FULL STABILITY IN OPTIMIZATION

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DEDICATION

To dad and mom

Trần Mạnh Phách and Thái Cẩm Thạch

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Chapter 1

Introduction

The concept of *Lipschitzian full stability* of local minimizers in general optimization problems was introduced by Levy, Poliquin and Rockafellar [31] to single out those local solutions, which exhibit "nice" stability properties under appropriate parameter perturbations. Roughly speaking, the properties postulated in [31] require that the local minimizer in question does not lose its uniqueness and evolves "proportionally" (in some Lipschitzian way) with respect to a certain class of two-parametric perturbations; see Chapter 3 for the precise formulations. The full stability notion of [31] extended the previous one of *tilt stability* introduced by Poliquin and Rockafellar [53], where such a behavior was considered with respect to one-parametric linear/tilt perturbations. Both stability notions in [31, 53] were largely motivated by their roles in the justification of numerical algorithms, particularly the stopping criteria, convergence properties, and robustness.

The first second-order characterizations of tilt stability were obtained by Poliquin and Rockafellar [53] via the second-order subdifferential/generalized Hessian of Mordukhovich [36] in the general framework of extended-real-valued prox-regular functions and by Bonnans and Shapiro [5] via a certain uniform second-order growth condition in the framework of conic programs with $\mathcal{C}^2$-smooth data. More recent developments on tilt stability for various classes of optimization problems in both finite and infinite dimensions can be found in [13, 14, 17, 34, 42, 43, 45, 46, 47].

Much less has been done for full stability. In the pioneering work by Levy, Poliquin and Rockafellar [31] this notion was characterized in terms of a partial modification of the second-order subdifferential from [36] for a class of parametrically prox-regular functions in the unconstrained format of optimization with extended-real-valued objectives. The calculus rules for this partial second-order subdifferential developed by Mordukhovich and Rockafellar [47] allowed them in the joint work with Sarabi [48] to derive constructive second-order characterizations of fully stable minimizers for various classes of constrained optimization problems in finite dimensions including those in nonlinear and extended nonlinear programming and mathematical programs with polyhedral constraints, which plays an essential role in their works. In these two papers some relationships of full stability to the classical Robinson's *strong regularity* [58] has been revealed in some special classes of optimization problem [47, 48]. It is important to emphasize that Robinson's strong regularity which relates to the local single-valuedness and Lipschitz continuity of solution maps to generalized equations is the key tool in developing qualitative and numerical results (e.g., Newton method) on variational inequalities and complementarity problems [16, 18, 24, 25, 33, 37]. Since full stability is a weaker property than strong regularity in constrained optimization (see our Section 5.5 for further details), studying this remarkable stability gives us a realistic hope to improve many well-known results or even establish new understanding on the aforementioned areas of optimization. This actually makes full stability on the call!

Developing a systematic study to full stability in the general framework of optimization problems and applying it to many significant classes of constrained optimization without polyhedricity assumption are two main purposes of the thesis. More specifically, we introduce a new notion so-called Hölderian full stability weaker than its Lipschitzian counterpart [31] and generate a *geometric dual-space approach* to both of them even in infinite-dimensional spaces. In contrast to [31], our approach does not appeal to *tangential approximations* of sets and functions while operating instead with intrinsically nonconvex-valued normal and coderivative mappings, which satisfy comprehensive calculus rules. This leads us to more direct and simple proofs with a variety of quantitative and qualitative characterizations of full and tilt stability. Furthermore, in this way we may relax the assumption of polyhedricity on the constraint sets [47, 48] in studying full stability of *mathematical programs* with $\mathcal{C}^2$-*smooth data* (including those of conic programming). It is worth mentioning that many remarkable classes such as *semidefinite programming* [65] and *second-order cone programming* do not enjoy the aforementioned polyhedral conditions.

Besides Chapter 2 which provides some preliminaries from variational analysis and generalized differentiations, the thesis contains two major parts. Part A is devoted to the recent developments in our joint papers [40, 44] on the theory of full stability in general infinite-dimensional optimization problems. While Part B focuses on several applications of full stability to constrained optimization. Part A begins with Chapter 3, in which we formulate the basic notions of Hölderian and Lipschitzian full stabilities and focus on second-order descriptions of these notions for the general class of parametrically prox-regular extended-real-valued functions. The work not only covers the original result [31] in finite-dimensional frameworks but also reveals many convenient characterizations of both types of full stability. Particularly, these characterizations are obtained in terms of a certain second-order growth condition as well as via second-order subdifferential constructions with precise relationships between the corresponding moduli.

In Chapter 4 we present many implementations of full stability to *parametric variational systems* including *generalized equations* introduced by Robinson in his landmark paper [56]. Nowadays, the latter becomes a core notion in variational inequalities and complementary problems [18], constrained optimization associated with Lagrange multipliers [5], etc. In [58] Robinson introduced another significant notation so-called strong regularity which ensures the existence of a Lipschitz continuous single-valued localization of the solution mapping. Full characterizations of strong regularity over polyhedral convex sets have been established by Dontchev and Rockafellar [15], in which Mordukhovich's coderivative criterion for *Lipschitz-like* property and second-order subdifferential are very essential. Whether their main result is still valid when replacing the polyhedral convex sets by other vital ones such as the set of *positive semidefinite matrices* or *second-order/Lorent/ice-cream cones* is still a big open question in the area. Partial answers can be founded in [4, 54]. In this chapter we will provide some closer looks to that question by developing a new approach to generalized equations via full stability and second-order theory. More generally, we study parametric variational systems, which particularly covers the so-called *quasivariational inequalities* [18] or even *hemivariational inequalities* [51] and establish new sufficient conditions for Hölder and Lipschitz continuity of the solution mapping to these systems in term of second-order subdifferentials.

Part B regarding applications to constrained optimization starts with Chapter 5 which addresses the conventional class of $\mathcal{C}^2$-smooth parametric optimization problems with constraints written in the form $g(x,p) \in \Theta$, where Θ is a closed and convex subset of a finite-dimensional space. The model is indeed one of the most general problems in constrained optimization [5]. Imposing the classical *Robinson constraint qualification* [60], we show that the continuity of the stationary mapping in Kojima's *strong stability* [23] can be strengthened to Hölder continuity with order $\frac{1}{2}$ by using Hölderian full stability. If in addition the constraint are $\mathcal{C}^2$-*reducible* and the optimal point is (partially) *nondegenerate* in the sense of [5], then we prove the equivalence of *Lipschitzian full stability* to Robinson's *strong regularity* of the associated variational inequality on Lagrange multipliers. Furthermore, the complete characterizations of full stability in Chapter 3 allow us to establish new characterizations to strong stability and also strong stability via verifiable conditions involving the second-order subdifferential (or the generalized Hessian) $\partial^2\delta_\Theta$ of the indicator function δ_Θ of Θ. Also in this chapter these results are specified for semidefinite programs and second-order cone programs, where $\Theta = S^m_+$ is the cone of all the $m \times m$ symmetric positive semidefinite matrices. Furthermore, we show that without nondegeneration condition the aforementioned equivalences are not valid anymore. More specifically, in the classical nonlinear programming when both *Mangasarian-Fromovitz constraint qualification* and *constant rank constraint qualification* are satisfied at the minimizer point, full stability is characterized by a new *uniform second-order sufficient condition* while both strong regularity and strong stability may be not fulfill. This allows us to conclude that strong regularity is always a stronger property than full stability in general. Chapter 6 ends part B with several applications to mathematical programs in infinite-dimensional spaces including *polyhedric constrained programs* [20, 35] and *optimal control of semilinear elliptic equation* [2, 5].

Chapter 2

Preliminary

2.1 Basic Notation

We begin with some standard notation in variational analysis and generalized differentiation; cf. [8, 38, 62]. Let X be a Banach space. Recall that X is *Asplund* if each of its separable subspaces has a separable dual. This subclass of Banach spaces is sufficiently large including, in particular, every reflexive space; see, e.g., [38] for more details and references. As usual, $\|\cdot\|$ stands for the norm in X and $\langle\cdot,\cdot\rangle$ indicates the canonical pairing between X and its topological dual X^* with w^* signifying the weak* convergence in X^* and cl^* standing for the weak* topological closure of a set. We denote by $I\!B$ and $I\!B^*$ the closed unit ball in the space in question and its dual space, respectively, with $I\!B_\eta(x) := x + \eta I\!B$ standing for the closed ball centered at x with radius $\eta > 0$.

Given a set-valued mapping $F\colon X \rightrightarrows Y$ between two Banach spaces X and Y, the notion

$$\mathop{\mathrm{Lim\,sup}}_{x\to\bar{x}} F(x) := \Big\{ y \in Y \Big|\ \exists \text{ sequences } x_k \to \bar{x},\ y_k \to y \text{ such that } y_k \in F(x_k) \text{ for all } k \in I\!N := \{1,2,\ldots\} \Big\} \tag{2.1}$$

signifies the *sequential Painlevé-Kuratowski outer/upper limit* of $F(x)$ as $x \to \bar{x}$. When $Y = X^*$, we denote

$$w^* - \mathop{\mathrm{Lim\,sup}}_{x\to\bar{x}} F(x) := \Big\{ x^* \in X^* \Big|\ \exists \text{ sequences } x_k \to \bar{x},\ x_k^* \xrightarrow{w^*} x^* \text{ such that } x_k^* \in F(x_k) \text{ for all } k \in I\!N := \{1,2,\ldots\} \Big\} \tag{2.2}$$

2.2 Convex Analysis

Let $f : X \to \overline{I\!R}$ be an extended-real-valued function, which is always assumed to be *proper*, i.e., $\mathrm{dom}\, f := \{x \in X \mid f(x) < \infty\} \neq \emptyset$. Recall first some constructions and facts from *convex analysis* needed in the dissertation; see, e.g., [3, 62, 68]. If f is convex, its (Fenchel) *conjugate* $f^* : X^* \to \overline{I\!R}$ is defined by

$$f^*(x^*) := \sup_{x \in X} \big\{\langle x^*, x\rangle - f(x)\big\} \quad \text{for all} \quad x^* \in X^*, \tag{2.3}$$

and its *convex subdifferential* (collection of subgradients) at $\bar{x} \in \mathrm{dom}\, f$ is given by

$$\partial f(\bar{x}) := \big\{x^* \in X^* \mid f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle \ \text{ for all } \ x \in X\big\}, \tag{2.4}$$

which can be equivalently represented via the conjugate function by $\big\{x^* \in X^* \mid \langle x^*, \bar{x}\rangle - f(\bar{x}) \geq f^*(x^*)\big\}$.

The *biconjugate* $f^{**} : X \to \overline{I\!R}$ of f is the conjugate of f^*, i.e., $(f^*)^*$. The following result [68] is useful for our subsequent considerations.

Lemma 2.1 (subdifferential duality). *Let $f : X \to \overline{I\!R}$ be a convex and l.s.c. function, and let f^* be its conjugate* (2.3). *Then we have the relationship*

$$x^* \in \partial f(x) \ \textit{ if and only if } \ x \in \partial f^*(x^*),$$

which implies that $\partial f^*(x^*) = \mathrm{argmin}_X\{f(x) - \langle x^*, x\rangle\}$ for any $x^* \in X^*$.

It is well-known that f^{**} coincides f when f is a proper lower semi-continuous (l.s.c.) convex function. Furthermore, we have

Lemma 2.2 (biconjugate inequality, [68, Theorem 2.3.4]). *Let $f : X \to \overline{I\!R}$ be a proper function. Then we have the inequality $f^{**} \le f$. Moreover, $f^{**} = f$ if and only if f is l.s.c. and convex.*

One of the most impressive results in convex analysis is the sum rule of convex subdifferentials.

Lemma 2.3 (sum rule, [68, Theorem 2.8.7]). *Let $f, g : X \to \overline{I\!R}$ be two proper l.s.c. convex functions. Suppose that there is some $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$ such that g is continuous at x_0. Then for any $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom} g$ we have the relationship*

$$\partial(f+g)(\bar{x}) = \partial f(\bar{x}) + \partial g(\bar{x}).$$

2.3 Basic Variational Geometry

Throughout this section, Ω is assumed to be a subset of a Banach space X. The notations $\operatorname{co}\Omega$, $\operatorname{span}\Omega$, $\operatorname{cl}\Omega$, $\operatorname{bd}\Omega$, $\operatorname{int}\Omega$ signify the *convex hull*, *span hull*, *closure*, *boundary*, and *interior* of Ω respectively. We write $x \xrightarrow{\Omega} \bar{x}$ to express the convergence relative to Ω in the sense that $x \to \bar{x}$ with $x \in \Omega$.

The thesis mainly concerns about the dual approaches in variational analysis, where notions of *normal cones* play essential roles in studying optimization problems with constraints. When Ω is convex, the *convex normal cone to* Ω at a point $\bar{x} \in \Omega$ is given by

$$N_\Omega(\bar{x}) = \big\{x^* \in X^* \big|\ \langle x^*, x - \bar{x}\rangle \le 0 \ \text{ for all } \ x \in X\big\}. \tag{2.5}$$

This is indeed the convex subdifferental (2.4) at $\bar{x} \in \Omega$ to the *indicator function* $\delta_\Omega(x)$, which is equal to 0 if $x \in \Omega$ and to ∞ otherwise. The *polar cone* of the convex normal cone is the *convex* *tangent cone* formulated by

$$T_\Omega(\bar{x}) := \big[N_\Omega(\bar{x})\big]^* = \operatorname*{Lim\,sup}_{t\downarrow 0} \frac{\Omega - \bar{x}}{t}, \tag{2.6}$$

where the "Limsup" is taken from (2.1) and the notion $A^* := \big\{a \in X|\ \langle a^*, a\rangle \le 0, a^* \in A\big\}$ means the polar cone of $A \subset X^*$.

When Ω is not convex, there are many appropriate generalized concepts of normal cones; see, e.g., the monographs [8, 9, 38, 62] for further details and discussions. Following [38] we define the ε-normals to Ω at $x \in \Omega$ by

$$\widehat{N}_\varepsilon(x;\Omega) := \left\{x^* \in X^*\big|\ \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \le \varepsilon\right\}. \tag{2.7}$$

When $\varepsilon = 0$, we simply denote $\widehat{N}(x;\Omega)$ for $\widehat{N}_0(x;\Omega)$ and call this set *regular normal cone* (known also as the Fréchet normal cone) to Ω at x. Then the *limiting normal cone* (known also as the general or basic normal cones and as the Mordukhovich normal cone) to Ω at some $\bar{x} \in \Omega$ is defined by

$$N(x;\Omega) := w^* - \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\Omega} \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x;\Omega). \tag{2.8}$$

When X is an Asplund space, this formula can be simplified [38, Theorem 2.35] by

$$N(x;\Omega) := w^* - \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(x;\Omega).$$

It is easy to check that both $\widehat{N}(\bar{x};\Omega)$ and $N(\bar{x};\Omega)$ are cones in X^*. However, the set $\hat{N}(\bar{x};\Omega)$ is convex, while $N(\bar{x};\Omega)$ is not in general. Furthermore, when Ω is convex, these two notations reduce to the convex normal cone in (2.5). In the thesis we also write $\widehat{N}_\Omega(\bar{x})$ and $N_\Omega(\bar{x})$ to represent $\widehat{N}(\bar{x};\Omega)$ and $N(\bar{x};\Omega)$, respectively.

One of the most meaningful properties of construction (2.8) is that it satisfies the *intersection rule*: given Ω_1 and Ω_2 two closed subsets of a finite dimensional space X with $\bar{x} \in \Omega_1 \cap \Omega_2$, then we have

$$N(\bar{x};\Omega_1 \cap \Omega_2) \subset N(\bar{x};\Omega_1) + N(\bar{x};\Omega_2)$$

provided that $N(\bar{x};\Omega_1) \cap (-N(\bar{x};\Omega_2) = \{0\}$. This result can be generalized to Asplund space under an additional hypothesis that either Ω_1 or Ω_2 satisfies the so-called *sequential normal compactness* at $\bar{x}$, which is strictly weaker than the *nonempty interior assumption*, i.e., either $\operatorname{int}\Omega_1 \neq \emptyset$ or $\operatorname{int}\Omega_2 \neq \emptyset$; see [38, Definition 1.20 and Corollary 3.37].

2.4 Subdifferential of Nonsmooth Functions and Coderivative of Set-Valued Mappings

In this section we recall several subdifferential constructions for nonsmooth functions. Let us start with *first-order subdifferential* for proper extended-real-valued functions $f\colon X \to \overline{I\!R}$ on Banach spaces assuming that f is *lower semicontinuous* (l.s.c.) around $\bar{x}$ from the domain $\operatorname{dom} f := \{x \in X |\ f(x) < \infty\}$. The *regular subdifferential* (known also as the presubdifferential and as the Fréchet or viscosity subdifferential) of f at $\bar{x} \in \operatorname{dom} f$ can be defined via the regular normal cone (2.7)

$$\widehat{\partial} f(\bar{x}) := \Big\{x^* \in X^* \Big| (x^*,-1) \in \widehat{N}\big((\bar{x}, f(\bar{x})); \operatorname{epi} f\big)\Big\},$$

where $\operatorname{epi} f := \big\{(x,r) \in X \times I\!R \big|\ f(x) \le r\big\}$ is the epigraph of f. Indeed, this construction can be expressed by another equivalent "explicit" form

$$\widehat{\partial} f(\bar{x}) := \Big\{x^* \in X^* \Big|\ \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \ge 0\Big\}. \tag{2.9}$$

The *limiting subdifferential* (known also as the general or basic subdifferential and as the Mordukhovich subdifferential) of f at $\bar{x} \in \operatorname{dom} f$ is denoted similarly by

$$\partial f(\bar{x}) := \Big\{x^* \in X^* \Big| (x^*, -1) \in N\big((\bar{x}, f(\bar{x})); \operatorname{epi} f\big)\Big\},$$

which can be also defined explicitly via the sequential outer limit (2.2) by

$$\partial f(\bar{x}) := w^* - \operatorname*{Lim\,sup}_{x \xrightarrow{f} \bar{x}} \widehat{\partial} f(x) \tag{2.10}$$

when X is an Asplund space, where the notation $x \xrightarrow{f} \bar{x}$ stands for the convergence relative to the function f, i.e., $x \to \bar{x}$ and $f(x) \to f(\bar{x})$.

Another construction important in our study is the *horizontal subdifferential* formulated by

$$\partial^\infty f(\bar{x}) := \Big\{x^* \in X^* \Big| (x^*, 0) \in N\big((\bar{x}, f(\bar{x})); \operatorname{epi} f\big)\Big\}. \tag{2.11}$$

Note that for convex functions f both regular and limiting subdifferentials reduce to the subdifferential of convex analysis (2.4) and that $\partial^\infty f(\bar{x}) = \{0\}$ if f is locally Lipschitzian around $\bar{x}$. Moreover, if the function f is Fréchet differentiable at $\bar{x}$, then $\hat{\partial} f(\bar{x}) = \nabla f(\bar{x})$; while if the function f is *strictly* Fréchet differentiable at $\bar{x}$, then $\partial f(\bar{x}) = \nabla f(\bar{x})$. In general the subgradient sets in (2.10) are nonconvex while they and the corresponding normal and coderivative constructions for sets and mappings (see below) enjoy *full calculi* based on *variational/extremal principles* of variational analysis; see [38, 62]. To illustrate, we state here the *sum rule* of this setting used frequently in our study. The more general result can be found in [38, Theorem 3.36].

Lemma 2.4 (sum rule, [38, Theorem 3.36]). *Let X be an Asplund space and let $f, g : X \to \overline{I\!R}$ be two proper extended-real-valued functions with $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom} g$. Suppose that g is Lipschitz continuous around $\bar{x}$. Then we have*

$$\partial(f+g)(\bar{x}) \subset \partial f(\bar{x}) + \partial g(\bar{x}).$$

Before going to *second-order subdifferential* constructions, let us recall some useful notation of *coderivatives*. Given a set-valued mapping $F : X \rightrightarrows Y$ between Asplund spaces with the domain $\operatorname{dom} F := \{x \in X |\ F(x) \neq \emptyset\}$ and the graph $\operatorname{gph} F := \{(x,y) \in X \times Y |\ y \in F(x)\}$ assumed to be locally closed around the points in question. The *regular coderivative* and the *mixed coderivative* of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ are defined, respectively, by

$$\widehat{D}^* F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*, -y^*) \in \widehat{N}((\bar{x},\bar{y}); \operatorname{gph} F)\big\}, \quad y^* \in Y^*, \tag{2.12}$$

$$D^*_M F(\bar{x},\bar{y})(y^*) := w^* - \mathop{\operatorname{Lim\,sup}}_{\substack{(x,y) \overset{\operatorname{gph} F}{\to} (\bar{x},\bar{y}) \\ z^* \to y^*}} \widehat{D}^* F(x,y)(z^*), \quad y^* \in Y^*, \tag{2.13}$$

where the convergence $z^* \to y^*$ is strong in Y^* while the sequential outer limit in (2.13) is taken by (2.2) in the weak* topology of X^*; cf. [38] for these and other coderivative constructions. We omit the subscript "M" in (2.13) when X is finite-dimensional as well as the the indication of $\bar{y} = F(\bar{x})$ in (2.12) and (2.13) when F is single-valued. It has been well recognized that the coderivatives (2.12) and (2.13) are appropriate tools for the study and characterizations of well-posedness and sensitivity in variational analysis; see [38, Chapter 4] or Section 2.5 below for more details and references.

It is worth noting that if $F : X \to Y$ is a single-valued $\mathcal{C}^1$ mapping around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, then we have $\widehat{D}^* F(\bar{x}, \bar{y})(y^*) = D^*_M F(\bar{x}, \bar{y})(y^*) = \nabla f(\bar{x})^* y^*$ for $y^* \in Y^*$.

The following *coderivative sum rule* in Asplund spaces is significant.

Lemma 2.5 (coderivative sum rules, [38, Theorem 1.62]). *Let X, Y be Asplund spaces, let $f : X \to Y$ be Fréchet differentiable at $\bar{x}$, and let $F : X \rightrightarrows Y$ be an arbitrary set-valued mapping such that $\bar{y} - f(\bar{x}) \in F(\bar{x})$. The following hold:*

(i) *For all $y^* \in Y^*$ one has*

$$\widehat{D}^*(f + F)(\bar{x}, \bar{y})(y^*) = \nabla f(\bar{x})^* y^* + \widehat{D}^* F(\bar{x}, \bar{y} - f(\bar{x}))(y^*).$$

(ii) *If f is strictly differentiable at $\bar{x}$, then for all $y^* \in Y^*$ one has*

$$D^*(f + F)(\bar{x}, \bar{y})(y^*) = \nabla f(\bar{x})^* y^* + D^* F(\bar{x}, \bar{y} - f(\bar{x}))(y^*).$$

In this dissertation we widely employ *second-order subdifferential constructions* obtained by the scheme initiated in [36]: take a coderivative of a first-order subdifferential mapping. The major one used below was introduced in [42] as follows; cf. also [21] for the case of set indicator functions. Given $f : X \to \overline{I\!R}$ with $\bar{x} \in \operatorname{dom} f$ and $\bar{x}^* \in \partial f(\bar{x})$, the *combined second-order subdifferential* of f at $\bar{x}$ relative to $\bar{x}^*$ is the set-valued mapping $\breve{\partial}^2 f(\bar{x}, \bar{x}^*) : X^{**} \rightrightarrows X^*$ with the values

$$\breve{\partial}^2 f(\bar{x}, \bar{x}^*)(u) := \big(\widehat{D}^* \partial f\big)(\bar{x}, \bar{x}^*)(u), \quad u \in X^{**}. \tag{2.14}$$

The *mixed second-order subdifferential* of f at $\bar{x}$ relative to $\bar{x}^*$ is the set-valued mapping $\partial^2_M f(\bar{x}, \bar{x}^*) : X^{**} \rightrightarrows X^*$ with the values

$$\partial^2_M f(\bar{x}, \bar{x}^*)(u) := \big(D^*_M \partial f\big)(\bar{x}, \bar{x}^*)(u) \text{ for all } u \in X^{**}. \tag{2.15}$$

Both constructions (2.14) and (2.15) reduce to that of [36] in finite dimensions. The mixed second-order subdifferential is introduced in [38, Definition 1.118] (together with the *normal* one not used in the paper) while the combined second-order subdifferential seems to be new in the literature. Note however that its finite-dimensional version with the normal cone $\partial f(\cdot) = N(\cdot; \Omega)$ in (2.15) has been recently used in [21, 22] for different purposes. The letter "M" in (2.15) is omitted if X is a finite-dimensional space.

When f is $\mathcal{C}^2$ around $\bar{x}$ with $\bar{x}^* = \nabla f(\bar{x})$, both $\breve{\partial}^2 f(\bar{x}, \bar{x}^*)(u)$ and $\partial^2_M f(\bar{x}, \bar{x}^*)(u)$ reduce to the classical single-valued Hessian operator:

$$\breve{\partial}^2 f(\bar{x}, \bar{x}^*)(u) = \partial^2_M f(\bar{x}, \bar{x}^*)(u) = \big\{\nabla^2 f(\bar{x})^* u\big\} \text{ for all } u \in X^{**},$$

where $\nabla^2 f(\bar{x})^* = \nabla^2 f(\bar{x})$ in the Hilbert space setting.

2.5 Stability of Set-Valued Mappings and Mordukhovich Criterion

This section is devoted to characterizations of well-posedness and sensitivity analysis of set-valued mappings (multifunctions) via coderivatives introduced in Section 2.4. Given $F : X \rightrightarrows Y$ be a set-valued mapping between two Banach spaces. We say F is *metrically regular* with modulus $K>0$ around $(\bar{x},\bar{y})\in\operatorname{gph}F$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ with

$$\operatorname{dist}\big(x;F^{-1}(y)\big)\le K\operatorname{dist}\big(y;F(x)\big)\ \text{ for all }\ x\in U\ \text{ and }\ y\in V, \tag{2.16}$$

where $\operatorname{dist}(x;\Omega)$ stands for the distance from x to Ω. The infimum of all the moduli K over (K,U,V) in (2.16), denoted by $\operatorname{reg}F(\bar{x},\bar{y})$, is called the *exact regularity bound.*

It has been well recognized that the concept of metric regularity is fundamental in nonlinear analysis and optimization and is used not only in theoretical studies but also in numerical methods. For the classical linear and smooth operators this properties go back to the *Banach-Schauder open mapping theorem* and *Lyusternik-Graves Theorem*, respectively. For closed and convex multifunctions (i.e., those with the closed and convex graph, this notion is characterized by the *Robinson-Ursescu theorem*, which says that F is metrically regular around $(\bar{x},\bar{y})\in\operatorname{gph}F$ if and only if $\bar{y}$ belongs to the interior of the range $\operatorname{rge}F:=\{y\in Y|\ y\in F(x),\ x\in X\}$; see, e.g., [57, 68].

A significant specification of (2.16) is studied in [15] under the name of "strong metric regularity". Recall first that $\widehat{F}$ is a *localization* of $F\colon X\rightrightarrows Y$ around $(\bar{x},\bar{y})\in\operatorname{gph}F$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that $\operatorname{gph}\widehat{F}=\operatorname{gph}F\cap(U\times V)$. Then F is *strongly metrically regular* around $(\bar{x},\bar{y})$ with modulus $\kappa>0$ if the inverse mapping F^{-1} admits a single-valued localization around $(\bar{y},\bar{x})$ that is Lipschitz continuous with modulus κ around $\bar{y}$. It is easy to check that F is strongly metrically regular at $(\bar{x},\bar{y})$ if and only if F is metrically regular around $(\bar{x},\bar{y})$ and F^{-1} has a single-valued Lipschitzian localization around $(\bar{y},\bar{x})$. Moreover, the domain of such a single-valued localization must be a neighborhood of $\bar{y}$.

Another important property of F relating to the metric regularity of the inverse mapping F^{-1} is the so-called Lipschitz-like (known also as pseudo-Lipschitz or Aubin) property. The multifunction F is Lipschitz-like around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with modulus L if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x) \cap V \subset F(u) + L\|x - u\|I\!B \text{ for all } x, u \in U. \tag{2.17}$$

It is worth mentioning that F is Lipschitz-like around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if and only if F^{-1} is metrically regular with the same modulus around $(\bar{y}, \bar{x}) \in \operatorname{gph} F^{-1}$.

Next we recall Mordukhovich's criterion [62, Theorem 9.40], which is a characterization of Lipschitz-like property in finite-dimensional spaces. The infinite-dimensional version of this result can be found in [38, Theorem 4.10].

Lemma 2.6 (Mordukhovich criterion, [62, Theorem 9.40]). *Suppose that* $\dim X, \dim Y < \infty$. *Then* F *is Lipschitz-like around* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *if and only if* $D^*F(\bar{x}, \bar{y})(0) = 0$.

Another stability important to our study is a parametric version of Lipschitz-like property. Given $F : X \times P \rightrightarrows Y$ is a set-valued mapping, where (P, d) is a metric space. When $(\bar{x}, \bar{y}, \bar{p}) \in \operatorname{gph} F$, we say F is Lipschitz-like around $(\bar{x}, \bar{y})$ with compatible parameterization by p around $\bar{p}$ with modulus L if there are neighborhoods U of $\bar{x}$, V of $\bar{p}$, and W of $\bar{y}$ such that

$$F(x, p) \cap W \subset F(u, p) + L\|x - u\|I\!B \text{ for all } x, u \in U,\ p \in V. \tag{2.18}$$

The last result of this section provides a *quantitative* characterization of the above *parametric* Lipschitz-like property of set-valued mappings with a *precise modulus* in infinite-dimensional spaces. The proof modifies a similar result in our recent paper [49].

Lemma 2.7 (quantitative coderivative characterization of parametric Lipschitz-like property with prescribed modulus). *Let* X, Y *be two Asplund spaces while* (P, d) *is a* *metric space. Given a set-valued mapping* $F : X \times P \rightrightarrows Y$ *with* $(\bar{x}, \bar{p}, \bar{y}) \in \operatorname{gph} F$. *Suppose that the graph* $\operatorname{gph} F$ *be locally closed around* $(\bar{x}, \bar{p}, \bar{y})$. *Then* F *is Lipschitz-like around* $(\bar{x}, \bar{y})$ *with compatible parameterization by* p *around* $\bar{p}$ *with modulus* $\mu > 0$ *if and only if there is some* $\eta > 0$ *such that*

$$\|x^*\| \leq \mu\|y^*\| \quad \textit{whenever} \quad x^* \in \widehat{D}^* F_p(x, y)(y^*),\ (x, p, y) \in \operatorname{gph} F \cap I\!\!B_\eta(\bar{x}, \bar{p}, \bar{y}), \tag{2.19}$$

where $F_p(x, y) := F(x, p, y)$.

Proof. If F is Lipschitz-like around $(\bar{x}, \bar{y})$ with compatible parameterization by p around $\bar{p}$ with modulus $\mu > 0$, then there are neighborhoods U of $\bar{x}$, V of $\bar{p}$, and W of $\bar{y}$ such that

$$F(x, p) \cap W \subset F(u, p) + \mu\|x - u\| I\!\!B \ \text{ for all } \ x, u \in U,\ p \in V. \tag{2.20}$$

Choose any $\eta > 0$ such that $I\!\!B_{2\eta}(\bar{x}, \bar{p}, \bar{y}) \subset U \times V \times W$ and pick any $(x, p, y) \in \operatorname{gph} F \cap I\!\!B_\eta(\bar{x}, \bar{p}, \bar{y})$. For any $x^* \in \widehat{D}^* F_p(x, y)(y^*)$ and $\varepsilon > 0$, it follows from (2.12) that there is some $\delta \in (0, \eta)$ such that

$$\langle x^*, u - x\rangle - \langle y^*, v - y\rangle \leq \varepsilon(\|u - x\| + \|v - y\|) \quad \text{for all} \quad (u, v) \in \operatorname{gph} F_p \cap I\!\!B_\delta(x, y). \tag{2.21}$$

By (2.20) with any $u \in I\!\!B_\gamma(x)$ with $\gamma := \min\{\delta, \delta(\mu)^{-1}\} > 0$ we may find some $v \in F(u, p)$ such that $\|v - y\| \leq \mu\|x - u\| \leq \mu\gamma \leq \delta$. This together with (2.21) gives us that

$$\langle x^*, u - x\rangle \leq \langle y^*, v - y\rangle + \varepsilon(\|u - x\| + \|v - y\|) \leq \mu\|y^*\| \cdot \|u - x\| + \varepsilon(\|u - x\| + \mu\|u - x\|),$$

which clearly implies $\|x^*\| \leq \mu\|y^*\| + \varepsilon(1 + \mu)$, since u is chosen arbitrarily on $I\!\!B_\gamma(x)$. Letting

$\varepsilon \downarrow 0$ gives us $\|x^*\| \le \mu\|y^*\|$ and thus ensures (2.19).

It remains to prove the converse implication. Without loss of generality assume that gph F is closed on $X \times P \times Y$ and (2.19) holds for some $\eta > 0$. It is easy to prove that if there is some neighborhood $U \times V \times W$ of $(\bar{x}, \bar{p}, \bar{y})$ satisfying

$$\text{dist}\,(y; F(x,p)) \le \mu \text{dist}\,(x, F_p^{-1}(y)) \quad \text{for all} \quad (x,p,y) \in U \times V \times W, \tag{2.22}$$

then F is Lipschitz-like around $(\bar{x}, \bar{y})$ with compatible parameterization by p around $\bar{p}$ with modulus μ. Indeed, suppose that (2.22) holds and take any $(x,p,y) \in U \times V \times W$ and $u \in U$, we obtain from (2.22) that

$$\text{dist}\,(y; F(u,p)) \le \mu \text{dist}\,(u, F_p^{-1}(y)) \le \mu\|u - x\|,$$

which implies the existence of some $z \in F(u,p)$ with $\|y - z\| \le \mu\|u - x\|$ and verifies (2.20).

Arguing by contradiction that (2.20) is not valid. The above claim tells us (2.22) is not satisfied too. For any $p \in \text{int} I\!\!B_\eta(\bar{p})$ the latter allows us to find $(\hat{x}, \hat{y}) \in \text{int}\, I\!\!B_{\frac{\eta}{8}}(\bar{x}, \bar{y})$ satisfying dist $(\hat{y}; F(\hat{x},p)) > \mu\,$dist $(\hat{x}; F_p^{-1}(\hat{y}))$ and such that $(2\mu + 1)\varepsilon < \frac{\eta}{4}$, where $\varepsilon :=$ dist $(\hat{x}; F_p^{-1}(\hat{y})) > 0$. To proceed, pick any $\nu \in (\mu, 2\mu)$ with

$$\text{dist}\,\big(\hat{y}; F(\hat{x},p)\big) > \nu\varepsilon > \mu\,\text{dist}\,\big(\hat{x}; F_p^{-1}(\hat{y})\big)$$

and for any $\alpha > 0$ find some $\widetilde{x} \in F_p^{-1}(\hat{y})$ satisfying

$$\|\widetilde{x} - \hat{x}\| \le \text{dist}\,\big(\hat{x}; F_p^{-1}(\hat{y})\big) + \alpha = \varepsilon + \alpha. \tag{2.23}$$

For the l.s.c. and bounded from below function $\varphi(x,y) := \|x-\hat{x}\| + \delta((x,y);\operatorname{gph} F_p)$ on the Asplund space $X\times Y$ we have that

$$\inf_{(x,y)\in X\times Y} \varphi(x,y) + \varepsilon + \alpha \ge \varphi(\widetilde{x},\hat{y}).$$

Applying the seminal Ekeland variational principle (see, e.g., [38, Theorem 2.26]) to the function φ with the new norm $\|(x,y)\|_\xi := \xi\|x\|+\|y\|$, $\xi>0$ on $X\times Y$ gives us $(x_0,y_0)\in\operatorname{gph} F_p$ satisfying

$$\begin{cases} \xi\|x_0-\widetilde{x}\| + \|y_0-\hat{y}\| \le \nu\varepsilon, \\ \|x_0-\hat{x}\| = \varphi(x_0,y_0) \le \varphi(\widetilde{x},\hat{y}) = \|\widetilde{x}-\hat{x}\|, \\ \displaystyle\inf_{(x,y)\in X\times Y} \varphi(x,y) + \frac{\varepsilon+\alpha}{\nu\varepsilon}(\xi\|x-x_0\| + \|y-y_0\|) \ge \varphi(x_0,y_0) = \|x_0-\hat{x}\|. \end{cases} \tag{2.24}$$

It follows that

$$\|y_0-\hat{y}\| \le \nu\varepsilon < \operatorname{dist}\,(\hat{y};F(\hat{x},p)),$$

which yields $y_0\notin F(\hat{x},p)$ and thus $x_0\ne\hat{x}$. Consider the l.s.c. functions on $X\times Y$ defined by

$$\varphi_1(x,y) := \|x-\hat{x}\|,\ \varphi_2(x,y) := \delta((x,y);\operatorname{gph} F_p), \ \text{ and }\ \varphi_3(x,y) := \frac{\alpha+\varepsilon}{\nu\varepsilon}(\xi\|x-x_0\|+\|y-y_0\|),$$

where two of them are Lipschitz continuous. Then for any $0<\beta<\operatorname{dist}(\hat{y};F(\hat{x},p))-\nu\varepsilon$ we employ [38, Lemma 2.32] (the basic fuzzy sum rule or subgradient description of the extremal principle) to the optimization problem in (2.24) and thus find $(x_i,y_i)\in I\!B_\beta(x_0,y_0)$ with $0<\beta<\|x_0-\hat{x}\|$ as $i=1,2,3$ such that $(x_2,y_2)\in\operatorname{gph} F_p$ and that

$$\begin{aligned} 0 \ &\in \hat{\partial}\varphi_1(x_1,y_1) + \hat{\partial}\varphi_2(x_2,y_2) + \hat{\partial}\varphi_3(x_3,y_3) + \xi I\!B_{X^*}\times\frac{\beta}{\nu\varepsilon}I\!B_{Y^*} \\ &\subset \hat{\partial}\|\cdot-\hat{x}\|(x_1)\times\{0\} + \widehat{N}\big((x_2,y_2);\operatorname{gph} F_p\big) + \frac{(\nu+1)\varepsilon+\alpha}{\nu\varepsilon}\xi I\!B_{X^*}\times\frac{\varepsilon+\alpha+\beta}{\nu\varepsilon}I\!B_{Y^*}. \end{aligned} \tag{2.25}$$

Note further that $\|x_1-\hat{x}\| \geq \|x_0-\hat{x}\|-\|x_1-x_0\| \geq \|x_0-\hat{x}\|-\beta > 0$ which means $x_1 \neq \hat{x}$. This together (2.25) allows us to find some $x_1^* \in X^*$ with $\|x_1^*\| = 1$ and $(x_3^*, y_3^*) \in \frac{(\nu+1)\varepsilon+\alpha}{\nu\varepsilon}\xi I\!B_{X^*} \times \frac{\varepsilon+\alpha+\beta}{\nu\varepsilon} I\!B_{Y^*}$ such that $(x_1^* - x_3^*, -y_3^*) \in \widehat{N}((x_2, y_2); \operatorname{gph} F_p)$, i.e., $x_1^* - x_3^* \in \widehat{D}^* F_p(x_2, y_2)(y_3^*)$. Furthermore, it follows from (2.23), (2.24), and the choice of ν that

$$\begin{aligned}
\|x_2-\bar{x}\|+\|y_2-\bar{y}\| &\leq \|x_2-x_0\|+\|x_0-\hat{x}\|+\|\hat{x}-\bar{x}\|+\|y_2-y_0\|+\|y_0-\hat{y}\|+\|\hat{y}-\bar{y}\| \\
&\leq (\|x_2-x_0\|+\|y_2-y_0\|)+(\|\hat{x}-\bar{x}\|+\|\hat{y}-\bar{y}\|)+\|x_0-\hat{x}\|+\nu\varepsilon \\
&\leq \beta+\frac{\eta}{4}+\|\widetilde{x}-\hat{x}\|+\nu\varepsilon \leq \beta+\frac{\eta}{4}+\varepsilon+\alpha+\nu\varepsilon \\
&\leq \frac{\eta}{4}+(2\mu+1)\varepsilon+\alpha+\beta \leq \frac{\eta}{2}+\alpha+\beta
\end{aligned}$$

With $\alpha, \beta > 0$ sufficiently small we have $(x_2, y_2) \in B_\eta(\bar{x}, \bar{y}) \cap \operatorname{gph} F_p$. Thanks to (2.19) we get that

$$\mu\frac{\varepsilon+\alpha+\beta}{\nu\varepsilon} \geq \mu\|y_3^*\| \geq \|x_1^*-x_3^*\| \geq 1-\frac{(\nu+1)\varepsilon+\alpha}{\nu\varepsilon}\xi$$

Letting $\alpha, \beta, \xi \downarrow 0$ gives us that $\mu \geq \nu$, which contradicts the choice of ν and completes the proof of the lemma. $\square$

2.6 Prox-regular Functions and Monotone Operators

Finally in this chapter we recall significant concepts of prox-regularity and subdifferential continuity of extended-real-valued functions taken from [31], where they are comprehensively studied in finite dimensions; cf. also the nonparametric versions in [52, 62]. A l.s.c. function $f : X \to \overline{I\!R}$ is *prox-regular* at $\bar{x} \in \operatorname{dom} f$ for $\bar{x}^* \in \partial f(\bar{x})$ if there are constants $r > 0$ and $\varepsilon > 0$ such that for all $x, u \in I\!B_\varepsilon(\bar{x})$ with $|f(u) - f(\bar{x})| \leq \varepsilon$ we have

$$f(x) \geq f(u) + \langle u^*, x-u\rangle - \frac{r}{2}\|x-u\|^2 \text{ whenever } u^* \in \partial f(u) \cap I\!B_\varepsilon(\bar{x}^*). \tag{2.26}$$

Further, f is *subdifferentially continuous* at $\bar{x} \in \operatorname{dom} f$ for $\bar{x}^* \in \partial f(\bar{x})$ if the function $(x, x^*) \mapsto f(x)$ is continuous relative to the subdifferential graph $\operatorname{gph} \partial f$ at $(\bar{x}, \bar{x}^*)$.

These notions have been also studied in the frameworks of Hilbert and more general Banach spaces; see, e.g., [6, 7]. When f is both prox-regular and subdifferentially continuous at $\bar{x}$ for $\bar{x}^* \in \partial f(\bar{x})$, it is easy to observe that the condition "$|f(u) - f(\bar{x})| \le \varepsilon$" can be omitted in the definition of prox-regularity. The class of prox-regular and subdifferentially continuous functions is rather broad including, in particular, *strongly amenable* functions in finite dimensions, l.s.c. convex functions in Banach spaces, etc.; see [7, 52, 62] for further details.

Moreover, in the general Banach space X it is easy to check that the *graph* of ∂f is *closed* near $(\bar{x}, \bar{x}^*)$ in the norm$\times$norm topology of $X \times X^*$ when f is prox-regular and subdifferentially continuous at $\bar{x}$ for $\bar{x}^*$.

Next we formulate a parametric version of prox-regularity introduced by Levy, Poliquin, and Rockafellar [31]. Given $f : X \times P \to \overline{I\!R}$ finite at $(\bar{x}, \bar{p})$ (P is a metric space) and given a partial limiting subgradient $\bar{x}^* \in \partial_x f(\bar{x}, \bar{p})$ of $f(\cdot, \bar{p})$ at $\bar{x}$, we say that f is *prox-regular* in x at $\bar{x}$ for $\bar{x}^*$ with *compatible parameterization* by p at $\bar{p}$ if there are neighborhoods U of $\bar{x}$, U^* of $\bar{x}^*$, and V of $\bar{p}$ along with numbers $\varepsilon > 0$ and $r > 0$ such that

$$\begin{gathered} f(x,p) \ge f(u,p) + \langle u^*, x - u \rangle - \tfrac{r}{2}\|x - u\|^2 \text{ for all } x \in U, \\ \text{when } u^* \in \partial_x f(u,p) \cap U^*,\ u \in U, \text{ and } f(u,p) \le f(\bar{x},\bar{p}) + \varepsilon. \end{gathered} \tag{2.27}$$

Further, f is *subdifferentially continuous* in x at $\bar{x}$ for $\bar{x}^*$ with compatible parameterization by p at $\bar{p}$ if the function $(x, p, x^*) \mapsto f(x, p)$ is continuous relative to $\operatorname{gph} \partial_x f$ at $(\bar{x}, \bar{p}, \bar{x}^*)$. In this case the constraint $f(u, p) \le f(\bar{x}, \bar{p}) + \varepsilon$ in (2.27) can be ignored. If the function f is both prox-regular and subdifferentially continuous in x at $\bar{x}$ for $\bar{x}^*$ with compatible parameterization by p at $\bar{p}$, we say for brevity that it is *parametrically continuously prox-regular* at $(\bar{x},\bar{p})$ for $\bar{x}^*$. In this case the graph of $\partial_x f$ is not automatically closed near $(\bar{x},\bar{p},\bar{x}^*)$ in the norm topology of $X \times P \times X^*$ anymore. However, it is closed under an additional condition; see our Section 3.2 below.

The following result established by Levy, Poliquin, and Rockafellar [31] is the key tool in employing prox-regularity to constrained optimization.

Proposition 2.8 (prox-regularity from amenability, [31, Proposition 2.2]). *Suppose that* $\dim X, \dim P < \infty$ *and that* $f : X \times P \to \overline{I\!R}$ *is* STRONGLY AMENABLE *in x at $\bar{x}$ with compatible parameterization by p at $\bar{p}$, in the sense that on some neighborhood of $(\bar{x},\bar{p})$ there is a composite representation $f(x,p) = g(F(x,p))$ in which $F : X \times P \to Y$ is a C^2 mapping to a finite-dimensional space Y and $g : Y \to \overline{I\!R}$ is a convex, proper, l.s.c. function for which* $F(\bar{x},\bar{p}) \in D := \operatorname{dom} g$ *and*

$$y^* \in N(F(\bar{x},\bar{p});D), \quad \nabla_x F(\bar{x},\bar{p})^* y^* = 0 \quad \Longrightarrow \quad y^* = 0.$$

Then for any $\bar{x}^* \in \partial_x f(\bar{x},\bar{p})$, we have f is parametrically continuously prox-regular at $(\bar{x},\bar{p})$ for $\bar{x}^*$. Furthermore, we also have

$$(0,p^*) \in \partial^\infty f(\bar{x},\bar{p}) \quad \Longrightarrow \quad p^* = 0.$$

In applications we usually use $g = \delta_D$ as the indicator function to a convex set D and thus f is the indicator function to $\Omega := \big\{(x,p) \in X \times P \,|\; F(x,p) \in D\big\}$; see Part B of the dissertation.

As demonstrated in [7, 31, 52, 62], the limiting subdifferential of prox-regular functions is strongly connected to *monotonicity*. Recall that a set-valued mapping $T : X \rightrightarrows X^*$ (or sometimes $T : X^* \rightrightarrows X$) is *monotone* if it satisfies the relationship

$$\langle x^* - u^*, x - u\rangle \geq 0 \text{ for all } (x, x^*), \ (u, u^*) \in \operatorname{gph} T.$$

In addition the mapping $T\colon X \rightrightarrows X^*$ is said to be *maximal monotone* if $T = Q$ for any monotone mapping $Q : X \rightrightarrows X^*$ with $\operatorname{gph} T \subset \operatorname{gph} Q$. It is well-known in convex analysis that the convex subdifferential of a convex function is a maximal monotone.

The mapping T is *locally monotone* around $(\bar{x}, \bar{x}^*) \in \operatorname{gph} T$ if it admits a monotone localization around this point. Moreover, T is *locally maximal monotone* around $(\bar{x}, \bar{x}^*) \in \operatorname{gph} T$ if there are neighborhoods U of $\bar{x}$ and U^* of $\bar{x}^*$ such that for any monotone mapping $S : X \rightrightarrows X^*$ with $\operatorname{gph} T \cap (U \times U^*) \subset \operatorname{gph} S$ we have the equality $\operatorname{gph} T \cap (U \times U^*) = \operatorname{gph} S \cap (U \times U^*)$.

The next result ensures the "positive-semidefinite" property of coderivatives of maximal monotone operators. It extends that of [53, Theorem 2.1] to the Hilbert space setting. In fact, the proof of [53, Theorem 2.1] can be easily modified for this case; see, e.g., [10, Lemma 5.2]. Here we present a new and simple proof in Hilbert space.

Lemma 2.9 (coderivatives of maximal monotone operators). *Let X be a Hilbert space, and $T : X \rightrightarrows X$ be a maximal monotone operator. Then for any pair $(\bar{x}, \bar{x}^*) \in \operatorname{gph} T$ we have that*

$$\langle u^*, u\rangle \geq 0 \ \textit{ whenever } \ u^* \in \widehat{D}^* T(\bar{x}, \bar{x}^*)(u). \tag{2.28}$$

Consequently, $\langle u^*, u\rangle \geq 0$ whenever $u^* \in D^*_M T(\bar{x}, \bar{x}^*)(u)$.

Proof. It is well known that for any $\lambda > 0$ the resolvent $R_\lambda = (I + \lambda T)^{-1}$ is nonexpansive with $\operatorname{dom} R_\lambda = H$ by the classical Minty theorem. Pick an arbitrary pair $(u, u^*) \in$

gph $\widehat{D}^*T(\bar{x},\bar{x}^*)$ and deduce from Lemma 2.5 that

$$-\lambda^{-1}u \in \widehat{D}^* R_\lambda(\bar{x}+\lambda\bar{x}^*,\bar{x})(-u^*-\lambda^{-1}u).$$

Since R_λ is nonexpansive, it follows from [38, Theorem 1.43] that $\|-\lambda^{-1}u\| \le \|-u^*-\lambda^{-1}u\|$, which clearly implies that

$$\lambda^{-2}\|u\|^2 \le \|-u^*-\lambda^{-1}u\|^2 = \|u^*\|^2 + 2\lambda^{-1}\langle u^*,u\rangle + \lambda^{-2}\|u\|^2$$

and yields in turn that $0 \le \lambda\|u^*\|^2 + 2\langle u^*,u\rangle$ for all $\lambda > 0$. Letting $\lambda \downarrow 0$ gives us that $\langle u^*,u\rangle \ge 0$, which is the claimed relationship (2.28). Similarly, by replacing $(\bar{x},\bar{x}^*)$ with any point $(x,x^*) \in \operatorname{gph} T$, we also have

$$\langle u^*,u\rangle \ge 0 \quad \text{whenever} \quad u^* \in \widehat{D}^*T(x,x^*)(u).$$

This fact easily implies the second conclusion of the lemma by passing to the limit as $(x,x^*) \to (\bar{x},\bar{x}^*)$ and using definition (2.13) of the mixed coderivative. $\square$

Part A: Theory

Chapter 3

Full Stability in Unconstrained Optimization

3.1 Overview

This chapter is devoted to studying the notion of *full Lipschitzian stability* introduced by Levy, Poliquin and Rockafellar [31] in the general extended-real-valued framework of parametric optimization and its new *Hölderian* counterpart. Differently from the finite-dimensional setting of [31], we consider here full stability in infinite-dimensional optimization, which allows us to cover, in particular, problems of optimal control in Chapter 6. On the other hand, most of the results obtained below are new even for Lipschitzian full stability in finite-dimensions.

Let us introduce some notions used broadly in the chapter. Given an extended-real-valued function $f\colon X \times P \to \overline{I\!R} := I\!R \cup \{\infty\}$ between an Asplund decision space X and a metric parameter space (P, d) with the nominal parameter value $\bar{p} \in P$, consider the optimization problem

$$\mathcal{P} \quad \text{minimize} \;\; f(x, \bar{p}) \;\; \text{over} \;\; x \in X \tag{3.1}$$

and its two-parameter perturbations constructed as

$$\mathcal{P}(x^*,p) \quad \text{minimize} \;\; f(x,p)-\langle x^*,x\rangle \;\; \text{over} \;\; x\in X \tag{3.2}$$

with the *basic* parameter perturbations $p\in P$ and the *tilt* ones $x^*\in X^*$. For $(\bar{x},\bar{p})\in \operatorname{dom} f$ and $\gamma>0$, associate with these data the following objects:

$$\begin{aligned} &m_\gamma(x^*,p):=\inf\big\{f(x,p)-\langle x^*,x\rangle\big|\ \|x-\bar{x}\|\le\gamma\big\},\\ &M_\gamma(x^*,p):=\operatorname{argmin}\big\{f(x,p)-\langle x^*,x\rangle\big|\ \|x-\bar{x}\|\le\gamma\big\},\\ &S(x^*,p):=\big\{x\in X\big|\ x^*\in\partial_x f(x,p)\big\}, \end{aligned} \tag{3.3}$$

where $\partial_x f$ stands for the partial limiting subdifferential (2.10) of f with respect to x.

Now we formulate the two main stability properties discussed above. The first (Lipschitzian) was introduced in [31] in finite-dimensional spaces with the modulus modification given in [42] while its Hölderian counterpart has been recently introduced in [40, 44].

Definition 3.1 (Lipschitzian and Hölderian full stability). *Given* $f\colon X\times P\to\overline{I\!R}$ *and a point* $(\bar{x},\bar{p})\in\operatorname{dom} f$ *in* (3.1) *with some nominal basic parameter* $\bar{p}\in P$*, we say that:*

(i) *The point* $\bar{x}$ *is a* LIPSCHITZIAN FULLY STABLE LOCAL MINIMIZER *of* $\mathcal{P}(\bar{x}^*,\bar{p})$ *in* (3.2) *corresponding to* $\bar{p}$ *and some tilt parameter* $\bar{x}^*\in X^*$ *with a modulus pair* $(\kappa,\ell)\in I\!R^2_>:=\{(a,b)\in I\!R^2|\ a>0,\ b>0\}$ *if there are a number* $\gamma>0$ *and a neighborhood* $U^*\times V$ *of* $(\bar{x}^*,\bar{p})$ *such that the mapping* $(x^*,p)\mapsto M_\gamma(x^*,p)$ *is single-valued on* $U^*\times V$ *with* $M_\gamma(\bar{x}^*,\bar{p})=\bar{x}$ *satisfying the Lipschitz condition*

$$\|M_\gamma(x_1^*,p_1)-M_\gamma(x_2^*,p_1)\|\le\kappa\|x_1^*-x_2^*\|+\ell d(p_1,p_2) \;\; \textit{for all} \;\; x_1^*,x_2^*\in U^*,\ p_1,p_2\in V \tag{3.4}$$

and that the function $(x^*,p)\mapsto m_\gamma(x^*,p)$ *is also Lipschitz continuous around* $(\bar v,\bar p)$.

(ii) *The point* $\bar x$ *is a* HÖLDERIAN FULLY STABLE LOCAL MINIMIZER *of problem* $\mathcal{P}(\bar x^*,\bar p)$ *with a modulus pair* $(\kappa,\ell)\in I\!R^2_>$ *if there is a number* $\gamma>0$ *such that the mapping* M_γ *is single-valued on some neighborhood* $U^*\times V$ *of* $(\bar x^*,\bar p)$ *with* $M_\gamma(\bar x^*,\bar p)=\bar x$ *and*

$$\|M_\gamma(x_1^*,p_1)-M_\gamma(x_2^*,p_2)\|\le\kappa\|x_1^*-x_2^*\|+\ell d(p_1,p_2)^{\frac{1}{2}}\quad\textit{for all}\quad x_1^*,x_2^*\in U^*,\ p_1,p_2\in V.\tag{3.5}$$

If the parameter p is ignored, both properties in Definitions 3.1 reduce to *tilt stability* introduced by Poliquin and Rockafellar in [53]. However, in the parameter-dependent case for f full Hölderian stability is *strictly weaker* than its Lipschitzian counterpart; moreover, the exponent $r=\frac{1}{2}$ in (3.1) is the *largest possible* exponent of Hölder continuity for M_γ. To demonstrate it, we borrow the following example by Robinson [57] designed for a different purpose.

Example 3.2 (Hölderian full stability is strictly weaker than Lipschitzian one). Consider the following parametric nonlinear program in $I\!R^2$:

$$\mathcal{P}(p)\qquad\text{minimize}\quad\left\{\begin{array}{l}f_0(x,p):=\dfrac{1}{2}\|x\|^2,\quad x\in I\!R^2,\ \text{ subject to}\\ g(x,p):=-(A+p_1I)x+\dfrac{1}{2}(2+p_1)a+p_2b\in I\!R^2_-\end{array}\right.\tag{3.6}$$

with the parameter $p=(p_1,p_2)\in I\!R^2$ and the data A, I, a, and b defined by

$$A:=\begin{bmatrix}1&1\\1&1\end{bmatrix},\quad I:=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad a:=\begin{bmatrix}1\\1\end{bmatrix},\quad b:=\begin{bmatrix}1\\-1\end{bmatrix}.$$

It is shown in [57] that for any p around $\bar p=(0,0)$ the unique local minimizer $x(p)$ of problem

$\mathcal{P}(p)$ is continuous but not Lipschitz continuous around $\bar{x} = (\frac{1}{2}, \frac{1}{2})$. This implies that Kojima's *strong stability* [23] cannot be strengthened to Robinson's *strong regularity* [58]. Note that Kojima's strong stability is equivalent to the uniform second-order growth condition in the sense of [5, Definition 5.16]; see [5, Proposition 5.37]. The latter verifies our USOGC in Definition 3.4 at $(\bar{x}, \bar{p}, \bar{x}^*)$ with $\bar{x}^* = (0,0)$ for the function $f(x,p) := f_0(x,p) + \delta_\Omega(x,p)$, where Ω denotes the feasible solution set in (3.6). It follows from Proposition 2.8 that BCQ (3.7) (or (3.8)) holds at $(\bar{x}, \bar{p})$. We get from Theorem 3.5 and Theorem 3.6 below that Hölderian full stability is valid for function f. Furthermore, it can be observed from the calculations in [57] that $M_\gamma(\bar{x}^*, \bar{p}) = \bar{x}$ and $M_\gamma(\bar{x}^*, p) = \frac{1}{2}a + \frac{p_1}{9}b$ with $p = \Big(p_1, \frac{p_1^2}{9}\Big)$ when $p_1 > 0$ is sufficiently small, where M_γ is taken from (3.3) with some $\gamma > 0$. This demonstrates the *failure of Lipschitzian full stability* and also shows that the Hölderian exponent $\frac{1}{2}$ in (5.10) *cannot be improved.*

To this end we mention the beautiful result by Gfrerer [19] showing that strong stability can be strengthened to Hölder continuity of local minimizers with the best possible exponent $r = \frac{1}{2}$ for a general class of parametric nonlinear programs with smooth data.

3.2 Second-order Characterizations of Hölderian Full Stability

We say that the *basic constraint qualification* (BCQ) holds at $(\bar{x}, \bar{p})$ if the epigraphical mapping

$$F\colon p \mapsto \operatorname{epi} f(\cdot, p) \text{ is Lipschitz-like around } \big(\bar{p}, \bar{x}, f(\bar{x}, \bar{p})\big) \tag{3.7}$$

in the sense of (2.17). As discussed in Section (2.5), in the case of Asplund parameter spaces P the introduced BCQ (3.7) can be characterized via the mixed coderivative (2.13) of F at the reference point $(\bar{p}, \bar{x}, f(\bar{x}, \bar{p}))$. If both X and P are finite-dimensional, this gives from the Mordukhovich's criterion (2.6) us the equivalent form of the basic constraint qualification (3.7) formulated in [31] as

$$\big[(0,p^*)\in\partial^\infty f(\bar{x},\bar{p})\big]\Longrightarrow p^*=0. \tag{3.8}$$

The following result provides a necessary condition for the basic constraint qualification (3.7).

Proposition 3.3 (consequence of BCQ). *The validity of BCQ* (3.7) *ensures the existence of neighborhoods U of $\bar{x}$ and V of $\bar{p}$ along with a number $\varepsilon>0$ such that*

$$\left.\begin{array}{l} x_1\in U, p_1,p_2\in V\\[2ex] f(x_1,p_1)\le f(\bar{x},\bar{p})+\varepsilon\end{array}\right\}\Longrightarrow \exists x_2\ \ \textit{with}\ \ \left\{\begin{array}{l}\|x_1-x_2\|\le cd(p_1,p_2),\\[2ex] f(x_2,p_2)\le f(x_1,p_1)+cd(p_1,p_2),\end{array}\right. \tag{3.9}$$

where $c>0$ is a modulus of the Lipschitz-like property in (3.7).

Furthermore, if f is parametrically subdifferentially continuous at $(\bar{x},\bar{p})$ for some $\bar{x}^*\in\partial_x f(\bar{x},\bar{p})$, then the graph $\operatorname{gph}\partial f$ *is closed around $(\bar{x},\bar{p},\bar{x}^*)$.*

Proof. Can be distilled from [31, Proposition 3.1 and 3.2] given in finite dimensions under (3.8). □

Now we define our basic *uniform second-order growth condition* for f in (3.1), which is a general version of that for $\mathcal{C}^2$ conic programs with respect to the $\mathcal{C}^2$-smooth parametrization introduced in [5, Definition 5.16] and reduces to [48, Definition 3.6] in finite dimensions.

Definition 3.4 (uniform second-order growth condition). *Taking $\bar{x}^*\in\partial_x f(\bar{x},\bar{p})$, we say the* UNIFORM SECOND-ORDER GROWTH CONDITION (USOGC) *holds at $(\bar{x},\bar{p},\bar{x}^*)$ with modulus $\kappa>0$ there are neighborhoods U of $\bar{x}$, U^* of $\bar{x}^*$, and V of $\bar{p}$ such that*

$$f(x,p)\ge f(u,p)+\langle x^*,x-u\rangle+\frac{1}{2\kappa}\|x-u\|^2\ \ \textit{for all}\ \ x\in U \tag{3.10}$$

whenever $(x^*, p, u) \in \operatorname{gph} S \cap (U^* \times V \times U)$ *with the mapping* S *defined in* (3.3).

The next theorem shows that USOGC (3.10) characterizes the *Hölder continuity* of the mapping $S(x^*, p)$ with respect to p and its *Lipschitzian continuity* with respect to x^*, proving also a *precise relationship* between the corresponding constants crucial for characterizing *full stability* in Section 4. When ignoring the parameter p this results reduces to [42, Theorem 3.2] and [13, Theorem 3.3] in finite dimensions. Note that a version of implication (iii)$\Longrightarrow$(i) below ensuring the Hölder continuity with respect to *both* parameters (x^*, p) follows from [5, Theorem 5.17] for $\mathcal{C}^2$ conic programs in Banach spaces.

Theorem 3.5 (Hölder continuity of the inverse subgradient mapping via USOGC). *Let* X *be an Asplund space and let* $\bar{x}^* \in \partial_x f(\bar{x}, \bar{p})$*. Assume that BCQ* (3.7) *holds at* $(\bar{x}, \bar{p}) \in \operatorname{dom} f$*. Then the following assertions are equivalent:*

(i) *We have* $\bar{x} \in M_\gamma(\bar{x}^*, \bar{p})$ *for some* $\gamma > 0$*, and there is a neighborhood* $U^* \times V \times U$ *of* $(\bar{x}^*, \bar{p}, \bar{x})$ *such that the mapping* S *from* (3.3) *admits a single-valued localization* ϑ *with respect to* $U^* \times V \times U$ *satisfying the Hölder continuity condition*

$$\|\vartheta(x_1^*, p_1) - \vartheta(x_2^*, p_2)\| \le \kappa \|x_1^* - x_2^*\| + \ell d(p_1, p_2)^{\frac{1}{2}} \tag{3.11}$$

for all $x_1^*, x_2^* \in U^*$ *and* $p_1, p_2 \in V$*, where* κ *and* ℓ *are positive constants.*

(ii) *We have* $\bar{x} \in M_\gamma(\bar{x}^*, \bar{p})$ *for some* $\gamma > 0$*, and there exist a neighborhood* $U^* \times V \times U$ *of* $(\bar{x}^*, \bar{p}, \bar{x})$ *and a constant* $\kappa > 0$ *the same as in* (3.11) *such that the mapping* S *admits a single-valued localization* ϑ *with respect to* $U^* \times V \times U$*, which is Lipschitz continuous in* x^* *uniformly in* p*, i.e.,*

$$\|\vartheta(x_1^*, p) - \vartheta(x_2^*, p)\| \le \kappa \|x_1^* - x_2^*\| \quad \text{for all} \;\; x_1^*, x_2^* \in U^* \;\; \text{and} \;\; p \in V. \tag{3.12}$$

(iii) *USOGC* (3.10) *holds at* $(\bar{x},\bar{p},\bar{x}^*)$ *with modulus* κ *taken from* (3.11) *and* (3.12).

Proof. Implication (i)$\Longrightarrow$(ii) is trivial. To verify (ii)$\Longrightarrow$(iii), find by (ii) a number $\kappa>0$ and a localization ϑ of S with respect to $U^*\times V\times U$ and then split the proof into the following claims.

Claim 1: *There exist numbers* $\alpha,\nu>0$ *satisfying*

$$f(x,\bar{p})\ge f(\bar{x},\bar{p})+\langle\bar{x}^*,x-\bar{x}\rangle+\alpha\|x-\bar{x}\|^2\ \ \textit{for all}\ \ x\in I\!B_{2\nu}(\bar{x}).\tag{3.13}$$

Arguing by contradiction, suppose that such α,ν do not exist and find $x_k\to\bar{x}$ and $\alpha_k\downarrow 0$ with

$$f(x_k,\bar{p})<f(\bar{x},\bar{p})+\langle\bar{x}^*,x_k-\bar{x}\rangle+\alpha_k\|x_k-\bar{x}\|^2,\quad k\in I\!N.$$

By $\bar{x}\in M_\gamma(\bar{x}^*,\bar{p})$ we get from here with $\varepsilon_k:=\alpha_k\|x_k-\bar{x}\|^2\downarrow 0$ as $k\to\infty$ that

$$\inf_{x\in I\!B_\gamma(\bar{x})}\Big\{f(x,\bar{p})-\langle\bar{x}^*,x-\bar{x}\rangle\Big\}=f(\bar{x},\bar{p})>f(x_k,\bar{p})-\langle\bar{x}^*,x_k-\bar{x}\rangle-\varepsilon_k,$$

Then Ekeland's variational principle gives us a sequence $\{u_k\}$ such that $\|u_k-x_k\|\le\sqrt{\varepsilon_k}$ and

$$\inf_{x\in I\!B_\gamma(\bar{x})}\Big\{f(x,\bar{p})-\langle\bar{x}^*,x-\bar{x}\rangle+\sqrt{\varepsilon_k}\|x-u_k\|\Big\}=f(u_k)-\langle\bar{x}^*,u_k-\bar{x}\rangle,$$

where $u_k\to\bar{x}$ as $k\to\infty$. Applying the generalized Fermat rule to the local minimizer u_k of the above optimization problem and using the sum rule in Lemma 2.4 for limiting subdifferential provide the inclusion

$$0\in\partial_x f(u_k,\bar{p})-\bar{x}^*+\sqrt{\varepsilon_k}I\!B_{X^*}.$$

Thus there exists $x_k^* \in \partial_x f(u_k,\bar p)$ with $\|x_k^* - \bar x^*\| \le \sqrt{\varepsilon_k}$ implying that $u_k = \vartheta(x_k^*,\bar p)$ for sufficiently large $k \in I\!N$. It follows from (3.12) that

$$\kappa\sqrt{\varepsilon_k} \ge \kappa\|x_k^* - \bar x^*\| \ge \|u_k - \bar x\| \ge \|x_k - \bar x\| - \|u_k - \bar x\| \ge \sqrt{\frac{\varepsilon_k}{\alpha_k}} - \sqrt{\varepsilon_k},$$

and so $\sqrt{\alpha_k} \ge (\kappa+1)^{-1}$, which contradicts the assumption on $\alpha_k \downarrow 0$ and thus justifies (3.13).

Claim 2: *With ν from* (3.13) *and $\delta > 0$ sufficiently small, for any $(x^*,p) \in I\!B_\delta(\bar x^*) \times I\!B_\delta(\bar p) \subset U^* \times V$ the element $u := S(x^*,p)$ belongs to $I\!B_\nu(\bar x)$ and is a unique minimizer of the problem:*

$$\text{minimize} \quad f(x,p) - \langle x^*, x\rangle \quad \text{subject to} \quad x \in I\!B_\nu(\bar x). \tag{3.14}$$

By (3.9) we find constants $c,\varepsilon > 0$ such that

$$\left.\begin{array}{l} x_1 \in I\!B_{2\nu}(\bar x), p_1, p_2 \in I\!B_\nu(\bar p) \\[2ex] f(x_1,p_1) \le f(\bar x,\bar p) + \varepsilon \end{array}\right\} \Longrightarrow \exists x_2 \text{ with } \left\{\begin{array}{l} \|x_1 - x_2\| \le cd(p_1,p_2), \\[2ex] f(x_2,p_2) \le f(x_1,p_1) + cd(p_1,p_2). \end{array}\right. \tag{3.15}$$

Suppose further that $0 < \delta < (3c)^{-1}\nu$. Then fix $p \in I\!B_\delta(\bar p)$ and show that $f(\cdot,p)$ is bounded from below on $I\!B_\nu(\bar x)$. Observe first that this assertion holds if $f(x,p)$ is uniformly bounded from below for any $x \in I\!B_\nu(\bar x)$ satisfying $f(x,p) \le f(\bar x,\bar p) + \varepsilon$. Indeed, for such x we get from (3.15) that there is some $v \in X$ such that $\|x - v\| \le cd(p,\bar p)$ and $f(x,p) \ge f(v,\bar p) - cd(p,\bar p)$, which implies that $\|v - \bar x\| \le \|x - v\| + \|x - \bar x\| \le c\delta + \nu < 2\nu$ and verifies by (3.13) the boundedness from below:

$$f(x,p) \ge f(v,\bar p) - cd(p,\bar p) \ge f(\bar x,\bar p) - \langle \bar x^*, v - \bar x\rangle - c\delta \ge f(\bar x,\bar p) - \|\bar x^*\|2\nu - c\delta.$$

Hence there exists a sequence $\{v_k\} \subset I\!\!B_\nu(\bar{x})$ with

$$\inf_{x \in I\!\!B_\nu(\bar{x})} \Big\{ f(x,p) - \langle x^*, x \rangle \Big\} + k^{-2} \geq f(v_k, p) - \langle x^*, v_k \rangle.$$

By Ekeland's variational principle, for $k \in I\!\!N$ find $w_k \in I\!\!B_\nu(\bar{x})$ such that $\|w_k - v_k\| \leq k^{-1}$ and

$$\inf_{x \in I\!\!B_\nu(\bar{x})} \Big\{ f(x,p) - \langle x^*, x \rangle + k^{-1}\|x - w_k\| \Big\} \geq f(w_k, p) - \langle x^*, w_k \rangle. \tag{3.16}$$

By (3.15) there is some $w \in X$ with $\|w - \bar{x}\| \leq cd(p, \bar{p}) \leq c\delta < \nu$ and $f(\bar{x}, \bar{p}) \geq f(w, p) - cd(p, \bar{p}) \geq f(w, p) - c\delta$. It follows from (3.16) that

$$\begin{aligned} f(\bar{x}, \bar{p}) + c\delta \quad & \geq f(w, p) \geq f(w_k, p) + \langle x^*, w - w_k \rangle - k^{-1}\|w - w_k\| \\ & \geq f(w_k, p) - (\|\bar{x}^*\| + \delta)2\nu - k^{-1}2\nu, \end{aligned} \tag{3.17}$$

which allows us to have $f(w_k, p) \leq f(\bar{x}, \bar{p}) + \varepsilon$ when ν is small. Then by (3.15) there is z_k such that $\|z_k - w_k\| \leq cd(p, \bar{p}) \leq c\delta$ and $f(w_k, p) \geq f(z_k, \bar{p}) - cd(p, \bar{p}) \geq f(z_k, \bar{p}) - c\delta$. This gives us $\|z_k - \bar{x}\| \leq c\delta + \|w_k - \bar{x}\| < 2\nu$, which being combined with (3.13) and (3.17) implies that

$$\begin{aligned} f(\bar{x}, \bar{p}) + c\delta \quad & \geq f(z_k, \bar{p}) - c\delta + \langle x^*, w - w_k \rangle - k^{-1}\|w - w_k\| \geq f(\bar{x}, \bar{p}) \\ & + \langle \bar{x}^*, z_k - \bar{x} \rangle + \alpha\|z_k - \bar{x}\|^2 - c\delta + \langle x^*, w - w_k \rangle - k^{-1}2\nu \quad \text{ and} \end{aligned}$$

$$\begin{aligned} 2c\delta + k^{-1}2\nu \quad & \geq \alpha\|z_k - \bar{x}\|^2 + \langle \bar{x}^* - x^*, z_k - \bar{x} \rangle + \langle x^*, z_k - \bar{x} + w - w_k \rangle \\ & \geq \alpha\|z_k - \bar{x}\|^2 - \|\bar{x}^* - x^*\| \cdot \|z_k - \bar{x}\| - \|x^*\|(\|z_k - w_k\| + \|w - \bar{x}\|) \\ & \geq \alpha\|z_k - \bar{x}\|^2 - \delta\|z_k - \bar{x}\| - (\|\bar{x}^*\| + \delta)(cd(p, \bar{p}) + cd(p, \bar{p})) \\ & \geq \alpha\|z_k - \bar{x}\|^2 - \delta\|z_k - \bar{x}\| - (\|\bar{x}^*\| + \delta)2c\delta, \end{aligned}$$

where α is taken from (3.13). For small δ the above inequalities yield $\|z_k - \bar{x}\| \le \frac{2\nu}{3}$ and so

$$\|w_k - \bar{x}\| \le \|z_k - \bar{x}\| + \|w_k - z_k\| \le \frac{2\nu}{3} + c\delta < \nu \ \text{ for large } \ k.$$

Applying the generalized Fermat rule to problem (3.16) at $w_k \in \operatorname{int} I\!B_\nu(\bar{x})$ and then using the sum rule for limiting subdifferential give us that

$$0 \in \partial_x f(w_k, p) - x^* + k^{-1} I\!B,$$

which allows us to find $w_k^* \in \partial_x f(w_k, p)$ such that $\|w_k^* - x^*\| \le k^{-1}$. Thus we get $\|w_k^* - \bar{x}^*\| \le \|x^* - \bar{x}^*\| + \|w_k^* - x^*\| \le \delta + k^{-1}$ and so $w_k^* \in U^*$ when k is sufficiently large while δ is small. It follows from the assumptions of (ii) that $w_k = \vartheta(w_k^*, p) \to u$. Hence the passage to the limit in (3.16) shows that $u = \vartheta(x^*, p) \in I\!B_\nu(\bar{x})$ is a unique minimizer of (3.14), which verifies the claim.

Claim 3: *USOGC* (3.10) *holds at* $(\bar{x}, \bar{p}, \bar{x}^*)$ *with modulus* κ. To justify it, define

$$g_p(x^*) := (f_p + \delta_{I\!B_\nu(\bar{x})})^*(x^*) = \sup_{x \in I\!B_\nu(\bar{x})} \big\{ \langle x^*, x \rangle - f_p(x) \big\} \ \text{ for } \ x^* \in X^*, \tag{3.18}$$

where δ, ν are taken from Claim 2. It is well known from convex analysis that (3.18) with $p \in I\!B_\delta(\bar{x})$ and $f_p := f(\cdot, p)$ is the (proper and convex) Fenchel conjugate of $f_p + \delta_{I\!B_\nu(\bar{x})}$. Denote $\vartheta_p := \vartheta(\cdot, p)$ and get from (3.14) that $g_p(x^*) = \langle x^*, \vartheta_p(x^*) \rangle - f_p(\vartheta_p(x^*))$ if $x^* \in I\!B_\delta(\bar{x}^*)$ and $p \in I\!B_\delta(\bar{x})$. Then

$$g_p(v^*) - g_p(x^*) \ge \big[\langle v^*, \vartheta_p(x^*) \rangle - f_p(\vartheta_p(x^*)) \big] - \big[\langle x^*, \vartheta_p(x^*) \rangle - f_p(\vartheta_p(x^*)) \big] = \langle v^* - x^*, \vartheta_p(x^*) \rangle$$

whenever $v^* \in X^*$, which implies that $\vartheta_p(x^*) \in \partial g_p(x^*)$. Moreover, it is easy to check from (3.14) that ϑ_p is monotone on $I\!B_\delta(\bar{x}^*) \times I\!B_\nu(\bar{x})$. The Lipschitz continuity of ϑ_p ensures its maximal monotone on this set and, by the monotonicity of the convex subdifferential, implies that

$$\operatorname{gph} \vartheta_p \cap \big(I\!B_\delta(\bar{x}^*) \times I\!B_\nu(\bar{x})\big) = \operatorname{gph} \partial g_p \cap \big(I\!B_\delta(\bar{x}^*) \times I\!B_\nu(\bar{x})\big).$$

Thus the subgradient mapping ∂g_p is single-valued and Lipschitz continuous on $I\!B_\delta(\bar{x}^*)$, which can be true only when g_p is Fréchet differentiable on $\operatorname{int} I\!B_\delta(\bar{x}^*)$ with $\partial g_p(\cdot) = \{\nabla g_p(\cdot)\}$ on this set.

Choose $\beta > 0$ with $\kappa\beta < \nu$ and $3\beta < \delta$. Define $\widetilde{U} := I\!B_{\kappa\beta}(\bar{x}) \subset I\!B_\nu(\bar{x})$, $U^* := I\!B_\beta(\bar{x}^*)$ and observe that $\nabla g_p(\widetilde{U}^*) \subset \widetilde{U}$. Picking $(u^*, u) \in \operatorname{gph} \vartheta_p \cap \big(\widetilde{U}^* \times \widetilde{U}\big) = \operatorname{gph} \nabla g_p \cap \big(\widetilde{U}^* \times \widetilde{U}\big)$ gives us

$$\begin{aligned} g_p(v^*) - g_p(u^*) - \langle u, v^* - u^* \rangle &= \int_0^1 \langle \nabla g_p(u^* + t(v^* - u^*)) - \nabla g_p(u^*), v^* - u^* \rangle dt \\ &= \int_0^1 t\kappa \|v^* - u^*\| \cdot \|v^* - u^*\| dt = \frac{\kappa}{2}\|v^* - u^*\|^2, \quad v^* \in I\!B_\delta(\bar{x}^*). \end{aligned}$$

Since $g_p(u^*) = \langle u^*, u \rangle - f_p(u)$ by (3.14), the above inequality implies that

$$g_p(v^*) \le -f_p(u) + \frac{\kappa}{2}\|v^* - u^*\| + \langle v^*, u \rangle + \delta_{I\!B_\delta(\bar{x}^*)}(v^*) \ \text{ for all } \ v^* \in X^*.$$

This gives us by the biconjugate inequality from [68, Theorem 2.3.1] that

$$\begin{aligned} f_p(x) &= f_p(x) + \delta_{I\!B_\nu(\bar{x})}(x) \ge (f_p + \delta_{I\!B_\nu(\bar{x})})^{**}(x) = g_p^*(x) \\ &\ge \sup_{v^* \in I\!B_\delta(\bar{x}^*)} \Big\{ \langle v^*, x \rangle - \frac{\kappa}{2}\|v^* - u^*\| - \langle v^*, u \rangle \Big\} + f_p(u) \\ &\ge \sup_{v^* \in I\!B_\delta(\bar{x}^*)} \Big\{ \langle v^* - u^*, x - u \rangle - \frac{\kappa}{2}\|v^* - u^*\| \Big\} + \langle u^*, x - u \rangle + f_p(u), \quad x \in \widetilde{U}. \end{aligned} \tag{3.19}$$

Now we consider the duality mapping $J(v) := \frac{1}{2}\partial(\|\cdot\|)^2(v)$ for $v \in X$ and recall that

$$J(v) = \big\{v^* \in X^* \big|\ \langle v^*, v\rangle = \|v\|^2 = \|v^*\|^2\big\} \neq 0 \ \text{ whenever } \ v \in X. \tag{3.20}$$

Select further $w^* \in J\big(\frac{1}{\kappa}(x-u)\big)$ and get from (3.20) that

$$\langle w^*, x-u\rangle - \frac{\kappa}{2}\|w^*\|^2 = \frac{1}{\kappa}\|x-u\|^2 - \frac{1}{2\kappa}\|x-u\|^2 = \frac{1}{2\kappa}\|x-u\|^2. \tag{3.21}$$

Moreover, it follows from (3.21) due to $u^* \in \widetilde{U}^*$ that

$$\|w^* + u^* - \bar{x}^*\| \le \|w^*\| + \|u^* - \bar{x}^*\| \le \frac{1}{\kappa}\|x-u\| + \beta \le \frac{1}{\kappa}2\kappa\beta + \beta = 3\beta$$

and thus $w^* + u^* \in I\!B_\delta(\bar{x}^*)$ by the choice of β. Combining this with (3.21) and (3.19) ensures that

$$f_p(x) \ge f_p(u) + \langle u^*, x-u\rangle + \frac{1}{2\kappa}\|x-u\|^2 \ \text{ for all } \ x \in \widetilde{U}$$

whenever $(u^*, p, u) \in \operatorname{gph}\vartheta \cap \big(\widetilde{U}^* \times I\!B_\delta(\bar{p}) \times \widetilde{U}\big) = \operatorname{gph} S \cap \big(\widetilde{U}^* \times I\!B_\delta(\bar{p}) \times \widetilde{U}\big)$. This verifies Claim 3 and completes the proof of implication (ii)$\Longrightarrow$(iii).

Next we justify the *converse implication* (iii)$\Longrightarrow$(ii). By (iii) find the neighborhood $U \times V \times U^*$ of $(\bar{x}, \bar{p}, \bar{x}^*)$ for which (3.10) holds. It is clear that $\bar{x} \in M_\gamma(\bar{x}^*, \bar{p})$ with any $\gamma > 0$ satisfying $I\!B_\gamma(\bar{x}) \subset U$. Define $\vartheta : U^* \times V \rightrightarrows U$ by $\operatorname{gph}\vartheta := \operatorname{gph} S \cap (U^* \times V \times U)$ and observe from (3.10) that ϑ is a single-valued and that for any $(x_1^*, p), (x_2^*, p) \in \operatorname{dom}\vartheta$ we have

$$\begin{cases} f(u_2, p) \ge f(u_1, p) + \langle x_1^*, u_2 - u_1\rangle + \frac{1}{2\kappa}\|u_2 - u_1\|^2, \\ f(u_1, p) \ge f(u_2, p) + \langle x_2^*, u_1 - u_2\rangle + \frac{1}{2\kappa}\|u_1 - u_2\|^2 \end{cases}$$

with $u_1 := \vartheta(x_1^*, p)$ and $u_2 := \vartheta(x_2^*, p)$. Adding these two inequalities gives us the estimates $\frac{1}{\kappa}\|u_2 - u_1\|^2 \le \langle x_2^* - x_1^*, u_2 - u_1\rangle \le \|x_2^* - x_1^*\| \cdot \|u_2 - u_1\|$, which imply in turn that

$$\|\vartheta(x_1^*, p) - \vartheta(x_2^*, p)\| \le \kappa\|x_1^* - x_2^*\| \ \text{ for all } \ (x_1^*, p), (x_2^*, p) \in \operatorname{dom} S \cap (U^* \times V). \quad (3.22)$$

To verify (ii), it suffices to show the existence of $\delta > 0$ such that

$$I\!B_\delta(\bar{x}^*) \times I\!B_\delta(\bar{p}) \subset \operatorname{dom} \vartheta. \quad (3.23)$$

We proceed similarly to the proof of (ii)$\Longrightarrow$(iii) observing first that the counterpart of Claim 1 is trivial in this case. As for Claim 2, the usage of (3.22) instead of (3.12) allows us to find $\delta, \nu > 0$ such that for any $(x^*, p) \in I\!B_\delta(\bar{x}^*) \times I\!B_\delta(\bar{p}) \subset U^* \times V$ there are sequences $(w_k, w_k^*) \in I\!B_\nu(\bar{x}) \times I\!B_\delta(\bar{x}^*)$ with $w_k^* \in \partial_x f(w_k, p)$, i.e., $w_k = \vartheta(x_k^*, p)$ and $w_k^* \to \bar{x}^*$. It follows from (3.22) that $\{w_k\}$ is a Cauchy sequence and thus converges to some $w \in I\!B_\nu(\bar{x})$. Since $(w_k^*, p, w_k) \in \operatorname{gph} \vartheta$, we get from (3.10) that $f(x, p) \ge f(w_k, p) + \langle w_k^*, x - w_k\rangle + \frac{1}{2\kappa}\|x - w_k\|^2$ for all $x \in U$, which implies by letting $k \to \infty$ that

$$f(x, p) \ge f(w, p) + \langle x^*, x - w\rangle + \frac{1}{2\kappa}\|x - w\|^2 \ \text{ whenever } \ x \in U.$$

It yields $x^* \in \partial_x f(w, p)$ by the Fermat rule and thus justifies (3.23).

To complete the proof of the theorem, it remains to show that (iii)$\Longrightarrow$(i) by continuing the proof (iii)$\Longrightarrow$(ii) above. Pick $(x_i^*, p_i, u_i) \in \operatorname{gph} \vartheta \cap \big(I\!B_\delta(\bar{x}^*) \times I\!B_\delta(\bar{p}) \times I\!B_\nu(\bar{x})\big)$ and deduce from (3.15) that there are x_i such that $\|x_i - \bar{x}\| \le c\delta < \nu$ and $f(\bar{x}, \bar{p}) \ge f(x_i, p_i) - cd(p_i, \bar{p})$, $i = 1, 2$.

This together with (3.10) gives us that

$$\begin{aligned} f(\bar{x},\bar{p})+c\delta &\geq f(x_i,p_i)\geq f(u_i,p_i)+\langle x_i^*,x_i-u_i\rangle+\frac{1}{2\kappa}\|x_i-u_i\|^2\\ &\geq f(u_i,p_i)-\|x_i^*\|\cdot\|x_i-u_i\|\geq f(u_i,p_i)-(\|\bar{x}^*\|+\delta)2\delta. \end{aligned}$$

Thus we get $f(u_i,p_i)\leq f(\bar{x},\bar{p})+\varepsilon$ when $\delta>0$ is small. By (3.15) there are v_1,v_2 such that

$$\begin{cases} \|u_2-v_1\|\leq cd(p_1,p_2), \|u_1-v_2\|\leq cd(p_1,p_2),\\ f(v_1,p_1)\leq f(u_2,p_2)+cd(p_1,p_2), f(v_2,p_2)\leq f(u_1,p_1)+cd(p_1,p_2). \end{cases} \tag{3.24}$$

It follows that $\|v_1-\bar{x}\|\leq\|u_2-v_1\|+\|u_2-\bar{x}\|\leq 2c\delta+\nu<2\nu$, which yields $v_1\in U$ and similarly $v_2\in U$ when ν is sufficiently small. Hence we obtain from (3.10) that

$$\begin{cases} f(v_1,p_1)\geq f(u_1,p_1)+\langle x_1^*,v_1-u_1\rangle+\frac{1}{2\kappa}\|v_1-u_1\|^2,\\ f(v_2,p_2)\geq f(u_2,p_2)+\langle x_2^*,v_2-u_2\rangle+\frac{1}{2\kappa}\|v_2-u_2\|^2. \end{cases}$$

Summing up these two inequalities and combining it with (3.24) give us that

$$\begin{aligned} 2cd(p_1,p_2) &\geq \langle x_1^*,v_1-u_1\rangle+\frac{1}{2\kappa}\|v_1-u_1\|^2+\langle x_2^*,v_2-u_2\rangle+\frac{1}{2\kappa}\|v_2-u_2\|^2\\ &\geq \langle x_1^*-x_2^*,u_2-u_1\rangle+\langle x_1^*,v_1-u_2\rangle+\langle x_2^*,v_2-u_1\rangle+\frac{1}{2\kappa}(\|v_1-u_2\|-\|u_1-u_2\|)^2\\ &\quad+\frac{1}{2\kappa}(\|v_2-u_1\|-\|u_1-u_2\|)^2\\ &\geq -\|x_1^*-x_2^*\|\cdot\|u_1-u_2\|-(\|\bar{x}^*\|+\delta)\|v_1-u_2\|-(\|\bar{x}^*\|+\delta)\|v_2-u_1\|\\ &\quad-\frac{1}{\kappa}(\|v_1-u_2\|+\|v_2-u_1\|)\|u_1-u_2\|+\frac{1}{\kappa}\|u_1-u_2\|^2\\ &\geq -\big(\|x_1^*-x_2^*\|+\frac{2c}{\kappa}d(p_1,p_2)\big)\|u_1-u_2\|-\frac{1}{\kappa}(\|\bar{x}^*\|+\delta)d(p_1,p_2)+\frac{1}{\kappa}\|u_1-u_2\|^2, \end{aligned}$$

which ensures the validity of the estimate

$$\frac{1}{\kappa}\|u_1-u_2\|^2-\big(\|x_1^*-x_2^*\|+\frac{2c}{\kappa}d(p_1,p_2)\big)\|u_1-u_2\|-2c(\|\bar{x}^*\|+\delta+1)d(p_1,p_2)\le 0.$$

Therefore we arrive at the relationships

$$\begin{aligned}\|u_1-u_2\| &\le \frac{\kappa}{2}\Bigg[\|x_1^*-x_2^*\|+\frac{2c}{\kappa}d(p_1,p_2)+\\ &\sqrt{\big(\|x_1^*-x_2^*\|+\frac{2c}{\kappa}d(p_1,p_2)\big)^2+\frac{8c}{\kappa}(\|\bar{x}^*\|+\delta+1)d(p_1,p_2)}\Bigg]\\ &\le \kappa\|x_1^*-x_2^*\|+2cd(p_1,p_2)+\sqrt{2c\kappa(\|\bar{x}^*\|+\delta+1)}d(p_1,p_2)^{\frac{1}{2}}\\ &\le \kappa\|x_1^*-x_2^*\|+2c\sqrt{2\delta}d(p_1,p_2)^{\frac{1}{2}}+\sqrt{2c\kappa(\|\bar{x}^*\|+\delta+1)}d(p_1,p_2)^{\frac{1}{2}}\\ &\le \kappa\|x_1^*-x_2^*\|+\big(2c\sqrt{2\delta}+\sqrt{2c\kappa(\|\bar{x}^*\|+\delta+1)}\big)d(p_1,p_2)^{\frac{1}{2}}.\end{aligned}$$

This together with (3.23) justifies (3.11) and thus completes the proof. □

The next theorem characterizes Hölderian full stability in (3.2) in term of USOGC when X is a Hilbert space.

Theorem 3.6 (characterizing Hölderian full stability via USOGC). *Let X be a Hilbert space. Assume that BCQ* (3.7) *is satisfied at $(\bar{x},\bar{p})\in\operatorname{dom} f$ and that f is parametrically subdifferentially continuous at $(\bar{x},\bar{p})$ for $\bar{x}^*\in\partial_x f(\bar{x},\bar{p})$. The following are equivalent:*

(i) *The point $\bar{x}$ is a Hölderian fully stable local minimizer of problem $\mathcal{P}(\bar{x}^*,\bar{p})$ with a modulus pair $(\kappa,\ell)\in I\!R^2_>$ and the function f is prox-regular in x at $\bar{x}$ for $\bar{x}^*$ with compatible parameterization by p at $\bar{p}$.*

(ii) *USOGC* (3.10) *holds at $(\bar{x},\bar{p},\bar{x}^*)$ with modulus $\kappa>0$.*

Proof. To justify (ii)$\Longrightarrow$(i), take neighborhoods U,U^*,V from Definition 3.4 and suppose without loss of generality that $I\!B_\gamma(\bar{x})=U$. It follows from (3.10) that f is parametrically prox-regular as claimed in (i) and that $M_\gamma(x^*,p) = S(x^*,p)$ for all $x^* \in U^*$ and $p \in V$. Applying now Theorem 3.5 ensures (i).

To prove the converse implication (i)$\Longrightarrow$(ii), we get from (5.10) that

$$\|M_\gamma(x^*,p) - \bar{x}\| = \|M_\gamma(x^*,p) - M_\gamma(\bar{x}^*,\bar{p})\| \le \kappa\|x^* - \bar{x}^*\| + \ell d(p,\bar{p})^{\frac{1}{2}} \text{ if } (x^*,p) \in U^* \times V$$

for the neighborhoods U^* and V from Definition 3.1(ii) and choose a neighborhood U of $\bar{x}$ so that $U \subset \operatorname{int} I\!B_\gamma(\bar{x})$ and $M_\gamma(x^*,p) \subset U$ for all $(x^*,p) \in U^* \times V$. This gives us $M_\gamma(x^*,p) \subset S(x^*,p) \cap U$. Suppose without loss of generality that the neighborhoods U, U^*, V agree with those in (2.27), where the inequality $f(u,p) \le f(\bar{x},\bar{p}) + \varepsilon$ is omitted by the parametric subdifferential continuity of f. Denoting by T be a localization of $\partial_x f$ relative to $U \times V \times U^*$ and then defining $T_p(\cdot) := T(\cdot,p)$ for $p \in V$, we conclude from (2.27) that $T_p + sI$ is strongly monotone in the Hilbert space X with the identity operator I. Thus $(T_p + sI)^{-1}$ is single-valued in its domain. It is easy to observe from (3.3) that $M(\cdot) := M_\gamma(\cdot,p)$ is also monotone for any $p \in V$. Since M is Hölder continuous on U^* and $M(U^*) \subset U$, it is maximal monotone relative to $U^* \times U$. Invoking [3, Theorem 20.21], consider the maximal monotone extension Ξ of M and get that

$$\operatorname{gph} \Xi^{-1} \cap (U \times U^*) = \operatorname{gph} M^{-1} \cap (U \times U^*) \subset \operatorname{gph} T_p \cap (U \times U^*). \tag{3.25}$$

Define further $J : X \times X \to X \times X$ by $J(x,y) := (y + sx, x)$ for $(x,y) \in X \times X$ and then $Z := J(U \times U^*)$. The classical open mapping theorem tells us that Z is a neighborhood of $(\bar{x}^* + s\bar{x}, \bar{x})$. Observe by (3.25) that

$$\operatorname{gph} (\Xi^{-1} + rI)^{-1} \cap Z = \operatorname{gph} (M^{-1} + sI)^{-1} \cap Z \subset \operatorname{gph} (T_p + sI)^{-1} \cap Z. \tag{3.26}$$

Picking $(u, u^*) \in \operatorname{gph} T_p$, we have $u = (T_p + sI)^{-1}(u^* + su)$ by the single-valuedness of the mapping $(T_p + sI)^{-1}$. The seminal Minty's theorem tells us the mapping $(\Xi^{-1} + sI)^{-1}$ is of full domain. Combining this with (3.26) yields $u = (\Xi^{-1} + sI)^{-1}(u^* + su)$ by $(u^* + su, u) \in Z$. Hence we get $(u^*, u) \in \operatorname{gph} \Xi \cap (U^* \times U) = \operatorname{gph} M \cap (U^* \times U)$. Since $M(u^*) \in \partial f_p^{-1}(u^*) \cap U$ for all $u^* \in U^*$, it implies that $M(u^*) = T_p^{-1}(u^*)$ and thus

$$\operatorname{gph} M^{-1} \cap (U \times U^*) = \operatorname{gph} T_p = \operatorname{gph} \partial f_p \cap (U \times U^*).$$

This implies by Theorem 3.5 that Hölder continuity of M_γ in (3.3) yields USOGC, which thus completes the proof of the theorem. $\square$

Now we are ready to derive the main result of this section, which gives a characterization of Hölderian full stability in term of the combined second-order subdifferential (2.14). The tilt stability (p-independent) version of this result has been recently established in [42, Theorem 4.3].

Theorem 3.7 (second-order subdifferential characterization of Hölderian full stability). *Let X be Hilbert while P is metric. Suppose that BCQ (3.7) holds at $(\bar{x}, \bar{p}) \in \operatorname{dom} f$ and that f is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $\bar{x}^* \in \partial_x f(\bar{x}, \bar{p})$. The following are equivalent:*

(i) *The point $\bar{x}$ is a Hölderian fully stable local minimizer of problem $\mathcal{P}(\bar{x}^*, \bar{p})$ in (3.2) with a modulus pair $(\kappa, \ell) \in \mathbb{R}^2_>$.*

(ii) *There are $\eta > 0$ such that for all $(x, p, x^*) \in \operatorname{gph} \partial_x f \cap \mathbb{B}_\eta(\bar{x}, \bar{p}, \bar{x}^*)$ we have*

$$\langle u^*, u \rangle \geq \frac{1}{\kappa} \|u\|^2 \quad \textit{whenever} \quad u^* \in \breve{\partial}^2 f_p(x, x^*)(u), \quad u \in X. \tag{3.27}$$

Proof. Assuming (i) and using Theorem 3.6, find a neighborhood $U \times U^* \times V$ of $(\bar{x}, \bar{x}^*, \bar{p})$

such that (3.10) holds. Define a single-valued mapping ϑ by $\mathrm{gph}\,\vartheta := \mathrm{gph}\,S \cap (U^* \times V \times U)$ with S from (3.3) and have by (3.10) that

$$\langle x^* - u^*, \vartheta(x^*,p) - \vartheta(u^*,p)\rangle \geq \kappa^{-1}\|\vartheta(x^*,p) - \vartheta(u^*,p)\|^2 \ \text{ for all } \ x^*, u^* \in U^*,\ p \in V.$$

This implies that the mappings $\vartheta_p := \vartheta(\cdot,p)$ and $\vartheta_p^{-1} - \kappa^{-1}I$ are monotone for any $p \in V$. In fact they are maximal monotone together with ϑ_p^{-1} due to the assumed Hölder continuity of ϑ. Denoting by Ξ_p the maximal monotone extension of $\vartheta_p^{-1} - \kappa^{-1}I$ as in [3, Theorem 20.21], we get that $\Xi_p + \kappa^{-1}I$ is also a maximal monotone with $\mathrm{gph}\,\vartheta_p^{-1} \subset \mathrm{gph}\,(\Xi_p + \kappa^{-1}I) \cap (U \times U^*)$. Since ϑ_p^{-1} is maximal monotone relative to U and U^*, it follows that

$$\mathrm{gph}\,\partial f_p \cap (U \times U^*) = \mathrm{gph}\,\vartheta_p^{-1} = \mathrm{gph}\,(\Xi_p + \kappa^{-1}I) \cap (U \times U^*).$$

Find further $\eta > 0$ satisfying $I\!B_{2\eta}(\bar{x},\bar{x}^*) \subset U \times U^*$ and get from (2.14) and Lemma 2.5 that

$$\breve{\partial}^2 f_p(x,x^*) = \widehat{D}^*\partial f_p(x,x^*) = \widehat{D}^* S_p^{-1}(x,x^*) = \widehat{D}^*\Xi_p(x,x^* - \kappa^{-1}x) + \kappa^{-1}I, \quad (x,x^*) \in I\!B_\eta(\bar{x},\bar{x}^*).$$

This gives us by Lemma 2.9 that $\langle u^* - \kappa^{-1}u, u\rangle \geq 0$ for any $u^* \in \breve{\partial}^2 f_p(x,x^*)(u)$, which justifies (3.27) and thus implication (i)$\Longrightarrow$(ii).

Conversely, assuming (ii) and employing the parametric continuous prox-regularity of f at $(\bar{x},\bar{p})$ for $\bar{x}^* \in \partial_x f(\bar{x},\bar{p})$ give us numbers $\varepsilon, r > 0$ such that for any $p \in I\!B_\varepsilon(\bar{p})$ we have

$$f_p(x) \geq f_p(u) + \langle u^*, x - u\rangle - \frac{r}{2}\|x - u\|^2 \ \text{ for all } \ u^* \in \partial f_p(u) \cap I\!B_\varepsilon(\bar{x}^*),\ x, u \in I\!B_\varepsilon(\bar{x}). \quad (3.28)$$

Define $g(x,p) := f(x,p) + \frac{s}{2}\|x - \bar{x}\|^2$ and $g_p(x) := g(x,p)$ for $x \in X$ and $p \in P$. Employing the limiting subdifferential sum rule Lemma 2.4 gives for any fix $p \in I\!B_\varepsilon(\bar p)$ gives us $\partial g_p = \partial f_p + s(I-\bar x)$. Define further $W := J(I\!B_\varepsilon(\bar x,\bar p,\bar x^*))$ with $J(x,p,x^*) := (x,p,x^*+s(x-\bar x))$ and note from the classical open mapping theorem that W contains a neighborhood of $(\bar x,\bar p,\bar x^*)$. Picking any $(u,p,u^*) \in \operatorname{gph}\partial_x g \cap W$, it follows from $u^* - s(u-\bar x) \in \partial f_p(u)$ and (3.28) that for any $x \in I\!B_\varepsilon(\bar x)$ we have

$$\begin{aligned} g_p(x) &= f_p(x) + \frac{s}{2}\|x-\bar x\|^2 \ge f_p(u) + \langle u^* - s(u-\bar x), x-u\rangle - \frac{r}{2}\|x-u\|^2 + \frac{s}{2}\|x-\bar x\|^2 \\ &= g_p(u) + \langle u^*, x-u\rangle + \frac{s}{2}\|x-u\|^2. \end{aligned} \tag{3.29}$$

Let us check that BCQ (3.7) holds for g at $(\bar x,\bar p,g(\bar x,\bar p))$. Indeed, since the set-valued mapping $F: p \mapsto \operatorname{epi} f(\cdot,p)$ is Lipschitz-like around $(\bar p,\bar x,f(\bar x,\bar p))$, there are constants $c,\eta>0$ such that

$$F(p_1)\cap I\!B_\eta\big((\bar x,f(\bar x,\bar p))\big) \subset F(p_2) + c d(p_1,p_2) I\!B_{X\times I\!R} \text{ for all } p_1,p_2 \in I\!B_\eta(\bar p). \tag{3.30}$$

Define $F_1 : p \mapsto \operatorname{epi} g(\cdot,p)$ and choose a neighborhood Z of $(\bar x, g(\bar x,\bar p))$ with $(x, s-\frac{r}{2}\|x-\bar x\|^2) \in I\!B_\eta((\bar x,f(\bar x,\bar p))$ as $(x,s)\in Z$. Picking any $p_1,p_2 \in I\!B_\eta(\bar p)$ and $(x_1,s_1) \in F_1(p_1)\cap Z$, observe that $(x_1, s_1 - \frac{r}{2}\|x_1-\bar x\|^2) \in F(p_1)\cap I\!B_\eta((\bar x,f(\bar x,\bar p)))$, and thus we have by (3.30) that

$$\|x_2-x_1\| + \big\|r_2 - s_1 + \tfrac{r}{2}\|x_1-\bar x\|^2\big\| \le c d(p_1,p_2) \text{ for some } (x_2,r_2)\in F(p_2). \tag{3.31}$$

Denoting $s_2 := r_2 + \frac{r}{2}\|x_2-\bar x\|^2$ yields $(x_2,r_2)\in F_1(p_2)$ and gives us together with (3.31) that

$$\begin{aligned} \|x_2-x_1\| + \|s_2-s_1\| &\le \|x_2-x_1\| + \big\|r_2-s_1+\tfrac{r}{2}\|x_1-\bar x\|^2\big\| + \big\|\tfrac{r}{2}\|x_2-\bar x\|^2 - \tfrac{r}{2}\|x_1-\bar x\|^2\big\| \\ &\le c d(p_1,p_2) + \tfrac{r}{2}\|x_2-x_1\|\big(\|x_2-\bar x\| + \|x_1-\bar x\|\big) \\ &\le c d(p_1,p_2) + \tfrac{r}{2} c d(p_1,p_2) 2\eta = c(1+r\eta)d(p_1,p_2), \quad \text{and so} \end{aligned}$$

$$F_1(p_1)\cap Z\subset F_1(p_2)+c(1+r\eta)d(p_1,p_2)I\!B_{X\times R}\ \text{ for all }\ p_1,p_2\in I\!B_\eta(\bar p),$$

which thus verifies BCQ (3.7) for the function g around $(\bar p,\bar x,g(\bar x,\bar p))$.

To proceed further, pick any $v^*\in\breve\partial^2 g_p(x,x^*)(v)$ with $v\in X$ and $(x,x^*)\in W$ and get from Lemma 2.5 that $v^*-sv\in\breve\partial^2 f_p(x,x^*_r(x-\bar x))(v)$. Since $(x,x^*-s(x-\bar x))=J^{-1}(x,x^*)\in I\!B_\varepsilon(\bar x)\times I\!B_\varepsilon(\bar x^*)$, it follows from (3.28) that $\langle v^*-sv,v\rangle\ge\kappa^{-1}\|v\|^2$, which yields

$$(s+\kappa^{-1})\|v\|^2\le\langle v^*,v\rangle\le\|v^*\|\cdot\|v\|$$

and hence $(s+\kappa^{-1})\|v\|\le\|v^*\|$. This together with Lemma 2.7 shows us that the mapping $S^g(x^*,p):=\big\{x\in X|\ x^*\in\partial_x g(x,p)\big\}$ is Lipschitz-like with modulus $(s+\kappa^{-1})^{-1}$ around $(\bar x^*,\bar x)$ with compatible parameterization in p around $\bar p$. Moreover, since g satisfies the uniform second-order growth condition (3.29), we get from Theorem 3.5 that S^g contains a single-valued localization ϑ^g around $(\bar x^*,\bar p,\bar x)$. Thus there is some $\delta>0$ such that

$$\|\vartheta^g(x_1^*,p)-\vartheta^g(x_2^*,p)\|\le(s+\kappa^{-1})^{-1}\|x_1^*-x_2^*\|\quad\text{for all}\quad x_1^*,x_2^*\in I\!B_\delta(\bar x^*,\bar p).$$

Thanks to Theorem 3.5 and (3.29), g satisfies the uniform second-order growth condition (3.10) at $(\bar x,\bar p\,\bar x^*)$ with modulus $(s+\kappa^{-1})^{-1}$, i.e., there are a neighborhood $(U\times V\times U^*)$ of $(\bar x,\bar p,\bar x^*)$ such that

$$g_p(x)\ge g_p(u)+\langle u^*,x-u\rangle+\frac{s+\kappa^{-1}}{2}\|x-u\|^2\quad\text{for all}\quad x\in U,(u,p,u^*)\in\operatorname{gph}\partial g_x\cap(U\times V\times U^*).$$

Since $g_p(x) = f_p(x) + \frac{s}{2}\|x - \bar{x}\|^2$, we easily deduce that

$$f_p(x) \geq f_p(u) + \langle u^*, x - u\rangle + \frac{1}{2\kappa}\|x - u\|^2 \quad \text{for all} \quad x \in U, (u, p, u^*) \in \operatorname{gph} \partial g_x \cap Z,$$

where $Z := J^{-1}(U \times V \times U^*)$, which is also a neighborhood of $(\bar{x}, \bar{p}, \bar{x}^*)$. Thanks to Theorem 3.6, the point $\bar{x}$ is a Hölderian fully stable local minimizer of problem $\mathcal{P}(\bar{x}^*, \bar{p})$. This completes the proof of the theorem. $\square$

3.3 Second-order Characterizations of Lipschitzian Full Stability

The following proposition shows that the Lipschitz continuity of m_γ is automatic under BCQ (3.7). In finite dimensions it is derived by a different way in the proof of [31, Proposition 3.5] under an additional assumption that $M_\gamma(\bar{x}^*, \bar{p}) = \bar{x}$ for some $\gamma > 0$.

Proposition 3.8 (Lipschitz continuity of the infimum function under BCQ.) *Let $\bar{x}$ be a local minimizer of $\mathcal{P}(\bar{x}^*, \bar{p})$ in* (3.2)*, and let BCQ* (3.7) *hold at $(\bar{x}, \bar{p})$. Then the infimum function m_γ in* (3.3) *is Lipschitz continuous around $(\bar{x}^*, \bar{p})$ for all $\gamma > 0$ sufficiently small.*

Proof. Take the neighborhoods U, V and the constants c, ε from Proposition 3.3 as a consequence of BCQ, and let $\delta, \gamma > 0$ be such that $(2c + 1)\delta \leq \gamma < \varepsilon$, $I\!B_\gamma(\bar{x}) \subset U$, and $I\!B_\gamma(\bar{p}) \subset V$. Pick arbitrary pairs $(x_1^*, p_1), (x_2^*, p_2) \in I\!B_\delta(\bar{x}^*) \times I\!B_\delta(\bar{p})$ and for any $\nu \in (0, \varepsilon - c\delta)$ take $x_1 \in I\!B_\gamma(\bar{x})$ such that $f(x_1, p_1) - \langle x_1^*, x_1\rangle \leq m_\gamma(x_1^*, p_1) + \nu$. By (3.9) find $u \in X$ with $f(u, p_1) \leq f(\bar{x}, \bar{p}) + cd(p_1, \bar{p})$ and $\|u - \bar{x}\| \leq cd(p_1, \bar{p}) \leq c\delta \leq \gamma$. Then we get subsequently

$$f(x_1, p_1) - \langle x_1^*, x_1\rangle - \nu \leq m_\gamma(x_1^*, p_1) \leq f(u, p_1) - \langle x_1^*, u\rangle \leq f(\bar{x}, \bar{p}) + cd(p_1, \bar{p}) - \langle x_1^*, u\rangle \quad \text{and}$$

$$f(x_1,p_1) \le f(\bar{x},\bar{p}) + cd(p_1,\bar{p}) + \langle x_1^*, x_1 - u\rangle + \nu \le f(\bar{x},\bar{p}) + c\delta + (\|\bar{x}^*\| + \delta)2\gamma + \nu < f(\bar{x},\bar{p}) + \varepsilon$$

for $\delta,\gamma,\nu > 0$ sufficiently small. By Proposition 3.3 again we find $x_2 \in X$ such that $\|x_2 - x_1\| \le cd(p_1,p_2) \le 2c\delta$ and $f(x_2,p_2) \le f(x_1,p_1) + cd(p_1,p_2)$. Hence $\|x_2 - \bar{x}\| \le \|x_1 - \bar{x}\| + 2c\delta \le \delta + 2c\delta \le \gamma$, which yields $x_2 \in I\!B_\gamma(\bar{x})$ and thus implies the inequalities

$$\begin{aligned} m_\gamma(x_2^*,p_2) - m_\gamma(x_1^*,p_1) &\le f(x_2,p_2) - \langle x_2^*, x_2\rangle - [f(x_1,p_1) - \langle x_1^*, x_1\rangle - \nu] \\ &\le cd(p_1,p_2) - \langle x_2^* - x_1^*, x_2\rangle + \langle x_1^*, x_1 - x_2\rangle + \nu \\ &\le cd(p_1,p_2) + \|x_2^* - x_1^*\|(\|\bar{x}\| + \delta) + (\|\bar{x}^*\| + \delta)\|x_1 - x_2\| + \nu \\ &\le cd(p_1,p_2) + \|x_2^* - x_1^*\|(\|\bar{x}\| + \delta) + (\|\bar{x}^*\| + \delta)cd(p_1,p_2) + \nu. \end{aligned}$$

Changing the role of (x_2^*,p_2) and (x_1^*,p_1) in the above expressions gives us that

$$\|m_\gamma(x_1^*,p_1) - m_\gamma(x_2^*,p_2)\| \le cd(p_1,p_2) + \|x_2^* - x_1^*\|(\|\bar{x}\| + \delta) + (\|\bar{x}^*\| + \delta)cd(p_1,p_2) + \nu$$

for small $\nu > 0$. Thus omitting ν justifies the Lipschitz continuity of m_γ on $I\!B_\delta(\bar{x}^*) \times I\!B_\delta(\bar{p})$. $\square$

The next result shows that the *Lipschitz continuity* of $\vartheta(x^*,p)$ from Theorem 3.5 with respect to *both* variables (x^*,p) can be equivalently described in via USOGC (3.10) and an additional Lipschitz-like condition, which is essential even for simple problems in $I\!R^2$; see Example 3.2.

Theorem 3.9 (Lipschitz continuity of the inverse subgradient mapping). *Let $\bar{x}^* \in \partial_x f(\bar{x},\bar{p})$, and let BCQ* (3.7) *hold at $(\bar{x},\bar{p})$. Then the following assertions are equivalent:*

(i) *We have $\bar{x} \in M_\gamma(\bar{x}^*,\bar{p})$ for some $\gamma > 0$, and there exist a neighborhood $U^* \times V \times U$ of $(\bar{x}^*,\bar{p},\bar{x})$ and a constant pair $(\kappa,\ell) \in I\!R^2_>$ such that the mapping S from* (3.7) *admits a single-*

valued localization ϑ with respect to $U^* \times V \times U$ satisfying the Lipschitz continuity condition

$$\|\vartheta(x_1^*, p_1) - \vartheta(x_2^*, p_2)\| \le \kappa\|x_1^* - x_2^*\| + \ell d(p_1, p_2) \quad \text{as} \quad x_1^*, x_2^* \in U^*,\ p_1, p_2 \in V. \tag{3.32}$$

(ii) *USOGC from Definition* 3.4 *holds at $(\bar{x}, \bar{p}, \bar{x}^*)$ with modulus κ and the graphical mapping*

$$G : p \mapsto \operatorname{gph} \partial_x f(\cdot, p) \quad \text{is Lipschitz-like around} \quad (\bar{p}, \bar{x}, \bar{x}^*). \tag{3.33}$$

Proof. It follows from Theorem 3.5 that the conditions in (i) ensures the validity of USOGC. To verify (i)$\Longrightarrow$(ii), it remains to show that these conditions imply (3.33) as well. We claim that

$$G(p_1) \cap (U \times U^*) \subset G(p_2) + \ell d(p_1, p_2) I\!B_{X \times X^*} \quad \text{for all} \quad p_1, p_2 \in V \tag{3.34}$$

with U, U^*, V, and ℓ taken from (i), which gives us the Lipschitz-like property by (2.17). To proceed, pick $(x_1, x_1^*) \in G(p_1) \cap (U \times U^*)$ and choose $x_2 := \vartheta(x_1^*, p_2) \in U$; so $(x_2, x_1^*) \in G(p_2)$. It follows from (3.32) that $\|x_1 - x_2\| \le \kappa d(p_1, p_2)$, which therefore justifies the validity of (3.34).

Now let us verify the converse implication (ii)$\Longrightarrow$(i). Theorem 3.5 tells us that S has a single-valued localization around $(\bar{x}^*, \bar{p}, \bar{x})$ satisfying (3.11). By (3.33) there exist a neighborhood $V_1 \times U_1 \times U_1^* \subset V \times U \times U^*$ of $(\bar{p}, \bar{x}, \bar{x}^*)$ and a number $c > 0$ such that

$$G(p_1) \cap (U_1 \times U_1^*) \subset G(p_2) + c d(p_1, p_2) I\!B_{X \times X^*} \quad \text{for all} \quad p_1, p_2 \in V_1, \tag{3.35}$$

where V, U, U^* are taken from Definition 3.4. Picking $(x_1^*, p_1, u_1), (x_2^*, p_2, u_2) \in \operatorname{gph} S \cap (U_1^* \times$

$V_1 \times U_1$), we find from (3.35) a pair $(u, x^*) \in G(p_2)$ such that

$$\|u_1 - u\| + \|x_1^* - x^*\| \leq cd(p_1, p_2). \tag{3.36}$$

By shrinking $U_1^* \times V_1 \times U_1$ if necessary, suppose that $(x^*, p_2, u) \in \operatorname{gph} S \cap (U^* \times V \times U)$. Then the assumed USOGC (3.10) provides the estimates

$$\begin{aligned} f(u, p_2) &\geq f(u_2, p_2) + \langle x_2^*, u - u_2 \rangle + \frac{1}{2\kappa}\|u - u_2\|^2, \\ f(u_2, p_2) &\geq f(u, p_2) + \langle x^*, u_2 - u \rangle + \frac{1}{2\kappa}\|u_2 - u\|^2, \end{aligned}$$

which ensure in turn that

$$\|x^* - x_2^*\| \cdot \|u - u_2\| \geq \langle x^* - x_2^*, u - u_2 \rangle \geq \kappa^{-1}\|u - u_2\|^2$$

and thus yield $\|x^* - x_2^*\| \geq \kappa^{-1}\|u - u_2\|$. Combining this with (3.36) gives us that

$$\begin{aligned} \|u_1 - u_2\| &\leq \|u_1 - u\| + \|u - u_2\| \leq cd(p_1, p_2) + \kappa\|x^* - x_2^*\| \\ &\leq cd(p_1, p_2) + \kappa\|x^* - x_1^*\| + \kappa\|x_1^* - x_2^*\| \\ &\leq cd(p_1, p_2) + \kappa cd(p_1, p_2) + \kappa\|x_1^* - x_2^*\| \\ &= \kappa\|x_1^* - x_2^*\| + c(\kappa + 1)d(p_1, p_2), \end{aligned}$$

i.e., (3.32) holds, and so we complete the proof of the theorem. $\square$

Note that condition (3.32) can be equivalently described via

$$(0, p^*) \in \big(D^*\partial_x f\big)(\bar{x}, \bar{p}, \bar{x}^*)(0) \Longrightarrow p^* = 0 \tag{3.37}$$

by the coderivative criterion Lemma 2.6 for the Lipschitz-like property discussed in Section 2 when X, P are both finite-dimensional spaces. It is also worth mentioning that the existence of a *Lipschitzian* single-valued localization of the inverse partial subgradient mapping S in (i) of Theorem 3.9 is known as the *partial strong metric regularity* (PSMR) of $\partial_x f$; see [48, Definition 3.4]. This is an appropriate version of the so-called "strong metric regularity" [16] for $\partial_x f$, which in turn is an abstract version of Robinson's strong regularity [58]. In this way the property considered in (i) of Theorem 3.5 can be viewed as a *Hölderian* counterpart of PSMR. Note that the Hölderian effect *disappears* and Theorems 3.5 and 3.9 are identical when f does not depend on p, i.e., we have only tilt perturbations in (3.2). In this case some versions of the obtained equivalence can be found in [13] in finite dimensions and in [14, 42] in Asplund spaces.

In a similar way we arrive at the following characterization of Lipschitzian full stability in (3.2).

Theorem 3.10 (characterizing Lipschitzian full stability via USOGC). *Let X be Hilbert while P is metric. Assume that BCQ* (3.7) *is satisfied at $(\bar{x}, \bar{p}) \in \operatorname{dom} f$ and that f is parametrically subdifferentially continuous at $(\bar{x}, \bar{p})$ for $\bar{x}^* \in \partial_x f(\bar{x}, \bar{p})$. The following are equivalent:*

(i) *The point $\bar{x}$ is a Lipschitzian fully stable local minimizer of problem $\mathcal{P}(\bar{x}^*, \bar{p})$ in* (3.2) *with a modulus pair $(\kappa, \ell) \in I\!R^2_>$ and the function f is prox-regular in x at $\bar{x}$ for $\bar{x}^*$ with compatible parameterization by p at $\bar{p}$.*

(ii) *USOGC* (3.10) *holds at $(\bar{x}, \bar{p}, \bar{x}^*)$ with modulus κ together with the Lipschitz-like condition in* (3.33).

Proof. It follows the proof of Theorem 3.6 with using Theorem 3.9 instead of Theorem 3.5.□

When both X and P are finite-dimensional, Theorem 3.10 reduces to the recent result of [48, Theorem 3.8], where the Lipschitz-like property in (ii) is replaced by a more restrictive condition.

As a consequence of Theorem 3.6 and other results above, we get the following characterization of Lipschitzian full stability for (3.2) in terms of the combined second-order subdifferential (2.14).

Corollary 3.11 (second-order subdifferential characterization of Lipschitzian full stability). *Let X be Hilbert while P is metric. Suppose that BCQ* (3.7) *holds at $(\bar{x},\bar{p})$ and that f is parametrically continuously prox-regular at $(\bar{x},\bar{p})$ for $\bar{x}^*\in\partial_x f(\bar{x},\bar{p})$. The following are equivalent:*

(i) *The point $\bar{x}$ is a Lipschitzian fully stable local minimizer of problem $\mathcal{P}(\bar{x}^*,\bar{p})$ in* (3.2)*.*

(ii) *Conditions* (3.33) *and* (3.27) *hold with some $\kappa,\eta>0$.*

Proof. If $\bar{x}$ is a Lipschitzian fully stable local minimizer of $\mathcal{P}(\bar{x}^*,\bar{p})$, then condition (3.33) holds by Theorem 3.10. The validity of (3.27) is proved in Theorem 3.7, and so we get (ii). Conversely, (ii) implies by Theorem 3.7 that $\bar{x}$ is a Hölderian fully stable local minimizer of $\mathcal{P}(\bar{x}^*,\bar{p})$. Employing Theorem 3.6 ensures that USOGC (3.10) holds at $(\bar{x},\bar{p},\bar{x}^*)$. Thus we get from Theorem 3.10 that $\bar{x}$ is a Lipschitzian fully stable local minimizer of $\mathcal{P}(\bar{x}^*,\bar{p})$ and complete the proof. $\square$

If P is Asplund, we have yet another second-order subdifferential characterization of Lipschitzian full stability in (3.2) implicitly involving subdifferentiation in p as well.

Theorem 3.12 (Lipschitzian full stability with Asplund parameter spaces). *Let X be Hilbert while P is Asplund. Suppose that BCQ* (3.7) *holds at $(\bar{x},\bar{p})$ and that f is parametrically continuously prox-regular at $(\bar{x},\bar{p})$ for $\bar{x}^*\in\partial_x f(\bar{x},\bar{p})$. The following are equivalent:*

(i) *The point $\bar{x}$ is a Lipschitzian fully stable local minimizer of problem $\mathcal{P}(\bar{x}^*,\bar{p})$ with a modulus pair $(\kappa,\ell)\in I\!R^2_>$.*

(ii) *Condition* (3.33) *holds and there are positive constants* κ *and* η *such that for all* $u \in X^{**}$ *and* $(x, p, x^*) \in \operatorname{gph} \partial_x f \cap I\!B_\eta(\bar{x}, \bar{p}, \bar{x}^*)$ *we have*

$$\langle u^*, u\rangle \geq \frac{1}{\kappa}\|u\|^2 \quad \textit{whenever} \quad (u^*, p^*) \in \widehat{D}^*\partial_x f(x, p, x^*)(u). \tag{3.38}$$

Proof. To justify (i)$\Longrightarrow$(ii), it suffices to prove by Corollary 3.11 that (3.27) implies (3.38). To proceed, pick $(u^*, p^*) \in \widehat{D}^*(\partial_x f)(x, p, x^*)(u)$ with $u \in X^{**}$ and $(x, p, x^*) \in \operatorname{gph} \partial_x f \cap I\!B_\eta(\bar{x}, \bar{p}, \bar{x}^*)$, where $\eta > 0$ is taken from (3.27). This yields by definition (2.12) that

$$\limsup_{(x_1, p_1, x_1^*) \overset{\operatorname{gph} \partial_x f}{\to} (x, p, x^*)} \frac{\langle u^*, x_1 - x\rangle + \langle p^*, p_1 - p\rangle - \langle u, x_1^* - x^*\rangle}{\|x_1 - x\| + \|p_1 - p\| + \|x_1^* - x^*\|} \leq 0.$$

Choosing $p_1 = p$ in the latter gives us $u^* \in \breve{\partial}^2 f_p(x, x^*)(u)$ and thus ensures (3.38) by (3.27).

Conversely, assume by (ii) that the mapping G in (3.33) is Lipschitz-like around $(\bar{p}, \bar{x}, \bar{x}^*)$ with modulus $\ell > 0$ and that inequality (3.38) is satisfied with some $\kappa, \eta > 0$. To get (i), we only need to verify by Corollary 3.11 that (3.27) holds. Pick any $u \in X^{**}$ and $u^* \in \breve{\partial}^2 f_p(x, x^*)(u)$ with $(x, p, x^*) \in \operatorname{gph} \partial_x f \cap I\!B_{\eta_1}(\bar{x}, \bar{p}, \bar{x}^*)$ for some $\eta_1 \in (0, \eta)$. There is nothing to do if $u = 0$. Since the combined second-order subdifferential $\breve{\partial}^2$ is homogeneous, suppose without loss of generality that $\|u^*\| + \|u\| \leq (2\ell)^{-1}$ and $u \neq 0$. Defining $\Omega_1 := \operatorname{gph} G$ and $\Omega_2 := \{p\} \times X \times X$ we get by (2.14) that $(0, u^*, -u) \in \widehat{N}((p, x, x^*); \Omega_1 \cap \Omega_2)$. It follows from the fuzzy intersection rule in [38, Lemma 3.1] that for any $0 < \varepsilon < \min\{\eta - \eta_1, \frac{1}{4(\ell+1)}\}$ there are $\lambda \geq 0$, $(p_i, x_i, x_i^*) \in \Omega_i \cap I\!B_\varepsilon(p, x, x^*)$, and $(p_i^*, u_i^*, u_i) \in P^* \times X \times X$ as $i = 1, 2$ satisfying $(p_i^*, u_i^*, -u_i) \in \widehat{N}((p_i, x_i, x_i^*); \Omega_i) + \varepsilon I\!B_{P^* \times X \times X}$ with

$$\lambda(0, u^*, -u) = (p_1^*, u_1^*, -u_1) + (p_2^*, u_2^*, -u_2) \quad \text{and} \quad \max\{\lambda, \|(p_2^*, u_2^*, -u_2)\|\} = 1. \tag{3.39}$$

By the construction of Ω_2 we get $\widehat{N}((p_2,x_2,x_2^*);\Omega_2)\subset P^*\times\{0\}\times\{0\}$ and so $\|u_2^*\|+\|u_2\|\le\varepsilon$. Furthermore, there is $(\bar{p}_1^*,\bar{u}_1^*,-\bar{u}_1)\in\hat{N}((p_1,x_1,x_1^*);\Omega_1)$ with $\|p_1^*-\bar{p}_1^*\|+\|u_1^*-\bar{u}_1^*\|+\|u_1-\bar{u}_1\|\le\varepsilon$. Then the Lipschitz-like property of G implies by [38, Theorem 1.43] that $\|\bar{p}_1^*\|\le\ell(\|\bar{u}_1^*\|+\|\bar{u}_1\|)$. This together with (3.39) ensures the relationships

$$\begin{aligned}\|p_2^*\| &= \|p_1^*\|\le\|\bar{p}_1^*\|+\varepsilon\le\ell(\|\bar{u}_1^*\|+\|\bar{u}_1\|)+\varepsilon\le\ell(\|u_1^*\|+\|u_1\|+\varepsilon)+\varepsilon\\ &\le \ell(\lambda\|u^*\|+\|u_2^*\|+\lambda\|u\|+\|u_2\|+\varepsilon)+\varepsilon\le\ell(\|u^*\|+\|u\|+2\varepsilon)+\varepsilon\le\frac{1}{2}+\varepsilon(2\ell+1),\end{aligned}$$

and hence $\|(p_2^*,u_2^*,-u_2)\|\le\frac{1}{2}+\varepsilon(2\ell+1)+\varepsilon<1$. Combining it with (3.39) gives us $\lambda=1$ and

$$(0,u^*,-u)=(p_1^*,u_1^*,-u_1)+(p_2^*,u_2^*,-u_2).\tag{3.40}$$

Noting that $(x_1,p_1,x_1^*)\in I\!B_{\varepsilon+\eta_1}(\bar{x},\bar{p},\bar{x}^*)\subset I\!B_\eta(\bar{x},\bar{p},\bar{x}^*)$, we get from (3.38) and the inclusion $(\bar{u}_1^*,\bar{p}_1^*)\in(\widehat{D}^*\partial_x f)(x_1,p_1,x_1^*)(\bar{u}_1)$ that $\langle\bar{u}_1^*,\bar{u}_1\rangle\ge\kappa\|\bar{u}_1\|^2$. This together with (3.40) yields that

$$\begin{aligned}\langle u^*,u\rangle &= \langle u^*,u_1+u_2\rangle\ge\langle u_1^*,u_1\rangle-\|u_2\|\cdot\|u^*\|\ge\langle\bar{u}_1^*,u_1\rangle+\langle u_1^*-\bar{u}_1^*+u_2^*,u_1\rangle-\varepsilon\|u^*\|\\ &\ge \langle\bar{u}_1^*,u_1\rangle-(\|u_1^*-\bar{u}_1^*\|+\|u_2^*\|)\|u_1\|-\varepsilon\|u^*\|\\ &\ge \langle\bar{u}_1^*,\bar{u}_1\rangle-\|\bar{u}_1-u_1\|\cdot\|\bar{u}_1^*\|-2\varepsilon\|u_1\|-\varepsilon\|u^*\|\\ &\ge \kappa^{-1}\|\bar{u}_1\|^2-\varepsilon(\|u^*\|+\|u^*-\bar{u}_1^*\|)-2\varepsilon(\|u\|+\|u-u_1\|)-\varepsilon\|u^*\|\\ &\ge \kappa^{-1}\|\bar{u}_1\|^2-\varepsilon\|u^*\|-\varepsilon^2-2\varepsilon\|u\|-2\varepsilon^2-\varepsilon\|u^*\|\\ &\ge \kappa^{-1}(\|u\|-\|u-\bar{u}_1\|)^2-2\varepsilon(\|u^*\|+\|u\|)-3\varepsilon^2\\ &\ge \kappa^{-1}\|u\|^2-2\kappa\|u-\bar{u}_1\|\|u\|-2\varepsilon(\|u^*\|+\|u\|)-3\varepsilon^2\\ &\ge \kappa^{-1}\|u\|^2-4\kappa\varepsilon\|u\|-2\varepsilon(\|u^*\|+\|u\|)-2\varepsilon^2.\end{aligned}$$

Letting $\varepsilon \downarrow 0$ gives us that $\langle u^*, u\rangle \geq \kappa^{-1}\|u\|^2$ and thus verifies (3.27). The proof is complete. $\square$

As mentioned above, the Lipschitz-like condition (3.33) can be expressed via the *coderivative criterion* (3.37) if P is Asplund. Furthermore, passing to the limit in (3.38) allows us to obtain the pointwise consequence of this condition via the *mixed coderivative* of $\partial_x f$ at $(\bar{x}, \bar{p}, \bar{x}^*)$.

The next approximation lemma is helpful in the proof of the pointwise characterizations of Lipschitzian full stability established in finite-dimensional spaces.

Lemma 3.13 (coderivative approximation). *Let X, P be two finite-dimensional spaces. Assume that condition* (3.33) *and the following inequality*

$$\|u^*\| \geq \mu\|u\| \quad \textit{whenever} \quad (u^*, p^*) \in D^*\partial_x f(\bar{x}, \bar{p}, \bar{v})(u) \tag{3.41}$$

hold with some $\mu > 0$. Then for any $\delta \in (0, \mu)$ there is $\eta > 0$ such that

$$\|u^*\| \geq (\mu-\delta)\|w\| \quad \textit{if} \quad u^* \in \widehat{D}^*\partial f_p(x, x^*)(u) \quad \textit{with} \quad (x, p, x^*) \in \operatorname{gph}\partial_x f \cap I\!B_\eta(\bar{x}, \bar{p}, \bar{x}^*). \tag{3.42}$$

Proof. Assuming (3.41), we first show that for any $\delta \in (0, \mu)$ there is $\nu > 0$ satisfying

$$\|u^*\| \geq (\mu-\delta)\|u\| \text{ if } (u^*, p^*) \in \widehat{D}^*\partial_x f(x, p, x^*)(u) \text{ with } (x, p, x^*) \in \operatorname{gph}\partial_x f \cap I\!B_\nu(\bar{x}, \bar{p}, \bar{x}^*). \tag{3.43}$$

By contradiction, find sequences $(x_k, p_k, x_k^*) \overset{\operatorname{gph}\partial_x f}{\longrightarrow} (\bar{x}, \bar{p}, \bar{x}^*)$ and $(u_k^*, p_k^*) \in \widehat{D}^*\partial_x f(x_k, p_k, x_k^*)(u_k)$ such that $\|u_k^*\| < (\mu-\delta)\|u_k\|$, which clearly implies that $u_k \neq 0$. Denoting $\bar{u}_k^* := u_k^*\|u_k\|^{-1}$, $\bar{p}_k^* := p_k^*\|u_k\|^{-1}$, and $\bar{u}_k := u_k\|u_k\|^{-1}$ gives us $(\bar{u}_k^*, \bar{p}_k^*) \in \widehat{D}^*\partial_x f(x_k, p_k, x_k^*)(\bar{u}_k)$ as $k \in I\!N$. Since (3.33) holds, the mapping $G : p \mapsto \operatorname{gph}\partial_x f(\cdot, p)$ is Lipschitz-like with some modulus $\ell > 0$. Then the result of [38, Theorem 1.43] tells us that $\|\bar{p}_k^*\| \leq \ell(\|\bar{u}_k^*\| + \|\bar{u}_k\|)$ for all k. It follows that $\|\bar{u}_k\| = 1$, $\|\bar{u}_k^*\| \le \mu - \delta$, and $\|\bar{p}_k^*\| \le \ell(\mu - \delta + 1)$. By passing to a subsequence, suppose that $(\bar{u}_k^*, \bar{p}_k^*, \bar{u}_k)$ converges to $(\bar{u}^*, \bar{p}^*, \bar{u})$ as $k \to \infty$. Hence $\|\bar{u}\| = 1$ and $(\bar{u}^*, \bar{p}^*) \in D^*\partial_x f(\bar{x}, \bar{p}, \bar{v})(\bar{u})$ with $\|\bar{u}^*\| \le (\mu - \delta)$, which contradicts (3.41) and thus verifies condition (3.43).

To justify further (3.42), take any $u^* \in \widehat{D}^*\partial f_p(x, x^*)(u)$ with $(x, p, x^*) \in \operatorname{gph} \partial_x f \cap I\!B_\eta(\bar{x}, \bar{p}, \bar{v})$ for some $\eta \in (0, \nu)$. Due to the homogeneity of $\widehat{D}^*$ we may assume that $\|u^*\| + \|u\| \le \frac{1}{2\ell}$. Defining $\Omega_1 := \operatorname{gph} G$ and $\Omega_2 := \{p\} \times X \times X$, observe that $(0, u^*, -u) \in \widehat{N}((p, x, x^*); \Omega_1 \cap \Omega_2)$. It follows from the fuzzy intersection rule in [38, Lemma 3.1] that for any $\varepsilon > 0$ there are $\lambda \ge 0$, $(p_i, x_i, x_i^*) \in \Omega_i \cap I\!B_\varepsilon(p, x, x^*)$, and $(p_i^*, u_i^*, -u_i) \in \widehat{N}\big((p_i, x_i, x_i^*); \Omega_i\big) + \varepsilon I\!B$ as $i = 1, 2$ such that

$$\lambda(0, u^*, -u) = (p_1^*, u_1^*, -u_1) + (p_2^*, u_2^*, -u_2) \quad \text{and} \quad \max\big\{\lambda, \ \|(p_2^*, u_2^*, -u_2)\|\big\} = 1. \tag{3.44}$$

The construction of Ω_2 yields $\widehat{N}((p_2, x_2, x_2^*); \Omega_2) \subset I\!R^d \times \{0\} \times \{0\}$ and thus $\|u_2^*\| + \|u_2\| \le \varepsilon$. Moreover, there is $(\bar{p}_1^*, \bar{u}_1^*, -\bar{u}_1) \in \widehat{N}((p_1, x_1, x_1^*); \Omega_1)$ satisfying $\|p_1^* - \bar{p}_1^*\| + \|u_1^* - \bar{u}_1^*\| + \|u_1 - \bar{u}_1\| \le \varepsilon$. The Lipschitz-like property of G with modulus ℓ ensures by [38, Theorem 1.43] that $\|\bar{p}_1^*\| \le \ell(\|\bar{u}_1^*\| + \|\bar{u}_1\|)$. This together with (3.44) gives us the relationships

$$\begin{aligned}
\|p_2^*\| &= \|p_1^*\| \le \|\bar{p}_1^*\| + \varepsilon \le \varepsilon + \ell(\|\bar{u}_1^*\| + \|\bar{u}_1\|) \le \varepsilon + \ell(\|u_1^*\| + \|u_1\| + \varepsilon) \\
&\le \ell(\|\lambda u^* - u_2^*\| + \|\lambda u - u_2\|) + (\ell + 1)\varepsilon \le \ell(\lambda\|u^*\| + \|u_2^*\| + \lambda\|u\| + \|u_2\|) + (\ell + 1)\varepsilon \\
&\le \ell\big(\lambda(\|u^*\| + \|u\|) + \varepsilon\big) + (\ell + 1)\varepsilon \le \ell(\|u^*\| + \|u\|) + (2\ell + 1)\varepsilon < \frac{1}{2} + (2\ell + 1)\varepsilon.
\end{aligned}$$

When $\varepsilon > 0$ is sufficiently small, we have $\|p_2^*\| < 1 - \varepsilon$ and so $\|(p_2^*, u_2^*, -u_2)\| < 1$. It follows from (3.44) that $\lambda = 1$. Combining this with (3.43) and (3.44) implies that

$$\begin{aligned}
\|u^*\| &= \|u_1^* + u_2^*\| \ge \|\bar{u}_1^*\| - \|\bar{u}_1^* - \bar{u}_1^*\| - \|\bar{u}_2^*\| \ge (\mu - \delta)\|\bar{u}_1\| - \varepsilon - \varepsilon \\
&\ge (\mu - \delta)(\|u\| - \|u_1 - \bar{u}_1\| - \|u_2\|) - 2\varepsilon \ge (\mu - \delta)\|u\| - 2\varepsilon(\mu - \delta) - 2\varepsilon.
\end{aligned}$$

Letting finally $\varepsilon \downarrow 0$ shows that $\|u^*\| \geq (\mu - \delta)\|u\|$ and thus ends the proof of the lemma.

We conclude this section by showing that, when both X and P are finite-dimensional, our results imply the characterization of Lipschitzian full stability closely related to [31, Theorem 2.3] established in a more involved approach. Note to this end that the full stability characterization of Corollary 3.11 via the combined second-order subdifferential of f with respect to the *decision variable only*, valid in the general infinite-dimensional setting, is new even in finite dimensions.

Theorem 3.14 (pointwise characterization of Lipschitzian fully stable minimizers via the limiting coderivative of the subdifferential). *Let X, P be two finite-dimensional spaces. Suppose that BCQ* (3.7) *holds at $(\bar{x}, \bar{p}) \in \operatorname{dom} f$ and that f is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $\bar{x}^* \in \partial_x f(\bar{x}, \bar{p})$. Consider the following statements:*

(i) *The point $\bar{x}$ is a Lipschitzian fully stable local minimizer of problem $\mathcal{P}(\bar{x}^*, \bar{p})$ with a modulus pair $(\kappa, \ell) \in I\!R^2_>$.*

(ii) *Condition* (3.33) *is satisfied and there is some $\mu > 0$ such that*

$$\inf\Big\{\langle u^*, u\rangle \Big|\, (u^*, p^*) \in D^*\partial_x f(\bar{x}, \bar{p}, \bar{x}^*)(u)\Big\} \geq \mu\|u\|^2, \quad u \in X. \tag{3.45}$$

Then implication (**i**) $\Longrightarrow$ (**ii**) *holds with $\mu = \kappa^{-1}$ while implication* (**ii**) $\Longrightarrow$ (**i**) *is satisfied with any $\kappa > \mu^{-1}$. Furthermore, the validity of* **(i)** *with some modulus pair $(\kappa, \ell) \in I\!R^2_>$ is equivalent to the fulfillment of condition* (3.33) *together with the positive-definiteness condition*

$$\inf\Big\{\langle u^*, u\rangle \Big|\, (u^*, p^*) \in D^*\partial_x f(\bar{x}, \bar{p}, \bar{x}^*)(u)\Big\} > 0, \quad u \in X, u \neq 0. \tag{3.46}$$

Proof. Assuming **(i)** implies by Theorem 3.12 that both conditions (3.33) and (3.38) hold. By a limiting process, we arrive at (3.45) with $\mu = \kappa^{-1}$, which verifies **(ii)**.

To justify the converse implication **(ii)$\Longrightarrow$(i)**, we proceed similarly to the proof of **(ii)$\Longrightarrow$(i)** in Theorem 3.7 with some modifications. Since f parametrically continuously prox-regular at $(\bar{x},\bar{p},\bar{v})$, inequality (3.28) holds for some $r,\varepsilon>0$. Defining $g(x,p):=f(x,p)+\frac{s}{2}\|x-\bar{x}\|^2$ for $x\in X$, $p\in P$ with some fixed $s>r$, we have $\partial_x g(x,p)=\partial_x f(x,p)+s(x-\bar{x})$. Moreover, the quadratic growth condition (3.29) is satisfied for $g_p(x):=g(x,p)$. Note further that $\partial^\infty f(\bar{x},\bar{p})=\partial^\infty g(\bar{x},\bar{p})$ and that $D^*\partial_x g(\bar{x},\bar{p},\bar{v})(w)=D^*\partial_x f(\bar{x},\bar{p},\bar{v})(w)+(sw,0)$ by [38, Theorem 1.62(ii)]. Since BCQ and condition (3.33) hold for the function f, both these conditions hold at the same point for the function g as well. By Theorem 3.9 condition (3.33) ensures that $\bar{x}$ is a Lipschitzian fully stable local minimizer of problem $\mathcal{P}(\bar{x}^*,\bar{p})$ with replacing f by g.

It follows from [38, Theorem 1.62(ii)] that the inclusion $(u^*,p^*)\in D^*\partial_x g(\bar{x},\bar{p},\bar{x}^*)(u)$ yields $(u^*-su,p^*)\in D^*\partial_x f(\bar{x},\bar{p},\bar{v})(u)$. Furthermore, by (3.45) we have $\langle u^*-su,u\rangle\ge\mu\|u\|^2$, which implies that

$$\|u^*\|\cdot\|u\|\ge\langle u^*,u\rangle\ge(s+\mu)\|u\|^2.$$

By Lemma 3.13 above, for any $\lambda\in(0,s+\mu)$ we find some $\eta>0$ such that

$$\|u^*\|\ge(s+\mu-\lambda)\|u\|\ \text{whenever}\ u^*\in\widehat{D}^*\partial g_p(x,x^*)(u)\ \text{ with }\ (x,p,x^*)\in\operatorname{gph}\partial_x g\cap I\!B_\eta(\bar{x},\bar{p},\bar{x}^*).$$

Following the last part in the proof of Theorem 3.7 we find some $\alpha>0$ such that for any $x\in I\!B_\alpha(\bar{x})$ the following inequality holds

$$f_p(x)\ge f_p(u)+\langle u^*,x-u\rangle+\frac{\mu-\lambda}{2}\|x-u\|^2\ \text{ if }\ (u,p,u^*)\in\operatorname{gph}\partial_x f\cap I\!B_\alpha(\bar{x},\bar{p},\bar{x}^*).\qquad(3.47)$$

This together with Theorem 3.9 tells us that $\bar{x}$ is a Lipschitzian fully stable local minimizer of

$P(\bar{x}^*, \bar{p})$ with modulus pair $((\mu - \lambda)^{-1}, \ell)$ for some $\ell > 0$. This verifies implication **(ii)$\Longrightarrow$(i)**.

Next we prove the equivalence between **(i)** with *some* modulus pair $(\kappa, \ell) \in I\!R^2_>$ and the validity of (3.46) together with (3.33). Note that **(i)** readily yields both conditions (3.33) and (3.46) by implication **(i)$\Longrightarrow$(ii)** proved above. To justify the converse, observe first that the validity of (3.46) and (3.33) (or (3.37)) ensures the condition

$$(0, p^*) \in D^*\partial_x f(\bar{x}, \bar{p}, \bar{x}^*)(u) \Longrightarrow (p^*, u) = 0,$$

which shows that $D^*S(\bar{x}^*, \bar{p}, \bar{x})(0) = (0, 0)$ for the mapping S from (3.3). By the Mordukhovich criterion (2.6) this tells that S is Lipschitz-like around $(\bar{x}^*, \bar{p}, \bar{x})$ with some modulus $\ell > 0$. Moreover, arguing as in the proof of **(ii)$\Longrightarrow$(i)** above when $\mu = 0$ shows that for each $\lambda \in (0, \min\{(5\ell)^{-1}, s\})$ there is some $\alpha > 0$ such that condition (3.47) holds with $\mu = 0$. Define $h(x, p) := f(x, p) + \lambda\|x - \bar{x}\|^2$ with $\partial h(x, p) = \partial f(x, p) + 2\lambda(x - \bar{x})$. This together with condition (3.47) with $\mu = 0$ implies the existence of $\delta > 0$ so small that the quadratic growth condition

$$h(x, p) \geq h(u, p) + \langle v, x - u\rangle + \frac{\lambda}{2}\|x - u\|^2 \text{ if } x \in I\!B_\delta(\bar{x}), (u, p, v) \in \operatorname{gph} \partial_x h \cap I\!B_\delta(\bar{x}, \bar{p}, \bar{v}) \tag{3.48}$$

is satisfied for h. Observe further that for any $(u^*, p^*) \in D^*\partial_x h(\bar{x}, \bar{p}, \bar{x}^*)(u)$ we get from [38, Theorem 1.62(ii)] that $(u^* - 2\lambda u, q) \in D^*\partial_x f(\bar{x}, \bar{p}, \bar{x}^*)(u)$ whenever $u \in X$, which reads as $(-u, p^*) \in \widehat{D}^*S(\bar{x}^*, \bar{p}, \bar{x})(-u^* + 2\lambda u)$. Since the mapping S is Lipschitz-like around $(\bar{x}^*, \bar{x})$ with modulus $\ell > 0$, we deduce from [38, Theorem 1.44] that $\ell\|u^* - 2\lambda u\| \geq \|u\| + \|p^*\|$. This ensures the fulfillment of the inequalities

$$\ell\|u^*\| \geq \ell\|u^* - 2\lambda u\| - 2\ell\lambda\|u\| \geq \|u\| + \|p^*\| - 2\ell\lambda\|u\| \geq (1 - 2\ell\lambda)(\|u\| + \|p^*\|),$$

which in turn allow us to arrive at the estimate

$$\|u^*\| \geq \frac{1-2\ell\lambda}{\ell}\|u\| \ \text{ for all } \ (u^*,p^*) \in D^*\partial_x h(\bar{x},\bar{p},\bar{x}^*)(u).$$

Employing this inequality together with Lemma 3.13 gives us a number $\eta > 0$ such that

$$\|u^*\| \geq \frac{1-3\ell\lambda}{\ell}\|u\| \ \text{ whenever } \ u^* \in \widehat{D}^*\partial h_p(x,x^*)(u) \ \text{ and } \ (x,p,x^*) \in \operatorname{gph}\partial_x h \cap I\!B_\eta(\bar{x},\bar{p},\bar{x}^*).$$

Following the last part in the proof of Theorem 3.7 again gives us the existence of some $\beta > 0$ such that

$$h_p(x) \geq h_p(u)+\langle x^*, x-u\rangle+\frac{1-3\ell\lambda}{2\ell}\|x-u\|^2 \ \text{ for all } \ x \in I\!B_\beta(\bar{x}), \ (u,p,x^*) \in \operatorname{gph}\partial_x h \cap I\!B_\beta(\bar{x},\bar{p},\bar{x}^*).$$

Since $f(x,p) = h(x,p) - \lambda\|x-\bar{x}\|^2$ and $\partial_x f(x,p) = \partial_x h(x,p) - 2\lambda(x-\bar{x})$, this easily implies that

$$f(x,p) \geq f(u,p) + \langle u^*, x-u\rangle + \frac{1-5\ell\lambda}{2\ell}\|x-u\|^2 \ \text{ for all } \ x \in I\!B_\beta(\bar{x}), \ (u,p,u^*) \in \operatorname{gph}\partial_x f \cap W_2,$$

where $W_2 := J_\lambda^{-1}(I\!B_\beta(\bar{x},\bar{p},\bar{x}^*)$ and $J_\lambda(x,p,x^*) := (u,p,x^*+2\lambda(x-\bar{x}))$ for all $(x,p,x^*) \in X \times P \times X$. Applying finally Theorem 3.9 with taking into account the choice of $\lambda < (5\ell)^{-1}$ verifies that $\bar{x}$ is the Lipschitzian fully stable local minimizer of $\mathcal{P}(\bar{v},\bar{p})$, which completes the proof of the theorem. $\square$

The following consequence of Theorem 3.14 is useful for our applications in Section 6.

Corollary 3.15 (another form of the pointwise characterization of Lipschitzian full stability). *In the setting of Theorem* 3.14 *we have the equivalent statements:*

(i) *The point $\bar{x}$ is a Lipschitzian fully stable local minimizer of problem $\mathcal{P}(\bar{x}^*, \bar{p})$.*

(ii) *Condition* (3.33) *is satisfied together with the inequality*

$$\inf\Big\{\langle u^*, u\rangle \big|\ (u^*, p^*) \in D^*\partial_x f(\bar{x}, \bar{p}, \bar{x}^*)(u)\Big\} > 0 \ \ \text{for all} \ \ u \neq 0, \tag{3.49}$$

where we use the convention that $\inf \emptyset := \infty$.

Proof. It is proved in Theorem 3.14 that **(i)** implies the existence of some $\mu > 0$ for which we have condition (3.45) that immediately implies (3.49). Conversely, the validity of (3.49) readily yields (3.46). Together with (3.33) it gives **(i)** by Theorem 3.14 and thus completes the proof of this corollary. □

Chapter 4

Sensitivity Analysis of Parametric Variational Systems

4.1 Overview

In this chapter we consider the so-called *generalized equations* in term of a inclusion

$$0 \in F(x) + T(x), \tag{4.1}$$

where $F : I\!R^n \to I\!R^n$ is a single-valued mapping and $T : I\!R^n \rightrightarrows I\!R^n$ is a set-valued mapping. The crucial notion of generalized equation above was introduced by Robinson in [56, 57] with an additional assumption that T is a *monotone operator*. In the case that the set-valued mapping T disappears, this inclusion reduces to the standard equation "$F(x) = 0$". Furthermore, "equation" (4.1) also includes the classical *variational inequality* introduced by Stamphacchia [63]

$$0 \in F(x) + N_K(x), \quad \text{or equivalently,} \quad \langle F(x), u - x\rangle \geq 0 \quad \text{for all} \quad u \in K \tag{4.2}$$

when T in (4.1) is the convex normal cone to a convex set $K \subset I\!R^n$, which is monotone operator. Variational inequalities (4.2) are well defined on infinite-dimensional spaces and have been known as one of the most powerful tools in deriving the existence of solutions to *partial differential equations*; see further details and discussions in [26]. Recent remarkable applications of variational inequalities to optimization can be found in the monograph [18].

In the landmark paper [58] Robinson introduced the notion of *strong regularity* for variational inequalities (4.2), which concerns about the Lipschitz continuous single-valued localization for solution maps to the *linearization* of (4.2). This allows one to obtain the similar properties for solution maps to the following *parametric* generalized equations

$$0 \in F(x,p) + N_K(x), \tag{4.3}$$

where $F : I\!R^n \times I\!R^d \to I\!R^n$ and $K \subset I\!R^n$ is still a convex set. It is important to emphasize that full characterizations including Lipschitz-like property of the solution map to the linearization of (4.2) to strong stability are obtained by Dontchev and Rockafellar provided that K is a *polyhedral*. Their work indeed reveals the essential of using second-order subdifferentials via a so-called *critical face* condition to study strong regularity. Without the polyhedricity assumption on K, which (second-order) condition can characterize strong stability of variational inequality (4.2) is still unknown. In Section 4.4 we provide a new sufficient condition for this property in terms of *positive definiteness* and second-order subdifferentials of the indicator δ_K for the nonpolyhedral convex set K.

In many practical models, e.g., [12, 18, 32, 66, 67] the parameter p appears in both the function F and the convex normal cone N_K, which can be formulated by another generalized equation

$$0 \in F(x,p) + N_{K(p)}(x), \tag{4.4}$$

where $K(p)$ is a convex subset of $I\!R^n$ depending on the parameter p. It seems that Robinson's strong regularity is not useful anymore, since the linearization of (4.4) can not remove p out of this generalized equation. It is worth noting that the assumption on convexity of the set $K(p)$ for (4.4) is broken in many significant frameworks of optimization [5, 24, 25, 29, 30], e.g., in parametric constrained problems the set $K(p)$ is usually known as $\big\{x \in I\!R^n |\ g(x,p) \in \Theta\big\}$, which is not convex in general; see our Chapter 5 for further details. The convex normal cone in (4.4) has to be replaced by other *nonconvex* constructions. This motivates us to a study of (4.4) with the limiting normal cone (2.8). Indeed, we formulate a more general problem

$$0 \in F(x,p,q) + \partial_x f(x,p), \tag{4.5}$$

where $F : X \times P \times Q \to X$ and $f : X \times P \to \overline{I\!R}$, and where X is a Hilbert space while (P, d_1) and (Q, d_2) are two metric spaces. Note that problem (4.5) reduces to (4.4) when $f(x,p) := \delta_{K(p)}(x)$ and F does not depend on parameter q. Moreover, if the parameter p is ignored in F, our model (4.5) covers many ones in [30, 32, 38, 66, 67] and strictly relates to the so-called *hemivariational inequalities* introduced by Panagiotopoulos [51]. Following [38], from now on we label (4.5) as *parametric variational systems*. The main purpose of the chapter is to study the stability of the solution mapping $S : P \times Q \rightrightarrows X$ defined by

$$S(p,q) := \big\{x \in X |\ 0 \in F(x,p,q) + \partial_x f(x,p)\big\}. \tag{4.6}$$

A great source for Lipschitz-like properties of this mapping can be found in the monograph [38, Chapter 4]. Here we focus our study on the single-valued Hölder/Lipschitz continuity on the mapping S. Most of results in this chapter are new when reducing the parametric variational systems (4.5) to (4.4) or even (4.3); see our Section 4.4 for further discussions.

Otherwise stated, the natural standing assumptions in this chapter are:

Standing assumption: Let $(\bar{x}, \bar{p}, \bar{q}) \in X \times P \times Q$ satisfy $\bar{x} \in S(\bar{p}, \bar{q})$. We always assume that

(A1) X is a Hilbert space while P and Q are two metric spaces with metrics d_1 an d_2 respectively.

(A2) F is Lipschitz continuous around $(\bar{x}, \bar{p}, \bar{q})$ uniformly in x around x, i.e., there exist a neighborhood $U \times V \times W$ of $(\bar{x}, \bar{p}, \bar{q})$ and some constant L such that for all $(x_1, p_1, q_1), (x_2, p_2, q_2) \in U \times V \times W$ we have

$$\|F(x_1, p_1, q_1) - F(x_2, p_2, q_2)\| \leq L\big[\|x_1 - x_2\| + d_1(p_1, p_2) + d_2(q_1, q_2)\big]. \tag{4.7}$$

(A3) The function f satisfies the basic constraint qualification (3.7) at $(\bar{x}, \bar{p})$.

4.2 Parametric Variational Systems with Differentiability

In addition to the standing assumption in Section 4.1, we assume in this section that

(A4) F is differentiable with respect to x and the Jacobian matrix $\nabla_x F(x, p, q)$ is continuous at $(\bar{x}, \bar{p}, \bar{q})$.

4.2.1 Hölder continuity of parametric variational systems

Let us start with a definition used broadly in this section.

Definition 4.1 (Hölder/Lipschitz continuous single-valued localization). *Let $S : P \times Q \rightrightarrows X$ be a set-valued mapping with $(\bar{p}, \bar{q}, \bar{x}) \in \operatorname{gph} S$. We say that S has a Hölder continuous single-valued localization with an order pair $(\alpha, \beta) \in I\!R^2_>$ and a modulus pair $(\kappa, \ell) \in I\!R^2_>$ around $(\bar{p}, \bar{q})$ for $\bar{x}$ if there is a neighborhood $U \times V \times W \subset X \times P \times Q$ of $(\bar{x}, \bar{p}, \bar{q})$ such that the localization*

ϑ of S relative to $(V \times W \times U)$ is single-valued and that

$$\|\vartheta(p_1,q_1)-\vartheta(p_2,q_2)\| \le \kappa d_1(p_1,p_2)^{\alpha}+\ell d_2(q_1,q_2)^{\beta} \quad \text{for all} \quad (p_1,q_1),(p_2,q_2)\in V\times W. \tag{4.8}$$

If in addition S is a single-valued mapping, it is simply said that S is Hölder continuous with order pair (α,β) and modulus pair (κ,ℓ) around $(\bar p,\bar q)$.

When inequality (4.8) holds with $\alpha=\beta=1$, we say S has a Lipschitz continuous single-valued localization with the modulus pair (κ,ℓ) around $(\bar p,\bar q)$ for $\bar x$. If S is a single-valued mapping, we just say S is Lipschitz continuous with modulus pair (κ,ℓ) around $(\bar p,\bar q)$.

The major result of this section is to obtain Hölder continuity of the solution mapping S in (4.6).

Theorem 4.2 (Hölder continuity of solution maps). *Let $(\bar x,\bar p,\bar q)\in X\times P\times Q$ satisfy $\bar x\in S(\bar p,\bar q)$. Suppose that f is parametrically continuously prox-regular at $(\bar x,\bar p)$ for $\bar x^* := -F(\bar x,\bar p,\bar q)$. Assume further that there are some $\kappa,\delta>0$ such that for all $(x,p,x^*)\in \operatorname{gph}\partial_x f\cap I\!B_{\delta}(\bar x,\bar p,\bar x^*)$, we have*

$$\langle \nabla_x F(\bar x,\bar p,\bar q)u,u\rangle+\inf\Big\{\langle u^*,u\rangle\Big|\ u^*\in \widehat{D}^*\partial f_p(x,x^*)(u)\Big\}\ge \frac{1}{\kappa}\|u\|^2 \quad \text{for all} \quad u\in X. \tag{4.9}$$

Then the solution mapping S in (4.6) has a Hölder continuous single-valued localization with order pair $(\frac{1}{2},1)$ around $(\bar p,\bar q)$ for $\bar x$.

The proof of this theorem is based on several lemmas constructed below. The first lemma follows the key idea of Robinson in [58] in order to establish the relationship between the parametric variational system (4.5) to its linearization.

Lemma 4.3 *Let $\bar{x} \in X$ satisfy $\bar{x} \in S(\bar{p}, \bar{q})$. Define $G : X \times P \rightrightarrows X$ by $G(x^*, p)$ the solution mapping of the linearized system*

$$x^* \in F(\bar{x}, \bar{p}, \bar{q}) + A(x - \bar{x}) + \partial_x f(x, p) \quad \textit{with} \quad A := \nabla_x F(\bar{x}, \bar{p}, \bar{q}), \tag{4.10}$$

i.e., $G(x^*, p) := \{x \in X |\ x^* \in F(\bar{x}, \bar{p}, \bar{q}) + A(x - \bar{x}) + \partial_x f(x, p)\}$ for all $(x^*, p) \in X \times P$. If G has a Hölder continuous single-valued localization with order pair $(1, \frac{1}{2})$ around $(0, \bar{p})$ for $\bar{x}$, then the map S in (4.6) also admits a Hölder continuous single-valued localization with order pair $(\frac{1}{2}, 1)$ around $(\bar{x}^*, \bar{p})$ for $\bar{x}$.

Proof. Suppose that G has a Hölder continuous single-valued localization ϑ with the order $(1, \frac{1}{2})$ around $(0, \bar{p})$ for $\bar{x}$. Then we find neighborhood $U \times V \times U^*$ of $(\bar{x}, \bar{p}, 0)$ and some $\kappa, \ell > 0$ such that $\operatorname{gph} G \cap (U^* \times V \times U) = \operatorname{gph} \vartheta$ and that

$$\|\vartheta(x_1^*, p_1) - \vartheta(x_2^*, p_2)\| \leq \kappa \|x_1^* - x_2^*\| + \ell d_1(p_1, p_2)^{\frac{1}{2}} \quad \text{for all} \quad x_1^*, x_2^* \in U^*,\ p_1, p_2 \in V. \tag{4.11}$$

Let us define $r(x, p, q) := F(\bar{x}, \bar{p}, \bar{q}) + A(x - \bar{x}) - F(x, p, q) \quad \text{for all} \quad (x, p, q) \in X \times P \times Q$. Note that $x \in S(p, q)$ if and only if $x \in G(r(x, p, q), p)$. Due to assumption (A4) for any $\varepsilon \in (0, \kappa^{-1})$, we may find $\rho, \eta > 0$ with $I\!B_\rho(\bar{x}) \times I\!B_\eta(\bar{p}) \times I\!B_\eta(\bar{q}) \subset U \times V \times Q$ such that the remainder $r(x, p, q)$ belongs to U^* for all $(x, p, q) \in I\!B_\rho(\bar{x}) \times I\!B_\eta(\bar{p}) \times I\!B_\eta(\bar{q})$ and that

$$\begin{aligned} &\|F(x_1, p, q) - F(x_2, p, q) - A(x_1 - x_2)\| \leq \varepsilon \|x_1 - x_2\| \quad \text{and} \\ &\kappa \left[\|F(\bar{x}, \bar{p}, \bar{q}) - F(\bar{x}, p, q)\| + \ell d_1(p, \bar{p})^{\frac{1}{2}} \right] \leq (1 - \kappa\varepsilon)\rho \end{aligned} \tag{4.12}$$

whenever $(x_1, p, q), (x_2, p, q) \in I\!B_\rho(\bar{x}) \times I\!B_\eta(\bar{p}) \times I\!B_\eta(\bar{q})$. Thus the map $\Phi_{(p,q)}(x) := \vartheta(r(x, p, q), p)$ is well-defined on $I\!B_\rho(\bar{x})$ for $(p, q) \in I\!B_\eta(\bar{p}) \times I\!B_\eta(\bar{q})$ fixed.

Now take any $x_1, x_2 \in I\!B_\rho(\bar{x})$, it follows from (4.8) and (4.12) that

$$\begin{aligned} \|\Phi_{(p,q)}(x_1) - \Phi_{(p,q)}(x_2)\| &\leq \kappa \|r(x_1, p, q) - r(x_2, p, q)\| \\ &= \kappa \|F(x_1, p, q) - F(x_2, p, q) - A(x_1 - x_2)\| \leq \kappa\varepsilon \|x_1 - x_2\|, \end{aligned} \tag{4.13}$$

which means that $\Phi_{(p,q)}(\cdot)$ satisfies the *contraction* condition. Moreover, we get from (4.11) that

$$\begin{aligned} \|\Phi_{(p,q)}(\bar{x}) - \bar{x}\| &= \|\vartheta(r(\bar{x}, p, q), p) - \vartheta(0, \bar{p})\| \leq \kappa \left[\|r(\bar{x}, p, q), q)\| + \ell d_1(p, \bar{p})^{\frac{1}{2}} \right] \\ &\leq \kappa \left[\|F(\bar{x}, \bar{p}, \bar{q}) - F(\bar{x}, p, q)\| + \ell d_1(p - \bar{p})^{\frac{1}{2}} \right] \leq (1 - \kappa\varepsilon)\rho, \end{aligned}$$

which implies that for any $x \in I\!B_\rho(\bar{x})$

$$\|\Phi_{(p,q)}(x) - \bar{x}\| \leq \|\Phi_{(p,q)}(x) - \Phi_{(p,q)}(\bar{x})\| + \|\Phi_{(p,q)}(\bar{x}) - \bar{x}\| \leq \kappa\varepsilon \|x - \bar{x}\| + (1 - \kappa\varepsilon)\rho \leq \rho.$$

This together with (4.13) shows that there is a unique fixed point $x(p, q)$ of $\Phi_{(p,q)}$ due to the well-known contraction principle. Observe that $x(p, q) = \vartheta(r(x(p, q), p, q), q)$ is indeed a single-valued localization of S with respect to $I\!B_\eta(\bar{p}) \times I\!B_\eta(\bar{q}) \times I\!B_\rho(\bar{x})$. It suffices to check the Hölder continuity of $x(p, q)$. Pick any (p_1, q_1), $(p_2, q_2) \in I\!B_\eta(\bar{p}) \times I\!B_\eta(\bar{q})$, with $x_1 := x(p_1, q_1)$ and $x_2 := x(p_2, q_2)$ we obtain from (4.7), (4.11), and (4.13) that

$$\begin{aligned} \|x_1 - x_2\| &= \|\vartheta(r(x_1, p_1, q_1), p_1) - \vartheta(r(x_2, p_2, q_2), p_2)\| \\ &\leq \kappa \|r(x_1, p_1, q_1) - r(x_2, p_2, q_2)\| + \ell d_1(p_1, p_2)^{\frac{1}{2}} \\ &\leq \kappa \|r(x_1, p_1, q_1) - r(x_2, p_1, q_1)\| + \kappa \|r(x_2, p_1, q_1) - r(x_2, p_2, q_2)\| + \ell d_1(p_1, p_2)^{\frac{1}{2}} \\ &\leq \kappa\varepsilon \|x_1 - x_2\| + \kappa \|F(x_2, p_1, q_1) - F(x_2, p_2, q_2)\| + \ell d_1(p_1, p_2)^{\frac{1}{2}} \\ &\leq \kappa\varepsilon \|x_1 - x_2\| + \kappa L d_1(p_1, p_2) + \kappa L d_2(q_1, q_2) + \ell d_1(p_1, p_2)^{\frac{1}{2}} \\ &\leq \kappa\varepsilon \|x_1 - x_2\| + \left[\kappa L \sqrt{2\eta} + \ell\right] d_1(p_1, p_2)^{\frac{1}{2}} + \kappa L d_2(q_1, q_2), \end{aligned}$$

which readily yields that

$$\|x_1 - x_2\| \leq \frac{\kappa L\sqrt{2\eta} + \ell}{1 - \kappa\varepsilon} d_1(p_1, p_2)^{\frac{1}{2}} + \frac{\kappa L}{1 - \kappa\varepsilon} d_2(q_1, q_2).$$

This ensures the Hölder continuity of $x(\cdot,\cdot)$ with an order pair $(\frac{1}{2}, 1)$ and thus completes the proof of this lemma. $\square$

Lemma 4.4 *Suppose that* (4.9) *holds with some* $\kappa > 0$ *and that the map* G *in* Lemma 4.3 *has a Hölder continuous single-valued localization* ϑ *with order pair* $(1, \frac{1}{2})$ *around* $(0, \bar{p})$ *for* $\bar{x}$. *Then there is some* $\ell > 0$ *such that the localization* ϑ *is Hölder continuous with an order pair* $(1, \frac{1}{2})$ *and a modulus pair* (κ, ℓ) *around* $(0, \bar{p})$.

Proof. To justify, suppose that $\operatorname{gph}\vartheta = \operatorname{gph} G \cap (U^* \times V \times U)$, where $U^* \times V \times U \subset X \times P \times X$ is a neighborhood of $(0, \bar{p}, \bar{x})$ and that

$$\|\vartheta(x_1^*, p_1) - \vartheta(x_2^*, p_2)\| \leq \nu\|x_1^* - x_2^*\| + \ell d(q_1, q_2)^{\frac{1}{2}} \quad \text{for all} \quad x_1^*, x_2^* \in U^* \times P, \tag{4.14}$$

where ν, ℓ are some positive constants. By shrinking U^*, V, U if necessary, due to assumption (A4) we may assume that $U \subset I\!B_\delta(\bar{x})$ and $x^* - A(x - \bar{x}) \in I\!B_\delta(\bar{x}^*)$ with $\bar{x}^* := -F(\bar{x}, \bar{p}, \bar{q})$ for all $x^* \in U^*$ and $x \in U$, where δ is found in (4.9). Fix any $p \in V$ and define $\vartheta_p(\cdot) := \vartheta(\cdot, p)$. For any $(x^*, x) \in \operatorname{gph}\vartheta_p$ and $u \in \widehat{D}^*\vartheta_p(x^*, x)(u^*)$ we get from (4.10) and

$$-u^* \in \widehat{D}^*\vartheta_p^{-1}(x, x^*)(-u) = \widehat{D}^* G_p^{-1}(x, x^*)(-u) = -A^*u + \widehat{D}^*\partial f_p(x, x^* - A(x - \bar{x}))(-u).$$

Due to the choice of U, V, U^*, note that $(x, x^* - A(x - \bar{x})) \in I\!B_\eta(\bar{x}, \bar{x}^*)$. It follows from (4.9) and the above inclusion that

$$\|u^*\| \cdot \|u\| \geq \langle -u^*, -u\rangle \geq \frac{1}{\kappa}\| - u\|^2 = \frac{1}{\kappa}\|u\|^2,$$

which implies $\kappa\|u^*\| \geq \|u\|$ for $u^* \in \widehat{D}^*\vartheta_p(x^*, x)(u)$. By Lemma 2.7 and inequality (4.14), we may find a new neighborhood $U_1 \times V_1 \times U_1^* \subset U \times V \times U^*$ of $(\bar{x}, \bar{p}, 0)$ such that $\vartheta(U_1^* \times V_1) \subset U_1$ and that

$$\|\vartheta(x_1^*, p) - \vartheta(x_2^*, p)\| \leq \kappa\|x_1^* - x_2^*\| \quad \text{for all} \quad x_1^*, x_2^* \in U_1^*,\ p \in V_1.$$

Hence for any $(x_1^*, p_1), (x_2^*, p_2) \in U_1^* \times V_1$ we deduce from the latter and (4.14) that

$$\begin{aligned}\|\vartheta(x_1^*, p_1) - \vartheta(x_2^*, p_2)\| &\leq \|\vartheta(x_1^*, p_1) - \vartheta(x_2^*, p_1)\| + \|\vartheta(x_2^*, p_1) - \vartheta(x_2^*, p_2)\| \\ &\leq \kappa\|x_1^* - x_2^*\| + \ell d_1(p_1, p_2)^{\frac{1}{2}},\end{aligned}$$

which completes the proof of the lemma. □

Lemma 4.5 *Suppose that condition* (4.9) *is satisfied with some* $\kappa > 0$. *Define* $A_t := \frac{1}{2}(A + A^*) + tB$ *with* $B := A - A^*$ *and*

$$G_t(x^*, p) := \big\{x \in I\!R^n \,|\, x^* \in F(\bar{x}, \bar{p}, \bar{q}) + A_t(x - \bar{x}) + \partial_x f(x, p)\big\}, \quad t \in [0, 1]. \tag{4.15}$$

Suppose further that there exists some $\tau \in [0, 1]$ *such that* G_τ *has a Hölder continuous single-valued localization with order pair* $(1, \frac{1}{2})$ *around* $(0, \bar{p})$ *for* $\bar{x}$. *Then this localization is Hölder continuous with the same order pair and a modulus pair* (κ, ℓ) *for some* $\ell > 0$. *Furthermore,* G_t *also has a Hölder continuous single-valued localization with order pair* $(1, \frac{1}{2})$ *and modulus pair* $(\kappa, 2\ell)$ *around* $(0, \bar{p})$ *for* $\bar{x}$ *whenever* $t \in [\tau, \tau + \frac{1}{2\kappa\|B\|})$ *with the convention that* $1/0 = \infty$.

Proof. Observe first that

$$\langle A_\tau, u\rangle = \frac{1}{2}\langle (A+A^*)u, u\rangle + \tau\langle Bu, u\rangle = \langle Au, u\rangle,$$

since $\langle Au, u\rangle = \langle A^*u, u\rangle$ for all $u \in X$. We deduce from (4.9) that

$$\langle A_\tau u, u\rangle + \inf\Big\{\langle u^*, u\rangle \Big|\ w^* \in D^*\partial f_p(x, x^*)(u)\Big\} \geq \frac{1}{\kappa}\|u\|^2 \tag{4.16}$$

for all $(x, p, x^*) \in \operatorname{gph}\partial_x f \cap I\!B_\delta(\bar{x}, \bar{p}, \bar{x}^*)$. Suppose that G_τ has a Hölder continuous single-valued localization ϑ_τ with order pair $(1, \frac{1}{2})$ around $(0, \bar{p})$ for $\bar{x}$. Applying Lemma 4.4 to G_τ allows us to find some $\ell > 0$ and a neighborhood $U \times V \times U^* \subset X \times P \times X$ of $(\bar{x}, \bar{p}, 0)$ such that $\vartheta_\tau : U^* \times V \to U$ is Hölder continuous with order pair $(1, \frac{1}{2})$ and modulus pair (κ, ℓ).

Pick any $t \in [\tau, \tau + \frac{1}{2\kappa\|B\|})$ and choose some $r > 0$ sufficiently small such that $I\!B_{r\kappa}(\bar{x}) \times I\!B_r(0) \times I\!B_{\frac{r^2\kappa^2}{\ell^2}}(\bar{p}) \subset U \times U^* \times V$. It is obvious that $\varepsilon := r\big(1 - \kappa(t-\tau)\|B\|\big) \in (0, r]$. Furthermore, for any fixed $x^* \in I\!B_{\frac{\varepsilon}{2}}(0) \subset U^*$ and $p \in I\!B_{\frac{\varepsilon^2\kappa^2}{4\ell^2}}(\bar{p})$ we denote

$$T(x) := \vartheta_\tau(x^* - (t-\tau)B(x - \bar{x}), p) \quad \text{for all} \quad x \in I\!B_{r\kappa}(\vartheta_\tau(0, p)).$$

This mapping is well-defined. Indeed, for all $x \in I\!B_{r\kappa}(\bar{x}) \subset U$ we have

$$\|x^* - (t-\tau)B(x - \bar{x})\| \leq \|x^*\| + (t-\tau)\|B\| \cdot \|x - \bar{x}\| < \varepsilon + (t-\tau)\|B\| r\kappa = r.$$

Moreover, we obtain from the Hölder continuity of ϑ_τ that

$$\|T(\bar{x}) - \bar{x}\| = \|\vartheta_\tau(x^*, p) - \vartheta_\tau(0, \bar{p})\| \leq \kappa\|x^*\| + \ell d_1(p, \bar{p})^{\frac{1}{2}} \leq r\kappa\big[1 - (t-\tau)\kappa\|B\|\big]. \tag{4.17}$$

Note further that the mapping $T : I\!B_{r\kappa}(\bar{x}) \to I\!B_{r\kappa}(\bar{x})$ is Lipschitz continuous with modulus $\kappa(t-\tau)\|B\|$. To justify this claim, take any $x_1, x_2 \in I\!B_{r\kappa}(\bar{x})$, we deduce from the Hölder continuity of ϑ_τ that

$$\begin{aligned} \|T(x_1) - T(x_2)\| &= \|\vartheta_\tau(x^* - (t-\tau)B(x_1 - \bar{x}), p) - \vartheta_\tau(x^* - (t-\tau)B(x_2 - \bar{x}), p)\| \\ &\le \kappa(t-\tau)\|Bx_1 - Bx_2\| \le \kappa(t-\tau)\|B\| \cdot \|x_1 - x_2\|. \end{aligned} \tag{4.18}$$

For any $x \in I\!B_{r\kappa}(\bar{x})$, combining (4.17) and (4.18) gives us that

$$\|T(x) - \bar{x}\| \le \|T(x) - T(\bar{x})\| + \|T(\bar{x}) - \bar{x}\| \le \kappa(t-\tau)\|B\| r\kappa + r\kappa\big[1 - (t-\tau)\kappa\|B\|\big] = r\kappa.$$

Applying the contraction mapping principle allows us to find a unique fixed point $u \in I\!B_{r\kappa}(\bar{x})$ of T, which means that $u = \vartheta_\tau(x^* - (t-\tau)B(u - \bar{x}), p)$, or equivalently, $u \in G_t(x^*, p) \cap I\!B_{r\kappa}(\bar{x})$. Thus there is a single-valued localization ϑ_t of G_t with respect to $I\!B_{\frac{\varepsilon}{2}}(0) \times I\!B_{\frac{\varepsilon^2\kappa^2}{4\ell^2}}(\bar{p}) \times I\!B_{r\kappa}(\bar{x})$. For any $(x_1^*, p_1), (x_2^*, p_2) \in I\!B_{\frac{\varepsilon}{2}}(\bar{z}) \times I\!B_{\frac{\varepsilon^2\kappa^2}{4\ell^2}}(\bar{q})$ we have

$$\begin{aligned} &\|\vartheta_t(x_1^*, p_1) - \vartheta_t(x_2^*, p_2)\| = \|T(\vartheta_t(x_1^*, p_1)) - T(\vartheta_t(x_2^*, p_2))\| \\ &= \|\vartheta_\tau(x_1^* - (t-\tau)B(\vartheta_t(x_1^*, p_1) - \bar{x}), p_1) - \vartheta_\tau(x_1^* - (t-\tau)B(\vartheta_t(x_1^*, p_2) - \bar{x}), p_2)\| \\ &\le \kappa\|x_1^* - x_2^*\| + \kappa(t-\tau)\|B\| \cdot \|\vartheta_t(x_1^*, p_1) - \vartheta_t(x_2^*, p_2)\| + \ell d_1(p_1, p_2)^{\frac{1}{2}}, \end{aligned}$$

which gives us that

$$\begin{aligned} \|\vartheta_t(x_1^*, p_1) - \vartheta_t(x_2^*, p_2)\| &\le \frac{\kappa}{1 - \kappa(t-\tau)\|B\|}\|x_1^* - x_2^*\| + \frac{\ell}{1 - \kappa(t-\tau)\|B\|} d_1(p_1, p_2)^{\frac{1}{2}} \\ &\le 2\kappa\|x_1^* - x_2^*\| + 2\ell d_1(p_1, p_2)^{\frac{1}{2}}, \end{aligned} \tag{4.19}$$

where the last inequality holds due to the choice of t that $\frac{1}{2} < 1 - \kappa(t-\tau)\|B\| \le 1$.

Moreover, similarly to (4.16) we get from (4.9) that

$$\langle A_t u, u\rangle + \inf\Big\{\langle u^*, u\rangle \Big|\ u^* \in D^*\partial f_p(x, x^*)(u)\Big\} \ge \frac{1}{\kappa}\|u\|^2$$

for all $(x, p, x^*) \in \operatorname{gph} \partial_x f \cap I\!B_\delta(\bar{x}, \bar{p}, 0)$. This together Lemma 4.4 and Hölder condition (4.19) gives us that the localization ϑ_t of G_t is Hölder continuous with order pair $(1, \frac{1}{2})$ and modulus pair $(\kappa, 2\ell)$. The proof of the lemma is complete. $\square$

Proof of Theorem 4.2. By using the notation A, A_t and G_t as in the statement of Lemma 4.5, we first observe that $A_0 = \frac{1}{2}(A + A^*)$ and that $F(\bar{x}, \bar{p}, \bar{q}) + A_0(x - \bar{x}) + \partial_x f(x, p) = \partial_x h(x, p)$ with

$$h(x, p) := \langle F(\bar{x}, \bar{p}, \bar{q}), (x - \bar{x})\rangle + \langle A(x - \bar{x}), x - \bar{x}\rangle + f(x, p) \quad \text{for all} \quad (x, p) \in X \times P.$$

Since f is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $\bar{x}^* := -F(\bar{x}, \bar{p}, \bar{q})$, h is also parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for 0. Moreover, by Lemma 2.5 we have $\widehat{D}^*\partial h_p(x^*, x^*)(u) = A_0 u + \widehat{D}^*\partial f_p(x, \hat{x}^*)(w)$ with $\hat{x}^* := x^* - F(\bar{x}, \bar{p}, \bar{q}) - A_0(x - \bar{x})$ for all $(x, p, x^*) \in \operatorname{gph} \partial_x h$. There is some $\eta > 0$ so small that $(x, p, \hat{x}^*) \in \operatorname{gph} \partial_x f \cap I\!B_\delta(\bar{x}, \bar{p}, \bar{x}^*)$ whenever $(x, p, x^*) \in \operatorname{gph} \partial_x h \cap I\!B_\eta(\bar{x}, \bar{p}, 0)$, where δ is found in (4.9). This together with (4.9) implies that

$$\begin{aligned}
\inf\Big\{\langle u^*, u\rangle \Big|\ u^* \in \widehat{D}^*\partial_x h_p(x, x^*)(u)\Big\} \quad &= \langle A_0 u, u\rangle + \inf\Big\{\langle u^*, u\rangle \Big|\ u^* \in \widehat{D}^*\partial_x f_p(x, \hat{x}^*)(u)\Big\} \\
&= \langle A u, u\rangle + \inf\Big\{\langle u^*, u\rangle \Big|\ u^* \in \widehat{D}^*\partial_x f_p(x, \hat{x}^*)(u)\Big\} \\
&\ge \frac{1}{\kappa}\|u\|^2.
\end{aligned}$$

By Theorem 3.5 and Theorem 3.7, we get from the latter that the map

$$(x^*, p) \mapsto \{x \in X | \, x^* \in \partial_x h(x, p)\},$$

which is G_0 admits a Hölder continuous single-valued localization with order pair $(1, \frac{1}{2})$ and modulus pair (κ, ℓ) for some $\ell > 0$ around $(0, \bar{p})$ for $\bar{x}$. Applying Lemma 4.5 a finite number of times gives us that $G_1 = G$ has a Hölder continuous single-valued localization with order pair $(1, \frac{1}{2})$ and modulus pair $(\kappa, 2^n \ell)$ for some $n \in I\!N$ around $(0, \bar{p})$ for $\bar{x}$. This together with Lemma 4.3 completes the proof of the theorem. □

4.2.2 Lipschitz continuity of parametric generalized equations

Recall that Hölderian full stability becomes its Lipschitzian counterpart when condition (3.33) is satisfied. The following result gives a similarity: the Hölder continuity in Theorem 4.2 turns into the Lipschitzian one if condition (3.33) is fulfill.

Theorem 4.6 (Lipschitz continuity of parametric generalized equations). *Let $\bar{x} \in X$ satisfy $\bar{x} \in S(\bar{p}, \bar{q})$. Assume that f is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $\bar{x}^* := -F(\bar{x}, \bar{p}, \bar{q})$. If both conditions* (3.33) *and* (4.9) *hold with some $\kappa > 0$, then the solution mapping S in* (4.6) *admits a Lipschitz continuous single-valued localization around $(\bar{p}, \bar{q})$ for $\bar{x}$.*

Similarly to the proof of Theorem 4.2, we prove this theorem by constructing several lemmas as follows.

Lemma 4.7 *Let $(\bar{x}, \bar{p}, \bar{q}) \in X \times P \times Q$ satisfy $\bar{x} \in S(\bar{p}, \bar{q})$ and let $G : I\!R^n \times I\!R^d \rightrightarrows I\!R^n$ be defined as in* Lemma 4.3. *If G admits a Lipschitz continuous single-valued localization at $(0, \bar{p})$ for $\bar{x}$, then the solution map S in* (4.6) *also has a Lipschitz continuous single-valued localization at $(\bar{p}, \bar{q})$ for $\bar{x}$.*

Proof. The proof of this lemma is quite similar to that of Lemma 4.3. Indeed, this is expected due to [58, Theorem 2.1]. We omit the details. □

The next lemma follows the spirit of Lemma 4.4 for Lipschitzian stability.

Lemma 4.8 *Let $(\bar{x}, \bar{p}, \bar{q}) \in X \times P \times Q$ satisfy $\bar{x} \in S(\bar{p}, \bar{q})$. Suppose that both conditions* (3.33) *and* (4.9) *hold with some $\kappa > 0$ and that the map G in* Lemma 4.4 *has a Lipschitz continuous single-valued localization ϑ at $(0, \bar{p})$ for $\bar{x}$. Then ϑ is Lipschitz continuous around $(0, \bar{p})$ with a modulus pair (κ, ℓ) for some $\ell > 0$.*

Proof. Suppose that ϑ are Lipschitz around $(\bar{z}, \bar{q})$. Hence there are $\mu, \ell > 0$ such that

$$\|\vartheta(x_1^*, p_1) - \vartheta(x_2^*, p_2)\| \le \ell\big[\|x_1^* - x_2^*\| + d_1(p_1, p_2)\big] \quad \text{for all} \quad (x_1^*, p_1), (x_2^*, p_2) \in I\!B_\mu(0, \bar{p}). \tag{4.20}$$

Moreover, it follows from Lemma 4.4 that the mapping ϑ is Hölder continuous around $(0, \bar{q})$ with order pair $(1, \frac{1}{2})$ and the modulus pair (κ, ℓ_1) for some $\ell_1 > 0$. Without loss of generality we assume that

$$\|\vartheta(x_1^*, p_1) - \vartheta(x_2^*, p_2)\| \le \kappa\|x_1^* - x_2^*\| + \ell_1 d_1(p_1, p_2)^{\frac{1}{2}} \quad \text{for all} \quad (x_1^*, p_1), (x_2^*, p_2) \in I\!B_\mu(\bar{z}, \bar{q}).$$

This together with (4.20) gives us that for any $(z_1, q_1), (z_2, q_2) \in I\!B_\mu(\bar{z}, \bar{q})$

$$\begin{aligned} \|\vartheta(x_1^*, p_1) - \vartheta(x_2^*, p_2)\| &\le \|\vartheta(x_1^*, p_1) - \vartheta(x_2^*, p_1)\| + \|\vartheta(x_2^*, p_1) - \vartheta(x_2^*, p_2)\| \\ &\le \kappa\|x_1^* - x_2^*\| + \ell d_1(p_1, p_2), \end{aligned}$$

which justifies that ϑ is Lipschitz continuous around $(0, \bar{q})$ with modulus pair (κ, ℓ). The proof is complete. □

The following lemma is a counterpart of Lemma 4.5 for Lipschitzian stability.

Lemma 4.9 *Let* $(\bar{x}, \bar{p}, \bar{q}) \in X \times P \times Q$ *satisfy* $\bar{x} \in S(\bar{p}, \bar{q})$*. Suppose that both conditions* (3.33) *and* (4.9) *are satisfied with some* $\kappa > 0$*. Suppose further that* G_τ *defined in* (4.15) *admits a Lipschitz continuous single-valued localization at* $(0, \bar{p})$ *for* $\bar{x}$*. Then this localization is Lipschitz continuous with a modulus pair* (κ, ℓ) *for some* $\ell > 0$*. Furthermore,* G_t *in* (4.15) *also has a Lipschitz continuous single-valued localization with modulus pair* $(\kappa, 2\ell)$ *at* $(0, \bar{p})$ *for* $\bar{x}$ *whenever* $t \in [\tau, \tau + \frac{1}{2\kappa\|B\|})$ *with the convention that* $1/0 = \infty$*.*

Proof. By employing Lemma 4.8 instead of Lemma 4.4, the proof of this lemma is very similar to the one of Lemma 4.5. We omit the details. □

Proof of Theorem 4.6. Recall the function

$$h(x, p) := \langle F(\bar{x}, \bar{p}, \bar{q}), x - \bar{x} \rangle + \frac{1}{2} \langle A(x - \bar{x}), x - \bar{x} \rangle + f(x, p)$$

used in the proof of Theorem 4.2. By applying Theorem 3.9 and Corollary 3.11 to h, we obtain that G_0 in (4.15) admits a Lipschitz continuous single-valued localization around $(0, \bar{q})$ for $\bar{x}$ with a modulus pair (κ, ℓ) for some $\ell > 0$. Employing Lemma 4.9 for a finite consecutive steps, we derive that G_1 has a Lipschitz continuous single-valued localization around $(0, \bar{p})$ for $\bar{x}$ with modulus pair $(\kappa, 2^n \ell)$ for some $n \in I\!N$. Combining this with Lemma 4.7 ensures that the solution map S in (4.6) admits a Lipschitz continuous single-valued localization around $(\bar{p}, \bar{q})$ for $\bar{x}$. The proof is complete. □

When X, P, Q are finite-dimensional spaces, we derive a point-based sufficient condition to the Lipschitz stability of a single-valued localization of the mapping S in (4.6).

Corollary 4.10 *Let* X, P, Q *be finite-dimensional spaces and let* $\bar{x} \in X$ *satisfy* $\bar{x} \in S(\bar{p}, \bar{q})$*. Assume that the function* f *is parametrically continuously prox-regular at* $(\bar{x}, \bar{p})$ *for* $\bar{x}^* :=$

$-F(\bar{x},\bar{p},\bar{q})$. *Suppose further that condition* (3.37) *and the following inequality*

$$\langle \nabla_x F(\bar{x},\bar{p},\bar{q})u,u\rangle + \inf\Big\{\langle u^*,u\rangle \big|\ (u^*,p^*)\in D^*\partial_x f(\bar{x},\bar{p},\bar{x}^*)(u)\Big\} > 0 \ \ \text{if} \ \ u\in X\setminus\{0\}, \quad (4.21)$$

hold. Then the solution mapping S *in* (4.6) *admits a Lipschitz continuous single-valued localization at* $(\bar{p},\bar{q})$ *for* $\bar{x}$.

Proof. To justify, it suffices to show that inequality (4.21) ensures (4.9). Indeed, we use again the function h in the proof of Theorem 4.6 above

$$h(x,p) := \langle F(\bar{x},\bar{p},\bar{q}), x-\bar{x}\rangle + \frac{1}{2}\langle A(x-\bar{x}),x-\bar{x}\rangle + f(x,p) \quad \text{for all} \quad (x,p)\in X\times P.$$

Observe that $\partial_x h(x,p) = F(\bar{x},\bar{p},\bar{q}) + \frac{1}{2}(A+A^*)(x-\bar{x}) + \partial_x f(x,p)$. Note further from Lemma 2.5 that

$$D^*\partial_x h(\bar{x},\bar{p},0)(u) = \frac{1}{2}(A+A^*)u + D^*\partial_x f(\bar{x},\bar{p},\bar{x}^*)(u),$$

which clearly implies that the validity of (3.37) and (4.21) is equivalent to the following

$$\begin{aligned} &\text{(i)} \quad (0,p^*)\in D^*\partial_x h(\bar{x},\bar{p},\bar{x}^*)(0) \Longrightarrow p^*=0,\\ &\text{(ii)} \quad \inf\Big\{\langle u^*,u\rangle |\ u^*\in D^*\partial_x h(\bar{x},\bar{p},0)(u)\Big\} > 0 \end{aligned}$$

for any $u\in X\setminus 0$. Thanks to Theorem 3.14 and Corollary 3.11 that these two conditions ensure the existence of some $\kappa,\delta>0$ such that

$$\inf\Big\{\langle u^*,u\rangle |\ u^*\in \widehat{D}^*\partial h_p(x,x^*)(u)\Big\} \ge \frac{1}{\kappa}\|u\|^2 \quad \text{for all} \quad (x,p,x^*)\in \operatorname{gph}\partial_x h\cap I\!B_\delta(\bar{x},\bar{p},0).$$

By Lemma 2.5 the latter easily verifies (4.9) and thus completes the proof of the lemma. $\square$

4.3 Parametric Variational Systems with Monotonicity

In this section we drop off the assumption (A4) on differentiability of F in Section 4.2.

4.3.1 Hölder continuity

Define the following the following mapping

$$P_\lambda(x^*,p) := \big\{x \in X \,|\, x^* \in \lambda\partial_x f(x,p) + x\big\}. \tag{4.22}$$

The key tool used in this part is the next proposition.

Proposition 4.11 *Let $(\bar{x},\bar{p},\bar{q}) \in X$ satisfy $\bar{x} \in S(\bar{p},\bar{q})$. Assume that f is parametrically continuously prox-regular at $(\bar{x},\bar{p})$ for $\bar{x}^* := -F(\bar{x},\bar{p},\bar{q})$ with respect to $r > 0$ in (2.27). Then for any $\lambda \in (0,r^{-1})$, the mapping P_λ admits a Hölder continuous single-valued localization Π_λ around $(\lambda\bar{x}^* + \bar{x},\bar{p})$ for $\bar{x}$ with order pair $(1,\frac{1}{2})$ and modulus pair $\big((1-r\lambda)^{-1},\ell\big)$ for some $\ell > 0$.*

Proof. Fix $\lambda \in (0,r^{-1})$ and define $k(x,p) := \lambda f(x,p) + \frac{1}{2}\|x\|^2$ for all $(x,p) \in X \times P$. Note from Lemma 2.5 that $\partial_x k(x,p) = \lambda\partial_x f(x,p) + x$ for all $(x,p) \in X \times P$. Due to the assumption of parametric continuous prox-regularity of f at $(\bar{x},\bar{p})$ for $\bar{x}^*$ with respect to $r > 0$ in (2.27), we find some neighborhood $U \times V \times U^*$ of $(\bar{x},\bar{p},\bar{x}^*)$ such that condition (2.27) is satisfied. Define $W := \lambda U^* + U$ as a neighborhood of $\hat{x}^* = \lambda\bar{x}^* + \bar{x}$. For any $(u,p,u^*) \in \operatorname{gph}\partial_x k \cap (U \times V \times W)$, we may assume from (2.27) that

$$f(x,p) \ge f(u,p) + \langle \lambda^{-1}(u^* - u), x - u\rangle - \frac{r}{2}\|x-u\|^2 \quad \text{for all} \quad x \in U,$$

which implies that

$$
\begin{aligned}
k(x,p) &= \lambda f(x,p) + \frac{1}{2}\|x\|^2 \geq \lambda f(u,p) + \langle u^* - u, x-u\rangle - \frac{r\lambda}{2}\|x-u\|^2 + \frac{1}{2}\|x\|^2 \\
&= k(u,p) - \frac{1}{2}\|u\|^2 + \langle u^*, x-u\rangle - \langle u, x-u\rangle - \frac{r\lambda}{2}\|x-u\|^2 + \frac{1}{2}\|x\|^2 \\
&= k(u,p) + \langle u^*, x-u\rangle + \frac{1-r\lambda}{2}\|x-u\|^2.
\end{aligned} \tag{4.23}
$$

This tells us that the function k satisfies the uniform second-order growth condition at $(\bar{x},\bar{p},\bar{x}^*)$ in (3.10). Moreover, the basic constraint qualification (3.7) holds for the function k at $(\bar{x},\bar{p})$, i.e., the mapping $p \mapsto \operatorname{epi} k(\cdot,p)$ is Lipschitz-like around $(\bar{p},(\bar{x},k(\bar{x},\bar{p}))$. The proof for this fact is similar to the one after (3.30). Employing Theorem 3.5 to (4.23) shows us that the mapping P_λ admits a Hölder continuous single-valued localization at $(\hat{x}^*,\bar{p})$ for $\bar{x}$ with order pair $(1,\frac{1}{2})$ and modulus pair $\big((1-r\lambda)^{-1},\ell\big)$ for some $\ell > 0$. This completes the proof of the proposition. $\square$

The following result generalizes [66, Theorem 2.1], which obtains Hölder continuity of solutions to a specific model of (4.5) with $f(x,p) = \delta_{K(p)}(x)$ and a closed *convex* set-valued mapping $K : P \rightrightarrows X$. In the latter case note that the assumption that K is Lipschitz-like around $(\bar{p},\bar{x})$ in [66, Definition 1.1] turns into BCQ of the function f at $(\bar{x},\bar{p})$ in (3.7). Here we derive a similar result for the parametric variational system (4.5) without any assumption on convexity.

Theorem 4.12 *Let $(\bar{x},\bar{p},\bar{q}) \in X$ satisfy $\bar{x} \in S(\bar{p},\bar{q})$. Suppose that f is parametrically continuously prox-regular at $(\bar{x},\bar{p})$ for $\bar{x}^* := -F(\bar{x},\bar{p},\bar{q})$ with respect to $r > 0$ in* (2.27)*. Suppose further that there exist $\kappa > r$ and a neighborhood $U \times V \times W$ of $(\bar{x},\bar{p},\bar{q})$ such that for any $x_1, x_2 \in U$ and $(p,q) \in V \times W$ we have*

$$
\langle F(x_1,p,q) - F(x_2,p,q), x_1 - x_2\rangle \geq \kappa\|x_1 - x_2\|^2. \tag{4.24}
$$

Then the solution mapping S in (4.6) *admits a Hölder continuous single-valued localization at* $(\bar{p},\bar{q})$ *for* $\bar{x}$ *with order pair* $(\frac{1}{2},1)$.

Proof. Proposition 4.11 allows us to find a Hölder continuous single-valued localization Π_λ of the mapping P_λ at $(\lambda\bar{x}^*+\bar{x},\bar{p})$ for $\bar{x}$ with order pair $(1,\frac{1}{2})$ and modulus pair $\big((1-r\lambda)^{-1},\ell\big)$ for some $\ell>0$. Thus we may find a neighborhood $(U_1\times V_1\times U_1^*)\subset U\times V\times X$ of $(\bar{x},\bar{p},\hat{x}^*)$ with $\hat{x}^*:=\lambda\bar{x}^*+\bar{x}$ such that $\operatorname{gph}\Pi_\lambda=\operatorname{gph}P_\lambda\cap(U_1^*\times V_1\times U_1)$ and that

$$\|\Pi_\lambda(x_1^*,p_1)-\Pi_\lambda(x_2^*,p_2)\|\le\frac{1}{1-r\lambda}\|x_1^*-x_2^*\|+\ell\|p_1-p_2\|^{\frac{1}{2}} \tag{4.25}$$

for all $(x_1^*,p_1),(x_2^*,p_2)\in U_1^*\times V_1$. Moreover, it is worth noting that $x\in S(p,q)$ if and only if

$$x-\lambda F(x,p,q)\in\lambda\partial_x f(x,p)+x,$$

which equivalently means that $x\in P_\lambda(x-\lambda F(x,p,q),p)$. Due to the Lipschitz continuity assumption (A2), we find a neighborhood $(U_2,V_2,W_2)\subset U\times V\times W$ of $(\bar{x},\bar{p},\bar{q})$ such that $(x-\lambda F(x,p,q),p)\in U_1^*\times V_1$ for all $(x,p,q)\in U_2\times V_2\times W_2$. Thus $x\in S(p,q)$ if and only if $x=\Pi_\lambda(x-\lambda F(x,p,q),p):=H_{(p,q)}(x)$ for $(x,p,q)\in U_2\times V_2\times W_2$.

Fix $(p,q)\in V_2\times W_2$, we claim that $H_{(p,q)}$ satisfies the contraction condition for some $\lambda\in(0,r^{-1})$. Indeed, for any $x_1,x_2\in U_2$ we obtain from (4.25) that

$$\begin{aligned}
&\|H_{(p,q)}(x_1)-H_{(p,q)}(x_2)\|^2=\|\Pi_\lambda(x_1-\lambda F(x_1,p,q),p)-\Pi_\lambda(x_2-\lambda F(x_2,p,q),p)\|^2\\
&\le\frac{1}{(1-r\lambda)^2}\|(x_1-x_2)-\lambda(F(x_1,p,q)-F(x_2,p,q))\|^2\\
&=\frac{1}{(1-r\lambda)^2}\left[\|x_1-x_2\|^2-2\lambda\langle F(x_1,p,q)-F(x_2,p,q),x_1-x_2\rangle+\lambda^2\|F(x_1,p,q)-F(x_2,p,q)\|^2\right]\\
&\le\frac{1}{(1-r\lambda)^2}\left[1-2\lambda\kappa+\lambda^2L^2\right]\|x_1-x_2\|^2=\left[1-\frac{\lambda\big(2(k-r)-\lambda(L^2-r^2)\big)}{(1-r\lambda)^2}\right]\|x_1-x_2\|^2.
\end{aligned}$$

Since $\kappa > r$, we can find some $\lambda \in (0, r^{-1})$ sufficiently small such that $2(\kappa - r) > \lambda(L^2 - r^2)$. By the above inequalities we have

$$\|H_{(p,q)}(x_1) - H_{(p,q)}(x_2)\| \le \alpha\|x_1 - x_2\|, \tag{4.26}$$

where $\alpha := \frac{\left[1-2\lambda\kappa+\lambda^2L^2\right]^{\frac{1}{2}}}{1-r\lambda} < 1$. Moreover, due to the continuity of the maps Π_λ and F we may find some $\delta, \eta > 0$ such that $I\!B_\delta(\bar{x}) \times I\!B_\eta(\bar{p}) \times I\!B_\eta(\bar{q}) \subset U_2 \times V_2 \times Q_2$ and that

$$\|H_{(p,q)}(\bar{x}) - \bar{x}\| = \|\Pi_\lambda(\bar{x} - \lambda f(\bar{x}, p, q), p) - \Pi_\lambda(\bar{x} - f(\bar{x}, \bar{p}, \bar{q}), \bar{p})\| \le \delta(1 - \alpha)$$

for all $(p, q) \in I\!B_\eta(\bar{p}) \times I\!B_\eta(\bar{q})$. This together with (4.26) shows that $H_{(p,q)}$ has a unique fixed point in $I\!B_\delta(\bar{x})$, which is called $x(p, q)$. Note that $\operatorname{gph} x(\cdot, \cdot) = \operatorname{gph} S \cap (I\!B_\eta(\bar{p}) \times I\!B_\eta(\bar{q}) \times I\!B_\delta(\bar{x}))$. It remains to check the Hölder continuity of this map. Indeed, take any $(p_1, q_1), (p_2, q_2) \in I\!B_\eta(\bar{p}, \bar{q})$, with $x_1 := x(p_1, q_1)$ and $x_2 := x(p_2, q_2)$ we obtain from (4.7), (4.25), and (4.26) that

$$\begin{aligned}
\|x_1 - x_2\| &= \|H_{(p_1,q_1)}(x_1) - H_{(p_2,q_2)}(x_2)\| \\
&\le \|H_{(p_1,q_1)}(x_1) - H_{(p_1,q_1)}(x_2)\| + \|H_{(p_1,q_1)}(x_2) - H_{(p_2,q_2)}(x_2)\| \\
&\le \alpha\|x_1 - x_2\| + \|\Pi_\lambda(x_2 - \lambda F(x_2, p_1, q_1), p_1) - \Pi_\lambda(x_2 - \lambda F(x_2, p_2, q_2), p_2)\| \\
&\le \alpha\|x_1 - x_2\| + \frac{1}{1 - r\lambda}\left[\|\lambda F(x_2, p_1, q_1) - \lambda F(x_2, p_2, q_2)\| + \ell\|p_1 - q_2\|^{\frac{1}{2}}\right] \\
&\le \alpha\|x_1 - x_2\| + \frac{1}{1 - r\lambda}\left[L d_1(p_1, p_2) + L d_2(q_1, q_2) + \ell d_1(p_1, p_2)^{\frac{1}{2}}\right] \\
&\le \alpha\|x_1 - x_2\| + \frac{1}{1 - r\lambda}\left[(L\sqrt{2\eta} + \ell) d_1(p_1, p_2)^{\frac{1}{2}} + L d_2(q_1, q_2)\right],
\end{aligned}$$

which clearly yields that

$$\|x_1 - x_2\| \le \frac{1}{(1 - \alpha)(1 - r\lambda)}\left[(L\sqrt{2\eta} + \ell) d_1(p_1, p_2)^{\frac{1}{2}} + L d_2(q_1, q_2)\right].$$

This verifies the Hölder continuity of $x(\cdot,\cdot)$ with the order pair $(\frac{1}{2},1)$ and thus completes the proof of the theorem. $\square$

Remark. The assumption $\kappa > r$ is essential in the theorem. Indeed, if $\kappa \le r$ choose $f(x,p) = \delta_{I\!R^2_+}(x) - \frac{r}{2}\|x\|^2$ and $F(x,p,q) = \kappa x + p + q$ for all $(x,p,q) \in I\!R^2 \times I\!R^2 \times I\!R^2$. It is clear that f is parametrically continuously prox-regular at $(\bar{x},\bar{p}) = (0,0) \in I\!R^2 \times I\!R^2$ for $\bar{x}^* = 0$ with respect to r in (2.27) and all conditions (A1), (A2), and (A3) are satisfied. Note further that inequality (4.24) holds and that $x \in S(p,q)$ if and only if $(r-\kappa)x - p - q \in N_{I\!R^2_+}(x)$ which equivalently means $(\kappa - r)x + p + q \in I\!R^2_+$ and $x \in I\!R^2_+$. Since $\kappa - r \le 0$, we get from the latter that $p + q \in I\!R^2_+$, which is not the case for all (p,q) around $(0,0) \in I\!R^2 \times I\!R^2$. Thus S does not admit a Hölder continuous single-valued localization around $(0,0) \in I\!R^2 \times I\!R^2$ for $\bar{x}$.

4.3.2 Lipschitz continuity

The following proposition is the counterpart of Proposition 4.11 for Lipschitz continuity.

Proposition 4.13 *Let $(\bar{x},\bar{p},\bar{q}) \in X$ satisfy $\bar{x} \in S(\bar{p},\bar{q})$. Assume that f is parametrically continuously prox-regular at $(\bar{x},\bar{p})$ for $\bar{x}^* := -F(\bar{x},\bar{p},\bar{q})$ with respect to $r > 0$ in* (2.27)*. Suppose further that condition* (3.33) *holds for the function f. Then for any $\lambda \in (0, r^{-1})$, the mapping P_λ admits a Lipschitzian continuous single-valued localization Π_λ around $(\lambda\bar{x}^* + \bar{x}, \bar{p})$ for $\bar{x}$ with order pair $(1, \frac{1}{2})$ and a modulus pair $\big((1 - r\lambda)^{-1}, \ell\big)$ for some $\ell > 0$.*

Proof. Following the proof of Proposition 4.11 ensures that the mapping $k(x,p) = \lambda g(x,p) + \frac{1}{2}\|x\|^2$ satisfies the uniform second-order growth condition (4.23) at $(\bar{x},\bar{p},\hat{x}^*)$ with $(\hat{x}^* := \lambda\bar{x}^* + \bar{x}$ and also the basic constraint qualification at $(\bar{x},\bar{p})$. Moreover, it is easy to check from condition (3.33) that the mapping $p \mapsto \operatorname{gph} \partial_x k(\cdot,p)$ is Lipschitz-like around $(\bar{p},\bar{x},\hat{x}^*)$.

By employing Theorem 3.9 to the function k at $(\bar{x},\bar{p})$ for $\hat{x}^* \in \partial_x k(\bar{x},\bar{p})$, we obtain that the mapping Π_λ admits a Lipschitz continuous single-valued localization at $(\bar{v}, \bar{q})$ for $\hat{v} := \lambda\bar{v} + \bar{x}$ with a modulus pair $(1 - r\lambda, \ell)$ for some $\ell > 0$. $\square$

The major theorem in this section is stated below. In the case of (4.4) with *closed convex* set-valued mapping $K : P \rightrightarrows X$, this result reduces to [12, Theorem 2.1] and to [67, Theorem 3.1] when changing the parameter p in the function F from (4.4) by q.

Theorem 4.14 *Let $(\bar{x}, \bar{p}, \bar{q}) \in X$ satisfy $\bar{x} \in S(\bar{p}, \bar{q})$. Suppose that f is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $\bar{x}^* := -F(\bar{x}, \bar{p}, \bar{q})$ with respect to $r > 0$ in* (2.27)*. Suppose further that both conditions* (3.33) *and* (4.24) *are satisfied with some $\kappa > r$. Then the solution mapping S admits a Lipschitz continuous single-valued localization at $(\bar{p}, \bar{q})$ for $\bar{x}$.*

Proof. The proof follows the lines in that of Theorem 4.12 by using Proposition 4.13 instead of Proposition 4.11. We skip the details. $\square$

4.4 Applications to Variational Inequalities

By dropping parameters p in (4.5) and considering f as an indicator of convex set K in X, we narrow our work to the variational inequalities (4.2). The section is devoted to some new sufficient conditions to strong regularity introduced by Robinson [58] as follows.

Definition 4.15 (strong regularity). *Let K be a closed convex set in X and let $F : X \to X$ be Fréchet differentiable at $\bar{x}$ with $0 \in F(\bar{x}) + N_K(\bar{x})$. We say the variational inequalities*

$$0 \in F(x) + N_K(x)$$

is STRONGLY REGULAR *at $\bar{x}$ if the inverse mapping of $T : X \rightrightarrows X$ defined by*

$$T(x) := F(\bar{x}) + \nabla F(\bar{x})(x - \bar{x}) + N_K(x), \quad x \in X \tag{4.27}$$

admits a Lipschitz continuous single-valued localization around $(0, \bar{x})$*.*

Theorem 4.16 (sufficient condition to strong regularity of variational inequalities). *Let* K *be a closed convex set in* X *and let* $F : X \to X$ *be Fréchet differentiable at* $\bar{x}$ *with* $0 \in F(\bar{x}) + N_K(\bar{x})$*. Then the variational inequality* (4.2) *is strongly regular at* $\bar{x}$ *provided that there exist some* $\delta, \kappa > 0$ *such that for all* $(x, x^*) \in \operatorname{gph} N_K \cap I\!B_\delta(\bar{x}, -F(\bar{x}))$ *we have*

$$\langle \nabla F(\bar{x})u, u\rangle + \inf\Big\{\langle u^*, u\rangle \Big|\ u^* \in \breve{\partial}^2 \delta_K(x, x^*)(u)\Big\} \geq \frac{1}{\kappa}\|u\|^2 \quad \textit{whenever} \quad u \in X.$$

If, in addition, X *is a finite-dimensional space, then the above condition can be replaced by*

$$\langle \nabla F(\bar{x})u, u\rangle + \inf\Big\{\langle u^*, u\rangle \Big|\ u^* \in \partial^2 \delta_K(\bar{x}, -F(\bar{x}))(u)\Big\} > 0 \quad \textit{whenever} \quad u \in X \setminus \{0\}. \quad (4.28)$$

Proof. Define $F(x, q) := F(\bar{x}) + \nabla F(\bar{x})(x - \bar{x}) - q$ for all $(x, q) \in X \times X$. Applying Theorem 4.2 to the case the parameter p is ignored in (4.4) and $f = \delta_K$, we have that the mapping

$$S(q) := \big\{x \in X | \in 0 \in F(x, q) + N_K(x)\big\}, \quad q \in X$$

has a Lipschitz continuous single-valued localization around $(0, \bar{x})$. Observe that $S(q) = T^{-1}(q)$, the latter ensures that the variational inequality (4.2) is strongly regular at $\bar{x}$.

When $\dim X < \infty$, instead of using Theorem 4.2 we employ Corollary 4.10 in the same situation and also obtain the strong regularity of the variational inequality (4.2) at $\bar{x}$. The proof is complete. $\square$

The following result is a simple consequence of the above theorem, in which the second part is similar to [5, Proposition 5.2] in finite-dimensional spaces.

Corollary 4.17 *Let X be a finite-dimensional space and $F : X \to X$ be differentiable at $\bar{x}$ with $0 \in F(\bar{x}) + N_K(\bar{x})$. Suppose that*

$$\langle \nabla F(\bar{x})u, u\rangle > 0 \quad \textit{for all} \quad u \in \operatorname{dom} \partial^2 \delta_K(\bar{x}, -F(\bar{x}))(\cdot), u \neq 0. \tag{4.29}$$

Then the variational inequality (4.3) *is strongly regular at $\bar{x}$.*

Consequently, the latter is also valid when replacing (4.29) *by the following condition*

$$\langle \nabla F(\bar{x})u, u\rangle > 0 \quad \textit{for all} \quad u \in \operatorname{span} T_K(\bar{x}) \cap F(\bar{x})^{\perp}, u \neq 0.$$

Proof. Since N_K is a maximal monotone operator, it follows from Lemma 2.9 that

$$\inf \Big\{ \langle u^*, u\rangle \big| \; u^* \in \partial^2 \delta_K(\bar{x}, -F(\bar{x}))(u) \Big\} > 0 \quad \text{for all} \quad u \in \operatorname{dom} \partial^2 \delta_K(\bar{x}, -F(\bar{x}))(\cdot).$$

This shows that condition (4.29) is sufficient for (4.28) and thus verifies the strong regularity of (4.3) by Theorem 4.16.

To justify the second claim of the corollary, it suffices to show that

$$\operatorname{dom} \partial^2 \delta_K(\bar{x}, -F(\bar{x}))(\cdot) \subset \operatorname{span} T_K(\bar{x}) \cap F(\bar{x})^{\perp}. \tag{4.30}$$

Indeed, pick any $u \in \operatorname{dom} \partial^2 \delta_K(\bar{x}, -F(\bar{x}))(\cdot)$, we find a sequence (u_k, x_k, v_k) such that $u_k \to u$ and $(x_k, v_k) \overset{\text{gph} N_K}{\to} (\bar{x}, -F(\bar{x}))$. Thanks to (6.5) in Chapter 6, we have $u_k \in -T_K(x_k) \cap v_k^{\perp}$. Hence, for each $k \in I\!N$ there exist sequences $y_{n_k} \to x_k$ and $t_{n_k} \downarrow 0$ such that $\frac{y_{n_k} - x_k}{t_{n_k}} \to u_k$ as $n \to \infty$. For each $k \in I\!N$ we find $z_k \in \{y_{n_k}\}$ and $\alpha_k \in \{t_{n_k}\}$ such that $z_k \to \bar{x}$, $\alpha_k \downarrow 0$, and that $\frac{z_k - x_k}{\alpha_k} - u_k \to 0$ as $k \to \infty$. It follows that

$$u = \lim_{k\to\infty} u_k = \lim_{k\to\infty} \frac{z_k - x_k}{\alpha_k} = \lim_{k\to\infty} \frac{(z_k - \bar{x}) - (x_k - \bar{x})}{\alpha_k} \in \text{cl}\left[T_K(\bar{x}) - T_K(\bar{x})\right] = \text{cl}\left[\text{span}\, T_K(\bar{x})\right]$$

with a note that $z_k - \bar{x}, x_k - \bar{x} \in T_K(\bar{x})$ for all k. Since $\text{span}\, T_K(\bar{x})$ is a subspace of the finite-dimensional space X, it is closed, i.e., $\text{cl}\left[\text{span}\, T_K(\bar{x})\right] = \text{span}\, T_K(\bar{x})$. Furthermore, since $\langle u_k, v_k \rangle = 0$, we derive $\langle u, F(\bar{x}) \rangle = 0$ when taking $k \to \infty$. This together with the above inclusion verifies (4.30) and thus completes the proof of the corollary. $\square$

Next let us discuss two particular (nonpolyhedral) cases of K, which are of high interests in optimization. The first case is for the set of symmetric positive matrices.

Corollary 4.18 *Let $X = S^n$ be the space of $n \times n$ symmetric matrices and let $K = S^n_+$ be the cone of $n \times n$ symmetric positive semidefinite matrices. Suppose that $F : S^n \to S^n$ is differentiable at B with $S^n_+ \ni F(B) \perp B \in S^n_+$. Then the variational inequality* (4.2)*, which is equivalent to the complementarity problem*

$$S^n_+ \ni F(C) \perp C \in S^n_+ \tag{4.31}$$

is strongly regular at B provided that

$$\langle \nabla F(B)U, U \rangle + 2\langle F(B), UB^{\dagger}U \rangle > 0 \quad \textit{for all} \quad U \in L(B), \tag{4.32}$$

where $A^{\dagger}$ is the Moore-Penrose pseudoinverse of B, and where $L(B)$ is defined by

$$L(B) := \left\{U \in S^n \mid P^*_{\beta} U P_{\gamma} = 0, P^* U P_{\gamma} = 0\right\},$$

with the matrix P taken from the eigenvalue decomposition (5.24) *for $A = B - F(B)$.*

Proof. Since S^n_+ is a *self-dual* convex cone, it is well-known that variational inequality (4.2) is equivalent to the complementarity problem (4.31). Moreover, it follows from Lemma 5.9 that $\operatorname{dom} \partial^2 \delta_{S^n_+}(B, -F(B)) = L(B)$ and that

$$\inf \Big\{ \langle C, U\rangle \big|\ C \in \partial^2 \delta_{S^n_+}(B, -F(B))(U) \Big\} = 2\langle F(B), UB^{\dagger}U\rangle \ \text{ for all } \ U \in \operatorname{dom} \partial^2 \delta_{S^n_+}(B, -F(B)).$$

This together with Theorem 4.16 ensures that the complementarity problem (4.31) is strongly regular at B and thus completes the proof of the corollary. $\square$

Let us complete the section by noting that full characterizations of strong regularity of the variational inequalities for the above specific case have been established by Pang, Sun and Sun [54, Theorem 17] in term of degree theory. Albeit our condition (4.32) is just sufficient for strong regularity, it seems to be more verifiable.

Part B: Applications

Chapter 5

Full Stability in Finite-Dimensional Constrained Optimization

5.1 Overview

This chapter concerns the study of the corresponding counterparts of both Hölderian and Lipschitzian full stability of local solutions to the following large class of problems in constrained optimization:

$$\widehat{\mathcal{P}} \quad \begin{cases} \text{minimize} \;\; \varphi(x,\bar{p}) \;\; \text{subject to } x \in X, \\ g(x,\bar{p}) \in \Theta, \end{cases} \tag{5.1}$$

where the cost function $\varphi : X \times P \to I\!R$ and the constrained mapping $g : X \times P \to Y$ are $\mathcal{C}^2$-smooth around the reference point $(\bar{x},\bar{p})$, where X, Y, P are finite-dimensional Euclidean space, and where Θ is a closed and convex subset of Y. Besides standard nonlinear programs (NLP), model (5.1) encompasses various problems of *conic programming* [5, 41] when the set Θ is a cone, mathematical programs with *polyhedral constraints* (MPPC) designated in [48] when Θ is a polyhedral set, etc. It is worth noting that, despite describing (5.1) in the classical *smooth and convex* terms, the progress in the study of full stability and related issues achieved in this and the subsequent sections are based on the results and methods of *nonsmooth variational* *analysis* developed above.

The main purpose of this chapter is to study full stability for optimization problem (5.1) and its correlations to other well-known concepts of stability such as Robinson's strong regularity and Kojima's strong stability. In this way we provide a lot of new understanding for those stabilities as well. In accordance with the the scheme of Section 3 the two-parameter perturbation of $\widehat{\mathcal{P}}$ in (5.1) reads as

$$\widehat{\mathcal{P}}(x^*,p) \quad \begin{cases} \text{minimize } \; \varphi(x,p)-\langle x^*,x\rangle \; \text{ subject to } x\in X, \\ g(x,p)\in\Theta \end{cases} \tag{5.2}$$

for any $(x^*,p)\in X\times P$. It can be written in the equivalent unconstrained format

$$\widehat{\mathcal{P}}(x^*,p) \quad \text{minimize } f(x,p)-\langle x^*,x\rangle \;\text{ with }\; f(x,p):=\varphi(x,p)+\delta_\Theta\big(g(x,p)\big),\; (x,p)\in X\times P \tag{5.3}$$

5.2 Full Stability, Strong Regularity, and Strong Stability in Constrained Optimization

To proceed with the study of full stability and related properties, recall that the *Robinson constraint qualification* (RCQ) holds in $\widehat{\mathcal{P}}$ at the point $\bar{x}$ with $g(\bar{x},\bar{p})\in\Theta$ if

$$0\in \operatorname{int}\big\{g(\bar{x},\bar{p})+\nabla_x g(\bar{x},\bar{p})X-\Theta\big\}. \tag{5.4}$$

As well known, RCQ (5.4) reduces to the classical *Mangasarian-Fromovitz constraint qualification* (MFCQ) defined later in (5.37) for NLP problems. If x is a local minimizer of $\widehat{\mathcal{P}}$ and *RCQ* is satisfied at x, then x is the *stationary point* meaning that there is some *Lagrange multiplier*

$\lambda \in Y^*$, the dual space of Y, such that

$$0 \in \nabla_x L(x, \bar{p}, \lambda) \quad \text{and} \quad \lambda \in N_\Theta(g(x, \bar{p})), \tag{5.5}$$

where $L(\cdot, \cdot, \cdot)$ is the usual *Lagrangian function* defined by

$$L(x, p, \lambda) := \varphi(x, p) + \langle \lambda, g(x, p) \rangle \quad \text{with} \quad (x, p) \in X \times P \quad \text{and} \quad \lambda \in Y^*. \tag{5.6}$$

The system in (5.5) can be written as the form of Robinson's *generalized equation* (GE) [58]:

$$0 \in \begin{bmatrix} \nabla_x L(x, \bar{p}, \lambda) \\ \\ -g(x, \bar{p}) \end{bmatrix} + \begin{bmatrix} 0 \\ \\ N_\Theta^{-1}(\lambda) \end{bmatrix}. \tag{5.7}$$

Note that x is a stationary point of $\widehat{\mathcal{P}}(x^*, p)$ if and only if $x^* \in \partial_x f(x, p)$ for (x, p) near $(\bar{x}, \bar{p})$ due to the validity of RCQ (5.4). Since RCQ is always satisfied in all the results below concerning the stability around $(\bar{x}, \bar{p})$, from now on we suppose without loss of generality that the latter equivalence holds for all x.

Let $\Phi : X \times Q \to I\!R$ and $G : X \times Q \to Y$, where Q is also a finite-dimensional Euclidean space. The pair $(\Phi(x, q), G(x, q))$ provides a $\mathcal{C}^2$-*smooth parameterization* of $(\varphi(x, \bar{p}), g(x, \bar{p}))$ at $\bar{q} \in Q$ if both mappings Φ and G are twice continuously differentiable with $\Phi(x, \bar{q}) = \varphi(x, \bar{p})$ and $G(x, \bar{q}) = g(x, \bar{p})$. Consider the following parametric optimization problem:

$$\widetilde{\mathcal{P}}(q) \quad \begin{cases} \text{minimize} \ \ \Phi(x, q) \ \ \text{subject to } x \in X, \\ \\ G(x, q) \in \Theta. \end{cases} \tag{5.8}$$

Observe that problem $\widehat{\mathcal{P}}(x^*, p)$ in (5.2) is a special form of $\widetilde{\mathcal{P}}(q)$ when $\Phi(x, q) = \varphi(x, p) - \langle x^*, x \rangle$

and $G(x,q) = g(x,p)$ for $q = (x^*,p) \in X \times P$ and $\bar{q} = (0,\bar{p})$. The next definition is taken from [5, Definition 5.16].

Definition 5.1 (uniform quadratic growth condition). *Let $\bar{x}$ be a stationary point of problem $\widehat{\mathcal{P}}$. The* UNIFORM QUADRATIC GROWTH CONDITION (UQGC) *holds at $\bar{x}$ with respect to a $\mathcal{C}^2$-smooth parameterization $(\Phi(x,q), G(x,q))$ of $(\varphi(x,\bar{p}), g(x,\bar{p}))$ at some $\bar{q} \in Q$ if there exist $\ell > 0$ and neighborhoods U of $\bar{x}$ and W of $\bar{q}$ such that for any $q \in W$ and any stationary $\bar{x}(q) \in U$ of problem $\widetilde{\mathcal{P}}(q)$ we have*

$$\Phi(x,q) \ge \Phi\big(\bar{x}(q), q\big) + \ell \|x - \bar{x}(q)\|^2 \quad \text{for all} \;\; x \in U, \; G(x,q) \in \Theta. \tag{5.9}$$

We say that UQGC (5.9) *holds at $\bar{x}$ if it holds for all $\mathcal{C}^2$-smooth parameterization of the pair $(\varphi(x,\bar{p}), g(x,\bar{p}))$.*

Our uniform second-order growth condition (3.10) for the function $f(x,p)$ defined in (5.3) can be viewed as the above UQGC at $\bar{x}$ with respect to the $\mathcal{C}^2$-smooth parameterization $(\varphi(x,p) - \langle x^*, p\rangle, g(x,p))$. It is shown in [5, Theorem 5.24] that under RCQ (5.4) the defined UQGC is equivalent to Kojima's strong stability [23] formulated in the first parts of the following definition taken from [5, Definition 5.33].

Definition 5.2 (strong stability). *We say that a stationary point $\bar{x}$ of problem $\widehat{\mathcal{P}}$ is* STRONGLY STABLE *with respect to a $\mathcal{C}^2$-smooth parameterization $(\Phi(x,q), G(x,q))$ of $(\varphi(x,\bar{p}), g(x,\bar{p}))$ at some $\bar{q}$ if there is a neighborhood $U \times Q$ of $(\bar{x}, \bar{q})$ such that for any $q \in Q$ the parametric problem $\widetilde{\mathcal{P}}(q)$ has a unique stationary point $\bar{x}(q) \in U$ such that the mapping $q \mapsto \bar{x}(q)$ is continuous on Q. If this holds for any $\mathcal{C}^2$-smooth parameterization of $(\varphi(x,\bar{p}), g(x,\bar{p}))$, we say that $\bar{x}$ is strongly stable. In the conditions above the mapping $q \mapsto \bar{x}(q)$ in Lipschitz continuous on Q, we* *speak about* Lipschitzian strong stability *of* $\bar{x}$.

Next we show that the continuity of the function $\bar{x}(q)$ in Definition 5.2 can be strengthened to *Hölderian continuity* with degree $\frac{1}{2}$ provided that $\bar{x}$ is a *local minimizer* of problem $\widehat{\mathcal{P}}$ under the validity of RCQ (5.4) at $\bar{x}$. This Hölder continuity can be treated as a natural counterpart of Hölderian full stability in the problem under consideration. In the case of NLP ($\Theta = \{0\} \times I\!R^l_-$), our result agrees with that by Gfrerer [19, Corollary 3.2] due to the fact that Kojima's strong stability is characterized by Robinson's *strong second-order sufficient condition* (SSOSC) [58]. Note further that the Hölder exponent $\frac{1}{2}$ is shown to the best possible for NLP; see Example 3.2.

Theorem 5.3 (strong stability and Hölder continuity). *Let $\bar{x}$ be a local minimizer of problem $\widehat{\mathcal{P}}$, and RCQ* (5.4) *holds at $\bar{x}$. Then the point $\bar{x}$ is strongly stable in the sense of Definition* 5.2 *if and only if for any $\mathcal{C}^2$-smooth parameterization $(\Phi(x,q), G(x,q))$ of $(\varphi(x,\bar{p}), g(x,\bar{p}))$ at some $\bar{q} \in I\!R^k$ there exist a neighborhood $U \times Q$ of $(\bar{x},\bar{q})$ and a constant $\kappa > 0$ such that for every $q \in Q$ the parametric problem $\widetilde{\mathcal{P}}(q)$ has a unique stationary point $\bar{x}(q) \in U$ satisfying the Hölder continuity property*

$$\|\bar{x}(q_1) - \bar{x}(q_2)\| \le \kappa \|q_1 - q_2\|^{\frac{1}{2}} \quad \text{whenever} \quad q_1, q_2 \in Q. \tag{5.10}$$

Proof. It is obvious that $\bar{x}$ is strongly stable if the function $\bar{x}(q)$ in Definition 5.2 satisfies the Hölderian continuity property (5.10). Conversely, suppose that the stationary point $\bar{x}$ is strongly stable. Take any $\mathcal{C}^2$-smooth parameterization $(\Phi(x,q), G(x,q))$ of $(\varphi(x,\bar{p}), g(x,\bar{p}))$ at some $\bar{q} \in Q$ with $(x,q) \in X \times Q$. Define $\Psi(x,w) := \Phi(x,q) - \langle x^*, x\rangle$ and $\mathcal{G}(x,z) := G(x,p)$ with $z = (q,x^*) \in Q \times X$. Note that $(\Psi, \mathcal{G})$ is also a $\mathcal{C}^2$-smooth parameterization of $(\varphi(x,\bar{p}), g(x,\bar{p}))$ at $\bar{z} := (\bar{q}, 0)$. Since $\bar{x}$ is strongly stable, it follows from [5, Theorem 5.34] that UQGC (5.9) holds at $\bar{x}$ with respect to the parameterization $(\Psi, \mathcal{G})$. By Definition 5.1 there exist $\ell > 0$ and neighborhoods U of $\bar{x}$ and $Z = W \times U^*$ of $\bar{z} = (\bar{q}, 0)$ such that for any $(q, x^*) \in W \times U^*$ and any stationary point $u \in U$ of the parametric problem $\widetilde{\mathcal{P}}(z)$ we have

$$\Phi(x,q) - \langle x^*, x\rangle \geq \Phi(u,q) - \langle x^*, u\rangle + \ell\|x-u\|^2 \text{ whenever } x \in X,\ G(x,q) \in \Theta. \tag{5.11}$$

Denoting $F(x,q) := \Phi(x,q) + \delta_\Theta(G(x,q))$, observe from Proposition 2.8 that this function is parametrically continuously prox-regular at $(\bar{x}, \bar{q})$ for $\bar{v} = 0 \in \partial_x F(\bar{x}, \bar{p})$ and that BCQ (3.7) (or (3.8)) holds for this function at $(\bar{x}, \bar{q})$ due to the validity of RCQ. Furthermore (5.11) tells us that the uniform second-order growth condition in (3.10) is satisfied for the function F around $(\bar{x}, \bar{p}, 0) \in \operatorname{gph} \partial_x F$. Applying Theorem 3.5 allows us to find $(\ell_1, \ell_2) \in I\!R^2_>$ and a neighborhood $U_1 \times W_1 \times U_1^* \subset U \times W \times U^*$ of $(\bar{x}, \bar{q}, 0)$ such that for any $(u_i, q_i, u_i^*) \in \operatorname{gph} \partial_x F \cap (U_1 \times W_1 \times U_1^*)$ with $i = 1, 2$ we have

$$\|u_1 - u_2\| \leq \ell_1\|u_1^* - u_2^*\| + \ell_2\|q_1 - q_2\|^{\frac{1}{2}}.$$

Put $u_1^* = u_2^* = 0$ and note that $u_1 = \bar{x}(q_1)$ and $u_2 = \bar{x}(q_2)$, which gives us the estimate

$$\|\bar{x}(q_1) - \bar{x}(q_2)\| \leq \ell_2\|q_1 - q_2\|^{\frac{1}{2}} \text{ for all } q_1, q_2 \in W_1.$$

This ensures (5.10) and thus completes the proof of the theorem. □

Observe from the proof of Theorem 5.3 that when $\bar{x}$ is a local minimizer of problem $\widehat{\mathcal{P}}$, Kojima's strong stability of $\bar{x}$ implies Hölderian full stability at the same point. However, the converse implication is not valid even in the NLP setting. Indeed, it is shown by in Section 5.4 that, under MFCQ and the well-known *constant rank constraint qualification* for NLP problems, Hölderian full stability and its Lipschitzian counterpart are the same due to the validity of

(3.37) (see Proposition 5.14 below) and can be characterized by a condition *strictly weaker* than SSOSC. Since SSOSC is equivalent to strong stability in this framework, we conclude that Hölderian full stability can not generally imply strong stability.

Another significant notion of variational analysis is Robinson's *strong regularity* for generalized equations introduced by his landmark paper [58]. We formulate it for the generalized equation (5.7) under consideration.

Definition 5.4 (strong regularity). *Let $(\bar{x}, \bar{\lambda})$ be a solution to the generalized equation* (5.7)*. We say that $(\bar{x}, \bar{\lambda})$ is* STRONGLY REGULAR *if there exist neighborhoods U of $0 \in X \times Y$ and V of $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ such that for every $\delta \in U$ the system*

$$\delta \in \begin{bmatrix} 0 \\ -g(\bar{x}, \bar{p}) \end{bmatrix} + \begin{bmatrix} \nabla^2_{xx} L(\bar{x}, \bar{p}, \bar{\lambda})(x - \bar{x}) + \nabla_x g(\bar{x}, \bar{p})^*(\lambda - \bar{\lambda}) \\ -\nabla_x g(x, \bar{p})(x - \bar{x}) \end{bmatrix} + \begin{bmatrix} 0 \\ N_\Theta^{-1}(\lambda) \end{bmatrix} \tag{5.12}$$

has a unique solution in V denoted by $\zeta(\delta)$ and that the mapping $\zeta : U \to V$ is Lipschitz continuous.

It can be deduced from [5, Theorem 5.24] that the strong stability of $(\bar{x}, \bar{\lambda})$ in (5.12) above is *equivalent* to UQGC (5.9) under the following two assumptions:

(A1) The set Θ is $\mathcal{C}^2$*-reducible* to a closed convex set K at $\bar{y} := g(\bar{x}, \bar{p})$, and the reduction is *pointed.* This means that there exist a neighborhood W of $\bar{y}$ and a $\mathcal{C}^2$-smooth mapping $h : W \to I\!R^k$ such that $\nabla h(\bar{y})$ is surjective, $\Theta \cap W = \{y \in W |\ h(y) \in K\}$, and the tangent cone $T_K(h(\bar{y}))$ defined in (2.6) is pointed.

(A2) The point $(\bar{x}, \bar{p})$ is *partially nondegenerate* for g with respect to Θ, i.e.,

$$\nabla_x g(\bar{x}, \bar{p})X + \text{lin}\big(T_\Theta\big(g(\bar{x}, \bar{p})\big)\big) = Y, \tag{5.13}$$

where $\mathrm{lin}\big(T_\Theta(g(\bar{x},\bar{p}))\big)$ is the largest linear subspace of the space Y that is contained in the classical tangent cone $T_\Theta(g(\bar{x},\bar{p}))$.

Note that the reducibility condition **(A1)** is satisfied for a great variety of convex sets Θ arising in important classes of problems in constrained optimization. This includes polyhedral sets [5, Example 3.139], the second-order (Lorentz, ice-cream) cone [4, Lemma 15], the cone of positive semidefinite symmetric matrices [5, Example 3.140], etc. In contrast, the nondegeneration condition **(A2)** is rather restrictive. In particular, for NLP problems it reduces to the classical *linear independence constraint qualification* (LICQ), in the case of MPPC problems (when Θ is a convex polyhedral) it agrees with the polyhedral constraint qualification (PCQ) introduced and studied in [48]; see also [5] for the versions of **(A2)** for other classes of problems in conic programming. Observe that for the general class of problems $\widehat{\mathcal{P}}$ in (5.1) the nondegeneration condition **(A2)** implies the Robinson constraint qualification (5.4) but clearly not vice versa.

Before deriving the main result of this section we present the following lemma, which is based on the second-order chain rule obtained recently in [39]. This lemma will allow us to make a bridge between general characterizations of Lipschitzian full stability in Section 4 and their applications to the class of constrained problem (5.1) with new links to strong stability and strong regularity.

Lemma 5.5 (limiting coderivative of partial subgradient mappings). *Let both conditions* **(A1)** *and* **(A2)** *be satisfied at $\bar{x}$, which is a stationary point of problem $\widehat{\mathcal{P}}$ from* (5.1) *in the sense that $0 \in \partial_x f(\bar{x},\bar{p})$ the partial subgradient mapping of the function f in* (5.3). *Then for all $w \in X$ the limiting coderivative of the partial subgradient mapping $\partial f_x(\bar{x},\bar{p})$ is represented* *by*

$$D^*\partial_x f(\bar{x},\bar{p},0)(u) = \Big(\nabla^2_{xx}L(\bar{x},\bar{p},\bar{\lambda})u, \nabla^2_{xp}L(\bar{x},\bar{p},\bar{\lambda})u\Big) + \nabla g(\bar{x},\bar{p})^* D^* N_\Theta(\bar{y},\bar{\lambda})\big(\nabla_x g(\bar{x},\bar{p})u\big) \tag{5.14}$$

with $\bar{y} := g(\bar{x},\bar{p})$*, where* L *is the Lagrangian* (5.6)*, and where* $\bar{\lambda} \in Y^*$ *is a unique solution of the system*

$$\nabla_x g(\bar{x},\bar{p})^*\lambda = -\nabla_x\varphi(\bar{x},\bar{p}) \ \ \text{and} \ \ \lambda \in N_\Theta(\bar{y}). \tag{5.15}$$

Consequently, the coderivative condition (3.37) *is satisfied for this function* f *with* $\bar{x}^* = 0$*.*

Proof. Applying the simple subdifferential sum rule to the function f in (5.3), we get from the stationary condition $0 \in \partial_x f(\bar{x},\bar{p})$ that $0 \in \nabla_x\varphi(x,p) + \partial_x\delta_\Theta(g(x,p))$. Furthermore, the coderivative sum rule from Lemma 2.5 and the second-order subdifferential definition (2.15) give us

$$D^*\partial_x f(\bar{x},\bar{p},0)(u) = \Big(\nabla^2_{xx}\varphi(\bar{x},\bar{p})u, \nabla^2_{xp}\varphi(\bar{x},\bar{p})u\Big) + D^*\partial_x(\delta_\Theta\circ g)\big(\bar{x},\bar{p},-\nabla_x\varphi(\bar{x},\bar{p})\big)(u) \tag{5.16}$$

for all $w \in X$. The assumed conditions **(A1)** and **(A2)** allow us to apply the second-order chain rule from [39, Theorem 3.6] to the composite function $\delta_\Theta \circ g$ and get in this way the equality

$$\begin{aligned} D^*\partial_x(\delta_\Theta\circ g)\big(\bar{x},\bar{p},-\nabla_x\varphi(\bar{x},\bar{p})\big)(u) = \ & \Big(\nabla^2_{xx}\langle\bar{\lambda},g\rangle(\bar{x},\bar{p})u, \nabla^2_{xp}\langle\bar{\lambda},g\rangle(\bar{x},\bar{p})u\Big) \\ & +\nabla g(\bar{x},\bar{p})^* D^* N_\Theta(\bar{y},\bar{\lambda})\big(\nabla_x g(\bar{x},\bar{p})u\big) \end{aligned}$$

for all $u \in X$, where $\bar{\lambda}$ solves the KKT system (5.15). This together with (5.16) justifies (5.14).

It remains to verify the validity of (3.37) for the function f with $\bar{x}^* = 0$. To proceed, pick any vector p^* with $(0, p^*) \in D^*\partial_x f(\bar{x}, \bar{p}, 0)(0)$ and get from (5.14) a unique vector $\bar{\lambda} \in Y^*$ satisfying (5.15) such that

$$(0, p^*) \in \nabla g(\bar{x}, \bar{p})^* D^* N_\Theta(\bar{y}, \bar{\lambda})(0).$$

This allows us to find $z \in D^* N_\Theta(\bar{y}, \bar{\lambda})(0)$ satisfying $0 = \nabla_x g(\bar{x}, \bar{p})^* z$ and $p^* = \nabla_p g(\bar{x}, \bar{p})^* z$. By the inclusion $\operatorname{gph} N_\Theta \supset \Theta \times \{0\}$, we get that $z \in N_\Theta(\bar{y})$ from $z \in D^* N_\Theta(\bar{y}, \bar{\lambda})(0)$. Since Θ is a closed convex set, it follows that $\langle z, y \rangle \le 0$ for all $y \in \operatorname{lin}\big(T_\Theta(\bar{y})\big) \subset T_\Theta(\bar{y})$. Due to (5.13) there exist $x \in X$ and $y \in \operatorname{lin}\big(T_\Theta(\bar{y})\big)$ satisfying $\nabla_x g(\bar{x}, \bar{p})x + y = z$. It leads us to

$$\|z\|^2 = \langle z, \nabla_x g(\bar{x}, \bar{p})x + y \rangle = \langle \nabla_x g(\bar{x}, \bar{p})^* z, x \rangle + \langle z, y \rangle \le 0 + 0 = 0,$$

which yields $z = 0$ and thus $p^* = 0$. This justifies (3.37) and completes the proof of the lemma.

Now we are ready to characterize Lipschitzian full stability of local minimizers in $\widehat{\mathcal{P}}$, which we understand in the sense of Definition 3.1(i) for problem $\widehat{\mathcal{P}}(0, \bar{p})$ in (5.3) with the extended-real-valued objective. The next major theorem not only provides a constructive second-order characterization of Lipschitzian full stability in $\widehat{\mathcal{P}}$ under assumptions **(A1)** and **(A2)** but also establishes its equivalence in this setting to the above notions of strong regularity and Lipschitzian strong stability and thus characterizes these notions as well. Note that the equivalence between assertions **(iii)** and **(iv)** of this theorem has been recently derived in [46, Theorem 6.10] for the case of tilt stability in conic programming when the parameter p is ignored.

Theorem 5.6 (equivalence between strong regularity and Lipschitzian full and strong stability for nondegenerate local minimizers and their second-order characterization). *Let $\bar{x}$ be a stationary point of problem $\widehat{\mathcal{P}}$ in* (5.1) *under the validity of RCQ* (5.4), *let*

$\bar{\lambda} \in Y^*$ *be the corresponding Lagrange multiplier from* (5.5)*, and let* $\bar{y} := g(\bar{x}, \bar{p})$*. Assume that the reducibility condition* **(A1)** *holds at* $\bar{x}$*. Then the following assertions are equivalent:*

(i) *The pair* $(\bar{x}, \bar{\lambda})$ *is a strongly regular solution to GE* (5.7)*, and* $\bar{x}$ *is a local minimizer of problem* $\widehat{\mathcal{P}}$*.*

(ii) *The nondegeneration condition* **(A2)** *holds, and the point* $\bar{x}$ *is a Lipschitzian strongly stable local minimizer of problem* $\widehat{\mathcal{P}}$*.*

(iii) *The nondegeneration condition* **(A2)** *holds, and the point* $\bar{x}$ *is a Lipschitzian fully stable local minimizer of problem* $\widehat{\mathcal{P}}$*.*

(iv) *The nondegeneration condition* **(A2)** *holds together with the second-order subdifferential condition*

$$\langle \nabla^2_{xx} L(\bar{x}, \bar{p}, \bar{\lambda})u, u\rangle + \inf\Big\{\langle u^*, \nabla_x g(\bar{x}, \bar{p})u\rangle \,\Big|\, u^* \in D^* N_\Theta(\bar{y}, \bar{\lambda})\big(\nabla_x g(\bar{x}, \bar{p})u\big)\Big\} > 0, \quad u \neq 0. \tag{5.17}$$

Proof. Since $\bar{x}$ is a stationary point of $\widehat{\mathcal{P}}$ at which RCQ (5.4) holds, we deduce from Proposition 2.8 that the function f in (5.3) is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $0 \in \partial_x f(\bar{x}, \bar{p})$ and that BCQ (3.7) holds at $(\bar{x}, \bar{p})$. Observe that implication **(ii)**$\Longrightarrow$**(i)** follows from [5, Theorem 5.35].

To verify next implication **(i)**$\Longrightarrow$**(iii)**, suppose that the point $(\bar{x}, \bar{\lambda})$ is strongly regular for the generalized equation (5.7) and get from [5, Theorem 5.24] that **(A2)** and UQGC (5.9) are satisfied at $\bar{x}$. Defining $\Phi(x, q) := \varphi(x, p) - \langle v, p\rangle$ and $G(x, q) := g(x, p)$ with $q = (x^*, p)$, note that $(\Phi(x, q), G(x, q))$ is a $\mathcal{C}^2$-smooth parameterization of $(\varphi(x, \bar{p}), g(x, \bar{p}))$ at $\bar{q} := (0, \bar{p})$. Then this UQGC allows us to find $\ell > 0$ as well as neighborhoods $U^* \times V$ of $\bar{q} = (0, \bar{p})$ and U of $\bar{x}$ such that for any $q = (v, p) \in U^* \times V$ there is a unique stationary point $\bar{x}(q) \in U$ of problem

$\widetilde{\mathcal{P}}(q)$ satisfying

$$\Phi(x,q) \geq \Phi(\bar{x}(q),q) + \ell\|x-\bar{x}(q)\|^2 \ \text{ for all } \ x \in U,\ G(x,q) \in \Theta. \tag{5.18}$$

Picking any $(u,p,u^*) \in \operatorname{gph}\partial_x f \cap (U \times V \times U^*)$, we have $u = \bar{x}(q)$. It gives us by (5.18) that

$$\varphi(x,p) - \langle u^*, x\rangle \geq \varphi(u,p) - \langle u^*, u\rangle + \ell\|x-u\|^2 \ \text{ for all } \ x \in U,\ g(x,p) \in \Theta.$$

This clearly implies the inequality

$$f(x,p) \geq f(u,p) + \langle u^*, x-u\rangle + \ell\|x-u\|^2 \ \text{ for all } \ x \in U,$$

which ensures in turn the uniform second-order growth condition (3.10). Taking into account that the coderivative condition (3.37) holds by Lemma 5.5 and then employing Theorem 3.10, we arrive at **(iii)**.

Let us now verify implication **(iii)**$\Longrightarrow$**(iv)**. Assuming **(iii)**, we deduce inequality (3.49) from Corollary 3.15. This together with the second-order representation (5.14) from Lemma 5.5 gives us that

$$\begin{aligned}
0 \ &< \ \inf\Big\{\langle u^*, u\rangle \big|\ u^* \in \nabla^2_{xx}L(\bar{x},\bar{p},\bar{\lambda})u + \nabla_x g(\bar{x},\bar{p})^* D^*N_\Theta(\bar{y},\bar{\lambda})\big(\nabla_x g(\bar{x},\bar{p})u\big)\Big\} \\
&= \ \langle \nabla^2_{xx}L(\bar{x},\bar{p},\bar{\lambda})u, u\rangle + \inf\Big\{\langle \nabla_x g(\bar{x},\bar{p})^* u^*, u\rangle \big|\ z \in D^*N_\Theta(\bar{y},\bar{\lambda})\big(\nabla_x g(\bar{x},\bar{p})u\big)\Big\} \\
&= \ \langle \nabla^2_{xx}L(\bar{x},\bar{p},\bar{\lambda})u, u\rangle + \inf\Big\{\langle u^*, \nabla_x g(\bar{x},\bar{p})u\rangle \big|\ u^* \in D^*N_\Theta(\bar{y},\bar{\lambda})\big(\nabla_x g(\bar{x},\bar{p})u\big)\Big\}
\end{aligned}$$

for any $u \neq 0$, which shows that condition (5.17) in **(iv)** holds.

To complete the proof of the theorem, it remains to verify implication **(iv)**$\Longrightarrow$**(ii)**. To this end we suppose that condition (5.17) holds and take any $\mathcal{C}^2$-smooth parameterization $(\Phi(x,q), G(x,q))$ of $(\varphi(x,\bar{p}), g(x,\bar{p}))$ at some $\bar{q} \in I\!R^k$. Observe that $\nabla_x \Phi(\bar{x},\bar{q}) = \nabla_x \varphi(\bar{x},\bar{p})$, $\nabla^2_{xx}\Phi(\bar{x},\bar{q}) = \nabla^2_{xx}\varphi(\bar{x},\bar{p})$, $\nabla_x G(\bar{x},\bar{q}) = \nabla_x g(\bar{x},\bar{p})$, and $\nabla^2_{xx}G(\bar{x},\bar{q}) = \nabla^2_{xx}g(\bar{x},\bar{p})$. By replacing φ by Φ and g by G, we get both conditions **(A1)** and **(A2)** for the pair (Φ, G) at $(\bar{x},\bar{q})$. Letting $F(x,q) := \Phi(x,q) + \delta_\Theta(G(x,q))$ and combining (5.17) with the second-order representation (5.14) from Lemma 5.5 give us that (3.37) is fulfilled for F at $(\bar{x},\bar{q},0)$ and that

$$\inf\Big\{\langle u^*, u\rangle \Big|\ (u^*, q^*) \in D^*\partial_x F(\bar{x},\bar{q},0)(u)\Big\} > 0 \ \text{ for all } \ u \neq 0.$$

Unifying this with Corollary 3.15 and Theorem 3.9 allows us to find a neighborhood $(U \times W \times U^*)$ of $(\bar{x},\bar{q},0)$ and a constant $\kappa > 0$ such that the mapping S in (3.3), while replacing f by F therein, admits a localization ϑ with respect to $U^* \times V \times U$ that satisfies the Lipschitz continuity condition

$$\|\vartheta(x_1^*, q_1) - \vartheta(x_2^*, q_2)\| \le \kappa\big(\|x_1^* - x_2^*\| + \|q_1 - q_2\|\big) \ \text{ for all } \ x_1^*, x_2^* \in U^* \ \text{ and } \ q_1, q_2 \in W. \tag{5.19}$$

Define $\bar{x}(q) := \vartheta(0,q)$ for all $q \in W$ and observe that $\bar{x}(q)$ is a unique stationary point of problem $\widetilde{\mathcal{P}}(q)$ in (5.8). Furthermore, for any $q_1, q_2 \in Q$ we get from (5.19) that

$$\|\bar{x}(q_1) - \bar{x}(q_2)\| \le \kappa\|q_1 - q_2\|,$$

which ensures the Lipschitz continuity of the function $\bar{x}(q)$ and thus verifies Lipschitzian strong stability in Definition 5.2. This completes the proof of the theorem.

Observe that another characterization of strong regularity from Definition 5.4 for the class of problems modeled as $\widehat{\mathcal{P}}$ in (5.1) via a second-order condition different from (5.17) has been obtained by Bonnans and Shapiro [5, Theorem 5.64] under a certain "strong extended polyhedricity condition," which is not assumed here. Our results in Theorem 5.6 establish the *equivalence* between *all the properties* considered there for the general class of problems $\widehat{\mathcal{P}}$ with new *second-order characterization* (5.17) involving the construction D^*N_Θ for the underlying convex set Θ. Calculating this second-order object for particular cases of Θ, we arrive at characterizations of the listed properties entirely in terms of the initial data of the mathematical programs. Let us discuss several remarkable classes in mathematical programming, important from both viewpoints of optimization theory and applications, in comparison with known results in this direction. Note that all the classes discussed below we have the validity of the reducibility condition **(A1)**

• *Nonlinear programming with $\mathcal{C}^2$-smooth data* (NLP). By using the Mordukhovich criterion (2.6) and the calculation of the second-order construction D^*N_Θ for the orthant $\Theta = \{0\} \times I\!R^l_-$, Dontchev and Rockafellar [16] proved the equivalence of strong regularity to the simultaneous fulfillment of the *LICQ* and *SSOSC* conditions; see also the discussions and references therein on related results in this vein. It has been recently shown in [48] that condition (5.17) reduces for NLPs to the classical SSOSC being equivalent under the validity of LICQ to Lipschitzian full stability of local minimizers for nonlinear programs.

• *Mathematical programs with polyhedral constraints* (MPPC). Based on the second-order calculus rules from [47] and the coderivative calculations from [16], Mordukhovich, Rockafellar and Sarabi [48] established for this class of optimization problems (5.1) with a polyhedral set Θ a complete characterization of Lipschitzian full stability via the *polyhedral second-order optimality condition* (PSSOC) as well as its equivalence to strong regularity under the *polyhedral constraint qualification*, which is an analog of **(A2)** in the MPPC setting. The aforementioned PSSOC is a MPCC counterpart of the classical SSOSC obtained in the scheme of (5.17).

• *Extended nonlinear programming* (ENLP). The same paper [48] presents a second-order characterization of Lipschitzian full stability for the class of ENLP problems introduced by Rockafellar [61] via a certain duality representation. The characterization is given in terms of the *extended strong second-order optimality condition*, which is an ENLP counterpart of SSOSC obtained in the scheme of (5.17).

• *Second-order cone programming* (SOCP). This subclass of conic programs corresponds to (5.1), where Θ is a product of the *Lorentz/ice-cream cones*; see [1] for more details and applications. A characterization of strong regularity of the associated GE (5.7) was given by Bonnans and Ramírez [4] via a SOCP counterpart of SSOSC. By using the calculation of D^*N_Θ obtained by Outrata and Ramírez [50], it can be shown that this condition reduces to (5.17) in the SOCP setting, which therefore provides by Theorem 5.6 a second-order characterization of Lipschitzian full stability for SOCPs under the nondegeneracy condition **(A2)**.

• *Semidefinite programming* (SDP). This major class of conic programs, corresponding to (5.1) with $\Theta = S^m_+$, has been highly recognized in optimization theory and applications; see, e.g., [64, 65] and the references therein. In [64] Sun obtained a characterization of strong regularity of the GE (5.7) associated with SDPs via a counterpart of SSOSC in this setting under the nondegeneracy condition **(A2)**. In Section 6 we show that this SDP version of SSOSC is indeed the same as our condition (5.17) and thus derive from Theorem 5.6 a constructive second-order characterization of full (as well as strong) Lipschitzian stability of locally optimal solutions to semidefinite programs entirely via the their initial data.

• *Other classes of mathematical programs.* Besides the classes of mathematical programs listed above, the second-order construction D^*N_Θ in (5.17) has been constructively calculated for the underlying sets Ω in (5.1), which are not in the discussed forms; see, e.g., [22, 40, 47] and the references therein. These results can be incorporated in the framework of (5.17) and thus allow us to provide via Theorem 5.6 complete characterizations of the equivalent stability properties **(i)–(iii)** entirely in terms of initial data of the corresponding mathematical programs under the nondegeneracy condition **(A1)**.

We conclude this section with a convenient second-order condition ensuring the validity of the equivalent stability properties in Theorem 5.6 and therefore their implementations for the particular classes of mathematical programs discussed above. Note that a related result in this vein for Robinson's strong regularity can be extracted from [5, Theorem 5.27 and Corollary 5.29] but under an additional assumption that Y^* has a "lattice structure" that is not the case here; cf. [5, Example 3.57].

Corollary 5.7 (sufficient second-order condition for the equivalent stability properties in mathematical programming). *Let $\bar{x}$ be a stationary point of problem $\widehat{\mathcal{P}}$ in* (5.1)*, and let the conditions* **(A1)** *and* **(A2)** *be satisfied. Assume in addition the second-order condition*

$$\langle \nabla^2_{xx} L(\bar{x}, \bar{p}, \bar{\lambda}) w, w \rangle > 0 \quad \text{whenever} \quad \nabla_x g(\bar{x}, \bar{p}) w \in \operatorname{dom} D^* N_\Theta(g(\bar{x}, \bar{p}), \bar{\lambda}), \; w \neq 0, \tag{5.20}$$

where $\bar{\lambda} \in Y^*$ solves the system in (5.15)*. Then all the properties* **(i)–(iii)** *of Theorem* 5.6 *hold.*

Proof. When $z \in D^* N_\Theta(g(\bar{x}, \bar{p}), \bar{\lambda})(\nabla_x g(\bar{x}, \bar{p}) w)$, we have $\langle z, \nabla_x g(\bar{x}, \bar{p}) w \rangle \geq 0$ by the maximal monotonicity of N_Θ and [53, Theorem 2.1]. This together with (5.20) verifies (5.17) and implies therefore that $\bar{x}$ is a Lipschitzian fully stable local minimizer of problem $\widehat{\mathcal{P}}$ due to Theorem 5.6. The other stability/regularity properties of that theorem follows from the established equivalence relationships.

5.3 Full Stability in Semidefinite Programming

In this section we develop constructive and nontrivial implementations of the results of Theorem 5.6 for problems of *semidefinite programming* formulated as follows:

$$\breve{\mathcal{P}} \quad \begin{cases} \text{minimize } \ \varphi(x,\bar{p}) \ \text{ subject to } \ x \in X, \\ g(x,\bar{p}) \in \Theta := S^m_+, \end{cases} \tag{5.21}$$

where $\varphi : X \times I\!R^d \to I\!R$ and $g : X \times I\!R^d \to Y := S^m$ are $\mathcal{C}^2$-smooth mappings, where S^m is the space of $m \times m$ symmetric matrices, and where S^m_+ is the cone of all the $m \times m$ positive semidefinite matrices in S^m. Note that the cone S^m_+ satisfies the reducibility assumption **(A1)** in Section 5; see, e.g., [5, Example 3.140]. The Robinson constraint qualification (5.4) is written for (5.21) as

$$0 \in \text{int}\left\{g(\bar{x},\bar{p}) + \nabla_x g(\bar{x},\bar{p})X - S^m_+\right\} \tag{5.22}$$

and the partial nondegeneracy condition (5.13) reduces to

$$\nabla_x g(\bar{x},\bar{p})X + \text{lin}\left(T_{S^m_+}(g(\bar{x},\bar{p}))\right) = S^m. \tag{5.23}$$

The main goal of this section is to derive a complete characterization of Lipschitzian full stability of local minimizers for (5.21) *entirely in terms of the initial data* (φ, g, S^m_+) of this problem.

Let $A, B \in S^m$ and $\lambda_1(A), \ldots, \lambda_m(A)$ be m eigenvalues of the matrix A with $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_m(A)$. Denote $\lambda(A) := (\lambda_1(A), \ldots, \lambda_m(A)) \in I\!R^m$ and by $\Lambda(A) := \text{diag}\,(\lambda(A))$ the diagonal matrix whose i-th diagonal entry is $\lambda_i(A)$. Recall the *eigenvalue decomposition* of

A is given by

$$A = P \begin{pmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & \Lambda_\beta & 0 \\ 0 & 0 & \Lambda_\gamma \end{pmatrix} P^* \quad \text{with} \quad P = [P_\alpha \quad P_\beta \quad P_\gamma], \tag{5.24}$$

where $\alpha := \{i|\ \lambda_i(A) > 0\}$, $\beta := \{i|\ \lambda_i(A) = 0\}$, $\gamma := \{i|\ \lambda_i(A) < 0\}$, and where P is some $m \times m$ orthogonal matrix. Furthermore, we use the *Frobenius inner product* between A and B:

$$\langle A, B\rangle := \mathrm{Tr}\,(A^*B),$$

where "Tr" denotes the trace of a matrix; thus the norm of $A \in S^m$ is $\|A\| = \sqrt{\mathrm{Tr}\,(A^*A)}$. With these constructions it is well known that the dual space of S^m reduces to S^m.

The next condition is taken from Sun [64, Definition 3.2].

Definition 5.8 (strong second-order sufficient condition for SDPs.) *Let $\bar{x}$ be a stationary point of $\breve{\mathcal{P}}$, and let the partial nondegeneration condition* (5.23) *be satisfied. We say that the* SDP-STRONG SECOND-ORDER SUFFICIENT CONDITION *(SDP-SSOSC) holds at $\bar{x}$ if*

$$\langle \nabla^2_{xx} L(\bar{x}, \bar{p}, \bar{\lambda})w, w\rangle - 2\langle \bar{\lambda}, d(w)g(\bar{x}, \bar{p})^\dagger d(w)\rangle > 0 \ \ \textit{for all} \ \ w \in \mathrm{app}\,(\bar{\lambda}) \setminus \{0\}, \tag{5.25}$$

where $\bar{\lambda}$ is the corresponding unique Lagrange multiplier, $d(w) := \nabla_x g(\bar{x}, \bar{p})w$, $g(\bar{x}, \bar{p})^\dagger$ is the MOORE-PENROSE PSEUDOINVERSE *of $g(\bar{x}, \bar{p})$, and where* $\mathrm{app}\,(\bar{\lambda})$ *is defined by*

$$\mathrm{app}\,(\bar{\lambda}) := \left\{ w \in X \,\middle|\, P_\beta^* d(w) P_\gamma = 0,\ P_\gamma^* d(w) P_\gamma = 0 \right\} \tag{5.26}$$

with the matrix P taken from (5.24) *for* $A = g(\bar{x}, \bar{p}) + \bar{\lambda}$.

Since we use this condition simultaneously with the nondegeneration assumption **(A2)** in Section 5, it makes sense to formulate the above condition under **(A2)**.

As discussed in [64, p. 768], the choice of an orthogonal matrix P satisfying the decomposition (5.24) with $A = g(\bar{x}, \bar{p}) + \bar{\lambda}$ does not affect the set app$\,(\bar{\lambda})$ in (5.26).

The following calculation of the second-order construction $D^*N_{S^m_+}$ is a reformulation of the recent result from Ding, Sun and Ye [11, Theorem 3.1].

Lemma 5.9 (second-order subdifferential calculation for SDPs). *For any* $(X, Y) \in \mathrm{gph}\, N_{S^m_+}$ *consider the the eigenvalue decomposition* (5.24) *of the matrix* $A = X + Y$. *Then we have* $Z \in D^*N_{S^p_+}(X, Y)(D)$ *if and only if* $Z = P\widetilde{Z}P^*$ *and* $D = P\widetilde{D}P^*$ *with*

$$\textbf{(i)} \qquad \widetilde{Z} = \begin{pmatrix} 0 & 0 & \widetilde{Z}_{\alpha\gamma} \\ 0 & \widetilde{Z}_{\beta\beta} & \widetilde{Z}_{\beta\gamma} \\ \widetilde{Z}_{\gamma\alpha} & \widetilde{Z}_{\gamma\beta} & \widetilde{Z}_{\gamma\gamma} \end{pmatrix} \ \textit{and}\ \widetilde{D} = \begin{pmatrix} \widetilde{D}_{\alpha\alpha} & \widetilde{D}_{\alpha\beta} & \widetilde{D}_{\alpha\gamma} \\ \widetilde{D}_{\beta\alpha} & \widetilde{D}_{\beta\beta} & 0 \\ \widetilde{D}_{\gamma\alpha} & 0 & 0 \end{pmatrix}, \tag{5.27}$$

$$\textbf{(ii)} \qquad \widetilde{Z}_{\beta\beta} \in D^*N_{S^{|\beta|}_+}(0,0)(\widetilde{D}_{\beta\beta}) \ \textit{and}\ \Sigma_{\alpha\gamma} \circ \widetilde{Z}_{\alpha\gamma} - (E_{\alpha\gamma} - \Sigma_{\alpha\gamma}) \circ \widetilde{D}_{\alpha\gamma} = 0, \tag{5.28}$$

where α, β, γ *are taken from* (5.24), $|\beta|$ *is the cardinality of the set* β, E *is a* $m \times m$ *matrix whose all the unit entries, "$\circ$" is the Hadamard product, and where the matrix* Σ *is defined by*

$$\Sigma_{ij} := \frac{\max\{\lambda_i(A), 0\} - \max\{\lambda_j(A), 0\}}{\lambda_i(A) - \lambda_j(A)}, \quad i, j = 1, \ldots, m, \tag{5.29}$$

with the convention that $0/0 := 1$.

Proof. Note that $Z \in D^*N_{S^p_+}(X, Y)(D)$ if and only if $(Z, -D) \in N_{\mathrm{gph}\, N_{S^p_+}}(X, Y)$. Employing [11, Theorem 3.1] verifies claimed representations in the lemma.

The next result is new and plays a crucial role in deriving the main theorem of this section presented below. This lemma provides a precise calculation of the second-order subdifferential condition (5.17) from Theorem 5.6 for the SDP model and shows that it reduces to the SDP-SSOSC condition from Definition 5.8.

Lemma 5.10 (second-order subdifferential condition for SDPs). *Let $\bar{x}$ be a stationary point of problem* (5.21)*, and let $\bar{\lambda}$ is a unique Lagrange multiplier of the corresponding KKT system* (5.5) *under the validity of the partial nondegeneration condition* (5.23)*. Then we have* $\operatorname{dom} D^*N_{S^m_+}(g(\bar{x},\bar{p}),\bar{\lambda})(d(\cdot)) = \operatorname{app}(\bar{\lambda})$ *and*

$$\inf\Big\{\langle Z, d(w)\rangle \Big|\ Z \in D^*N_{S^m_+}(g(\bar{x},\bar{p}),\bar{\lambda})(d(w))\Big\} = -2\langle\bar{\lambda}, d(w)g(\bar{x},\bar{p})^{\dagger}d(w)\rangle \quad \textit{if} \ \ w \in \operatorname{app}(\bar{\lambda}) \tag{5.30}$$

with $d(w) := \nabla_x g(\bar{x},\bar{p})w$. Consequently, the second-order subdifferential condition (5.17) *from Theorem* 5.6 *agrees with the SDP-SSOSC condition from Definition* 5.8.

Proof. We split the proof of this lemma into following two main steps.

Step 1. *We have that* $\operatorname{dom} D^*N_{S^m_+}(g(\bar{x},\bar{p}),\bar{\lambda})(d(\cdot)) \subset \operatorname{app}(\bar{\lambda})$ *and that the inequality "$\geq$" holds in* (5.30).

To show it, pick any $w \in \operatorname{dom} D^*N_{S^m_+}(g(\bar{x},\bar{p}),\bar{\lambda})(d(\cdot))$ and find $Z \in D^*N_{S^m_+}(g(\bar{x},\bar{p}),\bar{\lambda})(d(w))$. Let $A := g(\bar{x},\bar{p}) + \bar{\lambda}$, and let P be an orthogonal matrix satisfying (5.24). With $D := d(w)$ it follows from Lemma 5.9 that $Z = P\widetilde{Z}P^*$ and $D = P\widetilde{D}P^*$, where $\widetilde{Z}, \widetilde{D}$ are taken from (5.27). We get $\widetilde{D} = P^*DP$ and so

$$P^*_\beta D P_\gamma = 0 \ \ \text{and} \ \ P^*_\gamma D P_\gamma = 0,$$

which verifies that $w \in \operatorname{app}(\bar{\lambda})$ due to its expression in (5.26). It gives us the inclusion $\operatorname{dom} D^*N_{S^m_+}(g(\bar{x},\bar{p}),\bar{\lambda})(d(\cdot)) \subset \operatorname{app}(\bar{\lambda})$.

Furthermore, observe from (5.27) that

$$\begin{aligned}\langle Z, D\rangle &= \mathrm{Tr}(\mathrm{P}\widetilde{\mathrm{Z}}^*\mathrm{P}^*\mathrm{P}\widetilde{\mathrm{D}}\mathrm{P}^*) = \mathrm{Tr}(\widetilde{\mathrm{Z}}^*\widetilde{\mathrm{D}}\mathrm{P}^*\mathrm{P}) = \mathrm{Tr}(\widetilde{\mathrm{Z}}^*\widetilde{\mathrm{D}})\\ &= \mathrm{Tr}(\widetilde{\mathrm{Z}}^*_{\gamma\alpha}\widetilde{\mathrm{D}}_{\gamma\alpha}) + \mathrm{Tr}(\widetilde{\mathrm{Z}}^*_{\beta\beta}\widetilde{\mathrm{D}}_{\beta\beta}) + \mathrm{Tr}(\widetilde{\mathrm{Z}}^*_{\alpha\gamma}\widetilde{\mathrm{D}}_{\alpha\gamma})\\ &= \mathrm{Tr}(\widetilde{\mathrm{Z}}^*_{\beta\beta}\widetilde{\mathrm{D}}_{\beta\beta}) + 2\mathrm{Tr}(\widetilde{\mathrm{Z}}^*_{\alpha\gamma}\widetilde{\mathrm{D}}_{\alpha\gamma}).\end{aligned} \tag{5.31}$$

By (5.29) for any $i \in \alpha$ and $j \in \gamma$, we have $\Sigma_{ij} = \frac{\lambda_i(A)}{\lambda_i(A)-\lambda_j(A)}$, and thus (5.28) implies that

$$\frac{\lambda_i(A)}{\lambda_i(A)-\lambda_j(A)}\widetilde{Z}_{ij} + \frac{\lambda_j(A)}{\lambda_i(A)-\lambda_j(A)}\widetilde{D}_{ij} = 0,$$

which ensures therefore the equalities

$$\mathrm{Tr}(\widetilde{\mathrm{Z}}^*_{\alpha\gamma}\widetilde{\mathrm{D}}_{\alpha\gamma}) = \sum_{\mathrm{i}\in\alpha,\mathrm{j}\in\gamma}\widetilde{\mathrm{Z}}_{\mathrm{ij}}\widetilde{\mathrm{D}}_{\mathrm{ij}} = \sum_{\mathrm{i}\in\alpha,\mathrm{j}\in\gamma} -\frac{\lambda_{\mathrm{j}}(\mathrm{A})}{\lambda_{\mathrm{i}}(\mathrm{A})}\widetilde{\mathrm{D}}^2_{\mathrm{ij}}. \tag{5.32}$$

By the spectral decomposition (5.24) and the fact that $\bar{\lambda} \in N_{S^m_+}(g(\bar{x},\bar{p}))$, which actually means that $-\bar{\lambda} \in S^m_+$ and $\langle\bar{\lambda}, g(\bar{x},\bar{p})\rangle = 0$, we get the representations

$$g(\bar{x},\bar{p}) = P\begin{pmatrix}\Lambda_\alpha & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}P^* \quad\text{and}\quad \bar{\lambda} = P\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \Lambda_\gamma\end{pmatrix}P^*. \tag{5.33}$$

Hence the Moore-Penrose matrix $g(\bar{x},\bar{p})^\dagger$ is formulated in this case as

$$g(\bar{x},\bar{p})^\dagger = P\begin{pmatrix}\Lambda_\alpha^{-1} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}P^*.$$

This together with (5.33) gives us that

$$
\begin{aligned}
\langle\lambda, d(w)g(\bar{x},\bar{p})^{\dagger}d(w)\rangle &= \mathrm{Tr}\left[\mathrm{P}\begin{pmatrix}0&0&0\\0&0&0\\0&0&\Lambda_\gamma\end{pmatrix}\mathrm{P^*DP}\begin{pmatrix}\Lambda_\alpha^{-1}&0&0\\0&0&0\\0&0&0\end{pmatrix}\mathrm{P^*D}\right]\\
&= \mathrm{Tr}\left[\widetilde{\mathrm{D}}\begin{pmatrix}0&0&0\\0&0&0\\0&0&\Lambda_\gamma\end{pmatrix}\widetilde{\mathrm{D}}\begin{pmatrix}\Lambda_\alpha^{-1}&0&0\\0&0&0\\0&0&0\end{pmatrix}\right]\\
&= \mathrm{Tr}\left[\begin{pmatrix}0&0&\widetilde{D}_{\alpha\gamma}\Lambda_\gamma\\0&0&\widetilde{D}_{\beta\gamma}\Lambda_\gamma\\0&0&\widetilde{D}_{\gamma\gamma}\Lambda_\gamma\end{pmatrix}\begin{pmatrix}\widetilde{D}_{\alpha\alpha}\Lambda_\alpha^{-1}&0&0\\\widetilde{D}_{\beta\alpha}\Lambda_\alpha^{-1}&0&0\\\widetilde{D}_{\gamma\alpha}\Lambda_\alpha^{-1}&0&0\end{pmatrix}\right]\\
&= \mathrm{Tr}\big[\widetilde{\mathrm{D}}_{\alpha\gamma}\Lambda_\gamma\widetilde{\mathrm{D}}_{\gamma\alpha}\Lambda_\alpha^{-1}\big] = \sum_{\mathrm{i}\in\alpha,\mathrm{j}\in\gamma}\frac{\lambda_\mathrm{j}(\mathrm{A})}{\lambda_\mathrm{i}(\mathrm{A})}\widetilde{\mathrm{D}}_{\mathrm{ij}}^2.
\end{aligned}
$$

We obtain from this representation as well as (5.31) and (5.32) that

$$
\langle Z, D\rangle = \langle\widetilde{Z}_{\beta\beta}, \widetilde{D}_{\beta\beta}\rangle - 2\langle\lambda, d(w)g(\bar{x},\bar{p})^{\dagger}d(w)\rangle. \tag{5.34}
$$

Taking into account that the mapping $N_{S_+^{|\beta|}}$ is maximally monotone, it follows from (5.28) and [53, Theorem 2.1] that $\langle\widetilde{Z}_{\beta\beta}, \widetilde{D}_{\beta\beta}\rangle \geq 0$. This together with (5.34) verifies the inequality "$\geq$" in (5.30) for any $w \in \mathrm{dom}\, D^*N_{S_+^m}(g(\bar{x},\bar{p}),\bar{\lambda})(d(\cdot))$ and thus completes the proof of Step 1.

Step 2. *We have that* $\mathrm{app}\,(\bar{\lambda}) \subset \mathrm{dom}\, D^*N_{S_+^m}(g(\bar{x},\bar{p}),\bar{\lambda})(d(\cdot))$ *and that the inequality "$\leq$" holds in* (5.30).

To verify this, pick $w \in \mathrm{app}\,(\bar{\lambda})$ and define $D := d(w)$. It follows from (5.26) that $\widetilde{D} := P^*DP$ is of form (5.27). Observe from [11, Proposition 3.3], by choosing $\Xi_1 = E$ therein, that

$0 \in D^*N_{S_+^{|\beta|}}(0,0)(\widetilde{D}_{\beta\beta})$. By (5.28) find a matrix $\widetilde{Z}$ of form (5.27) satisfying (5.28) and $\widetilde{Z}_{\beta\beta} = 0$. With $Z := P\widetilde{Z}P^*$ it follows from Lemma 5.9 that $Z \in D^*N_{S_+^m}(g(\bar{x},\bar{p}),\bar{\lambda})(D)$. Thus we have $w \in \operatorname{dom} D^*N_{S_+^m}(g(\bar{x},\bar{p}),\lambda)(d(\cdot))$ and deduce from (5.34) that $\langle Z, D\rangle = -2\langle\lambda, d(w)g(\bar{x},\bar{p})^\dagger d(w)\rangle$. This also verifies the inequality "$\leq$" in (5.30) and thus completes the verification of the assertions claimed in Step 2.

Combining finally Step 1 and Step 2 allows us to obtain $\operatorname{dom} D^*N_{S_+^m}(g(\bar{x},\bar{p}),\bar{\lambda})(d(\cdot)) = \operatorname{app}(\bar{\lambda})$ and justify equality (5.25). Hence the second-order subdifferential condition (5.17) agrees with the SDP-SSOSC condition from Definition 5.8, which therefore completes the proof of the lemma.

This lemma together with our major results in Theorem 5.6 allows us not only to recover the equivalence between Robinson's strong regularity and the SDP-SSOSC condition from [64, Theorem 4.1] but also characterize Lipschitzian full stability and strong stability in the SDP.

Theorem 5.11 (second-order characterization of Lipschitzian full stability and equivalent properties for SDPs). *Let $\bar{x}$ be a stationary point of problem $\breve{\mathcal{P}}$ in* (5.21)*, and let $\bar{\lambda}$ be the corresponding Lagrange multiplier from* (5.5) *under the validity of RCQ* (5.22)*. The following assertions are equivalent:*

(i) *The point $(\bar{x},\bar{\lambda})$ is strongly regular for* (5.12)*, and $\bar{x}$ is a local minimizer of problem $\breve{\mathcal{P}}$.*

(ii) *The partial nondegeneration condition* (5.23) *holds, and the point $\bar{x}$ is Lipschitzian strongly stable local minimizer of problem $\breve{\mathcal{P}}$.*

(iii) *The partial nondegeneration condition* (5.23) *holds, and the point $\bar{x}$ is a Lipschitzian fully stable local minimizer of problem $\breve{\mathcal{P}}$.*

(iv) *Both conditions* (5.23) *and SDP-SSOSC from Definition* 5.8 *hold.*

Proof. It follows directly by combining Theorem 5.6 and Lemma 5.10.

5.4 Full Stability in Nonlinear Programming without LICQ

As demonstrated in Section 5.1, the partial nondegeneration condition (5.23) turns into the classical linear independence constraint qualification (LICQ) in nonlinear programming (NLP). In this case full stability has been characterized by SSOSC in the recent work [48]. Without LICQ which second-order condition distinguishes full stability NLP is in question. In this section we introduce a new condition so-called *uniform second-order sufficient condition* and show that it is a complete characterization of full stability under both MFCQ and *constant rank constraint qualification* (CRCQ). Note that the validity of both latter conditions are still weaker than LICQ. In the view of Theorem 5.6, this also tells us that full stability is a strictly weaker property than Robinson's strong regularity in Definition 5.4. Moreover, in the late Example 5.17 we show that full stability is different from Kojima's strong stability in Definition 5.2.

Consider the problem of nonlinear programming (NLP) given by:

$$\overline{\mathcal{P}} \quad \begin{cases} \text{minimize } \ \varphi(x,\bar{p}) \text{ subject to } \ x \in X, \\ g_i(x,\bar{p}) \le 0 \ \text{ for } \ i = 1,\ldots,m, \end{cases} \tag{5.35}$$

where $X = I\!R^n, P = I\!R^d$, and where all the functions $\varphi, g_i : X \times P \to \overline{I\!R}$, $i = 1,\ldots,m$, are $\mathcal{C}^2$ around the reference point $(\bar{x},\bar{p}) \in X \times P$. Define the set of feasible solutions

$$\Omega := \big\{(x,p) \in X \big|\, \varphi(x,p) \in I\!R^m_- \big\} \ \text{ with } \ g(x,p) := \big(g_1(x,p),\ldots,g_m(x,p)\big). \tag{5.36}$$

Recall the partial *Mangasarian-Fromovitz constraint qualification* (MFCQ) with respect to x holds at $(\bar{x},\bar{p}) \in \Omega$ if there is $d \in X$ such that

$$\langle \nabla_x g_i(\bar{x},\bar{p}), d\rangle < 0 \ \text{ for } \ i \in I(\bar{x},\bar{p}) := \big\{i \in \{1,\ldots,m\} \big|\, g_i(\bar{x},\bar{p}) = 0\big\}. \tag{5.37}$$

Note that this condition is equivalent to RCQ for NLP. The Lagrange function (5.6) becomes

$$L(x,p,\lambda)=\varphi(x,p)+\sum_{i=1}^{m}\lambda_i g_i(x,p)\ \text{ with }\ x\in I\!R^n,\ p\in I\!R^d,\ \text{ and }\ \lambda\in I\!R^m$$

and then define the set-valued mapping $\Psi: I\!R^n\times I\!R^d \rightrightarrows I\!R^n$ by

$$\Psi(x,p):=\big\{\nabla_x L(x,p,\lambda)\big|\ \lambda\in N\big(g(x,p);\Theta\big)\big\}\ \text{ with }\ \Theta:=I\!R^m_- \tag{5.38}$$

and g from (5.36). It is well known that $\partial_x f(x,p)=\Psi(x,p)$ for all (x,p) around $(\bar{x},\bar{p})$ under the validity of MFCQ (5.37), where f is defined in (5.3). Furthermore, every local minimizer $\bar{x}$ of $\overline{\mathcal{P}}(\bar{x}^*,\bar{p})$ for $\bar{x}^*\in\partial_x f(\bar{x},\bar{p})$ satisfies the *Karush-Kuhn-Tucker* (*KKT*) *system*

$$\bar{x}^*\in\nabla_x\varphi(\bar{x},\bar{p})+\nabla_x g(\bar{x},\bar{p})^*\lambda=\nabla_x L(\bar{x},\bar{p},\lambda)\ \text{ with some }\ \lambda\in N\big(g(\bar{x},\bar{p});\Theta\big), \tag{5.39}$$

where the set of *Lagrange multipliers* is represented by

$$\Lambda(\bar{x},\bar{p},\bar{x}^*):=\big\{\lambda\in I\!R^m_+\big|\ \bar{x}^*\in\nabla_x L(\bar{x},\bar{p},\lambda),\ \langle\lambda,\varphi(\bar{x},\bar{p})\rangle=0\big\}. \tag{5.40}$$

It follows from Proposition 2.8 that the validity of MFCQ (5.37) implies that BCQ (3.8) holds for the function f in (5.3) at $(\bar{x},\bar{p})$.

Let us further recall the following partial counterpart of the classical *strong second-order sufficient condition* (SSOSC) in nonlinear programming [58]: given $(\bar{x},\bar{p})\in\Omega$ and $\bar{x}^*\in\Psi(\bar{x},\bar{p})$ in (5.38), the *partial SSOSC* holds at $(\bar{x},\bar{p},\bar{x}^*)$ if for all $\lambda\in\Lambda(\bar{x},\bar{p},\bar{x}^*)$ we have

$$\langle u,\nabla^2_{xx}L(\bar{x},\bar{p},\lambda)u\rangle>0\ \text{ whenever }\ \langle\nabla_x\varphi_i(\bar{x},\bar{p}),u\rangle=0\ \text{ as }\ i\in I_+(\bar{x},\bar{p},\lambda),\ u\neq 0 \tag{5.41}$$

with the *strict complementarity* index set $I_+(\bar{x},\bar{p},\lambda) := \big\{i \in \{1,\ldots,m\}\big|\ \lambda_i > 0\big\}$.

Next we formulate a partial version of the uniform second-order sufficient condition (USOSC) to characterize tilt stability in nonlinear programming.

Definition 5.12 (uniform second-order sufficient condition) *We say that the* USOSC *with respect to x holds at $(\bar{x},\bar{p}) \in \Omega$ with $\bar{x}^* \in \Psi(\bar{x},\bar{p})$ if there are $\eta,\ell > 0$ such that*

$$\langle \nabla^2_{xx} L(x,p,\lambda)u,u\rangle \ge \ell\|u\|^2 \ \ \textit{for all}\ \ (x,p,x^*) \in \operatorname{gph}\Psi \cap I\!B_\eta(\bar{x},\bar{p},\bar{x}^*),\ \lambda \in \Lambda(x,p,x^*),$$
$$\langle \nabla_x \varphi_i(x,p),u\rangle = 0 \ \ \textit{as}\ \ i \in I_+(x,p,\lambda) \ \ \textit{and}\ \ \langle \nabla_x g_i(x,p),u\rangle \ge 0 \ \ \textit{as}\ \ i \in I(x,p)\setminus I_+(x,p,\lambda),$$

where the mapping Ψ and the set $\Lambda(x,p,x^*)$ are defined in (5.38) *and* (5.40)*, respectively.*

Next we show that, under the validity of the MFCQ (5.37), the partial SSOSC (5.41) implies the partial USOSC from Definition 5.12 at $(\bar{x},\bar{p}) \in \Omega$ with $\bar{x}^* \in \Psi(\bar{x},\bar{p})$.

Proposition 5.13 (SSOSC implies USOSC under MFCQ). *Let $(\bar{x},\bar{p}) \in \Omega$ satisfy* (5.39) *under the validity of MFCQ at $(\bar{x},\bar{p})$. Assume also that SSOSC* (5.41) *holds w.r.t. x at this point. Then USOSC from Definition* 5.12 *is satisfied w.r.t. x at $(\bar{x},\bar{p})$.*

Proof. Arguing by contradiction that the PUSOSC is not satisfied w.r.t. x at $(\bar{x},\bar{p})$ gives us the existence of sequences $(x_k,p_k,x_k^*) \overset{\operatorname{gph}\Psi}{\to} (\bar{x},\bar{p},\bar{x}^*)$, $\lambda_k \in \Lambda(x_k,p_k,x_k^*)$, and $u_k \in X$ with

$$\begin{aligned} &\langle \nabla^2_{xx} L(x_k,p_k,\lambda_k)u_k,u_k\rangle \le \tfrac{1}{k}\|u_k\|^2 \quad \text{whenever} \quad \langle \nabla_x \varphi_i(x_k,p_k),u_k\rangle = 0 \ \text{ for }\ i \in I_+(x_k,p_k,\lambda_k) \\ &\quad \text{and} \quad \langle \nabla_x \varphi_i(x_k,p_k),u_k\rangle \ge 0 \ \text{ for }\ i \in I(x_k,p_k)\setminus I_+(x_k,p_k,\lambda_k). \end{aligned} \tag{5.42}$$

With no loss of generality assume that $\|u_k\| = 1$ for all $k \in I\!N$. Since the MFCQ holds w.r.t. x at $(\bar{x},\bar{p})$, the Lagrange multipliers λ_k are bounded. By passing to subsequence, we may assume further that $u_k \to u$ with $\|u\| = 1$ and $\lambda_k \to \lambda$ with $\lambda \in \Lambda(\bar{x},\bar{p},\bar{x}^*)$. Moreover, observe that

$I_+(x_k, p_k, \lambda_k)$ is a subset of $I_+(\bar{x}, \bar{p}, \lambda)$ for sufficiently large k. Taking $k \to \infty$ in (5.42) gives us

$$\langle \nabla^2_{xx} L(x_k, p_k, \lambda_k)u, u\rangle \le 0 \quad \text{with} \quad \langle \nabla_x \varphi_i(\bar{x}, \bar{p}), u\rangle = 0 \ \text{ for } \ i \in I_+(\bar{x}, \bar{p}, \lambda),$$

which contradicts to SSOSC (5.41). The proof of the proposition is completed. $\square$

The last qualification condition needed in this section is the following partial version [55] of the *constant rank constraint qualification* (CRCQ) for NLP (5.35): the *partial CRCQ* with respect to x holds at $(\bar{x}, \bar{p}) \in \Omega$ if there is a neighborhood W of $(\bar{x}, \bar{p})$ such that for any subset J of $I(\bar{x}, \bar{p})$ the gradient family $\{\nabla_x g_i(x,p) \big|\ i \in J\}$ has the same rank in W. It occurs that the simultaneous fulfillment of the partial MFCQ and CRCQ ensures the Lipschitz-like property of graphical mapping G in (3.33) crucial for the (Lipschitzian) full stability results in Section 4.

Proposition 5.14 (graphical Lipschitz-like property under partial MFCQ and CRCQ). *Assume that both partial MFCQ and partial CRCQ conditions at $(\bar{x}, \bar{p}) \in \Omega$. Then, given any $\bar{x}^*$ from* (5.39)*, the Lipschitz-like property* (3.33) *holds around $(\bar{p}, \bar{x}, \bar{x}^*)$.*

Proof. Consider the parametric optimization problem

$$\mathcal{Q}(x^*, p) \qquad \begin{cases} \text{minimize } \ \frac{1}{2}\|x - x^*\|^2 \ \text{ subject to } x \in I\!\!R^n, \\ g_i(x,p) \le 0 \ \text{ for } \ i = 1, \ldots, m \ \text{ with some } \ x^* \in I\!\!R^n \end{cases}$$

and observe that $\bar{x}$ is the only minimizer of $\mathcal{Q}(\bar{x}, \bar{p})$. It is easy to check that the partial SSOSC (5.41) holds $(\bar{x}, \bar{p})$ for $\mathcal{Q}(\bar{x}, \bar{p})$. By [55, Theorem 2] we find neighborhoods U, U^* of $\bar{x}$ and V of $\bar{p}$ together with a Lipschitzian mapping $\pi : U^* \times V \to U$ such that $\pi(x^*, p)$ is the unique minimizer of $\mathcal{Q}(x^*, p)$ for all $(x^*, p) \in U^* \times V$. Define $S(p) := \{x \in I\!\!R^n |\ (x,p) \in \Omega\} = \{x \in I\!\!R^n |\ g(x,p) \in \Theta\}$ and observe that $\pi(x^*, p)$ is the projection from x^* to $S(p)$.

By the structures of g and Θ we see that the set $S(p)$ is *fully amenable* at any $x \in S(p)$ in the sense of [62, Definition 10.23] whenever $p \in V$. Invoking [29, Lemma 5], we obtain that

$$\pi(x^*,p)\cap U=\big(I+N(\cdot;S(p))\big)^{-1}(x^*)\cap U=\big(I+\partial_x\delta_\Omega(\cdot,p)\big)^{-1}(x^*)\cap U \tag{5.43}$$

for all $x^* \in U^*$ and $p \in V$. To verify (3.33) by the coderivative criterion in finite dimensions, we need to check (3.37) for f from (5.3). To proceed, observe by Lemma 2.5 that

$$\begin{aligned}\big(D^*\partial_x f\big)(\bar{x},\bar{p},\bar{x}^*)(0) &= D^*\big(\nabla_x\varphi+\partial_x\delta_\Omega\big)(\bar{x},\bar{p},\bar{x}^*)(0)=\big(D^*\partial_x\delta_\Omega\big)\big(\bar{x},\bar{p},\bar{x}^*-\nabla_x\varphi(\bar{x},\bar{p})\big)(0)\\ &=\big(D^*\partial_x\delta_\Omega\big)\big(\bar{x},\bar{p},\alpha(\bar{x}^*-\nabla_x\varphi(\bar{x},\bar{p}))\big)(0)\end{aligned}$$

for all $\alpha>0$. Fix $\alpha>0$ with $\bar{x}^*_\alpha := \alpha(\bar{x}^*-\nabla_x\varphi(\bar{x},\bar{p}))+\bar{x}\in U^*$ and observe by $\alpha(\bar{x}^*-\nabla_x\varphi(\bar{x},\bar{p}))\in N(\bar{x};S(\bar{p}))$ and (5.43) that $\pi(\bar{x}^*_\alpha,\bar{p})=\bar{x}$. By Lemma 2.5 and (5.43) we have

$$(0,p^*)\in\big(D^*\partial_x\delta_\Omega\big)\big(\bar{x},\bar{p},\alpha(\bar{x}^*-\nabla_x\varphi_0(\bar{x},\bar{p}))\big)(0)\Longrightarrow(0,p^*)\in D^*\pi(\bar{x}^*_\alpha,\bar{p},\bar{x})(0).$$

Since π is Lipschitz continuous on $U^*\times V$, this ensures together with [38, Theorem 1.44] that (3.37) holds, which thus completes the proof of the proposition. $\square$

Our next result shows that the partial USOSC from Definition 5.12 completely characterizes full stability in $\overline{\mathcal{P}}(x^*,p)$ under the validity of partial MFCQ and CRCQ.

Theorem 5.15 (second-order characterization of full stability under partial MFCQ and CRCQ). *Let $(\bar{x},\bar{p})\in\Omega$ and $\bar{x}^*\in\Psi(\bar{x},\bar{p})$ satisfy* (5.39)*, and let both partial MFCQ and CRCQ conditions hold at $(\bar{x},\bar{p})$. Then the following assertions are equivalent:*

(i) *The point $\bar{x}$ is a fully stable local minimizer of $\overline{\mathcal{P}}(\bar{x}^*,\bar{p})$.*

(ii) *The USOSC from Definition* 5.12 *holds at* $(\bar{x},\bar{p},\bar{x}^*)$.

Proof. Due to robustness of MFCQ (5.37), suppose that it holds together with CRCQ for all $(x,p)\in I\!B_\eta(\bar{x},\bar{p})$ with some small $\eta>0$. Pick any $u^*\in\breve{\partial}^2 f_p(x,x^*)(u)$ with $(x,p,x^*)\in\operatorname{gph}\partial_x f\cap I\!B_\eta(\bar{x},\bar{p},\bar{x}^*)$ and get from Lemma 2.5 that

$$\breve{\partial}^2 f_p(x,x^*)(u)=\nabla^2_{xx}\varphi(x,p)u+\breve{\partial}^2\delta_{S(p)}\big(x,x^*-\nabla_x\varphi(x,p)\big)(u) \tag{5.44}$$

with $S(p)=\{x\in I\!R^n|\ (x,p)\in\Omega\}$. Employing this together with the exact calculation of $\breve{\partial}^2\delta_{S(p)}(x,x^*-\nabla_x\varphi(x,p))(u)$ given in [21, Theorem 6] ensures that

$$u^*-\nabla^2_{xx}L(x,p,\lambda)u\in\mathcal{K}\big(x,x^*-\nabla_x\varphi(x,p)\big)^*\ \text{ and }\ -u\in\mathcal{K}\big(x,x^*-\nabla_x\varphi(x,p)\big) \tag{5.45}$$

for any $\lambda\in\Lambda(x,p,x^*)$, where $\mathcal{K}\big(x,x^*-\nabla_x\varphi(x,p)\big):=\widehat{N}\big(x;S(p)\big)^*\cap\big\{x^*-\nabla_x\varphi(x,p)\big\}^\perp$ is the corresponding *critical cone*. It follows from MFCQ (5.37) that

$$\widehat{N}\big(x;S(p)\big)^*=\big\{w\in X\big|\ \langle\nabla_x g_i(x,p),w\rangle\le 0,\ i\in I(x,p)\big\}.$$

By using this formula and the fact that $x^*-\nabla_x\varphi_0(x,p)=\sum_{i=1}^m\lambda_i\nabla_x\varphi_i(x,p)$ valid for any $\lambda\in\Lambda(x,p,x^*)$ by (5.39), we get the representation

$$-u\in\mathcal{K}\big(x,x^*-\nabla_x\varphi(x,p)\big)\Longleftrightarrow\begin{cases}\langle\nabla_x g_i(x,p),u\rangle=0\text{ if }i\in I_+(x,p,\lambda),\\ \langle\nabla_x g_i(x,p),u\rangle\ge 0\text{ if }i\in I(x,p)\setminus I_+(x,p,\lambda).\end{cases} \tag{5.46}$$

Assuming now (ii) and combining (5.44) with (5.45) and (5.46) gives us that

$$\langle u^*,u\rangle=\langle\nabla^2_{xx}L(x,p,\lambda)u,u\rangle+\langle u^*-\nabla^2_{xx}L(x,p,\lambda)u,u\rangle\ge\langle\nabla^2_{xx}L(x,p,\lambda)u,u\rangle\ge\ell\|u\|^2,$$

where $\ell > 0$ is from Definition 5.12. Then Corollary 3.11 and Proposition 5.14 ensure (i).

Conversely, assuming (i) implies by (5.44) and (5.45) that $u_0^* := \nabla_{xx}^2 L(x,p,\lambda)u + x^* - \nabla_x \varphi_0(x,p) \in \breve{\partial}^2 f_p(x,x^*)(u)$ if u satisfies (5.46). Since $u \in \{x^* - \nabla_x \varphi_0(x,p)\}^\perp$, we get from Corollary 3.11 that

$$\langle \nabla_{xx}^2 L(x,p,\lambda)u, u\rangle = \langle u_0^*, u\rangle \geq \kappa \|u\|^2$$

with $\kappa > 0$ from (3.27). Thus we conclude by (5.46) that the partial USOSC from Definition 5.12 holds, which completes the proof of the theorem. $\square$

When f does not depend on p, Theorem 5.15 recovers the characterization of tilt stability obtained in [43, Theorem 4.3]. For full stability we improve the recent result of [48, Corollary 6.8], where full stability in $\overline{\mathcal{P}}(x^*, \bar{p})$ is characterized via SSOSC (5.41) under the validity of the *linear independence constraint qualification* (LICQ), which implies both partial MFCQ and CRCQ and ensures that the partial USOSC agrees with its SSOSC counterpart. It is worth noting that Theorem 5.15 implies that SSOSC is a *sufficient* condition for full stability under the validity of both MFCQ and CRCQ. In fact this result can be distilled from [55, Theorem 2].

Moreover, Theorem 5.15 allows us to derive a stronger result about the uniqueness and Lipschitz continuity of local minimizers for $\overline{\mathcal{P}}(\bar{x}^*, p)$ with respect to the basic parameter p.

Corollary 5.16 (Lipschitz continuity of local minimizers with respect to the basic parameter). *Let $(\bar{x}, \bar{p}) \in \Omega$ and $\bar{x}^* \in I\!R^n$ satisfy (5.39). Assume that the partial MFCQ, CRCQ, and USOSC hold at $(\bar{x}, \bar{p})$. Then there are neighborhoods U of $\bar{x}$, V of $\bar{p}$ and a Lipschitz continuous mapping $x(\cdot) : V \to U$ such that $x(p)$ is a unique local solution to the problem $\overline{\mathcal{P}}(\bar{x}^*, p)$.*

Proof. By Theorem 5.15 the assumed partial MFCQ, CRCQ, and USOSC imply that $\bar{x}$ is a fully stable local minimizer of $\overline{\mathcal{P}}(\bar{x}^*,\bar{p})$. Define the mapping $x(p) := M_\gamma(\bar{x}^*,p)$ for all $p \in V$ with $U = I\!B_\gamma(\bar{x})$, where M_γ and V are taken from Definition 3.1. Then we get that $x(\cdot)$ is Lipschitz continuous on V and for each $p \in V$ it is a unique local solution to $\overline{\mathcal{P}}(\bar{x}^*,p)$. $\square$

We conclude this section by the following example showing that SSOSC is *not* a necessary condition for full stability under MFCQ and CRCQ and also that Corollary 5.16 is a strict improvement of [55, Theorem 2] obtained under SSOSC.

Example 5.17 (SSOSC is not necessary for full stability under partial MFCQ and CRCQ). Consider the two-parameter nonlinear problem in $I\!R^3$ given by

$$\begin{cases} \text{minimize } \ \varphi(x,p) - \langle x^*, x\rangle \ \text{ subject to} \\ g_1(x,p) := x_1 - x_3 - p_1 \le 0, \\ g_2(x,p) := -x_1 - x_3 + p_1 \le 0, \\ g_3(x,p) := x_2 - x_3 - p_2 \le 0, \\ g_4(x,p) := -x_2 - x_3 + p_2 \le 0, \\ x = (x_1,x_2,x_3), \ x^* = (x_1^*,x_2^*,x_3^*) \in I\!R^3, \ p = (p_1,p_2) \in I\!R^2, \end{cases} \tag{5.47}$$

where $\varphi(x,p) := x_3 + (\frac{1}{4}+p_2)x_1 + p_1x_2 + x_3^2 - x_1x_2$. It is easy to check that both partial MFCQ and CRCQ hold at $(\bar{x},\bar{p})$ with $\bar{x} = (0,0,0)$ and $\bar{p} = (0,0)$. Choosing $\bar{x}^* := (0,0,0) \in \Psi(\bar{x},\bar{p})$ and taking into account that $\|x^*\| < \frac{1}{12}$ and the definition of f in (5.3), we get the relationships

$$\begin{aligned} f(x,p) - \langle x^*, x\rangle &= x_3 + \frac{1}{4}x_1 + x_3^2 - (x_1-p_1)(x_2-p_2) + p_1p_2 - x_1^*x_1 - x_2^*x_2 - x_3^*x_3 \\ &\ge \frac{1}{3}x_3 + \Big(\frac{1}{4} - x_1^*\Big)x_1 + \frac{1}{3}x_3 - x_2^*x_2 + \Big(\frac{1}{3} - x_3^*\Big)x_3 + p_1p_2 \\ &\ge \Big(\frac{1}{4} - x_1^*\Big)(x_3 + x_1) + \frac{1}{3}|x_2 - p_2| - x_2^*(x_2 - p_2) - x_2^*p_2 + p_1p_2 \\ &\ge \Big(\frac{1}{4} - x_1^*\Big)p_1 - x_2^*p_2 + p_1p_2 \ \text{ for all } \ (x,p) \in \Omega. \end{aligned}$$

It follows from (3.3) that $M_\gamma(x^*,p)=\{(p_1,p_2,0)\}$ for all (x^*,p) around $(\bar{x}^*,\bar{p})$ and $\gamma>0$. Due to the validity of BCQ (3.8) under MFCQ and by Proposition 2.8 we have full stability at $\bar{x}$ in (5.47). Theorem 5.15 ensures USOSC is fulfilled at $(\bar{x},\bar{p},\bar{x}^*)$ in this example. To show that the partial SSOSC does not hold here, observe that $\lambda=(\frac{3}{8},\frac{5}{8},0,0)\in\Lambda(\bar{x},\bar{p},\bar{x}^*)$ and that the nonzero vector $u=(0,1,0)$ satisfies the equation

$$\langle\nabla_x g_i(\bar{x},\bar{p}),u\rangle=0 \;\text{ for }\; i\in I_+(\bar{x},\bar{p},\lambda).$$

Since $\langle\nabla^2_{xx}L(\bar{x},\bar{p},\lambda)u,u\rangle=0$, the partial SSOSC fails at this point. Observe finally that the Lipschitz continuous mapping $x(p)=(p_1,p_2,0)$ is a unique solution to (5.47), which confirms the result of Corollary 5.16 despite the failure of the partial SSOSC.

Note that the generalized equation/KKT system associated with problem (5.7) is *not strongly regular* in the sense of Definition 5.4 at the tilt-stable minimizer $\bar{x}$ and the corresponding Lagrange multiplier in Example 5.17. Indeed, the converse assertion ensures LICQ and thus contradicts [16, Theorem 6]. Observe also that we do *not* have *strong stability* in this example. Indeed, it has been well recognized (see the original version in [23, Theorem 7.2] and the improved one in [5, Proposition 5.37] with the references therein) that strong stability of NLP can be characterized, under the validity of MFCQ, via a uniform quadratic growth condition equivalent in this case to SSOSC. As shown in Example 5.17, SSOSC does not hold at the tilt-stable minimizer $\bar{x}$ in problem (5.47) while MFCQ is satisfied. Thus strong stability fails in this setting.

Chapter 6

Full Stability in Infinite-Dimensional Constrained Optimization

6.1 Orverview

6.2 Full Stability in Polyhedric Programming

The main goal of this section is to study full stability of the following mathematical program:

$$\check{\mathcal{P}} \quad \text{minimize } \; \varphi(x,\bar{p}) \; \text{ subject to } x \in K, \tag{6.1}$$

where the cost function $\varphi : X \times P \to \overline{I\!R}$ is $\mathcal{C}^2$ around the reference point $(\bar{x},\bar{p}) \in \operatorname{dom}\varphi$, K is a *closed* and *convex* subset of the *Hilbert* space X, and P is an *Asplund* space; these are our *standing assumptions* in this section. The corresponding two-parametric perturbation of (6.1) is defined by

$$\check{\mathcal{P}}(x^*,p) \quad \text{minimize } \; \varphi(x,p) - \langle x^*, x\rangle \; \text{ subject to } \; x \in K \tag{6.2}$$

with the tilt parameter $x^* \in X^*$ and the basic parameter $p \in P$. We say that $\bar{x} \in K$ is a *fully stable* local minimizer of problem $\widetilde{\mathcal{P}}(\bar{x}^*,\bar{p})$ if it is a fully stable local minimizer (in the Lipschitzian sense of Definition 3.1) of problem $\mathcal{P}(\bar{x}^*,\bar{p})$ in (3.2) with $f(x,p) := \varphi(x,p) + \delta_K(x)$. When the parameter p is ignored, a second-order characterization of tilt stability for problem (6.1) was established in [53, Theorem 4.5] in finite dimensions under the assumption that the constraint set K is polyhedral. Motivated by the application to optimal control given in Section 7 below, we derive here a new characterization of full (and hence tilt) stability in (6.2) for the case of *polyhedric* sets K in Hilbert spaces, which is a more general setting in comparison with [53] even in finite dimensions. Let us first recall the concept of polyhedricity introduced in [20, 35] and then widely applied in optimal control; see, e.g., [2, 5, 27] and the references therein.

Definition 6.1 (polyhedric sets). *Let K be a closed and convex subset of X. We say that K is* POLYHEDRIC *at $\bar{x} \in \Omega$ for $\hat{x}^* \in N(\bar{x}; K)$ if we have the representation*

$$\mathcal{K}(\bar{x}, \hat{x}^*) := T_K(\bar{x}) \cap \{\hat{x}^*\}^{\perp} = \mathrm{cl}\Big\{\mathcal{R}_K(\bar{x}) \cap \{\hat{x}^*\}^{\perp}\Big\} \tag{6.3}$$

of the corresponding critical cone $\mathcal{K}(\bar{x}, \hat{x}^*)$, where

$$\mathcal{R}_K(\bar{x}) := \bigcup_{t>0} \frac{K - \bar{x}}{t} \tag{6.4}$$

is the radial cone and $T_K(\bar{x}) := \mathrm{cl}^* \mathcal{R}_K(\bar{x})$ is the tangent cone to K at $\bar{x}$. If K is polyhedric at each $\bar{x} \in K$ for any $\hat{x}^* \in N(\bar{x}; K)$, we say that K is polyhedric.

It is easy to check that any polyhedral and also generalized polyhedral sets from [5, Definition 2.95] are polyhedric. However, the converse is not true; see, e.g., the set $K(a, b)$ in the control setting (6.25) below, which is neither polyhedral nor generalized polyhedral. The next theorem, important of its own sake, provides a precise calculation of the combined second-order subdifferential (2.14) for the indicator functions of polyhedric sets in Hilbert spaces. Note that calculations of this type but for the second-order subdifferential of [36] were done in [15] for polyhedral sets in finite dimensions.

Theorem 6.2 (combined second-order subdifferential of polyhedric sets). *For any*

$\bar{x} \in K$ and $\hat{x}^* \in N(\bar{x};K)$ we have the inclusion

$$\mathrm{dom}\,\breve{\partial}^2\delta_K(\bar{x},\hat{x}^*) \subset -\mathcal{K}(\bar{x},\hat{x}^*) = -T_K(\bar{x}) \cap \{\hat{x}^*\}^{\perp}. \tag{6.5}$$

If in addition K is polyhedric at $\bar{x} \in \Omega$ for $\hat{x}^*$, then

$$\breve{\partial}^2\delta_K(\bar{x},\hat{x}^*)(u) = \mathcal{K}(\bar{x},\hat{x}^*)^* \quad \textit{whenever} \quad u \in -\mathcal{K}(\bar{x},\hat{x}^*). \tag{6.6}$$

Proof. To justify (6.5) and the inclusion "$\subset$" in (6.6), pick any pair $(u^*,u) \in X \times X$ with $u^* \in \breve{\partial}^2\delta_K(\bar{x},\hat{x}^*)(u)$. It follows from definition (2.14) that

$$\limsup_{(x,x^*)\overset{\mathrm{gph}N(\cdot;K))}{\longrightarrow}(\bar{x},\hat{x}^*)} \frac{\langle u^*, x-\bar{x}\rangle - \langle u, x^*-\hat{x}^*\rangle}{\|x-\bar{x}\| + \|x^*-\hat{x}^*\|} \le 0. \tag{6.7}$$

Letting $x = \bar{x}$ in (6.7) gives us the inequality

$$\limsup_{x^*\overset{N(\bar{x};K)}{\to}\hat{x}^*} \frac{-\langle u, x^*-\hat{x}^*\rangle}{\|x^*-\hat{x}^*\|} \le 0$$

from which we conclude, since $N(\bar{x};K)$ is a convex cone, that

$$-u \in N(\bar{x};K)^* \cap \{\hat{x}^*\}^{\perp} = T_K(\bar{x}) \cap \{\hat{x}^*\}^{\perp} = \mathcal{K}(\bar{x},\hat{x}^*)$$

and thus get (6.5). Suppose further that the set K is polyhedric at $\bar{x} \in \Omega$ for $\hat{x}^*$ and pick any $0 \ne v \in \mathcal{K}(\bar{x},\hat{x}^*\}$. It follows from (6.3) and (6.4) that there are sequences of $t_k > 0$ and $v_k \to v$ such that $x_k := \bar{x} + t_k v_k \in K$ and $\langle \bar{x}^*, v_k\rangle = 0$ for all $k \in I\!N$. Since K is convex and $\bar{x} \in K$, we get $\alpha t_k v_k + \bar{x} = \alpha x_k + (1-\alpha)\bar{x} \in K$ for $\alpha \in [0,1]$. It allows us to assume without loss of generality that $t_k \downarrow 0$ as $k \to \infty$. Taking into account $\hat{x}^* \in N(\bar{x}; K)$ ensures that

$$\langle \hat{x}^*, x - x_k \rangle = \langle \hat{x}^*, x - \bar{x} \rangle - t_k \langle \hat{x}^*, v_k \rangle \le 0 \ \text{ for all } \ x \in K,$$

which yields $(x_k, \hat{x}^*) \overset{\text{gph}N(\cdot;K)}{\longrightarrow} (\bar{x}, \hat{x}^*)$. Replacing (x^*, x) in (6.7) by $(x_k, \hat{x}^*)$ gives us that $\langle u^*, v \rangle \le 0$. Thus we have $u^* \in \mathcal{K}(\bar{x}, \hat{x}^*)^*$ and justify the inclusion "$\subset$" in (6.6).

It remains to prove the opposite inclusion "$\supset$" in (6.6) when K is polyhedric at $\bar{x}$ for $\bar{x}^*$. The classical separation theorem tells us that

$$\Big(T_K(\bar{x}) \cap \{\bar{x}^*\}^\perp\Big)^* = \text{cl}^*\big(T_K(\bar{x})^* + \big\{\{\bar{x}^*\}^\perp\big\}^*\big) = \text{cl}^*\big(N(\bar{x}; K) + I\!R\{\bar{x}^*\}\big) = \text{cl}^*\big(N(\bar{x}; K) - I\!R_+\{\bar{x}^*\}\big).$$

Since the regular normal cone is closed in norm topology, we only need to check that

$$N(\bar{x}; K) - I\!R_+\{\bar{x}^*\} \subset \breve{\partial}^2 \delta_K(\bar{x}, \hat{x}^*)(u) \ \text{ for all } \ u \in -\mathcal{K}(\bar{x}, \hat{x}^*).$$

To proceed, pick any $u^* \in N(\bar{x}; K)$, $\beta \in I\!R_+$, and $u \in -\mathcal{K}(\bar{x}, \hat{x}^*)$ and observe that

$$\langle u^* - \beta \hat{x}^*, x - \bar{x} \rangle \le -\beta \langle \hat{x}^*, x - \bar{x} \rangle \le -\beta \langle \hat{x}^* - x^*, x - \bar{x} \rangle \ \text{ if } \ (x, x^*) \in \text{gph}\, N(\cdot; K). \tag{6.8}$$

Since $u \in -\mathcal{K}(\bar{x}, \hat{x}^*)$, it follows from (6.3) and (6.4) that there are sequences $t_k \downarrow 0$ and $u_k \to -u$ satisfying $\bar{x} + t_k u_k \in K$ and $\langle \hat{x}^*, u_k \rangle = 0$ for all $k \in I\!N$. Taking $(x, x^*) \in \text{gph}\, N(\cdot; K)$, we get

$$\begin{aligned}
\langle -u, x^* - \hat{x}^* \rangle &\le \|u + u_k\| \cdot \|x^* - \hat{x}^*\| + \langle u_k, x^* - \hat{x}^* \rangle = \|u + u_k\| \cdot \|x^* - \hat{x}^*\| + \langle u_k, x^* \rangle \\
&\le \|u + u_k\| \cdot \|x^* - \hat{x}^*\| + \frac{\langle \bar{x} + t_k u_k - x, x^* \rangle + \langle x - \bar{x}, x^* \rangle}{t_k} \\
&\le \|u + u_k\| \cdot \|x^* - \hat{x}^*\| + \frac{\langle x - \bar{x}, x^* \rangle}{t_k} \le \|u + u_k\| \cdot \|x^* - \hat{x}^*\| + \frac{\langle x - \bar{x}, x^* - \hat{x}^* \rangle}{t_k}
\end{aligned}$$

since $\langle \bar{x} + t_k u_k - x, x^* \rangle \le 0$ and $\langle x - \bar{x}, -\hat{x}^* \rangle \ge 0$. This together with (6.8) implies that

$$\langle u^* - \beta\hat{x}^*, x - \bar{x}\rangle - \langle u, x^* - \hat{x}^*\rangle \le (\beta + t_k^{-1})\|x - \bar{x}\| \cdot \|x^* - \hat{x}^*\| + \|u + u_k\| \cdot \|x^* - \hat{x}^*\|,$$

which in turn ensures the estimate

$$\limsup_{(x,x^*)\overset{\mathrm{gph}N(\cdot;K))}{\longrightarrow}(\bar{x},\hat{x}^*)} \frac{\langle u^* - \beta\hat{x}^*, x - \bar{x}\rangle - \langle u, x^* - \hat{x}^*\rangle}{\|x - \bar{x}\| + \|x^* - \hat{x}^*\|} \le \|u + u_k\| \text{ for all } k \in I\!N.$$

Letting $k \to \infty$ gives us that $u^* - \beta\bar{x}^* \in \breve{\partial}^2\delta_K(\bar{x}, \hat{x}^*)(u)$ and thus completes the proof. $\square$

The next theorem is the main result of this section, which contains a complete second-order characterization of full stability in polyhedric programming.

Theorem 6.3 (second-order characterization of full stability for polyhedric programs). *Let $\bar{x}^* \in \nabla_x\varphi(\bar{x}, \bar{p}) + N(\bar{x}; K)$ with $\bar{x} \in K$ and $\bar{p} \in P$. Consider the following statements:*

(i) *The point $\bar{x}$ is a fully stable local minimizer for $\check{\mathcal{P}}(\bar{x}^*, \bar{p})$ in* (6.2)*.*

(ii) *There are $\eta, \kappa > 0$ such that for each $(x, x^*) \in \mathrm{gph}\, N(\cdot; K) \cap I\!B_\eta(\bar{x}, \bar{x}^* - \nabla_x\varphi(\bar{x}, \bar{p}))$ we have*

$$\langle \nabla^2_{xx}\varphi(x, p)u, u\rangle \ge \kappa\|u\|^2 \;\; \textit{whenever} \;\; u \in \mathcal{K}(x, x^*) \tag{6.9}$$

via the critical cone from (6.3)*. Then* (ii) *is a sufficient condition for* (i)*. If in addition the set K is polyhedric, then* (ii) *is also necessary for the validity of* (i)*.*

Proof. Note first that $\partial_x f(x, p) = \nabla_x\varphi(x, p) + N(x; K)$ whenever $(x, p) \in K \times P$ for the function $f(x, p) = \varphi(x, p) + \delta_K(x)$, which is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$

for $\bar{x}^* \in \partial_x f(\bar{x}, \bar{p})$. Let us now check that BCQ (3.7) as well as the graphical Lipschitz-like condition (3.33) hold in this setting. To justify (3.7), take any $p_1, p_2 \in I\!B_\delta(\bar{p})$ and $(x_1, r_1) \in F(p_1) \cap I\!B_\delta(\bar{x}, \varphi(\bar{x}, \bar{p}))$ for small $\delta > 0$, where F is defined in (3.7). Then we have $(x_1, r_2) \in F(p_2)$ with $r_2 := r_1 + \ell\|p_1 - p_2\|$, where $\ell > 0$ is a Lipschitz constant of φ around $(\bar{x}, \bar{p})$. It follows that

$$(x_1, r_1) \in (x_1, r_2) + \ell\|p_1 - p_2\| I\!B_{X \times P},$$

which verifies BCQ (3.7). The Lipschitz-like property of G in (3.33) can be checked similarly.

To show next that (ii)$\Longrightarrow$(i), it suffices to verify condition (3.38) du to Theorem 3.12. To proceed, pick any $u^* \in (\widehat{D}^* \partial_x f)(x, p, x^*)(u)$ with $(x, p, x^*) \in \operatorname{gph} \partial_x f \cap I\!B_\nu(\bar{x}, \bar{p}, \bar{x}^*)$ with $\frac{\eta}{1+\ell} > \nu > 0$ sufficiently small. Observe from Lemma 2.5 that

$$\begin{aligned} \big(\widehat{D}^* \partial_x f\big)(x, p, x^*)(u) &= \big((\widehat{D}^* \nabla_x \varphi)(\cdot, \cdot) + N(\cdot; K)\big)(x, p, x^*)(u) \\ &= \nabla^2_{xx}\varphi(x, p)u + \breve{\partial}^2 \delta_K\big(x, x^* - \nabla_x \varphi(x, p)\big)(u). \end{aligned}$$

This gives us that $u^* - \nabla^2_{xx}\varphi(x, p)u \in \breve{\partial}^2 \delta_K(x, x^* - \nabla_x \varphi(x, p))(u)$, which in turn implies by (6.5) that $u \in -\mathcal{K}(x, x^* - \nabla_x \varphi(x, p))$. Since $N(\cdot; K)$ is maximal monotone, we get from Lemma 2.9 that $\langle u^* - \nabla^2_{xx}\varphi(x, p)u, u\rangle \geq 0$. Note further that $x^* - \nabla_x \varphi(x, p) \in N(x; K)$ and so

$$\|x^* - \nabla_x \varphi(x, p) - \bar{x}^* + \nabla_x \varphi(\bar{x}, \bar{p})\| \leq \|x^* - \bar{x}^*\| + \ell(\|x - \bar{x}\| + \|p - \bar{p}\|) \leq \nu + \ell\nu < \eta.$$

It follows from (6.9) that $\langle \nabla^2_{xx}\varphi(x, p)u, u\rangle = \langle -\nabla^2_{xx}\varphi(x, p)u, -u\rangle \geq \kappa\| - u\|^2 = \kappa\|u\|^2$, which yields

$$\langle u^*, u\rangle = \langle u^* - \nabla^2_{xx}\varphi(x, p)u, u\rangle + \langle \nabla^2_{xx}\varphi(x, p)u, u\rangle \geq \kappa\|u\|^2.$$

This ensures condition (3.38) in Theorem 3.12 and hence justifies the first part of the theorem.

To prove the second part, suppose now that K is polyhedric and that $\bar{x}$ is a fully stable local minimizer of $\widetilde{\mathcal{P}}(\bar{x}^*,\bar{p})$. Theorem 3.12 tells us that condition (3.38) holds for some $\kappa,\eta>0$. Picking any $u\in\mathcal{K}(x,x^*)$ with $(x,x^*)\in\operatorname{gph}N(\cdot;K)\cap I\!B_\nu(\bar{x},\bar{x}^*-\nabla_x\varphi(\bar{x},\bar{p}))$ with some $\frac{\eta}{1+\ell}>\nu>0$, we get that $x^*+\nabla_x\varphi(x,p)\in\partial_x f(x,p)$ and that

$$\|x^*+\nabla_x\varphi(x,p)-\bar{x}^*\|=\|x^*-\bar{x}^*+\nabla_x\varphi(\bar{x},\bar{p})\|+\|\nabla_x\varphi(x,p)-\nabla_x\varphi(\bar{x},\bar{p})\|\le\nu+\ell\nu<\eta,$$

which justifies the inclusion $(x,p,x^*+\nabla_x\varphi(x,p))\in\operatorname{gph}\partial_x f\cap I\!B_\eta(\bar{x},\bar{p},\bar{x}^*)$. Since $x^*\in\mathcal{K}(x,x^*)^*$, it follows from (6.6) and Lemma 2.5 that

$$\nabla^2_{xx}\varphi(x,p)u+x^*\in-\nabla^2_{xx}\varphi(x,p)(-u)+\breve{\partial}^2\delta_K(x,x^*)(-u)=\big(\widehat{D}^*\partial_x f\big)\big(x,p,x^*+\nabla_x\varphi(x,p)\big)(-u).$$

This together with (3.38) ensures that

$$\langle\nabla^2_{xx}\varphi(x,p)u,u\rangle=\langle-\nabla^2_{xx}\varphi(x,p)u+x^*,-u\rangle\ge\kappa\|u\|^2,$$

which verifies (6.9) and thus completes the proof of the theorem. $\square$

6.3 Full Stability in Optimal Control of Semilinear Elliptic PDEs

This section is devoted to applications of the infinite-dimensional results obtained in Section 6 to characterizing (Lipschitzian) full stability in optimal control problems governed by elliptic partial differential equations. More specifically, we consider the following control problem:

$$\text{minimize}\begin{cases}J(y,x):=\dfrac{1}{2}\displaystyle\int_\Omega\big(y(w)-\bar{p}(w)\big)^2dw+\dfrac{M}{2}\displaystyle\int_\Omega x(w)^2dw\\[2mm]\text{subject to}\quad x\in K,\end{cases}\tag{6.10}$$

where Ω is an open bounded subset of $I\!R^n$ as $n \le 3$ with $\mathcal{C}^2$-smooth boundary $\mathrm{bd}\,\Omega$, the number $M > 0$ is given, the target $\bar{p}$ belongs to the Asplund space $P := L^q(\Omega)$ for $q \in [2, \infty)$, K is a closed convex subset of the Hilbert *control space* $X := L^2(\Omega)$, and y is the corresponding solution to the Dirichlet problem for the *semilinear elliptic equation*

$$\begin{cases} -\Delta y + \phi(y) = x \ \text{ in } \ \Omega, \\ y = 0 \ \text{ on } \ \mathrm{bd}\,\Omega \end{cases} \tag{6.11}$$

with the Laplacian Δ. We refer the reader to the books [5, 27, 28] for the equations of this type, control problems for them, and various applications.

In what follows we assume that the function $\phi : I\!R \to I\!R$ in (6.11) is nondecreasing, Lipschitz continuous, and $\mathcal{C}^2$ on $I\!R$. It follows from [5, Proposition 6.12] that for each $x \in L^2(\Omega)$ equation (6.11) has a unique solution $y_x \in H^2(\Omega) \cap H_0^1(\Omega)$, where $H^2(\Omega) := W^{2,2}(\Omega)$ and $H_0^1(\Omega) := W_0^{1,2}(\Omega)$ are the classical Sobolev spaces. Thus the functional $J(y, x)$ can be understood as a function of one variable x. Define next the function $\varphi : X \times P \to I\!R$ depending on x and $p \in P$ by

$$\varphi(x,p) := \frac{1}{2}\int_\Omega \big(y_x(w) - p(w)\big)^2 dw + \frac{M}{2}\int_\Omega x(w)^2 dw \ \text{ for all } \ x \in L^2(\Omega),\ p \in L^q(\Omega) \tag{6.12}$$

and observe that problem (6.10) can be treated as $\widetilde{\mathcal{P}}$ in (6.1) with the cost function (6.12). The following result follows from [5, Proposition 6.13, Proposition 6.15, and Lemma 6.27].

Lemma 6.4 (well-posedness). *Under the assumptions imposed above the mapping $L^2(\Omega) \ni x \mapsto y_x \in H^2(\Omega) \cap H_0^1(\Omega)$ is $\mathcal{C}^2$. Moreover, the function φ in (6.12) is also $\mathcal{C}^2$ with*

$$\langle \nabla_{xx}^2 \varphi(x,p)u, u\rangle = \int_\Omega \Big[Mu(w)^2 + \big[1 - q_{x,p}(w)\phi''\big(y_x(w)\big)\big] z_u(w)^2\Big] dw, \tag{6.13}$$

where $q_{x,p} \in H^2(\Omega) \cap H_0^1(\Omega)$ *is a unique solution of the adjoint equation*

$$\begin{cases} -\Delta q + \phi'(y_x)q = y_x - p \quad in \ \ \Omega, \\ q = 0 \quad on \ \ \mathrm{bd}\,\Omega \end{cases} \tag{6.14}$$

while $z_u \in H^2(\Omega) \cap H_0^1(\Omega)$ *is a unique solution of the homogeneous one*

$$\begin{cases} -\Delta z + \phi'(y_x)z = u \quad in \ \ \Omega, \\ z = 0 \quad on \ \ \mathrm{bd}\,\Omega. \end{cases} \tag{6.15}$$

Considering the perturbed version $\check{\mathcal{P}}(x^*,p)$ of problem $\check{\mathcal{P}}$ in (6.1), we derive in the next theorem pointwise necessary and sufficient conditions for full stability of local minimizers in this PDE setting.

Theorem 6.5 (full stability for elliptic PDEs). *For the reference pair* $(\bar{x},\bar{p}) \in X \times P$*, fix* $\bar{x}^* \in L^2(\Omega)$ *satisfying* $\bar{x}^* \in q_{\bar{x},\bar{p}} + M\bar{x} + N(\bar{x};K)$ *and consider the following assertions:*

(i) *The point* $\bar{x}$ *is a fully stable local minimizer of* $\check{\mathcal{P}}(\bar{x}^*,\bar{p})$*.*

(ii) *With* $\hat{x}^* := \bar{x}^* - q_{\bar{x},\bar{p}} - M\bar{x}$ *we have*

$$\langle \nabla^2_{xx}\varphi(\bar{x},\bar{p})u,u\rangle > 0 \quad for\ all \ \ u \in \mathcal{H}(\bar{x},\hat{x}^*) \setminus \{0\}, \tag{6.16}$$

where $\mathcal{H}(\bar{x},\hat{x}^*)$ *is defined by the outer limit*

$$\mathcal{H}(\bar{x},\hat{x}^*) := \mathop{\mathrm{Lim\,sup}}_{(x,x^*) \overset{\mathrm{gph}\,N(\cdot;K)}{\longrightarrow} (\bar{x},\hat{x}^*)} \mathcal{K}(x,x^*) \tag{6.17}$$

of the critical cone (6.3)*. Then* (ii) *is a sufficient condition for* (i)*. If in addition* K *is polyhedric, then* (ii) *is also necessary for the validity of* (i)*.*

Proof. It follows from [5, Lemma 6.18] that $\nabla_x\varphi(\bar{x},\bar{p}) = q_{\bar{x},\bar{p}} + M\bar{x}$, and hence we have the inclusion $\bar{x}^* \in \nabla_x\varphi(\bar{x},\bar{p}) + N(\bar{x};K)$. By Theorem 6.3 it suffices to show that assertion (ii) is equivalent to (ii) in Theorem 6.3. To proceed, suppose that (6.9) holds with some $\kappa \in (0, M)$ and $\eta > 0$. Picking any $u \in \mathcal{H}(\bar{x},\hat{x}^*) \setminus \{0\}$, find from (6.17) sequences $(x_k, x_k^*) \in \operatorname{gph} N(\cdot;K)$, $u_k \in \mathcal{K}(x_k, x_k^*)$, and $p_k \in L^q(\Omega)$ satisfying $(x_k, x_k^*) \to (\bar{x},\hat{x}^*)$, $u_k \overset{w}{\to} u$, and $p_k \to \bar{p}$ as $k \to \infty$. Taking (6.9) and (6.13) into account, assume without loss of generality that

$$\int_\Omega \Big[Mu_k(w)^2 + \big[1 - q_k(w)\phi''(y_k(w))\big]z_k(w)^2\Big]dw \geq \kappa \int_\Omega u_k(w)^2 dw, \tag{6.18}$$

where $q_k := q_{x_k,p_k}$, $y_k := y_{x_k}$, and $z_k := z_{u_k}$. Employing now (6.15), Poincaré's inequality, and the fact that $\phi'(v) \geq 0$ gives us the estimates

$$\begin{aligned} \|z_k\|^2_{H^1_0(\Omega)} &\leq C\|\nabla z_k\|^2_{L^2(\Omega)} \leq C\int_\Omega \big(\|\nabla z_k(w)\|^2 + \phi'(y_k(w))z_k(w)^2\big)dw \\ &= C\int_\Omega u_k(w)z_k(w)dw \leq C\|u_k\|_{L^2(\Omega)}\|z_k\|_{L^2(\Omega)} \leq C\|u_k\|_{L^2(\Omega)}\|z_k\|_{H^1_0(\Omega)} \end{aligned} \tag{6.19}$$

with some $C > 0$. Since $u_k \overset{w}{\to} u$ in $L^2(\Omega)$, the sequence $\{\|u_k\|_{L^2(\Omega)}\}$ is bounded. It follows from (6.19) that $\{z_k\}$ is bounded in $H^1_0(\Omega)$. Hence there is a subsequence of $\{z_k\}$ satisfying

$$z_k \overset{w}{\to} z \ \text{ in } \ H^1_0(\Omega) \ \text{ and } \ z_k \to z \ \text{ in } \ L^2(\Omega) \ \text{ as } \ k \to \infty. \tag{6.20}$$

It follows from [5, Proposition 6.13] that $z_k = \nabla_x y_{x_k} u_k$ and that $\nabla_x y_{x_k} \to \nabla_x y_{\bar{x}}$ due to Lemma 6.4. We get from the latter, (6.20), and the convergence $u_k \overset{w}{\to} u$ that $z = \nabla_x y_{\bar{x}} u$, which yields $z = z_u$ by [5, Proposition 6.13]. Similarly to the case of $\{z_k\}$, the sequence $\{q_k\}$ also contains a subsequence converging to $q_{\bar{x},\bar{p}}$ in $L^2(\Omega)$. Moreover, the Sobolev embedding for $H^2(\Omega) \subset \mathcal{C}(\overline{\Omega})$ as $n \leq 3$ (where $\overline{\Omega}$ indicates the closure) tells us by [5, Lemma 6.14] and (6.20) that there is $c > 0$ such that

$$\|z_k\|_{\mathcal{C}(\overline{\Omega})} \le c\|z_k\|_{H^2(\Omega)} \le c^2\|z_k\|_{L^2(\Omega)} \le c^3.$$

Since $y_k \to y_{\bar{x}}$ in $H^2(\Omega) \cap H_0^1(\Omega)$ by Lemma 6.4, we have $y_k \to y_{\bar{x}}$ in $\mathcal{C}(\overline{\Omega})$ due to the aforementioned Sobolev embedding. It ensures that $\phi''(y_k) \to \phi''(y_{\bar{x}})$ in $\mathcal{C}(\overline{\Omega})$ as well, and thus

$$\begin{aligned}
&\Big|\int_\Omega (q_k(w) - q_{\bar{x},\bar{p}}(w))\phi''(y_k(w))z_k(w)^2 + \int_\Omega q_{\bar{x},\bar{p}}(w)(\phi''(y_k(w)) - \phi''(y_{\bar{x}}(w))z_k(w)^2\Big| \\
&\le c_1\|q_k - q_{\bar{x},\bar{p}}\|_{L^2(\Omega)}\|\phi''(y_k)\|_{\mathcal{C}(\overline{\Omega})}\|z_k\|^2_{\mathcal{C}(\overline{\Omega})} + \|q_{\bar{x},\bar{p}}\|_{\mathcal{C}(\overline{\Omega})}\|\phi''(y_k) - \phi''(y_{\bar{x}})\|_{\mathcal{C}(\overline{\Omega})}\|z_k\|^2_{\mathcal{C}(\overline{\Omega})} \\
&\le c_1^2c^3\|q_k - q_{\bar{x},\bar{p}}\|_{L^2(\Omega)} + c^3\|q_{\bar{x},\bar{p}}\|_{\mathcal{C}(\overline{\Omega})}\|\phi''(y_k) - \phi''(y_{\bar{x}})\|_{\mathcal{C}(\overline{\Omega})} \longrightarrow 0 \;\text{ as }\; k \to \infty
\end{aligned} \tag{6.21}$$

with some $c_1 > 0$. Furthermore, it follows from (6.20) that

$$\Big|\int_\Omega q_{\bar{x},\bar{p}}(w)\phi''(y_{\bar{x}}(w))(z_k(w)^2 - z_u(w)^2)dw\Big| \le \|q_{\bar{x},\bar{p}}\|_{\mathcal{C}(\overline{\Omega})}\|\phi''(y_{\bar{x}})\|_{\mathcal{C}(\overline{\Omega})}\|z_k - z_u\|_{L^2(\Omega)}\|z_k + z_u\|_{L^2(\Omega)},$$

which also converges to 0 as $k \to \infty$. This together with (6.18), (6.20), and (6.21) ensures that

$$\begin{aligned}
&\int_\Omega \big[1 - q_{\bar{x},\bar{p}}(w)\phi''(y_{\bar{x}}(w))\big]z_u(w)^2dw = \lim_{k\to\infty}\int_\Omega (1 - q_k(w)\phi''(y_k(w))z_k(w)^2dw \\
&\ge -\liminf_{k\to\infty}(M - \kappa)\|u_k\|^2_{L^2(\Omega)} \ge -(M - \kappa)\|u\|^2_{L^2(\Omega)},
\end{aligned} \tag{6.22}$$

which implies that $\langle\nabla^2_{xx}\varphi(\bar{x},\bar{p})u, u\rangle \ge \kappa\|u\|^2$ due to (6.13) and thus justifies (ii).

To prove the converse implication, suppose by contradiction that (6.16) holds while (6.9) does not. This gives us sequences $(x_k, x_k^*) \overset{\text{gph } N(\cdot;K)}{\longrightarrow} (\bar{x}, \hat{x}^*)$, $p_k \to \bar{p}$, and $u_k \in \mathcal{K}(x_k, x_k^*)$ such that

$$\langle\nabla^2_{xx}\varphi(x_k, p_k)u_k, u_k\rangle < k^{-1}\|u_k\|^2_{L^2(\Omega)} \;\text{ for all }\; k \in I\!N. \tag{6.23}$$

Assume without loss of generality that $\|u_k\|^2_{L^2(\Omega)} = 1$ and find a subsequence of $\{u_k\}$ with $u_k \overset{w}{\to} u \in L^2(\Omega)$ as $k \to \infty$. It follows from (6.17) that $u \in \mathcal{H}(\bar{x}, \bar{x}^*)$. Defining $q_k := q_{x_k,p_k}$, $y_k := y_{x_k}$, and $z_k := z_{u_k}$ and arguing similarly to (6.22) imply that

$$M + \int_\Omega \big(1 - q_{\bar{x},\bar{p}}\varphi''(y_{\bar{x}})\big) z_u(w)^2 dw = M + \lim_{k\to\infty} \int_\Omega \big(1 - q_k(w)\phi''(y_k(w))\big) z_k(w)^2 dw \le 0, \quad (6.24)$$

where the inequality follows from (6.13) and (6.23). If $u = 0$, then $z_u = 0$ by (6.14), which is not possible by (6.24) since $M > 0$. Hence $u \neq 0$, and by (6.24) and (6.13) we get $\langle \nabla^2_{xx}\varphi(\bar{x}, \bar{p})u, u\rangle \le 0$. This contradicts (6.16) and thus completes the proof of the theorem. $\square$

The next result provides an precise calculation of the outer limit in (6.17) for the critical cones generated by the constraint set K from (6.10) given by

$$K = K_{a,b} := \big\{x \in L^2(\Omega) \big|\ a \le x(w) \le b \ \text{ a.e. on } \ \Omega\big\} \quad (6.25)$$

with $-\infty \le a < b \le \infty$. It follows from [5, Proposition 6.33] that this set is *polyhedric* in $L^2(\Omega)$, which is not however polyhedral or generalized polyhedral. Note also that the *pointwise* magnitude control constraints of type (6.25) are typical in optimal control theory and its applications while being among the most difficult (“hard") in PDE control; see, e.g., [5, 27, 28].

Proposition 6.6 (limits of critical cones for pointwise constraint). *Let $\bar{x} \in K_{a,b}$, and let $\hat{x}^* \in N(\bar{x}; K)$ with K defined in* (6.25)*. Then the outer limit* (6.17) *is calculated by*

$$\mathcal{H}(\bar{x}, \hat{x}^*) = \big\{u \in L^2(\Omega) \big|\ u(w)\hat{x}^*(w) = 0 \ \ a.e. \ on \ \ \Omega\big\}. \quad (6.26)$$

Proof. To justify the inclusion "$\subset$" in (6.26), pick any $u \in \mathcal{H}(x, \hat{x}^*)$ and find sequences

$u_k \xrightarrow{w} u$ in $L^2(\Omega)$ and $(x_k, x_k^*) \overset{\operatorname{gph} N(\cdot;K)}{\longrightarrow} (\bar{x}; \hat{x}^*)$ as $k \to \infty$ with $u_k \in \mathcal{K}(x_k, x_k^*)$ for all $k \in I\!N$. It follows from the proof of [5, Proposition 6.33] that

$$\mathcal{K}(x_k, x_k^*) = \big\{ v \in T_K(x_k) \big|\ v(w)x_k^*(w) = 0 \ \text{ a.e. on } \ \Omega \big\},$$

which implies that $u_k(w)x_k^*(w) = 0$ a.e. on Ω. For any measurable subset $A \subset \Omega$ with the characteristic function χ_A, we get the relationships

$$\begin{aligned}
\Big| \int_A u(w)\hat{x}^*(w)dw \Big| &= \lim_{k\to\infty} \Big| \int_\Omega u_k(w)\hat{x}^*(w)\chi_A(w)dw \Big| \\
&\le \limsup_{k\to\infty} \Big| \int_\Omega u_k(w)x_k^*(w)\chi_A(w)dw \Big| + \Big| \int_\Omega u_k(w)\big(x_k^*(w) - \hat{x}^*(w)\big)\chi_A(w)dw \Big| \\
&= \limsup_{k\to\infty} \Big| \int_\Omega u_k(w)\big(x_k^*(w) - \hat{x}^*(w)\big)\chi_A(w)dw \Big| \\
&\le \limsup_{k\to\infty} \|u_k\|_{L^2(\Omega)} \|x_k^* - \hat{x}^*\|_{L^2(\Omega)} = 0,
\end{aligned}$$

where the last equality holds due to the convergence $x_k^* \to \hat{x}^*$ in $L^2(\Omega)$ and $u_k \xrightarrow{w} u$; the latter ensures the boundedness of $\{\|u_k\|^2_{L^2(\Omega)}\}$. Since $A \subset \Omega$ was chosen arbitrarily, this implies that $u(w)\hat{x}^*(w) = 0$ a.e. on Ω and thus justifies the inclusion "$\subset$" in (6.26).

To prove the converse inclusion, pick any u from the right-hand side set in (6.26). Define

$$u_1 := \begin{cases} \max\{0, -u(w)\} & \text{for } \ w \in \{\bar{x} = a\}, \\ \min\{0, -u(w)\} & \text{for } \ w \in \{\bar{x} = b\}, \\ 0 & \text{otherwise} \end{cases}$$

with $\{\bar{x} = a\} := \{w \in \Omega |\ \bar{x}(w) = a\}$. Denoting $u_2 := u + u_1$ and employing the formula

$$T_K(\bar{x}) = \big\{ v \in L^2(\Omega) \big|\ v(w) \ge 0 \ \text{ over } \ \{\bar{x} = a\} \ \text{ and } \ v(w) \le 0 \ \text{ over } \ \{\bar{x} = b\} \big\}$$

obtained in [5, Proposition 6.33], we get that $u_1, u_2 \in T_K(\bar{x})$. Note further that $u_1, u_2 \in \{\hat{x}^*\}^\perp$, which gives us $u_2, u_1 \in \mathcal{K}(\bar{x}, \hat{x}^*)$. Since $K_{a,b}$ is polyhedric, it follows from (6.3) that there are sequences $u_{1k} \to u_1$, $u_{2k} \to u_2$ and $t_{1k}, t_{2k} \downarrow 0$ such that $\bar{x} + t_{1k}u_{1k} \in K_{a,b}$, $\bar{x} + t_{2k}u_{2k} \in K_{a,b}$, and $u_{1k}, u_{2k} \in \{\hat{x}^*\}^\perp$. Defining $t_k := \min\{t_{1k}, t_{2k}\}$, we get from the convexity of $K_{a,b}$ that $x_k := \bar{x} + t_k u_{1k} \in K_{a,b}$ and $y_k := \bar{x} + t_k u_{2k} \in K_{a,b}$, which ensures by (6.4) that

$$u_{2k} - u_{1k} = \frac{y_k - x_k}{t_k} \in \mathcal{R}_K(x_k) \cap \{\hat{x}^*\}^\perp \subset \mathcal{K}(x_k, \hat{x}^*).$$

Since $u_{2k} - u_{1k} \to u_2 - u_1 = u$ and $x_k \to \bar{x}$ in $L^2(\Omega)$ as $k \to \infty$, we get from (6.17) that $u \in \mathcal{H}(\bar{x}, \hat{x}^*)$ and thus complete the proof of the proposition. $\square$

We conclude the section with the following straightforward consequence of the results above.

Corollary 6.7 (characterizing full stability for elliptic control problems with pointwise constraints). *Let K be given by* (6.25) *in the setting of Theorem* 6.5. *The following are equivalent:*

(i) *The point $\bar{x}$ is a fully stable local minimizer of $\check{\mathcal{P}}(\bar{x}^*, \bar{p})$.*

(ii) *With $\hat{x}^* := -\bar{x}^* - q_{\bar{x},\bar{p}} + M\bar{x}$ we have*

$$\langle \nabla^2_{xx}\varphi(\bar{x}, \bar{p})u, u\rangle > 0 \ \textit{ for all } \ u \neq 0 \ \textit{ with } \ u(w)\hat{x}^*(w) = 0 \ \textit{ a.e. } \ w \in \Omega. \tag{6.27}$$

Proof. Follows directly from Theorem 6.5 and Proposition 6.6. $\square$

Note that the obtained characterization (6.27) of full stability in Corollary 6.7 can be interpreted as the *positive definiteness* of the cost function *Hessian* $\nabla^2_{xx}\varphi(\bar{x}, \bar{p})$ on the subspace pointwise *orthogonal* to the *adjoint impulse* $\hat{x}^*$ generated by the reference local minimizer $\bar{x}$ of $\check{\mathcal{P}}(\bar{x}^*, \bar{p})$.

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ABSTRACT

FULL STABILITY IN OPTIMIZATION

by

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Advisor: Prof. Boris S. Mordukhovich

Major: Mathematics (Applied)

Degree: Doctor of Philosophy

The dissertation concerns a systematic study of full stability in general optimization models including its conventional Lipschitzian version as well as the new Hölderian one. We derive various characterizations of both Lipschitzian and Hölderian full stability in nonsmooth optimization, which are new in finite-dimensional and infinite-dimensional frameworks. The characterizations obtained are given in terms of second-order growth conditions and also via second-order generalized differential constructions of variational analysis. We develop effective applications of our general characterizations of full stability to parametric variational systems including the well-known generalized equations and variational inequalities. Many relationships of full stability with the conventional notions of strong regularity and strong stability are established for a large class of problems of constrained optimization with twice continuously differentiable data. Other applications of full stability to nonlinear programming, to semidefinite programming, and to optimal control problems governed by semilinear elliptic PDEs are also studied.

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Selected Publications

1. D. Drusvyatskiy, B. S. Mordukhovich and T. T. A. Nghia, Second-order growth, tilt stability, and metric regularity of the subdifferential, submitted (2013)
2. B. S. Mordukhovich and T. T. A. Nghia, Constraint qualifications and optimality conditions in nonlinear semi-infinite and infinite programs, *Math. Program.* **139** (2013), 271–300.
3. B. S. Mordukhovich and T. T. A. Nghia, DC approach to regularity of convex multifunctions with applications to infinite systems, *J. Optim. Theory Appl.* **12** (2012), 762–784.
4. B. S. Mordukhovich and T. T. A. Nghia, Full Lipschitzian and Holderian stability in optimization with applications to mathematical programming and optimal control, submitted (2013).
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6. B. S. Mordukhovich and T. T. A. Nghia, Second-order variational analysis and characterizations of tilt-stable optimal solutions in infinite-dimensional spaces, *Nonlinear Anal.* **86** (2013), 159–180.
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9. B. S. Mordukhovich, T. T. A. Nghia and R. T. Rockafellar, Full stability in finite-dimensional optimization, submitted (2013).
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