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Maximum Likelihood Estimation of the Kumaraswamy Exponential Distribution with Applications

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The Kumaraswamy exponential distribution, a generalization of the exponential, is developed as a model for problems in environmental studies, survival analysis and reliability. The estimation of parameters is approached by maximum likelihood and the observed information matrix is derived. The proposed models are applied to three real data sets.

Keywords: Information matrix, Maximum likelihood, Moment generating function.

Introduction

A random variable $X$ has the exponential distribution if its cumulative distribution function for $x > 0$ is given by

$$F(x) = 1 - e^{-\lambda x}$$  \hspace{1cm} (1)

where $\lambda > 0$ is a scale parameter, the probability density function is

$$f(x) = \lambda e^{-\lambda x}$$  \hspace{1cm} (2)

Using the Kumaraswamy link function by Cordeiro and de Castro (2011) given as

$$g(x) = a, b f(x)[F(x)]^{b-1} \left[1 - F(x)^b\right]^{a-1}$$  \hspace{1cm} (3)
By inserting (1) and (2) in (3) we have

\[
g(x) = a, b, \lambda \ell^{-\lambda x} \left(1 - \ell^{-\lambda x}\right)^{b-1} \left[1 - \left(1 - \ell^{-\lambda x}\right)^{b}\right]^{a-1}
\]

(4)

\[a, b, \lambda > 0\]

Another term of Kumaraswamy distribution can be obtained using the binomial series expansion. The Kumaraswamy exponential distribution in equation (4) can be expanded as follows:

\[
(1-m)^K = \sum_{j=1}^{K} (-1)^j \binom{K}{j} M^j
\]

(5)
as

\[
g(x) = a, b, \lambda \ell^{-\lambda x} \left(1 - \ell^{-\lambda x}\right)^{b-1} \sum_{j=1}^{K} (-1)^j \binom{K}{j} \left(1 - \ell^{-\lambda x}\right)^{bj}
\]

\[= a, b, \lambda \ell^{-\lambda x} \sum_{j=1}^{K} (-1)^j \binom{K}{j} \left(1 - \ell^{-\lambda x}\right)^{bj+b-1}\]

**Statistical inference**

Given a random variable \(X\) following equation (4), the likelihood function is obtained as

\[
L = a^n b^n \lambda^n \ell \prod_{i=1}^{n} \ell^{-\lambda x_i} \left(1 - \ell^{-\lambda x_i}\right)^{b-1} \left[1 - \left(1 - \ell^{-\lambda x_i}\right)^{b}\right]^{a-1}
\]

Taking log-likelihood of the above

\[
\log L = n \log a + n \log b + n \log \lambda - \lambda \sum_{i=1}^{n} x_i + (b-1) \sum_{i=1}^{n} \log \left(1 - \ell^{-\lambda x_i}\right)
\]

\[+ (a-1) \sum_{i=1}^{n} \log \left[1 - \left(1 - \ell^{-\lambda x_i}\right)^{b}\right]\]
MLE OF THE KUMARASWAMY DISTRIBUTION

\[
\frac{\partial \log L}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \log \left[1 - \left(1 - \ell^{-\lambda x}\right)^b \right] 
\]

\[
\frac{\partial \log L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \log \left(1 - \ell^{-\lambda x}\right) - (a-1) \sum_{i=1}^{n} \frac{\left(1 - \ell^{-\lambda x}\right)^b \log \left(1 - \ell^{-\lambda x}\right)}{1 - \left(1 - \ell^{-\lambda x}\right)^b} 
\]

\[
\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x + (b-1) \sum_{i=1}^{n} \frac{x \ell^{-\lambda x}}{1 - \ell^{-\lambda x}} - (a-1) b \sum_{i=1}^{n} \frac{x \ell^{-\lambda x} \left(1 - \ell^{-\lambda x}\right)^{b-1}}{1 - \left(1 - \ell^{-\lambda x}\right)^b} 
\]

**Fisher information**

\[
\frac{\partial^2 \log L}{\partial a^2} = -\frac{n}{a^2} 
\]

\[
\frac{\partial^2 \log L}{\partial b^2} = -\frac{n}{b^2} - (a-1) \sum_{i=1}^{n} \frac{\left(1 - \ell^{-\lambda x}\right)^b \left[\log \left(1 - \ell^{-\lambda x}\right)\right]^2}{\left[1 - \left(1 - \ell^{-\lambda x}\right)^b\right]^2} 
\]

\[
\frac{\partial^2 \log L}{\partial \lambda^2} = -\frac{n}{\lambda^2} - (b-1) \sum_{i=1}^{n} \frac{x}{\left(1 - \ell^{-\lambda x}\right)^2} - 
\]

\[
b(a-1) \sum_{i=1}^{n} \frac{x^2 \ell^{-\lambda x} \left(1 - \ell^{-\lambda x}\right)^{b-1}}{1 - \left(1 - \ell^{-\lambda x}\right)^b} \left[\frac{(b-1) \ell^{-\lambda x} \left(1 - \ell^{-\lambda x}\right)^{b-2}}{(1 - \ell^{-\lambda x})^{b-1}} - \frac{b \ell^{-\lambda x} \left(1 - \ell^{-\lambda x}\right)^{b-1}}{(1 - \ell^{-\lambda x})^b}\right] 
\]
\[ \frac{\partial^2 \log L}{\partial a \partial b} = -\left(1 - \ell^{-\lambda x}\right)^b \log \left(1 - \ell^{-\lambda x}\right) \]

\[ \frac{\partial^2 \log L}{\partial b \partial \lambda} = \sum_{i=1}^{n} \frac{x_i \ell^{-\lambda x_i}}{1 - \ell^{-\lambda x_i}} \]

\[ \frac{\partial^2 \log L}{\partial a \partial \lambda} = -b \sum_{i=1}^{n} \frac{x_i \ell^{-\lambda x_i} \left(1 - \ell^{-\lambda x_i}\right)^{b-1}}{1 - \left(1 - \ell^{-\lambda x_i}\right)^b} \]

**Application**

For the sake of numerical illustrations, the two data sets used by Raja and Mir (2011) are considered. The first data set is on the failure time of the conditioning system of an airplane and the second is the runs scored by a Cricketer in 27 innings at national level.

**Table 1. Descriptive Statistics on Failure Time on Conditional System**

<table>
<thead>
<tr>
<th>Min</th>
<th>Q1</th>
<th>Q2</th>
<th>Mean</th>
<th>Q3</th>
<th>Max</th>
<th>Var</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>12.5</td>
<td>22.0</td>
<td>59.6</td>
<td>83.0</td>
<td>261.0</td>
<td>5167.421</td>
</tr>
</tbody>
</table>

**Skewness**  
1.693605  

**Kurtosis**  
4.966655

**Table 2. Descriptive Statistics in runs scored by a Cricketer**

<table>
<thead>
<tr>
<th>Min</th>
<th>Q1</th>
<th>Q2</th>
<th>Mean</th>
<th>Q3</th>
<th>Max</th>
<th>Var</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.00</td>
<td>8.00</td>
<td>25.00</td>
<td>36.41</td>
<td>50.00</td>
<td>127.00</td>
<td>1149.02</td>
</tr>
</tbody>
</table>

**Skewness**  
1.124548  

**Kurtosis**  
3.492725
MLE OF THE KUMARASWAMY DISTRIBUTION

Table 3. Failure Time on Conditional System

<table>
<thead>
<tr>
<th>Model</th>
<th>Estimates</th>
<th>Statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Log-likelihood</td>
</tr>
<tr>
<td>Weibull</td>
<td>( \hat{\alpha} = 0.8536, \hat{\lambda} = 0.0183 )</td>
<td>-151.970</td>
</tr>
<tr>
<td>Lognormal</td>
<td>( \hat{\mu} = 3.358, \hat{\lambda} = 1.3190 )</td>
<td>151.621</td>
</tr>
<tr>
<td>Exponentiated Weibull</td>
<td>( \hat{\alpha} = 3.824, \theta = 0.1732, \hat{\delta} = 82.235 )</td>
<td>-149.567</td>
</tr>
<tr>
<td>Exponentiated Gumbel</td>
<td>( \hat{\alpha} = 1.9881, \hat{\lambda} = 49.0638 )</td>
<td>-148537</td>
</tr>
<tr>
<td>Exponentiated Lognormal</td>
<td>( \hat{\alpha} = 0.1542, \hat{\mu} = 3.1353, \hat{\delta} = 0.3648 )</td>
<td>-148.659</td>
</tr>
<tr>
<td>Lehman Type II Exponential</td>
<td>( \hat{\alpha} = 0.3439, \hat{\lambda} = 0.0057 )</td>
<td>-152.6097</td>
</tr>
<tr>
<td>Exponential</td>
<td>( \hat{\lambda} = 0.0168 )</td>
<td>-152.6297</td>
</tr>
<tr>
<td>Kumaraswamy Exponential Distribution</td>
<td>( \hat{\alpha} = 10.142, \hat{\beta} = 0.9129, \hat{\lambda} = 0.0005 )</td>
<td>-107/9653</td>
</tr>
</tbody>
</table>

Table 4. Runs Scored by a Cricketer

<table>
<thead>
<tr>
<th>Model</th>
<th>Estimates</th>
<th>Statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Log-likelihood</td>
</tr>
<tr>
<td>Gamma</td>
<td>( \hat{\alpha} = 0.7235, \hat{\lambda} = 0.0127 )</td>
<td>-125.654</td>
</tr>
<tr>
<td>Weibull</td>
<td>( \hat{\alpha} = 1.040, \hat{\lambda} = 36.985 )</td>
<td>-124.021</td>
</tr>
<tr>
<td>Lognormal</td>
<td>( \hat{\mu} = 3.0534, \lambda = 1.174 )</td>
<td>-125.059</td>
</tr>
<tr>
<td>Exponentiated exponential</td>
<td>( \hat{\alpha} = 0.8126, \lambda = 0.0153 )</td>
<td>-125.945</td>
</tr>
<tr>
<td>Exponentiated Lognormal</td>
<td>( \hat{\alpha} = 0.578, \hat{\mu} = 3.1836, \hat{\delta} = 0.7834 )</td>
<td>-125.965</td>
</tr>
<tr>
<td>Exponentiated Gumbel</td>
<td>( \hat{\alpha} = 1.873, \hat{\lambda} = 45.264 )</td>
<td>-124.843</td>
</tr>
<tr>
<td>Exponential</td>
<td>( \hat{\lambda} = 0.0275 )</td>
<td>-124.0589</td>
</tr>
<tr>
<td>Kumaraswamy Exponential Distribution</td>
<td>( \hat{\alpha} = 0.13006, \hat{\beta} = 0.9557, \hat{\lambda} = 0.00014 )</td>
<td>-108.7224</td>
</tr>
</tbody>
</table>
Conclusion

The probability density function of Kumaraswamy-exponential distribution was discussed and applied for two data sets. In first data set regarding failure times of the conditioning system of an aeroplane. Kumaraswamy exponential provided the best fit followed by exponentiated Gumbel. In second data set regarding runs scored by a cricketer Kumaraswamy exponential, Weibull and exponential distributions provided better fit.

References


