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Estimation Using Bivariate Extreme Ranked Set Sampling With Application To The Bivariate Normal Distribution

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In this article, the procedure of bivariate extreme ranked set sampling (BVERSS) is introduced and investigated as a procedure of obtaining more accurate samples for estimating the parameters of bivariate populations. This procedure takes its strength from the advantages of bivariate ranked set sampling (BVRSS) over the usual ranked set sampling in dealing with two characteristics simultaneously, and the advantages of extreme ranked set sampling (ERSS) over usual RSS in reducing the ranking errors and hence in being more applicable. The BVERSS procedure will be applied to the case of the parameters of the bivariate normal distributions. Illustration using real data is also provided.

Key words: Bivariate ranked set sampling; Efficiency; Ranked set sampling; Extreme ranked set sampling; Bivariate extreme ranked set sampling

Introduction

Ranked set sampling (RSS) was first suggested by McIntyre (1952) as a method for estimating pasture yields. The supporting mathematical theory was later provided by Takahasi and Wakimoto (1968). The RSS procedure consists of drawing *m* random samples of size *m* each from the population of interest, and ranking each of them by judgment with respect to (w.r.t.) the characteristic of interest. Then the *ith* smallest observation from the *ith* set is chosen for actual quantification. The RSS consists of these m selected units.

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Although only m units out of m^2 are chosen for quantification, all units contribute information to the m quantified ones. The entire cycle may be repeated, if necessary, *r* times to produce a RSS sample of size $n = rm$. The mean of the RSS sample, as an unbiased estimator of the population mean (μ), is found to have smaller variance than the mean of a simple random sample (SRS) of the same size.

For recent work, consult Patil et al. (1999), Al-Saleh and Al-Kadiri (2000), Al-Saleh and Samawi (2000), Chen (2000), Samawi (2001), Zheng and Al-Saleh (2002) and Al-Saleh and Al-Omari (2002).

The RSS procedure is rarely applicable with large set size m. Ranking a large set of elements is not possible without committing errors of ranking. Ranking errors can destroy the efficiency gain of using RSS instead of SRS. Extreme Ranked Set Sampling (ERSS), as introduced and investigated by Samawi et al. (1996), is a modified procedure of RSS that consists of choosing for quantification the first and the last (Judgment) ordered statistics. In other words, the ERSS procedure consists of drawing *m* random samples of size *m* each from the population.

Then, the smallest observation (identified by judgment) from each of the first $\frac{m}{2}$ sets and the largest observation for each of the last $\frac{m}{2}$ sets are chosen for actual quantification. The ERSS consists of these m selected units, assuming that m is even. It turns out that this procedure, besides being more applicable, can be more efficient than RSS procedure in case of uniform distributions and more efficient than SRS in case of symmetric distributions

A new RSS plan for multiple characteristics was introduced recently by Al-Saleh and Zheng (2002). For simplicity, they introduced the method for two characteristics and refer to it as *Bivariate Ranked Set Sampling* (BVRSS). It is believed that both characteristics will benefit from this scheme of BVRSS. There are situations, when several attributes are to be studied simultaneously using a single combined study rather than separate studies, one for each characteristics. For example, in situations where quantifications entail destruction of units as in uprooting of plants. Also, analytical procedures such as *spectroscopy* can be used to quantify several contaminants at once (Patil *et al.* 1994); also Mode et al. (1999) and Al-Saleh and Zheng (2002) for more applications.

Suppose (X,Y) is a bivariate random vector with pdf $f_{X,Y}(x,y)$. Let θ and μ be the means of X and Y, respectively. To obtain a BVRSS sample follow the five steps described below:

1) For a given set size m, a random sample of size m^4 is identified from the population and randomly allocated into m^2 pools of size m each, where each pool is a square matrix with m rows and m columns.

2) In the first pool, identify the minimum value by judgment w.r.t. the first characteristic, for each of the *m* rows.

3) For the m minima obtained in Step 2, choose the pair that corresponds to the minimum value of the second characteristic, identified by judgment, for actual quantification. This pair, which resembles the label $(1,1)$, is the first element of the BVRSS sample.

4) Repeat Steps 2 and 3 for the second pool, but the pair that corresponds to the first minimum value w.r.t. the first characteristic and the second minimum value w.r.t. the second characteristic is chosen for actual quantification. This pair resembles the label $(1,2)$.

5) The process continues until the label (m,m) is resembled from the m^2th (last) pool. This process produces a BVRSS sample of size m^2. If a sample of higher size is required, then the whole process can be repeated *r* times until the required size $n = 2rm$ is achieved. Note that although m^4 units are identified for the BVRSS sample, only m^2 are chosen for actual quantification. However all m^4 units contribute information to the m^2 quantified units.

In this article, the ERSS is combined with BVRSS to obtain a more applicable procedure namely the *Bivariate Extreme Ranked Set Sampling (BVERSS).* In section 2, the procedure is described and some fundamental properties will be given. Application to bivariate normal distribution is introduced in Section 3. Section 4 provides illustration to the procedure using real data set.

Methodology

Assume that (X,Y) is a bivariate random variable with the joint density function (p.d.f) $f_{X,Y}(x,y)$. To obtain a BVERSS follow the following steps:

1) For a given set size m, $4m$ random samples of size m each are drawn from the population.

2) For each of the first m samples drawn in (1), the minimum with respect to the *X*-characteristic is identified by Judgment. Among the m pairs identified in this step, the pair that corresponds to the minimum with respect to the Y -characteristic is identified. This pair is the first element in the BVERSS. This element is chosen for actual quantification.

3) For each of the second m samples drawn in (1), the minimum with respect to the *X*-characteristic is identified by Judgment. Among the m pairs identified in this step, the pair that corresponds to the maximum with respect to the Y -characteristic is identified. This pair is the second element in the BVERSS. This element is chosen for actual quantification.

4) For each of the third m samples drawn in (1), the maximum with respect to the *X*-characteristic is identified by Judgment. Among the m pairs identified in this step, the pair that corresponds to the minimum with respect to the Y -characteristic is identified. This pair is the third element in the BVERSS. This element is chosen for actual quantification.

5) For each of the fourth m samples drawn in (1), the maximum with respect to the *X*-characteristic is identified by Judgment. Among the m pairs identified in this step, the pair that corresponds to the maximum with respect to the Y -characteristic is identified. This pair is the fourth element in the BVERSS. This element is chosen for actual quantification.

The above 5 steps leads to a BVERSS of size 4. The above steps can be repeated, if necessary, r times to obtain a sample of size n = 4r.

Denote the elements obtained in the second step by $\left(X_{(1)j}, Y_{[1]j}\right)$; where for $j=1,2,....,m$, $X_{(1)j}$ denotes the minimum of the m elements in the *jth* set and $Y_{[1]j}$ is the corresponding Y -value, where the squared brackets is used here to denote the induced rank of Y by the actual rank of the X.

let $Y_{[1](1)} = \min_j (Y_{[1]j})$ and let $X_{(1)[1]}$ be the corresponding *X*-value then $\left(X_{(1)[1]}, Y_{[1](1)}\right)$ denotes the first element in the BVERSS. The other three elements of the first cycle are defined similarly and will be denoted by

$$\left(X_{(1)[m]}, Y_{[1](m)}\right),$$
$$\left(X_{(m)[1]}, Y_{[m](1)}\right),$$
$$\left(X_{(m)[m]}, Y_{[m](m)}\right).$$

Now for the *kth* cycle, let

$$\{\left(X_{(1)[1],k}, Y_{[1](1),k}\right), \left(X_{(1)[m],k}, Y_{[1](m),k}\right),$$
$$\left(X_{(m)[1],k}, Y_{[m](1),k}\right), \left(X_{(m)[m],k}, Y_{[m](m),k}\right)\}$$

be the chosen $BVERSS, k = 1,2,..., r$. $\left(X_{(1)[1],k}, Y_{[1](1),k}\right)$ are independent and identically distributed (iid) with common joint density $f_{X_{(1)[1]}, Y_{[1](1)}}$ given by

$$f_{X_{(1)[1]}, Y_{[1](1)}}(x, y) = f_{Y_{[1](1)}}(y) \frac{f_{X_{(1)}}(x)\, f_{Y|X}(y \mid x)}{f_{Y_{[1]}}(y)} \qquad 2.1$$

where $f_{X_{(1)}}(x)$ is the density of the first order statistics of an iid sample from the marginal density $f_X(x)$, given by

$$f_{X_{(1)}}(x) = m\left(1 - F_X(x)\right)^{m-1} f_X(x);$$

$$f_{Y_{[1]}}(y) = \int_{-\infty}^{\infty} f_{X_{(1)}}(x)\, f_{Y|X}(y \mid x) dx; \quad f_{Y_{[1](1)}}(y)$$

is the density of the first order statistics of an iid sample from $f_{Y_{[1]}}(y)$. Similarly, for the other three quantities the joint densities are respectively given by:

$$f_{X_{(1)[m]}, Y_{[1](m)}}(x, y) = f_{Y_{[1](m)}}(y) \frac{f_{X_{(1)}}(x)\, f_{Y|X}(y \mid x)}{f_{Y_{[1]}}(y)} \qquad 2.2$$

$$f_{X_{(m)[1]}, Y_{[m](1)}}(x, y) = f_{Y_{[m](1)}}(y) \frac{f_{X_{(m)}}(x)\, f_{Y|X}(y \mid x)}{f_{Y_{[m]}}(y)} \qquad 2.3$$

$$f_{X_{(m)[m]}, Y_{[m](m)}}(x, y) = f_{Y_{[m](m)}}(y) \frac{f_{X_{(m)}}(x)\, f_{Y|X}(y \mid x)}{f_{Y_{[m]}}(y)}$$

2.4 (Saleh and Zheng, 2002).

Note that if X and Y are uncorrelated then $X_{(1)[1],k} \underset{=}{d} X_{(1)[m]} \underset{=}{d} X_{(1)}$ and

$X_{(m)[1],k} \underset{=}{d} X_{(m)[m]} \underset{=}{d} X_{(m)}$. Similar statements can be said about the $Y's$. Thus, in this case the *BVERSS* is equivalent to an *ERSS* sample of size $4r$ from the *X*-population, and an *ERSS* sample of size $4r$ from the Y -population. This means that there is no gain of using *BVERSS* instead of using ERSS. However, there are situations when several characteristics are to be investigated simultaneously and using a single combined study rather than a separate study for each attribute. (Al-Saleh and Zheng 2002).

On the other extreme, if X and Y are perfectly correlated then $X_{(1)[1],k} \underset{=}{d} X_{(1)(1)}$; $X_{(1)[m],k} \underset{=}{d} X_{(1)(m)}$, $X_{(m)[1],k} \underset{=}{d} X_{(m)(1)}$; and $X_{(m)[m],k} \underset{=}{d} X_{(m)(m)}$. Thus, in this case for the first variable, the *BVERSS* is equivalent to two ERSS of size $2r$ each one from $f_{X_{(1)}}$ and the other from $f_{X_{(m)}}$. Similar statements can be said about the Y -variable. Therefore, the advantage of *BVERSS* over the (univariate) *ERSS* is obvious.

Let

$$\mu = E(X); \theta = E(Y); \sigma^2 = Var(X); \tau^2 = Var(Y) \text{ and } \rho = Corr(X,Y).$$

Assume that there is a *BVERSS* of size $n = 4r$ given by

$$\{ \left(X_{(1)[1],k}, Y_{[1](1),k}\right), \left(X_{(1)[m],k}, Y_{[1](m),k}\right), \left(X_{(m)[1],k}, Y_{[m](1),k}\right), \left(X_{(m)[m],k}, Y_{[m](m),k}\right) \}$$

Let

$$\overline{X}_k = \frac{X_{(1)[1],k} + X_{(1)[m],k} + X_{(m)[1],k} + X_{(m)[m],k}}{4}$$

then

$$\hat{\mu}_{BVERSS} = \frac{1}{r}\sum_{k=1}^{r} \overline{X}_k \quad 2.5$$

is an estimator of μ based on the *BVERSS*. Similarly $\hat{\theta}_{BVERSS}$ can be defined as an estimator of θ.

Now, let $\mu_{(1)[1]} = E(X_{(1)[1]})$; $\mu_{(1)[m]} = E(X_{(1)[m]})$; $\mu_{(m)[1]} = E(X_{(m)[1]})$; $\mu_{(m)[m]} = E(X_{(m)[m]})$; $\sigma^2_{(1)[1]} = Var(X_{(1)[1]})$; $\sigma^2_{(1)[m]} = Var(X_{(1)[m]})$; $\sigma^2_{(m)[1]} = Var(X_{(m)[1]})$; $\sigma^2_{(m)[m]} = Var(X_{(m)[m]})$.

Then,

$$E(\overline{X}_k) = \frac{\mu_{(1)[1]} + \mu_{(1)[m]} + \mu_{(m)[1]} + \mu_{(m)[m]}}{4}$$

and

$$Var(\overline{X}_k) = \frac{\sigma^2_{(1)[1]} + \sigma^2_{(1)[m]} + \sigma^2_{(m)[1]} + \sigma^2_{(m)[m]}}{16}$$

Hence,

$$E(\hat{\mu}_{BVERSS}) = E(\overline{X}_k) = \frac{\mu_{(1)[1]} + \mu_{(1)[m]} + \mu_{(m)[1]} + \mu_{(m)[m]}}{4} \quad 2.6$$

and

$$Var(\hat{\mu}_{BVERSS}) = \frac{Var(\overline{X}_k)}{r} = \frac{\sigma^2_{(1)[1]} + \sigma^2_{(1)[m]} + \sigma^2_{(m)[1]} + \sigma^2_{(m)[m]}}{16r}. \quad 2.7$$

Similar formulas can be obtained for $\hat{\theta}_{BVERSS}$.

Note that the performances of $\hat{\mu}_{BVERSS}$ and $\hat{\theta}_{BVERSS}$ depend on the properties of the joint distribution of *X* and *Y*. Though not explicitly seen in the above formula, the means and variances of the two estimators depend on the relation between the two variables; Values of $\mu_{(i)[j]}$ and $\sigma^2_{(i)[j]}$ depend on the joint distribution of *X* and *Y*.

Now assume that (X,Y) have the joint density $f_{X,Y}(x,y)$ which is symmetric in both variable around (μ,θ), i.e. $f_{X,Y}(x-\mu, y-\theta) = f_{X,Y}(-x+\mu, -y+\theta)$. Then each of *X* and *Y* has a symmetric marginal distribution. As result of that

$X_{(1)} - \mu \underset{=}{d} - X_{(m)} + \mu$ and $Y_{(1)} - \theta \underset{=}{d} - Y_{(m)} + \theta$. The following lemma summarizes other related results.

Lemma (1): Under the above assumptions exist

i. $X_{[1]} - \mu \underset{=}{d} - X_{[m]} + \mu$ & $Y_{[1]} - \theta \underset{=}{d} - Y_{[m]} + \theta$

ii. $X_{(1)[1]} - \mu \underset{=}{d} - X_{(m)[m]} + \mu$ & $Y_{[1](1)} - \theta \underset{=}{d} - Y_{[m](m)} + \theta$

iii. $X_{(1)[m]} - \mu \underset{=}{d} - X_{(m)(1)} + \mu$ & $Y_{[1](m)} - \theta \underset{=}{d} - Y_{[m](1)} + \theta$.

Proof: (i) Without loss of generality, assume that $\theta = \mu = 0$. Then exist

$$f_{Y_{[1]}}(y) = \int_{-\infty}^{\infty} f_{X_{(1)}}(x)\, f_{Y|X}(y \mid x)dx$$

$$\begin{aligned} f_{Y_{[1]}}(-y) &= \int_{-\infty}^{\infty} f_{X_{(1)}}(x)\, f_{Y|X}(-y \mid x)dx \\ &= \int_{-\infty}^{\infty} f_{X_{(1)}}(-x)\, f_{Y|X}(-y \mid -x)dx \\ &= \int_{-\infty}^{\infty} f_{X_{(m)}}(x)\, f_{Y|X}(y \mid x)dx \\ &= f_{Y_{[m]}}(y) \end{aligned}$$

For the other variable, the proof is similar.

(ii)

$$\begin{aligned} f_{Y_{[1](1)}}(y) &= m\left(1 - F_{Y_{[1]}}(y)\right)^{m-1} f_{Y_{[1]}}(y) \\ f_{Y_{[1](1)}}(-y) &= m\left(1 - F_{Y_{[1]}}(-y)\right)^{m-1} f_{Y_{[1]}}(-y) \\ &= m\left(F_{Y_{[m]}}(y)\right)^{m-1} f_{Y_{[m]}}(y) \end{aligned}$$

hence, $Y_{[1](1)} - \theta \underset{=}{d} - Y_{[m](m)} + \theta$.

From (2.1), $X_{(1)[1]} - \mu \underset{=}{d} - X_{(m)[m]} + \mu$ iff $Y_{[1](1)} - \theta \underset{=}{d} - Y_{[m](m)} + \theta$. (iii) follows similarly.

As a consequence of Lemma (1), the following properties of $\hat{\mu}_{BVERSS}$ and $\hat{\theta}_{BVERSS}$, which can be shown easily.

Lemma (2): Under the above assumptions exist

i. $\hat{\mu}_{BVERSS}$ and $\hat{\theta}_{BVERSS}$ are unbiased estimators of μ and θ; respectively.

ii. $Var(\hat{\mu}_{BVERSS}) = \dfrac{\sigma^2_{(1)[1]} + \sigma^2_{(1)[m]}}{8r}$;

$Var(\hat{\theta}_{BVERSS}) = \dfrac{\tau^2_{[1](1)} + \sigma^2_{[1](m)}}{8r}$.

Examples: (i) Assume that the marginal distribution of X is uniform on the interval $(0, \delta)$ Then it is straight forward to show that

$$\mu_{(1)} = \frac{\delta}{m+1}; \qquad \mu_{(m)} = \frac{m\delta}{m+1};$$

$$\sigma^2(1) = \sigma^2(m) = \frac{m\delta^2}{(m+1)^2(m+2)}.$$

Thus the efficiency of $\hat{\mu}_{ERSS}$ with respect to the mean $\overline{X}$ of a simple random sample of equivalent size is

$$eff\,(\hat{\mu}_{ERSS}; \overline{X}) = \frac{(m+2)(m+1)^2}{12m}.$$

This is the same quantity reported by Samawi et al. (1996). From this formula, $eff\,(\hat{\mu}_{ERSS}; \overline{X})$ is always larger than 1; its value for $m = 2, 4$ *and* 6 are, respectively, 1.50, 3.13 and 5.44. It can be shown that (with $\delta = 1$) that

$$\begin{aligned} f_{X_{(1)(1)}}(x) &= m^2(1-x)^{m-1}(1-(1-x)^m)^{m-1} \\ f_{X_{(1)(m)}}(x) &= m^2(1-x)^{m^2-1} \\ f_{X_{(m)(1)}}(x) &= m^2 x^{m-1}(1-x^m)^{m-1} \\ f_{X_{(m)(m)}}(x) &= m^2 x^{m^2-1} \end{aligned}$$

Thus, for any given m; the mean and variance of each of these can be obtained easily. In the best situation when $\rho = 1$, $eff\,(\hat{\mu}_{MVERSS}; \overline{X})$ can be obtained for any value of m. The values of this efficiency for $m = 2, 4$ *and* 6 are respectively 2.19; 4.07 and 5.95.

(ii) Assume that the marginal distribution of X is exponential with mean μ In this case, as shown numerically by Samawi et al. (1996), $eff\,(\hat{\mu}_{ERSS}; \overline{X})$ is decreasing in m; its values for $m=2,4$ *and* 6 are respectively 1.33; 1.17 *and* 0.75. It can be shown that (with $\delta=1$) that

$$f_{X_{(1)(1)}}(x)=m^2e^{-m^2x}$$

$$f_{X_{(1)(m)}}(x)=m^2e^{-mx}(1-e^{-mx})^{m-1}$$

$$f_{X_{(m)(1)}}(x)=m^2e^{-x}(1-e^{-x})^{m-1}(1-(1-e^{-x})^m)^{m-1}$$

$$f_{X_{(m)(m)}}(x)=m^2e^{-x}(1-e^{-x})^{m^2-1}$$

Thus, for any given m*;* the mean and variance of each of these can be obtained easily. In the best situation when $\rho=1$, $eff\,(\hat{\mu}_{BVERSS};\overline{X})$ can be obtained for any value of m. Using Scientific work place, $eff\,(\hat{\mu}_{BVERSS};\overline{X})$ was evaluated for some values of m. For $m=2,4$ *and* 6, the efficiency found to be respectively, 1.82*;* 1.36 *and* 0.78*:* Thus the estimator here doesn't perform well. Note that the distribution in this case is not symmetric and the estimator is biased. Next presented is a case of bivariate normal distribution.

Assume next that (X,Y) has the bivariate normal density given by

$$f_{X,Y}(x,y)=\frac{1}{2\pi\sigma\,\tau\sqrt{1-\rho^2}}\,e^{-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu}{\sigma}\right)^2+\left(\frac{y-\theta}{\tau}\right)^2+\left(\frac{x-\mu}{\sigma}\right)\left(\frac{y-\theta}{\tau}\right)\right)}$$

where, $\mu,\theta,\sigma^2,\tau^2$ *, and* ρ are, respectively, the mean of X, the mean of Y, the variance of X, the variance of Y, and the correlation between X and Y. Denote this bivariate normal density by $N_2(\mu,\theta,\sigma^2,\tau^2,\rho)$. Using (2.1) above, the joint density of $X_{(1)[1]},Y_{[1](1)}$ can be written as

$$f_{X_{(1)[m]},Y_{[1](m)}}(x,y)=f_{Y_{[1](m)}}(y)\frac{f_{X_{(1)}}(x)\,f_{Y|X}(y\,|\,x)}{f_{Y_{[1]}}(y)} \quad 3.1$$

$$=m^2(1-F_{Y_{[1]}}(y))^{m-1}(1-F_X(x))^{m-1}f_X(x)f_{Y|X}(y\,|\,x), \quad 3.2$$

and the joint density of $X_{(1)[m]},Y_{[1](m)}$ can be written as

$$f_{X_{(m)[1]},Y_{[m](1)}}(x,y)=f_{Y_{[m](1)}}(y)\frac{f_{X_{(m)}}(x)\,f_{Y|X}(y\,|\,x)}{f_{Y_{[m]}}(y)} \quad 3.3$$

$$=m^2(F_{Y_{[1]}}(y))^{m-1}(1-F_X(x))^{m-1}f_X(x)f_{Y|X}(y\,|\,x), \quad 3.4$$

For simplicity assume $\mu=\theta=0$ and $\sigma^2=\tau^2=1$, (easily one can go back to the general case), then it can be shown that $f_{Y_{[1]}}(y)=2\phi(x)\Phi(\frac{-\rho x}{\sqrt{2-\rho^2}})=f_{Y_{m1]}}(-y)$ where ϕ & Φ are, respectively, the density and the cumulative distribution function of the standard normal distribution. Hence,

$$f_{X_{(1)[1]},Y_{[1](1)}}(x,y)=$$
$$m^2(\Phi(-x))^{m-1}\phi_{X,Y}(x,y)$$
$$\left[\int_{-\infty}^{y}2\phi(z)\Phi\left(\frac{-\rho z}{\sqrt{2-\rho^2}}\right)dz\right]^{m-1}$$

$$f_{X_{(1)[m]},Y_{[1](m)}}(x,y)=$$
$$m^2(\Phi(-x))^{m-1}\phi_{X,Y}(x,y)$$
$$\left[1-\int_{-\infty}^{y}2\phi(z)\Phi\left(\frac{-\rho z}{\sqrt{2-\rho^2}}\right)dz\right]^{m-1}$$

and,

$$f_{X_{(1)[1]}}(x)=$$
$$m^2(\Phi(-x))^{m-1}\int_{-\infty}^{\infty}\left\{\phi_{X,Y}(x,y)\left[\int_{-\infty}^{y}2\phi(z)\Phi\left(\frac{-\rho z}{\sqrt{2-\rho^2}}\right)dz\right]^{m-1}\right\}dy$$

$$f_{X_{(1)[m]}}(x) = m^2 (\Phi(-x))^{m-1}$$

$$\int_{-\infty}^{\infty} \{ \phi_{X,Y}(x,y)$$

$$[\ 1-\int_{-\infty}^{y} 2\phi(z)\Phi(\frac{-\rho z}{\sqrt{2-\rho^2}})\ dz\ ]^{m-1} \}dy$$

The mean and variance of $X_{(1)[1]}$ *and* $X_{(1)[m]}$ can be evaluated numerically and hence the variance of the unbiased estimator $\hat{\mu}_{BVERSS}$ can be obtained. Its efficiency with respect to the sample mean of a simple random sample of equivalent sample size can be obtained. Since the efficiency depends on ρ, it will be denoted by $eff\,(\hat{\mu}_{BVERSS};\overline{X})$.

If $\rho = 0$, then $eff_0\,(\hat{\mu}_{BVERSS};\overline{X}) = eff\,(\hat{\mu}_{ERSS};\overline{X})$. In this case the efficiency was reported by Samawi et al. (1996). For $m = 2, 4$ *and* 6, it is, respectively, 1.47; 2.03; 2.39.

If $\rho = 1$, it can be shown that

$$f_{X_{(1)(1)}}(x) = m^2 \Phi^{m^2-1}(-x)\phi(x) \tag{3.9}$$

$$f_{X_{(1)(m)}}(x) = m^2 [\ 1-\Phi^m(-x)\ ]^{m-1} \Phi^{m-1}(-x)\phi(x). \tag{3.10}$$

The variance of $X_{(1)(1)}$ *and* $X_{(1)(m)}$ were obtained using *Scientific Work Place*. Based on these values $eff\,(\hat{\mu}_{BVERSS};\overline{X})$ is calculated for some values of m. The results are given in the following table.

Table 1. Efficiency of $eff_1\,(\hat{\mu}_{BVERSS};\overline{X})$ with respect to $\overline{X}$ for $\rho = 0,1$.

m	$\sigma^2_{(1)(1)}$	$\sigma^2_{(1)(m)}$	$eff_1\,(\hat{\mu}_{BVERSS};\overline{X})$	$eff_0\,(\hat{\mu}_{BVERSS};\overline{X}) = eff\,(\hat{\mu}_{ERSS};\overline{X})$
2	0.4989	0.4389	2.15	1.47
4	0.2949	0.1996	4.04	2.03
6	0.2344	0.1295	5.05	2.39

Table 2. Efficiency of $\hat{\mu}_{BVERSS}$ with respect to $\overline{X}$ based on 5000 simulation.

m	$\rho = 0.10$	$\rho = 0.30$	$\rho = 0.50$	$\rho = 0.70$	$\rho = 0.90$
2	1.49	1.51	1.59	1.74	1.98
4	2.10	2.12	2.33	2.62	3.34
6	2.34	2.54	2.82	3.45	4.42

Table 3. Efficiency of $\hat{\mu}_{BVRSS}$ with respect to $\overline{X}$ based on 5000 simulation.

m	$\rho = 0.00$	$\rho = 0.10$	$\rho = 0.30$	$\rho = 0.50$	$\rho = 0.70$	$\rho = 0.90$
2	1.47	1.49	1.51	1.59	1.74	1.98
4	2.30	2.54	2.60	2.74	3.13	4.17
6	3.05	3.16	3.43	3.67	4.84	6.92

For other values of ρ, the marginal densities of $X_{(1)[1]}$ *and* $X_{(1)[m]}$ given by (3.7 and 3.8) can't be simplified further. Based on simulation with 5000 replications, the efficiency for some values of ρ is given in Table 2.

Next, to compare *BVERSS* with the usual *BVRSS*, i.e. to find the efficiency of *BVRSS* with respect to SRS for estimating μ. This efficiency was calculated for $m = 2$ *and* 3 by Al-Saleh and Zheng (2002). Table 3 contains this efficiency for $m = 2,4$ *and* 6.

It is clear that when $m = 2$ then *BVERSS* is the same as *BVRSS*. Table 1 and Table 2 show that the *BVERSS* is substantial more efficient than *SRS* and comparing with Samawi et al. (1996), it is more efficient than ERSS and *RSS* in case of bivariate normal distribution. Also, the efficiency of *BVRSS* w.r.t. *SRS* is increasing with increasing the set size m and the correlation coefficient ρ. Although Table 3 shows that *BVRSS* is more efficient than our proposed *BVERSS* assuming no error in ranking, *BVERSS* still more practical than *BVRSS* and less prone to ranking error.

Results

The BVERSS estimation procedure is illustrated using a real data set which consists of the height (*Y)* and the diameter (*X*) at breast height of 399 trees. See Platt et al. (1988) for a detailed description of the data set. The summary statistics of the original data are reported in Table 4. Note that the correlation coefficient $\rho = 0.908$.

In this article, ranking is performed on the both variables exactly measured. However, in practice ranking is done before any actual quantification. Using a set size $m = 4$ and cycle size $r = 4$, bivariate *SRS*, *BVRSS* and *BVERSS* of size 16 are drawn. The analysis to the tree data showed that the distributions of *X* and *Y* have skewed to the right shape. So to compare between *BVERSS* and *BVRSS* the means for the transformed data by using the natural logarithm were estimated. Table 5 contains all the above proposed estimators using the drown samples. Also, provided are estimates for the efficiency based on 1000 repeated sampling.

Table 4. Summary statistics of trees data.

Variable	Mean	Variance
Height (*Y*) in feet	52.36	325.14
Diameter (*X*) in cm	20.84	310.11

Table 5. Results of the selected samples of transformed trees data.

Variable	Mean	*eff* $(BVRSS; SRS)$	*eff* $(BVERSS; SRS)$
Ln (Height (*Y*))	3.39	5.02	4.97
Ln (Diameter (*X*))	2.61	4.82	4.88

Conclusion

From the above results, support exists that *BVERSS* procedure can be, in some situations, much better than the bivariate *SRS*, *ERSS* and *RSS* (using concomitant variable) sampling methods for estimating the distribution means of multiple characteristics. Also, *BVERSS* provides unbiased estimators for distribution means in case of symmetric marginal distributions. Finally, *BVERSS* is more practical than *BVRSS* and less prone to ranking error.

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