5-1-2016

Almost Unbiased Estimator Using Known Value of Population Parameter(s) in Sample Surveys

Rajesh Singh  
*Department of Statistics, Banaras Hindu University Varanasi, rsinghstat@gmail.com*

S.B. Gupta  
*Department of Community Medicine, SRMS Institute of Medical Sciences, dr_sbgupta@rediffmail.com*

Sachin Malik  
*SRM University Delhi, Sonepat, Haryana, India, sachinkurava999@gmail.com*

Follow this and additional works at: [http://digitalcommons.wayne.edu/jmasm](http://digitalcommons.wayne.edu/jmasm)  
Part of the [Applied Statistics Commons](http://digitalcommons.wayne.edu/jmasm), [Social and Behavioral Sciences Commons](http://digitalcommons.wayne.edu/jmasm), and the [Statistical Theory Commons](http://digitalcommons.wayne.edu/jmasm)

**Recommended Citation**

Singh, Rajesh; Gupta, S.B.; and Malik, Sachin (2016) "Almost Unbiased Estimator Using Known Value of Population Parameter(s) in Sample Surveys," *Journal of Modern Applied Statistical Methods*: Vol. 15 : Iss. 1 , Article 30. Available at: [http://digitalcommons.wayne.edu/jmasm/vol15/iss1/30](http://digitalcommons.wayne.edu/jmasm/vol15/iss1/30)

This Regular Article is brought to you for free and open access by the Open Access Journals at DigitalCommons@WayneState. It has been accepted for inclusion in Journal of Modern Applied Statistical Methods by an authorized editor of DigitalCommons@WayneState.
Almost Unbiased Estimator Using Known Value of Population Parameter(s) in Sample Surveys

Rajesh Singh  
Banaras Hindu University  
Varanasi, India

S. B. Gupta  
SRMS Institute of Medical Sciences  
Uttar Pradesh, India

Sachin Malik  
SRM University Delhi  
Sonepat, Haryana, India

An almost unbiased estimator using known value of some population parameter(s) is proposed. A class of estimators is defined which includes Singh and Solanki (2012) and Sahai and Ray (1980), Sisodiya and Dwivedi (1981), Singh, Cauhan, Sawan, and Smarandache (2007), Upadhyaya and Singh (1984), Singh and Tailor (2003) estimators. Under simple random sampling without replacement (SRSWOR) scheme the expressions for bias and mean square error (MSE) are derived. Numerical illustrations are given.

Keywords: Auxiliary information, bias, mean square error, unbiased estimator

Introduction

The precision of the estimates of the population mean or total of the study variable $y$ can be considering improved by the use of known information on an auxiliary variable $x$ which is highly correlated with the study variable $y$. Consider a finite population $U = U_1, U_2, ..., U_N$ of $N$ units. Let $y$ and $x$ stand for the variable under study and auxiliary variable respectively. Let $(y_i, x_i)$, $i = 1, 2, ..., n$ denote the values of the units included in a sample $s_n$ of size $n$ drawn by simple random sampling without replacement (SRSWOR). The auxiliary information has been used in improving the precision of the estimate of a parameter (see Sukhatme, Sukhatme, Sukhatme, & Ashok (1984) and the references cited therein). Among many methods, the ratio and product methods of estimation are good illustrations in this context.

Rajesh Singh is an Assistant Professor in the Department of Statistics. Email him at rsinghstat@gmail.com. Dr. Gupta is in the Department of Community Medicine. Email at dr_sbgupta@rediffmail.com. Sachin Malik in the Department of Community Medicine. Email him at sachinkurava999@gmail.com.
In order to have a survey estimate of the population mean $\bar{Y}$ of the study character $y$, assuming the knowledge of the population mean $\bar{X}$ of the auxiliary character $x$, the well-known ratio estimator is

$$t_r = \frac{\bar{Y}}{\bar{X}} \tag{1}$$

Bahl and Tuteja (1991) suggested an exponential ratio type estimator as

$$t_{exp} = \bar{Y} \exp \left[ \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right] \tag{2}$$

Several authors have used prior value of certain population parameter(s) to find more precise estimates. Sisodiya and Dwivedi (1981), Sen (1978) and Upadhyaya and Singh (1984) used the known coefficient of variation (CV) of the auxiliary character for estimating population mean of a study character in ratio method of estimation. The use of prior value of coefficient of kurtosis in estimating the population variance of study character $y$ was first made by Singh et al. (1973) Later used by Singh and Kakran (1993) in the estimation of population mean of study character. Singh and Tailor (2003) proposed a modified ratio estimator by using the known value of correlation coefficient. Kadilar and Cingi (2006) and Singh, Pandey, and Hirano (2008) have suggested modified ratio estimators by using different pairs of known value of population parameter(s).

Under SRSWOR, an almost unbiased estimator for estimating $\bar{Y}$ is proposed. To obtain the bias and MSE,

$$\bar{Y} = \bar{Y} (1 + e_0), \quad \bar{X} = \bar{X} (1 + e_1),$$

such that $E(e_0) = E(e_1) = 0$.

$$E\left(e_0^2\right) = f_1 C_y^2, \quad E\left(e_1^2\right) = f_1 C_x^2, \quad E(e_0 e_1) = f_1 \rho C_y C_x$$

where
ALMOST UNBIASED ESTIMATOR OF POPULATION PARAMETER

\[ f_i = \left( \frac{1}{n} - \frac{1}{N} \right) \], \[ S_y^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} (y_i - \bar{Y})^2 \], \[ S_x^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} (x_i - \bar{X})^2 \]

\[ C_y = \frac{S_y}{\bar{Y}} \], \[ C_x = \frac{S_x}{\bar{X}} \], \[ K_i = \rho_{y_i} \left( \frac{C_y}{C_x} \right) \], \[ \rho_{yx} = \frac{S_{yx}}{S_y S_x} \], \[ S_{xx} = \frac{1}{(N-1)} \sum_{i=1}^{N} (y_i - \bar{Y})(x_i - \bar{X}) \]

The Proposed Estimator

Consider the following estimator

\[ t_1 = \bar{Y} \left( \frac{K_i \bar{X} + K_i K_3}{K_i \bar{X} + K_2 K_3} \right)^{\alpha} \]

(3)

The bias and MSE expressions of the estimator \( t_1 \) up to the first order of approximation are, respectively, given by

\[ B(t_1) = \bar{Y} f_i C_x^2 \left[ \frac{\alpha(\alpha+1) V_i^2}{2} - \alpha V_i K_x \right] \]

(4)

\[ \text{MSE}(t_1) = \bar{Y}^2 f_i \left[ C_y^2 + C_x^2 \left( \alpha^2 V_i^2 + 2V_i \alpha K_x \right) \right] \]

(5)

Following Singh and Solanki (2012), consider the following estimator

\[ t_2 = \bar{Y} \left[ 2 - \left( \frac{\bar{X}}{\bar{X}} \right)^{\beta} \exp \left[ \lambda \left( \frac{(K_i \bar{X} + K_3) - (K_i \bar{X} + K_5)}{(K_i \bar{X} + K_3) + (K_i \bar{X} + K_5)} \right) \right] \right] \]

(6)

The bias and MSE expressions of the estimator \( t_2 \) up to the first order of approximation are, respectively, given by

\[ B(t_2) = \bar{Y} f_i C_x^2 \left[ \frac{\lambda V_i \beta}{2} - \frac{\beta(\beta-1)}{2} - \frac{\lambda(\lambda+2) V_i^2}{8} - \beta K_x + \frac{\lambda V_i K_x}{2} \right] \]

(7)
\[ \text{MSE}(t_2) = \bar{Y}^2 f_1 \left[ \frac{C_2}{2} \left( \beta^2 + \frac{\lambda^2 \lambda V_2}{4} - \beta \lambda V_2 \right) - 2K_1C_2 \left( \beta - \frac{\lambda V_2}{2} \right) \right] \]  

\[ (8) \]

\( \alpha, \lambda, \) and \( \beta \) are suitable chosen constants. Also \( K_1, K_3, K_4, K_5 \) are either real numbers or function of known parameters of the auxiliary variable \( x \) such as \( C_3, \beta_2 (x), \rho_{xy} \) and \( K_1. K_2 \) is an integer which takes values +1 and -1 for designing the estimators and

\[ V_1 = \frac{K_1 \bar{X}}{K_1 \bar{X} + K_2 K_3} \]

\[ V_2 = \frac{K_4 \bar{X}}{K_4 \bar{X} + K_5} \]

The estimators \( t_1 \) and \( t_2 \) are biased estimators. In some applications bias is disadvantageous. Following these estimators we have proposed almost unbiased estimator of \( \bar{Y} \).

**Almost Unbiased Estimator**

Suppose

\[ t_0 = \bar{Y}_t = \bar{Y} \left( \frac{K_1 \bar{X} + K_2 K_3}{K_1 \bar{X} + K_2 K_3} \right)^\alpha \]

\[ t_2 = 2 - \left( \frac{\bar{X}}{\bar{X}} \right)^\beta \exp \left[ \lambda \left( \frac{K_4 \bar{X} + K_5}{K_4 \bar{X} + K_5} \right) - \left( \frac{K_4 \bar{X} + K_5}{K_4 \bar{X} + K_5} \right) \right] \]

such that \( t_0, t_1, t_2 \in W \), where \( W \) denotes the set of all possible estimators for estimating the population mean \( \bar{Y} \). By definition, the set \( W \) is a linear variety if

\[ t_p = \sum_{i=0}^{3} w_i t_i \in W \]  

\[ (9) \]

such that,

\[ \sum_{i=0}^{3} w_i = 1 \text{ and } w_i \in R \]  

\[ (10) \]
ALMOST UNBIASED ESTIMATOR OF POPULATION PARAMETER

where \( w_i \) \((i = 0, 1, 2, 3)\) denotes the constants used for reducing the bias in the class of estimators, \( H \) denotes the set of those estimators that can be constructed from \( t_i \) \((i = 0, 1, 2, 3)\) and \( R \) denotes the set of real numbers.

Expressing \( t_p \) in terms of \( e \)'s,

\[
t_p = \bar{Y} \left[ 1 + e_0 + w_1 \left( \frac{\alpha(\alpha+1)V_1^2e_i^2}{2} - \alpha V_1 e_1 - \alpha V_1 e_0 e_i \right) + w_2 \left( -\beta e_i - \frac{\beta(\beta-1)e_i^2}{2} + \frac{\lambda V_2^2 e_i}{2} + \frac{\lambda(\lambda+2)V_2^2 e_i^2}{8} - \beta e_0 e_i + \frac{\lambda V_2 e_0 e_i}{2} \right) \right]
\]

(11)

Subtracting \( \bar{Y} \) from both sides of equation (11) and then taking expectation of both sides, the bias of the estimator \( t_p \) is obtained up to the first order of approximation, as

\[
B(t_p) = \bar{Y}_i w_i C_x^2 \left( \frac{\alpha(\alpha+1)V_1^2}{2} - \alpha V_1 K_x \right) + \bar{Y}_i w_2 C_x^2 \left( \frac{\lambda V_2^2}{2} - \frac{\beta(\beta-1)}{2} - \frac{\lambda(\lambda+2)V_2^2}{8} - \beta K_x + \frac{\lambda V_2 K_x}{2} \right)
\]

(12)

From (11), we have

\[
(t_p - \bar{Y}) = \bar{Y} \left[ e_0 - w_i \alpha V_1 e_1 - w_2 \left( \beta e_i + \frac{\lambda V_2 e_i}{2} \right) \right]
\]

(13)

Squaring both sides of (13) and then taking expectation, the MSE of the estimator \( t_p \) up to the first order of approximation is obtained, as

\[
MSE(t_p) = \bar{Y}^2 f_i \left[ C_x^2 + C_x^2 \left( Q^2 - 2QK_x \right) \right]
\]

(14)

which is a minimum when

\[
Q = K_x
\]

(15)
where

\[ Q = w_1 \alpha V_1 + w_2 \left( \beta - \frac{\lambda V_2}{2} \right) \]  

(16)

Putting the value of \( Q = K \) in (14), the optimum value of estimator as \( t_p \) (optimum) is obtained. Thus, the minimum MSE of \( t_p \) is given by

\[ \text{min MSE}(t_p) = \bar{Y}^2 f_1 C_f^2 \left( 1 - \rho^2 \right) \]  

(17)

which is same as that of traditional linear regression estimator.

From (10) and (16), there are two equations and three unknowns. It is not possible to find the unique values for \( w_i \)'s, \( i = 0, 1, 2 \). In order to get unique values of \( w_i \)'s, impose the linear restriction

\[ \sum_{i=0}^{2} w_i B(t_i) = 0 \]  

(18)

where \( B(t_i) \) denotes the bias in the \( i \)th estimator.

Equations (10), (16) and (18) can be written in the matrix form as

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & \alpha V_1 & \beta - \frac{\lambda V_2}{2} \\
0 & B(t_1) & B(t_2)
\end{bmatrix}
\begin{bmatrix}
w_0 \\
w_1 \\
w_2
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
k_x \\
0
\end{bmatrix}
\]  

(19)

Using (19), the unique values of \( w_i \)'s, \( i = 0, 1, 2 \) are

\[
w_0 = w_1 = \frac{X_1 K_x A_i}{\alpha V_1 \left[ \alpha V_i A_2 - A_i X_1 \right]} + X_2
\]

\[
w_2 = \frac{K_x A_i}{\alpha V_1 \left[ \alpha V_i A_2 - A_i X_1 \right]}
\]
where,

\[
A_i = \frac{\alpha (\alpha + 1) V_i^2}{2} - \alpha V_i K_x
\]

\[
A_2 = \frac{\lambda V_2 \beta}{2} - \frac{\beta (\beta - 1)}{2} - \frac{\lambda (\lambda + 2) V_2^2}{8} - \beta K_x + \frac{\lambda V_2 K_x}{2}
\]

\[
X_1 = A_i \left[ \beta - \frac{\lambda V_2^2}{2} \right]
\]

\[
X_2 = \frac{K_x}{\alpha V_i}
\]

Use of these \( w_i \)'s, \( i = 0, 1, 2 \) remove the bias up to terms of order \( o (n^{-1}) \) at (9).

**Empirical Study**


**Data Statistics**

<table>
<thead>
<tr>
<th>Population</th>
<th>( N )</th>
<th>( n )</th>
<th>( \bar{Y} )</th>
<th>( \bar{X} )</th>
<th>( C_y )</th>
<th>( C_x )</th>
<th>( \rho_{yx} )</th>
<th>( \beta_2(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population 1</td>
<td>106</td>
<td>20</td>
<td>2212.59</td>
<td>27421.7</td>
<td>5.22</td>
<td>2.1</td>
<td>0.86</td>
<td>34.57</td>
</tr>
<tr>
<td>Population 2</td>
<td>20</td>
<td>8</td>
<td>19.55</td>
<td>18.8</td>
<td>0.355</td>
<td>0.394</td>
<td>-0.92</td>
<td>3.06</td>
</tr>
</tbody>
</table>

**Table 1. Values of \( w_i \)**

<table>
<thead>
<tr>
<th>( w_i )</th>
<th>Population 1</th>
<th>Population 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_0 )</td>
<td>2.104965</td>
<td>3.692323</td>
</tr>
<tr>
<td>( w_1 )</td>
<td>-6.48599</td>
<td>1.379436</td>
</tr>
<tr>
<td>( w_2 )</td>
<td>5.381022</td>
<td>-4.07176</td>
</tr>
</tbody>
</table>

The percent relative efficiencies (PRE) of various estimators of \( \bar{Y} \) are computed and presented in Table 2 below.
Table 2. PRE of different estimators of $\overline{Y}$ with respect to $\overline{y}$

<table>
<thead>
<tr>
<th>Choice of scalars</th>
<th>$w_0$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$K_3$</th>
<th>$K_4$</th>
<th>$K_5$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\lambda$</th>
<th>Estimator</th>
<th>PRE (POPI)</th>
<th>PRE (POPII)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\overline{y}$</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td>$t_R$</td>
<td>212.8</td>
<td>24.69</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-1</td>
<td>$t_{exp}$</td>
<td>53.94</td>
<td>583.07</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>$t_{1(1,0)}$</td>
<td>212.8</td>
<td>23.39</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-1</td>
<td>$t_{1(-1,0)}$</td>
<td>53.94</td>
<td>527.29</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$t_{2(1,1)}$</td>
<td>143.99</td>
<td>42.93</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$t_{2(1,-1)}$</td>
<td>306.54</td>
<td>14.63</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$t_{2(0,1)}$</td>
<td>72.12</td>
<td>348.58</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$t_{2(0,-1)}$</td>
<td>143.97</td>
<td>42.93</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>$t_p$</td>
<td>384.02</td>
<td>651.04</td>
</tr>
</tbody>
</table>

Proposed Estimators in Two Phase Sampling

When $\overline{X}$ is unknown, it is sometimes estimated from a preliminary large sample of size $n'$ on which only the characteristic $x$ is measured (for details see Singh et al., 2007). Then, a second phase sample of size $n$ ($n < n'$) is drawn on which both $y$ and $x$ characteristics are measured. Let $\overline{x} = \frac{1}{n'} \sum_{i=1}^{n'} x_i$ denote the sample mean of $x$ based on first phase sample of size $n'$, $\overline{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$ and $\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$, be the sample means of $y$ and $x$ respectively based on second phase of size $n$.

In two-phase sampling the estimator $t_p$ will take the following form

$$t_{pd} = \sum_{i=0}^{1} h_i t_{id} \in H$$  \hspace{1cm} (20)

Such that,

$$\sum_{i=0}^{1} h_i = 1 \text{ and } h_i \in R$$  \hspace{1cm} (21)
ALMOST UNBIASED ESTIMATOR OF POPULATION PARAMETER

where,

\[ t_{0d} = \bar{y}, \quad t_{1d} = \bar{y} \left( \frac{K_1 \bar{x} + K_2 K_3}{K_1 \bar{x} + K_2 K_3} \right)^m, \quad t_{2d} = \bar{y} \left\{ 2 - \left( \frac{\bar{x}}{\bar{x}'} \right)^q \exp \left[ \gamma \left( \frac{K_4 \bar{x} + K_5}{K_4 \bar{x} + K_5} \right) \right] \right\} \]

The bias and MSE expressions of the estimator \( t_{1d} \) and \( t_{2d} \) up to the first order of approximation are, respectively, given by

\[
B(t_{1d}) = \bar{y} \left[ \frac{m(m-1)R_i^2 f_2 C_i^2}{2} + \frac{m(m+1)R_i^2 f_1 C_i^2}{2} - mR_i f_1 K_i C_i^2 \right] \]

(22)

\[
\text{MSE}(t_{1d}) = \bar{y}^2 \left[ f_1 C_i^2 + mR_i f_1 K_i C_i^2 - 2mR_i K_i f_3 C_i^2 \right] \]

(23)

\[
B(t_{2d}) = \bar{y} \left[ -\frac{q(q-1)f_1 C_i^2}{2} + \frac{q(q+1)f_1 C_i^2}{2} + qf_2 K_i C_i^2 + q^2 f_2 C_i^2 + f_3 \gamma R_i K_i C_i^2 + f_3 \gamma R_i q C_i^2 \right] \]

(24)

\[
\text{MSE}(t_{2d}) = \bar{y}^2 \left[ f_1 C_i^2 + L_1 f_3 C_i^2 \right] \]

(25)

where

\[
R_1 = \frac{K_1 \bar{x}}{K_1 \bar{x} + K_2 K_3} \quad R_2 = \frac{K_4 \bar{x}}{2(K_4 \bar{x} + K_5)} \quad L_1 = q - \gamma A_2 \]

(26)
Expressing (20) in terms of \( e \)'s,
\[
t_{pd} = \bar{Y} \left[ 1 + e_s + w_i \left( \frac{m(m+1)R e_i^2 - mR e_i - mR e_i e_i' + mR e_e e_i' + \frac{m(m-1)R e_i^2 - mR e_i e_i'}{2}}{2} \right) \right] \\
+ w_i \left( -q e_e - \frac{q(q-1)e_i^2}{2} + q e_e' + q e_e' - q \frac{(q+1)e_i^2}{2} - \gamma R_2 (e_i' - e_i) + \gamma R_2 (e_i' - e_i) - q e_e e_i \right)
\]

Subtracting \( \bar{Y} \) from both sides of equation (22) and then taking expectation of both sides, the bias of the estimator \( t_{pd} \) is obtained up to the first order of approximation, as
\[
B(t_{pd}) = \bar{Y} [B(t_{1d}) + B(t_{2d})] \tag{27}
\]
also,
\[
(t_{pd} - \bar{Y}) = \bar{Y} \left[ e_0 + w_i [mR e_i' - mR e_i] + w_i (-q e_e + q e_e' - \gamma R_2 e_i' + \gamma R_2 e_i) \right] \tag{28}
\]

Squaring both sides of (28) and then taking expectation, the MSE of the estimator \( t_{pd} \) is obtained up to the first order of approximation, as
\[
\text{MSE}(t_{pd}) = \bar{Y}^2 f_1 C_y^2 + L_2^2 f_3 C_x^2 - 2L_2 f_3 K_x C_x^2 \tag{29}
\]
It is a minimum when
\[
L_2 = K_x \tag{30}
\]
where
\[
L_2 = h_m R_1 + h_2 (q - \gamma R_2) \tag{31}
\]
Putting the value of \( L_2 = K_x \) in (28), the optimum value of estimator as \( t_{pd} \) (optimum) is obtained. Thus, the minimum MSE of \( t_{pd} \) is given by
\[
\text{min.MSE}(t_{pd}) = \bar{Y}^2 C_y^2 \left( f_1 - f_3 \rho_{yx}^2 \right). \tag{32}
\]
ALMOST UNBIASED ESTIMATOR OF POPULATION PARAMETER

which is same as that of traditional linear regression estimator.

From (21) and (31), there are two equations in three unknowns. It is not possible to find the unique values for \( h_i \)’s, \( i = 0, 1, 2 \). In order to get unique values of \( h_i \)’s, impose the linear restriction

\[
\sum_{i=0}^{2} h_i B(t_i) = 0
\]

(33)

where \( B(t_i) \) denotes the bias in the \( i \)th estimator.

Equations (21), (31) and (33) can be written in the matrix form as

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & mR_1 & q - \gamma R_2 \\
0 & B(t_{id}) & B(t_{2d})
\end{pmatrix}
\begin{bmatrix}
h_0 \\
h_1 \\
h_2
\end{bmatrix}
= \begin{bmatrix}
1 \\
K_x \\
0
\end{bmatrix}
\]

(34)

Using (34), we get the unique values of \( h_i \)’s, \( i = 0, 1, 2 \) as

\[
\begin{align*}
h_0 &= 1 - h_1 - h_2 \\
h_1 &= \frac{k_x}{mR_1} - \frac{N_1 K_x (q - \gamma R_2)}{N_1 q - mR_1 N_2 - N_1 \gamma R_2} \\
h_2 &= \frac{K_x N_1}{[N_1 q - mR_1 N_2 - N_1 \gamma R_2]}
\end{align*}
\]

where,

\[
N_1 = \frac{m(m-1)R_1^2 f_2 C_x^2}{2} + \frac{m(m+1)R_1^2 f_1 C_x^2}{2} - m^2 R_1^2 f_2 C_x^2 + mR_1 f_3 K_x C_x^2
\]

\[
N_2 = \left[-\frac{q(q-1)f_x C_x^2}{2} + \frac{q(q+1)f_x C_x^2}{2} + q f_x K_x C_x^2 + q^2 f_x C_x^2 + f_x \gamma R_x K_x C_x^2 + f_x \gamma R_x q C_x^2 \right]
\]

Use of these \( h_i \)’s, \( i = 0, 1, 2 \) remove the bias up to terms of order \( o(n^{-1}) \) at (20).
References


Appendix A.

Some members (ratio-type) of the class $t_1$ when $w_0 = 0$, $w_1 = 1$, $w_2 = 0$, $\alpha = 1$

<table>
<thead>
<tr>
<th></th>
<th>$K_1$</th>
<th>$K_3$</th>
<th>PRE's $K_2 = 1$</th>
<th>PRE's $K_2 = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$C_x$</td>
<td></td>
<td>212.80</td>
<td>212.82</td>
</tr>
<tr>
<td>1</td>
<td>$\beta_2(x)$</td>
<td></td>
<td>212.60</td>
<td>213.02</td>
</tr>
<tr>
<td>$\beta_2(x)$</td>
<td>$C_x$</td>
<td></td>
<td>212.81</td>
<td>212.81</td>
</tr>
<tr>
<td>$C_x$</td>
<td>$\beta_2(x)$</td>
<td></td>
<td>212.71</td>
<td>212.91</td>
</tr>
<tr>
<td>1</td>
<td>$\rho_{yx}$</td>
<td></td>
<td>212.81</td>
<td>212.82</td>
</tr>
<tr>
<td>$N\bar{X}$</td>
<td>$S_x$</td>
<td></td>
<td>212.81</td>
<td>212.81</td>
</tr>
<tr>
<td>$N\bar{X}$</td>
<td>$f$</td>
<td></td>
<td>212.80</td>
<td>212.82</td>
</tr>
<tr>
<td>$\beta_2(x)$</td>
<td>$K_x$</td>
<td></td>
<td>212.60</td>
<td>213.02</td>
</tr>
<tr>
<td>$N$</td>
<td>$K_x$</td>
<td></td>
<td>212.81</td>
<td>212.81</td>
</tr>
<tr>
<td>$N$</td>
<td>1</td>
<td></td>
<td>212.71</td>
<td>212.91</td>
</tr>
<tr>
<td>$N$</td>
<td>$C_x$</td>
<td></td>
<td>212.81</td>
<td>212.82</td>
</tr>
<tr>
<td>$N$</td>
<td>$\rho_{yx}$</td>
<td></td>
<td>212.81</td>
<td>212.81</td>
</tr>
<tr>
<td>$N$</td>
<td>$S_x$</td>
<td></td>
<td>212.80</td>
<td>212.82</td>
</tr>
<tr>
<td>$N$</td>
<td>$f$</td>
<td></td>
<td>212.60</td>
<td>213.02</td>
</tr>
<tr>
<td>$N$</td>
<td>$g = 1-f$</td>
<td></td>
<td>212.81</td>
<td>212.81</td>
</tr>
<tr>
<td>$N$</td>
<td>$K_x$</td>
<td></td>
<td>212.71</td>
<td>212.91</td>
</tr>
<tr>
<td>$n$</td>
<td>$\rho_{yx}$</td>
<td></td>
<td>212.81</td>
<td>212.82</td>
</tr>
<tr>
<td>$n$</td>
<td>$S_x$</td>
<td></td>
<td>212.81</td>
<td>212.81</td>
</tr>
<tr>
<td>$n$</td>
<td>$f$</td>
<td></td>
<td>212.80</td>
<td>212.82</td>
</tr>
<tr>
<td>$n$</td>
<td>$g = 1-f$</td>
<td></td>
<td>212.60</td>
<td>213.02</td>
</tr>
<tr>
<td>$n$</td>
<td>$K_x$</td>
<td></td>
<td>212.81</td>
<td>212.81</td>
</tr>
<tr>
<td>$\beta_2(x)$</td>
<td>$\bar{X}$</td>
<td></td>
<td>212.71</td>
<td>212.91</td>
</tr>
<tr>
<td>$N\bar{X}$</td>
<td>$\bar{X}$</td>
<td></td>
<td>212.81</td>
<td>212.82</td>
</tr>
<tr>
<td>$N$</td>
<td>$\bar{X}$</td>
<td></td>
<td>212.81</td>
<td>212.81</td>
</tr>
<tr>
<td>$n$</td>
<td>$\bar{X}$</td>
<td></td>
<td>212.81</td>
<td>212.81</td>
</tr>
</tbody>
</table>
Appendix B.

Some members (product-type) of the class $t_1$ when $w_0 = 0$, $w_1 = 1$, $w_2 = 0$, $\alpha = -1$

<table>
<thead>
<tr>
<th>$K_1$</th>
<th>$K_3$</th>
<th>PRE's $K_2 = 1$</th>
<th>PRE's $K_2 = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$C_x$</td>
<td>550.91</td>
<td>501.92</td>
</tr>
<tr>
<td>1</td>
<td>$\beta_2(x)$</td>
<td>646.03</td>
<td>314.33</td>
</tr>
<tr>
<td>$\beta_2(x)$</td>
<td>$C_x$</td>
<td>535.22</td>
<td>519.18</td>
</tr>
<tr>
<td>$C_x$</td>
<td>$\beta_2(x)$</td>
<td>582.35</td>
<td>91.18</td>
</tr>
<tr>
<td>1</td>
<td>$\rho_{yx}$</td>
<td>466.00</td>
<td>579.15</td>
</tr>
<tr>
<td>$N\bar{X}$</td>
<td>$S_x$</td>
<td>528.52</td>
<td>526.06</td>
</tr>
<tr>
<td>$N\bar{X}$</td>
<td>$f$</td>
<td>527.30</td>
<td>527.28</td>
</tr>
<tr>
<td>$\beta_2(x)$</td>
<td>$K_x$</td>
<td>510.01</td>
<td>543.74</td>
</tr>
<tr>
<td>$N$</td>
<td>$K_x$</td>
<td>527.15</td>
<td>527.43</td>
</tr>
<tr>
<td>$N$</td>
<td>1</td>
<td>530.39</td>
<td>524.16</td>
</tr>
</tbody>
</table>

| $N$     | $C_x$   | 528.52          | 526.03          |
| $N$     | $\rho_{yx}$ | 524.41          | 530.15          |
| $N$     | $S_x$   | 549.55          | 503.49          |
| $N$     | $f$     | 527.53          | 527.06          |
| $N$     | $g = 1 - f$ | 530.16          | 524.40          |

| $N$     | $K_x$   | 524.70          | 529.87          |
| $n$     | $\rho_{yx}$ | 520.05          | 534.38          |
| $n$     | $S_x$   | 579.44          | 465.58          |
| $n$     | $f$     | 527.88          | 526.71          |
| $n$     | $g = 1 - f$ | 534.42          | 520.01          |

| $n$     | $K_x$   | 520.77          | 533.69          |
| $\beta_2(x)$ | $\bar{X}$ | 622.76          | 146.09          |
| $N\bar{X}$ | $\bar{X}$ | 530.39          | 524.16          |
| $N$     | $\bar{X}$ | 580.14          | 464.59          |
| $n$     | $\bar{X}$ | 632.80          | 363.64          |
Appendix C.

Some members (product-type) of the class $t_2$ when $w_0 = 0, w_1 = 0, w_2 = 1$

<table>
<thead>
<tr>
<th>$K_4$</th>
<th>$K_5$</th>
<th>PRE’S</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>$K_3$</td>
<td>$(\beta = -1, \lambda = -1)$</td>
</tr>
<tr>
<td>1</td>
<td>$C_x$</td>
<td>358.00</td>
</tr>
<tr>
<td>1</td>
<td>$\beta_2(x)$</td>
<td>423.38</td>
</tr>
<tr>
<td>$\beta_2(x)$</td>
<td>$C_x$</td>
<td>351.94</td>
</tr>
<tr>
<td>$C_x$</td>
<td>$\beta_2(x)$</td>
<td>357.48</td>
</tr>
<tr>
<td>1</td>
<td>$\rho_{yx}$</td>
<td>324.10</td>
</tr>
<tr>
<td>$N\bar{X}$</td>
<td>$S_x$</td>
<td>349.09</td>
</tr>
<tr>
<td>$N\bar{X}$</td>
<td>$f$</td>
<td>348.58</td>
</tr>
<tr>
<td>$\beta_2(x)$</td>
<td>$K_x$</td>
<td>341.45</td>
</tr>
<tr>
<td>$N$</td>
<td>$K_x$</td>
<td>348.52</td>
</tr>
<tr>
<td>$N$</td>
<td>1</td>
<td>349.89</td>
</tr>
<tr>
<td>$N$</td>
<td>$C_x$</td>
<td>349.09</td>
</tr>
<tr>
<td>$N$</td>
<td>$\rho_{yx}$</td>
<td>347.37</td>
</tr>
<tr>
<td>$N$</td>
<td>$S_x$</td>
<td>358.21</td>
</tr>
<tr>
<td>$N$</td>
<td>$f$</td>
<td>348.68</td>
</tr>
<tr>
<td>$N$</td>
<td>$g = 1 - f$</td>
<td>349.79</td>
</tr>
<tr>
<td>$N$</td>
<td>$K_x$</td>
<td>347.49</td>
</tr>
<tr>
<td>$n$</td>
<td>$\rho_{yx}$</td>
<td>345.56</td>
</tr>
<tr>
<td>$n$</td>
<td>$S_x$</td>
<td>372.38</td>
</tr>
<tr>
<td>$n$</td>
<td>$f$</td>
<td>348.82</td>
</tr>
<tr>
<td>$n$</td>
<td>$g = 1 - f$</td>
<td>351.60</td>
</tr>
<tr>
<td>$\beta_2(x)$</td>
<td>$\bar{X}$</td>
<td>345.86</td>
</tr>
<tr>
<td>$N\bar{X}$</td>
<td>$\bar{X}$</td>
<td>349.89</td>
</tr>
</tbody>
</table>

In addition to above estimators a large number of estimators can also be generated from the proposed estimators just by putting different values of constants $w_i$’s, $h_i$’s $K_1$, $K_2$, $K_3$, $K_4$, $K_5$, $\alpha$, $\beta$ and $\lambda$. 

615