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Approximate Bayesian Confidence Intervals for The Mean of a Gaussian Distribution Versus Bayesian Models

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This study obtained and compared confidence intervals for the mean of a Gaussian distribution. Considering the square error and the Higgins-Tsokos loss functions, approximate Bayesian confidence intervals for the mean of a normal population are derived. Using normal data and SAS software, the obtained approximate Bayesian confidence intervals were compared to a published Bayesian model. Whereas the published Bayesian method is sensitive to the choice of the hyper-parameters and does not always yield the best confidence intervals, it is shown that the proposed approximate Bayesian approach relies only on the observations and often performs better.

Key words: Estimation; loss functions; confidence intervals, statistical analysis.

Introduction

A significant amount of research in Bayesian analysis and modeling has been published during the last twenty-five years. Bayesian analysis implies the exploitation of suitable prior information and the choice of a loss function in association with Bayes' Theorem. It rests on the notion that a parameter within a model is not merely an unknown quantity, but behaves as a random variable that follows some distribution. In the area of life testing, it is realistic to assume that a life parameter is stochastically dynamic. This assertion is supported by the fact that the complexity of electronic and structural systems is likely to cause undetected component interactions resulting in an unpredictable fluctuation of the life parameter.

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Although no specific analytical procedure exists which identifies the appropriate loss function to be used, the most commonly used is the square error loss function. One reason for selecting this loss function is due to its analytical tractability in Bayesian modeling and analysis.

The square error loss function places a small weight on estimates near the parameter's true value and proportionately more weight on extreme deviations from the true value. The square error loss is defined as follows:

$$L_{SE}(\hat{\theta},\theta)=\left(\hat{\theta}-\theta\right)^2.$$

This study considers a widely used and useful underlying model, the normal underlying model, which is characterized by

$$f\ (x)=\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2};$$

$$-\infty \prec x \prec \infty, -\infty \prec \mu \prec \infty, \sigma \succ 0. \quad (1)$$

Employing the square error loss function along with a normal prior, Fogel (1991) obtained the following Bayesian confidence interval for the mean of the normal probability density function:

$$L_B = \frac{\mu_1 \sigma^2 / n + + \bar{x}\tau^2}{\tau^2 + \sigma^2 / n} - Z_{\alpha/2} \frac{\tau\sigma / \sqrt{n}}{\sqrt{\tau^2 + \sigma^2 / n}} \tag{2}$$

$$U_B = \frac{\mu_1 \sigma^2 / n + + \bar{x}\tau^2}{\tau^2 + \sigma^2 / n} + Z_{\alpha/2} \frac{\tau\sigma / \sqrt{n}}{\sqrt{\tau^2 + \sigma^2 / n}} \tag{3}$$

where the mean and variance of the selected normal prior are respectively denoted by μ_1 and τ^2.

This study employs the square error and the Higgins-Tsokos loss functions to derive approximate Bayesian confidence intervals for the normal population mean. Obtained confidence bounds are then compared with their Bayesian counterparts corresponding to (3).

Methodology

Considering the normal density function (2), to derive approximate Bayesian confidence intervals for the mean of a normal distribution, results obtained on approximate Bayesian confidence intervals for the variance of a Gaussian distribution are used (Camara, 2003). The loss functions used are the square error loss function (1), and the Higgins-Tsokos loss function.

The Higgins-Tsokos loss function places a heavy penalty on extreme over- or under-estimation. That is, it places an exponential weight on extreme errors. The Higgins-Tsokos loss function is defined as follows:

$$L_{HT}(\hat{\theta}, \theta) = \frac{f_1 e^{f_2(\hat{\theta} - \theta)} + f_2 e^{-f_1(\hat{\theta} - \theta)}}{f_1 + f_2} - 1,$$

$$f_1, f_2 \succ 0. \tag{4}$$

The use of these loss functions (1) and (4), along with suitable approximations of the Pareto prior, led to the following approximate Bayesian confidence bounds for the variance of a normal population (Camara, 2003). For the square error loss function:

$$L_{o^2(SE)} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 2 - 2\ln(\alpha / 2)}$$

$$U_{\sigma^2(SE)} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 2 - 2\ln(1 - \alpha / 2)} \tag{5}$$

For the Higgins-Tsokos loss function:

$$U_{o^2(HT)} = \frac{1}{\frac{n - 1 - 2Ln(1 - \alpha / 2)}{\sum_{i=1}^{n} (x_i - \bar{x})^2} - G(x)} \tag{6}$$

$$f_1 \prec \frac{(x - \mu)^2}{2},$$

where

$$G(x) = \frac{1}{f_1 + f_2} Ln\left(\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2 + f_2}{\sum_{i=1}^{n} (x_i - \bar{x})^2 - f_1} \right). \tag{7}$$

Using the above approximate Bayesian confidence intervals for a normal population variance (5) (6) along with

$$\sigma^2 = E(X^2) - \mu^2, \tag{8}$$

the following approximate Bayesian confidence intervals for the mean of a normal population can easily be derived for a strictly positive mean.

The approximate Bayesian confidence interval for the normal population mean corresponding to the square error loss is:

$$L_{\mu(SE)} = \left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1} + \bar{x}^2 - \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-2-2\ln(1-\alpha/2)} \right)^{0.5}$$

$$U_{\mu(SE)} = \left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1} + \bar{x}^2 - \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-2-2\ln(\alpha/2)} \right)^{0.5} \qquad (9)$$

The approximate Bayesian confidence interval for the normal population mean corresponding to the Higgins-Tsokos loss function is:

$$U_{\mu(HT)} = \left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1} + \bar{x}^2 - \frac{1}{H_2(x)} \right)^{0.5} \qquad (10)$$

where

$$H_1(x) = \frac{n-1-2Ln(1-\alpha/2)}{\sum_{i=1}^{n}(x_i-\bar{x})^2} - G(x) \qquad (11)$$

$$H_2(x) = \frac{n-1-2Ln(\alpha/2)}{\sum_{i=1}^{n}(x_i-\bar{x})^2} - G(x) \qquad (12)$$

$$L_{\mu(HT)} = \left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1} + \bar{x}^2 - \frac{1}{H_1(x)} \right)^{0.5}$$

With (9),(10), (11), (12) and a change of variable, approximate Bayesian Confidence intervals are easily obtained when $\mu \le 0$.

Results

To compare the Bayesian model (3) with the approximate Bayesian models (9 & 10), samples obtained from normally distributed populations (Examples 1, 2, 3, 4, 7) as well as approximately normal populations (Examples 5, 6) were considered. SAS software was employed to obtain the normal population parameters corresponding to each sample data set. For the Higgins-Tsokos loss function, f1 = 1 and f2 = 1 were considered.

Example 1

Data Set: 24, 28, 22, 25, 24, 22, 29, 26, 25, 28, 19, 29 (Mann, 1998, p. 504).

Normal population distribution obtained with SAS:

$$N(\mu = 25.083, \sigma = 3.1176),$$
$$\bar{x} = 25.08333,\ s^2 = 9.719696.$$

Table 1a: Approximate Bayesian Confidence Intervals for the Population Mean Corresponding Data Set 1

C.L. %	Approximate Bayesian Bounds (SE)	Approximate Bayesian Bounds (HT)
80	25.0683-25.1311	25.0730-25.1158
90	25.0661-25.1437	25.0683-25.1311
95	25.0650-25.1543	25.0661-25.1437
99	25.0641-25.1734	25.0643-25.1660

Table 1b: Bayesian Confidence Intervals for the Population Mean Corresponding Data Set 1

	Bayesian C. I. I	Bayesian C. I. II
C.L. %	Bayesian Bounds $\mu_1 = 2, \tau = 1$	Bayesian Bounds $\mu_1 = 25, \tau = 10$
80	13.8971-15.6097	23.9353-26.2300
90	13.6496-15.8572	23.6037-26.5617
95	13.4422-16.0646	23.3258-26.8395
99	13.0275-16.4793	22.7701-27.3953

Example 2

Data Set: 13, 11, 9, 12, 8, 10, 5, 10, 9, 12, 13 (Mann, 1998, p. 504).

Normal population distribution obtained with SAS:

$$N(\boldsymbol{\mu} = 10.182, \boldsymbol{\sigma} = 2.4008),$$
$$\bar{x} = 10.181812,\ s^2 = 5.763636.$$

Table 2a: Approximate Bayesian Confidence Intervals for the Population Mean Corresponding to Data Set 2

C.L. %	Approximate Bayesian Bounds (SE)	Approximate Bayesian Bounds (HT)
80	10.1575-10.2565	10.1652-10.2330
90	10.1538-10.2756	10.1575-10.2565
95	10.1520-10.2914	10.1538-10.2756
99	10.1506-10.3194	10.1506-10.3194

Table 2b: Bayesian Confidence Intervals for the Population Mean Corresponding to Data Set 2

C.L. %	Bayesian C. I. I	Bayesian C. I. II
	Bayesian Bounds $\mu_1 = 2, \tau = 1$	Bayesian Bounds $\mu_1 = 25, \tau = 10$
80	6.6182-8.1193	9.3349-11.1832
90	6.4013-8.3363	9.0678-11.4503
95	6.2195-8.5180	8.8440-11.6741
99	5.8560-8.8816	8.3964-12.1217

Example 3

Data Set: 16, 14, 11, 19, 14, 17, 13, 16, 17, 18, 19, 12 (Mann, 1998, p. 504).

Normal population distribution obtained with SAS:

$$N(\boldsymbol{\mu} = 15.5, \boldsymbol{\sigma} = 2.6799),$$
$$\bar{x} = 15.5,\ s^2 = 7.181818.$$

Table 3a: Approximate Bayesian Confidence Intervals for the Population Mean Corresponding to Data Set 3

C.L. %	Approximate Bayesian Bounds (SE)	Approximate Bayesian Bounds (HT)
80	15.4820-15.5570	15.4877-15.5388
90	15.4794-15.5721	15.4820-15.5570
95	15.4781-15.5847	15.4794-15.5721
99	15.4770-15.6075	15.4773-15.5986

Table 3b: Bayesian Confidence Intervals for the Population Mean Corresponding to Data Set 3

C.L. %	Bayesian C. I. I	Bayesian C. I. II
	Bayesian Bounds $\mu_1 = 2, \tau = 1$	Bayesian Bounds $\mu_1 = 25, \tau = 10$
80	9.6623-11.2287	14.5692-16.5438
90	9.4359-11.4551	14.2839-16.8292
95	9.2462-11.6448	14.0447-17.0683
99	8,8668-12.0242	13.5665-17.5465

Example 4

Data Set: 27, 31, 25, 33, 21, 35, 30, 26, 25, 31, 33, 30, 28 (Mann, 1998, p. 504).

Normal population distribution obtained with SAS:

$$N(\boldsymbol{\mu} = 28.846, \boldsymbol{\sigma} = 3.9549),$$
$$\bar{x} = 28.846153,\ s^2 = 15.641025.$$

Table 4a: Approximate Bayesian Confidence Intervals for the Population Mean Corresponding to Data Set 4

C.L. %	Approximate Bayesian Bounds (SE)	Approximate Bayesian Bounds (HT)
80	28.8270-28.9087	28.8330-28.8884
90	28.8242-28.9256	28.8270-28.9087
95	28.8228-28.9400	28.8242-28.9256
99	28.8217-28.9663	28.8220-28.9560

Table 5a: Approximate Bayesian Confidence Intervals for the Population Mean Corresponding to Data Set 5

C.L. %	Approximate Bayesian Bounds (SE)	Approximate Bayesian Bounds (HT)
80	43.5794-43.6703	43.5858-43.6169
90	43.5764-43.6902	43.5794-43.6703
95	43.5749-43.7074	43.5764-43.6902
99	43.5738-43.7395	43.5741-43.7268

Table 4b: Bayesian Confidence Intervals for the Population Mean Corresponding to Data Set 4

C.L. %	Bayesian C. I. I	Bayesian C. I. II
	Bayesian Bounds $\mu_1 = 2, \tau = 1$	Bayesian Bounds $\mu_1 = 25, \tau = 10$
80	13.2394-15.1312	27.4048-30.1961
90	12.9659-15.4047	27.0014-30.5995
95	12.7369-15.6337	26.6634-30.9375
99	12.2787-16.0919	25.9873-31.6135

Table 5b: Bayesian Confidence Intervals for the Population Mean Corresponding to Data Set 5

C.L. %	Bayesian C. I. I	Bayesian C. I. II
	Bayesian Bounds $\mu_1 = 2, \tau = 1$	Bayesian Bounds $\mu_1 = 25, \tau = 10$
80	14.8305-16.9204	41.4441-45.0272
90	14.5285-17.2225	40.9263-45.5450
95	14.2754-17.4756	40.4924-45.9789
99	13.7692-17.9817	39.6246-46.8467

Example 5

Data Set: 52, 33, 42, 44, 41, 50, 44, 51, 45, 38,37,40,44, 50, 43 (McClave & Sincich, p. 301).

Normal population distribution obtained with SAS:

$$N(\mu = 43.6, \sigma = 5.4746),$$
$$\bar{x} = 43.6,\ s^2 = 29.971428.$$

Example 6

Data Set: 52, 43, 47, 56, 62, 53, 61, 50, 56, 52, 53, 60, 50, 48, 60, 5543 (McClave & Sincich, p. 301).

Normal population distribution obtained with SAS:

$$N(\mu = 53.625, \sigma = 5.4145)$$
$$\bar{x} = 53.625,\ s^2 = 29.316666.$$

Table 6a: Approximate Bayesian Confidence Intervals for the Population Mean Corresponding to Data Set 6

C.L. %	Approximate Bayesian Bounds (SE)	Approximate Bayesian Bounds (HT)
80	53.6098-53.6779	53.6145-53.6602
90	53.6076-53.6932	53.6098-53.6779
95	53.6065-53.7064	53.6076-53.6932
99	53.6056-53.7315	53.6058-53.7216

Table 7a: Approximate Bayesian Confidence Intervals for the Population Mean Corresponding to Data Set 7

C.L. %	Approximate Bayesian Bounds (SE)	Approximate Bayesian Bounds (HT)
80	82.7072-83.4808	82.7539-83.2572
90	82.6856-83.6884	82.7072-83.4808
95	82.6751-83.8815	82.6856-83.6884
99	82.6669-84.2823	82.6690-83.7173

Table 6b: Bayesian Confidence Intervals for the Population Mean Corresponding to Data Set 6

C.L. %	Bayesian C. I. I	Bayesian C. I. II
	Bayesian Bounds $\mu_1 = 2, \tau = 1$	Bayesian Bounds $\mu_1 = 25, \tau = 10$
80	19.1978-21.2568	51.3930-54.8269
90	18.9002-21.5544	50.8967-55.3232
95	18.6508-21.8038	50.4808-55.7391
99	18.1521-22.3024	49.6492-56.5707

Table 7b: Bayesian Confidence Intervals for the Population Mean Corresponding to Data Set 7

C.L. %	Bayesian C. I. I	Bayesian C. I. II
	Bayesian Bounds $\mu_1 = 2, \tau = 1$	Bayesian Bounds $\mu_1 = 25, \tau = 10$
80	3.2940-5.8132	63.0810-75.4828
90	2.9299-6.17740	61.2886-77.2752
95	2.6248-6.4824	59.7868-78.7770
99	2.0147-7.0926	56.7833-81.7806

Example 7

Data Set: 50, 65, 100, 45, 111, 32, 45, 28, 60, 66, 114, 134, 150, 120, 77, 108, 112, 113, 80, 77, 69, 91, 116, 122, 37, 51, 53, 131, 49, 69, 66, 46, 131, 103, 84, 78 (SAS Data).

Normal population distribution obtained with SAS:

$$N(\boldsymbol{\mu} = 82.861, \boldsymbol{\sigma} = 33.226)$$

$$\bar{x} = 82.8611,\ s^2 = 1103.951587$$

All seven Examples show that the proposed approximate Bayesian confidence intervals contain the population mean. The Bayesian model, however, does not always contain the population mean.

Conclusion

In this study, approximate Bayesian confidence intervals for the mean of a normal population under two different loss functions were derived and compared with a published Bayesian model (Fogel, 1991). The loss functions employed were the square error and the Higgins-Tsokos loss functions. The following conclusions are based on results obtained:

1. The Bayesian model (3) used to construct confidence intervals for the mean of a normal population does not always yield the best coverage accuracy. Each of the obtained approximate Bayesian confidence intervals contains the population mean and performs better than its Bayesian counterparts.
2. Bayesian models are generally sensitive to the choice of hyper-parameters. Some values arbitrarily assigned to the hyper-parameters may lead to a very poor estimation of the parameter(s) under study. In this study some values assigned to the hyper-parameters led to confidence intervals that do not contain the normal population mean.
3. Contrary to the Bayesian model (3), which uses the Z-table, both the approach employed in this study and our approximate Bayesian models rely only on observations.
4. With the proposed approach, approximate Bayesian confidence intervals for a normal population mean are easily obtained for any level of significance..
5. The approximate Bayesian approach under the popular square error loss function does not always yield the best approximate Bayesian results: The Higgins-Tsokos loss function performs better in the examples presented.

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