Journal of Modern Applied Statistical Methods

Volume 14 | Issue 2 Article 10

11-1-2015

New Entropy Estimators with Smaller Root Mean Squared Error

Amer Ibrahim Al-Omari
Al al-Bayt University, Mafraq, Jordan, alomari_amer@yahoo.com

New Entropy Estimators with Smaller Root Mean Squared Error

Cover Page Footnote
The author thanks the editor and the referees for their helpful and valuable comments that substantially improved this paper.

Journal of Modern Applied Statistical Methods
November 2015, Vol. 14, No. 2, 88-109.

ISSN 1538 − 9472

New Entropy Estimators with Smaller Root Mean Squared Error

Amer Ibrahim Al-Omari
Al al-Bayt University
Mafraq, Jordan

New estimators of entropy of continuous random variable are suggested. The proposed estimators are investigated under simple random sampling (SRS), ranked set sampling (RSS), and double ranked set sampling (DRSS) methods. The estimators are compared with Vasicek (1976) and Al-Omari (2014) entropy estimators theoretically and by simulation in terms of the root mean squared error (RMSE) and bias values. The results indicate that the suggested estimators have less RMSE and bias values than their competing estimators introduced by Vasicek (1976) and Al-Omari (2014).

Keywords: Shannon entropy; simple random sampling, ranked set sampling; double ranked set sampling; root mean square error.

Introduction

The ranked set sampling was first suggested by McIntyre (1952) to estimate a mean of pasture and forage yields. It is a cost efficient sampling procedure alternative to the commonly used simple random sampling scheme. The RSS is useful in situations where the visual ordering of a set of units can be done easily, but the exact measurement of the units is difficult or expensive.

Let the variable of interest X has a probability density function (pdf) $g(x)$ and a cumulative distribution function (cdf) $G(x)$, with mean μ and variance σ^2. Let $g_{(i:n)}(x)$ and $G_{(i:n)}(x)$ be the pdf and cdf of the ith order statistic, $X_{(i:n)}$, $(1 \le i \le n)$ of a random sample of size n. The pdf and the cdf of $X_{(i:n)}$, respectively, are given by

$$g_{(i:n)}(x) = n\binom{n-1}{i-1}G^{i-1}(x)\left[1-G(x)\right]^{n-i} g(x), \quad -\infty < x < \infty$$

Amer Ibrahim Al-Omari is Faculty of Science, in the Department of Mathematics. Email at: alomari_amer@yahoo.com.

and

$$G_{(i:n)}(x)=\sum_{j=1}^{n}\binom{n}{j}G^{j}(x)\left[1-G(x)\right]^{n-j},\ -\infty<x<\infty\ ,$$

with mean $\mu_{(i:n)}=\int_{-\infty}^{\infty}x\,g_{(i:n)}(x)\,dx$ and variance $\sigma_{(i:n)}^{2}=\int_{-\infty}^{\infty}\left(x-\mu_{(i:n)}\right)^{2}g_{(i:n)}(x)\,dx$.

The ranked set sampling method can be describes as follows:

Step 1. Randomly select n^2 units from the target population.

Step 2. Allocate the n^2 selected units randomly into n sets, each of size n.

Step 3. Without yet knowing any values for the variable of interest, rank the units within each set with respect to a variable of interest. This may be based on a personal professional judgment or based on a concomitant variable correlated with the variable of interest.

Step 4. The sample units are selected for actual measurement by including the ith smallest ranked unit of the ith sample (i = 1, 2, …, n).

Step 5. Repeat Steps 1 through 4 for r cycles to obtain a sample of size nr for actual measurement.

It is of interest to note here that even if n^2 units are selected from the population, but only n of them are measured for comparison with a simple random sampling of the same size n.

Let the measured RSS units are denoted by $X_{1(1:n)}, X_{2(2:n)}, \ldots, X_{n(n:n)}$. The RSS estimator of the population mean is defined as $\bar{X}_{RSS}=\frac{1}{n}\sum_{i=1}^{n}X_{i(i:n)}$. Takahasi and Wakimoto (1968) provided the mathematical theory of the RSS and showed that

$$g(x)=\frac{1}{n}\sum_{i=1}^{n}g_{(i:n)}(x),\ \mu=\frac{1}{n}\sum_{i=1}^{n}\mu_{(i:n)},\ \mathrm{Var}\left(\bar{X}_{RSS}\right)=\frac{\sigma^{2}}{n}-\frac{1}{n^{2}}\sum_{i=1}^{n}\left(\mu_{i(i:n)}-\mu\right)^{2}.$$

Al-Saleh and Al-Kadiri (2000) suggested double ranked set sampling (DRSS) method for estimating the population mean to increase the efficiency of the estimators for fixed sample size. The DRSS method can be described as:

Step 1. Randomly choose n^2 samples of size n each from the target population.

Step 2. Apply the RSS method described above on the n^2 samples in Step 1. This step yields n samples of size n each.

Step 3. Reapply the RSS method again on the n samples obtained in Step 2 to obtain a sample of size n from the DRSS data. The cycle can be repeated r times if needed to obtain a sample of size rn units.

Let X be a continuous random variable with probability density function $g(x)$ and cumulative distribution function $G(x)$. The entropy $H[g(x)]$ of the random variable is defined by Shannon (1948a, 1948b) as

$$H\left[g(x)\right] = -\int_{-\infty}^{\infty} g(x)\log\left[g(x)\right]dx. \tag{1}$$

The problem of entropy estimation of a continuous random variable is considered by many authors. Vasicek's (1976) suggested an estimator of entropy based on spacing's as

$$H\left[g(x)\right] = \int_{0}^{1}\log\left(\frac{dG^{-1}(p)}{dp}\right)dp, \tag{2}$$

where the estimation is found by replacing the distribution function $G(x)$ by the empirical distribution function $G_n(x)$, and using the difference operator instead of the differential operator. Then the derivative $\frac{d}{dp}G^{-1}(p)$ is estimated by a function of the order statistics.

Let $X_1, X_2, \ldots, X_n$ be a simple random sample of size n from $G(x)$ and $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ be the order statistics of the sample. Then Vasicek's (1976) estimator of $H[g(x)]$ is defined as

$$HV_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right\} \tag{3}$$

where $m < n/2$ is a positive integer known as the window size, $X_{(i-m)} = X_{(1)}$ if $i \le m$, and $X_{(i+m)} = X_{(n)}$ if $i \ge n - m$. He proved that $HV_{mn} \xrightarrow{P.} H\left[g\left(x\right)\right]$ as $n \to \infty,\ m \to \infty$, and $\frac{m}{n} \to 0$.

Van Es (1992) suggested an estimator of entropy based on spacings as

$$HVE_{mn} = \frac{1}{n-m}\sum_{i=1}^{n-m}\left\{\frac{n+1}{m}\left(X_{(i+m)} - X_{(i)}\right)\right\} + \sum_{k=m}^{n}\frac{1}{k} + \log\left(m\right) - \log\left(n+1\right) \tag{4}$$

and proved the consistency and the asymptotic normality of the estimator under some conditions.

Ebrahimi, Pflughoeft, and Soofi (1994) adjusted the weights of Vasicek (1976) estimator to have a smaller weights and proposed an entropy estimator given by

$$HE_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{\theta_i m}\left(X_{(i+m)} - X_{(i-m)}\right)\right\} \tag{5}$$

where

$$\theta_i = \begin{cases} 1 + \dfrac{i-1}{m}, & 1 \le i \le m, \\ 2, & m+1 \le i \le n-m, \\ 1 + \dfrac{n-i}{m}, & n-m+1 \le i \le n, \end{cases}$$

where $X_{(i-m)} = X_{(1)}$ for $i \le m$ and $X_{(i+m)} = X_{(n)}$ for $i \ge n - m$. Ebrahimi et al. (1994) showed by simulation that their estimator has a smaller bias and mean squared error than Vasicek (1976) estimator. Also, they proved that

$$HE_{mn} \xrightarrow{P.} H\left[g\left(x\right)\right] \text{ as } n \to \infty,\ m \to \infty,\ m/n \to 0.$$

Noughabi and Noughabi (2013) suggested a new estimator of entropy of an unknown continuous probability density function as

$$HNN_{mn} = -\frac{1}{n}\sum_{i=1}^{n} \log\left\{s_i(n,m)\right\}, \tag{6}$$

where

$$s_i(n,m) = \begin{cases} \hat{g}\left(X_{(i)}\right), & 1 \le i \le m, \\ \dfrac{2m/n}{X_{(i+m)} - X_{(i-m)}}, & m+1 \le i \le n-m, \\ \hat{g}\left(X_{(i)}\right), & n-m+1 \le i \le n, \end{cases}$$

and $\hat{g}(X_i) = \frac{1}{nh}\sum_{j=1}^{n} k\left(\frac{X_i - X_j}{h}\right)$, where h is bandwidth and k is a kernel function satisfies $\int_{-\infty}^{\infty} k(x)\,dx = 1$. They proved that $HNN_{mn} \xrightarrow{P} H\left[g(x)\right]$ as $n \to \infty$, $m \to \infty$, $m/n \to 0$. Note that the kernel function in Noughabi and Noughabi (2013) is selected to be the standard normal distribution and the bandwidth h is chosen to be $h = 1.06sn^{-1/5}$, where s is the sample standard deviation.

To estimate the entropy $H[g(x)]$ of an unknown continuous probability density function $g(x)$, Noughabi and Arghami (2010) suggested an entropy estimator given by

$$HN_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{n}{c_i m}\left(X_{(i+m)} - X_{(i-m)}\right)\right\} \tag{7}$$

where

$$c_i = \begin{cases} 1, & 1 \le i \le m, \\ 2, & m+1 \le i \le n-m, \\ 1, & n-m+1 \le i \le n, \end{cases}$$

and $X_{(i\text{-m})} = X_{(1)}$ if $i \le m$ and $X_{(i\text{+m})} = X_{(n)}$ for $i \ge n - m$.

Correa (1995) suggested a modified entropy estimator to have smaller mean squared error in the form

$$HC_{mn} = -\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{\sum_{j=i-m}^{i+m}(j-i)\left(X_{(j)}-\bar{X}_{(i)}\right)}{n\sum_{j=i-m}^{i+m}\left(X_{(j)}-\bar{X}_{(i)}\right)^2}\right), \tag{8}$$

where $\bar{X}_{(i)} = \frac{1}{2m+1}\sum_{j=i-m}^{i+m} X_{(j)}$.

Al-Omari (2014) suggested three estimators of entropy of an unknown continuous probability density function $g(x)$ using SRS, RSS, and DRSS methods. Based on SRS his first suggested estimator is defined as

$$AHESRS_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{\omega_i m}\left(X_{(i+m)}-X_{(i-m)}\right)\right\} \tag{9}$$

where $X_{(i\text{-m})} = X_{(1)}$ for $i \le m$, $X_{(i\text{+m})} = X_{(n)}$ for $i \ge n - m$, and

$$\omega_i = \begin{cases} 1+\frac{1}{2}, & 1 \le i \le m, \\ 2, & m+1 \le i \le n-m, \\ 1+\frac{1}{2}, & n-m+1 \le i \le n, \end{cases} \tag{10}$$

The second and third estimators suggested by Al-Omari (2014), based on RSS and DRSS respectively, are given by

$$AHERSS_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{\omega_i m}\left(X^{*}_{(i+m)}-X^{*}_{(i-m)}\right)\right\} \tag{11}$$

and

$$AHEDRSS_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{\omega_i m}\left(X^{**}_{(i+m)}-X^{**}_{(i-m)}\right)\right\} \tag{12}$$

where $X^{*}_{(i-m)} = X^{*}_{(1)}$ for $i \le m$ and $X^{*}_{(i+m)} = X^{*}_{(n)}$ for $i \ge n - m$, and

$X^{**}_{(i-m)} = X^{**}_{(1)}$ for $i \le m$ and $X^{**}_{(i+m)} = X^{**}_{(n)}$ for $i \ge n - m$.

For more about entropy estimators, see Choi, Kim, and Song (2004), Park, Park (2003), Goria, Leonenko, Mergel, and Novi Inverardi (2005) and Choi (2008).

The remaining part of this paper is organized as follows. The suggested entropy estimators are given in the section, "Proposed Estimators". Next, a simulation study is conducted to compare the new estimators with their counterparts suggested by Vasicek (1976) and Al-Omari (2014). Finally, some conclusions and suggestions for further works.

The proposed estimators

The coefficient of the entropy estimators in Ebrahimi et al. (1994), Noughabi and Arghami (2010), and Al-Omari (2014) are adjusted. Let $X_1^{(0)}, X_2^{(0)}, \ldots, X_n^{(0)}$ be a simple random sample of size n from $G(x)$. Based on SRS the first suggested estimator is given by

$$SHESRS_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{n}{\varepsilon_i m}\left(X^{(0)}_{(i+m)} - X^{(0)}_{(i-m)}\right)\right\} \tag{13}$$

where

$$\varepsilon_i = \begin{cases} 1+\dfrac{1}{4}, & 1 \le i \le m, \\ 2, & m+1 \le i \le n-m, \\ 1+\dfrac{1}{4}, & n-m+1 \le i \le n, \end{cases} \tag{14}$$

$X^{(0)}_{(i-m)} = X^{(0)}_{(1)}$ for $i \le m$ and $X^{(0)}_{(i+m)} = X^{(0)}_{(n)}$ for $i \ge n - m$. Comparing (3) with (13), we have

$$
\begin{aligned}
SHESRS_{mn} &= \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{\varepsilon_i m}\left(X_{(i+m)}^{(0)} - X_{(i-m)}^{(0)}\right)\right\} \\
&= HVSRS_{mn} + \frac{1}{n}\sum_{i=1}^{n}\log\frac{2}{\varepsilon_i} \\
&= HVSRS_{mn} + \frac{2m}{n}\log\frac{8}{5}
\end{aligned} \tag{15}
$$

Let $X_{(1:n)}^{(1)}, X_{(2:n)}^{(1)}, \ldots, X_{(n:n)}^{(1)}$ be a RSS of size n, Vasicek (1976) entropy estimator using RSS as considered by Mahdizadeh (2012) is given by

$$
HVRSS_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{2m}\left(X_{(i+m)}^{(1)} - X_{(i-m)}^{(1)}\right)\right\} \tag{16}
$$

Based on the RSS units $X_{(1:n)}^{(1)}$, $X_{(2:n)}^{(1)}$, …, $X_{(n:n)}^{(1)}$, the second suggested entropy estimator is

$$
SHERSS_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{\varepsilon_i m}\left(X_{(i+m)}^{(1)} - X_{(i-m)}^{(1)}\right)\right\} \tag{17}
$$

where ε_i is defined as in (14), and $X_{(i-m)}^{(1)} = X_{(1)}^{(1)}$ for $i \le m$ and $X_{(i+m)}^{(1)} = X_{(n)}^{(1)}$ for $i \ge n - m$. Comparing (16) with (17) to have

$$
\begin{aligned}
SHERSS_{mn} &= \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{\varepsilon_i m}\left(X_{(i+m)}^{(1)} - X_{(i-m)}^{(1)}\right)\right\} \\
&= HVRSS_{mn} + \frac{1}{n}\sum_{i=1}^{n}\mathrm{Log}\frac{2}{\varepsilon_i} \\
&= HVRSS_{mn} + \frac{2m}{n}\log\frac{8}{5}
\end{aligned} \tag{18}
$$

Assume that $X_{(1:n)}^{(2)}, X_{(2:n)}^{(2)}, \ldots, X_{(n:n)}^{(2)}$ is a DRSS sample of size n. The third suggested entropy estimator has the form

$$SHEDRSS_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{\varepsilon_i m}\left(X_{(i+m)}^{(2)} - X_{(i-m)}^{(2)}\right)\right\} \tag{19}$$

where ε_i is defined as in (14), and $X_{(i-m)}^{(2)} = X_{(1)}^{(2)}$ for $i \le m$ and $X_{(i+m)}^{(2)} = X_{(n)}^{(2)}$ for $i \ge n-m$. Based on DRSS method Mahdizadeh (2012) showed that Vasicek (1976) estimator will be

$$SHEDRSS_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{2m}\left(X_{(i+m)}^{(2)} - X_{(i-m)}^{(2)}\right)\right\} \tag{20}$$

Comparing (19) with (20) to get

$$\begin{aligned} SHEDRSS_{mn} &= \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{\varepsilon_i m}\left(X_{(i+m)}^{(2)} - X_{(i-m)}^{(2)}\right)\right\} \\ &= HVDRSS_{mn} + \frac{1}{n}\sum_{i=1}^{n}\log\frac{2}{\varepsilon_i} \\ &= HVDRSS_{mn} + \frac{2m}{n}\log\frac{8}{5} \end{aligned} \tag{21}$$

Remark 1: The entropy $H\left(f_n^{ME}\right)$ of an empirical maximum entropy density f_n^{ME} which is related to $HVSRS_{1n}$ and $SHESRS_{1n}$ can be computed following Theil (1980) as:

$$\begin{aligned} H\left(f_n^{ME}\right) &= HVSRS_{1n} + \frac{2-2\log 2}{n} \\ &= SHESRS_{1n} - \frac{2}{n}\log\frac{8}{5} + \frac{2-2\log 2}{n} \\ &= SHESRS_{1n} + \frac{2}{n}\left(1-\log\frac{4}{5}\right) \end{aligned} \tag{22}$$

Remark 2: If $n \to \infty$ in (22), then $H\left(f_n^{ME}\right) = SHESRS_{1n}$.

In the following two theorems, we compared the suggested estimators with Vasicek (1967) and Al-Omari (2014).

Theorem 1: The suggested estimators have the following properties:

a) Let $X_1^{(0)}, X_2^{(0)}, \ldots, X_n^{(0)}$ be SRS of size n, then $SHESRS_{mn} > HVSRS_{mn}$.

b) Let $X_{(1)}^{(1)}, X_{(2)}^{(1)}, \ldots, X_{(n)}^{(1)}$ be a RSS of size n, then $SHERSS_{mn} > HVRSS_{mn}$.

c) Let $X_{(1)}^{(2)}, X_{(2)}^{(2)}, \ldots, X_{(n)}^{(2)}$ be a DRSS of size n, then $SHEDRSS_{mn} > HVDRSS_{mn}$.

Proof: The proof of (a), (b), (c), is straightforward by using (15), (18), (21), respectively, where $\frac{2m}{n}\log\frac{8}{5} > 0$.

In the following theorem, we compare our suggested entropy estimators with their competitors in Al-Omari (2014).

Theorem 2: Based on the suggested estimators and Al-Omari (2014) entropy respectively, we have

$$SHEj_{mn} > AHEj_{mn},\ j = \text{SRS, RSS, DRSS}.$$

Proof: Compare (9) with (13) based on SRS to obtain

$$SHESRS_{mn} - AHESRS_{mn} = \frac{2m}{n}\log\frac{6}{5},$$

and since $\frac{2m}{n}\log\frac{6}{5} > 0$, then the case of SRS holds. Also, compare (11) with (17) based on RSS, and (12) with (19) using DRSS to complete the proof of this theorem.

The following theorem proves the consistency of the suggested estimators $SHESRS_{mn}$, $SHERSS_{mn}$, and $SHEDRSS_{mn}$.

Theorem 3: Let Ω be the class of continuous densities with finite entropies and let $X_1, X_2, \ldots, X_n$ be a random sample from $g \in \Omega$. If $n \to \infty$, $m \to \infty$, $m/n \to 0$, then $SHEj_{mn}$, (j = SRS, RSS, DRSS) converges in probability to $H[g(x)]$.

Proof: Based on the simple random sampling, from (15) we have

$$SHESRS_{mn} = HVSRS_{mn} + \frac{2m}{n}\log\frac{8}{5},$$

and Vasicek (1976) showed that $HVSRS_{mn}$ converges in probability to $H[g(x)]$ and since $\frac{2m}{n}\log\frac{8}{5}$ converges to zero as n goes to infinity, then we proved the case of the SRS. Follow the same approach and use (18) and (21) to prove the theorem for RSS and DRSS estimators, respectively.

Methodology

Simulation study

A simulation was conducted to investigate the performance of the suggested entropy estimators with Vasicek (1976) and Al-Omari (2014) entropy estimators using sampling methods considered in this study. The comparison is based on the root mean squared errors (RMSEs) and bias values of the estimators for 10000 samples generated from the uniform, exponential and the standard normal distributions using SRS, RSS and DRSS methods. The selection of the optimal values of the window size of m for a given value n is as yet an open problem in the entropy estimation. Therefore, we used the heuristic formula $m = \sqrt{n} + 0.5$ suggested by Wieczorkowski and Grzegorzewski (1999) to select m and to compute the RMSEs of entropy estimators. In this study, we considered the sample and window sizes as given in Table 1.

Table 1. The sample and window sizes considered in this simulation

Sample size	$n = 10$	$n = 20$	$n = 30$
Window size	$1 \le m \le 5$	$1 \le m \le 10$	$1 \le m \le 15$

Also, the performance of the RMSE of the suggested estimators for samples generated from the uniform, exponential and standard normal distributions is evaluated based on the quantity

$$Q_N = \frac{HVj_{mn} - N}{HVj_{mn}} \times 100,\ N = SHEj_{mn}, AHEj_{mn},\quad j = SRS, RSS, DRSS.$$

The results are summarized in Tables 2-6. Also, we compared the suggested estimators of entropy with their competitors suggested by Al-Omari (2014) and the results presented in Table 7 are taken from Al-Omari (2014).

Based on these results observe the following.

- The suggested entropy estimators using SRS, RSS and DRSS methods are more efficient than their competitors HV_{mn} based on the same method for all cases considered in this study. As an example, from Table 3, with $n = 10$ and $m = 3$ for the exponential distribution with $H[g(x)] = 1$ using RSS method, the RMSE and bias value of $SHERSS_{mn}$ are 0.230412 and -0.052759 compared to 0.401125 and -0.332760 the RMSE and bias of $HVRSS_{mn}$.
- The $SHEDRSS_{mn}$ is superior to the other suggested estimators, $SHERSS_{mn}$ and $SHESRS_{mn}$ under the uniform, exponential and normal distributions. From Table 1, consider the case of $n = 20$ and $m = 4$ under the uniform distribution when $H[g(x)] = 0$, it can be noted that the RMSE values of $SHEDRSS_{mn}$, $SHERSS_{mn}$, and $SHESRS_{mn}$ are 0.052373, 0.068747 and 0.114983, respectively.
- The nature of the underlying distribution as well as the value of $H[g(x)]$ affect on the efficiency of the estimator using the same method. As an example, the $Q_{SHERSS_{mn}}$ values with $n = 30$ and $m = 3$ for the uniform, exponential, and the standard normal distributions are 95.39025, 31.76442 and 32.75544, respectively. However, the values of $Q_{SHE_{mn}}$ for the uniform distribution with $H[g(x)] = 0$ are superior to their counterparts for the exponential and normal distributions.
- Finally, the suggested entropy estimators are found to be more efficient than their competitors in Al-Omari (2014) entropy estimators using SRS, RSS and DRSS schemes for the same window and sample sizes. For illustration, assume that $n = 30$ and $m = 8$ when the underlying distribution is the standard normal, from Table 4, the RMSE of $SHERSS_{mn}$ is 0.120242 compared to 0.157726 which is the RMSE of $AHERSS_{mn}$ as shown in Table 7.

Table 2. The Monte Carlo RMSEs and bias values of HV_{mn} and SHE_{mn} for the uniform distribution with $H[g(x)] = 0$.

		SRS					RSS				
		HV_{mn}		SHE_{mn}		$Q_{SHE_{mn}}$	HV_{mn}		SHE_{mn}		$Q_{SHE_{mn}}$
n	***m***	**Bias**	**RMSE**	**Bias**	**RMSE**		**Bias**	**RMSE**	**Bias**	**RMSE**	
10	1	-0.519826	0.569537	-0.430151	0.490404	13.89427	-0.396308	0.443439	-0.303703	0.361606	22.63043
	2	-0.415135	0.452358	-0.226627	0.290240	35.83843	-0.304078	0.329233	-0.116915	0.172961	90.35100
	3	-0.422613	0.453818	-0.135797	0.213148	53.03227	-0.327681	0.343991	-0.045891	0.114159	201.3262
	4	-0.458940	0.487054	-0.080015	0.179669	63.11107	-0.371538	0.383103	0.004574	0.093383	310.24920
	5	-0.502063	0.527918	-0.032713	0.167982	68.18029	-0.425903	0.436521	0.042936	0.105150	315.14120
20	1	-0.393900	0.418346	-0.349192	0.376728	9.94822	-0.343340	0.365754	-0.294874	0.320679	14.05611
	2	-0.271880	0.290818	-0.177492	0.204940	29.52981	-0.217937	0.233026	-0.125116	0.150017	55.33306
	3	-0.253931	0.270200	-0.112786	0.145519	46.14397	-0.205321	0.216879	-0.063859	0.093348	132.33380
	4	-0.260596	0.274678	-0.074069	0.114983	58.13898	-0.214042	0.222524	-0.026611	0.068747	223.68540
	5	-0.276800	0.288985	-0.043624	0.095299	67.02286	-0.235141	0.242179	0.000439	0.052744	359.15930
	6	-0.299321	0.310256	-0.017934	0.085705	72.37604	-0.258899	0.264554	0.022973	0.059480	344.77810
	7	-0.322084	0.332301	0.005663	0.082331	75.22397	-0.285310	0.290156	0.043299	0.067712	328.51490
	8	-0.348254	0.357901	0.028228	0.087902	75.43958	-0.314138	0.318471	0.061191	0.081194	292.23460
	9	-0.374620	0.383864	0.048022	0.097710	74.54567	-0.343410	0.347711	0.079914	0.096721	259.49900
	10	-0.402840	0.411741	0.066866	0.108377	73.67836	-0.371780	0.375737	0.097578	0.112133	235.08160
30	1	-0.352853	0.368369	-0.323835	0.340961	7.44037	-0.319230	0.333509	0.288992	0.305176	9.28415
	2	-0.223356	0.235685	-0.161288	0.178121	24.42412	-0.190866	0.201625	-0.127419	0.142794	41.19991
	3	-0.197719	0.208362	-0.104892	0.124359	40.31589	-0.165182	0.173360	-0.070574	0.088725	95.39025
	4	-0.196240	0.205882	-0.071025	0.093814	54.43312	-0.162899	0.169841	-0.038020	0.061304	177.04720
	5	-0.202003	0.210395	-0.046135	0.075603	64.06616	-0.172441	0.178293	-0.014997	0.046725	281.57950
	6	-0.213804	0.221385	-0.024700	0.063205	71.45019	-0.185622	0.190458	0.002250	0.043550	337.33180
	7	-0.226688	0.233521	-0.007941	0.057695	75.29344	-0.200036	0.204048	0.018588	0.045106	352.37440
	8	-0.242599	0.248992	0.007775	0.057090	77.07155	-0.217704	0.221309	0.033174	0.051831	326.98190
	9	-0.259471	0.265356	0.022036	0.060359	77.25358	-0.235661	0.238850	0.046793	0.060639	293.88840

Table 2 continued on next page

	10	-0.276934	0.282548	0.036215	0.067383	76.15167	-0.254437	0.257257	0.058627	0.069646	269.37800
	11	-0.295302	0.300725	0.049094	0.074862	75.10616	-0.273700	0.276336	0.072000	0.081003	241.14290
	12	-0.313803	0.319255	0.062218	0.085295	73.28311	-0.293398	0.295911	0.083363	0.091704	222.68060
	13	-0.332279	0.337432	0.075374	0.095536	71.68733	-0.311978	0.341101	0.095165	0.102770	231.90720
	14	-0.351090	0.356205	0.087783	0.106535	70.09166	-0.332096	0.334518	0.106272	0.113446	194.86980
	15	-0.370555	0.375518	0.099545	0.116477	68.98231	-0.352077	0.354327	0.118516	0.125081	183.27800

Table 3. The Monte Carlo RMSEs and bias values of HV_{mn} and SHE_{mn} for the exponential distribution with $H[g(x)] = 1$.

		SRS					RSS				
		HV_{mn}		SHE_{mn}		$Q_{SHE_{mn}}$	HV_{mn}		SHE_{mn}		$Q_{SHE_{mn}}$
n	***m***	**Bias**	**RMSE**	**Bias**	**RMSE**		**Bias**	**RMSE**	**Bias**	**RMSE**	
10	1	-0.552032	0.677001	-0.457584	0.600041	11.36778	-0.430553	0.505229	-0.342184	0.432785	14.33884
	2	-0.442683	0.571820	-0.253108	0.442568	22.60362	-0.337494	0.404667	-0.148595	0.269907	33.30146
	3	-0.435444	0.561640	-0.154607	0.391369	30.31675	-0.332760	0.401125	-0.052759	0.230412	42.55855
	4	-0.451545	0.575390	-0.076188	0.371210	35.48550	-0.348029	0.420617	0.025378	0.233566	44.47062
	5	-0.469437	0.597761	0.005489	0.372418	37.69784	-0.366628	0.445977	0.101893	0.270512	39.34396
20	1	-0.414064	0.490107	-0.360711	0.445976	9.00436	-0.357765	0.398661	-0.312513	0.358752	10.01076
	2	-0.285717	0.376086	-0.193143	0.310495	17.44043	-0.234959	0.280262	-0.140851	0.207405	25.99603
	3	-0.260773	0.351341	-0.122104	0.272095	22.55530	-0.213397	0.261261	-0.072871	0.165700	36.57683
	4	-0.256116	0.352810	-0.067569	0.251502	28.71461	-0.210620	0.259248	-0.017564	0.152350	41.23388
	5	-0.262412	0.358638	-0.022414	0.244018	31.95980	-0.214122	0.265246	0.022190	0.156584	40.96650
	6	-0.265650	0.360325	0.016823	0.248330	31.08166	-0.218028	0.272315	0.061287	0.174543	35.90401
	7	-0.266934	0.365008	0.055461	0.256349	29.76894	-0.224596	0.282196	0.103601	0.200858	28.82323
	8	-0.273952	0.377519	0.100674	0.274582	27.26671	-0.232629	0.293062	0.145963	0.231970	20.84610
	9	-0.280123	0.381968	0.143573	0.293999	23.03046	-0.236125	0.302083	0.188596	0.267430	11.47135
	10	-0.285183	0.391290	0.179545	0.322338	17.62171	-0.238413	0.310922	0.231203	0.303760	2.30347
30	1	-0.367058	0.423423	-0.332016	0.394742	6.77360	-0.332526	0.361491	-0.303272	0.334033	7.59576

Table 3 continued on next page

2	-0.233677	0.306086	-0.173511	0.262016	14.39791	-0.203455	0.236001	-0.137679	0.182964	22.47321
3	-0.202277	0.281503	-0.108684	0.223191	20.71452	-0.170859	0.207468	-0.078000	0.141567	31.76442
4	-0.194424	0.275072	-0.067472	0.207505	24.56339	-0.160246	0.199410	-0.036059	0.123278	38.17863
5	-0.191705	0.272356	-0.033792	0.197718	27.40457	-0.159714	0.200465	-0.002510	0.122595	38.84469
6	-0.186870	0.272196	0.000772	0.195841	28.05148	-0.158702	0.202869	0.027994	0.128086	36.86270
7	-0.191094	0.275374	0.029066	0.198154	28.04186	-0.161705	0.206226	0.059517	0.141042	31.60804
8	-0.195662	0.280589	0.056849	0.208607	25.65389	-0.164468	0.212265	0.085540	0.160732	24.27767
9	-0.196983	0.282040	0.088082	0.220610	21.78060	-0.165511	0.217222	0.115128	0.182796	15.84830
10	-0.197171	0.283394	0.115949	0.235447	16.91885	-0.167152	0.220237	0.144441	0.205632	6.63149
11	-0.198853	0.286241	0.142656	0.253233	11.53154	-0.173076	0.229318	0.172966	0.220033	4.04896
12	-0.204089	0.293653	0.171742	0.274080	6.66535	-0.171555	0.232740	0.200259	0.214615	7.78766
13	-0.202908	0.298108	0.204980	0.228389	23.38717	-0.176996	0.240454	0.231487	0.232102	3.47343
14	-0.205700	0.300842	0.232277	0.290007	3.60156	-0.176922	0.244541	0.262425	0.211142	13.65780
15	-0.210699	0.305809	0.258234	0.300011	1.89595	-0.177959	0.248760	0.291253	0.239115	3.87723

Table 4. The Monte Carlo RMSEs and bias values of HV_{mn} and SHE_{mn} for the standard normal distribution and $H[g(x)] = 1.419$.

		SRS					RSS				
		HV_{mn}		SHE_{mn}		$Q_{SHE_{mn}}$	HV_{mn}		SHE_{mn}		$Q_{SHE_{mn}}$
n	***m***	**Bias**	**RMSE**	**Bias**	**RMSE**		**Bias**	**RMSE**	**Bias**	**RMSE**	
10	1	-0.598925	0.676499	-0.499469	0.434171	35.8208955	-0.484489	0.549750	-0.388446	0.466743	15.09905
	2	-0.521455	0.591007	-0.335907	0.436633	26.1205028	-0.422169	0.471157	-0.238609	0.320258	32.02733
	3	-0.563002	0.623188	-0.275063	0.382983	38.5445484	-0.462240	0.504378	-0.181597	0.269765	46.51531
	4	-0.610651	0.663364	-0.236072	0.351842	46.9609445	-0.523019	0.557792	-0.149270	0.244690	56.13239
	5	-0.671777	0.719069	-0.200702	0.325688	54.7069892	-0.584483	0.614209	-0.111978	0.218489	64.42758
20	1	-0.435480	0.483459	-0.380981	0.434171	10.1948666	-0.382986	0.420310	-0.335512	0.377639	10.15227
	2	-0.327145	0.375798	-0.231087	0.296133	21.1988888	-0.275716	0.313472	-0.182040	0.234712	25.12505
	3	-0.317948	0.364927	-0.175301	0.251511	31.0790925	-0.268657	0.304811	-0.125104	0.189103	37.96057

Table 4 continued on next page

	4	-0.327070	0.372436	-0.143556	0.230357	38.1485678	-0.285331	0.318855	-0.098619	0.172598	45.86944
	5	-0.352658	0.395796	-0.117332	0.215233	45.6202185	-0.305555	0.337744	-0.073404	0.160748	52.40537
	6	0.375996	0.416964	-0.098719	0.204234	51.0187930	-0.335066	0.365185	-0.051912	0.152608	58.21077
	7	-0.404050	0.442997	-0.083445	0.199295	55.0121107	-0.363782	0.391748	-0.036080	0.148138	62.18538
	8	-0.439618	0.475094	-0.061765	0.187822	60.4663498	-0.395221	0.421583	-0.020165	0.147835	64.93336
	9	-0.467134	0.500777	-0.043230	0.186628	62.7323140	-0.428042	0.451680	-0.006860	0.144519	68.00412
	10	-0.496926	0.527456	-0.029603	0.178984	66.0665534	-0.454818	0.477152	0.009882	0.145955	69.41121
30	1	-0.378860	0.413455	-0.346828	0.384885	6.91006276	-0.343626	0.370512	-0.313688	0.342854	7.464805
	2	-0.259105	0.299687	-0.196988	0.246877	17.6217187	-0.226914	0.255947	-0.163491	0.201857	21.13328
	3	-0.236758	0.277238	-0.145212	0.203905	26.4512801	-0.204698	0.234358	-0.108571	0.157593	32.75544
	4	-0.234369	0.275867	-0.108651	0.179817	34.8175026	-0.204765	0.234413	-0.081230	0.140863	39.90820
	5	-0.244288	0.283027	-0.088572	0.166051	41.3303324	-0.214434	0.243683	-0.056181	0.127184	47.80760
	6	-0.255248	0.293332	-0.068084	0.157937	46.1575962	-0.227340	0.255901	-0.038603	0.122294	52.21043
	7	-0.269724	0.305134	-0.048333	0.151084	50.4860160	-0.241325	0.268228	-0.021655	0.120957	54.90516
	8	-0.285713	0.321039	-0.036608	0.151194	52.9047873	-0.254983	0.282376	-0.008427	0.120242	57.41777
	9	-0.304064	0.337563	-0.020683	0.147718	56.2398723	-0.274697	0.301420	0.010331	0.123468	59.03789
	10	-0.320051	0.352764	-0.009717	0.148068	58.0263292	-0.295057	0.319933	0.018501	0.125482	60.77866
	11	-0.339131	0.369866	0.005731	0.147483	60.1252886	-0.314201	0.339141	0.030498	0.129224	61.89667
	12	-0.361226	0.392070	0.016315	0.149674	61.8246742	-0.333173	0.356224	0.042772	0.133458	62.53537
	13	-0.382347	0.410463	0.027129	0.152493	62.8485393	-0.353582	0.375170	0.053690	0.138200	63.16337
	14	-0.400618	0.428008	0.039711	0.155154	63.7497430	-0.375752	0.397462	0.064272	0.140967	64.53321
	15	-0.423597	0.449968	0.048426	0.156576	65.2028590	-0.394363	0.414605	0.072957	0.147206	64.49488

Table 5. The Monte Carlo RMSEs and bias values of HV_{mn} and SHE_{mn} for the uniform distribution with $H[g(x)] = 0$ and exponential distribution with $H[g(x)] = 1$ using DRSS.

		SRS					RSS				
		HV_{mn}		SHE_{mn}		$Q_{SHE_{mn}}$	HV_{mn}		SHE_{mn}		$Q_{SHE_{mn}}$
n	***m***	**Bias**	**RMSE**	**Bias**	**RMSE**		**Bias**	**RMSE**	**Bias**	**RMSE**	

Table 5 continued on next page

10	1	-0.327408	0.369593	-0.230787	0.285326	22.7999448	-0.365854	0.425279	-0.267318	0.345821	18.68373
	2	-0.260621	0.278731	-0.071592	0.121826	56.2926262	-0.288898	0.340618	-0.101687	0.207273	39.14796
	3	-0.296104	0.306116	-0.014117	0.078474	74.3646199	-0.300393	0.351750	-0.018027	0.181245	48.47335
	4	-0.346305	0.352712	0.029482	0.073990	79.0225453	-0.322839	0.377437	0.056521	0.201436	46.63056
	5	-0.404121	0.409902	0.065862	0.095121	76.7942093	-0.335248	0.399189	0.134718	0.252269	36.80462
20	1	-0.308453	0.329353	-0.260588	0.285042	13.4539537	-0.329105	0.363241	-0.278366	0.317530	12.58421
	2	-0.189231	0.202666	-0.095093	0.119561	41.0058915	-0.204908	0.240316	-0.112945	0.168444	29.90729
	3	-0.182095	0.191163	-0.041993	0.071976	62.3483624	-0.191216	0.228320	-0.050530	0.133863	41.37044
	4	-0.197693	0.204342	-0.010391	0.052373	74.3699288	-0.190904	0.229986	-0.003685	0.126728	44.89752
	5	-0.220876	0.225845	0.012711	0.049477	78.0924971	-0.197900	0.239789	0.036502	0.139896	41.65871
	6	-0.247733	0.251580	0.035133	0.056178	77.6699261	-0.207032	0.251002	0.078413	0.161731	35.56585
	7	-0.275808	0.278919	0.053697	0.068101	75.5839509	-0.209883	0.258152	0.118656	0.192217	25.54115
	8	-0.303823	0.306608	0.071232	0.082285	73.1628007	-0.218701	0.271560	0.158069	0.224230	17.42893
	9	-0.333903	0.336495	0.089491	0.098489	70.7309172	-0.223692	0.278728	0.200103	0.262984	5.648518
	10	-0.363272	0.365731	0.106408	0.114566	68.6747910	-0.228126	0.290431	0.244783	0.283888	2.252859
30	1	-0.298092	0.312767	-0.267592	0.283216	9.44824742	-0.308011	0.331033	-0.278838	0.304383	8.050557
	2	-0.170745	0.180210	-0.107748	0.122162	32.2113090	-0.182416	0.207785	-0.118447	0.154790	25.50473
	3	-0.146113	0.153646	-0.052193	0.070391	54.1862463	-0.152039	0.180708	-0.059074	0.114805	36.46933
	4	-0.149143	0.154886	-0.023125	0.047458	69.3593998	-0.145325	0.176699	-0.019990	0.102139	42.19605
	5	-0.159888	0.164564	-0.003052	0.038571	76.5617024	-0.146632	0.179028	0.009230	0.105307	41.17847
	6	-0.174419	0.178204	0.013102	0.038421	78.4398779	-0.149443	0.184598	0.038407	0.115953	37.18621
	7	-0.191854	0.194940	0.027534	0.046606	76.0921309	-0.150245	0.188158	0.068588	0.133307	29.15156
	8	-0.209886	0.212509	0.040817	0.052754	75.1756396	-0.153441	0.194332	0.095598	0.152215	21.67270
	9	-0.229010	0.231261	0.052824	0.061955	73.2099230	-0.157250	0.199936	0.123844	0.175122	12.41097
	10	-0.248006	0.249993	0.065446	0.072283	71.0859904	-0.162854	0.208891	0.151295	0.198703	4.877185
	11	-0.267506	0.269188	0.077163	0.082922	69.1955065	-0.163540	0.213175	0.182129	0.207543	2.641961
	12	-0.287408	0.289018	0.088169	0.093391	67.6867877	-0.167660	0.221482	0.207757	0.202062	8.768207
	13	-0.307160	0.308699	0.100118	0.104801	66.0507485	-0.171024	0.225764	0.239466	0.211883	6.148456
	14	-0.327370	0.328890	0.111085	0.115458	64.8946456	-0.170880	0.232977	0.268159	0.210502	9.646875
	15	-0.346997	0.348439	0.122960	0.126985	63.5560313	-0.169873	0.235173	0.299068	0.210721	10.397450

Table 6. The Monte Carlo RMSEs and bias values of HV_{mn} and SHE_{mn} for the standard normal distribution and $H[g(x)] = 1.419$.

		HV_{mn}		SHE_{mn}		$Q_{SHE_{mn}}$
n	**m**	**Bias**	**RMSE**	**Bias**	**RMSE**	
10	1	-0.415021	0.472162	-0.316672	0.385139	18.43075
	2	-0.373395	0.412666	-0.186378	0.256423	37.86185
	3	-0.427401	0.459119	-0.143329	0.218981	52.30409
	4	-0.492911	0.518275	-0.115918	0.202153	60.99503
	5	-0.554351	0.577281	-0.084253	0.181100	68.62880
20	1	-0.350703	0.383160	-0.303014	0.340790	11.05804
	2	-0.245907	0.277809	-0.152363	0.200155	27.95230
	3	-0.246496	0.276941	-0.104439	0.162172	41.44168
	4	-0.262789	0.290545	-0.078826	0.147712	49.16037
	5	-0.291340	0.317967	-0.055774	0.138687	56.38321
	6	-0.316105	0.341597	-0.037661	0.134214	60.70984
	7	-0.349246	0.373132	-0.021199	0.132559	64.47397
	8	-0.384526	0.406764	-0.008681	0.134158	67.01822
	9	-0.416151	0.436696	0.006082	0.132054	69.76066
	10	-0.445901	0.465518	0.023744	0.134764	71.05074
30	1	-0.321940	0.345223	-0.292331	0.318084	7.861300
	2	-0.206709	0.231560	-0.143028	0.177006	23.55934
	3	-0.187163	0.212774	-0.094482	0.138090	35.10015
	4	-0.190073	0.215577	-0.066854	0.122350	43.24534
	5	-0.199843	0.224569	-0.044224	0.111818	50.20773
	6	-0.214636	0.239021	-0.025579	0.108667	54.53663
	7	-0.231613	0.255278	-0.012061	0.108224	57.60543
	8	-0.247340	0.271084	0.001734	0.109348	59.66269
	9	-0.268298	0.291044	0.014961	0.113895	60.86674
	10	-0.286538	0.308661	0.027278	0.118811	61.50761
	11	-0.305310	0.326485	0.040250	0.123778	62.08769
	12	-0.324892	0.346062	0.051274	0.129747	62.50759
	13	-0.343097	0.363236	0.061548	0.135452	62.70964
	14	-0.369990	0.388586	0.070900	0.140756	63.77739
	15	-0.387740	0.406081	0.080947	0.145418	64.18990

Table 7. The Monte Carlo RMSEs and bias values of $AHEj_{mn}$, j = SRS, RSS, DRSS (Al-Omari, 2014).

		$AHESRS_{mn}$		Q_{AHESRS}	$AHERSS_{mn}$		Q_{AHERSS}	$AHEDRSS_{mn}$		$Q_{AHEDRSS}$
n	***m***	**Bias**	**RMSE**		**Bias**	**RMSE**		**Bias**	**RMSE**	
Uniform distribution with $H[g(x)] = 0$										
10	2	-0.298609	0.350332	22.554260	-0.189664	0.228762	30.516686	-0.145388	0.176159	36.799638
	3	-0.249056	0.298944	34.126897	-0.154894	0.186380	45.818350	-0.122180	0.144286	52.865580
20	4	-0.144016	0.167779	38.917933	-0.100304	0.118284	46.844385	-0.082268	0.096978	52.541328
	5	-0.133179	0.157805	45.393360	-0.091608	0.108584	55.163743	-0.077708	0.091093	59.665700
30	7	-0.092957	0.109089	53.285144	-0.066053	0.077716	61.912883	-0.058041	0.067650	65.297015
	8	-0.089259	0.105818	57.501446	-0.064713	0.076188	65.573926	-0.056421	0.065369	69.239420
Exponential distribution with $H[g(x)] = 1$										
10	2	-0.323532	0.483573	15.432654	-0.220406	0.315220	22.103853	-0.173991	0.251460	26.175364
	3	-0.265713	0.443276	21.074710	-0.159787	0.276197	31.144406	-0.128545	0.223802	36.374698
20	4	0.141143	0.279706	20.720501	-0.098056	0.179990	30.572271	-0.075338	0.179771	21.833938
	5	0.118697	0.271887	24.189015	-0.072456	0.172661	34.905333	-0.052175	0.145269	39.417988
30	7	-0.058550	0.205261	25.461009	-0.027194	0.130283	36.825134	-0.046556	0.115023	38.868929
	8	-0.036080	0.200329	28.604115	-0.010631	0.136358	35.760488	-0.001239	0.120306	38.092543
Standard normal distribution with $H[g(x)] = 1.419$										
10	2	-0.409842	0.496627	15.969354	-0.308706	0.375690	20.262250	-0.262149	0.316029	23.417728
	3	-0.386562	0.468471	24.826698	-0.291133	0.353844	29.845470	-0.254450	0.303820	33.825435
20	4	-0.214227	0.279269	25.015573	-0.168035	0.219922	31.027583	-0.148107	0.194728	32.978368
	5	-0.205782	0.272804	31.074594	-0.160392	0.213700	36.727225	-0.145734	0.191755	39.693427
30	7	-0.132038	0.196792	35.506368	-0.105796	0.158654	40.851067	-0.095517	0.143483	43.793433
	8	-0.129915	0.193509	39.724146	-0.102504	0.157726	44.143270	-0.094560	0.145579	46.297458

Conclusion

Three entropy estimators are suggested using SRS, RSS, and DRSS methods. The consistency of these estimators is proved as well as some properties are reported. Based on theoretical and numerical comparisons the suggested entropy estimators are more efficient than Vasicek (1976) and Al-Omari (2014) entropy estimators. However, the suggested estimators of entropy in this paper can be extended by considering other sampling methods such as the multistage RSS and median RSS methods.

Acknowledgements

The author thanks the referees for their helpful and valuable comments that substantially improved this paper.

References

Al-Omari, A. I. (2014). Estimation of entropy using random sampling. *Journal of Computation and Applied Mathematics*, *261*, 95-102. doi:10.1016/j.cam.2013.10.047

Al-Saleh, M. F. & Al-Kadiri, M. A. (2000). Double ranked set sampling. *Statistics and Probability Letters*, *48*(2), 205-212. doi:10.1016/S0167-7152(99)00206-0

Choi, B. (2008). Improvement of goodness of fit test for normal distribution based on entropy and power comparison. *Journal of Statistical Computation and Simulation*, *78*(9), 781-788. doi:10.1080/00949650701299451

Choi, B., Kim, K., & Song, S. H. (2004). Goodness of fit test for exponentiality based on Kullback-Leibler information. *Communication in Statistics-Simulation and Computation*, *33*(2), 525–536. doi:10.1081/SAC-120037250

Goria, M. N., Leonenko, N. N., Mergel, V. V., & Novi Inverardi, P. L. (2005). A new class of random vector entropy estimators and its applications in testing statistical hypotheses. *Journal of Nonparametric Statistics*, *17*(3), 277-297. doi:10.1080/104852504200026815

Correa, J. C. (1995). A new estimator of entropy. *Communication in Statistics-Theory Methods*, *24*(10), 2439-2449. doi:10.1080/03610929508831626

Ebrahimi, N., Pflughoeft, K., & Soofi, E. S. (1994). Two measures of sample entropy. *Statistics & Probability Letters*, *20*(3), 225-234. doi:10.1016/0167-7152(94)90046-9

Mahdizadeh, M. (2012). On the use of ranked set samples in entropy based test of fit for the Laplace distribution. *Revista Colombiana de Estadística*, *35*(3), 443-455.

McIntyre, G. A. (1952). A method for unbiased selective sampling using ranked sets. *Australian Journal of Agricultural Research*, *3*(4), 385-390. doi:10.1071/AR9520385

Noughabi, H. A. & Noughabi, R. A. (2013). On the entropy estimators. *Journal of Statistical Computation and Simulation*, *83*(4), 784-792. doi:10.1080/00949655.2011.637039

Noughabi, H. A. & Arghami, N. R. (2010). A new estimator of entropy. *Journal of the Iranian Statistical Society*, *9*(1), 53-64.

Park, S. & Park, D. (2003). Correcting moments for goodness of fit tests based on two entropy estimates. *Journal of Statistical Computation and Simulation*, *73*(9), 685-694. doi:10.1080/0094965031000070367

Shannon, C. E. (1948a). A mathematical theory of communications. *Bell System Technical Journal 27*(3), 379-423. doi:10.1002/j.1538-7305.1948.tb01338.x

Shannon, C. E. (1948b). A mathematical theory of communications. *Bell System Technical Journal 27*(4), 623-656. doi:10.1002/j.1538-7305.1948.tb00917.x

Takahasi, K. & Wakimoto, K. (1968). On the unbiased estimates of the population mean based on the sample stratified by means of ordering. *Annals of the Institute of Statistical Mathematics*, *20*(1), 1-31. doi:10.1007/BF02911622

Theil, J. (1980). The entropy of maximum entropy distribution. *Economics Letters*, *5*(2), 145–148. doi:10.1016/0165-1765(80)90089-0

Van Es, B. (1992). Estimating functionals related to a density by class of statistics based on spacings. *Scandinavian Journal of Statistics*, *19*(1), 61-72.

Vasicek, O. (1976). A test for normality based on sample entropy. *Journal of the Royal Statistical Society, B*, *38*, 54-59.

Wieczorkowski, R. & Grzegorzewsky, P. (1999). Entropy estimators - improvements and comparisons. *Communication in Statistics-Simulation and Computation*, *28*(2), 541-567. doi:10.1080/03610919908813564