \theta_j^2 V\left(m\right) - 2R' \hat{\theta}_j \text{Cov}\left(\bar{y}, m\right) \leq 0 \]

\[ \Rightarrow R^2 \theta_j^2 V\left(m\right) \leq 2R' \hat{\theta}_j \text{Cov}\left(\bar{y}, m\right) \]

\[ \Rightarrow \text{Cov}\left(\bar{y}, m\right) \geq \frac{R' \theta_j V\left(m\right)}{2}, j = 1, 2, 3, 4, 5 \]

\[ \Rightarrow \text{Cov}\left(\bar{y}, m\right) \geq \frac{\bar{Y} M \hat{\theta}_j C_{mm}}{2} \]

\[ \Rightarrow 2C_{jm} \geq \hat{\theta}_j C_{mm}, j = 1, 2, 3, 4, 5 \]
Comparison with that of Ratio Estimator

Consider \( \text{MSE}\left( \hat{Y}_{SPj} \right) \leq \text{MSE}\left( \hat{Y}_R \right) \)

\[
\Rightarrow V(\bar{y}) + \theta_j^2 V(m) - 2R'\theta_j \text{Cov}(\bar{y},m) \leq V(\bar{y}) + \theta^2 V(\bar{x}) - 2R\text{Cov}(\bar{y},\bar{x})
\]

\[
\Rightarrow R^2 \theta_j^2 V(m) - 2R'\theta_j \text{Cov}(\bar{y},m) \leq R^2 V(\bar{x}) - 2R\text{Cov}(\bar{y},\bar{x})
\]

\[
\Rightarrow R^2 \theta_j^2 V(m) - R^2 V(\bar{x}) \leq 2R'\theta_j \text{Cov}(\bar{y},m) - 2R\text{Cov}(\bar{y},\bar{x})
\]

\[
\Rightarrow R^2 \theta_j^2 V(m) - R^2 V(\bar{x}) \leq 2R'\theta_j \text{Cov}(\bar{y},m) - 2R\text{Cov}(\bar{y},\bar{x})
\]

\[
\Rightarrow \frac{\bar{Y}^2}{M^2} \theta_j^2 V(m) - 2\frac{\bar{Y}^2}{X^2} V(\bar{x}) \leq 2\left\{ \theta_j \frac{\text{Cov}(\bar{y},m)}{YM} - \frac{\text{Cov}(\bar{y},\bar{x})}{YX} \right\}
\]

\[
\Rightarrow \theta_j^2 C_{mm} - C_{xx} \leq 2\left\{ \theta_j \left[ C_{ym} - C_{yx} \right] \right\} ; j = 1, 2, 3, 4, 5
\]

Comparison with that of Modified Ratio Estimators

Consider \( \text{MSE}\left( \hat{Y}_{SPj} \right) \leq \text{MSE}\left( \hat{Y}_{RMi} \right) \)

\[
\Rightarrow V(\bar{y}) + \theta_j^2 V(m) - 2R'\theta_j \text{Cov}(\bar{y},m) \leq V(\bar{y}) + \theta_i^2 V(\bar{x}) - 2R\theta_i \text{Cov}(\bar{y},\bar{x})
\]

\[
\Rightarrow R^2 \theta_j^2 V(m) - 2R'\theta_j \text{Cov}(\bar{y},m) \leq R^2 \theta_i^2 V(\bar{x}) - 2R\theta_i \text{Cov}(\bar{y},\bar{x})
\]

\[
\Rightarrow R^2 \theta_j^2 V(m) - R^2 \theta_i^2 V(\bar{x}) \leq 2R'\theta_j \text{Cov}(\bar{y},m) - 2R\theta_i \text{Cov}(\bar{y},\bar{x})
\]

\[
\Rightarrow \frac{\bar{Y}^2}{M^2} \theta_j^2 V(m) - \frac{\bar{Y}^2}{X^2} \theta_i^2 V(\bar{x}) \leq 2\frac{\bar{Y}^2}{YM} \theta_j \text{Cov}(\bar{y},m) - 2\frac{\bar{Y}^2}{X} \theta_i \text{Cov}(\bar{y},\bar{x})
\]
\[ \Rightarrow \theta_j^2 \frac{V(m)}{M^2} - \frac{\theta_i^2 V(\bar{x})}{X^2} \leq 2 \left\{ \theta_j \frac{Cov(\bar{y}, m)}{YM} - \theta_i \frac{Cov(\bar{y}, \bar{x})}{YX} \right\} \]

\[ \Rightarrow \theta_j^2 C_{mn} - \theta_j^2 C_{xx} \leq 2 \left\{ \theta_j C_{ym} - \theta_j C_{xx} \right\}; i, j = 1, 2, 3, 4, 5 \]

**Comparison with that of Linear Regression Estimator**

Consider \( MSE\left(\hat{Y}_{sr}\right) \leq V\left(\hat{Y}_{lr}\right) \)

\[ \Rightarrow V(\bar{y}) + R^2 \theta_j^2 V(m) - 2R'\theta_j Cov(\bar{y}, m) \leq V(\bar{y})(1 - \rho^2) \]

\[ \Rightarrow R^2 \theta_j^2 V(m) - 2R'\theta_j Cov(\bar{y}, m) \leq -V(\bar{y}) \left[ \frac{Cov(\bar{y}, \bar{x})^2}{V(\bar{x})} \right] \]

\[ \Rightarrow 2R'\theta_j Cov(\bar{y}, m) - R^2 \theta_j^2 V(m) \geq \frac{[Cov(\bar{y}, \bar{x})]^2}{V(\bar{x})}; j = 1, 2, 3, 4, 5 \]

\[ \Rightarrow 2 \frac{\bar{y}}{M} \theta_j Cov(\bar{y}, m) - \frac{\bar{y}^2}{M^2} \theta_j^2 V(m) \geq \frac{[Cov(\bar{y}, \bar{x})]^2}{V(\bar{x})} \]

\[ \Rightarrow 2\bar{Y}^2 \theta_j^2 C_{ym} - \bar{Y}^2 \theta_j^2 C_{mm} \geq \frac{[Cov(\bar{y}, \bar{x})]^2}{V(\bar{x})} \]

\[ \Rightarrow 2\theta_j^2 C_{ym} - \theta_j^2 C_{mm} \geq \frac{[C_{ym}^2]}{C_{xx}}; j = 1, 2, 3, 4, 5 \]
Comparison with that of Median Based Ratio Estimator

Consider \( \text{MSE}\left( \hat{\bar{y}}_{spj} \right) \leq \text{MSE}\left( \hat{\bar{y}}_{M} \right) \)

\[
\Rightarrow V(\bar{y}) + R' \hat{\theta}_j^2 V(m) - 2R' \hat{\theta}_j \text{Cov}(\bar{y}, m) \leq V(\bar{y}) + R'\hat{\theta}_j^2 V(m) - 2R' \text{Cov}(\bar{y}, m)
\]

\[
\Rightarrow R' \hat{\theta}_j^2 V(m) - 2R' \hat{\theta}_j \text{Cov}(\bar{y}, m) \leq R'\hat{\theta}_j^2 V(m) - 2R' \text{Cov}(\bar{y}, m)
\]

\[
\Rightarrow R' \hat{\theta}_j^2 V(m) - R'\hat{\theta}_j^2 V(m) \leq 2R'\hat{\theta}_j \text{Cov}(\bar{y}, m) - 2R' \text{Cov}(\bar{y}, m)
\]

\[
\Rightarrow R'V(m) (\hat{\theta}_j^2 - 1) \leq 2(\hat{\theta}_j - 1) \text{Cov}(\bar{y}, m)
\]

\[
\Rightarrow R'V(m) (\hat{\theta}_j - 1)(\hat{\theta}_j + 1) \leq 2(\hat{\theta}_j - 1) \text{Cov}(\bar{y}, m)
\]

\[
\Rightarrow \text{Cov}(\bar{y}, m) \leq \frac{R'(\hat{\theta}_j + 1)V(m)}{2} \quad \text{Since } \hat{\theta}_j < 1; \ j = 1, 2, 3, 4, 5
\]

\[
\Rightarrow \text{Cov}(\bar{y}, m) \leq \frac{\bar{Y}M(\hat{\theta}_j + 1)C_{mm}'}{2} \quad \text{Since } \hat{\theta}_j < 1
\]

\[
\Rightarrow 2C_{ym}' \leq (\hat{\theta}_j + 1)C_{mm}', \ j = 1, 2, 3, 4, 5
\]