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Discrete Littlewood-Paley-Stein Theory And Wolff Potentials On Homogeneous Spaces And Multi-Parameter Hardy Spaces

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DISCRETE LITTLEWOOD-PALEY-STEIN THEORY
AND WOLFF POTENTIALS ON HOMOGENEOUS SPACES
AND MULTI-PARAMETER HARDY SPACES

by

YAYUAN XIAO

DISSERTATION

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of Wayne State University,

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Approved by:

Advisor

Date

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DEDICATION

To my family

Xiang Xiao, Shouren Long, and Jie Long
ACKNOWLEDGEMENTS

Over the past five years, I have received help, support and encouragement from many people.

First, I am deeply indebted to my advisor Professor Guozhen Lu. It has been my great honor and pleasure to work with you. I have always been inspired and motivated by your unsurpassed knowledge and your passion for research. You ongoing advice has been invaluable on both an academic and a personal level. Without your excellent guidance, persistent help, unwavering support and encouragement, this dissertation would not have been possible. I believe what I learned from you will continue to benefit me throughout my life.

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you were my surrogate family when I broke my leg.

Finally, I would like to dedicate this dissertation to my dearest mother Xiang Xiao, my father Shouren Long, and my sister Jie Long. You have given me unconditional and unlimited love and have stood by me through the good and the bad times.
PREFACE

This dissertation consists of two parts:

• **Part I** We establish a new atomic decomposition of the multi-parameter Hardy spaces of homogeneous type and obtain the associated $H^p - L^p$ and $H^p - H^p$ boundedness criterions for singular integral operators. On the other hand, we compare the Wolff and Riesz potentials on spaces of homogenous type, followed by a Hardy-Littlewood-Sobolev type inequality. Then we drive integrability estimates of positive solutions to the Lane-Emden type integral systems on spaces of homogeneous type.

• **Part II** We establish a $(p, 2)$-atomic decomposition of the Hardy space associated with different homogeneities for $0 < p \leq 1$. In addition, we characterize the dual spaces of the weighted multi-parameter Hardy spaces associated with Zygmund dilations, i. e. $(H^p_2(w))^* = CMO^p_2(w)$ for $w \in A_\infty(\mathcal{Z})$ and $0 < p \leq 1$. 
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Part II

4 The atomic decomposition of Hardy spaces associated with different homogeneities

4.1 Introduction and statements of main results

4.2 Proof of the atomic decomposition of $H^p_{com}(\mathbb{R}^m)$

5 The duality theorem of multi-parameter Hardy spaces associated with Zygmund dilation

5.1 Introduction and statements of main results

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5.4 Proof of the main duality theorem of $H^p_Z(w)$

References

Abstract

Autobiographical Statement
1 Introduction

1.1 Background and main questions

For a set $\mathcal{X}$, we say that a function $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a quasi-metric on $\mathcal{X}$ if it satisfies

(i) $\rho(x_1, x_2) = \rho(x_2, x_1)$;

(ii) $\rho(x_1, x_2) = 0$ if and only if $x_1 = x_2$;

(iii) $\rho(x_1, x_2) \leq C[\rho(x_1, x_3) + \rho(x_2, x_3)]$, where $C \in [1, \infty)$ is a constant independent of $x_1, x_2$ and $x_3$.

Let $B(x, t)$ denote the ball $\{y \in \mathcal{X} : \rho(x, y) < t\}$ for all $x \in \mathcal{X}$ and $t > 0$, then such quasi-metric $\rho$ defines a topology on $\mathcal{X}$, for which the balls $B(x, t)$ form a basis. Let $\mu$ be a nonnegative measure satisfying the doubling property, i.e.,

(iv) for all $t > 0$, there exists some constant $C$ such that $\mu(B(x, 2t)) \leq C\mu(B(x, t))$;

then the set $\mathcal{X}$ together with a quasi-metric $\rho$ and a nonnegative doubling measure $\mu$ on $\mathcal{X}$, $(\mathcal{X}, \rho, \mu)$, is called a space of homogeneous type, which was first introduced by R. Coifman and G. Weiss in [CW1] in order to extend the theory of Calderón-Zygmund singular integrals on $\mathbb{R}^n$ to a more general setting.

The project of developing theory of spaces of homogeneous type has received much attention due to its own difficulty, interest and applications. It has developed in significantly
in the past four decades, there are many monographs and surveys available in the literature, among them we mention [ABI], [Cm], [CW2], [DH], [FS], [N].

On the other hand, The Hardy spaces $H^p$ are important objects in classical harmonic analysis. For $p = 1$ (or $p = \infty$), the Hardy space $H^1$ (or its dual space) appears as a natural substitute of the classical Lebesgue space $L^1$ (or $L^\infty$). For $1 < p < \infty$, $\|f\|_{H^p} \sim \|f\|_p$ is well known as the Littlewood-Paley-Stein theory which implies that $H^p = L^p$. For $0 < p < 1$, while $L^p$ have some undesirable properties, the $H^p$ are much better behaved.

One of the principal interests of $H^p$ theory is that it gives a natural extension of the results for singular integrals (originally developed for $L^p$, $p > 1$) to $0 < p \leq 1$. Broadly speaking, the $L^p (1 < p < \infty)$ boundedness theorems for singular integrals may be extended to $H^p$ for all $0 < p \leq 1$. Therefore, one part of our research focus on using the discrete Littlewood-Paley theory to study the boundedness of singular integral operators on Hardy spaces of homogeneous type.

In Chapter 2, we first introduce the multi-parameter Hardy space of homogeneous type $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$. By using Journe’s covering lemma for spaces of homogeneous type, we derive a new atomic decomposition of $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ which converges in both the classical Lebesgue spaces $L^q$ (for $1 < q < \infty$) and Hardy spaces $H^p$ (for $0 < p \leq 1$). As an application, we prove boundedness criterions of operators from $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ to $L^p(\mathcal{X}_1 \times \mathcal{X}_2)$ and from $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ to itself for $0 < p \leq 1$.

In Chapter 3, we get a Hardy-Littlewood-Sobolev type inequality on space of homogeneous type by comparing the associated Wolff and Riesz potentials. After that, by using the regularity lifting method, we derive integrability estimates of positive solutions to Lane-Emden type integral system on spaces of homogeneous type.
At last, we would like to point it out that the spaces of homogeneous type include the classical Euclidean space $\mathbb{R}^n$, compact Lie groups, $C^\infty$ manifolds with doubling volume measures for geodesic balls, Carnot-Caratheodory spaces, nilpotent Lie groups such as the Heisenberg group, and many other cases, so all the above results can be applied to these cases.

1.2 Some properties of spaces of homogeneous type

For any space of homogeneous type $(X, \rho, \mu)$, R. A. Macias and C. Segovia [MS] have proved that the quasi-metric $\rho$ can be replaced by another $\rho^*$ such that $\rho^* \sim \rho$ and $\rho^*$ yields the same topology on $X$ as $\rho$. Moreover, let $B(x, t)$ denote the ball defined by $\rho^*$, $\{y \in X : \rho^*(x, y) \leq t\}$, then for all $0 < t < \infty$, there exists some $d > 0$ such that

$$\mu(B(x, t)) \sim t^d,$$

and there exists a constant $A > 0$ such that $\rho^*$ has the following regularity property

$$|\rho^*(x_1, x_2) - \rho^*(x_3, x_2)| \leq A\rho^*(x_1, x_3)^\theta[\rho^*(x_1, x_2) + \rho^*(x_3, x_2)]^{1-\theta}$$

for some regularity exponent $\theta \in (0, 1)$ and all $x_1, x_2, x_3 \in X$.

Therefore, a formal definition of homogeneous space in the sense of R. Coifman and G. Wiess can be given as follows.

**Definition 1.1.** Let $d > 0$ and $\theta \in (0, 1)$. A space of homogeneous type $(X, \rho, \mu)_{d, \theta}$ is a set $X$ equipped with a quasi-metric $\rho$ and a nonnegative measure $\mu$ on $X$, and there exists a
constant $A > 0$ such that for all $x_1, x_2, x_3 \in \mathcal{X}$ and $0 < t < \text{diam} \mathcal{X},$

$$\mu(B(x, t)) \sim t^d$$

(1.1)

and

$$|\rho(x_1, x_2) - \rho(x_3, x_2)| \leq A\rho(x_1, x_3)^{\theta}[\rho(x_1, x_2) + \rho(x_3, x_2)]^{1-\theta}. \quad (1.2)$$

1.3 Dyadic cubes on spaces of homogeneous type

For spaces of homogeneous type, an analogue of the grid of Euclidean dyadic cubes was given independently by M. Christ [Cm] and E. Sawyer and R. Wheeden [SW] as follows.

Lemma 1.2 (Dyadic cubes on homogeneous spaces [SW]). Let $(\mathcal{X}, \rho, \mu)_{d, \theta}$ be a space of homogeneous type. Then for every integer $k \in \mathbb{Z}_+$, there exists a collection of open subsets

$$\{Q^k_{\tau} \subseteq \mathcal{X} : \tau \in I_k\},$$

where $I_k$ denotes some index set depending on $k$, and positive constants $C_1, C_2$ such that

(i) $\mu(\{\mathcal{X} \setminus \cup Q^k_{\tau}\}) = 0$;

(ii) If $l \geq k$, then for all $\tau' \in I_l$ and $\tau \in I_k$ either $Q^l_{\tau'} \subseteq Q^k_{\tau}$ or $Q^l_{\tau'} \cap Q^k_{\tau} = \emptyset$;

(iii) If $l < k$, for each $\tau \in I_k$, there is a unique $\tau' \in I_l$ such that $Q^k_{\tau} \subseteq Q^l_{\tau'}$, $\text{diam}(Q^k_{\tau}) \leq C_12^{-k}$, and each $Q^k_{\tau}$ contains some ball $B(z^k_{\tau}, C_22^{-k})$.

With the settings defined above, in the following, we say that a cube $Q \subseteq \mathcal{X}$ is a dyadic cube if $Q = Q^k_{\tau}$ for some $k \in \mathbb{Z}_+$ and $\tau \in I_k$, denote it by $\text{diam}(Q) \sim 2^{-k}$. 
1.4 Notational index

The following notations will be frequently used in the rest of Part I.

- $\rho$: a quasi-metric.

- $\mu$: a nonnegative doubling measure.

- $(X, \rho, \mu)$ or $(X, \rho, \mu)_{d, \theta}$: a space of homogeneous type.

- $B(x, t)$: the ball centered at $x$ and of radius $t$.

- $\Omega$: an open and bounded domain in $\mathbb{R}^n$ or in $X$.

- $\mathcal{M}(\Omega)$: the set of all maximal dyadic rectangles contained in $\Omega$.

- $\mathcal{M}_i(\Omega)$: the set of all dyadic rectangles contained in $\Omega$ and maximal in the direction of $x_i$.

- $M_s f$: the strong maximal function of $f$.

- $q'$: the conjugate index of the index $1 \leq q \leq \infty$, that is, $\frac{1}{q} + \frac{1}{q'} = 1$.

- $Q^k_\tau$: a dyadic cube with $\text{diam}(Q) \sim 2^{-k}$ (See Lemma 1.2).

- $Q^k, \nu$: all dyadic cubes $Q^{k+j}_{\nu} \subseteq Q^k_\tau$ for a fixed positive integer $j$, $\nu = 1, 2, ..., N(k, \tau)$.

- $y^{k, \nu}_\tau$: a point in $Q^{k, \nu}_\tau$. 
2 Atomic decomposition of Multi-parameter Hardy spaces of Homogeneous Type

2.1 Introduction and Statements of Main results

It is well known that the elements in the classical Hardy space $H^p(\mathbb{R}^n)$ can be decomposed as the sum of an appropriate class of simple functions, that is, “atoms”. The “atoms” play an important role in proving the boundedness of operators on Hardy spaces by verifying their actions on such building blocks (see, for example, Coifman [CO] Coifman-Weiss [CW2], Grafakos [G], Latter [La], Lu [Lu], Meyer [M], Meyer-Coifman [MC], Stein [St], etc.).

In general, if a linear operator is bounded on the space of all atoms which is dense in Hardy space, it can be extended to a bounded operator on the whole Hardy space. However, this boundedness principle is not always true. M. Bownik [B] gave an example shows the boundedness principle is broken when considering $(1, \infty)$-atoms on $H^1(\mathbb{R}^n)$. Therefore, we need to proceed with caution when the above principle is used. And it is meaningful to ask under what circumstances the above fundamental principle can be applied. In [MSV], S. Meda, P. Sjogren and M. Vallarino have proved that this boundedness criterion holds on $H^1(\mathbb{R}^n)$ for $(1, q)$ atoms for $1 < q < \infty$ (see also [HZ] and [YZ] for related results). Furthermore, this criterion also holds on $H^p(\mathbb{R}^n)$ for $0 < p < 1$ when applying this principle for $(p, \infty)$-atoms as shown by Ricci and Verdera [RV].

For the multi-parameter Hardy spaces, A. Chang and R. Fefferman ([CF1], [CF2], [CF3]) developed the product Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ theory. In [CF2], they proved the following

**Theorem.** Let $0 < p < \infty$. $f \in H^p(\mathbb{R} \times \mathbb{R})$ if and only if $f(x, y) = \sum \lambda_k a_k(x, y)$ where
\[ \sum_{k} |\lambda_k|^p < \infty \] and each \( a_k(x, y) \) is a \((p, 2)\)-atom, that is, each \( a_k(x, y) \) is supported in an open set \( \Omega \) with finite measure and satisfies the following properties:

\[ \|a_k\|_2 \leq |\Omega|^{1/2-1/p}; \]

each \( a_k(x, y) \) can be further decomposed by

\[ a_k(x, y) = \sum_{R \in \Omega} a_R(x, y) \]

where \( R = I \times J \subset \Omega \) are dyadic rectangles in \( \mathbb{R}^2 \), and each \( a_R(x, y) \) satisfies

\[ \int_I a_R(x, y)x^\alpha dx = \int_J a_R(x, y)y^\beta dy = 0 \]

for \( 0 \leq |\alpha|, |\beta| \leq N_p = [2/p - 4/3] \). Moreover, \( a_R \) is a \( C^n (\eta \leq N_p + 1) \) function satisfying

\[ |\frac{\partial^{\eta}}{\partial x^\eta} a_R(x, y)| \leq d_R|I|^{-\eta}, \quad |\frac{\partial^{\eta}}{\partial y^\eta} a_R(x, y)| \leq d_R|J|^{-\eta} \]

with

\[ \sum_{R \in \mathcal{M}(\Omega)} |R|d_R^2 \leq |\Omega|^{1-2/p}, \]

where \( \mathcal{M}(\Omega) \) is the set of all rectangles in \( \Omega \) which are maximal in both directions of \( x \) and \( y \).

The key tool that A. Chang and R. Fefferman used to prove the above \((p, 2)\)-atomic decomposition is the classical version of continuous Calderón’s identity on the product space.
However, for $1 < q < \infty, q \neq 2$, the $(p, q)$-atomic decomposition on product Hardy spaces $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ cannot be established by using the classical Calderón’s identity. Therefore, the $(p, q)$ atomic decomposition for the product Hardy spaces becomes interesting for $q \neq 2$. This has been recently carried out by Han, Lu, and Zhao in [HLZk]. They established the $(p, q)$-product atoms on the multi-parameter Hardy spaces $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $0 < p \leq 1$ and $1 < q < \infty$ by using discrete Littlewood-Paley analysis and the discrete Calderón’s identity.

Since the spaces of homogeneous type is a generalized extension of the Euclidean spaces, it is natural to consider the atomic decomposition of Hardy spaces of homogeneous type.

We will derive a new $(p, q)$-atomic decomposition on the multi-parameter Hardy space $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ for $0 < p \leq 1$ and all $1 < q < \infty$, where $\mathcal{X}_1 \times \mathcal{X}_2$ is the product of two homogeneous type spaces in the sense of Coifman and Weiss ([CW1]). The series in $(p, 2)$-atomic decomposition in [CF1] converges only in the sense of distributions. But the decomposition we get converges in both $L^q(\mathcal{X}_1 \times \mathcal{X}_2)$ ($1 < q < \infty$) and $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ ($0 < p \leq 1$). As an application, we prove that an operator $T$, which is bounded on $L^q(\mathcal{X}_1 \times \mathcal{X}_2)$ for some $1 < q < \infty$, is bounded from $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ to $L^p(\mathcal{X}_1 \times \mathcal{X}_2)$ if and only if $T$ is bounded uniformly on all $(p, q)$-product atoms in $L^p(\mathcal{X}_1 \times \mathcal{X}_2)$. The similar boundedness criterion from $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ to $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ is also obtained. The main idea is establishing the Journé’s covering lemma for spaces of homogeneous type, and using the Littlewood-Paley theory and a new discrete Calderón reproducing formulas on product spaces of homogeneous type (see [HL3]) to derive a $(p, q)$-atomic decomposition for $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$. Then by the fact that $L^q(\mathcal{X}_1 \times \mathcal{X}_2) \cap H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ is dense in $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ for $0 < p \leq 1 < q < \infty$ and $\|f\|_{L^q(\mathcal{X}_1 \times \mathcal{X}_2)} \leq C\|f\|_{H^p(\mathcal{X}_1 \times \mathcal{X}_2)}$ (see [HLLW]), we get the boundedness criterion of operators on $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$. We would like to point out that this method is quite different from the classical product theory in Euclidean spaces.
(see [CF2], [Fr1], and [St]), which is not suitable for the Hardy space $H^p$ $(0 < p \leq 1)$ on product spaces of homogeneous type. This method also works for the atomic decomposition of other Hardy spaces such as Hardy spaces associated with two different homogeneities, which we show in Chapter 4. For more general applications of the discrete Littlewood-Paley-Stein theory to the multi-parameter Hardy space theory on Carnot-Carathéodory spaces and product spaces of homogeneous type, please see [HLL2].

For $i = 1, 2$, let $(X_i, \rho_i, \mu_i)_{d_i, \theta_i}$ be a space of homogeneous type, and $\rho_i$ satisfies (1.2) with $A$ replaced by $A_i$. Then $R = Q_1 \times Q_2 \subset X_1 \times X_2$ is said to be a dyadic rectangle in product spaces of homogeneous type if $Q_1$ and $Q_2$ are dyadic cubes in $X_1$ and $X_2$ respectively, with $\text{diam} Q_1 \sim 2^{-k_1}$ and $\text{diam} Q_2 \sim 2^{-k_2}$ for some $k_1, k_2 \in \mathbb{Z}$.

Now we introduce the approximation to identity on the space of homogeneous type.

**Definition 2.1 (approximation to the identity [HS]).** Let $(\mathcal{X}, \rho, \mu)_{d, \theta}$ be a space of homogeneous type. For $\epsilon \in (0, \theta]$, we call a sequence of linear operators $\{S_k\}_{k \in \mathbb{Z}}$ as an approximation to the identity of order $\epsilon$ on $\mathcal{X}$ if there exists $C_3 > 0$ such that for all $k \in \mathbb{Z}$, the kernel of $S_k$, $S_k(x, y_1)$, are functions from $\mathcal{X} \times \mathcal{X}$ into $\mathbb{C}$ satisfying that for all $x_1, x_2, y_1$ and $y_2 \in \mathcal{X}$,

1. $|S_k(x_1, y_1)| \leq C_3 \left(\frac{2^{-k\epsilon}}{2^{-k} + \rho(x_1, y_1)}\right)^d \epsilon$;

2. $|S_k(x_1, y_1) - S_k(x_2, y_1)| \leq C_3 \left(\frac{\rho(x_1, x_2)}{2^{-k} + \rho(x_1, y_1)}\right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x_1, y_1))^{d+\epsilon}}$ for $\rho(x_1, x_2) \leq \frac{1}{2A}(2^{-k} + \rho(x_1, y_1))$;
\begin{align}
(3) \quad |S_k(x_1, y_1) - S_k(x_1, y_2)| & \leq C_3 \left( \frac{\rho(y_1, y_2)}{2^{-k} + \rho(x_1, y_1)} \right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x_1, y_1))^{d+\epsilon}} \\
& \quad \text{for } \rho(y_1, y_2) \leq \frac{1}{2A} (2^{-k} + \rho(x_1, y_1)); \\
(4) \quad \|[S_k(x_1, y_1) - S_k(x_1, y_2)] - [S_k(x_2, y_1) - S_k(x_2, y_2)]\| & \leq C_3 \left( \frac{\rho(x_1, x_2)}{2^{-k} + \rho(x_1, y_1)} \right)^\epsilon \\
& \quad \times \left( \frac{\rho(y_1, y_2)}{2^{-k} + \rho(x_1, y_1)} \right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x_1, y_1))^{d+\epsilon}} \\
& \quad \text{for } \rho(x_1, x_2) \leq \frac{1}{2A} (2^{-k} + \rho(x_1, y_1)) \text{ and } \rho(y_1, y_2) \leq \frac{1}{2A} (2^{-k} + \rho(x_1, y_1)); \\
(5) \quad \int_X S_k(x_1, y_1) d\mu(y_1) & = 1; \\
(6) \quad \int_X S_k(x_1, y_1) d\mu(x_1) & = 1.
\end{align}

Moreover, we call a sequence of linear operators \(\{S_k\}_{k \in \mathbb{Z}}\) as an approximation to the identity of order \(\epsilon \in (0, \theta]\) having compact support if there exist constants \(C_4, C_5 > 0\) such that for all \(k \in \mathbb{Z}\), the kernel of \(S_k, S_k(x_1, y_1)\), are functions from \(X \times X\) into \(\mathbb{C}\) satisfying (1)-(6) and

\begin{align}
(7) \quad S_k(x_1, y_1) & = 0 \text{ if } \rho(x_1, y_1) \geq C_4 2^{-k} \text{ and } \|S_k\|_{L^\infty(X \times X)} \leq C_5 2^{kd}; \\
(8) \quad |S_k(x_1, y_1) - S_k(x_2, y_1)| & \leq C_5 2^{k(d+\epsilon)} \rho(x_1, x_2)^\epsilon; \\
(9) \quad |S_k(x_1, y_1) - S_k(x_1, y_2)| & \leq C_5 2^{k(d+\epsilon)} \rho(y_1, y_2)^\epsilon; \\
(10) \quad \|[S_k(x_1, y_1) - S_k(x_1, y_2)] - [S_k(x_2, y_1) - S_k(x_2, y_2)]\| & \leq C_5 2^{k(d+2\epsilon)} \rho(x_1, x_2)^\epsilon \rho(y, y')^\epsilon \\
& \quad \text{for all } x_1, x_2, y_1 \text{ and } y_2 \in X.
\end{align}
Remark. By Coifman’s construction in [DJS], one can construct an approximation to the identity of order \( \theta \) having compact support satisfying the above Definition 2.1.

To introduce the multi-parameter Hardy space of homogeneous type \( H^p(X_1 \times X_2) \), we first need to introduce the space of test functions on the product space of homogeneous type \( X_1 \times X_2 \).

**Definition 2.2.** ([HLβ]) For \( i = 1, 2 \), fix \( \gamma_i > 0 \) and \( \beta_i > 0 \). A function \( f \) defined on \( X_1 \times X_2 \) is said to be a test function of type \((\beta_1, \beta_2, \gamma_1, \gamma_2)\) centered at \((x_0, y_0) \in X_1 \times X_2\) with width \( r_1, r_2 > 0 \) if for all \( x, x' \in X_1 \), and \( y, y' \in X_2 \), \( f \) satisfies the following conditions:

(i) \[ |f(x, y)| \leq C \frac{r_1^{\gamma_1}}{(r_1 + \rho_1(x, x_0))^{d_1 + \gamma_1}} \frac{r_2^{\gamma_2}}{(r_2 + \rho_2(y, y_0))^{d_2 + \gamma_2}} \]

for \( \rho_1(x, x') \leq \frac{1}{2A_1}[r_1 + \rho_1(x, x_0)] \);

(ii) \[ |f(x, y) - f(x', y)| \leq C \left( \frac{\rho_1(x, x')}{r_1 + \rho_1(x, x_0)} \right)^{\beta_1} \frac{r_1^{\gamma_1}}{(r_1 + \rho_1(x, x_0))^{d_1 + \gamma_1}} \frac{r_2^{\gamma_2}}{(r_2 + \rho_2(y, y_0))^{d_2 + \gamma_2}} \]

for \( \rho_2(y, y') \leq \frac{1}{2A_2}[r_2 + \rho_2(y, y_0)] \);

(iii) \[ |f(x, y) - f(x, y')| \leq C \frac{r_1^{\gamma_1}}{(r_1 + \rho_1(x, x_0))^{d_1 + \gamma_1}} \left( \frac{\rho_2(y, y')}{r_2 + \rho_2(y, y_0)} \right)^{\beta_2} \frac{r_2^{\gamma_2}}{(r_2 + \rho_2(y, y_0))^{d_2 + \gamma_2}} \]

for \( \rho_1(x, x') \leq \frac{1}{2A_1}[r_1 + \rho_1(x, x_0)] \) and \( \rho_2(y, y') \leq \frac{1}{2A_2}[r_2 + \rho_2(y, y_0)] \);

(iv) \[ |[f(x, y) - f(x', y)] - [f(x, y') - f(x', y')]| \leq C \left( \frac{\rho_1(x, x')}{r_1 + \rho_1(x, x_0)} \right)^{\beta_1} \frac{r_1^{\gamma_1}}{(r_1 + \rho_1(x, x_0))^{d_1 + \gamma_1}} \]

\[ \times \left( \frac{\rho_2(y, y')}{r_2 + \rho_2(y, y_0)} \right)^{\beta_2} \frac{r_2^{\gamma_2}}{(r_2 + \rho_2(y, y_0))^{d_2 + \gamma_2}} \]

for \( \rho_1(x, x') \leq \frac{1}{2A_1}[r_1 + \rho_1(x, x_0)] \) and \( \rho_2(y, y') \leq \frac{1}{2A_2}[r_2 + \rho_2(y, y_0)] \);
(v) $\int_{X_1} f(x, y) \, d\mu_1(x) = 0$ for all $y \in X_2$;

(vi) $\int_{X_2} f(x, y) \, d\mu_2(y) = 0$ for all $x \in X_1$.

If $f$ is a test function of type $(\beta_1, \beta_2, \gamma_1, \gamma_2)$ centered at $(x_0, y_0) \in X_1 \times X_2$ with width $r_1$, $r_2 > 0$, we write $f \in G(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ and we define the norm of $f$ by

$$\|f\|_{G(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)} = \inf\{C : (i), (ii), (iii) and (iv) hold\}.$$

If $\beta_1 = \beta_2 = \beta$ and $\gamma_1 = \gamma_2 = \gamma$, we will then simply write $f \in G(x_0, y_0; r_1, r_2; \beta; \gamma)$. And we denote by $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ the class of $G(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ with $r_1 = r_2 = 1$ for fixed $(x_0, y_0) \in X_1 \times X_2$. Then if $\beta_1 = \beta_2 = \beta$ and $\gamma_1 = \gamma_2 = \gamma$, we will simply write $f \in G(\beta; \gamma)$.

**Remark.** It is easy to see that $G(x_1, y_1; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2) = G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with an equivalent norm for all $(x_1, y_1) \in X_1 \times X_2$. We can easily check that the space $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is a Banach space. Also, we denote by $(G(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ its dual space which is the set of all linear functionals $L$ from $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ to $\mathbb{C}$ with the property that there exists $C \geq 0$ such that for all $f \in G(\beta_1, \beta_2; \gamma_1, \gamma_2)$,

$$|L(f)| \leq C\|f\|_{G(\beta_1, \beta_2; \gamma_1, \gamma_2)}.$$

Clearly, for all $h \in (G(\beta_1, \beta_2; \gamma_1, \gamma_2))'$, $(h, f)$ is well defined for all $f \in G(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ with $(x_0, y_0) \in X_1 \times X_2$, $r_1 > 0$ and $r_2 > 0$. By the same reason as the case of one-parameter spaces, we denote by $\hat{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ the completion of the space $G(\epsilon_1; \epsilon_2)$ in $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ when $0 < \beta_1$, $\gamma_1 < \epsilon_1$ and $0 < \beta_2$, $\gamma_2 < \epsilon_2$. 

Next we recall Littlewood-Paley theorem on product spaces of homogeneous type, which can be stated as follows.

**Lemma 2.3.** ([HL3]) For $i = 1, 2$, let $\epsilon_i \in (0, \theta_i]$, $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$ be an approximation to the identity of order $\epsilon_i$ on $\mathcal{X}_i$, and $D_{k_i} = S_{k_i} - S_{k_i-1}$ for all $k_i \in \mathbb{Z}$. If $1 < p < \infty$, then there is a constant $C_p > 0$ such that for all $f \in L^p(\mathcal{X}_1 \times \mathcal{X}_2)$,

$$
C_p^{-1} \| f \|_{L^p(\mathcal{X}_1 \times \mathcal{X}_2)} \leq \| g_2(f) \|_{L^p(\mathcal{X}_1 \times \mathcal{X}_2)} \leq C_p \| f \|_{L^p(\mathcal{X}_1 \times \mathcal{X}_2)},
$$

where $g_q(f)$ for $q \in (0, \infty)$ is called the discrete Littlewood-Paley $g$-function on $\mathcal{X}_1 \times \mathcal{X}_2$ defined by

$$
g_q(f)(x_1, x_2) = \left\{ \sum_{k_1=\infty}^{\infty} \sum_{k_2=\infty}^{\infty} |D_{k_1}D_{k_2}(f)(x_1, x_2)|^q \right\}^{1/q}
$$

for all $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$.

Now we can introduce the multi-parameter Hardy spaces of homogeneous type $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ for some $p \leq 1$ and establish their $(p, q)$-atomic decomposition characterization.

**Definition 2.4.** For $i = 1, 2$, let $\epsilon_i \in (0, \theta_i]$, $\{D_{k_i}\}_{k_i \in \mathbb{Z}}$ be the same as in Lemma 2.3,

$$
\max \left\{ \frac{d_1}{d_1 + \epsilon_1}, \frac{d_2}{d_2 + \epsilon_2} \right\} < p < \infty
$$

and

$$
d_i(1/p - 1)_+ < \beta_i, \gamma_i < \epsilon_i. \tag{2.1}
$$
The multi-parameter Hardy spaces of homogeneous type $H^p(X_1 \times X_2)$ is the set defined by

$$\left\{ f \in \left( \hat{G}(\theta_1, \theta_2; \gamma_1, \gamma_2) \right)^\prime : \|g_2(f)\|_{L^p(X_1 \times X_2)} < \infty \right\},$$

and we define

$$\|f\|_{H^p(X_1 \times X_2)} = \|g_2(f)\|_{L^p(X_1 \times X_2)},$$

where $g_2(f)$ is the discrete Littlewood-Paley square function defined as in Lemma 2.3.

**Remark.** Here the definition of $H^p(X_1 \times X_2)$ is independent of the choice of the approximation to identity, see [HL3] for the proof.

We now can give the definition of $(p, q)$-atoms of $H^p(X_1 \times X_2)$ as follows. For the convenience, in the following, we use $C$ to denote all constants only dependent on $X_1$ and $X_2$, which may vary from line to line.

**Definition 2.5.** For $0 < p \leq 1$ and $0 < q < \infty$, a function $a(x_1, x_2)$ on $X_1 \times X_2$ is called a $(p, q)$-product atom of $H^p(X_1 \times X_2)$, if it satisfies the following conditions:

1. $\text{supp } a \subset \Omega$, where $\Omega$ is an open set in $X_1 \times X_2$ with finite measure;

2. $\|a\|_{L^q(X_1 \times X_2)} \leq \mu(\Omega)^{1/q - 1/p}$, where $\mu = \mu_1 \times \mu_2$.

Moreover, $a$ can be decomposed into rectangle $(p, q)$-atoms $a_R$ associated to the dyadic rectangle $R = Q_1 \times Q_2$ with $\text{diam}Q_1 \sim 2^{-k_1}$ and $\text{diam}Q_2 \sim 2^{-k_2}$ for some $k_1, k_2 \in \mathbb{Z}_+$, which is supported in $B_1(z_1, C2^{-k_1}) \times B_2(z_2, C2^{-k_2})$, where $z_i$ is the center of $Q_i$ for $i = 1, 2$. To be specify,
(3a) For $2 \leq q < \infty$, $a = \sum_{R \in \mathcal{M}(\Omega)} a_R$, and
\[
\left\{ \sum_{R \in \mathcal{M}(\Omega)} \|a_R\|_{L^q(\mathcal{X}_1 \times \mathcal{X}_2)}^q \right\}^{1/q} \leq \mu(\Omega)^{1/q-1/p}.
\]

Here and in the sequel, $\mathcal{M}(\Omega)$ is the set of all maximal dyadic rectangles contained in $\Omega$ in both directions of $x_1$ and $x_2$, that is, $\mathcal{M}(\Omega) = \{ R' \subset \Omega : R' = Q'_1 \times Q'_2, \text{diam}Q'_i \sim 2^{-k'_i} \text{ for some } k'_i \in \mathbb{Z}_+ \text{ and } Q'_i \text{ is not contained in any other dyadic cube } Q \in \Omega \cap \mathcal{X}_i \text{ for } i = 1, 2 \}$. And
\[
\tilde{\Omega} = \{(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 : M_s(\chi_{\tilde{\Omega}})(x_1, x_2) > C \},
\]

Where
\[
M_s f(x_1, x_2) = \sup_R \frac{1}{\mu(R)} \int_R |f(x_1, x_2)| d\mu(x_1, x_2)
\]
is the strong maximal function, $\mu = \mu_1 \times \mu_2$, and the above supremum is taken among all dyadic rectangles $R$ in $\mathcal{X}_1 \times \mathcal{X}_2$, $C$ is a small enough positive constant only depend on $\mathcal{X}_1$ and $\mathcal{X}_2$.

(3b) For $1 < q < 2$, $a = \sum_{R \in \mathcal{M}_1(\Omega)} a_R + \sum_{R \in \mathcal{M}_2(\Omega)} a_R$, and for any $\delta > 0$, there exists a constant $C_{\delta, q} > 0$, where $C_{\delta, q}$ only depends on $\delta$ and $q$, such that
\[
\left\{ \sum_{R \in \mathcal{M}_1(\Omega)} \gamma_2^{-\delta} \|a_R\|_{L^q(\mathcal{X}_1 \times \mathcal{X}_2)}^q + \sum_{R \in \mathcal{M}_2(\Omega)} \gamma_1^{-\delta} \|a_R\|_{L^q(\mathcal{X}_1 \times \mathcal{X}_2)}^q \right\}^{1/q} \leq C_{\delta, q} \mu(\Omega)^{1/q-1/p}.
\]

Here and in the sequel, $\mathcal{M}_1(\Omega)$ is the set of all dyadic rectangles contained in $\Omega$ and maximal in the direction of $x_1$ and $\mathcal{M}_2(\Omega)$ is defined similarly. $\gamma_1$ is defined by $\gamma_1(R) = \ldots$
where $R = Q_1 \times Q_2 \subset M_2(\Omega)$ and $\hat{Q}_1 = \hat{Q}_1(Q_2)$ be the “longest” dyadic cube containing $Q_1$ such that $\mu(\hat{Q}_1 \times Q_2 \cap \Omega) > \frac{1}{2} \mu(\hat{Q}_1 \times Q_2)$. $\gamma_2(R)$ is similarly defined.

(4) For all $x_1 \in X_1$,

$$\int_{X_2} a_R(x_1, x_2) \, d\mu_2(x_2) = 0$$

and for all $x_2 \in X_2$,

$$\int_{X_1} a_R(x_1, x_2) \, d\mu_1(x_1) = 0.$$

Note that for $0 < p \leq 1 < q < \infty$, $\|f\|_{L^p} \leq C \|f\|_{H^p}$ for $f \in L^q \cap H^p$ and $L^q(X_1 \times X_2) \cap H^p(X_1 \times X_2)$ is dense in $H^p(X_1 \times X_2)$ (see [HLLW]). Therefore, it is sufficient to consider the atomic decomposition in the subspace $L^q(X_1 \times X_2) \cap H^p(X_1 \times X_2)$. Then One of our main results, atomic decomposition in terms of $(p, q)$-atoms for the multi-parameter product Hardy space of homogeneous type is as follows:

**Theorem 2.6.** For $i = 1, 2$, let $\epsilon_i \in (0, \theta_i]$, $f \in H^p(X_1 \times X_2) \cap L^q(X_1 \times X_2)$, and

$$0 < \max \left\{ \frac{d_1}{d_1 + \epsilon_1}, \frac{d_2}{d_2 + \epsilon_2} \right\} < p \leq 1 < q < \infty.$$

Then $f \in \left( G(\beta_1, \beta_2, \gamma_1, \gamma_2) \right)'$ for some $\beta_i$, $\gamma_i$ satisfying (2.1) for $i = 1, 2$, and there is a sequence of numbers, $\{\lambda_k\}_{k \in \mathbb{Z}}$, and a sequence of $(p, q)$-atoms of $H^p(X_1 \times X_2)$, $\{a_k\}_{k \in \mathbb{Z}}$, such that $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$ and

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k,$$

where the series converges to $f$ in both $H^p(X_1 \times X_2)$ and $L^q(X_1 \times X_2)$ norms. Moreover, in
In this case,
\[ \|f\|_{H^p(X_1 \times X_2)} \sim \inf \left\{ \left[ \sum_{k=-\infty}^{\infty} |\lambda_k|^p \right]^{1/p} \right\}, \]
where the infimum is taken over all the decompositions as above.

### 2.2 Journé’s Covering Lemma for spaces of homogeneous type

To prove Theorem 2.6, we need to establish Journé’s covering lemma in the setting of spaces of homogeneous type.

For \( i = 1, 2 \), let \( \{Q_{I_i}^k \subset X_i : \tau_i \in I_{k_i}\} \) be the same as in Lemma 1.2, where \( k_i \in \mathbb{Z}\); \( \Omega \subset X_1 \times X_2 \) be an open set with the finite measure and \( \mathcal{M}_i(\Omega) \) be the same in Definition 2.5, that is the family of dyadic rectangles \( R \subset \Omega \) which are maximal in the direction of \( x_i \). In what follows, we denote by \( R = Q_1 \times Q_2 \) any dyadic rectangle of \( X_1 \times X_2 \). Given \( R = Q_1 \times Q_2 \in \mathcal{M}_i(\Omega) \), let \( \tilde{Q}_2 = \tilde{Q}_2(Q_1) \) be the “longest” dyadic cube containing \( Q_2 \) such that
\[ \mu(Q_1 \times \tilde{Q}_2 \cap \Omega) > \frac{1}{2} \mu(Q_1 \times \tilde{Q}_2); \tag{2.2} \]
and given \( R = Q_1 \times Q_2 \in \mathcal{M}_2(\Omega) \), let \( \tilde{Q}_1 = \tilde{Q}_1(Q_2) \) be the “longest” dyadic cube containing \( Q_1 \) such that
\[ \mu(\tilde{Q}_1 \times Q_2 \cap \Omega) > \frac{1}{2} \mu(\tilde{Q}_1 \times Q_2). \tag{2.3} \]

If \( Q_i = Q_{I_i}^{k_i} \subset X_i \) for some \( k_i \in \mathbb{Z} \) and some \( \tau_i \in I_{k_i} \), \( (Q_i)_k \) for \( k \in \mathbb{N} \) is used to denote any dyadic cube \( Q_{I_i}^{k_{i-k}} \) containing \( Q_{I_i}^{k_i} \) and \( (Q_i)_0 = Q_i \), where \( i = 1, 2 \). Also, let \( w(x) \) be any increasing function such that \( \sum_{j=0}^{\infty} jw(C_6 2^{-j}) < \infty \), where \( C_6 > 0 \) is any given constant. In particular, we may take \( w(x) = x^{\theta} \) for any \( \theta > 0 \).
The main idea of the following variant of Journé’s covering lemma in the setting of spaces of homogeneous type comes from Pipher [P].

**Lemma 2.7.** Assume that \( \Omega \subset \mathcal{X}_1 \times \mathcal{X}_2 \) is an open set with finite measure. Let all the notation be the same as above. Then

\[
\sum_{R = Q_1 \times Q_2 \in \mathcal{M}_1(\Omega)} \mu(R) w \left( \frac{\mu_2(Q_2)}{\mu(Q_2)} \right) \leq C \mu(\Omega) \tag{2.4}
\]

and

\[
\sum_{R = Q_1 \times Q_2 \in \mathcal{M}_2(\Omega)} \mu(R) w \left( \frac{\mu_1(Q_1)}{\mu(Q_1)} \right) \leq C \mu(\Omega). \tag{2.5}
\]

**Proof.** We only verify (2.4) and the proof of (2.5) is similar. Let \( R = Q_1 \times Q_2 \in \mathcal{M}_2(\Omega) \) and for \( k \in \mathbb{N} \), let

\[
A_{Q_1,k} = \bigcup \left\{ Q_2 : Q_1 \times Q_2 \in \mathcal{M}_2(\Omega) \text{ and } \hat{Q}_1 = (Q_1)_{k-1} \right\}. \tag{2.6}
\]

Then

\[
\sum_{R = Q_1 \times Q_2 \in \mathcal{M}_2(\Omega)} \mu(R) w \left( \frac{\mu_1(Q_1)}{\mu(Q_1)} \right) \tag{2.7}
\]

\[
= \sum_{R = Q_1 \times Q_2 \in \mathcal{M}_2(\Omega)} \mu_1(Q_1) \mu_2(Q_2) w \left( \frac{\mu_1(Q_1)}{\mu(Q_1)} \right)
\]

\[
= \sum_{\{Q_1 : Q_1 \times Q_2 \in \mathcal{M}_2(\Omega)\}} \mu_1(Q_1) \sum_{k=1}^{\infty} \sum_{\{Q_2 : Q_2 \in A_{Q_1,k}\}} \mu_2(Q_2) w \left( \frac{\mu_1(Q_1)}{\mu(Q_1)} \right)
\]

\[
\leq \sum_{\{Q_1 : Q_1 \times Q_2 \in \mathcal{M}_2(\Omega)\}} \mu_1(Q_1) \sum_{k=1}^{\infty} w \left( C_0 2^{-k} \right) \sum_{\{Q_2 : Q_2 \in A_{Q_1,k}\}} \mu_2(Q_2)
\]

\[
= \sum_{\{Q_1 : Q_1 \times Q_2 \in \mathcal{M}_2(\Omega)\}} \mu_1(Q_1) \sum_{k=1}^{\infty} w \left( C_0 2^{-k} \right) \mu_2(A_{Q_1,k}).
\]
since \( \{Q_2 : Q_2 \in A_{Q_1,k}\} \) are disjoint by their “maximality”, where \( C_6 > 0 \) depends only on the doubling measure \( \mu_1 \) and the constants \( C_1 \) and \( C_2 \) in Lemma 1.2 for \( X_1 \).

Set

\[
E_{Q_1}(\Omega) = \bigcup \{Q_2 : Q_1 \times Q_2 \subset \Omega\}.
\]

If \( x_2 \in A_{Q_1,k} \), then there is some dyadic cube \( Q_1 \times Q_2 \in \mathcal{M}_2(\Omega) \) and some \( k \in \mathbb{N} \) such that \( x_2 \in Q_2 \) and \( \tilde{Q}_1 = (Q_1)_{k-1} \) by (2.6). By (2.3) and the maximality of \( \tilde{Q}_1 \), we have

\[
\mu \left( (Q_1)_{k-1} \times Q_2 \cap \Omega \right) > \frac{1}{2} \mu \left( (Q_1)_{k-1} \times Q_2 \right)
\]

and

\[
\mu \left( (Q_1)_k \times Q_2 \cap \Omega \right) \leq \frac{1}{2} \mu \left( (Q_1)_k \times Q_2 \right),
\]

which implies that

\[
\mu \left( (Q_1)_k \times Q_2 \cap (Q_1)_k \times E_{Q_1,k} \right) \leq \frac{1}{2} \mu \left( (Q_1)_k \times Q_2 \right)
\]

and further

\[
\mu \left( (Q_1)_k \times Q_2 \cap E_{(Q_1)_k} \right) \leq \frac{1}{2} \mu \left( (Q_1)_k \times Q_2 \right).
\]

Therefore,

\[
\mu_2 \left( Q_2 \cap E_{(Q_1)_k} \right) \leq \frac{1}{2} \mu_2 (Q_2),
\]

which in turn tells us that

\[
\mu_2 \left( Q_2 \cap E_{(Q_1)_k} \right) > \frac{1}{2} \mu_2 (Q_2), \tag{2.8}
\]
where \((E_{(Q_1)_k})^c = X_2 \setminus E_{(Q_1)_k}\). From (2.8), it follows that

\[
M_2 \left( \chi_{E_{Q_1 \setminus E_{(Q_1)_k}}} \right) (x_2) > \frac{1}{2}
\]

and therefore

\[
A_{Q_1,k} \subset \left\{ x_2 \in X_2 : M_2 \left( \chi_{E_{Q_1 \setminus E_{(Q_1)_k}}} \right) (x_2) > \frac{1}{2} \right\},
\]

which implies that

\[
\mu_2 \left( A_{Q_1,k} \right) \leq \mu_2 \left( \left\{ x_2 \in X_2 : M_2 \left( \chi_{E_{Q_1 \setminus E_{(Q_1)_k}}} \right) (x_2) > \frac{1}{2} \right\} \right)
\]

\[
\leq C \mu_2 \left( E_{Q_1 \setminus E_{(Q_1)_k}} \right). \tag{2.9}
\]

Combining (2.7) with (2.9) yields that

\[
\sum_{R=Q_1 \times Q_2 \in M_2(\Omega)} \mu(R) \left( \frac{\mu_1(Q_1)}{\mu_1(Q)} \right) w \left( \frac{\mu_1(Q_1)}{\mu_1(Q)} \right)
\]

\[
\leq C \sum_{\{Q_1 : Q_1 \times Q_2 \in M_2(\Omega)\}} \mu_1(Q_1) \sum_{k=1}^{\infty} w \left( C_0 2^{-k} \right) \mu_2 \left( E_{Q_1 \setminus E_{(Q_1)_k}} \right)
\]

\[
\leq C \sum_{\{Q_1 : Q_1 \times Q_2 \in M_2(\Omega)\}} \mu_1(Q_1) \sum_{k=1}^{\infty} w \left( C_0 2^{-k} \right)
\]

\[
\times \left\{ \mu_2 \left( E_{Q_1 \setminus E_{(Q_1)_1}} \right) + \cdots + \mu_2 \left( E_{(Q_1)_k \setminus E_{(Q_1)_k}} \right) \right\}
\]

\[
\leq C \sum_{\{Q_1 : Q_1 \times Q_2 \in M_2(\Omega)\}} \mu_1(Q_1) \sum_{k=1}^{\infty} w \left( C_0 2^{-k} \right)
\]

\[
\times \sum_{\{Q_0 \text{ dyadic cube : } Q_1 \subseteq Q_0 \in (Q_1)_k \}} \mu_2 \left( E_{Q_0 \setminus E_{(Q_0)_1}} \right).
\]
Lemma 2.8. (See [HLY, HLβ]) For \( i = 1, 2 \), let \( \epsilon_i \in (0, \theta_i] \), \( \{S_{ki}\}_{k_i \in \mathbb{Z}} \) be an approximation to the identity of order \( \epsilon_i \), \( D_{ki} = S_{ki} - S_{k_{i-1}} \) for \( k_i \in \mathbb{Z} \), \( \{Q_{\tau_1}^{k_1, \nu_1} : k_1 \in \mathbb{Z}, \tau_1 \in I_{k_1}, \nu_1 = 1, \ldots, N(k_1, \tau_1)\} \) and \( \{Q_{\tau_2}^{k_2, \nu_2} : k_2 \in \mathbb{Z}, \tau_2 \in I_{k_2}, \nu_2 = 1, \ldots, N(k_2, \tau_2)\} \) respectively be the dyadic cubes of \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) defined in Lemma 1.2 with \( j_1, j_2 \in \mathbb{N} \) large enough. Then there are families of linear operators \( \{\mathcal{D}_{ki}\}_{k_i \in \mathbb{Z}} \) on \( \mathcal{X}_1 \) such that for all \( f \in G(\beta_1, \beta_2; \gamma_1, \gamma_2) \) with \( \beta_i, \gamma_i \in (0, \epsilon_i) \) for \( i = 1, 2 \), and any point \( y_{\tau_1}^{k_1, \nu_1} \in Q_{\tau_1}^{k_1, \nu_1} \) and \( y_{\tau_2}^{k_2, \nu_2} \in Q_{\tau_2}^{k_2, \nu_2} \),
\[ f(x_1, x_2) = \sum_{k_1 = -\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1 = 1}^{N(k_1, \tau_1)} \sum_{k_2 = -\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2 = 1}^{N(k_2, \tau_2)} \mu_1(Q_{k_1, \nu_1}^{k_1}) \mu_2(Q_{k_2, \nu_2}^{k_2}) \times D_{k_1}(x_1, y_{\tau_1}^{k_1, \nu_1}) D_{k_2}(x_2, y_{\tau_2}^{k_2, \nu_2}) \bar{D}_{k_1} \bar{D}_{k_2}(f)(y_{\tau_1}^{k_1, \nu_1}, y_{\tau_2}^{k_2, \nu_2}), \]

where the series converge in the norm of both the space \( G(\beta_i', \beta'_i; \gamma'_i, \gamma'_i) \) with \( \beta'_i \in (0, \beta_i) \) and \( \gamma'_i \in (0, \gamma_i) \) for \( i = 1, 2 \), and \( L^p(X_1 \times X_2) \) with \( p \in (1, \infty) \).

Now we can first establish the atomic decomposition into \((p, q)\)-atoms for \( 0 < p < 1 < q < \infty \), namely Theorem 2.6.

Let \( f \in H^p(X_1 \times X_2) \), then by Definition 2.4, \( f \in \left( \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \right)' \) for some \( \beta_i, \gamma_i \) satisfying (2.1) for \( i = 1, 2 \). We will use Lemma 2.8 to get the atomic decomposition of \( f \).

For any \( k \in \mathbb{Z} \), let

\[ \Omega_k = \{(x_1, x_2) \in X_1 \times X_2 : g_2(f)(x_1, x_2) > 2^k\} \]

and

\[ \widetilde{\Omega}_k = \{(x_1, x_2) \in X_1 \times X_2 : M_s(\chi_{\Omega_k})(x_1, x_2) > C\} \]

with a small enough constant \( C \) only depending on \( X_i, i = 1, 2 \), here and in the sequel, \( M_s \)

is the strong Hardy-Littlewood maximal function on \( X_1 \times X_2 \) defined as in Definition 2.5.

Then, the \( L^q(X_1 \times X_2) \)-boundedness of \( M_s \) (see [DH]) implies that \( \mu(\widetilde{\Omega}_k) \leq C \mu(\Omega_k) \).

Let \( \mathcal{R} \) be the set of all dyadic rectangles of \( X_1 \times X_2 \), that is

\[ \mathcal{R} = \{R = Q_1 \times Q_2 : Q_1 \text{ and } Q_2 \text{ are dyadic cubes, respectively, of } X_1 \text{ and } X_2\} \]
and for $k \in \mathbb{Z}$,

$$
\mathcal{R}_k = \left\{ R \in \mathcal{R} : \mu(R \cap \Omega_k) > \frac{1}{2} \mu(R) \text{ and } \mu(R \cap \Omega_{k+1}) \leq \frac{1}{2} \mu(R) \right\}.
$$

Obviously, for any $R \in \mathcal{R}$, there is a unique $k \in \mathbb{Z}$ such that $R \in \mathcal{R}_k$. Thus, we can reclassify the set of all dyadic rectangles in $\mathcal{X}_1 \times \mathcal{X}_2$ by

$$
\bigcup_{R \in \mathcal{R}} R = \bigcup_{k \in \mathbb{Z}} \bigcup_{R \in \mathcal{R}_k} R. \quad (2.10)
$$

In what follows, for $i = 1, 2$, if $Q_{k_i}$ is a dyadic cube and $\text{diam}Q_{k_i} \sim 2^{-k_i}$, we rewrite $D_{k_i}$ and $\overline{D}_{k_i}$, respectively, by $D_{Q_{k_i}}$ and $\overline{D}_{Q_{k_i}}$. And denote by $y_{k_i}$ a point in $Q_{k_i}$. Then, by lemma 2.8, we have

$$
f(x_1, x_2) = \sum_{k_1 = -\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1 = 1}^{N(k_1, \tau_1)} \sum_{k_2 = -\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2 = 1}^{N(k_2, \tau_2)} \mu_1(Q_{\tau_1}^{k_1, \nu_1}) \mu_2(Q_{\tau_2}^{k_2, \nu_2})
$$

$$
\times D_{Q_{k_1}}(x_1, y_{k_1}^{\tau_1}) D_{Q_{k_2}}(x_2, y_{k_2}^{\tau_2}) \overline{D}_{Q_{k_1}}(y_{k_1}^{\tau_1}, y_{k_1}) \overline{D}_{Q_{k_2}}(y_{k_2}^{\tau_2}, y_{k_2}) \quad (2.11)
$$

$$
= \sum_{k_1 = -\infty}^{\infty} \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \mu_1(Q_{k_1}) \mu_2(Q_{k_2}) D_{Q_{k_1}}(x_1, y_{k_1}) D_{Q_{k_2}}(x_2, y_{k_2}) \overline{D}_{Q_{k_1}}(y_{k_1}) \overline{D}_{Q_{k_2}}(y_{k_2})
$$

$$
= \sum_{k = -\infty}^{\infty} \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \mu(R) D_{Q_{k_1}}(x_1, y_{k_1}) D_{Q_{k_2}}(x_2, y_{k_2}) \overline{D}_{Q_{k_1}}(y_{k_1}) \overline{D}_{Q_{k_2}}(y_{k_2})
$$

$$
= \sum_{k = -\infty}^{\infty} \lambda_k a_k(x_1, x_2),
$$
where

\[ a_k(x_1, x_2) = \frac{1}{\lambda_k} \sum_{R=Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \mu(R) D_{Q_{k_1}}(x_1, y_{k_1}) D_{Q_{k_2}}(x_2, y_{k_2}) \overline{D}_{Q_{k_1}} D_{Q_{k_2}}(f)(y_{k_1}, y_{k_2}), \]

and when \( 2 \leq q < \infty \) we let

\[ \lambda_k = C \left\| \left\{ \sum_{R=Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} |D_{Q_{k_1}} D_{Q_{k_2}}(f)(y_{k_1}, y_{k_2})|^2 \chi_R(\cdot, \cdot) \right\} \right\|_q^{1/2} \mu(\Omega_k)^{\frac{1}{p} - \frac{1}{q}}, \]

while \( 1 < q < 2 \) we let,

\[ \lambda_k = C \left\| \left\{ \sum_{R=Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} |D_{Q_{k_1}} D_{Q_{k_2}}(f)(y_{k_1}, y_{k_2})|^2 \chi_R(\cdot, \cdot) \right\} \right\|_q^{1/2} \mu(\Omega_k)^{\frac{1}{p} - \frac{1}{2}}. \]

We now verify that \( \{\lambda_k\}_{k \in \mathbb{Z}} \) and \( \{a_k\}_{k \in \mathbb{Z}} \) satisfy the requirement of the Theorem 2.6. First, note that in the above expressions we have set

\[ \tilde{\Omega}_k = \{(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 : M_s(\chi_{\Omega_k})(x_1, x_2) > C\}, \]

where \( C \) is only dependent on \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) and is chosen to be small enough. It is easy to check that \( \text{supp} a_k \subset \tilde{\Omega}_k \), since \( R \in \mathcal{R}_k \) implies \( R \in \tilde{\Omega}. \) Thus \( a_k \) is supported in an open set, and hence satisfies (1) of Definition 2.5.

To see that \( a_k \) satisfies (2) of Definition 2.5, let \( h \in L^q'(\mathcal{X}_1 \times \mathcal{X}_2) \cap L^2(\mathcal{X}_1 \times \mathcal{X}_2) \), where
\[
\frac{1}{q} + \frac{1}{q'} = 1. \text{ By Hölder inequality and Lemma 2.7, we have }
\]

\[
\begin{align*}
\left\| \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \mu(R) D_{Q_{k_1}}(x_1, y_{k_1}) D_{Q_{k_2}}(x_2, y_{k_2}) \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2}) \right\|_q \\
= \sup_{\|h\|_{q'} \leq 1} \left\| \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \mu(R) D_{Q_{k_1}}(x_1, y_{k_1}) D_{Q_{k_2}}(x_2, y_{k_2}) \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2}), h \right\|_q \\
\leq \sup_{\|h\|_{q'} \leq 1} \int_{X_1 \times X_2} \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} D_{Q_{k_1}} D_{Q_{k_2}}(h)(y_{k_1}, y_{k_2}) \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2}) \\
\chi_R(y_1, y_2) d\mu_1(y_1) d\mu_2(y_2) \\
\leq \sup_{\|h\|_{q'} \leq 1} \left\| \left\{ \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \left\| D_{Q_{k_1}} D_{Q_{k_2}}(h)(y_{k_1}, y_{k_2}) \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2}) \right\|^2 \chi_R(\cdot, \cdot) \right\|_q^{1/2} \right\|_{q'}^{1/2} \\
\leq \sup_{\|h\|_{q'} \leq 1} \left\| \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} D_{Q_{k_1}} D_{Q_{k_2}}(h)(y_{k_1}, y_{k_2}) \chi_R(\cdot, \cdot) \right\|^2 \right\|_{q'}^{1/2} \\
\leq C \sup_{\|h\|_{q'} \leq 1} \left\| \left\{ \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2}) \right\|^2 \chi_R(\cdot, \cdot) \right\|_{q'}^{1/2} \\
= C \left\| \left\{ \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2}) \right\}^2 \chi_R(\cdot, \cdot) \right\|_{q'}^{1/2}.
\end{align*}
\]
Then the above estimate yields that when $2 \leq q < \infty$,

\[
\|a_k\|_q = \left( C \left\{ \sum_{R=Q_k1 \times Q_k2 \in \mathcal{R}_k} |\overline{D}_{Q_k1} \overline{D}_{Q_k2} (f)(y_{k1}, y_{k2})|^2 \chi_{\cdot} \right\}^{1/2} \mu(\widetilde{\Omega}_k)^{\frac{1}{p} - \frac{1}{q}} \right)^{-1}
\]

\[
\times \left( \sum_{R=Q_k1 \times Q_k2 \in \mathcal{R}_k} \mu(R) D_{Q_k1} (x_1, y_{k1}) D_{Q_k2} (x_2, y_{k2}) \overline{D}_{Q_k1} \overline{D}_{Q_k2} (f)(y_{k1}, y_{k2}) \right)^{1/2} \mu(\widetilde{\Omega}_k)^{\frac{1}{2} - \frac{1}{2}}
\]

\[
\leq \mu(\widetilde{\Omega}_k)^{\frac{1}{2} - \frac{1}{2}} \sum_{R=Q_k1 \times Q_k2 \in \mathcal{R}_k} \mu(R) D_{Q_k1} (x_1, y_{k1}) D_{Q_k2} (x_2, y_{k2}) \overline{D}_{Q_k1} \overline{D}_{Q_k2} (f)(y_{k1}, y_{k2}) \right)^{1/2} \mu(\widetilde{\Omega}_k)^{\frac{1}{2} - \frac{1}{2}}
\]

Note that $a_k$ is supported in $\widetilde{\Omega}_k$. Thus if $1 < q < 2$, the similar estimate and the definition of $\lambda_k$ yield

\[
\|a_k\|_q = \left( C \left\{ \sum_{R=Q_k1 \times Q_k2 \in \mathcal{R}_k} |\overline{D}_{Q_k1} \overline{D}_{Q_k2} (f)(y_{k1}, y_{k2})|^2 \chi_{\cdot} \right\}^{1/2} \mu(\widetilde{\Omega}_k)^{\frac{1}{p} - \frac{1}{q}} \right)^{-1}
\]

\[
\times \left( \sum_{R=Q_k1 \times Q_k2 \in \mathcal{R}_k} \mu(R) D_{Q_k1} (x_1, y_{k1}) D_{Q_k2} (x_2, y_{k2}) \overline{D}_{Q_k1} \overline{D}_{Q_k2} (f)(y_{k1}, y_{k2}) \right)^{1/2} \mu(\widetilde{\Omega}_k)^{\frac{1}{2} - \frac{1}{2}}
\]

\[
\leq \mu(\widetilde{\Omega}_k)^{\frac{1}{2} - \frac{1}{2}} \sum_{R=Q_k1 \times Q_k2 \in \mathcal{R}_k} \mu(R) D_{Q_k1} (x_1, y_{k1}) D_{Q_k2} (x_2, y_{k2}) \overline{D}_{Q_k1} \overline{D}_{Q_k2} (f)(y_{k1}, y_{k2}) \right)^{1/2} \mu(\widetilde{\Omega}_k)^{\frac{1}{2} - \frac{1}{2}}
\]

which implies that $a_k$ satisfies the size condition (2) of $(p,q)$-atoms.

To verify that $a_k$ satisfies the condition (3) and (4) of Definition 2.5, note that if $R \in \mathcal{R}_k$,
then \( R \subset \tilde{\Omega}_k \). From this, it is easy to see that we can further decompose \( a_k(x_1, x_2) \) into

\[
a_k(x_1, x_2) = \frac{1}{\lambda_k} \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \mu(R) D_{Q_{k_1}}(x_1, y_{k_1}) D_{Q_{k_2}}(x_2, y_{k_2}) \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2})
\]

\[
= \frac{1}{\lambda_k} \sum_{\tilde{R} \in \mathcal{M}(\tilde{\Omega}_k)} \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \mu(R) D_{Q_{k_1}}(x_1, y_{k_1}) D_{Q_{k_2}}(x_2, y_{k_2}) \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2})
\]

\[
= \sum_{\tilde{R} \in \mathcal{M}(\tilde{\Omega}_k)} \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \alpha_{\tilde{R}}(x_1, x_2).
\]

Let \( \tilde{R} = Q_1 \times Q_2 \) with \( \text{diam}Q_1 \sim 2^{-k_1} \) and \( \text{diam}Q_2 \sim 2^{-k_2} \) and \( z_i \) be the center of \( Q_i \) with \( i = 1, 2 \). Then \( k_i' \leq k_i \) for \( i = 1, 2 \). From this, it is easy to verify that

\[
\text{supp } a_{\tilde{R}} \subset B_1(z_1, C^2k_1') \times B_2(z_2, C^2k_2').
\]

Obviously, we have that for all \( x_2 \in \mathcal{X}_2 \),

\[
\int_{\mathcal{X}_1} \alpha_{\tilde{R}}(x_1, x_2) \, d\mu_1(x_1) = 0,
\]

and for all \( x_1 \in \mathcal{X}_1 \),

\[
\int_{\mathcal{X}_2} \alpha_{\tilde{R}}(x_1, x_2) \, d\mu_2(x_2) = 0.
\]

Then it remains to show (3). To see that when \( 2 \leq q < \infty \), \( \alpha_{\tilde{R}} \) satisfies the estimate of
(3a), by the same proof for the estimate of \( \|a_k\|_q \), we have

\[
\|a_R\|_q \leq \frac{C}{\lambda_k} \left\| \sum_{R = Q_k \times Q_k \in \mathcal{R}_k} |\mathcal{D}_{Q_k_1} \mathcal{D}_{Q_k_2}(f) (y_{k_1}, y_{k_2})|^2 \chi_R(\cdot, \cdot) \right\|_q^{1/2},
\]

hence, the fact that \( 2 \leq q < \infty \) and the definition of \( \lambda_k \) yield

\[
\sum_{\tilde{R} \in \mathcal{M}(\tilde{\Omega}_k)} \|a_{\tilde{R}}\|_q \leq \|a_k\|^q,
\]

which, by the estimate (2) for \( a_k \), implies that \( a_k \) satisfies (3a) of Definition 2.5. When \( 1 < q < 2 \), we have

\[
\sum_{R \in \mathcal{M}_1(\tilde{\Omega}_k)} \sum_{R \in \mathcal{M}_1(\tilde{\Omega}_k)} \gamma_2^{-\delta} \|a_R\|^q_{L^q(X_1 \times X_2)} \\
\leq C \left\| \sum_{R = Q_k \times Q_k \in \mathcal{R}_k} |\mathcal{D}_{Q_k_1} \mathcal{D}_{Q_k_2}(f) (y_{k_1}, y_{k_2})|^2 \chi_R(\cdot, \cdot) \right\|_q^{1/2} \\
\leq \frac{C}{\lambda_k^q} \sum_{R \in \mathcal{M}_1(\tilde{\Omega}_k)} \gamma_2^{-\delta} (R) \mu(R)^{1-\frac{2}{q}} \left\{ \int_{X_1 \times X_2} \sum_{R = Q_k \times Q_k \in \mathcal{R}_k} |\mathcal{D}_{Q_k_1} \mathcal{D}_{Q_k_2}(f) (y_{k_1}, y_{k_2})|^2 \chi_R(x_1, x_2) d\mu_1(x_1) d\mu_2(x_2) \right\}^{q/2} \\
\leq \frac{C}{\lambda_k^q} \left\{ \sum_{R \in \mathcal{M}_1(\tilde{\Omega}_k)} \gamma_2^{-\delta' (R)} \mu(R) \right\}^{1-\frac{2}{q}} \left\{ \int_{X_1 \times X_2} \sum_{R = Q_k \times Q_k \in \mathcal{R}_k} |\mathcal{D}_{Q_k_1} \mathcal{D}_{Q_k_2}(f) (y_{k_1}, y_{k_2})|^2 \chi_R(x_1, x_2) d\mu_1(x_1) d\mu_2(x_2) \right\}^{q/2} \\
\leq C_{q, \delta} \mu(\tilde{\Omega}_k)^{1-\frac{2}{q}} \mu(\tilde{\Omega}_k)^\frac{q}{2} \leq C_{q, \delta} \mu(\tilde{\Omega}_k)^{1-\frac{2}{q}},
\]

where the last inequality follows from Lemma 2.7. The other summation in (3b) can be
proved by the same manner. This shows that \( a_k \) satisfies (3b) of Definition 2.5.

Note that by the maximal theorem \( \mu(\tilde{\Omega}_k) \leq C\mu(\Omega_k) \). Since if \( (x_1, x_2) \in R \in \mathcal{R}_k \) then
\[
M_s \left( \chi_{R \cap \tilde{\Omega}_k \setminus \Omega_{k+1}} \right) (x_1, x_2) > \frac{1}{2}, \text{ we have } \chi_R(x_1, x_2) \leq 2M_s \left( \chi_{R \cap \tilde{\Omega}_k \setminus \Omega_{k+1}} \right) (x_1, x_2).
\]
Thus, by the Fefferman-Stein vector valued inequality, for all \( 1 < q < \infty \),
\[
\sum_{k=-\infty}^{\infty} \left\{ \sum_{R=Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} |\overline{D}_{Q_{k_1}} D_{Q_{k_2}} (f)(y_{k_1}, y_{k_2})|^2 \chi_R(\cdot, \cdot) \right\}^{q/2} \leq C \int_{\chi_1 \times \chi_2} \left\{ \sum_{R=Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} |\overline{D}_{Q_{k_1}} D_{Q_{k_2}} (f)(y_{k_1}, y_{k_2})|^2 \chi_R(x_1, x_2) \right\}^{q/2} d\mu_1(x_1) d\mu_2(x_2)
\leq C \int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} \left\{ \sum_{R=Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} |\overline{D}_{Q_{k_1}} D_{Q_{k_2}} (f)(y_{k_1}, y_{k_2})|^2 \chi_R(x_1, x_2) \right\}^{q/2} d\mu_1(x_1) d\mu_2(x_2)
\leq C 2^{kp} \mu(\tilde{\Omega}_k).
\]

Therefore, when \( 2 \leq q < \infty \), we have
\[
\sum_{k=-\infty}^{\infty} |\lambda_k|^p = \sum_{k=-\infty}^{\infty} \left\{ \sum_{R=Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} |\overline{D}_{Q_{k_1}} D_{Q_{k_2}} (f)(y_{k_1}, y_{k_2})|^2 \chi_R(\cdot, \cdot) \right\}^{1/2} \left\| \frac{\mu(\tilde{\Omega}_k)}{\mu(\tilde{\Omega}_k)^{1-\frac{p}{q}}} \right\|_q^{1/p}
\leq C \sum_{k=-\infty}^{\infty} 2^{kp} \mu(\tilde{\Omega}_k)^{\frac{q}{p}} \mu(\tilde{\Omega}_k)^{1-\frac{p}{q}} = C \sum_{k=-\infty}^{\infty} 2^{kp} \mu(\tilde{\Omega}_k)
\leq C \sum_{k=-\infty}^{\infty} 2^{kp} \mu(\Omega_k) \leq C \sum_{k=-\infty}^{\infty} 2^{kp} \mu(\Omega_k \setminus \Omega_{k+1})
\leq C \|g_2(f)\|_{L^p(\chi_1 \times \chi_2)}^p,
\]
and when $1 < q < 2$,

$$
\sum_{k=-\infty}^{\infty} |\lambda_k|^p = \sum_{k=-\infty}^{\infty} \left\| \sum_{R=Q_k \times Q_k \in \mathcal{R}_k} |\overline{D}_{Q_k_1} \overline{D}_{Q_k_2}(f)(y_{k_1}, y_{k_2})|^2 \chi_{R}(\cdot, \cdot) \right\|_q^{1/2} \left\| \Omega_k \right\|^{1 - \frac{p}{2}} \\
\leq C \sum_{k=-\infty}^{\infty} 2^{kp} \mu(\Omega_k)^{\frac{p}{2}} \mu(\Omega_k)^{1 - \frac{p}{2}} = C \sum_{k=-\infty}^{\infty} 2^{kp} \mu(\Omega_k) \\
\leq C \|g_2(f)\|_{L^p(X_1 \times X_2)}^p,
$$

which is a desired estimate. Finally, note the fact that the atomic decomposition converges in $L^q(X_1 \times X_2)$ follows from the same proof of the convergence of Lemma 2.1 in [HL3]. This ends the proof of Theorem 2.6.

### 2.4 Boundedness criterions of operators

For an operator on multi-parameter Hardy spaces of homogeneous type, by considering its action on $(p, q)$-atoms, we are able to prove a uniform boundedness criterion as follows.

**Theorem 2.9.** Suppose that $T$ is a bounded linear operator on $L^q(X_1 \times X_2)$ for some $1 < q < \infty$. Let $\epsilon_i \in (0, \theta_i]$ and

$$
\max \left\{ \frac{d_1}{d_1 + \epsilon_1}, \frac{d_2}{d_2 + \epsilon_2} \right\} < p \leq 1.
$$

Then

(1) $T$ is bounded from $H^p(X_1 \times X_2)$ to $L^p(X_1 \times X_2)$ if and only if $\|Ta\|_{L^p(X_1 \times X_2)} \leq C$ for all $(p, q)$-atoms of $H^p(X_1 \times X_2)$;
(2) \( T \) is bounded on \( H^p(\mathcal{X}_1 \times \mathcal{X}_2) \) if and only if \( \|Ta\|_{H^p(\mathcal{X}_1 \times \mathcal{X}_2)} \leq C \) for all \((p, q)\)-atoms of \( H^p(\mathcal{X}_1 \times \mathcal{X}_2) \), where the constant \( C \) is independent of \( a \).

To prove Theorem 2.9, we first claim that for any \((p, q)\)-atom \((0 < p \leq 1 < q < \infty)\) of \( H^p(\mathcal{X}_1 \times \mathcal{X}_2) \), \( a \), there is a constant \( C > 0 \) such that

\[
\|g_2(a)\|_{L^p(\mathcal{X}_1 \times \mathcal{X}_2)} \leq C,
\]

where \( g_2 \) is the discrete Littlewood-Paley square function on \( \mathcal{X}_1 \times \mathcal{X}_2 \) defined in Lemma 2.3.

We give the outline of the proof of the claim that if \( a \) is an \((p, q)\)-product atom for \( 1 < q < 2 \), then \( \|a\|_{H^p(\mathcal{X}_1 \times \mathcal{X}_2)} \leq C \), where \( C \) is a constant independent of \( a \). The proof for this fact when \( q > 2 \) is easier and we omit it here.

In fact, to show \( \|a\|_{H^p(\mathcal{X}_1 \times \mathcal{X}_2)} \leq C \), it suffices to show \( \|g_2(a)\|_{L^p(\mathcal{X}_1 \times \mathcal{X}_2)} \leq C \).

Recall that \( a \) is an atom supported in \( \Omega \) satisfying conditions (1), (2) in Definition 2.5 and the following (i.e. (3b) in Definition 2.5):

\[
a = \sum_{R \in \mathcal{M}_1(\Omega)} a_R + \sum_{R \in \mathcal{M}_2(\Omega)} a_R,
\]

and for any \( \delta > 0 \), there exists a constant \( C_{q, \delta} \) which only depends on \( q \) and \( \delta \), and \( a_R \) satisfying (4), such that

\[
\left\{ \sum_{R \in \mathcal{M}_1(\Omega)} \gamma_2^{-\delta} \|a_R\|^q_{L^q(\mathcal{X}_1 \times \mathcal{X}_2)} + \sum_{R \in \mathcal{M}_2(\Omega)} \gamma_1^{-\delta} \|a_R\|^q_{L^q(\mathcal{X}_1 \times \mathcal{X}_2)} \right\}^{1/q} \leq C_{q, \delta} \mu(\Omega)^{1/q - 1/p}.
\]
We will follow the similar outline as given on page 120 of [Fr1]. Let

$$\tilde{\Omega} = \{(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 : M_s(\chi_{\Omega})(x_1, x_2) > C\},$$

and $\tilde{\Omega} = \overline{\tilde{\Omega}}$. Then by Holder inequality and the boundedness of $g_2$ on $L^q(\mathcal{X}_1 \times \mathcal{X}_2)$ in Lemma 2.3, we have

$$\int_{\tilde{\Omega}} |g_2(a)|^p d\mu_1(x_1) d\mu_2(x_2) \leq \left\{ \int_{\tilde{\Omega}} \|g_2(a)\|^q d\mu_1(x_1) \mu_2(x_2) \right\}^{\frac{p}{q}} \mu(\tilde{\Omega})^{1-\frac{p}{q}}.$$

To estimate $\int_{\tilde{\Omega}} |g_2(a)|^p d\mu_1(x_1) d\mu_2(x_2)$, let $R = Q_1 \times Q_2 \subset \mathcal{M}_2(\Omega)$, $Q$ be the “longest” dyadic cube containing $Q_1$ such that $\mu(Q \times Q_2 \cap \Omega) > \frac{1}{2} \mu(Q \times Q_2)$, $\tilde{Q}$ be the double of $Q$ and $(\tilde{Q})^c$ be the complement of $\tilde{Q}$, we have

$$\int_{(\tilde{Q})^c \times \mathcal{X}_2} |g_2(a_R)|^p d\mu_1(x_1) \mu_2(x_2) \leq C (\gamma_1(R))^{-\delta} \|a_R\|^p_q \mu(R)^{1-\frac{p}{q}}.$$

Summing over $R$ gives

$$\sum_{R \in \mathcal{M}_2(\Omega)} (\gamma_1(R))^{-\delta} \|a_R\|^p_q \mu(R)^{1-\frac{p}{q}} \leq \left\{ \sum_{R \in \mathcal{M}_2(\Omega)} (\gamma_1(R))^{-\delta} \|a_R\|^q_q \right\}^{\frac{p}{q}} \left\{ \sum_{R \in \mathcal{M}_2(\Omega)} (\gamma_1(R))^{-\delta''} \mu(R) \right\}^{1-\frac{p}{q}} \leq C,$$

where $\delta'$, $\delta''$ are constants only dependent on $\delta$, $q$, $p$ and $\mathcal{X}_1$. Here the last inequality above from the condition (3b) of $(p,q)$-atom $a$ in Definition 2.5 and Lemma 2.7.

Hence for any $(p,q)$-atom $a$, there exists a constant $C$, such that $\|a\|_{H^p(\mathcal{X}_1 \times \mathcal{X}_2)} \leq C$. Then
we only need to prove the "if" part of Theorem 2.9. If \( \|Ta\|_{L^p(X_1 \times X_2)} \leq C \) uniformly on all \((p,q)\)-atoms of \( H^p(X_1 \times X_2) \) in \( L^p(X_1 \times X_2) \), then by Theorem 2.6, for \( f \in H^p(X_1 \times X_2) \cap L^q(X_1, X_2) \),

\[
Tf = \sum_k \lambda_k Ta_k.
\]

Since \( T \) is bounded on \( L^q(X_1 \times X_2) \), and \( f = \sum_k \lambda_k a_k \) on \( L^q(X_1 \times X_2) \). Thus

\[
\|Tf\|_p^p \leq \sum_k |\lambda_k|^p \|Ta_k\|_p^p \leq C^p \sum_k |\lambda_k|^p \leq C \|f\|_{H^p}^p.
\]

(2) If \( \|Ta\|_{H^p(X_1 \times X_2)} \leq C \) uniformly on all \((p,q)\)-atoms of \( H^p(X_1 \times X_2) \) in \( H^p(X_1 \times X_2) \), then by Theorem 1.1, for \( f \in H^p(X_1 \times X_2) \cap L^q(X_1, X_2) \),

\[
\|Tf\|_{H^p(X_1 \times X_2)}^p \leq \sum_k |\lambda_k|^p \|Ta_k\|_{H^p(X_1 \times X_2)}^p \leq C^p \sum_k |\lambda_k|^p \leq C \|f\|_{H^p}^p.
\]

Since \( H^p(X_1 \times X_2) \cap L^q(X_1 \times X_2) \) is dense in \( H^p(X_1 \times X_2) \), the proof of Theorem 2.9 is complete.

3 Wolff potentials and regularity of solutions to integral systems on spaces of homogenous type

3.1 Introduction and statements of main results

Wolff potentials on \( \mathbb{R}^n \) were originally studied by Hedberg and Wolff [HW]: Given \( \omega \in M^+(\mathbb{R}^n) \), the class of all positive locally finite Borel measure on \( \mathbb{R}^n \), the (continuous) Wolff
potential $W_{\alpha,p}\omega(x)$ for $\alpha > 0$ and $p > 1$ is defined as

$$W_{\alpha,p}\omega(x) = \int_0^\infty \left[ \frac{\omega(B_t(x))}{t^{n-\alpha p}} \right]^{\frac{1}{p'-1}} \frac{dt}{t}$$

for $x \in \mathbb{R}^n$, where $\omega(B_t(x)) = \int_{B_t(x)} d\omega$ and $p'$ is the conjugate index of $p$. They also introduced the discrete version of Wolff potentials as

$$W_{\alpha,p}^{D}\omega(x) = \sum_{Q \in \mathbb{D}} \left[ \frac{\omega(Q)}{|Q|^{\frac{1}{1-\alpha p}/n}} \right]^{\frac{1}{p'-1}} \chi_Q(x),$$

where $\mathbb{D}$ is the set of all the dyadic cubes $Q \subseteq \mathbb{R}^n$ and $|Q|$ denotes its volume. We define the (continuous) Riesz potential of $\omega$ for $0 < \lambda < n$ as

$$I_\lambda \omega(x) = c \int_{\mathbb{R}^n} |x-y|^\lambda d\omega = \int_0^\infty \frac{\omega(B_t(x))}{t^{n-\lambda}} \frac{dt}{t}.$$

It is evident that $I_\lambda \omega = W_{\frac{n}{\lambda},2}\omega$, and the discrete version of Riesz potentials can be similarly established. Wolff’s theorem ([HW], see also §4.5 in [AH]) states

**Theorem 3.1** (Wolff’s theorem). Let $\alpha > 0$, $1 < p < \infty$, $0 < \alpha p < n$ and $\omega \in \mathcal{M}^+(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} W_{\alpha,p}^{D}\omega(x) d\omega \simeq \int_{\mathbb{R}^n} (I_\alpha \omega(x))^{p'} dx.$$

The brilliant work [HW] of Hedberg and Wolff was originally carried out to fill the gap in the study of Sobolev spaces, however it also has important applications in other areas. Here we mention some interesting examples among them. Note that if $u \geq 0$ is measurable on $\mathbb{R}^n$, then $d\omega = u dx \in \mathcal{M}^+(\mathbb{R}^n)$. 
Example 1.

\[ u(x) = W_{2,2}(u^{\frac{n+\lambda}{n-\lambda}})(x) = I_\lambda(u^{\frac{n+\lambda}{n-\lambda}})(x), \]

and its corresponding semilinear partial differential equation

\[ (-\Delta)^\lambda u = u^{\frac{n+\lambda}{n-\lambda}}. \]

This family of equations are closely related to optimizers of sharp Hardy-Littlewood-Sobolev inequality. See [FL, Lu] for the study of this inequality, and [HLZi] provides some recent results about sharp HLS inequalities on homogeneous spaces of Heisenberg type.

Example 2 (p-Laplacian equations).

\[ u(x) = W_{1,p}(u^q)(x), \]

and its corresponding p-Laplacian equation

\[ -\Delta_p u = -div(\nabla u|\nabla u|^{p-2}) = u^q. \]

Example 3 (Hessian equations).

\[ u(x) = W_{2,2k+1}(u^q)(x). \]

\[ ^i\text{We use HLS to denote Hardy-Littlewood-Sobolev in the following content.} \]
and its corresponding \( k \)-Hessian equation

\[
F_k[-u] = u^q.
\]

Phuc and Verbitsky [PV1] studied Examples 2 and 3, based on systematic use of Wolff potentials. They gave the existence and pointwise estimate of the positive solutions, in terms of the corresponding Wolff potentials. Recently, Ma, Chen and Li [MCL] proved regularity for positive solutions of an integral system associated with Wolff potentials. In this chapter, we shall concentrate on some analogous results on homogeneous spaces, and first record truncated version of Wolff potentials defined above for \( 0 < r \leq \infty \) as

\[
W^r_{\alpha,p}(x) = \int_0^r \left[ \frac{\omega(B_t(x))}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t},
\]

thus \( W^r_{\alpha,p} \) and \( W_{\alpha,p} \) coincide when \( r = \infty \).

**Proposition 3.2** ([PV1]). Let \( \alpha > 0 \), \( 1 < p < \infty \), \( q > p - 1 \), \( \omega \in M^+(\mathbb{R}^n) \) and \( 0 < r \leq \infty \), then the following quantities are equivalent.

\[
\| W^r_{\alpha,p}(\omega) \|_{L^1(\omega)}^{q-1} = \int_{\mathbb{R}^n} \left\{ \int_0^r \left[ \frac{\omega(B_t(x))}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right\}^q \, dx,
\]

\[
\| W^r_{\alpha,p}(\omega) \|_{L^q(dx)}^q = \int_{\mathbb{R}^n} \left\{ \int_0^r \left[ \frac{\omega(B_t(x))}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right\}^q \, dx,
\]

\[
\| W^r_{\alpha,p}(\omega) \|_{L^q(dx)}^q = \int_{\mathbb{R}^n} \left[ \int_0^r \frac{\omega(B_t(x))}{t^{n-\alpha p}} \frac{dt}{t} \right]^{q-1} \, dx.
\]

**Remark.** In Proposition 3.2, (3.1) \( \simeq (3.3) \) is the truncated version of Wolff’s theorem, while
we call \((3.2) \simeq (3.3)\) a HLS type inequality.

In the following sections of this chapter, we will switch our attention to spaces of homogeneous type. We extend \((3.2) \simeq (3.3)\) in Proposition 3.2 to spaces of homogeneous type, followed by an associated HLS inequality for Wolff potentials on spaces of homogenous type.

We define the continuous truncated version of Wolff potentials on spaces of homogeneous type for \(\omega \in M^+(\mathcal{X})\) as

\[
W_{\alpha,p}^r \omega(x) = \int_0^r \left[ \frac{\omega(B_t(x))}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \frac{t^{\frac{1}{p-1}}}{t} dt.
\]

One can similarly define the continuous version \(W_{\alpha,p} \omega = W_{\alpha,p}^\infty \omega\) and the discrete version \(W_{\alpha,p}^D \omega\), using the dyadic construction on spaces of homogeneous type by Christ [Cm] and Sawyer and Wheeden [SW] (see in Lemma 1.2).

One of our main results about the Wolff potentials on spaces of homogeneous type is as follows. Similar result on Garnot groups of arbitrary steps has also been independently obtained by N. Phuc and I. Verbitsky in [PV2].

**Theorem 3.3.** Let \(\alpha > 0, 1 < p < \infty, q > p - 1, \omega \in M^+(\mathcal{X})\) and \(0 < r \leq \infty\), then

\[
\|W_{\alpha,p}^r \omega\|_{L^q(d\mu)}^q = \int_\mathcal{X} \left\{ \int_0^r \left[ \frac{\omega(B_t(x))}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \frac{t^{\frac{1}{p-1}}}{t} dt \right\}^q d\mu \quad (3.4)
\]

\[
\simeq \|I_{\alpha,p}^r \omega\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} = \int_\mathcal{X} \left[ \int_0^r \frac{\omega(B_t(x))}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \frac{t^{\frac{2}{p-1}}}{t} dt \right]^{\frac{q}{p-1}} d\mu. \quad (3.5)
\]

We point out that Wolff’s theorem on spaces of homogeneous type, i.e., the parallel result of \((3.1) \simeq (3.3)\) in Proposition 3.2 on homogeneous spaces was proved by Cascante and Ortega (Theorems 2.7 and 3.1 in [CO]). By a HLS inequality proved by Sawyer and
Wheeden [SW] (see also Sawyer, Wheeden and Zhao [SWZ]) for Riesz potentials on spaces of homogeneous type (i.e., fractional integrals, and they proved weighted version therein), it is not difficult (We also provide the proof in the next section.) to derive the following HLS type inequality for Wolff potentials.

**Theorem 3.4** (HLS type inequality for Wolff potentials). *Let* \( \alpha > 0, 1 < p < \infty, q > p - 1 \) *and* \( \alpha p < N \). *If* \( f \in L^s(d\mu) \) *for* \( s > 1 \), *then*

\[
\| W_{\alpha,p}(f) \|_{L^q(d\mu)} \leq C \| f \|_{L^s(d\mu)}^{\frac{1}{q-1}},
\]

*where* \( \frac{p-1}{q} = \frac{1}{s} - \frac{\alpha p}{N} \).

We apply this inequality to study a Lane-Emden type integral system, that is,

\[
\begin{align*}
  u &= W_{\alpha,p}(v^{q_2}), \\
  v &= W_{\alpha,p}(u^{q_1}),
\end{align*}
\]

under the (critical) condition

\[
\frac{p-1}{q_1 + p - 1} + \frac{p-1}{q_2 + p - 1} = \frac{N - \alpha p}{N},
\]

and when \( u = v \) and \( q_1 = q_2 = q \), (3.6) is reduced to

\[
 u = W_{\alpha,p}(u^q),
\]

which is the Lane-Emden type integral equation, and deduces Examples 1, 2 and 3 above on
homogeneous spaces, given special pairs of $\alpha$ and $p$. Our main regularity theorems state

**Theorem 3.5** (Integrability estimates). Let $\alpha > 0$, $1 < p \leq 2$, $\alpha p < N$, and $q_1, q_2 > 1$, assume that $(u, v)$ is a pair of positive solutions of (3.6) and (3.7) satisfying $(u, v) \in L^{q_1+p-1}(d\mu) \times L^{q_2+p-1}(d\mu)$, then $(u, v) \in L^{s_1}(d\mu) \times L^{s_2}(d\mu)$ for all $s_1$ and $s_2$ such that

$$
\frac{1}{s_1} \in \left(0, \frac{p}{q_1 + p - 1}\right) \cap \left(-\frac{1}{q_2 + p - 1} + \frac{1}{q_1 + p - 1}, \frac{p - 1}{q_2 + p - 1} + \frac{1}{q_1 + p - 1}\right)
$$

and

$$
\frac{1}{s_2} \in \left(0, \frac{p}{q_2 + p - 1}\right) \cap \left(-\frac{1}{q_1 + p - 1} + \frac{1}{q_2 + p - 1}, \frac{p - 1}{q_1 + p - 1} + \frac{1}{q_2 + p - 1}\right).
$$

**Theorem 3.6** ($L^\infty$ estimates). Under the same conditions in Theorem 3.5, $u$ and $v$ are both uniformly bounded on $\mathcal{X}$.

### 3.2 Comparison of Wolff and Reize potentials

For $\alpha > 0$, $1 < p < \infty$ and $\omega \in \mathcal{M}^+(\mathcal{X})$, we define the discrete Wolff potentials on homogeneous space $\mathcal{X}$ by

$$
W_{\alpha,p}^D \omega(x) = \sum_{k} \sum_{\text{diam}(Q) \sim 2^{-k}} \left[ \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \chi_Q(x).
$$

and when $\alpha = \lambda/2$ and $p = 2$, the discrete Riesz follows as

$$
I_\lambda^D \omega(x) = \sum_{k} \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\lambda}{N}}} \chi_Q(x),
$$
Next we will prove the discrete version of Theorem 3.3, that is

**Theorem 3.7** (Discrete version of Theorem 3.3). Let \( \alpha > 0 \), \( 1 < p < \infty \), \( q > p-1 \) and \( \omega \in \mathcal{M}^+(\mathcal{X}) \), then

\[
\| W_{\alpha,p}^D \omega \|_{L^q(d\mu)}^q = \int_{\mathcal{X}} \left\{ \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \left[ \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{q}}} \right] \frac{1}{\mu(Q)^{\frac{1-\alpha p}{q}}} \chi_Q(x) \right\}^q d\mu \tag{3.8}
\]

\[
\simeq \| I_{\alpha p}^D \omega \|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} = \int_{\mathcal{X}} \left[ \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{q}}} \chi_Q(x) \right]^{\frac{q}{p-1}} d\mu. \tag{3.9}
\]

In order to prove Theorem 3.7, discrete version of Theorem 3.3, we need to introduce an equivalent recording of discrete Riesz potentials.

**Lemma 3.8.** Assume the same conditions in Theorem 3.7, define

\[
\Lambda(\omega, \mu) := \int_{\mathcal{X}} \left[ \sup_{k \in \mathbb{Z}_+, \text{diam}(Q) \sim 2^{-k}} \frac{1}{\mu(Q)^{1-\frac{\alpha p}{q}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{\frac{q}{p-1}} d\mu.
\]

Then we have

\[
\Lambda(\omega, \mu) \simeq \| I_{\alpha p}^D \omega \|_{L^{\frac{q}{p-1}}(d\mu)}. \tag{3.10}
\]

**Proof of Lemma 3.8.**

- \( \Lambda(\omega, \mu) \lesssim \| I_{\alpha p}^D \omega \|_{L^{\frac{q}{p-1}}(d\mu)} \)

We need dyadic Hardy-Littlewood maximal function \( M^d \) on \( \mathcal{X} \), which is defined for all
\( \nu \in M^+(\mathcal{X}) \) by

\[
M^d(\nu)(x) = \sup_{x \in Q} \frac{\nu(Q)}{\mu(Q)}.
\]

Since for \( d\nu = |f|d\mu \), the operator \( M^d \) is bounded on \( L^{q/p}(\mu) \) for \( q > p - 1 \), (See, e.g. Theorem 3.1(c) in [ABI].) we have

\[
\Lambda(\omega, \mu) = \int_{\mathcal{X}} \left[ \sup_{k \in \mathbb{Z}_+, \text{diam}(Q) \sim 2^{-k}, x \in Q} \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q)(1 - \frac{2p}{q})} \right]^\frac{q}{p-1} d\mu
\leq \int_{\mathcal{X}} M^d \left[ \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)(1 - \frac{2p}{q})} \right]^\frac{q}{p-1} d\mu
\leq C \left\| I_{op}^D \omega \right\|_{L^{\frac{q}{p-1}}(d\mu)}^\frac{q}{p-1},
\]

which finishes the proof \( \Lambda(\omega, \mu) \lesssim \left\| I_{op}^D \omega \right\|_{L^{\frac{q}{p-1}}(d\mu)}^\frac{q}{p-1} \).

- \( \Lambda(\omega, \mu) \gtrsim \left\| I_{op}^D \omega \right\|_{L^{\frac{q}{p-1}}(d\mu)}^\frac{q}{p-1} \)

First we show that for all \( x \in \mathcal{X} \),

\[
\left[ \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \omega(Q) \chi_Q(x) \right]^\frac{q}{p-1}
\leq \frac{q}{p-1} \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \omega(Q) \chi_Q(x) \left[ \sum_{Q' \subseteq Q} \omega(Q') \chi_{Q'}(x) \right]^{\frac{q}{p-1}-1}.
\]

in three cases.

**Case I:** If

\[
\sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \omega(Q) \chi_Q(x) \leq \infty.
\]

Note that for a fixed \( x \in \mathcal{X} \), the dyadic cubes containing \( x \) form a nested family of
cubes. Hence using the elementary $b^t - a^t \leq t(b-a)b^{t-1}$ for $0 \leq a \leq b$ and $1 \leq t < \infty$, we have

$$
\left[ \sum_{Q' \subseteq Q} \omega(Q') \chi_{Q'}(x) \right]^{\frac{q}{p-1}} - \left[ \sum_{Q' \subseteq Q} \omega(Q') \chi_{Q'}(x) \right]^{\frac{q}{p-1}} \leq \frac{q}{p-1} \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \omega(Q) \chi_Q(x) \left[ \sum_{Q' \subseteq Q} \omega(Q') \chi_{Q'}(x) \right]^{\frac{q}{p-1} - 1}.
$$

From this (3.11) follows by a telescoping sum argument, taking the sums of both sides over all dyadic cubes $Q$ that contain $x$.

**Case II:** If

$$
\sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \omega(Q) \chi_Q(x) = \infty,
$$

but

$$
\sum_{Q \subseteq Q_0} \omega(Q) \chi_Q(x) \leq \infty
$$

for some (and hence every) dyadic cube $Q_0$ which contains $x$, then (3.11) follows by the same argument as in Case I taking the sums over all $Q \subseteq Q_0$ and then letting $\mu(Q_0) \to \infty$.

**Case III:** If

$$
\sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \omega(Q) \chi_Q(x) = \infty,
$$

but

$$
\sum_{Q \subseteq Q_0} \omega(Q) \chi_Q(x) = \infty
$$

for some $Q_0$, then both side of (3.10) are obviously infinite. This completes the proof.
of (3.11).

Next we use induction on $\frac{q}{p-1} > 1$ to prove

$$
\|I_{\alpha}^D \omega\|_{L^{\frac{q}{p}}(d\mu)}^{\frac{q}{p-1}} \leq C \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{q}}} \left[ \frac{1}{\mu(Q)^{1-\frac{\alpha p}{q}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{\frac{q}{p-1} - 1},
$$

where $C$ only depends on $X$, $p$ and $q$.

**Step 1:** To verify (3.12) is true if $1 < \frac{q}{p-1} \leq 2$. By (3.11),

$$
\int_X \left[ \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{q}}} \chi_Q(x) \right] \frac{d\mu}{\mu(Q)^{1-\frac{\alpha p}{q}}},
$$

then by Hölder’s inequality with exponents $\frac{p-1}{q-p+1}$ and $\frac{p-1}{2p-2-q}$, we have

$$
\int_Q \left[ \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{q}}} \chi_{Q'}(x) \right] \frac{d\mu}{\mu(Q)^{1-\frac{\alpha p}{q}}},
$$

$$
\leq \left[ \int_Q \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{q}}} \chi_{Q'}(x) d\mu \right]^{\frac{q-p+1}{p-1}} \mu(Q)^{\frac{2p-2-q}{p-1}},
$$

$$
\leq \left[ \int_Q \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{q}}} \chi_{Q'}(x) d\mu \right]^{\frac{q-1}{p-1} - 1} \left[ \frac{1}{\mu(Q)} \right]^{\frac{q-p+1}{p-1} - 1} \mu(Q),
$$
\[
\begin{align*}
&= \mu(Q) \left[ \frac{1}{\mu(Q)} \int_Q \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1 - \frac{q}{p}}} \chi_{Q'}(x) d\mu \right]^{q \over p - 1 - 1} \\
&\leq \mu(Q) \left[ \frac{1}{\mu(Q)} \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1 - \frac{q}{p}}} \right]^{q \over p - 1 - 1} \\
&\leq \mu(Q) \left[ \frac{1}{\mu(Q)^{1 - \frac{q}{p}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{q \over p - 1 - 1}.
\end{align*}
\]

Therefore,

\[
\|I_{\alpha p}^D \omega\|_{L^{q \over p - 1}(d\mu)} \leq \frac{q}{p - 1} \sum_k \frac{\omega(Q)}{\mu(Q)^{1 - \frac{q}{p}}} \int_Q \left[ \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1 - \frac{q}{p}}} \chi_{Q'}(x) \right]^{q \over p - 1 - 1} d\mu
\]

\[
\leq \frac{q}{p - 1} \sum_k \frac{\omega(Q)}{\mu(Q)^{1 - \frac{q}{p}}} \left[ \frac{1}{\mu(Q)^{1 - \frac{q}{p}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{q \over p - 1 - 1},
\]

which means (3.12) holds for \(1 < \frac{q}{p - 1} \leq 2\).

**Step 2:** Given an integer \(m \geq 2\), we assume that (3.12) holds for any \(\frac{q}{p - 1} \leq m\), then and we show that it also holds for \(\frac{q}{p - 1} \leq m + 1\).

By (3.11) and the induction hypothesis, we have

\[
\|I_{\alpha p}^D \omega\|_{L^{q \over p - 1}(d\mu)} \leq \frac{q}{p - 1} \left( \frac{q}{p - 1} - 1 \right) \sum_k \frac{\omega(Q)}{\mu(Q)^{1 - \frac{q}{p}}} \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1 - \frac{q}{p}}} \left[ \frac{1}{\mu(Q')^{1 - \frac{q}{p}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right]^{q \over p - 1 - 2}
\]
\[ C \frac{q(q-p+1)}{(p-1)^2} \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{np}{m}}} \left[ \frac{1}{\mu(Q')^{1-\frac{np}{m}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right]^{\frac{q}{p-1} - 2} \sum_{Q' \subseteq Q} \frac{\omega(Q)}{\mu(Q)^{1-\frac{np}{m}}} \leq C \frac{q(q-p+1)}{(p-1)^2} \int_X \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{np}{m}}} \chi_{Q'}(x) \left[ \frac{1}{\mu(Q')^{1-\frac{np}{m}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right]^{\frac{q}{p-1} - 2} \times \left[ \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{np}{m}}} \chi_Q(x) \right] d\mu. \]

Note that \( \frac{q}{p-1} - 1 > m - 1 \geq 2 \), by Hölder’s inequality with exponents \( \frac{q}{p-1} - 1 = \frac{q-p+1}{p-1} \)

and \( \frac{q-p+1}{q-2p+2} \), we have

\[ \sum_k \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{np}{m}}} \chi_{Q'}(x) \left[ \frac{1}{\mu(Q')^{1-\frac{np}{m}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right]^{\frac{q}{p-1} - 2} \leq \sum_k \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-\frac{np}{m}}} \chi_{Q'}(x) \left[ \frac{1}{\mu(Q')^{1-\frac{np}{m}}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right]^{\frac{q}{p-1} - 1} \left[ \frac{q-p+1}{q-2p+2} \right] \times \left[ \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{np}{m}}} \chi_Q(x) \right]^{\frac{q-p+1}{q-2p+2}}. \]

Therefore,

\[ \| f_d^{\alpha p} \|_{L^p(d\mu)}^{\frac{q}{p}} \]
\[
\leq C^q \frac{(q-p+1)}{(p-1)^2} \int_X \left[ \sum_{k \text{ diam}(Q) \sim 2^{-k}} \sum_{k'} \frac{\omega(Q)}{\mu(Q)^{1-p}} \chi_Q(x) \right] \left[ \sum_{k' \text{ diam}(Q') \sim 2^{-k'}} \sum_{k''} \frac{\omega(Q')}{\mu(Q')^{1-p}} \chi_{Q'}(x) \right]^{\frac{p-1}{q-p+1}} \frac{q-2p+2}{q-p+1} d\mu \\
\times \left[ \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-p}} \chi_{Q'}(x) \left( \frac{1}{\mu(Q')^{1-p}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right)^{\frac{q}{p-1}-1} \right]^{\frac{q}{q-p+1}} d\mu \\
\leq C^q \frac{(q-p+1)}{(p-1)^2} \int_X \left[ \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-p}} \chi_{Q'}(x) \right] \left[ \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-p}} \chi_{Q'}(x) \right]^{\frac{p-1}{q-p+1}} \frac{q-2p+2}{q-p+1} d\mu.
\]

By using Hölder’s inequality with exponents \( \frac{q-p+1}{p-1} \) and \( \frac{q-p+1}{q-2p+2} \) again, we have

\[
\left\| I_{\alpha p}^D \omega \right\|_{L^{\frac{q}{p-1}}(d\mu)} \leq C^q \frac{(q-p+1)}{(p-1)^2} \int_X \left[ \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-p}} \chi_{Q'}(x) \right]^{\frac{p-1}{q-p+1}} \frac{q}{q-p+1} d\mu \\
\times \left[ \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-p}} \chi_{Q'}(x) \left( \frac{1}{\mu(Q')^{1-p}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right)^{\frac{q}{p-1}-1} \right]^{\frac{q}{q-p+1}} d\mu \\
= C^q \frac{(q-p+1)}{(p-1)^2} \left[ \left\| I_{\alpha p}^D \omega \right\|_{L^{\frac{q}{p-1}}(d\mu)} \right]^{\frac{p-1}{q-p+1}} \frac{q}{q-p+1} d\mu \times \left[ \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-p}} \chi_{Q'}(x) \left( \frac{1}{\mu(Q')^{1-p}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right)^{\frac{q}{p-1}-1} \right]^{\frac{q}{q-p+1}} d\mu \\
\leq C^q \frac{(q-p+1)}{(p-1)^2} \left[ \left\| I_{\alpha p}^D \omega \right\|_{L^{\frac{q}{p-1}}(d\mu)} \right]^{\frac{p-1}{q-p+1}} \frac{q}{q-p+1} d\mu \times \left[ \sum_{k'} \sum_{\text{diam}(Q') \sim 2^{-k'}} \frac{\omega(Q')}{\mu(Q')^{1-p}} \left( \frac{1}{\mu(Q')^{1-p}} \sum_{Q'' \subseteq Q'} \omega(Q'') \right)^{\frac{q}{p-1}-1} \right]^{\frac{q}{q-p+1}} d\mu.
\]
From the above inequality it follows that (3.12) holds for \( m < \frac{q}{p-1} \leq m + 1 \), where \( C \) only depends on \( \mathcal{X}, p \) and \( q \), and then (3.12) is verified for every \( 1 < \frac{q}{p-1} < \infty \).

With the help of (3.12), we compute

\[
\left\| I^D_{\alpha p} \omega \right\|_{L^{\frac{q}{p-1}}(d\mu)} \leq C \int_{\mathcal{X}} \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1 - \frac{ap}{N}}} \chi_Q(x) \left[ \frac{1}{\mu(Q)^{1 - \frac{aq}{N}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{\frac{q}{p-1}-1} d\mu
\]

\[
\leq C \int_{\mathcal{X}} \left[ \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1 - \frac{ap}{N}}} \chi_Q(x) \right] \left[ \sup_{k, \text{diam}(Q) \sim 2^{-k}, x \in Q} \frac{1}{\mu(Q)^{1 - \frac{aq}{N}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{\frac{q}{p-1}-1} d\mu
\]

\[
\leq C \left( \left\| I^d_{\alpha p} \omega \right\|_{L^{\frac{q}{p-1}}(d\mu)} \right)^{\frac{p-1}{q}} \left[ \Lambda(\omega, \mu) \right]^{1 - \frac{p-1}{q}},
\]

where the last estimate we have used Hölder’s inequality with exponents \( \frac{q}{p-1} \) and \( \frac{q}{q-p+1} \).

Thus

\[
\left\| I^D_{\alpha p} \omega \right\|_{L^{\frac{q}{p-1}}(d\mu)} \leq C^{\frac{q}{q-p+1}} \Lambda(\omega, \mu) \lesssim \Lambda(\omega, \mu),
\]

and the proof of Lemma 3.8 is completed. \( \square \)

Next we will use Lemma 3.8 to prove Theorem 3.7.

**Proof of Theorem 3.7.**

- \( \left\| W^D_{\alpha,p} \omega \right\|_{L^q(d\mu)} \gtrsim \left\| I^D_{\alpha p} \omega \right\|_{L^{\frac{q}{p-1}}(d\mu)} \)}
It becomes obvious once one notices that

\[
\|I^{\omega}_{\alpha p}\|_{L^{p-1}(d\mu)} \lesssim \Lambda(\omega, \mu)
\]

\[
= \int_X \left[ \sup_{k \in \mathbb{Z}_+} \frac{1}{\mu(Q)} \frac{1}{(\mu(Q))^{1-\frac{\alpha p}{N}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{\frac{q}{p-1}} d\mu
\]

\[
\leq \int_X \left\{ \sum_k \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \right\}^{\frac{q}{p-1}} \chi_Q(x) d\mu
\]

\[
= \|W^{\omega}_{\alpha p}\|_{L^{q}(d\mu)}^{q}.
\]

\[\bullet \|W^{D}_{\alpha p}\|_{L^{q}(d\mu)}^{q} \lesssim \|I^{D}_{\alpha p}\|_{L^{p-1}(d\mu)}^{\frac{q}{p-1}}.\]

The proof of this direction follows the same line to that given in [PV1]. We only show \(p > 2\), since \(\|W^{D}_{\alpha p}\|_{L^{q}(d\mu)}^{q} \lesssim \|I^{D}_{\alpha p}\|_{L^{p-1}(d\mu)}^{\frac{q}{p-1}}\) is trivial when \(p \leq 2\) by using Minkowski’s inequality. Write \(t = \frac{p-1}{p-2}\) and \(0 < \varepsilon < \frac{\alpha p}{(p-1)n}\), then \(t' = p - 1 > 1\) and

\[-t \left( 1 - \frac{\alpha p}{N} \right) \frac{1}{p-1} + t - t\varepsilon > 1.\]

By Hölder’s inequality, we have

\[
\sum_{Q' \subseteq Q} \frac{\omega(Q')^{\frac{1}{p-1}}}{\mu(Q')^{(1-\frac{\alpha p}{N})\frac{1}{p-1}+1-\varepsilon}}
\]

\[
= \sum_{Q' \subseteq Q} \left[ \frac{\omega(Q')^{\frac{1}{p-1}}}{\mu(Q')^{\frac{1}{p-1}+\varepsilon}} \right] \mu(Q')^{-(1-\frac{\alpha p}{N})\frac{1}{p-1}+1-\varepsilon}
\]

\[
\leq \left[ \sum_{Q' \subseteq Q} \omega(Q')^{\frac{1}{p-1}} \mu(Q')^{\varepsilon t'} \right]^{\frac{1}{t'}} \left[ \sum_{Q' \subseteq Q} \mu(Q')^{-(1-\frac{\alpha p}{N})\frac{1}{p-1}+1-t\varepsilon} \right]^{\frac{1}{t}}
\]
\[ \leq C \omega(Q)^{\frac{1}{p-1}} \mu(Q)^{\epsilon} \mu(Q)^{-(1-\frac{ap}{N}) \frac{1}{p-1} + 1-\epsilon} \]
\[= C \frac{\omega(Q)^{\frac{1}{p-1}}}{\mu(Q)^{(1-\frac{ap}{N}) \frac{1}{p-1} - 1}}. \]

Therefore,

\[
\| W_{\alpha,p}^{D} \omega \|_{L^{q}(d\mu)}^{q} \leq C \sum_{k} \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)^{\frac{1}{p-1}}}{\mu(Q)^{(1-\frac{2p}{N}) \frac{1}{p-1} + q-2} \left[ \frac{\omega(Q)^{\frac{1}{p-1}}}{\mu(Q)^{(1-\frac{ap}{N}) \frac{1}{p-1} - 1}} \right]^{q-1}} \]
\[= C \sum_{k} \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)^{\frac{q}{p-1}}}{\mu(Q)^{(1-\frac{2p}{N}) \frac{q}{p-1} - 1}} \]
\[= C \int_{\mathcal{X}} \left[ \sum_{k} \sum_{\text{diam}(Q) \sim 2^{-k}} \frac{\omega(Q)^{\frac{q}{p-1}}}{\mu(Q)^{(1-\frac{2p}{N}) \frac{q}{p-1}}} \chi_{Q}(x) \right]^{\frac{q}{p-1}} d\mu \]
\[\approx \| I_{\alpha,p}^{D} \omega \|_{L^{q}(\mu)}^{q}, \]

which completes the proof of the Theorem 3.4.

\[\square\]

### 3.3 Proof of HLS inequality

Theorem 3.3 follows evidently from its discrete counterpart, and we give a short proof of the HLS type inequality for Wolff potentials in Theorem 3.4.

**Proof of Theorem 3.4.** From [SW], one have for Riesz potentials

\[ \| I_{\lambda}(f) \|_{L^{q}(d\mu)} \leq C \| f \|_{L^{r}(d\mu)}, \]
where $1 < s \leq q < \infty$, $0 < \lambda < N$, $\frac{1}{q} = \frac{1}{s} - \frac{\lambda}{N}$ and $f \in L^s(d\mu)$. Thus by taking $\alpha > 0$, $1 < p < \infty$ and $\lambda = \alpha p$, we have

$$\|I_{\alpha p}(f)\|_{L^{\frac{q}{q-1}}(d\mu)} \leq C\|f\|_{L^s(d\mu)},$$

where

$$\frac{p-1}{q} = \frac{1}{s} - \frac{\alpha p}{N}.$$  

Then by comparison of Wolff and Riesz potentials in Theorem 3.3, we arrive at

$$\|W_{\alpha,p}(f)\|_{L^s(d\mu)} \leq C\|I_{\alpha p}(f)\|_{L^{\frac{q}{q-1}}(d\mu)} \leq C\|f\|_{L^s(d\mu)},$$

and Theorem 3.4 is verified. \qed

### 3.4 Proof of the integrability and $L^\infty$ estimates

In this section, we prove regularity estimates in Theorems 3.5 and 3.6. The tool is regularity lifting, and let us begin with setting the frame, that is, suppose $V$ is a topological vector space with two extended norms,

$$\| \cdot \|_X, \| \cdot \|_Y : V \to [0, \infty],$$

let $X := \{ v \in V : \|v\|_X < \infty \}$ and $Y := \{ v \in V : \|v\|_Y < \infty \}$. The operator $T : X \to Y$ is said to be contracting if

$$\|Tf - Th\|_Y \leq \eta \|f - h\|_X,$$
∀ f, h ∈ X and some 0 < η < 1. And T is said to be shrinking if

\[ \| Tf \|_Y \leq \theta \| f \|_X, \]

∀ f ∈ X and some 0 < θ < 1.

**Remark.** It is obvious that for a linear operator T, these two conditions above are equivalent. Thus the following theorem is also true for linear shrinking operators.

**Theorem 3.9** (Regularity lifting by contracting operators ([HaL, MCL])). Let T be a contracting operator from X to itself and from Y to itself, and assume that X, Y are both complete. If f ∈ X, and there exists g ∈ Z := X ∩ Y such that f = Tf + g in X, then f ∈ Z.

Now we can prove Theorem 3.5 by using the above lifting Theorem. Without causing any confusion, we simply denote \( \| \cdot \|_{L^p(d\mu)} \) by \( \| \cdot \|_q \), and \( L^q(d\mu) \) by \( L^q \) in the following proof.

**Proof of Theorem 3.5.** For a fixed real number \( a > 0 \), define

\[
v_a(u) = \begin{cases} 
  v(x) & \text{if } |v(x)| > a, \text{ or } |x| > a, \\
  0 & \text{otherwise.}
\end{cases}
\]

Let \( v_b(u) = v(u) - v_a(u) \), and similarly we define \( u_a \) and \( u_b \), then \( v_b \) and \( u_b \) are uniformly bounded by \( a \) in \( B_a(0) \) obviously. It is evident that \( v_a \cdot v_b = 0 \) and \( v^r = (v_a + v_b)^r = v_a^r + v_b^r \) for all \( r > 0 \). Define the linear operator \( T_1 \),

\[
T_1 h(x) = \int_0^\infty \left[ \frac{\int_{B_t(x)} v_{q_2} d\mu}{\mu(B_t(x))^{1 - \frac{ap}{N}}} \right]^{2 - \frac{p}{q}} \left[ \frac{\int_{B_t(x)} v_{q_2 - 1} h d\mu}{\mu(B_t(x))^{1 - \frac{ap}{N}}} \right] \frac{dt}{t}.
\]
Since $u$ satisfies (3.6), $u = W_{\alpha,p}(v^{q_2})$, we have

\[
u(x) = W_{\alpha,p}(v^{q_2})(x)
= \int_0^\infty \left[ \frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{q}} t^{1-\frac{\alpha p}{q}}} \right]^{\frac{2-p}{p-1}} \frac{\int_{B_t(x)} v^{q_2}_a + v^{q_2}_b d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{q}}} dt

= T_1 v(x) + \int_0^\infty \left[ \frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{q}}} \right]^{\frac{2-p}{p-1}} \frac{\int_{B_t(x)} v^{q_2}_b d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{q}}} dt

:= T_1 v(x) + F(x),
\]

and thus $u = T_1 v + F$, in which

\[
F(x) = \int_0^\infty \left[ \frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{q}}} \right]^{\frac{2-p}{p-1}} \frac{\int_{B_t(x)} v^{q_2}_b d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{q}}} dt.
\]

Similarly, we define

\[
T_2 h(x) = \int_0^\infty \left[ \frac{\int_{B_t(x)} u^{q_1} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{q}}} \right]^{\frac{2-p}{p-1}} \frac{\int_{B_t(x)} u^{q_1-1} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{q}}} dt
\]

and

\[
G(x) = \int_0^\infty \left[ \frac{\int_{B_t(x)} u^{q_1} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{q}}} \right]^{\frac{2-p}{p-1}} \frac{\int_{B_t(x)} u^{q_1}_b d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{q}}} dt.
\]

Then we have $v = T_2 u + G$. Define the operator $T(f, g) = (T_1 g, T_2 f)$, equip the product space $L^{q_1+p-1} \times L^{q_2+p-1}$ with norm $\| (f, g) \|_{q_1+p-1, q_2+p-1} = \| f \|_{q_1+p-1} + \| g \|_{q_2+p-1}$, and $L^{s_1} \times L^{s_2}$ with norm $\| (f, g) \|_{s_1, s_2} = \| f \|_{s_1} + \| g \|_{s_2}$. It is easy to see they are both complete under these norms respectively.

Thus we immediately observe that $(u, v)$ solves the equation $(f, g) = T(f, g) + (F, G)$. In
order to apply regularity lifting by contracting operators (Theorem 3.9), we fix the indices $s_1$ and $s_2$ satisfying

$$\frac{1}{s_1} - \frac{1}{s_2} = \frac{1}{q_1 + p - 1} - \frac{1}{q_2 + p - 1}. \quad (3.13)$$

Note that the interval conditions in Theorem 3.5 guarantee the existence of such pairs $(s_1, s_2)$. Then to arrive at the conclusion that $(f, g) \in L^{s_1} \times L^{s_2}$, we need to verify the following conditions, for sufficiently large $a$. (Here $T$ is linear, by the remark above we only need to verify that it is shrinking.)

1. $T$ is shrinking from $L^{q_1 + p - 1} \times L^{q_2 + p - 1}$ to itself.

2. $T$ is shrinking from $L^{s_1} \times L^{s_2}$ to itself.

3. $(F, G) \in L^{q_1 + p - 1} \times L^{q_2 + p - 1} \cap L^{s_1} \times L^{s_2}$, i.e., $F \in L^{q_1 + p - 1} \cap L^{s_1}$ and $G \in L^{q_2 + p - 1} \cap L^{s_2}$.

(1). $T$ is shrinking from $L^{q_1 + p - 1} \times L^{q_2 + p - 1}$ to itself.

First, we show that $\|T_1 h\|_{q_1 + p - 1} \leq \frac{1}{2}\|h\|_{q_2 + p - 1}$ for all $h \in L^{q_2 + p - 1}$. By choosing $\frac{1}{2-p}$ and $\frac{1}{p-1}$ as two conjugate indices in Hölder’s inequality, we have

$$|T_1 h(x)| \leq \left[ \int_0^\infty \left[ \frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{2q_2}{q}}} \right]^{\frac{2-p}{p-1}} dt \left[ \frac{\int_{B_t(x)} v^{q_2-1} h d\mu}{\mu(B_t(x))^{1-\frac{2q_2-1}{q}}} \right]^{\frac{1}{p-1}} \right]^{1-p}$$

$$= \left[ W_{\alpha,p}(v^q(x)) \right]^{2-p} \left[ W_{\alpha,p}(u^{q_2-1}|h|(x)) \right]^{p-1}$$

$$= u^{2-p}(x) \left[ W_{\alpha,p}(u^{q_2-1}|h|(x)) \right]^{p-1}.$$
Thus, applying Hölder’s inequality again,

\[
\|T_1 h\|_{q_1+p-1} \\
\leq \|u^{2-p}\|_{\frac{q_1+p-1}{2-p}} \left[ W_{\alpha,p}(v_a^{q_1-1}|h|) \right]^{p-1}_s \\
= \|u\|_{q_1+p-1}^{2-p} \|W_{\alpha,p}(v_a^{q_2-1}|h|)\|_{s(p-1)}^{p-1} \\
\leq C \|u\|_{q_1+p-1}^{2-p} \|v_a^{q_2-1}|h|\|_{q_2+p-1}^{q_2+p-1} \\
\leq C \|u\|_{q_1+p-1}^{2-p} \|v_a^{q_2-1}|h|\|_{q_2+p-1}^{q_2+p-1} \\
= C \|u\|_{q_1+p-1}^{2-p} \|v_a\|_{q_2+p-1}^{q_2+p-1} |h|_{q_2+p-1},
\]

in which we used HLS type inequality for Wolff potentials in Theorem 3.4 and have

\[
\frac{1}{q_1 + p - 1} = \frac{2 - p}{q_1 + p - 1} + \frac{1}{s}
\]

and

\[
\frac{1}{s} = \frac{q_2}{q_2 + p - 1} - \frac{\alpha p}{N},
\]

which is ensured by the condition (3.7). Thus we choose \(a\) sufficiently large that

\[
C \|u\|_{q_1+p-1}^{2-p} \|v_a\|_{q_2+p-1}^{q_2-1} \leq \frac{1}{2},
\]

since \(u \in L^{q_1+p-1}\) and \(v \in L^{q_2+p-1}\). Then \(\|T_1 h\|_{q_1+p-1} \leq \frac{1}{2}||h||_{q_2+p-1}\) is verified. Similarly we can prove that \(\|T_2 h\|_{q_2+p-1} \leq \frac{1}{2}||h||_{q_1+p-1}\) for all \(h \in L^{q_1+p-1}\) by choosing \(a\) large enough.
Combining them together, we have no difficulty to get

\[ \|T(f, g)\|_{q_1+p-1, q_2+p-1} \]

\[ = \|T_1 g\|_{q_1+p-1} + \|T_2 f\|_{q_2+p-1} \]

\[ \leq \frac{1}{2}(\|g\|_{q_2+p-1} + \|f\|_{q_1+p-1}) \]

\[ = \frac{1}{2}||(f, g)||_{q_1+p-1, q_2+p-1}, \]

and this shows that \( T \) is shrinking from \( L^{q_1+p-1} \times L^{q_2+p-1} \) to itself.

(2). \( T \) is shrinking from \( L^{s_1} \times L^{s_2} \) to itself.

We use the same tool as we did in (1), that is, HLS type inequality for Wolff potentials in Theorem 3.4 with assistance of Hölder’s inequality, by properly choosing the indices. Here, we prove that \( \|T_2 h\|_{s_2} \leq \frac{1}{2} \|h\|_{s_1} \) first,

\[ \|T_2 h\|_{s_2} \]

\[ \leq \|v^{2-p}\|_{q_2+p-1}^{\frac{1}{2-p}} \left\| W_{\alpha, p}(u_{a}^{q_1-1}|h|)^{p-1} \right\|_{t_1} \]

\[ = \|v\|_{q_2+p-1}^{2-p} \left\| W_{\alpha, p}(u_{a}^{q_1-1}|h|)^{p-1} \right\|_{t_1(p-1)} \]

\[ \leq C\|v\|_{q_2+p-1}^{2-p} \|u_{a}^{q_1-1}|h|\|_{t_2} \]

\[ \leq C\|v\|_{q_2+p-1}^{2-p} \|u_{a}^{q_1-1}\|_{q_1+p-1} \|h\|_{s_1} \]

\[ = C\|v\|_{q_2+p-1}^{2-p} \|u_{a}^{q_1-1}\|_{q_1+p-1} \|h\|_{s_1}. \]
in which we choose \( a \) sufficiently large such that

\[
C \|v\|^{2-p}_{q_2+p-1} \|u_0\|^{q_1-1}_{q_1+p-1} \leq \frac{1}{2},
\]

since \( v \in L^{n+p-1} \) and \( u \in L^{n+p-1} \). Thus, \( \|T_2 h\|_{s_2} \leq \frac{1}{2} \|h\|_{s_1} \) for all \( h \in L^{s_1} \). The indices \( s_1, s_2, t_1 \) and \( t_2 \) above satisfy

\[
\frac{1}{s_2} = \frac{2-p}{q_2+p-1} + \frac{1}{t_1},
\]

\[
\frac{1}{t_2} = \frac{q_1-1}{q_1+p-1} + \frac{1}{s_1}
\]

and by (3.13) and (3.7),

\[
\frac{1}{t_1} = \frac{1}{s_2} - \frac{2-p}{q_2+p-1} = \frac{1}{s_1} - \frac{1}{q_1+p-1} + \frac{1}{q_2+p-1} - \frac{2-p}{q_2+p-1} = \frac{1}{s_1} - \frac{1}{q_1+p-1} + \frac{p-1}{q_2+p-1} = \frac{1}{s_1} - \frac{1}{q_1+p-1} + \frac{N-\alpha p}{N} - \frac{p-1}{q_1+p-1} = \frac{1}{s_1} + \frac{q_1-1}{q_1+p-1} - \frac{\alpha p}{N} = \frac{1}{t_2} - \frac{\alpha p}{N},
\]

which ensures us to use HLS type inequality for Wolff potentials in Theorem 3.4, and we need

\[
\frac{1}{t_2} = \frac{q_1-1}{q_1+p-1} + \frac{1}{s_1} < 1,
\]
that is
\[ \frac{1}{s_1} < \frac{p}{q_1 + p - 1}. \]

Similarly we estimate \( T_1 \) for \( h \in L^{s_2} \) if
\[ \frac{1}{s_2} < \frac{p}{q_2 + p - 1}, \]
and easily pass the results to \( L^{s_1} \times L^{s_2} \), i.e.,
\[ \|T(f, g)\|_{s_1, s_2} \leq \frac{1}{2} \|(f, g)\|_{s_1, s_2}, \]
which shows that \( T \) is shrinking from \( L^{s_1} \times L^{s_2} \) to itself.

(3). \( F \in L^{q_1 + p - 1} \cap L^{s_1} \) and \( G \in L^{q_2 + p - 1} \cap L^{s_2} \).

We only estimate \( F \), one notices that \( v_b \) is uniformly bounded by \( a \) in \( B_a(0) \), thus \( v_b \in L^{q_2 + p - 1} \cap L^{s_2} \). Because \( T_1 \) is bounded from \( L^{q_2 + p - 1} \) to \( L^{q_1 + p - 1} \) by (1), then \( F = T_1 v_b \in L^{q_1 + p - 1} \).

Because \( T_1 \) is bounded from \( L^{s_2} \) to \( L^{s_1} \) by (2), then \( F = T_1 v_b \in L^{s_1} \), and we conclude \( F \in L^{q_1 + p - 1} \cap L^{s_1} \).

Applying regularity lifting we finish the proof of Theorem 3.5.

Now we are able to prove \( L^\infty \) estimate.

Proof of Theorem 3.6. It is sufficient to show for \( u \), then the estimate of \( v \) can be proved.
similarly. For any \( x \in \mathcal{X} \), we divide

\[
\begin{align*}
  u(x) &= W_{\alpha,p}(v^{q_2})(x) \\
  &= \int_0^1 \left[ \frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} + \int_1^\infty \left[ \frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \\
  &= I_1(x) + I_2(x),
\end{align*}
\]

in which the first integral

\[
I_1(x) = \int_0^1 \left[ \frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t}
\]

\[
\leq \int_0^1 \left[ \frac{\left( \int_{B_t(x)} 1^{s'} d\mu \right)^{\frac{1}{s'}} \left( \int_{B_t(x)} v^{q_2 s} d\mu \right)^{\frac{1}{s}}}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t}
\]

\[
\leq \|v\|_{q_2s}^{\frac{q_2}{q_2 s}} \int_0^1 \left[ \mu(B_t(x)) \right]^{\frac{1}{p-1} \left( \frac{1}{s'} - 1 + \frac{\alpha p}{N} \right)} \frac{dt}{t}
\]

\[
\leq \|v\|_{q_2s}^{\frac{q_2}{q_2 s}} \int_0^1 t^{\frac{N}{s'} - 1} \left( \frac{1}{s'} - 1 + \frac{\alpha p}{N} \right)^{-1} dt
\]

\[
\leq C_1,
\]

as we choose \( s \) such that \( \|v\|_{q_2s} < \infty \) and \( \frac{1}{s'} - \frac{N - \alpha p}{N} > 0 \), that is, \( \frac{1}{q_2 s} < \frac{\alpha p}{q_2 N} \). By integrability estimate of \( v \) in Theorem 3.5, we only need to check

\[
\frac{\alpha p}{q_2 N} > -\frac{1}{q_1 + p - 1} + \frac{1}{q_2 + p - 1},
\]

this is plain by a simple computation.

We notice that \( C_1 \) is independent of \( x \). To estimate the second integral \( I_2 \), given \( \delta > 0 \),
for all $y \in X$ such that $d(x, y) \leq \delta$, thus we have $d(z, y) \leq k_1(d(z, x) + d(x, y)) \leq k_1(t + \delta)$ for all $z \in B_t(x)$. (Recall the definition of quasi-metric on homogeneous spaces.) We compute

$$I_2(x) = \int_1^\infty \left[ \frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1 - \frac{ap}{N}}} \right]^\frac{1}{p-1} \frac{dt}{t} \leq \int_1^\infty \left[ \frac{\int_{B_{k_1(t+\delta)}(y)} v^{q_2} d\mu}{\mu(B_{k_1(t+\delta)}(y))^{1 - \frac{ap}{N}}} \right]^\frac{1}{p-1} \frac{1}{\mu(B_t(x))^{1 - \frac{ap}{N}}} \frac{dt}{t} \leq \int_1^\infty \left[ \frac{\int_{B_{k_1(t+\delta)}(y)} v^{q_2} d\mu}{\mu(B_{k_1(t+\delta)}(y))^{1 - \frac{ap}{N}}} \right]^\frac{1}{p-1} \frac{k_1(t+\delta)}{t} \frac{dt}{k_1(t+\delta)} \leq k_1^{N-ap+1} (1 + \delta)^{N-ap+1} \int_{k_1(1+\delta)}^\infty \left[ \frac{\int_{B_{k_1(t+\delta)}(y)} v^{q_2} d\mu}{\mu(B_{k_1(t+\delta)}(y))^{1 - \frac{ap}{N}}} \right]^\frac{1}{p-1} \frac{dt}{k_1(t+\delta)} \leq k_1^{N-ap} (1 + \delta)^{N-ap+1} \int_{k_1(1+\delta)}^\infty \left[ \frac{\int_{B_t(y)} v^{q_2} d\mu}{\mu(B_t(y))^{1 - \frac{ap}{N}}} \right]^\frac{1}{p-1} \frac{dt}{t} \leq C_2 W\alpha,p(v^{q_2})(y) = C_2 u(y),$$

in which $C_2$ is independent of $x$ and $y$. Thus, combining $I_1$ and $I_2$, we have

$$u(x) \leq C_1 + C_2 u(y),$$
for any $x$ and $y$ such that $d(x, y) \leq \delta$. $s$-th powering and integrating both sides,

$$
\int_{B_{\delta}(x)} u^s d\mu \leq \int_{B_{\delta}(x)} (C_1 + C_2u(y))^s d\mu \lesssim C\|u\|_s^s
$$

by choosing $s > 1$ in the integrability interval such that $\|u\|_s < \infty$. Then we finish $L^\infty$ estimate by noticing that $C$ is independent of $x$. 

$\Box$
Part II

4 The atomic decomposition of Hardy spaces
associated with different homogeneities

4.1 Introduction and statements of main results

For all functions and operators defined on $\mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R}$ with $x = (x', x_m)$ where $x' \in \mathbb{R}^{m-1}$ and $x_m \in \mathbb{R}$, we denote $|x|_e = (|x'|^2 + |x_m|^2)^{\frac{1}{2}}$ and $|x|_h = (|x'|^2 + |x_m|)^{\frac{1}{2}}$.

Let $K_1 \in L^1_{\text{loc}}(\mathbb{R}^m \setminus \{0\})$ and satisfying

$$\left| \frac{\partial^\alpha}{\partial x'^\alpha} K_1(x) \right| \leq A |x|_e^{-m-|\alpha|} \text{ for all } |\alpha| \geq 0$$

and

$$\int_{r<|x|_e<R} K_1(x) dx = 0$$

for all $0 < r < R < \infty$. We say that the operator $T_1$ defined by

$$T_1(f)(x) = \text{p.v.}(K_1 * f)(x)$$

is a Calderón-Zygmund singular integral operator associated with isotropic homogeneity.

Let $K_2 \in L^1_{\text{loc}}(\mathbb{R}^m \setminus \{0\})$ and satisfying

$$\left| \frac{\partial^\alpha}{\partial (x')^\alpha} \frac{\partial^\beta}{\partial (x_m)^\beta} K_2(x) \right| \leq B |x|_h^{-m-1-|\alpha|-2\beta} \text{ for all } |\alpha| \geq 0, \beta \geq 0$$
and
\[ \int_{r < |x| < R} K_2(x) \, dx = 0 \]
for all \( 0 < r < R < \infty \). We say that the operator \( T_2 \) defined by 
\[ T_2(f)(x) = \text{p.v.}(K_2 * f)(x) \]
is a Calderón-Zygmund singular integral operator associated with non-isotropic homogeneity.

It is well known that both \( T_1 \) and \( T_2 \) are bounded on \( L^p(\mathbb{R}^m) \) for \( 1 < p < \infty \) and of weak type \((1, 1)\). In addition, \( T_1 \) is bounded on the classical isotropic Hardy space, i.e., the classical Hardy space \( H^p(\mathbb{R}^m) \) introduced in [FeS], and \( T_2 \) is bounded on the non-isotropic Hardy spaces \( H^p_{\text{non}}(\mathbb{R}^m) \). Consider the composition of these two Calderón-Zygmund operators which arise from the \( \bar{\partial} \)- Neumann problem, D. H. Phong, E. M. Stein [PS] show that \( T_1 \circ T_2 \) is of weak-type \((1, 1)\), which answered the question asked by Rivière in [WW]. However, \( T_1 \circ T_2 \) is bounded neither on the classical Hardy space \( H^p(\mathbb{R}^m) \) nor the non-isotropic Hardy space \( H^p_{\text{non}}(\mathbb{R}^m) \). Therefore, Y. Han, C. Lin, G. Lu, Z. Ruan and E. Sawyer in [HLLRS] develop a new Hardy space theory and prove that the composition \( T_1 \circ T_2 \) is bounded on these new Hardy spaces. In this chapter we will establish the atomic decomposition of these new Hardy spaces associated with different homogeneities which are defined as follows.

Let \( \psi^{(1)} \in \mathcal{S}(\mathbb{R}^m) \) with \( \text{supp} \, \hat{\psi}^{(1)} \subseteq \{ (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \frac{1}{2} \leq |\xi'|, |\xi_m| \leq 2 \} \) and
\[
| \sum_{j \in \mathbb{Z}} | \hat{\psi}^{(1)}(2^{-j} \xi', 2^{-j} \xi_m) |^2 = 1 \quad \text{for all} \quad (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} \setminus \{(0, 0)\},
\]
\( \psi^{(2)} \in \mathcal{S}(\mathbb{R}^m) \) with \( \text{supp} \, \hat{\psi}^{(2)} \subseteq \{ (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \frac{1}{2} \leq |\xi'|, |\xi_m| \leq 2 \} \) and
\[
| \sum_{k \in \mathbb{Z}} | \hat{\psi}^{(2)}(2^{-k} \xi', 2^{-2k} \xi_m) |^2 = 1 \quad \text{for all} \quad (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} \setminus \{(0, 0)\}.
\]
For \( j, k \in \mathbb{Z} \), let 
\[
\psi^{(1)}_j(x) = \psi^{(1)}_j(x', x_m) = 2^{jm} \psi^{(1)}_j(2^j x', 2^j x_m),
\]
\[
\psi^{(2)}_k(x) = \psi^{(2)}_k(x', x_m) = 2^{j(m+1)} \psi^{(2)}_k(2^k x', 2^{2k} x_m),
\]
and \( \psi_{j,k}(x) = \psi^{(1)}_j(x) * \psi^{(2)}_k(x) \). Then a discrete Littlewood-Paley-Stein square function \( g^d_{\psi, \text{com}} \) is defined by

\[
g^d_{\psi, \text{com}}(f)(x', x_m) = \left\{ \sum_{j,k \in \mathbb{Z}} \sum_{(l', l_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}} |(\psi_{j,k} * f(2^{-(j\wedge k)} l', 2^{-(j\wedge 2k)} l_m))|^{2} \chi_I(x') \chi_J(x_m) \right\}^{\frac{1}{2}},
\]

where \( I \) are dyadic cubes in \( \mathbb{R}^{m-1} \) and \( J \) are dyadic intervals in \( \mathbb{R} \) with the side length \( \ell(I) = 2^{-(j\wedge k)} \) and \( \ell(J) = 2^{-(j\wedge 2k)} \), and the left lower corners of \( I \) and the left end points of \( J \) are \( 2^{-(j\wedge k)} l' \) and \( 2^{-(j\wedge 2k)} l_m \), respectively.

Let \( \mathcal{S}_0(\mathbb{R}^m) = \{ f \in \mathcal{S}(\mathbb{R}^m) : \int_{\mathbb{R}^m} f(x) x^\alpha dx = 0 \text{ for any } |\alpha| \geq 0 \} \). Now we can define the Hardy spaces associated with two different homogeneities by the following

**Definition 4.1.** Let \( 0 < p \leq 1 \). \( H^p_{\text{com}}(\mathbb{R}^m) = \{ f \in \mathcal{S}_0^c(\mathbb{R}^m) : g^d_{\psi, \text{com}}(f) \in L^p(\mathbb{R}^m) \} \). If \( f \in H^p_{\text{com}}(\mathbb{R}^m) \), the norm of \( f \) is defined by \( \| f \|_{H^p_{\text{com}}(\mathbb{R}^m)} = \| g^d_{\psi, \text{com}}(f) \|_{L^p(\mathbb{R}^m)} \).

\( H^p_{\text{com}}(\mathbb{R}^m) \) is independent of the choice of the function \( \psi^{(1)} \) and \( \psi^{(2)} \) and thus it is well-defined. Moreover, for all \( 0 < p < \infty \), we have \( \| g^d_{\psi, \text{com}}(f) \|_{L^p} \sim \| f \|_{L^p} \). In fact, it can also be shown that \( \| g^d_{\psi, \text{com}}(f) \|_{L^p} \sim \| g_{\text{com}}(f) \|_{L^p} \) holds for all \( 0 < p < \infty \) by a similar argument in [FJ], where \( g_{\text{com}}(f)(x) = \left\{ \sum_{j,k} |\psi_{j,k} * f(x)|^2 \right\}^{\frac{1}{2}} \).

Now we can introduce the \((p, 2)\)-atom of \( H^p_{\text{com}}(\mathbb{R}^m) \) for \( 0 < p \leq 1 \).

**Definition 4.2.** A function \( a(x', x_m) \) on \( \mathbb{R}^{m-1} \times \mathbb{R} \) is called a \((p, 2)\)-atom of \( H^p_{\text{com}}(\mathbb{R}^m) \) for \( 0 < p \leq 1 \), if it satisfies

1. \( \text{supp } a \subset \Omega \), where \( \Omega \) is an open set of \( \mathbb{R}^m \) with finite measure;
(2) \( \|a\|_{L^2(\mathbb{R}^m)} \leq |\Omega|^{\frac{1}{2} - \frac{1}{p}} \). Moreover, \( a \) can be further decomposed into rectangle atom \( a_R \) associated with the rectangle \( R = I \times J \subset \mathbb{R}^{m-1} \times \mathbb{R} \). To be precise,

(3) \[
a = \sum_{R=I \times J \in \mathcal{M}(\Omega)} a_R \quad \text{and} \quad \left\{ \sum_{R \in \mathcal{M}(\Omega)} \|a_R\|_{L^2(\mathbb{R}^m)}^2 \right\}^{\frac{1}{2}} \leq |\Omega|^{\frac{1}{2} - \frac{1}{p}}.
\]

(4) For all \( x' \in \mathbb{R}^{m-1} \),

\[
\int_{\mathbb{R}} a_R(x', x_m) dx_m = 0
\]

and for all \( x_m \in \mathbb{R} \),

\[
\int_{\mathbb{R}^{m-1}} a_R(x', x_m) dx' = 0.
\]

**Theorem 4.3.** For \( 0 < p \leq 1 \) and \( f \in L^2(\mathbb{R}^m) \cap H^p_{\text{com}}(\mathbb{R}^m) \), there is a sequence of numbers, \( \{\lambda_k\}_{k \in \mathbb{Z}} \), and a sequence of \((p,2)\)-atoms of \( H^p_{\text{com}}(\mathbb{R}^m) \), \( \{a_k\}_{k \in \mathbb{Z}} \), such that

\[
\left( \sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{\frac{1}{p}} \leq C \|f\|_{H^p_{\text{com}}(\mathbb{R}^m)}
\]

with the constant \( C \) independent of \( f \) and

\[
f = \sum_{k=-\infty}^{\infty} \lambda_k a_k,
\]

where the series converges to \( f \) in both the \( L^2(\mathbb{R}^m) \) and \( H^p_{\text{com}}(\mathbb{R}^m) \) norms.
4.2 Proof of the atomic decomposition of $H_{com}^p(\mathbb{R}^m)$

For $i = 1, 2$, let $\phi^{(i)} \in \mathcal{S}(\mathbb{R}^m)$ with $\text{supp } \phi^{(i)} \subseteq B(0, 1)$,

$$
\sum_{j \in \mathbb{Z}} |\hat{\phi}^{(1)}(2^{-j}\xi)|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}^m \setminus \{0\},
$$

and

$$
\sum_{k \in \mathbb{Z}} |\hat{\phi}^{(2)}(2^{-k}\xi', 2^{-2k}\xi_m)|^2 = 1 \quad \text{for all } (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} \setminus \{(0, 0)\}.
$$

Moreover,

$$
\int_{\mathbb{R}^m} \phi^{(1)}(x) x^\alpha dx = 0 \quad \text{for all } |\alpha| \leq 10M
$$

and

$$
\int_{\mathbb{R}^m} \phi^{(2)}(x) x^\beta dx = 0 \quad \text{for all } |\beta| \leq 10M,
$$

where $M$ is a fixed large positive integer depending on $p$. Set $\phi_{j,k} = \phi^{(1)}_j * \phi^{(2)}_k$, where $\phi^{(1)}_j(x) = 2^{jm} \phi^{(1)}(2^j, x)$ and $\phi^{(2)}_k(x', x_m) = 2^{k(m+1)} \phi^{(2)}(2^k x', 2^{2k} x_m)$. To show Theorem 4.3, we need the following two lemmas.

**Lemma 4.4 ([HLLRS]).** For any $f \in L^2(\mathbb{R}^m) \cap H_{com}^p(\mathbb{R}^m)$, there exists $\tilde{f} \in L^2(\mathbb{R}^m) \cap H_{com}^p(\mathbb{R}^m)$ such that for a sufficiently large $N \in \mathbb{N}$,

$$
f(x', x_m) = \sum_{j,k \in \mathbb{Z}} \sum_{I=(l', l_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}} |I||J| \phi_{j,k}(x' - 2^{-j+k-N} l', x_m - 2^{-j+2k-N} l_m) \\
\times (\phi_{j,k} * \tilde{f})(2^{-j+k-N} l', 2^{-j+2k-N} l_m),
$$

where the series converges in $L^2$, $I$ are dyadic cubes in $\mathbb{R}^{m-1}$ and $J$ are dyadic intervals in
with the side length $\ell(I) = 2^{-(j \wedge k) - N}$ and $\ell(J) = 2^{-(j \wedge 2k) - N}$, and the left lower corners of $I$ and the left end points of $J$ are $2^{-(j \wedge k) - N}l'$ and $2^{-(j \wedge 2k) - N}l_m$, respectively. Moreover,

$$\|f\|_{L^2(\mathbb{R}^m)} \approx \|	ilde{f}\|_{L^2(\mathbb{R}^m)},$$

and

$$\|f\|_{H^p_{\text{com}}(\mathbb{R}^m)} \approx \|	ilde{f}\|_{H^p_{\text{com}}(\mathbb{R}^m)}.$$}

**Lemma 4.5 ([HLLRS]).** Let $0 < p \leq 1$ and all the notation be the same as in Lemma 4.4. Then for $f \in L^2(\mathbb{R}^m) \cap H^p_{\text{com}}(\mathbb{R}^m)$,

$$\|f\|_{H^p_{\text{com}}} \approx \left\| \sum_{j,k \in \mathbb{Z}} \sum_{(l', l_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}} |(\phi_{j,k} \ast f)(2^{-j \wedge k - N}l', 2^{-j \wedge 2k - N}l_m)|^2 \chi_{I \times J} \right\|_{L^p}^{1/2}.$$

Now we can prove Theorem 4.3.

**Proof.** For any $f \in L^2(\mathbb{R}^m) \cap H^p_{\text{com}}(\mathbb{R}^m)$, let $\phi_{j,k}, \tilde{f}, I$ and $J$ are the same as in Lemma 4.4. For any $i \in \mathbb{Z}$, set

$$\Omega_i = \{(x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \bar{g}_{\phi}^d(f)(x', x_m) > 2^i\},$$

and

$$\mathcal{B}_i = \{(j, k, l) : |(I \times J) \cap \Omega_i| > \frac{1}{2}|I||J|, |(I \times J) \cap \Omega_{i+1}| \leq \frac{1}{2}|I||J|\},$$
where

\[
\cd_{\phi}(f)(x', x_m) = \left\{ \sum_{j,k \in \mathbb{Z}} \sum_{(l', l_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}} |(\phi_{j,k} \ast \tilde{f}(2^{-(j \land k) - N} l', 2^{-(j \land k) - N} l_m))|2 \chi_I(x') \chi_J(x_m) \right\}^{1/2},
\]

and for fixed \( N > 0, I \subset \mathbb{R}^{m-1}, J \subset \mathbb{R} \) are dyadic cubes (intervals) determined by \( j, k \in \mathbb{Z} \) and \( l = (l', l_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z} \) as in Lemma 4.4, that is, \( \ell(I) = 2^{-(j \land k) - N} \) and \( \ell(J) = 2^{-(j \land k) - N} \), \( 2^{-(j \land k) - N} l' \) and \( 2^{-(j \land k) - N} l_m \) are the left lower corners of \( I \) and \( J \), respectively.

By Lemma 4.4, we can write

\[
f(x', x_m) = \sum_i \sum_{(j,k,l) \in B_i} |I| |J| \phi_{j,k}(x' - 2^{-(j \land k) - N} l', x_m - 2^{-(j \land k) - N} l_m)
\times (\phi_{j,k} \ast \tilde{f})(2^{-(j \land k) - N} l', 2^{-(j \land k) - N} l_m)
= \sum_i \lambda_i a_i(x', x_m),
\]

where

\[
a_i(x', x_m) = \frac{1}{\lambda_i} \sum_{(j,k,l) \in B_i} |I| |J| \phi_{j,k}(x' - 2^{-(j \land k) - N} l', x_m - 2^{-(j \land k) - N} l_m)
\times (\phi_{j,k} \ast \tilde{f})(2^{-(j \land k) - N} l', 2^{-(j \land k) - N} l_m)
\]

and

\[
\lambda_i = \left\| \left\{ \sum_{(j,k,l) \in B_i} |(\phi_{j,k} \ast \tilde{f})(2^{-(j \land k) - N} l', 2^{-(j \land k) - N} l_m)|^2 \chi_I \chi_J \right\}^{1/2} \right\|_2 \left\| \Omega_i \right\|^{1/2},
\]

where

\[
\tilde{\Omega}_i = \{ x = (x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : M_s(\chi_{\Omega_i})(x) > \frac{1}{2^{N+1}} \}.
\]

Note that \( \text{supp } \phi^{(1)} \subseteq B(0, 1) \) and \( \text{supp } \phi^{(2)} \subseteq B(0, 1), \phi_{j,k}(x) = \phi^{(1)}_{j,k}(x), \phi^{(2)}_{j,k}(x) = \phi^{(1)}_{j,k}(x'), \phi_{j,k}(x) = 2^{jm} \phi^{(1)}_{j,k}(2^j x', 2^j x_m), \) and \( \phi^{(2)}_{j,k}(x) = \phi^{(2)}_{j,k}(2^j x', 2^j x_m) = 2^{k(m+1)} \phi^{(2)}_{j,k}(2^j x', 2^j x_m) \), then
for any \( j, k \in \mathbb{Z} \), \( \phi_{j,k} \) is supported in \( B(0, 2^{-j^{\wedge}k}) \times B(0, 2^{-j^{\wedge}2k}) \subset \mathbb{R}^{m-1} \times \mathbb{R} \).

Since for \( (j, k, l) \in B_i \), we have \(|(I \times J) \cap \Omega_i| > \frac{1}{2} |I||J| \Rightarrow 2^N(I \times J) \subset \tilde{\Omega}_i \), this implies \( a_i \) is supported in

\[
\bigcup_{(j,k,l) \in B_i} (2^{-(j^{\wedge}k)}Nl' + B(0, 2^{-(j^{\wedge}k)}), 2^{-(j^{\wedge}2k)}Nl_m + B(0, 2^{-(j^{\wedge}2k)})) \subset \bigcup_{(j,k,l) \in B_i} 2^N(I \times J) \subset \tilde{\Omega}_i,
\]

and hence \( a_i \) satisfies (1) in Definition 4.2.

To see that \( a_i \) satisfies (2) in Definition 4.2, let \( h \in L^2(\mathbb{R}^m) \), by H"older's inequality, Lemma 4.4, and Lemma 4.5, we have

\[
\left\| \sum_{(j,k,l) \in B_i} |I||J| \phi_{j,k}(x' - 2^{-j^{\wedge}k-Nl'}, x_m - 2^{-j^{\wedge}2k-Nl_m})(\phi_{j,k} * \tilde{f})(2^{-j^{\wedge}k-Nl'}, 2^{-j^{\wedge}2k-Nl_m}) \right\|_2
\]

\[
= \sup_{\|h\|_{L^2} \leq 1} \left\| \sum_{(j,k,l) \in B_i} |I||J| \phi_{j,k}(x' - 2^{-j^{\wedge}k-Nl'}, x_m - 2^{-j^{\wedge}2k-Nl_m})(\phi_{j,k} * \tilde{f})(2^{-j^{\wedge}k-Nl'}, 2^{-j^{\wedge}2k-Nl_m}), h > \right\|
\]

\[
= \sup_{\|h\|_{L^2} \leq 1} \int_{\mathbb{R}^{m-1} \times \mathbb{R}} \int_{I \times J} \sum_{(j,k,l) \in B_i} \phi_{j,k}(x' - 2^{-j^{\wedge}k-Nl'}, x_m - 2^{-j^{\wedge}2k-Nl_m})(\phi_{j,k} * \tilde{f})(2^{-j^{\wedge}k-Nl'}, 2^{-j^{\wedge}2k-Nl_m})
\]

\[
h(x', x_m) dy' dy_m dx' dx_m
\]

\[
= \sup_{\|h\|_{L^2} \leq 1} \int_{I \times J} \sum_{(j,k,l) \in B_i} \phi_{j,k} * h(2^{-j^{\wedge}k-Nl'}, 2^{-j^{\wedge}2k-Nl_m})(\phi_{j,k} * \tilde{f})(2^{-j^{\wedge}k-Nl'}, 2^{-j^{\wedge}2k-Nl_m}) dy' dy_m
\]

\[
\leq \sup_{\|h\|_{L^2} \leq 1} \left\| \sum_{(j,k,l) \in B_i} |\phi_{j,k} * h(2^{-j^{\wedge}k-Nl'}, 2^{-j^{\wedge}2k-Nl_m})|^2 \chi_{I \times J} \right\|_{L^2}^{1/2}
\]

\[
\times \left\| \sum_{(j,k,l) \in B_i} |\phi_{j,k} * \tilde{f}(2^{-j^{\wedge}k-Nl'}, 2^{-j^{\wedge}2k-Nl_m})|^2 \chi_{I \times J} \right\|_{L^2}^{1/2}
\]

\[
\leq C \sup_{\|h\|_{L^2} \leq 1} \left\| \tilde{g}_h^{(j)}(h) \right\|_2 \left\| \sum_{(j,k,l) \in B_i} |\phi_{j,k} * \tilde{f}(2^{-j^{\wedge}k-Nl'}, 2^{-j^{\wedge}2k-Nl_m})|^2 \chi_{I \times J} \right\|_{L^2}^{1/2}
\]

\[
\leq C \left\| \sum_{(j,k,l) \in B_i} |\phi_{j,k} * \tilde{f}(2^{-j^{\wedge}k-Nl'}, 2^{-j^{\wedge}2k-Nl_m})|^2 \chi_{I \times J} \right\|_{L^2}^{1/2}.
\]
The above estimate implies the size condition (2) of $a_i$, since

$$\|a_i\|_2 = \left(\left\{\sum_{(j,k,l) \in B_i} |(\phi_{j,k} * \tilde{f})(2^{-j/2k-N} l', 2^{-j/2k-2N} l_m)|^2 \chi_{I \times J}\right\}^{1/2} \left|\tilde{\Omega}_i\right|^{1/p-\frac{1}{2}}\right)^{-1} \times \left\{\sum_{(j,k,l) \in B_i} |(\phi_{j,k} * \tilde{f})(2^{-j/2k-N} l', 2^{-j/2k-2N} l_m)|^2 \chi_{I \times J}\right\}^{1/2} \leq |\tilde{\Omega}_i|^{1/2-\frac{1}{p}}. $$

To verify $a_i$ satisfies conditions (3) and (4), note that if $(j,k,l) \in B_i$, then $R = I \times J \subseteq \tilde{\Omega}_i$ and there exists a $\tilde{R} \in \mathcal{M}(\tilde{\Omega}_i)$ such that $R \subseteq \tilde{R}$. Therefore, we can further decompose $a_i$ into

$$a_i(x', x_m) = \sum_{\tilde{R} \in \mathcal{M}(\tilde{\Omega}_i)} a_{\tilde{R}}(x', x_m),$$

where

$$a_{\tilde{R}}(x', x_m) = \frac{1}{\lambda_i} \sum_{(j,k,l) \in B_i, R = I \times J \subset \tilde{R} \in \mathcal{M}(\tilde{\Omega}_i)} |I||J| \phi_{j,k}(x' - 2^{-j/2k-N} l', x_m - 2^{-j/2k-2N} l_m)$$

$$\times (\phi_{j,k} * \tilde{f})(2^{-j/2k-N} l', 2^{-j/2k-2N} l_m).$$

We can see that $\text{supp } a_{\tilde{R}} \subset \sum_{(j,k,l) \in B_i} 2^{N}(I \times J) \subset 2^{N} \tilde{R}$ and the cancellation conditions (4) follow directly from the conditions on $\phi^{(1)}$ and $\phi^{(2)}$. On the other hand, by the same proof for the estimate of $\|a_i\|_2$, we have

$$\|a_{\tilde{R}}\|_2 \leq \frac{C}{\lambda_i} \left\{\sum_{(j,k,l) \in B_i, R = I \times J \subset \tilde{R} \in \mathcal{M}(\tilde{\Omega}_i)} |(\phi_{j,k} * \tilde{f})(2^{-j/2k-N} l', 2^{-j/2k-2N} l_m)|^2 \chi_{I \times J}\right\}^{1/2}. $$
Therefore, by the definition of \( \lambda_i \) we have \( \sum_{\tilde{R} \in \mathcal{M}(\tilde{\Omega}_i)} \|a_{\tilde{R}}\|_2 \leq \|a_i\|_2 \leq |\tilde{\Omega}_i|^{\frac{1}{2} - \frac{1}{p}}. \)

Note that by the maximal theorem \( |\tilde{\Omega}_i| \leq C|\Omega| \). Since if \((j, k, l) \in \mathcal{B}_i \) and \( x = (x', x_m) \) belongs to the corresponding \( R = I \times J \), then \( M_s(\chi_{R \cap \tilde{\Omega}_i \setminus \Omega_{i+1}}(x', x_m) > \frac{1}{2} \), we have \( \chi_R(x', x_m) \leq 2M_s(\chi_{R \cap \tilde{\Omega}_i \setminus \Omega_{i+1}})(x', x_m) \). Thus,

\[
\left\| \left\{ \sum_{(j, k, l) \in \mathcal{B}_i} |(\phi_{j,k} * \tilde{f})(2^{-j \wedge k - N} l', 2^{-j \wedge 2k - N} l_m)|^2 \chi_{I \times J} \right\}^{1/2} \right\|_2^2 \\
\leq C \int_{\mathbb{R}^{m-1} \times \mathbb{R}} \sum_{(j, k, l) \in \mathcal{B}_i} |(\phi_{j,k} * \tilde{f})(2^{-j \wedge k - N} l', 2^{-j \wedge 2k - N} l_m)|^2 M_s(\chi_{R \cap \tilde{\Omega}_i \setminus \Omega_{i+1}})(x', x_m) \, dx' \, dx_m \\
\leq C \int_{\tilde{\Omega}_i \setminus \Omega_{i+1}} \sum_{(j, k, l) \in \mathcal{B}_i} |(\phi_{j,k} * \tilde{f})(2^{-j \wedge k - N} l', 2^{-j \wedge 2k - N} l_m)|^2 \chi_R(x', x_m) \, dx' \, dx_m \\
\leq C 2^i |\tilde{\Omega}_i|.
\]

Therefore, by the definition of \( \Omega_i \), we have

\[
\sum_{i=-\infty}^{\infty} |\lambda_i|^p \leq C \sum_{i=-\infty}^{\infty} 2^i |\tilde{\Omega}_i|^\frac{p}{2} |\tilde{\Omega}_i|^{1 - \frac{p}{2}} \\
= C \sum_{i=-\infty}^{\infty} 2^i |\tilde{\Omega}_i| \leq C \sum_{i=-\infty}^{\infty} 2^i |\Omega_i| \\
\leq C \sum_{i=-\infty}^{\infty} 2^i |\Omega_i \setminus \Omega_{i+1}| \\
\leq C \|\tilde{g}_\phi(f)\|_p \leq C \|f\|_{H_{\text{com}}^p(\mathbb{R}^m)}.
\]

\[\square\]

For a \( L^2 \)-bounded linear operator on \( H_{\text{com}}^p(\mathbb{R}^m) \), consider its action on \((p, 2)\)-atoms, we have the following boundedness criterion.
Theorem 4.6. Let $T$ is bounded linear operator on $L^2(\mathbb{R}^m)$, then $T$ is bounded from $H_{com}^p(\mathbb{R}^m)$ to $L^p(\mathbb{R}^m)$ if and only if $\|Ta\|_{L^p(\mathbb{R}^m)} \leq C$ for all $(p, 2)$ atoms of $H_{com}^p(\mathbb{R}^m)$.

Here we omit the proof of Theorem 4.6 because it is same with the proof of Theorem 2.9.

5 The duality theorem of weighted multi-parameter Hardy spaces associated with Zygmund dilation

5.1 Introduction and statements of main results

The celebrated $H^1(\mathbb{R}^n) - BMO(\mathbb{R}^n)$ duality theorem was proved by C. Fefferman and Stein [Fc, FcS] in one-parameter case. In multi-parameter setting, S-Y. A. Chang and R. Fefferman [CF1, CF3] proved that the dual space of the product $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ is the product $BMO(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ using the bi-Hilbert transform.

Among the multi-parameter analysis, the Zygmund dilations are the simplest after pure product space dilations. (See R. Fefferman’s survey [Fr2].) Recently, Y. Han and G. Lu [HL2, HL3] developed a unified approach of multi-parameter Hardy space theory using the discrete multi-parameter Littlewood-Paley-Stein analysis, and the $H^p_Z - CMO^p_Z$ duality theorem (Theorem 1.6 in [HL2]) is one of their subsequent work, where $H^p_Z$ is the multi-parameter Hardy space associated with Zygmund dilations and $CMO^p_Z$ is the Carleson measure spaces associated with Zygmund dilations.

We will characterize the dual spaces of the weighted multi-parameter Hardy spaces associated with Zygmund dilations, that is, $(H^p_Z(\omega))^* = CMO^p_Z(\omega)$ for all $0 < p \leq 1$ and $\omega \in A_\infty(\mathcal{Z})$. Such Carleson measure spaces $CMO^p_Z(\omega)$ play the same role as the John-
Nirenberg \(BMO\) spaces in the duality \(H^1(\mathbb{R}^n) - BMO(\mathbb{R}^n)\) in the one-parameter setting.

Let us first establish the preliminaries for Zygmund dilations and recall the related background briefly. In \(\mathbb{R}^3\), the Zygmund dilation is given by \(\rho_{s,t}(x, y, z) = (sx, ty, stz)\) for \(s, t > 0\), and the maximal operator associated to Zygmund dilations is defined by

\[
M_Z f(x, y, z) = \sup_{Q \in \mathcal{R}_Z} \frac{1}{|Q|} \int_Q |f|,
\]

where \(\mathcal{R}_Z\) is the class of rectangles whose sides are parallel to the axes and have side lengths of the form \(s, t,\) and \(st\). As a special case of Córdoba’s solution \([Ca]\) of Zygmund’s conjecture, the operator \(M_Z\) is bounded from the Orlicz space \(L \log^+ L(Q_1)\) to weak \(L^1(Q_1)\). \((Q_1\) is the unit cube in \(\mathbb{R}^3).\) The weighted \(L^p\) boundedness of \(M_Z\) for \(1 < p < \infty\) was proved by R. Fefferman \([Fr1]\), see also \([FP]\) and various generalizations in \([JT]\).

Write \(\mathcal{S}(\mathbb{R}^n)\) as the space of Schwartz functions in \(\mathbb{R}^n\). The test function defined on \(\mathbb{R}^3\) is given by

\[
\psi(x, y, z) = \psi^{(1)}(x)\psi^{(2)}(y, z),
\]

where \(\psi^{(1)} \in \mathcal{S}(\mathbb{R})\) and \(\psi^{(2)} \in \mathcal{S}(\mathbb{R}^2)\) satisfy

\[
\sum_{j \in \mathbb{Z}} |\hat{\psi}^{(1)}(2^{-j}\xi_1)|^2 = 1 \text{ for all } \xi_1 \in \mathbb{R} \setminus \{0\},
\]

\[
\sum_{j \in \mathbb{Z}} |\hat{\psi}^{(2)}(2^{-k}\xi_2, 2^{-k}\xi_3)|^2 = 1 \text{ for all } (\xi_2, \xi_3) \in \mathbb{R}^2 \setminus \{(0, 0)\},
\]
and the moment conditions

$$\int_{\mathbb{R}} x^\alpha \psi^{(1)}(x) dx = \int_{\mathbb{R}^2} y^\beta z^\gamma \psi^{(2)}(y, z) dy dz = 0$$

for all integers $\alpha, \beta, \gamma \geq 0$. By taking Fourier transform, it is easy to see the continuous version of Calderón’s identity

$$f(x, y, z) = \sum_{j,k} \psi_{j,k} * \psi_{j,k} * f(x, y, z), \quad (5.2)$$

where

$$\psi_{j,k}(x, y, z) = 2^{-2(j+k)} \psi^{(1)}(2^j x) \psi^{(2)}(2^k y, 2^{j+k} z), \quad (5.3)$$

and the series converges in $L^2$. Ricci and Stein [RS] introduced the what is now called Ricci-Stein singular integral operator $T_Z$ as $T_Z = K * f$, and

$$K(x, y, z) = \sum_{j,k} 2^{-2(j+k)} \psi_{j,k} \left( \frac{x}{2^j}, \frac{y}{2^k}, \frac{z}{2^{j+k}} \right),$$

where the functions $\psi_{j,k}$ are test functions in $\mathcal{S}(\mathbb{R}^3)$. They also gave the $L^p$ ($1 < p < \infty$) boundedness of the operator $T_Z$. The weighted $L^p$ boundedness theorem was proved by R. Fefferman and Pipher (Theorem 2.4 in [FP]) when $w \in A_p(Z)$. The authors in [HL2] proved that both the convolution and non-convolution type Ricci-Stein operators are bounded on $H^p_Z$ and $BMO_Z$. While the other paper [HLX2] will show the boundedness result on weighted $H^p_Z$ spaces when $w \in A_\infty(Z)$, it is interesting to note that we only require $w \in A_\infty(Z)$ which

---

The multi-parameter Hardy space associated with Zygmund dilations $H^p_Z$ is defined in the following content, see [HL2] for more information, where one can also find a nice historical note in the introductory section.
is much weaker than the usual requirement \( w \in A_1 \) for boundedness of singular integral operators on weighted Hardy spaces. Using the \( A_\infty \) weight to consider the boundedness of singular integrals on weighted multiparameter Hardy spaces seems to be first used in [DHLW]. (See also [R] for the case of more parameters.)

Now we define the Littlewood-Paley-Stein square function of \( f \) associated with the Zygmund dilation,

\[
g_Z(f)(x,y,z) = \left\{ \sum_{j,k} |\psi_{j,k} \ast f(x,y,z)|^2 \right\}^{1/2}.
\]

(5.4)

From Ricci and Stein’s \( L^p \) boundedness of the operator \( T_Z \), together with the \( L^2 \) convergence of Calderón’s identity, one can obtain the \( L^p \) estimate of \( g_Z \) as \( \| g_Z(f) \|_{L^p} \approx \| f \|_{L^p} \) for \( 1 < p < \infty \). Precisely, there exist two constants \( C_1, C_2 > 0 \) independent of \( f \) such that

\[
C_1 \| f \|_{L^p} \leq \| g_Z(f) \|_{L^p} \leq C_2 \| f \|_{L^p}.
\]

(5.5)

To pass these Littlewood-Paley-Stein analysis to Hardy spaces and weighted Hardy spaces, we need to introduce a proper distribution space.

**Definition 5.1.** A Schwartz test function \( f(x,y,z) \) defined on \( \mathbb{R}^3 \) is said to be a product test function on \( \mathbb{R} \times \mathbb{R}^2 \) if \( f \in \mathcal{S}(\mathbb{R}^3) \) and

\[
\int_{\mathbb{R}} x^\alpha f(x,y,z)dx = \int_{\mathbb{R}^2} y^\beta z^\gamma f(x,y,z)dydz = 0
\]

for all indices \( \alpha, \beta, \) and \( \gamma \) of nonnegative integers.

If \( f \) is a product test function on \( \mathbb{R} \times \mathbb{R}^2 \), we denote \( f \in \mathcal{S}_Z(\mathbb{R}^3) \) and the norm of \( f \) is defined by the norm of Schwartz test function. We denote the dual of \( \mathcal{S}_Z(\mathbb{R}^3) \) by \( (\mathcal{S}_Z(\mathbb{R}^3))' \).
Since the functions $\psi_{j,k}$ constructed above belong to $S_\mathcal{Z}(\mathbb{R}^3)$, so the Littlewood-Paley-Stein square function $g_\mathcal{Z}$ can be defined for all distributions in $(S_\mathcal{Z}(\mathbb{R}^3))'$. Thus for $0 < p < \infty$, the multi-parameter Hardy space associated with Zygmund dilations is defined as

$$H^p_\mathcal{Z}(\mathbb{R}^3) = \{f \in (S_\mathcal{Z}(\mathbb{R}^3))': g_\mathcal{Z}(f) \in L^p(\mathbb{R}^3)\},$$

and $H^p_\mathcal{Z}(\mathbb{R}^3) = L^p(\mathbb{R}^3)$ for $1 < p < \infty$ follows immediately from (1.5) above. See the work [HL2] for the thorough study of such $H^p_\mathcal{Z}$ spaces including the duality theory and boundedness of convolution and non-convolution operators.

Given $1 < p < \infty$, a nonnegative function $\omega$ on $\mathbb{R}^3$ is said to belong to $A_p(\mathcal{Z})$ if

$$\sup_{Q \in K_\mathcal{Z}} \left( \frac{1}{|Q|} \int_Q \omega \right) \left( \frac{1}{|Q|} \int_Q \omega^{-\frac{1}{p-1}} \right)^{p-1} = \|\omega\|_{A_p(\mathcal{Z})} < \infty.$$

When $p = 1$, $\omega \in A_1(\mathcal{Z})$ if there exists $C > 0$ such that $M_\mathcal{Z}(\omega)(x) \leq C \omega(x)$ for almost every $x \in \mathbb{R}^3$. Finally, we define

$$A_\infty(\mathcal{Z}) = \bigcup_{1 \leq p < \infty} A_p(\mathcal{Z}).$$

Notice that if $\omega \in A_\infty(\mathcal{Z})$, then $\omega \in A_{q_\omega}(\mathcal{Z})$, where $q_\omega = \inf\{q : \omega \in A_q(\mathcal{Z})\}$. Now let us introduce the two spaces that we study.

**Definition 5.2** ($H^p_\mathcal{Z}(\omega)$). Let $0 < p < \infty$ and $\omega \in A_\infty(\mathcal{Z})$, the multi-parameter Hardy space associated with the Zygmund dilation is defined as $H^p_\mathcal{Z}(\omega) = \{f \in (S_\mathcal{Z}(\mathbb{R}^3))': g_\mathcal{Z}(f) \in L^p_\omega\}$. If $f \in H^p_\mathcal{Z}(\omega)$, the norm of $f$ is defined by $\|f\|_{H^p_\mathcal{Z}(\omega)} = \|g_\mathcal{Z}(f)\|_{L^p_\omega}.$
Definition 5.3 \((CMO^p_Z(\omega))\). Let \(0 < p \leq 1\), \(\omega \in A^\infty(Z)\) and \(\psi_{j,k}\) be the same as in (1.3), we say that \(f \in CMO^p_Z(\omega)\) if \(f \in (S_Z(\mathbb{R}^3))'\) with the finite norm \(\|f\|_{CMO^p_Z(\omega)}\) defined by

\[
\sup_{\Omega} \left\{ \frac{1}{\omega(\Omega)^{\frac{1}{p}-1}} \sum_{j,k} \sum_{I \times J \times R \subseteq \Omega} |\psi_{j,k} * f(x_{I}, y_{J}, z_{R})|^2 \frac{|I \times J \times R|^2}{\omega(I \times J \times R)} \right\}^{\frac{1}{2}}
\]

for all open sets \(\Omega\) in \(\mathbb{R}^3\) with finite weighted measures and any fixed points \(x_{I}, y_{J}, \) and \(z_{R}\) in \(I \subseteq \mathbb{R}, J \subseteq \mathbb{R}, \) and \(R \subseteq \mathbb{R}\), where \(I, J, \) and \(R\) are dyadic intervals with interval-length \(\ell(I) = 2^{-j-N}, \ell(J) = 2^{-k-N},\) and \(\ell(R) = 2^{-j-k-2N}\) for a fixed large positive integer \(N\).

Remark. In Definitions 5.2 and 5.3 above, the definitions of \(H^p_Z(\omega)\) and \(CMO^p_Z(\omega)\) involve \(\psi_{j,k}\), to show these definitions are well defined, we need to prove that they are independent of the choice of functions \(\psi_{j,k}\). Precisely, we use sup-inf comparison principle of first kind as Theorem 5.7 to show that \(H^p_Z(\omega)\) is well-defined. While we state sup-inf comparison principle of second kind as one of our major theorems below, to prove that \(CMO^p_Z(\omega)\) is well-defined.

Theorem 5.4 (Sup-inf comparison principle of second kind). Let \(0 < p \leq 1\) and \(\omega \in A^\infty(Z)\), suppose \(\psi^{(1)} \in S(\mathbb{R}), \psi^{(2)} \in S(\mathbb{R}^2), \phi^{(2)} \in S(\mathbb{R}^2),\) and \(\psi_{j,k}, \phi_{j,k}\) satisfy the condition in (1.3). Then for \(f \in (S_Z(\mathbb{R}^3))'\),

\[
\sup_{\Omega} \left\{ \frac{1}{\omega(\Omega)^{\frac{1}{p}-1}} \sum_{j,k} \sum_{I \times J \times R \subseteq \Omega} \sup_{u \in I, v \in J, w \in R} |\psi_{j,k} * f(u, v, w)|^2 \frac{|I \times J \times R|^2}{\omega(I \times J \times R)} \right\}^{\frac{1}{2}} \\
\approx \sup_{\Omega} \left\{ \frac{1}{\omega(\Omega)^{\frac{1}{p}-1}} \sum_{j,k} \sum_{I \times J \times R \subseteq \Omega} \inf_{u \in I, v \in J, w \in R} |\phi_{j,k} * f(u, v, w)|^2 \frac{|I \times J \times R|^2}{\omega(I \times J \times R)} \right\}^{\frac{1}{2}}
\]

where \(I \subseteq \mathbb{R}, J \subseteq \mathbb{R}, \) and \(R \subseteq \mathbb{R}\) are dyadic intervals with interval-length \(\ell(I) = 2^{-j-N},\)
\(\ell(J) = 2^{-k-N},\) and \(\ell(R) = 2^{-j-k-2N}\) for a fixed large positive integer \(N\), and \(\Omega\) are all open
sets in $\mathbb{R}^3$ with finite weighted measures.

Then we state that the space $CMO^p_\mathcal{Z}$ is exactly the dual space of $H^p_\mathcal{Z}(\omega)$ for $0 < p \leq 1$.

**Theorem 5.5** $(H^p_\mathcal{Z}(\omega) - CMO^p_\mathcal{Z}(\omega))$. Let $0 < p \leq 1$ and $\omega \in A_\infty(\mathcal{Z})$. Then $(H^p_\mathcal{Z}(\omega))^* = CMO^p_\mathcal{Z}(\omega)$, namely the dual space of $H^p_\mathcal{Z}(\omega)$ is $CMO^p_\mathcal{Z}(\omega)$. More precisely, if $g \in CMO^p_\mathcal{Z}(\omega)$, the map $\ell_g$ given by $\ell_g(f) = \langle f, g \rangle$, defined initially for $f \in S_\mathcal{Z}(\mathbb{R}^3)$, extends to a continuous linear functional on $H^p_\mathcal{Z}(\omega)$ with $\|\ell_g\| \approx \|g\|_{CMO^p_\mathcal{Z}(\omega)}$. Conversely, for every $\ell \in (H^p_\mathcal{Z}(\omega))^*$ there exists some $g \in CMO^p_\mathcal{Z}(\omega)$ so that $\ell = \ell_g$. In particular, $(H^1_\mathcal{Z}(\omega))^* = BMO_\mathcal{Z}(\omega)$.

In Section 5.2, we collect several known results on the discrete Calderón’s identity and sup-inf comparison principle of first kind which are used to prove that $H^p_\mathcal{Z}(\omega)$ is well-defined. In Chapter 6, we show the well-definition of $CMO^p_\mathcal{Z}(\omega)$ using sup-inf comparison of second kind and almost orthogonality estimate. While Chapter 7 is devoted to prove the duality theory Theorem 5.5.

We shall point out in the end of the introduction that the main tool in this part, the discrete multi-parameter Littlewood-Paley-Stein analysis, is a relatively unified theory with a whole scheme, some theorems and lemmas we use here originate from the work [HL2]. An interested reader should consult the papers [HL1, HL2, HL3, HLL1] and related works mentioned therein. (See also [DHLW] and [R] where some nice application of the discrete Littlewood-Paley-Stein analysis was given in weighted setting.)

### 5.2 Discrete Calderón identity

To show the definition of $H^p_\mathcal{Z}(\omega)$ is independent of the choice of functions $\psi_{j,k}$ and thus well defined in Definition 5.2, we need to recall some results associated with the Zygmund
dilation. First, we require the discrete version of Calderón’s identity.

**Theorem 5.6** (Discrete Calderón’s identity). Suppose that $\psi_{j,k}$ are the same as in (5.3). Then

$$f(x, y, z) = \sum_{j,k} \sum_{I,J,R} |I||J||R| \tilde{\psi}_{j,k}(x, y, z, x_I, y_J, z_R)(\psi_{j,k} * f)(x_I, y_J, z_R), \quad (5.6)$$

converges in the norm of $S_Z(\mathbb{R}^3)$ and in the dual space $(S_Z(\mathbb{R}^3))'$, where $\tilde{\psi}_{j,k}(x, y, z, x_I, y_J, z_R) \in S_Z(\mathbb{R}^3)$, $I \subseteq \mathbb{R}$, $J \subseteq \mathbb{R}$, and $R \subseteq \mathbb{R}$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large integer $N$, and $x_I, y_J, z_R$ are any fixed points in $I, J, R$, respectively.

The complete proof is contained in §2.2 of [HL2], for the reader’s convenience, we provide a sketch of the proof here. An observation shows that the continuous version of Calderón’s identity (5.2) converges in the norm of $S_Z(\mathbb{R}^3)$ and in the dual space $(S_Z(\mathbb{R}^3))'$. Then it can be decomposed in dyadic form and we only need to estimate the remainder term as the difference. The explicit expression of $\tilde{\psi}_{j,k}(x, y, z, x_I, y_J, z_R)$ can also be found in [HL2].

The discrete Calderón’s identity enables us to derive the following weighted version sup-inf comparison principle of first kind, whose proof is included in [HLX2] (Theorem 1.1). It is an extension of the non-weighted one first derived in [HL2].

**Theorem 5.7** (Sup-inf comparison principle of first kind). Let $0 < p < \infty$ and $\omega \in A_\infty(\mathcal{Z})$, suppose $\psi^{(1)}, \phi^{(1)} \in S(\mathbb{R})$, $\psi^{(2)}, \phi^{(2)} \in S(\mathbb{R}^2)$, and $\psi_{j,k}, \phi_{j,k}$ satisfy the condition in (1.3).
Then for $f \in (S_Z(\mathbb{R}^3))'$,

\[
\left\| \sum_{j,k} \sum_{I,J,R} \sup_{u,v \in I, v \in J, w \in R} \left| \psi_{j,k}^* f(u,v,w) \right|^2 \chi_I(z) \chi_J(z) \chi_R(z) \right\|_{L^p_\omega} \approx \left\| \sum_{j,k} \sum_{I,J,R} \inf_{u,v \in I, v \in J, w \in R} \left| \phi_{j,k}^* f(u,v,w) \right|^2 \chi_I(z) \chi_J(z) \chi_R(z) \right\|_{L^p_\omega},
\]

(5.7)

where $I \subseteq \mathbb{R}$, $J \subseteq \mathbb{R}$, and $R \subseteq \mathbb{R}$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large positive integer $N$, $\chi_I$, $\chi_J$, and $\chi_R$ are indicator functions of $I$, $J$, and $R$, respectively.

From this sup-inf comparison principle, we introduce the discrete Littlewood-Paley-Stein square function

\[
g_d^Z(f)(x,y,z) = \left\{ \sum_{j,k} \sum_{I,J,R} \left| (\psi_{j,k}^* f)(x_I, y_J, z_R) \right|^2 \chi_I(x) \chi_J(y) \chi_R(z) \right\}^{1/2},
\]

(5.8)

where we admit all the settings in Theorem 5.7, and the $H^p_Z(\omega)$ norm of $f$ can be characterized using a discrete form

\[
\|f\|_{H^p_Z(\omega)} \approx \|g_d^Z(f)\|_{L^p_\omega}.
\]

Thus, we conclude that $H^p_Z(\omega)$ is well-defined by Theorem 5.7.

### 5.3 Sup-inf comparison principle of second kind

The purpose of this section is to get the sup-inf comparison principle of second kind, i.e., Theorem 5.4, to ensure that the space $CMO^p_Z(\omega)$ in Definition 5.3 is well-defined. First, we recall an “almost orthogonality lemma”, and refer the reader to Corollary 2.6 in [HL2] for
its detailed proof.

**Lemma 5.8** (Almost orthogonality estimate). If $\psi, \phi \in S_\mathbb{Z}(\mathbb{R}^3)$, define

$$
\psi_{t,s}(x, y, z) = t^{-2}s^{-2}\psi\left(\frac{x}{t}, \frac{y}{s}, \frac{z}{ts}\right)
$$

and $\phi_{t',s'}$ is defined similarly. Then, for any positive integers $L$ and $M$, there exists a constant $C = C(L, M) > 0$ such that

$$
|\psi_{t,s} \ast \phi_{t',s'}(x, y, z)| \leq C(\frac{t}{t'} \wedge \frac{t'}{t})^L(\frac{s}{s'} \wedge \frac{s'}{s})^L\frac{(t \vee t')^M}{(t \vee t' + |x|)^{M+1}}\frac{(s \vee s')^M}{(s \vee s' + |y| + |z|)^{M+2}}
$$

where $t^* = t$ if $s > s'$, $t^* = t'$ if $s \leq s'$, $t \wedge s = \min(t,s)$, and $t \vee s = \max(t,s)$.

Together with the discrete Calderón identity and some geometric properties of multi-parameter rectangles, Theorem 5.4 can be proved by a delicate study of the Zygmund rectangles.

**Proof of Theorem 5.4.** For simplicity, we denote $f_{j,k} = f_Q$, where $Q = I \times J \times R \subseteq \mathbb{R}^3$, $I \subseteq \mathbb{R}$, $J \subseteq \mathbb{R}$, and $R \subseteq \mathbb{R}$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large positive integer $N$. While $x_I$, $y_J$, and $z_R$ are any fixed points in $I$, $J$, and $R$, respectively.

Then, we can rewrite the discrete Calderón identity on $(S_\mathbb{Z}(\mathbb{R}^3))'$ as

$$
f(x, y, z) = \sum_{i,j} \sum_{Q=I' \times J' \times R'} |I'||J'||R'\tilde{\phi}_{Q'}(x, y, z, x_I', y_J', z_R')(\phi_{Q'} \ast f)(x_I', y_J', z_R').
$$
Thus, for all \((x, y, z) \in Q\),

\[
\psi_Q \ast f(x, y, z) = \sum_{i,j} \sum_{Q' = I' \times J' \times R'} |Q'| \psi_Q \ast \tilde{\phi}_{Q'}(x, y, z, x_{I'}, y_{J'}, z_{R'}) (\phi_{Q'} \ast f)(x_{I'}, y_{J'}, z_{R'}),
\]

where \(I' \subseteq \mathbb{R}\), \(J' \subseteq \mathbb{R}\), and \(R' \subseteq \mathbb{R}\) are dyadic intervals with interval-length \(\ell(I') = 2^{-j' - N}\), \(\ell(J') = 2^{-k' - N}\), and \(\ell(R') = 2^{-j' - k' - 2N}\) for a fixed large positive integer \(N\). While \(x_{I'}, y_{J'},\) and \(z_{R'}\) are any fixed points in \(I', J',\) and \(R',\) respectively.

From the almost orthogonality estimates (5.9) in Lemma 5.8, by choosing \(t = 2^{-j'}\), \(t' = 2^{-j'}\), \(s = 2^{-k'}\), and \(s' = 2^{-k'}\),

\[
|\psi_Q \ast f(x, y, z)|^2 \leq C \sum_{Q' = I' \times J' \times R'} |Q'| \left( \frac{|I|}{|I'|} \wedge \frac{|I|}{|I'|} \right)^L \left( \frac{|J|}{|J'|} \wedge \frac{|J|}{|J'|} \right)^L \left( \frac{|I| \vee |I'|}{|I| \vee |I'| + d(I, I')} \right)^{M+1} \times \frac{(|J| \vee |J'|)^M}{t^*(|J| \vee |J'| + d(J, J') + d(R, R'))^{M+2}} |\phi_{Q'} \ast f(x_{I'}, y_{J'}, z_{R'})|^2,
\]

where \(|Q'| = |I'| \cdot |J'| \cdot |R'|\), \(t^* = |I|\) when \(|J| \geq |J'|\), and \(t^* = |I'|\) when \(|J| < |J'|\), the constant \(C\) depends only on \(M, L,\) and functions \(\psi\) and \(\phi\). Write

\[
P_Q = \sup_{x \in I, y \in J, z \in R} |\psi_Q \ast f(x, y, z)|^2,
\]

and

\[
F_Q = \inf_{x \in I, y \in J, z \in R} |\phi_Q \ast f(x, y, z)|^2.
\]
Since $x_I, y_J, \text{ and } z_R$ in (3.2) are arbitrary in $I', J'$ and $R'$, we have

$$
\sum_{Q \subseteq \Omega} \frac{|I \times J \times R|^2}{\omega(I \times J \times R)} P_Q \leq C \sum_{Q \subseteq Q' \subseteq \Omega} \tilde{r}(Q, Q') P(Q, Q') \frac{|I' \times J' \times R'|^2}{\omega(I' \times J' \times R')} F_{Q'},
$$

(5.11)

where

$$
\tilde{r}(Q, Q') = \left( \frac{|I|}{|I'|} \wedge \frac{|I'|}{|I|} \right)^{L-2} \left( \frac{|J|}{|J'|} \wedge \frac{|J'|}{|J|} \right)^{L-2} \left( \frac{|R|}{|R'|} \wedge \frac{|R'|}{|R|} \right)^{-2} \frac{\omega(I' \times J' \times R')}{\omega(I \times J \times R)},
$$

and

$$
P(Q, Q') = \frac{(|I| \vee |I'|)^M + d(I, I')}{(|I| \vee |I'| + d(I, I'))^M} \frac{(|J| \vee |J'|)^M + d(J, J')}{(|J| \vee |J'| + d(J, J'))^M} \frac{|R| \vee |R'|}{t^*(|J| \vee |J'|) + t^*d(J, J') + d(R, R')}.
$$

Since $\omega \in A_\infty(\mathcal{Z})$ and $Q \subseteq Q'$, there exists $q_\omega$ and $1 < q_\omega < \infty$ such that

$$
\frac{\omega(I' \times J' \times R')}{\omega(I \times J \times R)} \leq C \left( \frac{|I' \times J' \times R'|}{|I \times J \times R|} \right)^{q_\omega}.
$$

Thus,

$$
\tilde{r}(Q, Q') \leq r(Q, Q'),
$$

where

$$
r(Q, Q') = \left( \frac{|I|}{|I'|} \wedge \frac{|I'|}{|I|} \right)^{L-q_\omega-2} \left( \frac{|J|}{|J'|} \wedge \frac{|J'|}{|J|} \right)^{L-q_\omega-2} \left( \frac{|R|}{|R'|} \wedge \frac{|R'|}{|R|} \right)^{-q_\omega-2}.
$$
We have

\[
\frac{1}{\omega(\Omega)^{\frac{\alpha}{p} - 1}} \sum_{Q \subseteq \Omega} P_Q \frac{|Q|^2}{\omega(Q)} \leq C \frac{1}{\omega(\Omega)^{\frac{\alpha}{p} - 1}} \sum_{Q \subseteq Q' \subseteq \Omega} r(Q, Q') P(Q, Q') \cdot F_{Q'} \frac{|Q'|^2}{\omega(Q')}.
\] (5.12)

To show Theorem 5.4, we only need to estimate the right hand side of (5.12). That is, to prove that it can be controlled by

\[
\sup_{\Omega^*} \frac{1}{\omega(\Omega^*)^{\frac{\alpha}{p} - 1}} \sum_{Q' \subseteq \Omega^*} F_{Q'} \frac{|Q'|^2}{\omega(Q')}.
\]

For \(i, l \geq 0\), set

\[
\Omega^{i,l} = \bigcup_{Q = I \times J \times R \subseteq \Omega} 3(2^i I \times 2^l J \times 2^{i+l} R).
\]

Then, write

\[
B_{0,0} = \{Q' = I' \times J' \times R' : 3Q' \cap \Omega^{0,0} \neq \emptyset\},
\]
and for \(i, l \geq 1\),

\[
B_{i,0} = \{Q' = I' \times J' \times R' : 3(2^i I' \times J' \times 2^l R') \cap \Omega^{i,0} \neq \emptyset, 3(2^{i-1} I' \times J' \times 2^{i+1} R') \cap \Omega^{i,0} = \emptyset\},
\]

\[
B_{0,l} = \{Q' = I' \times J' \times R' : 3(I' \times 2^l J' \times 2^l R') \cap \Omega^{0,l} \neq \emptyset, 3(I' \times 2^{l-1} J' \times 2^{l-1} R') \cap \Omega^{0,l} = \emptyset\},
\]

\[
B_{i,l} = \{Q' = I' \times J' \times R' : 3(2^i I' \times 2^l J' \times 2^{i+l} R') \cap \Omega^{i,l} \neq \emptyset, 3(2^{i-1} I' \times 2^{l-1} J' \times 2^{i+l-2} R') \cap \Omega^{i,l} = \emptyset\}.
\]
Note that since \( \bigcup Q' = \bigcup_{i,l \geq 0} Q' \in B_{i,l} \), the right hand of (5.12) can be bounded by

\[
\frac{1}{\omega(\Omega)^{\frac{2}{p} - 1}} \sum_{Q' \subseteq \Omega} \left[ \sum_{Q' \in B_{0,0}} + \sum_{i \geq 1} \sum_{Q' \in B_{i,0}} + \sum_{l \geq 1} \sum_{Q' \in B_{0,l}} + \sum_{i,l \geq 1} \sum_{Q' \in B_{i,l}} \right] \times r(Q, Q') P(Q, Q') \cdot F_{Q'} \left( \frac{|Q'|^2}{\omega(Q')} \right)
\]

\[
\triangleq I + II + III + IV.
\]

Here we only show the estimate of \( I \), then the estimates for the other three can follow similarly. Notice that if \( Q' \in B_{0,0} \), then \( 3Q' \cap \Omega^{0,0} \neq \emptyset \). Let

\[
\mathcal{F}^{0,0}_h = \{ Q' \in B_{0,0} : |3Q' \cap \Omega^{0,0}| \geq \frac{1}{2^h} |3Q'| \},
\]

\[
\mathcal{D}^{0,0}_h = \mathcal{F}^{0,0}_h \setminus \mathcal{F}^{0,0}_{h-1},
\]

and

\[
\Omega^{0,0}_h = \bigcup_{Q' \in \mathcal{D}^{0,0}_h} Q',
\]

where \( h \geq 0 \) and \( \mathcal{F}^{0,0}_{-1} = \emptyset \). Without loss of generality we may assume that for any open set \( \Omega \subset \mathbb{R}^3 \),

\[
\sum_{Q = I \times J \times R \subseteq \Omega} \frac{|I \times J \times R|^2}{\omega(I \times J \times R)} F_Q \leq C \omega(\Omega)^{\frac{2}{p} - 1}. \tag{5.14}
\]

Since \( \bigcup_{h \geq 0} \mathcal{D}^{0,0}_h = B_{0,0} \), we have

\[
I \leq \frac{1}{\omega(\Omega)^{\frac{2}{p} - 1}} \sum_{h} \sum_{Q' \in \mathcal{D}^{0,0}_h} \sum_{Q \subseteq \Omega} r(Q, Q') P(Q, Q') \cdot F_{Q'} \left( \frac{|Q'|^2}{\omega(Q')} \right). \tag{5.15}
\]
To estimate (5.15), for each $Q' \in B_{0,0}$ and $i', l', v' \geq 1$, we decompose \( \{Q \subset \Omega \} \) into 8 pieces as follows,

\[
A_{0,0,0}(Q') = \{Q \subset \Omega : d(I, I') \leq |I| \lor |I'|, \ d(J, J') \leq |J| \lor |J'|, \ d(R, R') \leq |R| \lor |R'| \},
\]

\[
A_{v,0,0}(Q') = \{Q \subset \Omega : 2^{v-1}(|I| \lor |I'|) < d(I, I') \leq 2^{v}(|I| \lor |I'|), \ d(J, J') \leq |J| \lor |J'|,
\]
\[
d(R, R') \leq |R| \lor |R'| \},
\]

\[
A_{0,v,0}(Q') = \{Q \subset \Omega : d(I, I') \leq |I| \lor |I'|, \ 2^{v-1}(|J| \lor |J'|) < d(J, J') \leq 2^{v}(|J| \lor |J'|),
\]
\[
d(R, R') \leq |R| \lor |R'| \},
\]

\[
A_{0,0,v}(Q') = \{Q \subset \Omega : d(I, I') \leq |I| \lor |I'|, \ d(J, J') \leq |J| \lor |J'|,
\]
\[
2^{v-1}(|R| \lor |R'|) < d(R, R') \leq 2^{v}(|R| \lor |R'|) \},
\]

\[
A_{v,v,0}(Q') = \{Q \subset \Omega : 2^{v-1}(|I| \lor |I'|) < d(I, I') \leq 2^{v}(|I| \lor |I'|), \ d(J, J') \leq |J| \lor |J'|,
\]
\[
2^{v-1}(|J| \lor |J'|) < d(J, J') \leq 2^{v}(|J| \lor |J'|), \ d(R, R') \leq |R| \lor |R'| \},
\]

\[
A_{v,0,v}(Q') = \{Q \subset \Omega : 2^{v-1}(|I| \lor |I'|) < d(I, I') \leq 2^{v}(|I| \lor |I'|), \ d(J, J') \leq |J| \lor |J'|,
\]
\[
2^{v-1}(|J| \lor |J'|) < d(J, J') \leq 2^{v}(|J| \lor |J'|), \ d(R, R') \leq |R| \lor |R'| \},
\]

\[
A_{0,v,v}(Q') = \{Q \subset \Omega : d(I, I') \leq |I| \lor |I'|, \ 2^{v-1}(|J| \lor |J'|) < d(J, J') \leq 2^{v}(|J| \lor |J'|),
\]
\[
2^{v-1}(|J| \lor |J'|) < d(J, J') \leq 2^{v}(|J| \lor |J'|), \ d(R, R') \leq |R| \lor |R'| \},
\]

\[
A_{v,v,v}(Q') = \{Q \subset \Omega : 2^{v-1}(|I| \lor |I'|) < d(I, I') \leq 2^{v}(|I| \lor |I'|), \ d(J, J') \leq |J| \lor |J'|,
\]
\[
2^{v-1}(|J| \lor |J'|) < d(J, J') \leq 2^{v}(|J| \lor |J'|), \ d(R, R') \leq |R| \lor |R'| \}.
\]
Then, (5.15) becomes

\[
I \leq \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q' \in \mathcal{D}_h^{0,0}} \sum_{Q \subset \Omega} r(Q, Q') P(Q, Q') \cdot m_Q \frac{|Q'|^2}{\omega(Q')}
\]

\[
= \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q' \in \mathcal{D}_h^{0,0}} \left( \sum_{Q \in A_{0,0,0}(Q')} + \sum_{Q_{i',0,0}(Q')} \sum_{i' \geq 1} + \sum_{Q_{i',i',0}(Q')} \sum_{i',i' \geq 1} \right)
\]

\[
\cdot r(Q, Q') P(Q, Q') \cdot F_{Q'} \frac{|Q'|^2}{\omega(Q')}
\]

\[
\triangleq I_1 + \cdots + I_8.
\]

In the following proof, we will give the estimates for \(I_1\) and \(I_4\) separately and the estimates for \(I_2, I_3, I_5, I_6, I_7,\) and \(I_8\) can be showed similarly. (i). To estimate

\[
I_1 = \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q' \in \mathcal{D}_h^{0,0}} \sum_{Q \in A_{0,0,0}(Q')} r(Q, Q') P(Q, Q') \cdot F_{Q'} \frac{|Q'|^2}{\omega(Q')}, \tag{5.16}
\]

we divide \(\{Q \in A_{0,0,0}(Q')\}\) into 6 cases, and note that \(3Q \cap 3Q' \neq \emptyset\) for \(Q \in A_{0,0,0}(Q').\)

**Case 1.** \(|I'| \geq |I|, |J'| \leq |J|,\) and \(|R'| \leq |R|\).

We will use a similar idea of analyzing geometric properties of the intervals (such analysis is similar to what was used in [CF2] in a less complicated situation). Since

\[
\frac{|I|}{|3I'|} |3Q'| = |I| \times |3J'| \times |3R'| \leq 9 |3Q \cap 3Q'| \leq |3Q' \cap \Omega^{0,0}| < \frac{9}{2h-1} |3Q'|,
\]

then \(|I| \leq 2^{-h+5}|I'|\) and thus \(|I'| \sim 2^{5-5+n}|I|\) for some \(n \geq 0.\) Moreover, for each given such
$n$, the number of such $I$’s is no more than $5 \cdot 2^{h-5+n}$. As for $J$, we have $|J| \sim 2^m|J'|$ for some $m \geq 0$ and for each $m$, the number of such $J$’s is no more than 5. Since $|R| = |I| \times |J|$ and $|R'| = |I'| \times |J'|$, we have $|R| \sim 2^{-(h-5+n)2^m}|R'|$. Note that $|R| \geq |R'|$, thus $m > h - 5 + n$. Furthermore, for each fixed $n$ and $m$, the number of such $R$’s is no more than 5. Thus,

$$
\sum_{\text{Case 1}} r(Q,Q') P(Q,Q') \\
\leq C \sum_{\text{Case 1}} \left( \frac{|I|}{|I'|} \right)^{L-q_w-2} \left( \frac{|J|}{|J'|} \right)^{L-q_w-2} \left( \frac{|R|}{|R'|} \right)^{-q_w-2} \frac{|R|}{|I||J|} \\
\leq C \sum_{n \geq 0} \sum_{m \geq 0} 2^{(5-h-n)(L-q_w-2)} \cdot 2^{-m(L-q_w-2)} \cdot 2^{(h-5+n-m)(-q_w-2)} \cdot 2^n \\
\leq C \sum_{n \geq 0} \sum_{m \geq 0} 2^{-hL} \cdot 2^{5L} \cdot 2^{-m(L-2q_w-4)} \cdot 2^{-4n} \\
\leq C 2^{-hL}.
$$

**Case 2:** $|I'| \geq |I|$, $|J'| \leq |J|$, and $|R'| \geq |R|$.

Since

$$
\frac{|I||R|}{|3I||3R'|} |3Q'| = |I| \times |3J'| \times |R| \leq 3|3Q' \cap 3Q| \leq 3|3Q' \cap \Omega^{0,0}| < \frac{3}{2^{h-1}} |3Q'|
$$

then $|I'||R'| \sim 2^{h-5+n}|I||R|$. As for $J$, $|J| \sim 2^m|J'|$. So for each $m$, the number of such $J$’s is no more than 5. Noting that $|R| = |I| \times |J|$ and $|R'| = |I'| \times |J'|$, we have $|I'||I'||J'| \sim 2^{h-5+n}|I||J|$, which yields that $|I'|^2 \sim 2^{h-5+n+m}|I|^2$, that is, $|I'| \sim 2^{(h-5+n+m)/2}|I|$. Hence for each $n$ and $m$, the number of such $I$’s is less than $5 \cdot 2^{(h+m+n)/2}$. Since we can obtain that $|R'| \sim 2^{(h-5+n-m)/2}|R|$, and $|R'| \geq |R|$, we have $m \leq h - 5 + n$. For each fixed $n$ and $m$, the
The number of such $R$'s is less than $5 \cdot 2^{(h-5+n-m)/2}$. Thus,

$$\sum_{\text{Case 2}} r(Q, Q') P(Q, Q') \leq C \sum_{\text{Case 2}} \left( \frac{|I|}{|I'|} \right)^{L-q_\omega-2} \left( \frac{|J|}{|J'|} \right)^{L-q_\omega-2} \left( \frac{|R|}{|R'|} \right)^{-q_\omega-2} \frac{|R'|}{|I||J|} \leq C \sum_{n \geq 0} \sum_{m \geq 0} 2^{2\left(5-h-n-m)(L-q_\omega-2)\right)} \cdot 2^{-m(L-q_\omega-2)} \cdot 2^{2\left(5-h-n-m)(-q_\omega-2)\right)} \cdot 2^{h+n} \leq C 2^{\left(\frac{h}{2}-q_\omega-3\right)}.$$

**Case 3:** $|I'| \leq |I|$, $|J'| \geq |J|$, and $|R'| \leq |R|$.

This can be handled in a way similar to that of Case 1, and we have

$$\sum_{\text{Case 3}} r(Q, Q') P(Q, Q') \leq C 2^{-hL}.$$

**Case 4:** $|I'| \leq |I|$, $|J'| \geq |J|$, and $|R'| \geq |R|$.

This can be handled in a similar way to that of Case 2, and we have

$$\sum_{\text{Case 4}} r(Q, Q') P(Q, Q') \leq C 2^{-h\left(\frac{h}{2}-q_\omega-2\right)}.$$

**Case 5:** $|I'| \geq |I|$, $|J'| \geq |J|$, and thus $|R'| \geq |R|$.

Since

$$|I| \times |J| \times |R| \leq |3Q' \cap 3Q| \leq |3Q' \cap \Omega^{0,0}| < \frac{1}{2^{h-1}} |3Q'|,$$

then $|Q'| \sim 2^{h-1+n}|Q|$ for some $n \geq 0$. Note that for each $n$, the number of such $Q$'s is less
than \((2^n)^3 = 2^{3n}\). More precisely, we have \((|I'||J'|)^2 \sim 2^{h-1+n(|I||J|)^2}. Thus,

\[
\sum_{\text{Case 5}} r(Q, Q') P(Q, Q') 
\leq C \sum_{\text{Case 5}} \left( \frac{|I||J|}{|I'||J'|} \right)^{L-q_o-2} \left( \frac{|R|}{|R'|} \right)^{-q_o-2} \left( \frac{|R'|}{|I'||J'|} \right) 
\leq C \sum_{n \geq 0} 2^{-h-L-q_o-2} \cdot 2^{-h_L+1+n} \cdot 2^{3n} 
= C \sum_{n \geq 0} 2^{-h\left(\frac{L}{2} - q_o - 2\right)} \cdot 2^{\frac{L}{2} - q_o - 2} \cdot 2^{-n\left(\frac{L}{2} - q_o - 5\right)} 
\leq C 2^{-h\left(\frac{L}{2} - q_o - 2\right)}.
\]

**Case 6:** \(|I'| \leq |I|, |J'| \leq |J|, and thus \(|R'| \leq |R|\).

Since

\[|I'| \times |J'| \times |R'| \leq |3Q' \cap \Omega^0,0| < \frac{1}{2^{n-1} |3Q'|},\]

then we can see that in this case, \(h\) must be less than 3. From \(|I'| \leq |I|\), we have \(|I| \sim 2^n |I'|\) for some \(n \geq 0\) and for each given such \(n\), the number of such \(I\)’s is less than 5. Similarly, from \(|J'| \leq |J|\), we have \(|J| \sim 2^n |J'|\) and for each \(m\), the number of such \(J\)’s is less than 5.

Hence we have \(|R| \sim 2^{n+m} |R'|\), and for each \(n\) and \(m\), the number of such \(R\)’s is less than 5. Thus,

\[
\sum_{\text{Case 6}} r(Q, Q') P(Q, Q') 
\leq C \sum_{\text{Case 6}} \left( \frac{|I'||J'|}{|I||J|} \right)^{L-q_o-2} \left( \frac{|R'|}{|R|} \right)^{-q_o-2} \left( \frac{|R|}{|I||J|} \right) 
\leq C \sum_{n \geq 0} \sum_{n \geq 0} (2^{-n-m})^{L-q_o-2} \cdot (2^{-n-m})^{-q_o-2} \cdot 2^{-mL} 
= C \sum_{n \geq 0} \sum_{n \geq 0} 2^{-n(L-2q_o-4)} \cdot 2^{-m(2L-2q_o-4)} \leq C.
\]
Before we combine these 6 cases above, observe that since $\Omega_{h}^{0,0} \leq Ch2^h|\Omega^{0,0}|$, $|\Omega^{0,0}| \leq C|\Omega|$, and $\omega \in A_\infty(\mathcal{Z})$, which is a doubling measure, together with (3.6), we have

$$\sum_{h} 2^{-h\left(\frac{k}{2} - q - 3\right)} \omega(\Omega_{h}^{0,0}) \frac{2}{p-1}$$

$$\leq \sum_{h} 2^{-h\left(\frac{k}{2} - q - 3\right)} \omega(Ch^22^h\Omega^{0,0}) \frac{2}{p-1}$$

$$\leq C\omega(\Omega^{0,0}) \frac{2}{p-1} \leq C\omega(\Omega) \frac{2}{p-1}.$$

Thus, combining the above 6 cases, $I_1$ in (5.16) can be estimated as

$$I_1 \leq \frac{1}{\omega(\Omega)^{\frac{2}{p-1}}} \sum_{h} \sum_{Q' \in \mathcal{D}_{h}^{0,0}} \left( \sum_{\text{Case 1}} + \cdots + \sum_{\text{Case 5}} \right) r(Q, Q') P(Q, Q') \cdot F_{Q'} \frac{|Q'|^2}{\omega(\Omega')}

+ \frac{1}{\omega(\Omega)^{\frac{2}{p-1}}} \sum_{h} \sum_{Q' \in \mathcal{D}_{h}^{0,0}} r(Q, Q') P(Q, Q') \cdot F_{Q'} \frac{|Q'|^2}{\omega(\Omega')}

\leq C \frac{1}{\omega(\Omega)^{\frac{2}{p-1}}} \sum_{h} \sum_{Q' \in \mathcal{D}_{h}^{0,0}} 2^{-h\left(\frac{k}{2} - q - 3\right)} \cdot F_{Q'} \frac{|Q'|^2}{\omega(\Omega')}

+ \frac{1}{\omega(\Omega)^{\frac{2}{p-1}}} \sum_{h=0}^{3} \sum_{Q' \in \mathcal{D}_{h}^{0,0}} F_{Q'} \frac{|Q'|^2}{\omega(\Omega')}

\leq C \frac{1}{\omega(\Omega)^{\frac{2}{p-1}}} \sum_{h} \sum_{Q' \in \mathcal{D}_{h}^{0,0}} \omega(\Omega_{h}^{0,0})^{\frac{2}{p-1}} \frac{1}{\omega(\Omega_{h}^{0,0})^{\frac{2}{p-1}}} \sum_{Q' \subseteq \Omega_{h}^{0,0}} F_{Q'} \frac{|Q'|^2}{\omega(\Omega')}

+ \frac{1}{\omega(\Omega)^{\frac{2}{p-1}}} \sum_{h=0}^{3} \omega(\Omega_{h}^{0,0})^{\frac{2}{p-1}} \frac{1}{\omega(\Omega_{h}^{0,0})^{\frac{2}{p-1}}} \sum_{Q' \subseteq \Omega_{h}^{0,0}} F_{Q'} \frac{|Q'|^2}{\omega(\Omega')}

\leq C \frac{1}{\omega(\Omega)^{\frac{2}{p-1}}} \sum_{h} \sum_{Q' \in \mathcal{D}_{h}^{0,0}} 2^{-h\left(\frac{k}{2} - q - 3\right)} \omega(\Omega_{h}^{0,0})^{\frac{2}{p-1}} \sup_{\Omega^*} \frac{1}{\omega(\Omega^*)^{\frac{2}{p-1}}} \sum_{Q' \subseteq \Omega^*} F_{Q'} \frac{|Q'|^2}{\omega(\Omega')}

+ \frac{1}{\omega(\Omega)^{\frac{2}{p-1}}} \sum_{h=0}^{3} (h^22^h)^{\frac{2}{p-1}} \omega(\Omega)^{\frac{2}{p-1}} \sup_{\Omega^*} \frac{1}{\omega(\Omega^*)^{\frac{2}{p-1}}} \sum_{Q' \subseteq \Omega^*} F_{Q'} \frac{|Q'|^2}{\omega(\Omega')}

\leq C \sup_{\Omega^*} \frac{1}{\omega(\Omega^*)^{\frac{2}{p-1}}} \sum_{Q' \subseteq \Omega^*} F_{Q'} \frac{|Q'|^2}{\omega(\Omega')}.$$
where we choose $L$ large enough, and the estimate of $I_1$ is finished. Next we move our attention to the estimate of $I_4$.

(iv) To estimate

$$I_4 = \frac{1}{\omega(\Omega)^{\frac{d}{p}-1}} \sum_{h} \sum_{Q' \in D_h^{0,0}, v' \geq 1} \sum_{Q \in A_{0,0,v'}(Q')} r(Q, Q') P(Q, Q') \cdot F_{Q'} \frac{|Q'|^2}{\omega(Q')} \tag{5.17}$$

similar to what we did in (i), we divide $\{Q \in A_{0,0,v'}(Q')\}$ into 6 cases for each $v' \geq 1$.

**Case 1:** $|I'| \geq |I|$, $|J'| \leq |J|$, and $|R'| \leq |R|$.

Note that in this case, $3(I' \times J' \times R') \cap 3(I \times J \times 2^v R) \neq \emptyset$. Since

$$\frac{|I|}{|3I'|} |3Q'| = |I| \times |3J'| \times |3R'| \leq |3Q' \cap \Omega^{0,0}| < \frac{1}{2^{h-1}} |3Q'|,$$

then $|I'| \sim 2^{h-1+n}|I|$ for some $n \geq 0$, and for each $n$, the number of such $I$'s is no more than $5 \cdot 2^n$. As for $J$, $|J| \sim 2^m |J'|$. And for each $m$, the number of such $J$'s is no more than 5. Note that $2^{v'-1}|R| < d(R, R') \leq 2^{v'} |R|$, which yields that $3R' \cap 3 \cdot 2^v R \neq \emptyset$. Moreover, from $|R| \sim 2^{-(h-1+n)} 2^m |R'|$ and $|R| \geq |R'|$, we have $m > h - 1 + n$ and for each fixed $v'$, $n$ and $m$, the number of such $R$'s is less than $5 \cdot 2^{v'}$. Thus,
\[
\begin{align*}
&\leq C \sum_{n \geq 0} \sum_{m \geq 0} 2^{-(h-1+n)(L-q_\omega-2)} \cdot 2^{(h-1+n-m)(-q_\omega-2)} \cdot 2^{-m(L-q_\omega-2)} \cdot 2^n \cdot 2^{-v(M+2)} \\
&\leq C \sum_{n \geq 0} \sum_{m \geq 0} 2^{-hL} \cdot 2^{-m(L-2q_\omega-4)} \cdot 2^{-n(L-1)} \cdot 2^L \cdot 2^{-v(M+2)} \\
&\leq C 2^{-hL} 2^{-v(M+2)}.
\end{align*}
\]

**Case 2:** \(|I'| \geq |I|, |J'| \leq |J|, \text{ and } |R'| \geq |R|.

Note that in this case, \(3(I' \times J' \times 2^v R') \cap 3(I \times J \times R) \neq \emptyset\). Since

\[
\frac{|I||R|}{|3I||3R|} |3Q'| = |I| \times |3J'| \times |R| \leq |3Q' \cap \Omega^0| < \frac{1}{2^{h-1}} |3Q'|,
\]

then \(|I'||R'| \sim 2^{h-1+n}|I||R|\). As for \(J, |J| \sim 2^m|J'|\). So for each \(m\), the number of such \(J\)'s is no more than 5. Noting that \(|R| = |I| \times |J|\) and \(|R'| = |I'| \times |J'|\), we have \(|I'||I'||J'| \sim 2^{h-1+n}|I||J|\), which yields that \(|I'|^2 \sim 2^{h-1+n+m}|I|^2\), that is, \(|I'| \sim 2^{(h-1+n+m)/2}|I|\). Hence for each \(n\) and \(m\), the number of such \(I\)'s is less than \(5 \cdot 2^{(h+m+n)/2}\). Also we can obtain that \(|R'| \sim 2^{(h-1+n-m)/2}|R|\). Since \(|R'| \geq |R|\), we have \(m \leq h-1+n\). Moreover, note that \(2^{v'-1}|R'| < d(R, R') \leq 2^{v'}|R'|\), which yields that \(3 \cdot 2^{v'} R' \cap 3R \neq \emptyset\). For each fixed \(v', n\) and \(m\), the number of such \(R\)'s is less than \(5 \cdot 2^{(h+n-m)/2v'}\). Thus,

\[
\sum_{\text{Case 2}} r(Q, Q') P(Q, Q') \leq C \sum_{\text{Case 1}} \left( \frac{|I|}{|I'|} \right)^{L-q_\omega-2} \left( \frac{|J|}{|J'|} \right)^{L-q_\omega-2} \left( \frac{|R'|}{|R|} \right)^{-q_\omega-2} \cdot \frac{|I||R'|}{(|J| + 2^{v'-1}|R'|)^{M+1}} \frac{|R'|}{|I| + 2^{v'-1}|R'|}.
\]
\[
\leq C \sum_{n \geq 0} \sum_{m \geq 0} 2^{-\frac{h+n}{2} (L-q_\omega-2)} \cdot 2^{-m(L-q_\omega-2)} \cdot 2^{-\frac{h+n-m}{2} (-q_\omega-2)}
\]
\[
\cdot 2^{h+n} 2^{v'(M+2)}
\]
\[
= C \sum_{n \geq 0} \sum_{m \geq 0} 2^{-h\left(\frac{h}{2} - q_\omega - 1\right)} \cdot 2^{-m\left(\frac{3h}{2} - q_\omega - 4\right)} \cdot 2^{-n\left(\frac{h}{2} - q_\omega - 1\right)} \cdot 2^{\frac{h-1}{2} - v'(M+2)}
\]
\[
\leq C 2^{-h\left(\frac{h}{2} - q_\omega - 1\right)} 2^{v'(M+1)}.
\]

**Case 3:** \( |I'| \leq |I|, |J'| \geq |J|, \) and \( |R'| \leq |R| \).

This can be handled similarly as Case 1, and we have

\[
\sum_{\text{Case 3}} r(Q, Q') P(Q, Q') \leq C 2^{-hL} 2^{-v'(M+2)}.
\]

**Case 4:** \( |I'| \leq |I|, |J'| \geq |J|, \) and \( |R'| \geq |R| \).

This can be handled similarly as Case 2, and we have

\[
\sum_{\text{Case 4}} r(Q, Q') P(Q, Q') \leq C 2^{-h\left(\frac{h}{2} - q_\omega - 1\right)} 2^{-v'(M+1)}.
\]

**Case 5:** \( |I'| \geq |I|, |J'| \geq |J|, \) and thus \( |R'| \geq |R| \).

Since

\[
|I| \times |J| \times |R| \leq |3Q' \cap \Omega^0| < \frac{1}{2^{h-1}} |3Q'|,
\]

then \( |Q'| \sim 2^{h-1+n}|Q| \). And for each \( v' \) and \( n \), the number of such \( Q \)'s is less than \( 2^{v'}(2^n)^3 \).
More precisely, we have $(|I'||J'|)^2 \sim 2^{h-1+n}(|I||J|)^2$. Thus,

$$\sum_{\text{Case 5}} r(Q, Q') P(Q, Q') \leq C \sum_{\text{Case 5}} \left( \frac{|I||J|}{|I'||J'|} \right)^{L-q_\omega-2} \left( \frac{|R|}{|R'|} \right)^{-q_\omega-2} \frac{(|J'|)^{M+1}}{(|J'| + \frac{2^{\nu|-1|R'|}}{|J'|})^{M+1}} \frac{|R'|}{|I'||J'| + 2^{\nu-1}|R'|} \leq C \sum_{n \geq 0} 2^{-\frac{(h-1+n)}{2}} \cdot 2^{\frac{(h-1+n)}{2}} \cdot 2^{3n} \cdot 2^{-\nu(M+2)} \leq C 2^{-h} \cdot 2^{-n} \cdot 2^{-\frac{L}{2} - q_\omega - 2} \cdot 2^{-\nu(M+1)} \leq C 2^{-h} \cdot 2^{-\nu(M+1)}.$$

**Case 6:** $|I'| \leq |I|$, $|J'| \leq |J|$, and thus $|R'| \leq |R|$. Since

$$|I'| \times |J'| \times |R'| \leq |3Q' \cap \Omega^{0,0}| < \frac{1}{2^{h-1}} |3Q'|,$$

then we can see that in this case, $h$ must be less than 3. And from $|I'| \leq |I|$, we have $|I| \sim 2^n |I'|$ and for each $n$, the number of such $I$ is less than 5. Similarly, from $|J'| \leq |J|$, we have $|J| \sim 2^m |J'|$ and for each $m$, the number of such $J$ is less than 5. Hence we have
$|R| \sim 2^{n+m}|R'|$, and for each $v', n, m$, the number of such $R$ is less than $5 \cdot 2^{v'}$. Thus,

$$
\sum_{\text{Case 6}} r(Q, Q') P(Q, Q') \leq C \sum_{n \geq 0} \sum_{m \geq 0} 2^{-n(L-q_\omega-2)} \cdot 2^{-m(L-q_\omega-2)} \cdot 2^{(-n-m)(-q_\omega-2)} \cdot 2^{v'} \cdot 2^{-v'(M+2)} \leq C 2^{-v'(M+1)}.
$$

Thus, combining the above 6 cases, by choosing $L$ and $M$ large enough, $I_4$ in (5.17) becomes

$$
I_4 \leq C \sup_{\Omega^*} \frac{1}{\omega(\Omega^*)^\frac{2}{p-1}} \sum_{Q' \subseteq \Omega^*} F_{Q'} \frac{|Q'|^2}{\omega(Q')}.
$$

Using the same techniques, we are able to control the 8 integrates for $I$, therefore give the estimate for $I$, that is,

$$
I \leq C \sup_{\Omega^*} \frac{1}{\omega(\Omega^*)^\frac{2}{p-1}} \sum_{Q' \subseteq \Omega^*} F_{Q'} \frac{|Q'|^2}{\omega(Q')}.
$$

Without any difficulty, $II, III, IV$ in (5.13) can be calculated similarly and bounded by the right hand side of the above inequality. Hence the proof of the theorem 5.4 is complete.

Finally we show that $CMO^p_\Omega(\omega)$ is well defined as a corollary of sup-inf comparison principle of second kind.
Corollary 5.9. The definition of $\text{CMO}_Z^p(\omega)$ in Definition 5.3 is independent of the choice of $\psi_{j,k}$, therefore it is well-defined.

5.4 Proof of the duality theorem

In this section, we prove the $(H_p^p(\omega))^* - CMO_Z^p(\omega)$ duality theorem, i.e., Theorem 5.5, and we need to introduce two sequence spaces.

Definition 5.10 $(s_Z^p(\omega)$ and $c_Z^p(\omega))$. Let $\omega \in A_{\infty}(\mathbb{Z})$, $j, k \in \mathbb{Z}$, and $I \subseteq \mathbb{R}$, $J \subseteq \mathbb{R}$, and $R \subseteq \mathbb{R}$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large positive integer $N$. The sequence $s = \{s_{I \times J \times R}\}$ is said to be in the sequence space $s_Z^p(\omega)$ if

$$
\|s\|_{s_Z^p(\omega)} = \left\| \sum_{j,k} \sum_{I \times J \times R} |s_{I \times J \times R}|^2 |I|^{-1} |J|^{-1} |R|^{-1} \chi_I(x) \chi_J(y) \chi_R(z) \right\|_{L_p^p(\omega)} < \infty,
$$

and the sequence $t = \{t_{I \times J \times R}\}$ is said to be in the sequence space $c_Z^p(\omega)$ if

$$
\|t\|_{c_Z^p(\omega)} = \sup_{\Omega} \left\{ \frac{1}{\omega(\Omega)^{\frac{1}{p}}} \sum_{j,k} \sum_{I \times J \times R \subseteq \Omega} |t_{I \times J \times R}|^2 \frac{|I \times J \times R|^2}{\omega(I \times J \times R)} \right\}^{\frac{1}{2}} < \infty,
$$

for all open sets $\Omega$ in $\mathbb{R}^3$ with finite weighted measures, and $I \times J \times R$ run over all dyadic cubes with side-lengths defined above.

We now derive the following duality theorem for these sequence spaces.

Theorem 5.11 $(s_Z^p(\omega) - c_Z^p(\omega))$. $(s_Z^p(\omega))^* = c_Z^p(\omega)$, precisely, let $\omega \in A_{\infty}(\mathbb{Z})$ and $0 < p \leq 1$, the map which maps $s = \{s_{I \times J \times R}\}$ to $< s, t > = \sum_{I \times J \times R} s_{I \times J \times R} \bar{t}_{I \times J \times R}$ defines a continuous
linear functional on $s^p_2(\omega)$ with operator norm $\|t\|_{(s^p_2(\omega))^*} \approx \|t\|_{c^p_2(\omega)}$, and every $\ell \in (s^p_2(\omega))^*$ is of this form for some $t \in c^p_2(\omega)$.

**Proof of Theorem 5.11.** First we show that $c^p_2(\omega) \subseteq (s^p_2(\omega))^*$. Suppose $t = \{t_{I \times J \times R}\} \in c^p_2(\omega)$ and $s = \{s_{I \times J \times R}\} \in s^p_2(\omega)$, set

$$h(x, y, z) = \left\{ \sum_{j,k} \sum_{I \times J \times R} |s_{I \times J \times R}|^2 |I|^{-1} |J|^{-1} |R|^{-1} \chi_I(x) \chi_J(y) \chi_R(z) \right\}^{\frac{1}{2}},$$

which means $\|s\|_{s^p_2(\omega)} = \|h\|_{L^p_\omega}$. Then we write $\Omega_i = \{(x, y, z) : h(x, y, z) > 2^i\}$, and

$$B_i = \{I \times J \times R : \omega(I \times J \times R \cap \Omega_i) > \frac{1}{2} \omega(I \times J \times R), \omega(I \times J \times R \cap \Omega_{i+1}) \leq \frac{1}{2} \omega(I \times J \times R)\}.$$

Thus,

$$\sum_{j,k} \sum_{I \times J \times R} s_{I \times J \times R} \bar{t}_{I \times J \times R} = \sum_{i} \sum_{I \times J \times R \in B_i} s_{I \times J \times R} \bar{t}_{I \times J \times R}.$$

Note that $0 < p \leq 1$, by Cauchy-Schwartz’s inequality,

$$\left| \sum_{i} \sum_{I \times J \times R \in B_i} s_{I \times J \times R} \bar{t}_{I \times J \times R} \right| \leq \sum_{i} \left( \sum_{I \times J \times R \in B_i} |s_{I \times J \times R}|^2 \omega(I \times J \times R) \right)^{\frac{1}{2}} \left( \sum_{I \times J \times R \in B_i} |\bar{t}_{I \times J \times R}|^2 \omega(I \times J \times R) \right)^{\frac{1}{2}}.$$
\[ \left\{ \sum_i \left( \sum_{I \times J \times R \in B_i} |s_{I \times J \times R}|^2 \omega(I \times J \times R) \right)^{\frac{p}{2}} \left( \sum_{I \times J \times R \in B_i} |t_{I \times J \times R}|^2 \frac{|I \times J \times R|}{\omega(I \times J \times R)} \right)^{\frac{p}{2}} \right\}^{\frac{1}{p}} \]

\[ \leq C \| t \|_{e_2^p(\omega)} \left\{ \sum_i \omega(\Omega_i)^{1-\frac{p}{2}} \left( \sum_{I \times J \times R \in B_i} |s_{I \times J \times R}|^2 \omega(I \times J \times R) \right)^{\frac{p}{2}} \right\}^{\frac{1}{p}}. \]

Where the last inequality follows from the fact that if \( I \times J \times R \in B_i \), then there exists \( 0 < \theta < 1 \) such that

\[ I \times J \times R \subseteq \{(x, y, z) : M_{Z}(\chi_{\Omega_i})(x, y, z) > \theta\} \triangleq \bar{\Omega}_i, \]

together with \( \omega(\bar{\Omega}_i) \leq C \omega(\Omega_i) \), imply

\[ \left( \sum_{I \times J \times R \in B_i} |t_{I \times J \times R}|^2 \frac{|I \times J \times R|}{\omega(I \times J \times R)} \right)^{\frac{1}{2}} \leq C \| t \|_{e_2^p(\omega)} \omega(\Omega_i)^{\frac{1-\frac{p}{2}}{2}}. \]

We claim for now

\[ \sum_{I \times J \times R \in B_i} |s_{I \times J \times R}|^2 \omega(I \times J \times R) \frac{|I \times J \times R|}{|I \times J \times R|} \leq C 2^{|\Omega_i|} \omega(\Omega_i). \quad (5.20) \]

Assume this claim for the moment, then

\[ \left| \sum_i \sum_{I \times J \times R \in B_i} s_{I \times J \times R} t_{I \times J \times R} \right| \]

\[ \leq C \| t \|_{e_2^p(\omega)} \left( \sum_i 2^{|\Omega_i|} \omega(\Omega_i) \right)^{\frac{1}{p}} \]

\[ \leq C \| t \|_{e_2^p(\omega)} \| h \|_{L_p} \]

\[ \leq C \| t \|_{e_2^p(\omega)} \| s \|_{e_2^p(\omega)}. \]
therefore \( c^p_\omega(\omega) \subseteq (s^p_\omega(\omega))^* \). To show the claim (5.20), it is sufficient to prove

\[
\sum_{I \times J \times R \in B_i} |s_{I \times J \times R}|^2 \omega(I \times J \times R) \leq C \int_{\Omega_i \setminus \Omega_{i+1}} h^2(x, y, z) \omega dxdydz
\]

because

\[
\int_{\Omega_i \setminus \Omega_{i+1}} h^2(x, y, z) \omega dxdydz \leq 2^{2i+1} \omega(\tilde{\Omega}_i) \leq C 2^i \omega(\Omega_i).
\]

However,

\[
\int_{\Omega_i \setminus \Omega_{i+1}} h^2(x, y, z) \omega dxdydz = \int_{\Omega_i \setminus \Omega_{i+1}} \sum_{I \times J \times R} |s_{I \times J \times R}|^2 |I|^{-1} |J|^{-1} |R|^{-1} \chi_I(x) \chi_J(y) \chi_R(z) \omega dxdydz
\]

\[
\geq \sum_{I \times J \times R \in B_i} |s_{I \times J \times R}|^2 \frac{\omega((\tilde{\Omega}_i \setminus \Omega_{i+1}) \cap (I \times J \times R)) \omega(I \times J \times R)}{|I \times J \times R|}
\]

\[
\geq \frac{1}{2} \sum_{I \times J \times R \in B_i} |s_{I \times J \times R}|^2 \omega(I \times J \times R) \left( \frac{\omega(\Omega_i \setminus I \times J \times R)}{|I \times J \times R|} \right),
\]

since for \( I \times J \times R \in B_i \),

\[
\omega((\tilde{\Omega}_i \cap I \times J \times R)) > \frac{1}{2} \omega(I \times J \times R),
\]

and

\[
\omega((\Omega_{i+1} \cap I \times J \times R)) \leq \frac{1}{2} \omega(I \times J \times R).
\]

Then \( I \times J \times R \in \tilde{\Omega}_i \), hence

\[
\omega((\tilde{\Omega}_i \setminus \Omega_{i+1}) \cap (I \times J \times R)) > \frac{1}{2} \omega(I \times J \times R).
\]
The claim is verified.

Next we shall prove that \((s'_Z(\omega))^* \subseteq c'_Z(\omega)\). Let \(\ell \in (s'_Z(\omega))^*\), then there exists some \(t = \{t_{I \times J \times R}\}\) such that \(\forall s = \{s_{I \times J \times R}\} \in s'_Z(\omega)\),

\[
\ell(s) = \sum_{I \times J \times R} s_{I \times J \times R} \bar{t}_{I \times J \times R}.
\]

For an open set \(\Omega\) in \(\mathbb{R}^3\) with \(\omega(\Omega) < \infty\), define

\[
\|s\|_{s'_Z, \Omega(\omega)} = \left\{ \int_{\Omega} \left( \sum_{I \times J \times R \subseteq \Omega} |s_{I \times J \times R}|^2 |I|^{-1} |J|^{-1} |R|^{-1} \chi_I(x) \chi_J(y) \chi_R(z) \right)^{\frac{p}{2}} \omega dxdydz \right\}^{\frac{1}{p}},
\]

and

\[
\|s\|_{\ell'^2_Z, \Omega(\omega)} = \left( \sum_{I \times J \times R \subseteq \Omega} |s_{I \times J \times R}|^2 \frac{\omega(I \times J \times R)}{|I \times J \times R|} \right)^{\frac{1}{2}}.
\]

Then, by Hölder’s inequality,

\[
\|s\|_{s'_Z, \Omega(\omega)} = \left\{ \int_{\Omega} \left( \sum_{I \times J \times R \subseteq \Omega} |s_{I \times J \times R}|^2 |I|^{-1} |J|^{-1} |R|^{-1} \chi_I(x) \chi_J(y) \chi_R(z) \right)^{\frac{p}{2}} \omega dxdydz \right\}^{\frac{1}{p}} \leq \omega(\Omega)^{\frac{1}{p} - \frac{1}{2}} \left\{ \int_{\Omega} \sum_{I \times J \times R \subseteq \Omega} |s_{I \times J \times R}|^2 |I|^{-1} |J|^{-1} |R|^{-1} \chi_I(x) \chi_J(y) \chi_R(z) \omega dxdydz \right\}^{\frac{1}{2}} \leq \omega(\Omega)^{\frac{1}{p} - \frac{1}{2}} \left( \sum_{I \times J \times R \subseteq \Omega} |s_{I \times J \times R}|^2 \frac{\omega(I \times J \times R)}{|I \times J \times R|} \right)^{\frac{1}{2}} = \omega(\Omega)^{\frac{1}{p} - \frac{1}{2}} \|s\|_{\ell'^2_Z, \Omega(\omega)}.
\]
Thus, we compute

\[
\left\{ \frac{1}{\omega(\Omega)^{\frac{1}{p} - 1}} \sum_{j,k} \sum_{I \times J \times R \subseteq \Omega} \frac{|I \times J \times R|^2}{\omega(I \times J \times R)} \right\}^{\frac{1}{2}} \\
= \frac{1}{\omega(\Omega)^{\frac{1}{p} - \frac{1}{2}}} \sup_{\|s\|_{L_2,\Omega} \leq 1} \left| \sum_{I \times J \times R \subseteq \Omega} s_{I \times J \times R} \right| \\
\leq \frac{1}{\omega(\Omega)^{\frac{1}{p} - \frac{1}{2}}} \sup_{\|s\|_{L_2,\Omega} \leq 1} \|t\|_{(s^p_{L_2}(\omega))^*} \|s_{I \times J \times R}\|_{s^p_{L_2,\Omega}(\omega)} \\
= \|t\|_{(s^p_{L_2}(\omega))^*} \sup_{\|s\|_{L_2,\Omega} \leq 1} \frac{1}{\omega(\Omega)^{\frac{1}{p} - \frac{1}{2}}} \|s_{I \times J \times R}\|_{s^p_{L_2,\Omega}(\omega)} \\
\leq \|t\|_{(s^p_{L_2}(\omega))^*}
\]

for all \(\Omega\). Therefore, by taking the superium, \(t \in c^p_{L_2}(\omega)\) and \(\|t\|_{c^p_{L_2}(\omega)} \leq \|t\|_{(s^p_{L_2}(\omega))^*}\), which implies \((s^p_{L_2}(\omega))^* \subseteq c^p_{L_2}(\omega)\), and thus the proof of Theorem 5.11 is complete.

\[\square\]

In order to pass the duality theory from sequence spaces to \(H^p_{L_2}(\omega)\) and \(CMO^p_{L_2}(\omega)\), we need the following lemmas.

**Lemma 5.12.** Given large positive integer \(N\) and integers \(j, k, j', k' \in \mathbb{Z}\). Let \(I, J, R, I', J', R' \subseteq \mathbb{R}\) are dyadic intervals with interval-length \(\ell(I) = 2^{-j-N}\), \(\ell(J) = 2^{-k-N}\), \(\ell(R) = 2^{-j-k-2N}\), \(\ell(I') = 2^{-j'-N}\), \(\ell(J') = 2^{-k'-N}\), and \(\ell(R') = 2^{-j'-k'-2N}\). Let \(\{a_{I', J', R'}\}\) be any given sequence, \(x_{I'} \in I', y_{J'} \in J',\) and \(z_{R'} \in R'\) be any points. Then for any \(u, u^* \in I, v, v^* \in J, w, w^* \in R\)
we have

\[
\sum_{I', J', R'} 2^{-(j' \wedge j') M_1 + 2^{-k \wedge k' M_2}} |I'| |J'| |R'| (2^{-j' M_1} + |u - x_{I'}|)^{1+M_1} 2^{-j' (2^{-k M_1} + |v - y_{J'}| + |w - z_{R'}|)^{2+M_2}} |a_{I', J', R'}| \\
\leq C^2 \frac{M_Z(\sum_{I', J', R'} |a_{I', J', R'}| \chi_{I'} \chi_{J'} \chi_{R'})(u^*, v^*, w^*)}{r^{1/r}} ,
\]

where \( j^* = j \) if \( k < k' \), and \( j^* = j' \) if \( k \geq k' \). \( M_Z \) is the maximal operator associated with Zygmund dilations defined in (5.1), and \( \max\{\frac{2}{1+M_1}, \frac{2}{2+M_2}\} < r \leq 1 \). The summation is taken for all \( I', J', R' \) with the fixed side-length. \( \tau \) is defined as follows,

\[
\tau = \begin{cases} 
(\frac{2}{r} - 2)(j' + k' - j - k) & \text{if } j < j' \text{ and } k < k', \\
(\frac{2}{r} - 1)(j' - j) & \text{if } j < j' \text{ and } k \geq k', \\
j - j' + (\frac{2}{r} - 2)(k' - k) & \text{if } j \geq j' \text{ and } k < k', \\
0 & \text{if } j \geq j' \text{ and } k \geq k'.
\end{cases}
\]

The detailed proof of Lemma 5.12 can be found in [HL2].

**Lemma 5.13.** Let \( \omega \in A_\infty (\mathbb{Z}) \), \( j, k \in \mathbb{Z} \), \( \psi_{j,k} \) be same as in (5.3) and \( I \subseteq \mathbb{R} \), \( J \subseteq \mathbb{R} \), \( R \subseteq \mathbb{R} \) are dyadic intervals with interval-length \( \ell(I) = 2^{-j - N}, \ell(J) = 2^{-k - N} \), and \( \ell(R) = 2^{-j-k-2N} \) for a fixed large positive integer \( N \). Define a map \( S \) on \( (S_2(\mathbb{R}^3))' \) by

\[
h(f) = \left\{ |I|^{\frac{1}{2}} |J|^{\frac{1}{2}} |R|^{\frac{1}{2}} \psi_{j,k} * f(x_I, y_J, z_R) \right\}.
\]
For any sequence \( s = \{s_{I \times J \times R}\} \), we define the map \( T \) by

\[
T(s) = \sum_{j,k} \sum_{I \times J \times R} s_{I \times J \times R} \left| I \right|^\frac{1}{2} \left| J \right|^\frac{1}{2} \left| R \right|^\frac{1}{2} \tilde{\psi}_{j,k}(x, y, z, x_I, y_J, z_R),
\]

where \( \tilde{\psi}_{j,k} \) are same as in the discrete Calderón’s identity in Theorem 5.6.

Then, \( S \) is bounded from \( H^p_\omega \) to \( s^p_\omega \), and from \( CMO^p_\omega \) to \( c^p_\omega \). While \( T \) is bounded from \( s^p_\omega \) to \( H^p_\omega \), and from \( c^p_\omega \) to \( CMO^p_\omega \). Moreover, \( T \circ S \) is the identity map on both \( H^p_\omega \) and \( CMO^p_\omega \).

Proof of Lemma 5.13. If \( f \in H^p_\omega \), then by the definition of \( H^p_\omega \) in Definition 5.2 together with discrete Littlewood-Paley-Stein square function (5.8),

\[
\|h(f)\|_{s^p_\omega} = \left\| \sum_{j,k} \sum_{I \times J \times R} |h(f)_{I \times J \times R}|^2 |I|^{-1} |J|^{-1} |R|^{-1} \chi_I(x) \chi_J(y) \chi_R(z) \right\|_{L^p_\omega}^{\frac{1}{2}}
\]

\[
= \left\| \sum_{j,k} \sum_{I \times J \times R} |(\psi_{j,k} \ast f)(x_I, y_J, z_R)|^2 \chi_I(x) \chi_J(y) \chi_R(z) \right\|_{L^p_\omega}^{\frac{1}{2}}
\]

\[
\leq C \|f\|_{H^p_\omega}.
\]

Similarly, by the aid of sup-inf comparison principle of second kind in Theorem 5.4, we can show \( S \) is bounded from \( CMO^p_\omega \) to \( c^p_\omega \).

To show \( T \) is bounded from \( s^p_\omega \) to \( H^p_\omega \), by using almost orthogonality estimate in
Lemma 5.8, we get

\[
\sum_{j',k'} \sum_{l',k',R'} |\psi_{j',k'} * T(s)(x_{l'}, y_{l'}, z_{l'})|^2 \chi_{P}(x)\chi_{R'}(z)
\]

\[
= \sum_{j',k'} \sum_{l',k',R'} \left( \sum_{j,k} s_{I \times J \times R} |I|^{-\frac{1}{2}} |J|^{-\frac{1}{2}} |R|^{-\frac{1}{2}} \psi_{j,k}(x, y, z, x_{l}, y_{l}, z_{l}) \right) \sum_{s_{i \times j \times k}} (x_{l'}, y_{l'}, z_{l'})^2 \chi_{P}(x)\chi_{R'}(z)
\]

\[
\leq C \sum_{j,k} 2^{-|j-j'|L} 2^{-|k-k'|L} 2^{j} |J|^{-\frac{1}{2}} |J|^{-\frac{1}{2}} |R|^{-\frac{1}{2}} \left( \sum_{l,I,J,R} |I|^{-\frac{1}{2}} |J|^{-\frac{1}{2}} |R|^{-\frac{1}{2}} s_{I \times J \times R} \chi_{I \chi J \chi R} \right) \]

in which we applied Lemma 5.12 to get the last inequality, and use the weighted inequalities for vector-valued maximal operator associated with Zygmund dilations, we will be able to derive that

\[
\|T(s)\|_{H_{p}^{p}(\omega)}
\]

\[
= \left\| \left\{ \sum_{j',k'} \sum_{l',k',R'} |(\psi_{j',k'} * T(s))(x_{l'}, y_{l'}, z_{l'})|^2 \chi_{P}(x)\chi_{R'}(z) \right\} \right\|_{L_{p}^{p}}^{\frac{1}{2}}
\]

\[
\leq C \left\| \left\{ \sum_{j,k} 2^{-|j-j'|L} 2^{-|k-k'|L} 2^{j} |J|^{-\frac{1}{2}} |J|^{-\frac{1}{2}} |R|^{-\frac{1}{2}} \left( \sum_{l,I,J,R} |I|^{-\frac{1}{2}} |J|^{-\frac{1}{2}} |R|^{-\frac{1}{2}} s_{I \times J \times R} \chi_{I \chi J \chi R} \right) \right\} \right\|_{L_{p}^{p}}^{\frac{1}{2}}
\]

\[
\leq C \left\| \left\{ \sum_{j,k} \sum_{l,I,J,R} s_{I \times J \times R} |I|^{-1} |J|^{-1} |R|^{-1} \chi_{I \chi J \chi R} \right\} \right\|_{L_{p}^{p}}^{\frac{1}{2}}
\]

\[
= C \|s\|_{s_{p}^{p}(\omega)}
\]

Similarly, we can prove \(T\) is bounded from \(c_{p}^{p}(\omega)\) to \(CMO_{p}^{p}(\omega)\), and it is evident that \(T \circ S\) is the identity map on \(H_{p}^{p}(\omega)\) and \(CMO_{p}^{p}(\omega)\). \(\square\)
Combining Theorem 5.11 and Lemma 5.13, we are able to prove Theorem 5.5.

Proof of Theorem 5.5. First, if \( g \in CMO_Z^p(\omega) \), the map \( \ell_g \) is given by \( \ell_g(f) = \langle f, g \rangle \) for \( f \in S_Z(\mathbb{R}^3) \)

\[
\ell_g(f) = | \langle f, g \rangle |
\]

\[
= | \langle \sum \sum |I||J||R|\tilde{\psi}_{j,k}(x, y, z, x_I, y_J, z_R)(\psi_{j,k} * f)(x_I, y_J, z_R), g \rangle |
\]

\[
= | \langle S(f), S(g) \rangle |
\]

\[
\leq \|S(f)\|_{s_Z^p(\omega)} \|S(g)\|_{c_Z^p(\omega)}
\]

\[
\leq C \|f\|_{H_Z^p(\omega)} \|g\|_{CMO_Z^p(\omega)}.
\]

Since \( S_Z(\mathbb{R}^3) \) is dense in \( H_Z^p(\omega) \) (see [HLX2]), Hahn-Banach Theorem implies that the map \( \ell_g = \langle f, g \rangle \) can be extended to a continuous linear functional on \( H_Z^p(\omega) \), and \( \|\ell_g\| \leq C\|g\|_{CMO_Z^p(\omega)} \).

Conversely, for every \( \ell \in (H_Z^p(\omega))^* \), consider \( \ell_T = \ell \circ T \) defined on \( s_Z^p(\omega) \), and for every \( s \in s_Z^p(\omega) \),

\[
|\ell_T(s)| = |\ell(T(s))| \leq \|\ell\| \|T(s)\|_{H_Z^p(\omega)} \leq C \|\ell\| \|s\|_{s_Z^p(\omega)},
\]

which implies \( \ell_T \in (s_Z^p(\omega))^* \), then by Theorem 5.11, there exists \( t = \{t_{I \times J \times R}\} \in c_Z^p(\omega) \) such that

\[
\ell_T(s) = \langle s, t \rangle = \sum_{I \times J \times R} s_{I \times J \times R} \ell_{I \times J \times R},
\]

for all \( s \in s_Z^p(\omega) \), and

\[
\|t\|_{c_Z^p(\omega)} \approx \|\ell_T\| \leq C \|\ell\|.
\]
From Lemma 5.13, $T \circ S$ is the identity map on $H^p_\omega$, thus $\ell = \ell \circ T \circ S = \ell_T \circ S$, and

$$\ell(f) = \ell_T(S(f)) = \langle S(f), t \rangle = \langle f, g \rangle,$$

in which $g = T(t)$. This shows $\ell = \ell_g$ for $g \in CMO^p_\omega$, and $\|g\|_{CMO^p_\omega} \leq C\|t\|_{c^p_\omega} \leq C\|\ell\|$, which completes the proof of Theorem 5.5.
REFERENCES


ABSTRACT

DISCRETE LITTLEWOOD-PALEY-STEIN THEORY
AND WOLFF POTENTIALS ON HOMOGENEOUS SPACES
AND MULTI-PARAMETER HARDY SPACES

by

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Part I Let $\mathcal{X}, \mathcal{X}_1$ and $\mathcal{X}_2$ be the spaces of homogeneous type, by using the discrete harmonic analysis, we

- derive a new $(p, q)$-atomic decomposition on the multi-parameter Hardy space $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ for $0 < p \leq 1$ and all $1 < q < \infty$, where this decomposition converges in both $L^q(\mathcal{X}_1 \times \mathcal{X}_2)$ (for $1 < q < \infty$) and Hardy space $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ (for $0 < p \leq 1$).

- prove that an operator $T$, which is bounded on $L^q(\mathcal{X}_1 \times \mathcal{X}_2)$ for some $1 < q < \infty$, is bounded from $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ to $L^p(\mathcal{X}_1 \times \mathcal{X}_2)$ if and only if $T$ is bounded uniformly on all $(p, q)$-product atoms in $L^p(\mathcal{X}_1 \times \mathcal{X}_2)$. The similar boundedness criterion from $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ to $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ is also obtained.

- compare the Wolff and Riesz potentials on $\mathcal{X}$ and get an associated Hardy-Littlewood-Sobolev type inequality. Applying this inequality, we derive integrability estimates of positive solutions to the Lane-Emden type integral system on $\mathcal{X}$.

Part II By applying the discrete Littlewood-Paley-Stein analysis, we establish a $(p, 2)$-atomic decomposition of Hardy spaces associated with different homogeneities. In addition, we prove the duality theorem of weighted multi-parameter Hardy spaces associated with Zygmund dilations, i.e., $(H^p_Z(\omega))^* = CMO^p_Z(\omega)$ for $0 < p \leq 1$. Our theorems extend the weighted Hardy spaces the $H^p_Z - CMO^p_Z$ duality established in [HL2] for non-weighted multi-parameter Hardy spaces associated with the Zygmund dilation.
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