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The President's Problem

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This Algorithms and Code is brought to you for free and open access by the Open Access Journals at DigitalCommons@WayneState. It has been accepted for inclusion in Journal of Modern Applied Statistical Methods by an authorized editor of DigitalCommons@WayneState.
A solution is offered in response to a complex combination problem challenged by Blom, Englund, and Sandell (1998). The problem is to determine the probability that a random permutation of the word BILLCLINTON has no equal neighbors.

Key words: Combinatorics, equal neighbors, random permutations, run

Introduction
The problem is to determine the probability that a random permutation of the word BILLCLINTON has no equal neighbors. Choose an initial order of the letters in the word Billclinton, for example, IINNLLLBTBCO. The problem is solved in three steps: Start with IINN, insert LLL, then insert B, T, C, and finally, insert O.

Methodology
Let $X_1$ be the number of equal neighbors in a random permutation of the four letters IINN. To obtain the solution, the probability function $P(X_1 = k)$ is needed. First, determine the probability function of the total number of runs (see references). Consider a random permutation of $m$ 1’s and $n$ 0’s. Denote by $U$ the number of runs of 1’s and by $V$ the number of runs of 0’s. The probability function $P(U = r, V = s)$ is needed. (Note that when $|r - s| > 1$, $P(U = r, V = s) = 0$.) The $m$ 1’s can be partitioned into $r$ groups in $\binom{m-1}{r-1}$ ways. Similarly, the $n$ 0’s can be partitioned into $s$ groups in $\binom{n-1}{s-1}$ ways. It is known that the number of permutations with $r$ 1- runs and $s$ 0-runs is the product of these two binomial coefficients when $|r - s| = 1$, and twice that product when $r = s$. Since the total number of permutations is $\binom{m+n}{m}$, obtained is

$$P(U = r, V = s) = \frac{(m-1)(n-1)}{\binom{m+n}{m}} \frac{r-1}{s-1}$$

(1.1)

for $r = 1, 2, \ldots, m$ and $s = 1, 2, \ldots, n$ such that $|r - s| = 1$. If $r = s$, then

$$P(U = r, V = r) = \frac{2(m-1)(n-1)}{\binom{m+n}{m}} \frac{r-1}{r-1}$$

(1.2)

Step 1. Consider the permutation of IINN, we know that $m = n = 2$ and Table 1 gives the probabilities $P(U = r, V = s)$ for this case.

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Table 1: m=n=2 Probability Function P(U=r,V=s).

<table>
<thead>
<tr>
<th>r/s</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2/6</td>
<td>1/6</td>
</tr>
<tr>
<td>2</td>
<td>1/6</td>
<td>2/6</td>
</tr>
</tbody>
</table>

The probability function \( P(U + V = k) \) of the total number of runs is obtained from the previous distribution (1.1) by summation. The result when \( m=n=2 \) is given in Table 2.

Table 2: m=n=2 Probability Function P(U+V=k) of the Total Number of Runs

<table>
<thead>
<tr>
<th>k</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(U+V=k)</td>
<td>2/6</td>
<td>2/6</td>
<td>2/6</td>
</tr>
</tbody>
</table>

Let \( W \) be the number of equal neighbors in the random permutation. The relation between runs and equal neighbors is \( W = m+n-U-V \). Hence, when \( m=n=2 \), the probability of 2 equal neighbors is equal to the probability 2/6 of two runs, the probability of 1 equal neighbors is 2/6, and so on. Therefore, \( X_1 = 4 - (U + V) \). The probability function of \( X_1 \) required for the solution of the Statistician problem is given in Table 3.

Table 3: Probability Function of \( X_1 \).

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X_1 = i)</td>
<td>2/6</td>
<td>2/6</td>
<td>2/6</td>
</tr>
</tbody>
</table>

Step 2. Insert LLL. Let \( X_2 \) be the number of equal neighbors among the seven letters, IIINNLLL, obtained. Since

\[
P(X_2 = j) = \sum_i P(X_2 = j|X_1 = i)P(X_1 = i).
\]

(1.3)

Consider \( j=0,1,2,3,4 \). Four letters B,T,C, and O can be inserted in the case of “IIINNLLL” to get no equal neighbors. The conditional probabilities required are given in Table 4.

Table 4: Some Conditional Probabilities \( P(X_2 = j|X_1 = i) \)

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>60/210</td>
<td>120/210</td>
<td>30/210</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>36/210</td>
<td>72/210</td>
<td>78/210</td>
<td>24/210</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>18/210</td>
<td>48/210</td>
<td>78/210</td>
<td>48/210</td>
<td>18/210</td>
<td>0</td>
</tr>
</tbody>
</table>

All the probabilities given in Table 4 were calculated. Suppose that \( X_1 = 0 \), for example, ININ, the first L is inserted, the second L, and then the third L. It does not matter to insert the first L at any place, for example, if the first L is inserted to the right end of ININ, i.e., ININL, then !!!IN!!!IL! is obtained (where “!” represent the space to insert the second L). It is convenient to use a tree diagram to do illustration; see Figure 1.

Step 3. Insert B,T,C, and O (The order to insert).

Let \( X_3 \) be the number of equal neighbors among the eleven letters obtained.

\[
P(X_3 = 0) = \sum_{j=0}^{4} P(X_3 = 0|X_2 = j)P(X_2 = j)
\]

(1.4)

When \( X_2 = 0 \), for example, LILINLN, it does not matter where to insert B,T,C,O, there are no equal neighbors. Therefore, \( P(X_3 = 0|X_2 = 0) = 1 \).

When \( X_2 = 1 \), for example, LIIINNLN,

\[
P(X_3 = 0|X_2 = 1) = \frac{1}{8} + \frac{7}{8} \left( \frac{1}{9} + \frac{1}{9} + \frac{8}{10} + \frac{9}{10} \times \frac{1}{11} \right) = \frac{2880}{7920}
\]

(see Figure 2 - 4).

Conclusion

Thus, the solution to the Presidents problem, the probability that a random permutation of the word BILLCLINTON has no equal neighbors, is 39/110.
Figure 1: Tree diagram for calculation of the conditional probability $p(X_2 = i|X_1 = 0)$.

Therefore, $P(X_2 = 2|X_1 = 0) = \frac{2}{6} \times \frac{3}{7} = \frac{6}{42} = \frac{30}{210}$, $P(X_2 = 1|X_1 = 0) = \frac{4}{6} \times \frac{4}{7} + \frac{4}{6} \times \frac{4}{7} = \frac{24}{42} = \frac{120}{210}$, and $P(X_2 = 0|X_1 = 0) = \frac{4}{6} \times \frac{3}{7} = \frac{12}{42} = \frac{60}{210}$.

Suppose that $X_1 = 1$, an example where this occurs is INNI. We obtain $X_2 = 0$ by separating the pair NN with the first L, the second L or the third L inserted, but not both or all three. It is convenient to use a tree diagram; see Figure 2. We obtain

$P(X_2 = 0|X_1 = 1) = \frac{1}{5} \times \frac{4}{6} \times \frac{3}{7} + \frac{4}{5} \times \frac{3}{6} \times \frac{1}{7} + \frac{4}{5} \times \frac{1}{6} \times \frac{3}{7} = \frac{36}{210}$,

$P(X_2 = 1|X_1 = 1) = \frac{1}{5} \times \frac{4}{6} \times \frac{4}{7} \times \frac{1}{5} \times \frac{2}{6} \times \frac{4}{7} \times \frac{2}{5} \times \frac{1}{6} \times \frac{3}{7} \times \frac{4}{5} \times \frac{3}{6} \times \frac{2}{7} \times \frac{4}{5} \times \frac{1}{6} \times \frac{4}{7} = \frac{72}{210}$,

$P(X_2 = 2|X_1 = 1) = \frac{1}{5} \times \frac{2}{6} \times \frac{3}{7} + \frac{4}{5} \times \frac{2}{6} \times \frac{3}{7} + \frac{4}{5} \times \frac{3}{6} \times \frac{4}{7} = \frac{78}{210}$, and

$P(X_2 = 3|X_1 = 1) = \frac{4}{5} \times \frac{2}{6} \times \frac{3}{7} = \frac{24}{210}$. 
Figure 2: Tree diagram for calculation of the conditional probability $P(X_2 = i|X_1 = 1)$.

Suppose that $X_1 = 2$, an example is III NN. Refer to Figure 3.
The diagram illustrates the calculation of the conditional probability $P(X_2 = k | X_1 = 2)$ for different values of $X_2$. Each branch represents a possible outcome with a probability attached to it. For example, when $X_1 = 2$, $X_2 = 4$ occurs with a probability of $3/7$, and $X_2 = 3$ occurs with a probability of $2/7$, and so on. The outcomes are given in a readable format such as 'II NN LLL'.
\[ P(X_2 = 0|X_1 = 2) = \frac{3}{5} \times \frac{2}{6} \times \frac{1}{7} + \frac{2}{5} \times \frac{3}{6} \times \frac{1}{7} + \frac{2}{5} \times \frac{1}{6} \times \frac{3}{7} = \frac{18}{210}, \]
\[ P(X_2 = 1|X_1 = 2) = \frac{3}{5} \times \frac{2}{6} \times \frac{2}{7} + \frac{2}{5} \times \frac{3}{6} \times \frac{2}{7} + \frac{2}{5} \times \frac{1}{6} \times \frac{3}{7} + \frac{2}{5} \times \frac{1}{6} \times \frac{4}{7} = \frac{48}{210}, \]
\[ P(X_2 = 2|X_1 = 2) = \frac{3}{5} \times \left( \frac{2}{6} \times \frac{1}{7} + \frac{2}{6} \times \frac{4}{7} \right) + \frac{2}{5} \times \left( \frac{2}{6} \times \frac{3}{7} + \frac{2}{6} \times \frac{3}{7} \right) = \frac{78}{210}, \]
\[ P(X_2 = 3|X_1 = 2) = \frac{3}{5} \times \left( \frac{2}{6} \times \frac{2}{7} + \frac{2}{6} \times \frac{4}{7} \right) + \frac{2}{5} \times \frac{2}{6} \times \frac{3}{7} = \frac{48}{210}, \text{ and } P(X_2 = 4|X_1 = 2) = \frac{3}{5} \times \frac{2}{6} \times \frac{3}{7} = \frac{18}{210}. \]

Based on (1.3) and combining Table 3 and Table 4, we obtain
\[ P(X_2 = 0) = \sum_{i=0}^{2} P(X_2 = 0|X_1 = i) P(X_1 = i) = \frac{2}{6} \times \left( \frac{60}{210} + \frac{36}{210} + \frac{18}{210} \right) = \frac{228}{1260}. \] Similarly,
\[ P(X_2 = 1) = \frac{2}{6} \times \left( \frac{120 + 72 + 48}{210} \right) = \frac{480}{1260}, \] \[ P(X_2 = 2) = \frac{2}{6} \times \left( \frac{30 + 78 + 78}{210} \right) = \frac{372}{1260}, \]
\[ P(X_2 = 3) = \frac{2}{6} \times \frac{24 + 48}{210} = \frac{144}{1260}, \text{ and } P(X_2 = 4) = \frac{2}{6} \times \frac{18}{210} = \frac{36}{1260}. \]

Table 5: The Probability Function of \( X_2 \)

<table>
<thead>
<tr>
<th>( j )</th>
<th>( P(X_2 = j) )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_2 = 1 )</td>
<td>228/1260</td>
<td>480/1260</td>
<td>372/1260</td>
<td>144/1260</td>
<td>36/1260</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4: Tree diagram for calculation of the conditional probability \( P(X_3 = 0|X_2 = 1) \).

When \( X_2 = 2 \), for example, III LLNL, \( P(X_3 = 0|X_2 = 2) = \frac{2}{8} \times \left( \frac{1}{9} + \frac{8}{9} \times \frac{1}{10} + \frac{8}{9} \times \frac{9}{10} \times \frac{1}{11} \right) \)
\[ + \frac{6}{8} \times \left( \frac{2}{9} \times \frac{1}{10} + \frac{2}{9} \times \frac{9}{10} \times \frac{1}{11} \right) + \frac{6}{8} \times \frac{7}{9} \times \frac{2}{10} \times \frac{1}{11} = \frac{864}{7920} \] (see Figure 5).
Figure 5: Tree diagram for calculation of the conditional probability $P(X_3 = 0|X_2 = 2)$.

When $X_2 = 3$, for example, $NN\ LL\ III$, 

$$P(X_3 = 0|X_2 = 3) = \frac{3}{8} \times \left( \frac{2}{9} \times \frac{1}{10} + \frac{2}{9} \times \frac{9}{10} \times \frac{1}{11} + \frac{7}{9} \times \frac{2}{10} \times \frac{1}{11} \right) + \frac{5}{8} \times \frac{3}{9} \times \frac{2}{10} \times \frac{1}{11} = \frac{192}{7920}$$

(see Figure 6).
When $X_2 = 4$, for example, $\text{NN II LLL}$, $P(X_3 = 0|X_2 = 4) = \frac{4}{8} \times \frac{3}{9} \times \frac{2}{10} \times \frac{1}{11} = \frac{24}{7920}$ (see Figure 7).

This is the answer to the Bill Clinton (President) problem.
References


