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Processing random signals in neuroscience, electrical engineering and operations research

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DEDICATION

This dissertation is dedicated to my heroes and loved ones—my late father Subramanian whose passion for science and love of knowledge were limitless, my mother Meenakshi who remains a lifelong learner and was my first teacher, my wife Lisa Leanne whose ongoing intellectual accomplishments outstrip her ability to keep up with herself, who is the very embodiment of the scientific humanist, whose graciousness and generosity of spirit are unbounded, my brother Sankar who is my role model, the Renaisscance man *par excellence*, equally fluent in Bach, the Grateful Dead, Camus, and complex epileptic conditions, equally comfortable embracing the absurd and celebrating the sublime, my brother Shiv who is a remarkable combination of brilliance, compassion, and sensitivity, my children Erin, Andrea, Xander and Jarrod who keep my feet anchored firmly to the ground and are a daily reminder that it would be a lonely and less complete existence without them, my late father-in-law James Conrad Howe, whose dignified life reflected the rich tapestry of American society and its possibilities, whose kindness and gentleness I fondly remember, and my mother-in-law Suzanne Howe whose tenacity and courage are admirable. Thank you all for bringing so much joy into my life. And Jim and Sue—Thank you for Lisa.

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I am grateful to Professor Marek Czosnyka, *University of Cambridge*, Neuroscience Group, UK, for introducing me to the key issues in cerebrospinal fluid (CSF) dynamics and hydrocephalus, for making data available from infusion studies conducted at Addenbrooke's Hospital, UK, to members of the Cambridge Neuroscience Group for their input on a presentation made to them in May 2008, to participants of the 53rd Society for Research in Hydrocephalus and Spina Bifida (SRHSB) Belfast conference in northern Ireland in July 2009 for their input on my presentation, to participants of the International Society for Hydrocephalus and Cerebrospinal Fluid (ISHCSF)-sponsored Hydrocephalus 2009 Conference in Baltimore, MD, for their comments on my presentation, to the late Professor Anthony Marmarou for his insightful suggestions on the model of CSF dynamics that extends his own fundamental model, developed in Chapters 2 and 3 of this dissertation, to Professor Roger Bayston, *University of Nottingham*, UK, for pointing out the implications of this extended model for shunts, and to Professor James P. McAllister II, Department of Neurosurgery, *University of Utah* and *Wayne State University*, for actively encouraging this research. A very special thanks to Mrs. Rosemary Batchelor, Hon Secretary, SRHSB, for her tireless efforts at connecting me with the most relevant people working on CSF dynamics, hydrocephalus and shunts throughout the world.

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If I be permitted a concluding philosophical muse: as in life, so in research—both are shaped by the intangible, and Time, that ultimate intangible,

has indelibly stamped this research. My first doctorate was considered innovative research, but, fueled by need and necessity, partly reflected the haste of a young man eager to get on with his career—a personal assessment of my own work, driven by the heightened objectivity of self-appraisal lent by Time. My second doctorate, driven by want rather than need, and scientific curiosity rather than necessity, is a reflection of my deepest intellectual passions—science, mathematics and technology, passions that I have lovingly nurtured throughout the years separating the two Ph.Ds. I could ask nothing more of Father Time.

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CHAPTER 1

INTRODUCTION TO NOISE

This dissertation studies noise in electrical engineering, biomedical engineering, and operations research through mathematical models that describe, explain, predict and control dynamic phenomena. Noise is modeled through Brownian Motion and the research problems are mathematically addressed by different versions of a generalized Langevin equation.

The lead article in the November 2005 issue of *IEEE Signal Processing Magazine* [1] was devoted to the history of noise, for, as noted by the author: "“Noise,” as an idea, a subject, a field, an instrument, came upon the scene with a power and swiftness that transformed all of science and our views of the nature of matter. At birth, it solved the major issue of its time, perhaps, the greatest idea of all time—the existence of atoms.”

Noise is pervasive across disciplines and its sources are diverse. A partial and far from exhaustive list of sources includes device noise, random environmental variation, noise arising from unknown, unmeasurable or unobserved variables and noise due to variables that are intentionally omitted from the mathematical model to promote analytical tractability. Concrete examples of noise are abundant: fluctuations generated by thermal, Johnson and shot noise in electrical and electronic devices [2, 3, 4], random environmental variation in population biology [5] and noise that is instrumental in the firing of neurons in the brain [6]. Noise due to unknown, unobservable or omitted variables is ubiquitous in operations research models of marketing and finance phenomena [7, 8]. A very important type of noise called Brownian Motion is vital to the functioning of protein machines called molecular motors that enable mobility in living organisms [9]. Brownian Motion regarded as a stochastic process is the prototypical continuous-time probabilistic model for describing noise [10], and provides the foundations for a calculus to analyze dynamic systems driven by noise.

Noise merits significant academic attention and scientific interest because it spans multiple disciplines in fundamental ways, provides a common language for scholarly discourse across disparate fields, and has spawned a sophisticated mathematical framework for the analysis of challenging problems across these disciplines. In particular, the solutions of these problems have significant implications for the improvement of processes, systems and devices in electrical engineering. Therefore, contributions to noise are expected to advance theory and practice in electrical engineering as well as related academic disciplines that are influenced by developments in electrical engineering.

1.1 History of noise

As a branch of scholarly inquiry, noise has great intellectual appeal because it raises unsolved problems, whose solutions demand blending ideas from the engineering, physical, mathematical, biological and statistical disciplines. Indeed, the history of research on noise reveals many fruitful exchanges of ideas and mutually reinforcing discoveries among these disciplines. Robert Brown's [11] biological observations of pollen particles suspended in water stimulated the development of a mathematical model of noise called Brownian Motion [12, 13], and Brownian Motion provides an accurate probabilistic model for the movements of gas molecules [14], cyclotron motion [14], the motion of self-propelled flagellated bacteria [15], the action of molecular motors [9] that facilitate directed movement in living organisms, and fluctuations of stock prices [8].

Einstein realized that Brownian Motion could settle one of the great controversies of his time—the existence of atoms [1]—and was the first to study the connection between atomic fluctuations and Brownian Motion. Five years before Einstein published his paper, Bachelier [12] had already anticipated many of Einstein's [13] results in the completely different context of fluctuations of stock prices in financial markets. Thus, from its very inception, Brownian Motion has been intimately intertwined with *both* basic and applied science. Haw [16] identifies four stages in the historical development of "Brownian Motion science," characterized by discovery, observation, theoretical prediction and quantitative confirmation and, according to him, we are currently living in the fourth stage, that of application.

A history of noise would be incomplete without a discussion of fuzzy logic, a field born in electrical engineering and related to yet distinct from classical probability theory. Fuzzy concepts arise from a conceptually different source of noise—that arising from human interactions with systems. For example, efficient, accurate and relevant information retrieval is a topic of central importance to researchers in all disciplines. But both the terms "information" and "access" are *ambiguous* and unstable theoretical objects [17]. Ying [18] has rigorously established the mathematical conditions under which fuzzy methodologies (such as fuzzy control) are expected to outperform or work no better than classical control. We regard classical and fuzzy methods as *complementary rather than mutually exclusive* or intellectually dissonant approaches. Particularly in systems in which the human element plays a role, fuzzy control appears capable of capturing the vague, linguistically imprecise and often ambiguous knowledge that nevertheless enables humans to perform tasks that pose formidable challenges to machines based on the principles of classical control [19]. Thus, the practical merits of fuzzy control are undeniable.

1.2 The significance of noise in engineering and science

Noise is pervasive throughout science and engineering and has enabled the solution of truly grand problems such as the origin of the universe through the discovery, by Penzias and Wilson at Bell Labs, of the isotropic noise called three-degree (Kelvin) blackbody radiation [1].

The topic of fluctuations and noise is subtle because, contrary to intuition, noise is *not* always a hindrance. Even though the extensive literature on communications and filtering [20, 21, 22, 23] shows that noise has traditionally been regarded as an unwanted nuisance, more recent discoveries have placed noise in a central position as an agent for active *improvement* of the performance of computational algorithms, natural systems and engineered devices. These discoveries show that noise is sometimes helpful. For instance, noise is explicitly used in random search algorithms and simulated annealing techniques [19] to avoid entrapment in local optima. Indeed, noise can sometimes suggest solutions to problems where it superficially *appears* to play no role. For example, this author derived a noise model to compute an accurate approximation for the frequently occurring Spence integral in Feynman diagrams that are extensively used in quantum electrodynamics [24]. An intriguing example of the unexpected effects of noise is provided by the phenomenon of stochastic resonance [25] in which the signal to noise ratio actually *improves* with additional noise under some circumstances. In neurobiology, stochastic resonance is helpful in aiding transduction across neurons [26] and in the efficient encoding of information in cochlear implants [27]. In chapter 4, this research will provide another example of how noise can be harnessed to improve the steady-state performance of a system.

1.2.1 Noise in neuroscience

A neuroscience phenomenon of focal interest in this dissertation is the noise influencing the hydrodynamics of cerebrospinal fluid flow. Cerebrospinal fluid (CSF) plays a key role in protecting the brain and shielding it from injuries and shocks. Intracranial dynamics, driven by the circulation of CSF, is important because it plays a central role in healthy brain function, and abnormalities in CSF dynamics can lead to multiple complications such as, among other things, hydrocephalus [28]. Intracranial pressure (ICP) is derived from cerebral blood and CSF circulatory dynamics, and clinical measurements of ICP over time show fluctuations around the deterministic time path predicted by the mathematical model [29]. Noise causes deviations of the predicted ICP from the actual ICP level [30]. One source of noise is the influence of relevant factors that have been omitted from the mathematical model, and a closely related source of noise is the idealized abstraction of the real world that is captured in the model. Because a mathematical model is an abstraction of reality, it is based on simplifying assumptions that are approximations to reality. In the case of CSF dynamics, these assumptions include [31]: (1) CSF is an incompressible Newtonian fluid, (2) CSF is produced by the choroid plexus at a constant rate, (3) CSF absorption occurs by a “valve” mechanism into the capillaries through the arachnoid granulations, (4) brain compliance reflects the volume storage capacity of tissue as the CSF pressure changes, (5) the Monroe-Kellie doctrine that the intracranial space has a fixed volume, (6) CSF flow dynamics are accurately captured by an analogy to an electrical circuit; specifically the Marmarou [32] model abstracts the CSF system as an electrical circuit consisting of a nonlinear capacitor (storage mechanism), resistor (area of CSF absorption), and so on [33]. Every one of these assumptions is an approximation to reality and becomes a source of noise that hampers deterministic attempts at modeling CSF dynamics. Noise due to sources other than idealized assumptions and omitted factors also influence CSF flow dynamics and these are discussed in chapter 2. Other important neurological phenomena such as the firing of neurons involve noise in fundamental ways [34] but are outside the scope of this research. This dissertation will develop a stochastic differential equation to model the noise influencing the hydrodynamics of cerebrospinal fluid flow, derive results of clinical significance from it and use it to offer a fresh perspective on an ongoing neuroscience controversy. Additionally, the new stochastic model of cerebrospinal fluid flow dynamics provides the mathematical basis for an automatic nonlinear regulator to keep the ICP within safe limits in patients suffering from hydrocephalus.

1.2.2 Noise in electrical engineering

It was the vacuum tube that initiated the study of noise in electrical engineering, and it is electrical engineering that has contributed more than any other discipline to the study of noise in both theory and practice [1]. Noise was the key player in the technology of the vacuum tube and later in semiconductor devices [1]. Modern communication theory is based on stochastic processes because the very concept of information transmission is rooted in probabilistic considerations [1]. Noise occupies center stage in signal processing models. Noise is present in the fluctuating current in a circuit with an inductance L in which the applied emf is a thermal noise voltage arising in the resistance [35]. Noise arises in the analysis of dynamically nonlinear translinear circuits [36]. In power systems, voltage and power consumption in gas discharge lamps show fluctuations due to thermal noise [37]. Noise fundamentally influences the degradation dynamics of ultra-thin gate oxides in MOS capacitors subjected to mechanical stress [38, 39]. Noise has important implications for nanotechnology, a field rapidly increasing in importance because molecular electronics, microelectromechanical devices, microscopic pumps and motors are achieving ever increasing degrees of physical realization [40, 9, 41]. Consequently, the scale of electronic manufacturing is becoming ever smaller, thereby making the engineering relevance of noise increasingly prominent. Specifically, Brownian Motion is linked to nanotechnology through its implications for cell motility [42], because Brownian Motion appears to be instrumental in the functioning of molecular motors, sometimes described as nature's nanomachines [43], that are the enablers of motility. In particular, as noted in Sharma and Mittal [44], researchers in the nanosciences are now increasingly interested in the development and applications of Brownian Motion models. This dissertation develops an algorithm to solve a general stochastic differential equation applicable to a large class of signal processing models and provides a basis to study noise in power systems, and the degradation dynamics of ultra-thin gate oxides in MOS capacitors. The methods developed to address these issues are applicable with only slight modification to future research on Brownian motors in nanotechnology.

1.2.3 Noise in operations research

Noise is a major issue in operations research models of industrial and business processes. Operations research is a broad field covering many disciplines such as marketing communications, financial engineering, industrial engineering, game theory, inventory theory and production management. Noise has a prominent presence in every one of these disciplines. Noise is typically called uncertainty in operations research but we will, for the sake of continuity of exposition, prefer the term noise throughout this dissertation. Marketing communications design is an important specialty within operations research and is concerned with making optimal decisions about multiple media or channels of communications. Many technological developments in electrical engineering have had direct and profound impact on the marketing communications process. New technologies have increased the number of channels of communication between the firm and its customers. The combination of new communication technologies, market response dynamics and market response noise makes profit maximization of a firm's marketing communications process a complex stochastic control problem. Market response is noisy because it is influenced by many factors over which the firm lacks control such as the state of the economy or technological developments. The optimal amount to spend on marketing communications requires careful mathematical analysis because it depends on the interaction of many factors. Answers to even simple questions are far from obvious. The issues here are: How much should the firm spend, given a profit maximization objective, and how should it spread the optimal amount over time? The first is an optimal budgeting issue and the second an optimal scheduling (or timing) issue. Is it better to spread the budget evenly over time, to decrease spending over time, to increase spending over time, or to do something more elaborate? The issue is important because large sums of money are at stake and suboptimal spending patterns significantly reduce profitability.

The salvage value associated with a dynamic process is the value associated with the final level of the state variable of interest at the end of the planning horizon. Many unaddressed issues remain and in particular, the influences of salvage value and uncertainty have received scant attention in the extant literature. Salvage value constraints are the boundary conditions in a finite-horizon stochastic control problem. The influence of salvage value on optimal advertising remains an open question and existing research is based on deterministic dynamic optimization models, which, by their very nature, rule out market noise. Yet market noise is important because stochastic, not deterministic, market response is the norm [7, 45]. The objective of this part of the dissertation is to provide a rigorous analysis of the joint influence of salvage value constraints and market response uncertainty upon the structure and pattern of dynamic spending of a key marketing communications instrument.

1.3 Mathematical treatment of noise

The general mathematical treatment of deterministic systems evolving dynamically under the influence of a set of control variables is the Newtonian framework. The future history of the system is perfectly predictable from the current conditions (current value of the state variable and other relevant data such as values of the control variables). In the terminology of dynamical system theory, a tangent field defines a *flow* in classical phase space. At every point in the phase space and at all times, the tangent vector uniquely and deterministically prescribes the direction of flow for the point.

1.3.1 Dynamical systems and noise

Using standard mathematical notation, X(t) denotes the state of the system, **u(t)** the control vector, f(X, **u**, t) the tangent field, σ(X, **u**, t) the infinitesimal (local) standard deviation parameter capturing the intensity (amplitude) of random fluctuations, all at time “t.” In the *absence* of fluctuations, σ(X, **u**, t) = 0, and such a system would enjoy the classical Newtonian description,

$$\frac{dX}{dt} = f(X, u, t) \tag{1.1}$$

in which X(t) could be (for instance) the position of a particle at time “t.” The above equation then defines the tangent field and its action maps the initial state X(0) into the state X(t) at time “t.” The tangent field f(X, **u**, t) defines a *flow* in classical phase space as a function of the current state of the system. In the presence of noise, a natural generalization of equation (1.1) is

$$\frac{dX}{dt} = f(X, u, t) + \sigma(X, u, t)\varepsilon(t) \tag{1.2}$$

in which ε(t) is a zero-mean, uncorrelated stochastic process, called white noise in the engineering literature [46]. The parameter σ governs the intensity or amplitude of the white noise process.

The distinguishing feature of white noise is that E[ε(t) ε(s)] = 0 for t ≠ s, where E denotes the expectation operator. This restriction leads to a Dirac Delta function representation for the covariance function of white noise [47], a feature that is equivalently mirrored by its constant spectral density. Consequently the tangent vector defined by equation (2) cannot be interpreted as a velocity vector in the traditional mathematical sense because (i) a Dirac Delta function is not a well-defined mathematical function and (ii) a constant spectral density necessarily implies an *infinite* power signal [2].

Thus, in the presence of noise, the picture of the smooth deterministic flow captured in equation (1.1) changes significantly. (A) First, the tangent vector is no longer a mathematically well-defined function; however it can be given a rigorous mathematical meaning within the theory of generalized functions. (B) Second, the flow becomes stochastic and the future history of the system is predictable only in a probabilistic sense. (C) Third, these probability distributions change over time, and the manner in which they change is described by the Fokker-Planck partial differential equations. The theory of stochastic differential equations accommodates all these facts and facilitates analytical treatment of stochastic dynamic systems [48].

Equation (1.2) can be formulated as a stochastic differential equation (SDE) by exploiting a fundamental connection between white noise and Brownian Motion. Stated informally, this connection is simply that smoothed white noise *is* Brownian Motion (strictly speaking, the equivalence is in the *almost sure* sense, i.e. with probability one). Integrating a process effectively smoothens it. Thus we rewrite the random differential equation (2) as a controlled stochastic differential equation:

$$dX = f(X, u, t)dt + \sigma(X, u, t)\, dW(t) \tag{1.3}$$

in which the dW term is identified with the error process ε_t via the fundamental relationship

$$W(t) = \int_0^t \varepsilon_s \, ds \tag{1.4}$$

The precise meaning of equation (1.3) will become clear after discussing Brownian Motion and the Langevin equation. The general Langevin equation for this dissertation is presented in section1.5. Equations (1.3) and (1.4) show that Brownian Motion is the key to modeling noise in dynamical systems. Brownian Motion and its modeling significance are the subject of subsection 1.3.2.

1.3.2 Brownian Motion

Brownian Motion is important not only because it is the foundation for stochastic differential equations, but also because it has major engineering and scientific significance. In electrical engineering, Brownian Motion plays a key role in the operation of semiconductor devices. The two main noise phenomena in semiconductor materials are generation-recombination noise and velocity-fluctuation or diffusion noise, and the latter is associated with Brownian Motion [49] of the free carriers (classical picture) or electron-phonon and electron-impurity scattering (quantum picture). Thus Brownian Motion is a mathematically powerful model for noise in electrical systems and electronic devices [2].

Brownian Motion is basic to Life itself, because it is responsible for our very ability to breathe! For it is Brownian Motion in its manifestation as random thermal motion that sustains the Boltzmann distribution of air molecules at various heights in our atmosphere: without the fluctuations of Brownian Motion, air molecules would fall to the floor [50], that is to say, they would fall to the lowest ground level in the gravitational potential field of our planet. Furthermore, the most fundamental processes of Life are driven by DNA and soluble proteins, whose operation is closely linked to Brownian Motion for the following reason. DNA and soluble proteins are macromolecules that form colloidal suspensions in water [50], and “to study colloidal suspensions is in many respects to study the consequences of Brownian Motion” [16].

Diffusion is the aggregate macroscopic result of the Brownian Motion of each individual molecule. Diffusion is responsible for transport phenomena, the development of membrane potentials, electrical resistance in circuits, friction and viscosity. The basic diffusion equation governing the concentration of a substance c(x, t) (the number of molecules per unit volume at location “x” at time “t”), is written as [15]:

$$\frac{\partial c}{\partial t} = D\,\frac{\partial^2 c}{\partial x^2} \tag{1.5}$$

where D is the diffusion constant defined to be $D = \delta^2/2\Delta t$, where δ is the size of the step taken by the Brownian particle every Δt time units. Diffusion in three dimensions is governed by the same equation in terms of the Laplacian operator ∇^2, where $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$. The diffusion equation in three dimensions is then:

$$\frac{\partial c}{\partial t} = D\,\nabla^2 c \tag{1.6}$$

Equation (1.6) is intimately connected to electrical engineering—at steady-state, the time-independent form of equation (1.6) is simply $\nabla^2 c = 0$, which is Laplace’s equation for the electrostatic potential in charge-free space [51]. A second connection between equation (1.6) and engineering is that it is the classic heat equation [51].

A particularly important interpretation of the concentration c(x, t) is as a probability density p(x, t), the probability of finding the Brownian particle near “x” at time “t,” and in that case, the evolution of the probability density function of Brownian Motion over time is given by:

$$\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2} \tag{1.7}$$

Equation (1.7) is an example of a Fokker-Planck equation (FPE) [48]. FPEs are partial differential equations describing the temporal evolution of the probability density function of a stochastic process. Given the SDE for a phenomenon, the associated FPE can be written immediately by inspection. Thus, FPEs provide an alternative way of studying the stochastic evolution of a system without using stochastic differential equations. However, the scope of FPEs is curtailed by the difficulty of solving PDEs. Stochastic differential equations (SDEs) driven by Brownian Motion vastly expand one’s modeling power and are used extensively in this research. Used judiciously as complementary analytical tools, SDEs and FPEs provide a powerful modeling methodology, as will be seen in chapter 2. The stochastic model for CSF flow dynamics in chapter 2 is formulated as a SDE and its steady-state distribution is derived by solving the associated FPE.

1.3.3 The Langevin equation

Next, the Langevin equation and the modeling philosophy known as the Langevin approach in Physics [3] is described. It will be seen that equation (1.3) effectively generalizes the Langevin equation to cover all cases of practical interest in this research.

The Langevin equation was proposed to model the motion of a Brownian particle immersed in a fluid. A Brownian particle is anything small enough to be affected by the thermal motion of the molecules or atoms of a fluid such as water, air or the bloodstream. For example, protein molecules and molecular motors are Brownian particles. Newton's equation F = ma, where "m" is the mass and "a" the acceleration of a particle, can be written as $\dot{V} = \frac{F}{m}$ where $\dot{V} = \frac{dV}{dt}$, and V(t) is the velocity of the particle at time "t." Because the motion of a Brownian particle is highly irregular and unpredictable, Paul Langevin modeled the specific impulse $\frac{F}{m}$ in Newton's law as the sum of a viscous drag force $-\gamma V$ plus random fluctuations L(t) [14]. The Langevin equation is then:

$$\dot{V} = -\gamma V + L(t) \tag{1.7}$$

The assumptions on the random error term are [3]:

(I) L(t) is irregular and completely unpredictable

(II) E{L(t)} = 0

(III) L(t) is caused by random collisions with the individual molecules of the surrounding fluid and it varies rapidly. This assumption is expressed by the requirement

$E\{L(t)L(t')\} = \sigma^2\delta(t-t')$, where $\delta(t)$ is the Dirac delta function.

Van Kampen [3] adds a fourth assumption:

(IV) L(t) has a Gaussian distribution.

However, in a very readable paper, Gillespie [52] shows that the fourth assumption is in fact a delightfully satisfying *consequence* of coupling the first three assumptions with two natural requirements--the Markovian nature (i.e.

memorylessness) of most physical processes and a self-consistency property of all reasonably behaved stochastic processes. Any force that can be written as the sum of a linear damping term and an irregular force with the properties (I) – (IV) above is called a Langevin force.

The Langevin approach to modeling is the following [3]:

a) Write the deterministic macroscopic equation of motion of the system;

b) Add a Langevin force with the four properties mentioned above

c) Adjust σ^2 so that the stationary distribution reproduces the correct mean square fluctuations as known from statistical mechanical or other considerations

This approach is followed throughout the dissertation; the last step is not undertaken because the nature of the problems do not permit it.

1.4 Objectives

The modeling significance of the Langevin equation has grown well beyond the physical sciences to encompass biology, engineering, and operations research. The general nonlinear form of the Langevin equation [3] is

$$\dot{X} = A(X) + C(X)\,L(t) \tag{1.8}$$

Written in the mathematically rigorous notation of stochastic differentials, the above general Langevin equation is

$$dX = A(X)\,dt + C(X)\,dW \tag{1.9}$$

With the incorporation of control variables, the general Langevin equation in this research is:

$$dX = f(X,u)dt + \sigma(X,u)\, dW(t) \qquad (1.10)$$

All the phenomena in this dissertation fit into the framework of the general Langevin equation (1.91) in which the drift and infinitesimal standard deviation are functions of the state and, possibly, of a control vector.

It was established in section 1.2 that noise raises significant problems in neuroscience, electrical engineering and operations research. Specifically, extant mathematical models of cerebrospinal fluid flow dynamics are deterministic and hence incapable of accounting for the noise that is intrinsic to the ICP waveform—generated by CSF flow dynamics—as is clearly revealed in experiments. By incorporating noise in a classic neuroscience model of CSF dynamics, this research makes contributions to an important area of neuroscience by deriving clinically relevant probabilities that will improve the treatment of hydrocephalus. The analytical results suggest new hypotheses that can be tested in the laboratory by experimental researchers. The stochastic model for CSF flow dynamics serves as the basis for the development of an automatic nonlinear regulator for the ICP. The automatic regulator has far reaching implications because controlling ICP to keep it within safe limits is important not only for patients suffering from hydrocephalus but also in the treatment of other disorders related to the brain such as brain injury and brain trauma.

SDEs are ubiquitous in electrical engineering and provide the natural mathematical framework for signal processing models but nonlinear SDEs are very difficult or impossible to solve. This research develops a new algorithm to quasi-analytically solve a class of SDEs with polynomial drift, and shows that the results are applicable in an approximate sense to SDEs with a completely general continuous drift. Furthermore, the algorithm can be used to control the system behavior at steady-state. Thermal noise affects the operation of gas discharge lamps, and while there is an existing stochastic model of this phenomenon, several unaddressed issues remain that need to be resolved. This research provides a basis to generalize existing results on gas discharge lamps and derives new probabilistic results that are useful in assessing their performance. Noise affects the degradation dynamics of ultra-thin metal oxides in MOS capacitors, and, although an existing stochastic model describes this phenomenon, the statistical quality of the estimators of that model leave room for improvement, and, based on this research, continuous-time estimators can be derived to inform future research in that area.

In operations research, models of marketing communications response, existing models ignore noise—a significant limitation since prescriptions for optimal budgeting and its temporal allocation are dependent on realistic models of market response. This research extends a classic model of marketing communications response and analytically derives the optimal budgeting and temporal allocation for an important marketing communications instrument through stochastic optimal control.

Guided by the above considerations, the objectives of this research are to:

a) Extend and improve the applicability of a classic model of CSF flow dynamics by incorporating noise;

b) Derive clinically relevant probabilities for ICP regulation to facilitate dynamic risk management of patients;

c) Use the extended model to resolve a neuroscience controversy;

d) Develop a new algorithm to solve SDEs with polynomial drift;

e) Use the new algorithm to improve system performance at steady state;

f) Improve budgeting and temporal allocation of marketing communications by solving a stochastic optimal control problem for a marketing communications instrument.

1.5 Novelty and significance

This research contributes by mathematically modeling noise that has been ignored in prior work on dynamic phenomena in neuroscience, electrical engineering and operations research in which noise plays a key role.

a) The stochastic model of CSF flow dynamics strengthens the classic model in the field by extending it to accommodate noise;

b) By doing so, it enables the computation of clinically relevant probabilities of critical events which facilitates dynamic risk management of patients;

c) Uses the stochastic model of CSF flow dynamics in a novel way to resolve a neuroscience controversy;

d) Develops a new algorithm to solve SDEs with polynomial drift which can be used to approximately solve SDEs with arbitrary continuous drift ;

e) Though noise is generally considered a nuisance, the new algorithm exploits noise in a novel way to improve system performance at steady state;

f) Proves that observed differences in temporal allocation of marketing communications need not be driven solely by differences in market response function effects as was previously thought—in fact the observed differences in temporal allocations are optimal for the *same* response function due to differences in assumptions about the boundary value.

CHAPTER 2

CEREBROSPINAL FLUID DYNAMICS

This chapter forms the heart of the dissertation. It uses electrical engineering methods—stochastic differential equations and an electrical circuit analogy—to model the circulatory flow of an important fluid within the brain. Techniques of neuroscience, brain physics, electrical engineering and mathematics are united to solve the problem.

2.1 Background

2.1.1 CSF Flow, Intracranial dynamics and pressure

The central nervous system consists of the brain and the spinal cord which are surrounded by a clear fluid excreted by the choroid plexus, called cerebrospinal fluid (CSF), which protects the brain from external pressure and shocks [1]. Intracranial dynamics, driven by the circulation of CSF, are important because CSF protects the brain from injury, contains nutrients enabling normal functioning of the brain and, transports waste products away from the surrounding tissues. Intracranial dynamics play a central role in healthy brain function because disturbances in the internal fluid environment of the skull can lead to multiple complications such as, among other things, hydrocephalus [1]. Much more is involved in hydrocephalus than a simple disorder of CSF circulation [2]; it is considered a complex spectrum of diseases, primarily defined by perturbation of the cranial contents—operationalized as CSF volume—and the intracranial pressure [3]. Given the complex nature of hydrocephalus, we define hydrocephalus as a disease associated with disturbances in the CSF dynamics, as in [1].

The CSF must be absorbed in order to prevent the brain from expanding uncontrollably [1]. CSF is absorbed by the arachnoid villi, small leafed channels, and emptied into the superior sagittal sinus, which is the major venous pathway exiting the brain. Intracranial pressure (ICP) is derived from cerebral blood and CSF circulatory dynamics. As the CSF leaves the ventricles within the brain, through the aqueduct of sylvius, and seeps into the arachnoid villi, it encounters impedance to its flow, and this resistance is responsible for the development of ICP [1]. Experimental evidence compellingly validates that, over a large range of pressures, brain compliance is not constant [4]. Marmarou [5] postulated a hyperbolic compliance function that decreases as the pressure increases, which coupled with other assumptions to be described below, led to a nonlinear ordinary differential equation for the variation of ICP over time. The Marmarou model [5] is fundamental in mathematical pressure-volume models of CSF dynamics. It uses an electrical circuit analogy to relate pressure and volume through an exponential relationship.

2.1.2 An electrical circuit analogy for CSF flow

In his approach to modeling CSF dynamics, Marmarou [5] conceptualized the relationship between flow and pressure drop in the same way that Ohm's law relates the current to voltage in an electric circuit. Large fluid spaces such as ventricles or subarachnoid spaces are modeled as lumped parameter compartments, and narrow conduits between them give rise to resistance or impedance [3]. The current source reflects formation of CSF; resistor with diode captures unilateral absorption of CSF into the sagittal sinuses; capacitor and voltage source models the nonlinear compliance of CSF space. P_{ss} is the pressure in the sagittal sinuses. P_0 is the reference pressure. See Figure 2.1 below for the circuit [2].

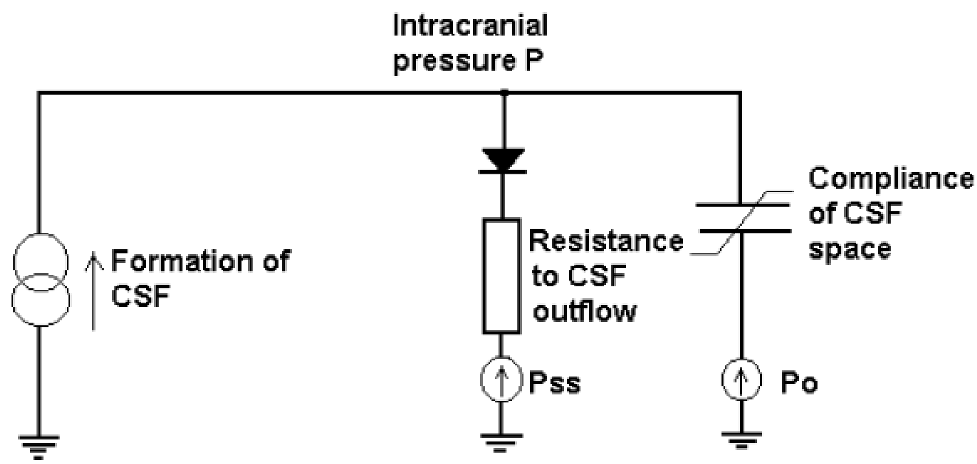

Figure 2.1 Electrical circuit analogy for CSF flow dynamics. Reproduced from [2] with permission.

2.2 The Marmarou model for CSF dynamics

While the Marmarou model has deservedly remained the mainstay of quantitative modeling of the dynamics of CSF flow, its deterministic nature prevents taking full advantage of the information in real ICP measurements, because deterministic models average over all possible fluctuations of real data. The ICP waveform contains additional information that is ignored by the time-averaged ICP mean value [6]. We draw upon the fundamental principles of modeling cerebrospinal fluid dynamics explicated in [2] to develop the deterministic Marmarou model.

2.2.1 The deterministic Marmarou model

Our starting point is Marmorou's model [5, 7] of pressure-volume compensation, which was subsequently modified in [8] and [9]. Central to the development of the Marmarou model is a conservation law. Conservation laws are ubiquitous in physics [10]. The Marmarou model represents CSF flow dynamics through a conservation equation relating the production of CSF to its storage and reabsorption [2].

$$\text{Production of CSF} = \text{storage of CSF} + \text{reabsorption of CSF} \tag{2.1}$$

Next, reabsorption is proportional to the differential between CSF pressure (p) and pressure in the sagittal sinuses (p_{ss}):

$$\text{reabsorption} = \frac{p - p_{ss}}{R} \tag{2.2}$$

p_{ss} is considered a constant parameter, determined by central venous pressure. The coefficient R is the resistance to CSF reabsorbtion or outflow, measured in units of mmHg mL^{-1} min. Storage of CSF is proportional to the cerebrospinal compliance C, measured in units of mL $mmHg^{-1}$.

$$\text{storage} = C\frac{dp}{dt} \tag{2.3}$$

The compliance of the cerebrospinal space is inversely proportional to the differential of CSF pressure p and the reference pressure p_0 [8, 11], and is considered the most important law of cerebrospinal dynamic compensation [2]:

$$C = \frac{1}{E(p - p_0)} \tag{2.4}$$

The coefficient E is called the cerebral elasticity (or elastance coefficient) and has the units mL^{-1} [12]. Next, by exploiting an analogy between an electrical model of CSF compensation, as described in section 21.2 based on [5] and [2], the deterministic description of the dynamics of CSF flow are given by:

$$\frac{1}{E(p - p_0)}\frac{dp}{dt} + \frac{p - p_b}{R} = I(t) \tag{2.5}$$

where I(t) is the rate of external volume addition and p_b is a baseline pressure. The circuit diagram, reproduced from [2], is shown in Figure 1. An electrical circuit analogy is also used in [13] and [14] to study the dynamics of ICP in the ventricular compartments. The reference pressure parameter p_0 is sometimes taken to be zero, as for example, in [5] because, as noted in [2], the significance of p_0 is unclear. Consequently, we assume $p_0 = 0$, which results in the equation:

$$\frac{1}{Ep}\frac{dp}{dt} + \frac{p - p_b}{R} = I(t) \tag{2.6}$$

2.3 Incorporating noise into Marmarou model

2.3.1 Importance of modeling noise in CSF dynamics

Broadly construed, noise arises from variations in factors that influence the observed outcome—which is the ICP in this paper— but that have been omitted from the mathematical model, and from factors affecting the observed outcome that are beyond the experimenter's control. Noise causes deviations of the predicted ICP from the actual ICP level. Factors uncontrolled by the experimenter include thermal fluctuations, body movement and breathing. Because a mathematical model is an abstraction of reality, it is based on simplifying assumptions, as listed in [3]. The Marmarou model abstracts the CSF system as an electrical circuit consisting of a nonlinear capacitor (storage mechanism), resistor (area of CSF absorption), and so on [15]. There remains substantial uncertainty regarding the average rate of CSF production [16]. Realistic estimates of the mechanical properties of the living human brain are hard to discover [15]. The compliance is not an appropriate indicator of the brain's elastic properties [14]. Shunts, used in the treatment of hydrocephalus, can be dramatically improved by more accurate modeling of the CSF dynamics. Shunts providing continuous CSF drainage are the ideal [17], and nonlinear control theory can be used to design an automatic controller for a shunt that provides continuous drainage. But in order to design a stable controller to facilitate a shunt with continuous drainage, we need a model of CSF drainage that either incorporates factors omitted in extant models, or that accounts for the noise caused by the omission. Our objective is to incorporate noise into the dynamics of CSF flow. The effect of noise on the ICP waveform is discernible in Figure 2.2, which shows the fluctuations of the ICP around the deterministic path predicted by the deterministic Marmarou model.

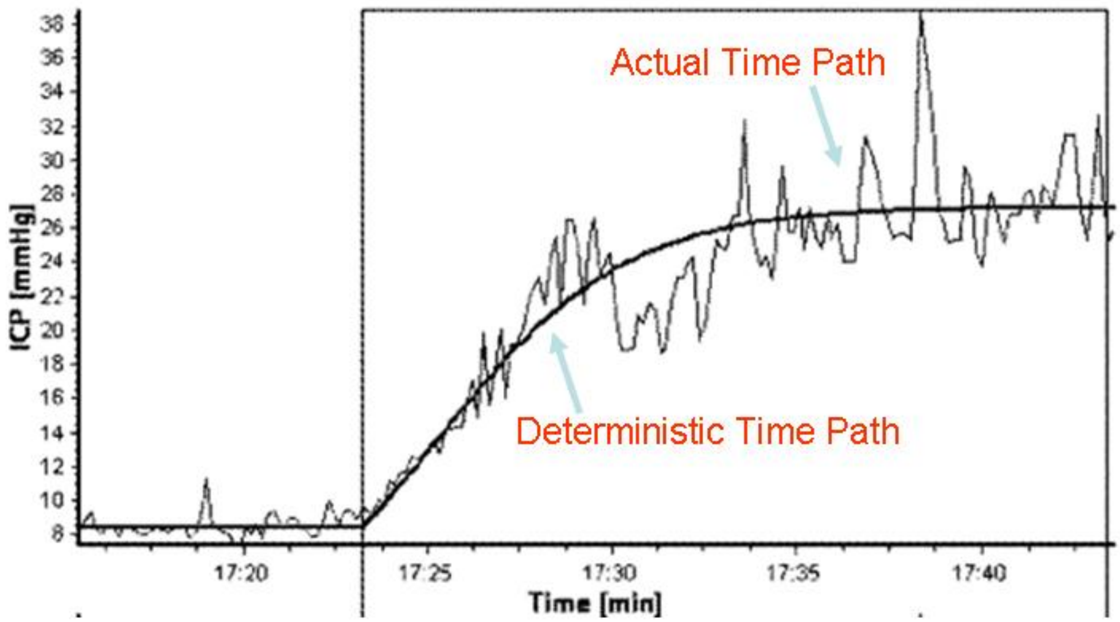

Figure 2.2 Comparison of actual path of ICP with the path predicted by the deterministic Marmarou model. Reproduced from [1] with permission.

2.3.2 Ito and Stratanovich stochastic differential equations

Visual examination of the time-series of ICP recordings shows that the fluctuations are smooth (unlike electrons in a wire which generate shot noise, characterized by jumps [17]), and therefore continuous state space Markov processes are appropriate to capture the noisy dynamics of CSF flow. A large class of Markov processes can be represented by SDEs, and here a methodological choice must be made—noisy dynamic processes can be represented by stochastic differential equations of the Ito type or the Stratonovich type which correspond to two different ways of introducing noise into a dynamic system. A central difference between the two is that the Stratonovich SDE uses the usual deterministic calculus whereas the Ito SDE requires a completely new stochastic calculus. Extensive conceptual, empirical and philosophical discussions of this issue exist in the literature on mathematical models of electrical, biological and physical phenomena [19, 20, 21]. The overwhelming majority of these discussions conclude that Ito processes, generated by stochastic differential equations of the Ito type, are superior to Stratonovich processes, generated by stochastic differential equations of the Stratonovich type [22, 23]. Ito [24] extended standard deterministic calculus to a "stochastic calculus" applicable to functions of a wide class of continuous-time random processes, known as Ito processes. Given the SDE for the process under consideration, a result called Ito's Lemma yields the SDE driving the dynamics of a general transformation of the original process [24]. This utilitarian result allows deducing the stochastic properties of considerably complex models driven by Ito processes [23]. An essential property of Ito processes is that nonlinear functions of Ito processes remain Ito processes—a property called closure under nonlinear transformations, indispensable for practical reasons. From an empirical standpoint, a compelling advantage of Ito processes is that they often yield very precise statistical specifications for estimation [23]. An attractive property of Ito processes—on theoretical, mathematical, practical and computational grounds—is that they are Markov processes. Finally, the Ito calculus has been extended to embrace general martingale processes [25]—a development that permits joint consideration of both smooth noise and noise that occurs in jumps. Thus our modeling framework can accommodate neurological phenomena requiring noise that encompasses *both* smooth and jumpy variations in the state of the system, such as the firing of neurons [26].

2.3.3 The stochastic Marmarou model

Given all these considerations, we modeled the fluctuations in CSF dynamics through an Ito stochastic differential equation. First we introduced noise into equation (6) through a white noise process $\varepsilon(t)$ with intensity parameter σ, which by definition satisfies the following properties: $E[\varepsilon(t)] = 0$, and $E[\varepsilon(t)\,\varepsilon(s)] = 0$, whenever $t \neq s$. The notation E[.] denotes the expectation operator, which, when applied to a random quantity such as $\varepsilon(t)$, signifies the value of $\varepsilon(t)$ on average. Thus $E[\varepsilon(t)] = 0$ signifies that the average value or mean of the random error at time “t” is zero, and this is a standard assumption in the literature on modeling noisy phenomena. The term $E[\varepsilon(t)\,\varepsilon(s)]$ is the expectation operator applied to the product of random errors at two different times ‘s’ and ‘t;’ technically it denotes the covariance between the errors at two different times. In this case, because of the zero-mean assumption, it also denotes the correlation between $\varepsilon(t)$ and $\varepsilon(s)$; and so the property $E[\varepsilon(t)\,\varepsilon(s)] = 0$ means that the errors at two different times are *uncorrelated*, which substantively means that an error at one point in time does not influence the error at another point in time. This too is a standard assumption in the dynamic modeling literature.

$$\frac{1}{Ep}\frac{dp}{dt} + \frac{p - p_b}{R} = I(t) + \sigma\varepsilon(t) \qquad (2.7)$$

Next we exploited the fundamental relationship between a white noise process $\varepsilon(t)$ and a Brownian motion process W(t): $W(t) = \int_{s=0}^{s=t} \varepsilon(s)\,ds$, which, when written in differential notation, yields $dW = \varepsilon(t)\,dt$. Therefore,

$$\frac{1}{Ep}\frac{dp}{dt} + \frac{p - p_b}{R} = I(t) + \sigma\frac{dW}{dt} \tag{2.8}$$

Rearranging the above terms yields our final model, which we will call the stochastic Marmarou model.

$$dp = \{Ep\,I(t) - \frac{Ep(p - p_b)}{R}\}dt + \sigma Ep\,dW \tag{2.9}$$

Note that in order for equation (9) to be dimensionally consistent, the unit of σ is mL/min. Because the ‘input’ I(t) is the infusion rate which is under direct experimental control, therefore, in the language of control theory, I(t) is a ‘control’ variable. In the infusion studies conducted at Addenbrookes’s Hospital in Cambridge, UK, I(t) is maintained at a constant rate of 1.5 mL/min. However, factors not within the experimenter’s control also influence the input flow rate. In addition to the infusion rate of the experimenter which influences CSF formation, CSF is produced inside the brain, but much about its production remains unknown at the present time. Currently, there are no direct methods to measure the CSF production rate over short periods of time. Globally, the average secretion rate—used as a proxy for the production—is 0.35 mL/min with a 95% confidence range of 0.27 mL/min to 0.45 mL/min [2]. The lack of precise knowledge about the CSF production rate and the unmeasured factors that influence it are sources of noise in the total CSF formation rate. Consequently the stochastic Marmarou model may be conceptualized as the classic Marmarou model with a noisy input flow rate that reflects uncertainty about CSF formation.

The deterministic Marmarou model is contained in the final model displayed above—it surfaces when σ = 0 mL/min, which precludes noise, and consequently produces only the mean ICP value. The general model with $\sigma \neq 0$ mL/min reproduces the fluctuations inherent in the time-path of real measurements of ICP—information which is discarded by the deterministic Marmarou model. Figure 2.3 compares the fluctuating path, similar to the actual noisy ICP data, reproduced by the stochastic Marmarou model with the path predicted by the deterministic Marmarou model.

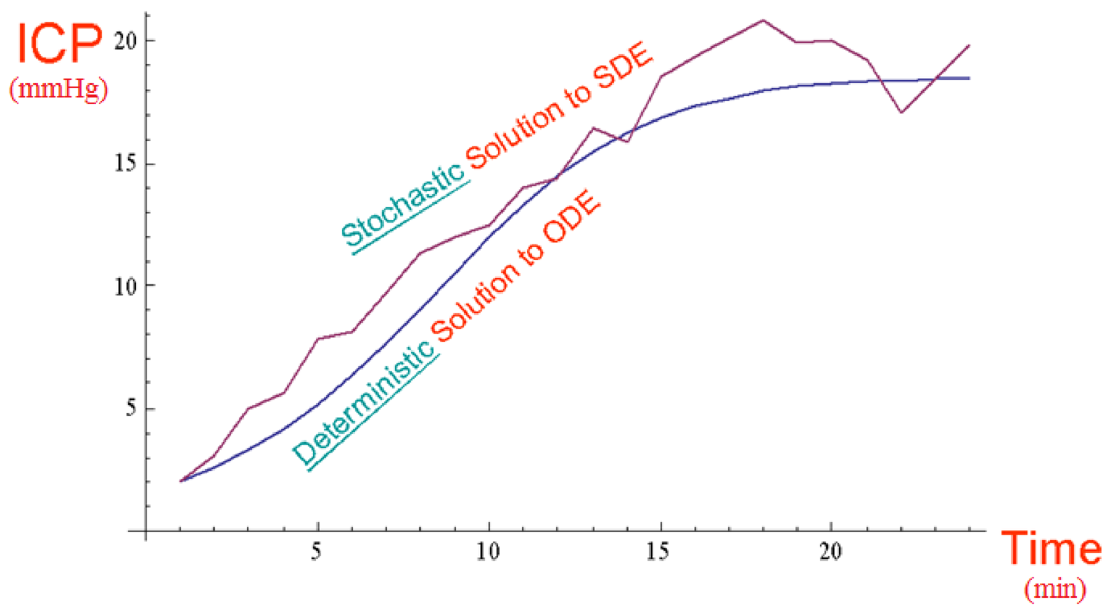

Figure 2.3 Comparison of deterministic and stochastic Marmarou model solutions.

The mathematical structure of the Marmarou et al. [5] equation is the classic Verhulst logistic model, ubiquitous in biological growth and saturation phenomena [27]. The mathematical form of equation (9) is the stochastic logistic model and it is the natural stochastic extension [28, 29] of the Verhulst logistic model.

2.3.4 Clinical significance of the stochastic Marmarou model

By building the fluctuations right into the dynamics of the model structure, the stochastic model makes full use of the information in the variations of the ICP waveform. From this additional information, the time-varying probability distributions of the ICP waveform can be extracted, and it is these latter quantities that enable computation of the probabilities of clinically relevant events. It is the knowledge of these probabilities of clinically relevant events that facilitate dynamic risk management of the patient. Conceptually, the average value of p(t) at any given time 't' is the average ICP at that time in an ensemble of patients with a similar CSF flow profile, as reflected in the values of the CSF flow parameters.

2.4 Analysis of the stochastic Marmarou model

In the remaining subsections, we will display the exact analytical solution to the stochastic Marmarou model and derive insights from the solution into the influence of noise on the ICP at each point in time, and on average. Under the normal conditions described in [2], biological processes will settle down to a steady state after the transients have died out. In the deterministic Marmarou model [5], the steady state (equilibrium) is found by setting the time rate of change of the ICP equal to zero. What is the corresponding steady-state concept for a stochastic process? The stochastic counterpart to the time-independent steady-state level of the ICP is the time-independent probability distribution of the ICP, and the equilibrium probability distribution is to the stochastic environment as the stable equilibrium point is to the deterministic one [30]. We derive the equilibrium probability distribution for the ICP, and from it, draw conclusions for the influence of CSF flow parameters and noise intensity upon the average steady-state ICP level. We compute a measure relevant to the treatment and control of hydrocephalus: given the current value of the patient's ICP, what is the probability that it will exceed a critical high level? And how is that probability influenced by neurological characteristics of the patient such as their resistance to CSF flow and the noise intensity of the fluctuations in CSF formation rate which in turn drives the fluctuations in their ICP?

2.4.1 Clinically relevant transition probabilities

The mathematical formulation of the problem posed in the previous paragraph is: given that a patient's ICP is currently x mmHg, where x is an arbitrary value, what is the probability that the ICP will exceed a critical threshold 'b' (mmHg) at a future time? Mathematically stated: given that $p(s) = x$, find the following transition probability— $P[p(t) > b \mid p(s) = x]$, $t > s$. Simple though the question seems, finding the answer requires computing the conditional probability distributions of the CSF process. Since the conditional probability distributions follow the Fokker-Planck partial differential equation, the problem is non-trivial, but Karlin and Taylor [31] circumvent the difficulty by solving a boundary-value problem associated with this dynamically changing probability. They show that the required probability satisfies a nonlinear ordinary differential equation which must be solved subject to two conditions on the probability that are natural consequences of the current ICP level when it is at one of the two extreme points of the range of ICP values under consideration. It is these conditions that give rise to the term ‘boundary value problem.’

2.4.2 Solution with constant infusion

For a constant infusion rate I, equation (9) is explicitly solvable in closed-form as shown below. Given any initial ICP value “p_0” (mmHg) at time t = 0, the future ICP value at any time “t” is given by:

$$p(t) = \frac{\exp\left[\left\{E\left(I+\frac{p_b}{R}\right)-\frac{\sigma^2E^2}{2}\right\}t+\sigma E\,W(t)\right]}{\frac{1}{p_0}+\frac{E}{R}\int_0^t \exp\left[\left\{E\left(I+\frac{p_b}{R}\right)-\frac{\sigma^2E^2}{2}\right\}s+\sigma E\,W(s)\right]ds} \tag{2.10}$$

Proof

The Marmarou model with a constant infusion rate is

$$dp = \{EIp - \frac{Ep(p-p_b)}{R}\}dt + \sigma Ep\,dW \tag{2.11}$$

This may be rewritten in the form of the stochastic logistic model

$$dp = \left(\frac{E}{R}\right)p\left(RI + p_b - p\right)dt + \sigma Ep\,dW \tag{2.12}$$

It is shown in [28] that the solution to the stochastic logistic model is

$$X(t) = \frac{\exp\left[\left\{rK-\frac{\sigma^2}{2}\right\}t+\sigma W(t)\right]}{\frac{1}{x_0}+r\int_0^t \exp\left[\left\{rK-\frac{\sigma^2}{2}\right\}s+\sigma W(s)\right]ds} \tag{2.13}$$

Identification of the parameters “r” with (E/R), ‘K’ with (RI + p_b) and σ with σE produces the claimed solution.

Discussion

Note that the solution to the stochastic Marmarou model is found through an “integrating factor” which involves an integration constant, the evaluation of which necessitates a unit of 1/min unit for the 2 inside the exponent of the exponential function. The noise intensity parameter σ and the Brownian Motion process W(t) in the solution show the explicit influence of noise on the evolution of the ICP, underscoring the importance of modeling the noise in the clinical ICP data. In addition to the practical utility of offering a closed-form analytical solution, this result has value for another reason: it shows explicitly that noise cannot be averaged away when the process is nonlinear. If the Brownian motion process W(t) entered the solution for p(t) in an additive linear way, its effect would disappear on average. But the Brownian motion process enters the solution in a highly nonlinear fashion, making it impossible to average out its effect to zero. Finally, the solution depends upon the noise intensity parameter σ in a mathematically continuous way, a fact that is meaningful because the result shows that the solution to the deterministic Marmarou model [5] emerges as the special case corresponding to σ = 0 mL/min, and so, it is natural to ask if the simpler deterministic model would suffice when the noise intensity is small. Should the influence of noise be negligible in a particular case, the value of σ will be very small, and, because of the mathematical continuity in its dependence upon σ, the stochastic solution will be very close to the deterministic solution in such a case, and we may use the simpler deterministic model with confidence. However, the stochastic model is preferable in general for two reasons: it captures the dynamics of the ICP data better than the deterministic model when the noise intensity is larger, and furthermore, the stochastic model characterizes the risk profile of the patient probabilistically. Almost tautologically, the deterministic model cannot evaluate the risks due to the errors that are an inseparable part of medical data because deterministic modeling philosophy sees the future as completely predictable from the present situation. These considerations suggest that, from a conservative modeling perspective, incorporating the influence of noise into the dynamics is conceptually more defensible.

In principle, the solution contains all the transient probability distributions of the ICP process that characterize it on its way to equilibrium. In practice, mathematical difficulties may make these transient distributions hard to extract from the solution. But we can still compute the probability of the critical events by using a methodology that does not depend on that knowledge. And we can still draw useful information about the nature of the process at steady-state. Next, we find the steady-state probability distribution of the ICP process.

2.4.3 Steady state probability distribution of ICP

The steady-state probability distribution of the ICP is gamma with the parameters shown p.149 in [29], and will exist provided that the noise intensity parameter σ satisfies the condition: $\sigma^2 < \frac{2(RI + p_b)}{RE}$.

Proof

The transition probability function satisfies the Fokker-Planck partial differential equation, which at steady-state, becomes an ordinary differential equation (ODE). Let $\phi(x,t,y,s) = P[p(t) \in (x, x+dx) \,|\, p(s) = y]$ be the transition density. Then, ϕ(x, t, y, s) satisfies the Fokker-Planck partial differential equation, hereafter abbreviated (FPE), stated in [29] for a general vector stochastic process, and here specialized to a scalar process for the stochastic Marmarou model:

$$\frac{\partial \phi}{\partial t} = -\frac{\partial [f(p)\phi]}{\partial p} + \frac{1}{2}\frac{\partial^2 [g(p)^2 \phi]}{\partial p^2} \tag{2.14}$$

subject to

ϕ(x, t, y, s) = δ(x – y), where δ(x – y) is the generalized Dirac-delta function centered at y. The FPE shows that the transition probabilities vary over time during the transient phase, but at steady-state, the probability transition functions are time-independent, and consequently $\frac{\partial \phi}{\partial t} = 0$. The Fokker-Planck partial differential equation then becomes an ordinary differential equation which may be solved to find the steady-state distribution of the ICP process p(t). The ODE is shown below:

$$\frac{d}{dp}\left[-f(p)\phi + \frac{1}{2}\frac{d[g(p)^2 \phi]}{dp} \right] = 0 \tag{2.14}$$

Solving the time-independent Fokker-Planck ODE yields the Gamma distribution with parameters that are shown in [29]. It is shown in [29] that the steady-state distribution for the SDE $dX = aX(1 - \frac{X}{K})dt + bX\,dW$ will exist provided that $b^2 < 2a$. The condition for the steady-state distribution of the ICP stated in the paper follows upon appropriate identification of the parameters of the stochastic Marmarou model with those of the model in [29].

Discussion

The integration required to evaluate the normalization constant generates a unit of 1/min for the 2 in the above inequality. The mean and variance of the gamma distribution are not independent parameters as they are for the normal distribution. Unlike the normal distribution in which the mean is the location parameter and the variance is the shape parameter, neither of the parameters of the gamma distribution is a pure location or pure shape parameter. Thus, the two parameters that characterize the gamma distribution *jointly* determine its location and shape, because the mean and variance of this distribution are functions of *both* the parameters. In practice, biological phenomena will converge to a steady-state, but nonetheless it is important to check that the technical condition stated for the existence of a steady-state distribution is satisfied by realistic values of the Marmarou model parameters. We obtained typical values of R, E, p_0, p_b and I from a combination of [2] and private communication with Dr. M. Czosnyka, yielding R = 7 mmHgmL^{-1} min (reported values range between 6 and 10 mmHgmL^{-1} min), p_0 = 0 mmHg and p_b = 8 mmHg. Elevated elasticity is reported to be E > 0.18 ml^{-1} and the rate of infusion is I = 1.5 mLmin^{-1} [2]. The value of E was taken to be E = 0.15 mL^{-1}, based on private communication with Dr. Czosnyka, and this value came from data gathered in infusion studies conducted at Addenbrooke's Hospital. p_b is a baseline pressure which is different for each individual patient. Based on the ICP plots in [2], we set p_b = 8 mmHg. This value is close to the average p_b across all infusion studies conducted at Addenbrooke's Hospital which Dr. Czosnyka, in private communication, reported to be 6 mmHg. While the authors solved the deterministic Marmarou model for the general case of $p_0 \neq 0$ mmHg in [2], and found that the average value of p_0 in the infusion studies was $p_0 = 4$ (private communication with Dr. Czosnyka), the non-zero p_0 case is currently not analytically solvable for the stochastic Marmarou model. Our $p_0 = 0$ mmHg assumption is consistent with [5], in which the authors ignore the reference pressure. However, we acknowledge that the non-zero p_0 case is an important issue in mathematical modeling of hydrocephalus—and, in Chapter 4, we develop a quasi-analytical algorithm based on a stochastic exponential transform to solve the stochastic Marmarou model for non-zero p_0. A typical value for σ is difficult to find since the input flow rate of CSF is not accessible to direct observation—only the fluctuations in the ICP are observable. We estimated σ roughly as, $\sigma = 0.33*I=0.33*1.5$ mL/min = 0.5 mL/min, so that the flow fluctuations are 33% (1/3) of the flow rate. This is a rough estimate—the choice of a typical value for σ is unclear. Because of the uncertainty and approximations involved in the estimate, we did the computations in which σ was fixed, not just at $\sigma = 0.5$ mL/min, but also at values of σ lower as well as higher than 0.5 mL/min in order to check the robustness of the conclusions. We have reported the results for $\sigma = 0.5$ mL/min and for $\sigma = 0.8$ mL/min. The results of the computations of the probability as a function of R are robust across a wide range of σ, so that, even though the estimate of σ is only approximate, we can be reasonably confident about the conclusions of the analysis. To examine the influence of σ itself on the risk probability, we computed the probability across a wide range of σ as shown in Figure 2.4 in subsection 2.4.6—again, in an effort to reduce the impact of our imprecise knowledge of σ upon the findings..

While our estimate of σ is only approximate, we note that there is imprecision and uncertainty about *all* the parameter estimates, especially that of the CSF flow resistance 'R.' Estimation methods specifically developed for dynamic models are needed. In this chapter, the primary objectives were to introduce SDE methodology to CSF research, demonstrate its analytical power, and show its clinical usefulness in dynamic risk management. Consequently, we used existing typical estimates of the model parameters even though some of them are imprecise and approximate. With σ = 0.5 mL/min, the condition for the existence of a steady-state probability distribution is met with ease.

Our next three results are motivated by the following considerations. A larger cerebrospinal fluid resistance R tends to increase ICP by increasing the pressure due to the circulatory CSF component. This is a direct consequence of Davson's equation [6]: ICP_{CSF} = (resistance to CSF outflow) x (CSF formation) + (pressure in sagittal sinus). This naturally leads to the following questions. How will the intensity of the fluctuations influence the relationship between resistance and ICP? The same relationship may hold on average, but, as anticipated in the solution to the stochastic Marmarou model, it may be moderated by the noise intensity parameter because of the nonlinearity of the ICP process. How will the intensity of fluctuations affect the average steady-state ICP—is the average steady-state ICP smaller or larger when the intensity of fluctuations increases? Finally, will the intensity of fluctuations attenuate or amplify the effect of resistance to CSF flow on the average steady-state ICP?

2.4.4 Relating average ICP to CSF flow resistance

The average steady-state ICP, denoted by μ, increases with the cerebrospinal fluid resistance R—thus the relationship between R and ICP holds on average.

Discussion

The steady-state probability distribution of ICP is gamma with the parameters shown in the previous subsection. From well-known properties of the gamma distribution, it follows that the steady-state mean ICP level μ is given by: $\mu = (RI + p_b) - \frac{RE\sigma^2}{2}$. Therefore, $\frac{\partial\mu}{\partial R} = I - \frac{\sigma^2 E}{2}$. From the expression for $\frac{\partial\mu}{\partial R}$, it is clear that the average ICP level does indeed increase with R, provided that $\frac{\sigma^2 E}{2} < I$. This condition is satisfied, using the values of the parameters in the previous subsection. Thus, the increasing relationship between the actual ICP level and the cerebrospinal fluid resistance, predicted by Davson’s equation when ICP is conceptualized as a deterministic process, also holds *on average* at steady-state when ICP is modeled as a stochastic process.

2.4.5 Relating average ICP to noise intensity

The average steady-state ICP level, decreases with the intensity of fluctuations, measured by the infinitesimal variance parameter σ^2.

Discussion

From the relationship derived in the previous subsection, $\mu = (RI + p_b) - \frac{R\sigma^2 E}{2}$, it is clear that μ decreases as σ^2 increases. A larger noise intensity corresponds to greater variation in the CSF input flow rate which translates into greater variation in ICP, and these larger fluctuations could cause the average ICP level to increase, decrease or remain unaffected. The nonlinear influence of the parameters of CSF flow dynamics on ICP level turns out to reduce the average ICP value when the fluctuations in ICP are greater. This is an outcome that one would expect to find when steady-state has been achieved—when the transition probabilities have settled down to constant levels so that the probability distribution of ICP is no longer changing over time. This mathematical finding could be tested by separating a random sample of patients into two groups, such that one group has more variability in its ICP levels (due to higher variability in its CSF input flow rate) than the other group, and then conducting a statistical test of significance—such as a t-test—on the difference in mean ICP levels in these two groups at steady-state.

2.4.6 Effect of noise intensity on ICP-R relationship

The resistance increases the ICP on average by a smaller amount when the intensity of fluctuations is higher.

Discussion

From $\mu = (RI + p_b) - \frac{R\sigma^2 E}{2}$, it is clear that a higher σ^2 will dampen the effect of the cerebrospinal resistance on the average steady-state ICP level. This is an outcome that one would expect to find at steady-state. The mathematical finding could be tested by separating a random sample of patients into two groups, such that one group has more variability in its ICP levels than the other group (due to higher variability in its CSF input flow rate), and then correlating the mean ICP level with the cerebrospinal resistance in each group at steady-state. According to the mathematics, the correlation should be smaller in the group with more variable ICP. Given the linear relationship between the steady-state mean and the cerebrospinal resistance, a simple correlation coefficient such as the Pearson product moment should suffice.

Next we turn our attention to dynamic management of the patient's risk. Risk may be quantified in terms of the probability of the onset of some critical event, say the ICP exceeding a dangerously high level. Given the current value of the patient's ICP, what is the probability that it will exceed a high level? Such a probability is intrinsically dynamic because it depends upon the patient's current condition (their current ICP), the dynamics of the patient's CSF flow and the noise intensity σ^2. We want to understand how the probability is influenced by important clinical characteristics of the patient such as their resistance to CSF flow, and by the noise intensity.

2.4.7 Transition probabilities as solutions of boundary value problems

Given that the current ICP is x mmHg, where $0 \le x \le b$, let u(x) denote the probability of reaching the level b. Then u(x) satisfies the following nonlinear differential equation, which must be solved subject to the two conditions on u(x) at x = 0 and at x = b:

$$\frac{du}{dx}\{EIx - \frac{Ex(x-p_b)}{R}\} + \frac{d^2u}{dx^2}\frac{\sigma^2E^2x^2}{2} = 0 \qquad (2.15)$$
$$u(0)=0,\ u(b)=1$$

The conditions on u(x) at the two corners x = 0 and x = b make this a two-point boundary value problem. The solution is given in terms of the scaling function S(x):

$$u(x) = \frac{S(x)-S(0)}{S(b)-S(0)}\text{, where } S(x) = \int^{x} s(\eta)\,d\eta\text{, and}$$

$$s(x) = \exp\left[-\int^{x}\frac{2\left[EI\xi - \frac{E\xi(\xi-p_b)}{R}\right]}{\sigma^2E^2\xi^2}d\xi\right] \qquad (2.16)$$

The integrals defining s(x) and S(x) are indefinite at the lower end because the final answer is unaffected by its choice. For our clinical applications, it is natural to take the lower end point to be zero.

Proof

Consider the stochastic differential equation dX = μ(X) dt + σ(X) dW. Given that the current value of X(t) is x, where 0≤x ≤ b, let u(x) denote the probability of reaching the level b. It is shown in [31] that u(x) satisfies the nonlinear ordinary differential equation: Consider the stochastic differential equation dX = μ(X) dt + σ(X) dW. Given that the current value of X(t) is x, where 0≤x ≤ b, let u(x) denote the probability of reaching the level b. It is shown in [31] that u(x) satisfies the nonlinear ordinary differential equation:

$$\frac{du}{dx}\mu(x) + \frac{d^2u}{dx^2}\frac{\sigma^2(x)}{2} = 0 \qquad (2.17)$$
$$u(0)=0,\ u(b)=1$$

Therefore, in the context of this paper, u(x) satisfies the following ODE,

$$\frac{du}{dx}\{EIx - \frac{Ex(x-p_b)}{R}\} + \frac{d^2u}{dx^2}\frac{\sigma^2E^2x^2}{2} = 0 \qquad (2.18)$$
$$u(0)=0,\ u(b)=1$$

Using Green's function methods, the solution to the above boundary value problem is given in terms of an important quantity called the scaling function S(x) [31]:

$$u(x) = \frac{S(x)-S(0)}{S(b)-S(0)}\text{, where } S(x) = \int^{x} s(\eta)\,d\eta\text{, and}$$

$$s(x) = \exp\left[-\int^{x}\frac{2\mu(\xi)}{\sigma^2(\xi)}d\xi\right] \qquad (2.19)$$

Identification of the parameters with those of the stochastic Marmarou model, $\mu(p) = \left(\frac{E}{R}\right)p\left(RI + p_b - p\right),\ \sigma^2(p) = \left(\sigma Ep\right)^2$ immediately yields the claimed result.

Discussion

While the above representation is, in principle, a closed-form analytical solution, it is, in practice, a *quasi-analytical* solution because the integral that defines s(x) cannot be obtained in closed-form. However, that is no limitation because we can integrate it numerically after substituting the empirically established values of the parameters. We used the parameter values shown in the subsection "Steady-state Probability Distribution of ICP." We took the critical level 'b' to be 40 mmHg, based on the clinical finding reported in [2] that patients were able to tolerate increases in ICP up to 40-50 mmHg. Our rationale was that, from a clinical perspective, a conservative approach to patient management would be consistent with assessing the probability of reaching the lower end point of the 40–50 mmHg range that patients are able to tolerate. Thus our critical event is defined as “reaching an ICP of 40 mmHg.” In order to understand how the probability is influenced by the noise intensity parameter σ, we computed the probability over a range of σ = 0.4 mL/min to σ =1.3 mL/min. For each value of σ in this range, we solved the boundary-value problem to find the probability of reaching 40 mmHg. Furthermore, in order to understand the influence of the patient’s initial condition upon the probability of the critical event, we repeated this set of computations for three different starting levels of ICP; the curve shown in Figure 2.4 is for a starting level of ICP of 35 mmHg.

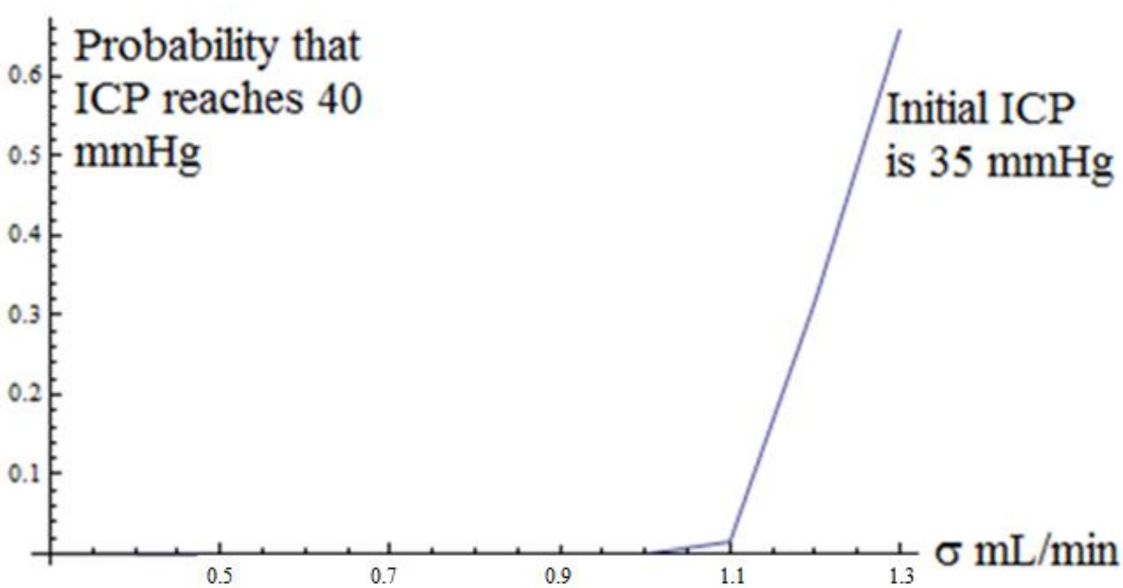

Figure 2.4 Probability that ICP reaches 40 mmHg as a function of noise intensity parameter σ.

In order to understand how the probability is influenced by the resistance to CSF outflow R, we computed the probability over a range of R = 4 $mmHgmL^{-1}$ min to R =12 $mmHgmL^{-1}$ min. For each value of R in this range, we solved the boundary-value problem to find the probability of reaching 40 mmHg. Again, we repeated this set of computations for three different starting levels of ICP; the curve shown in Figure 2.5 is for a starting level of ICP of 35 mmHg. Across the three initial levels of ICP, the curves have a similar shape and are simply translated vertically.

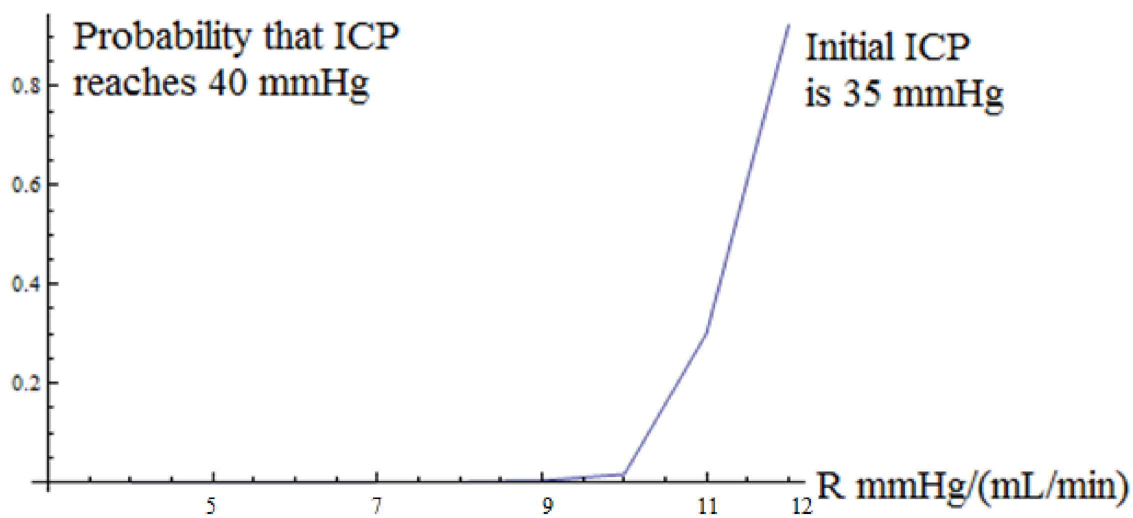

Figure 2.5 Probability that ICP reaches 40 mmHg as a function of resistance to CSF flow parameter R.

Figures 2.4 and 2.5 show that the probabilities increase at an increasing rate (convex functions). Furthermore, they tell an interesting neurological story—namely, that the probabilities of the critical events exhibit strong *threshold* effects. In Figure 2.4, below a critical level of noise intensity, the probabilities are very low—almost zero—but beyond a threshold value of σ = 1.1 mL/min in Figure 2.4, they rise steeply. In Figure 2.5, as R (mmHgmL^{-1}min) varies from 4 to below 10, the probabilities are almost zero, but beyond R = 10 mmHgmL^{-1} min, they rise dramatically. Furthermore, at low levels of noise intensity, the probabilities remain close to zero *throughout* the range of R (mmHgmL^{-1}min) from 4 to 12. But as σ increases to 0.8 mL/min—the value assumed for it in Figure 2.5—R has a strong effect on the probability beyond the critical threshold of 10 mmHgmL^{-1}min. The clinical significance of these findings is that erratic fluctuations in ICP (caused by a larger input flow rate noise intensity σ) will significantly increase the patient's risk, as measured by the probability of the critical event. Because the risk increases rapidly beyond the threshold value of σ, these results suggest that an essential component of risk management is to carefully minimize erratic fluctuations in the patient's CSF input flow rate at all times. Finally, Figure 6 shows the probability of the critical event as a function of both R and σ in a three-dimensional plot, starting at an ICP level of 35 mmHg.

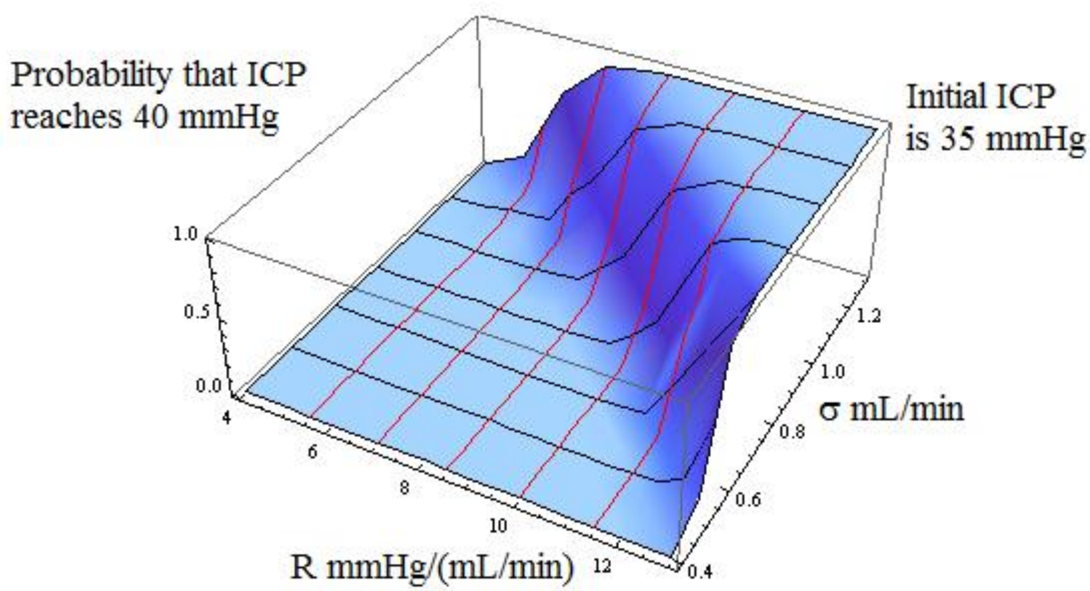

Figure 2.6 Probability that ICP reaches 40 mmHg as a function of resistance to CSF flow parameter R, and noise intensity parameter σ.

The two-dimensional surface shows the value of the probability for each combination of values of R and σ. To facilitate interpretation of the surface, we used a mesh in which the dark lines are the probability plots as a function of R and the red lines are the probability plots as a function of σ. Figure 6 shows very clearly that threshold effects are sensitive to both R and σ, and beyond the threshold, the probabilities asymptotically approach one.

2.5 Conclusions

The stochastic generalization of the Marmarou model offers a tractable analytical description of the noisy ICP dynamics and yields insights into the impact of noise. The SDE offers a rigorous analytical framework to study issues of clinical interest and neurological significance such as the patient's risk. A key clinical implication is that fluctuations in the CSF formation rate—which increase the fluctuations in ICP— should be minimized to lower the patient's risk. The stochastic differential equation framework, in conjunction with nonlinear control theory, can be used to develop a nonlinear automatic controller to regulate shunts to facilitate continuous CSF drainage.

CHAPTER 3

REFERENCE PRESSURE IN CEREBROSPINAL FLUID DYNAMICS

Mathematical models of pressure-volume compensation have played a central role in hydrocephalus research because they enhance understanding of the circulatory dynamics of cerebrospinal fluid (CSF), thereby improving treatment of hydrocephalus, a condition caused by excessive accumulation of CSF in the brain. The classic model in mathematical research on hydrocephalus was described and generalized in Chapter 2 to accommodate noise in the dynamics of CSF flow. The generalized model results in a nonlinear stochastic differential equation (SDE). In this chapter, we will use the generalized model to partially resolve an ongoing controversy over the appropriate value of the reference pressure parameter in the classic model.

3.1 Background

Intracranial dynamics are driven by the circulation of CSF, and the circulatory dynamics of CSF, in conjunction with cerebral blood, results in intracranial pressure (ICP). Cranial contents are operationalized by the volume of CSF, and mathematical pressure-volume models of cerebrospinal fluid relate CSF volume to ICP. In the literature on mathematical pressure-volume models, the Marmarou model [32, 53] remains the classic, and is given by the nonlinear ordinary differential equation (ODE):

$$\frac{1}{E(p - p_0)}\frac{dp}{dt} + \frac{p - p_b}{R} = I(t) \tag{3.1}$$

The parameter p_0 refers to the reference pressure about which substantial controversy exists in the literature.

3.1.1 The reference pressure controversy

The Marmarou [32, 53] ODE shown in equation (3.1) relates the intracranial pressure p(t) to the infusion rate I(t). The coefficient E is called the cerebral elasticity (or elastance coefficient) and has the units ml^{-1} [54], p_b is a baseline pressure (units mmHg), p_0 is the reference pressure (units mmHg) parameter, often taken to be zero because, as noted in [54], the significance of p_0 is unclear. Marmarou [32] himself implicitly set the reference pressure parameter p_0 to zero by simply ignoring it in his model. Under that assumption, equation (3.1) has the structure of a homogeneous logistic differential equation, described by the ODE:

$$\frac{dx}{dt} = (ax^2 + bx) \, , \tag{3.2}$$

The Marmarou model performs well in practice, continues to be extensively used and is now considered a classic in the mathematical modeling of cerebrospinal fluid (CSF) dynamics [57]. However, there is controversy over the reference pressure parameter p_0, upon whose value disagreement prevails among different reseachers. While Marmarou [32] himself took the reference pressure to be zero, non-zero values of reference pressure have in fact been used by other reseachers. For example, Czosnyka et al. [54] take the reference pressure to be the pressure inside the dural sinuses. Furthermore, in infusion studies conducted at Addenbrooke's Hospital at Cambridge, UK, the average value of p_0 in the infusion studies is reported to be p_0 = 4 (private communication with Dr. Czosnyka). Wirth and Sobey [56] also argue that the reference pressure should be non-zero. Indeed, based on their model for cerebral compliance, Wirth and Sobey [56] suggest a very definite value for the reference pressure. Under their reasoning, Wirth and Sobey [56] suggest that the reference pressure should be neither zero nor the pressure inside the dural sinuses but instead should be determined by the exact mathematical relationship:

$$p_0 = p_s + \Delta p_s - K \tag{3.3}$$

In equation (3), p_s is the dural sinus pressure, Δp_s is the pressure drop through the bridging veins, and K is the elastic modulus of a collapsed vessel. According to Wirth and Sobey's [56] reasoning, typical values of the parameters are: $p_s \sim 5$ mm Hg, , $\Delta p_s \sim 1$ mm Hg, and $K \sim 0.5$ to 1 mm Hg, therefore the reference pressure p_0 would range from values of 5.5 to 5 mm Hg. While Wirth and Sobey [56] disagree with Czosnyka et al. [54] in that they urge that the reference pressure should *not* be set equal to the sagittal sinus pressure, they acknowledge that certain values of Δp_s and K could produce a value for p_0 roughly equal to the sagittal pressure p_s. Thus we are left with the following possibilities for the reference pressure p_0: $p_0 = 0$, $p_0 = p_s$, and p_0 is determined by equation (3.3), which, for some values of of Δp_s and K, could yield $p_0 \sim$ ps. In either of the latter two cases, the fundamental point is that $p_0 \neq 0$.

3.1.2 An approach to resolving the controversy

We offer a possible resolution of this controversy through a mathematical generalization of the classic Marmarou [32] model. The generalization [30, 57] is motivated by the intrinsic fluctuations in intracranial (ICP) level around the deterministic path predicted by the Marmarou [32] model—these fluctuations are due to noise arising from various sources such as the measurement device, breathing and movement of the patient, to the approximations and idealizations made in developing the Marmarou [32] model, and to other factors that affect ICP that have been omitted from Marmarou's [32] deterministic model. As established in Chapter 2, one way of accounting for the effect of the noise in observed ICP measurements in pressure-volume compensation studies is to model the evolution of ICP as a stochastic differential equation, hereafter abbreviated SDE [30, 57]. Our strategy is based on the mathematical finding that the two cases $p_0 = 0$ and $p_0 \neq 0$ have very different implications for the nature of the solutions of the SDE, and in particular, that in the latter case we cannot mathematically rule out, with probability one, the possibility of unbounded values for the state variable.

3.1.3 The inhomogeneous stochastic logistic model

SDEs play an important role in electrical engineering and the neurosciences because they arise in a natural way as models of dynamic phenomena influenced by fluctuations. As Karlin and Taylor [8] note, they enjoy a great advantage over discrete models in that the continuous time formulation of SDEs frequently permits explicit answers to substantively important questions, even though such answers would be inaccessible in the discrete formulation of the same problem. It will be shown in this chapter that the SDE formulation suggests a novel resolution of the reference pressure controversy in cerebrospinal fluid dynamics research.

We consider a class of stochastic differential equations that are pervasive in neuroscience and engineering. This class of equations encompasses a large number of phenomena. To free the rest of the exposition from intrusive detail, we will henceforth focus on that part of the mathematical *structure* of the Marmarou model that is most relevant to settling the controversy about whether $p_0 = 0$ or $p_0 \neq 0$.

As noted earlier, under the hypothesis that $p_0 = 0$, the deterministic Marmarou model has the structure of equation (3.2), which is the homogeneous logistic equation. The first step is to extend equation (3.2) to make it stochastic. Considered the natural stochastic extension of the logistic model, the stochastic logistic model has been successfully used to model the noise in the degradation dynamics of progressive breakdown in ultrathin films in metal-oxide-semiconductor (MOS) capacitors [39]. This results in the following structure for the stochastic Marmarou model, which was analyzed in Chapter 2:

$$dX = (aX^2 + bX)dt + \sigma X dW \tag{3.4}$$

Under the hypothesis that $p_0 \neq 0$, the stochastic Marmarou model has the following structure:

$$dX = (aX^2 + bX + c)dt + \sigma X dW \tag{3.5}$$

In both equations (3.4) and (3.5), a, b and c are constants. The drift term is called the Marmarou model, considered the fundamental model in the study of the brain physics of cerebrospinal fluid dynamics [32, 53]. The infinitesimal variance structure extends the deterministic Marmarou model to take fluctuations in fluid flow into account. Mathematical modelers in neuroscience assume c = 0, so that the drift term in the Marmarou model is homogeneous, but this decision is grounded in the currently incomplete understanding of CSF flow dynamics rather than upon a clear scientific rationale. Our analytical results show that there are in fact strong mathematical reasons to support homogeneity of the drift term, thereby replacing an ad hoc decision by rational justification. The parameter c distinguishes between the competing hypotheses $p_0 = 0$ and $p_0 \neq 0$, because $c = -(I + p_b/R)Ep_0/R$; consequently the hypothesis $p_0 = 0$ implies that $c = 0$, and the hypothesis $p_0 \neq 0$ implies that $c \neq 0$. In the definition of c, p_b is a baseline pressure, R is the resistance to cerebrospinal fluid flow, and E is the cerebral elasticity. The class of phenomena described by Equation (5) includes, as special cases, models of many important phenomena in neuroscience and electrical engineering. Clearly, on both theoretical and practical grounds, these facts warrant a rigorous study of this class of stochastic differential equations.

If equation (3.4) is the correct description of CSF dynamics, then the hypothesis $p_0 = 0$ is supported; if equation (3.5) is the correct description of CSF dynamics, then the hypothesis $p_0 \neq 0$ is supported. To decide between these competing descriptions, we derive the conditions under which the existence of positive and non-explosive solutions to these SDEs is guaranteed. The logic is that a valid description of the evolution of ICP—which reflects CSF dynamics—must *minimally* generate predictions that are positive and do not grow unboundedly large. The question regarding the model that provides a better statistical fit to the ICP data is important but not considered here—for, regardless of the statistical fit, however measured, a model that does not guarantee *positive* and *finite* values for ICP must be viewed with caution from a theoretical perspective.

3.2 Analysis of the inhomogeneous stochastic logistic model

The solution of nonlinear SDEs raises daunting mathematical obstacles and most nonlinear SDEs defy closed-form analytical representations of their solutions. Nonlinear SDEs that satisfy the reducibility criterion [58] can be solved through Ito's lemma after discovering a suitable transformation of the state variable. The class of SDEs with quadratic drift and infinitesimal standard deviation proportional to the state variable is completely solvable in closed-form for the special case in which $c = 0$ so that the drift is homogeneous—this was done in Chapter 2. Standard text-book solutions for the homogeneous case are available, for example, in Oksendal [10] and Gard [58]. The inhomogeneous case $c = 0$ is challenging, because the SDE fails the reducibility criterion, as shown in Gard [58].

Because there is no explicit closed-form solution for SDEs with inhomogeneous quadratic drift and infinitesimal standard deviation proportional to the state variable, the existence question becomes imperative: under what conditions do solutions to these systems exist? Will a solution, when it exists, be unique so that we may rest assured that it is the only solution? Two additional issues of practical consequence and engineering relevance remain. Positive solutions are frequently desirable since the state variable in many practical applications has no meaning if it is negative. And finally, the engineering importance of solutions that remain finite with probability one is obvious. Thus we ask: will the solutions be positive and non-explosive? The rest of the chapter is devoted to answering these questions.

Before embarking on the mathematical details needed to answer these questions, a simple example will illustrate the subtlety of existence, positivity and non-explosion issues for this class of SDEs, so pervasive in applications. Consider the special case in this class defined by a = c = 0 so that the stochastic system is:

$$dX_t = bX_t dt + \sigma X_t dW_t \tag{3.6}$$

Even a tiny change of the coefficient ‘b’ in the above system will result in remarkably different sample path behavior. In the above system, if, $0 < b < \sigma^2/2$ then $X_t \to 0$ as $t \to \infty$; but changing the coefficient ‘b’ to ‘$b + \sigma^2/2$’ forces $X_t \to \infty$ as $t \to \infty$ [59]. If we choose σ to be an arbitrarily small quantity t , it is clear that, even though the ‘b’ coefficients for two processes differ by an arbitrarily small amount, the asymptotic behavior of those two processes will be strikingly different from each other, forbidding explosions in one case while permitting them in the other. The somewhat elaborate mathematical machinery that follows reflects the delicateness of these issues for the yet broader class of SDEs that contains this simple example as a special case.

3.2.1 Existence, positivity and non-explosion

Let (Ω, $\mathcal{F}$, P) be a complete probability space on which the following are defined:

(i)$W = \{W_t : 0 \le t \le T\}$, a standard one-dimensional Wiener process, and

(ii)ξ, a random variable

We assume that ξ is independent of the Wiener process W. For each t, define the σ -field $\mathcal{F}_t = \sigma\{\xi_s, W_s, 0 \le s \le t\} \vee$ (all P −null sets in F). It is clear that the filtration ($\mathcal{F}_t$) satisfies the usual conditions, and W is an $\mathcal{F}_t$ -adapted Wiener process.

Let $a(x) = ax^2 + bx + c$ where $a < 0$, $b > 0$,and $c \geq 0$. Let $\sigma(x) = \sigma x$ where $\sigma > 0$. Consider the stochastic differential equation:

$$X_t = \xi + \int_0^t a(X_s)ds + \int_0^t \sigma X_s dW_s \tag{3.7}$$

where $\xi \geq 0$ with $\xi \in L^4(P)$. The stochastic integral on the right side of (3.7) is taken in the sense of Itô.

Definition 3.1. A process $X = \{X_t\}$, $t \in [0,T]$, defined on (Ω, $\mathcal{F}$, P) is called a *strong solution* of the stochastic differential equation (3.1) if the following assertions hold:

1. X is $\mathcal{F}_t$ -adapted with continuous sample paths.
2. $\int_0^T (|a(X_t)| + |\sigma X_t|^2)\ dt < \infty$ a.s.
3. For each $t \in [0,T]$, $X_t = \xi + \int_0^t a(X_s)ds + \int_0^t \sigma X_s dW_s$ almost surely

Definition 3.2. The stochastic differential equation (3.1) with initial condition $X_0 = \xi$ has a unique strong solution if for any two strong solutions $X = \{X_t\}$ and $Y = \{Y_t\}$ on (Ω, $\mathcal{F}$, P), one has $P\{\omega : X_t(\omega) = Y_t(\omega) \forall t \in [0,T]\} = 1$.

The notion of uniqueness given above is called *strong uniqueness* of solutions which is especially attractive from an engineering perspective since it ties the solution concept directly back to applications and–where necessary–simulations. In this section, we will show the existence, uniqueness, non-negativity, and non-explosion of strong solutions to the SDE (3.1). Our proof is essentially self-contained, and is presented in several steps.

STEP 1: Define the functions

$$a_n(x) = a((x \wedge n) \vee (-n))$$

$$\sigma_n(x) = \sigma((x \wedge n) \vee (-n))$$

for any fixed integer n > 0, where the notation $u \wedge v$ stands for min{u, v} and $u \wedge v$ = max{u, v}. Consider the associated SDE given by

$$X_t^{(n)} = \xi_n + \int_0^t a_n(X_s^{(n)})\,ds + \int_0^t \sigma_n(X_s^{(n)})\,dW_s \tag{3.8}$$

where $\xi_n = (\xi \wedge n) \vee (-n)$. The coefficients in (3.2) are bounded (for any large fixed n) and Lipschitz-continuous. Therefore, by the standard theory of SDEs, there exists a unique strong solution of the equation (3.2).

Define $\tau_n =$ inf $\{t \geq 0 : |X_t^n|$ > n }, and $Y_t = X_{t \wedge \tau_n}^n$. Though the Y_t process depends on n, we have not displayed it to simplify the notation. It is clear that Y_t solves the equation (3.1) till time τ_n.

STEP 2: We next claim that, that $Y_t \geq 0$ almost surely for all t. To prove this, define

$$\alpha(x) = \begin{cases} a_n(x) & \text{if } x \geq 0 \\ a_n(0) & \text{if } x < 0 \end{cases}$$

Likewise, define

$$\beta(x)=\begin{cases}\sigma_n(x) & \text{if } x\geq 0\\ \sigma_n(0) & \text{if } x<0\end{cases}$$

Let Z_t be the solution of the equation

$$Z_t=\xi_n+\int_0^t \alpha(Z_s)ds+\int_0^t \beta(Z_s)dW_{s.}$$

Fix any ε > 0. Define

$S_1=$ inf $\{t : Z_t <$ -ε $\}$ and $S_2=$ inf $\{t\geq S_1 : Z_t=0\}$.

For all ω such that $S_1(\omega) < \infty$, it follows that for any $t \in (S_1(\omega), S_2(\omega))$,

$$Z_t=Z_{S_1}+\int_{S_1}^t \alpha(Z_s)ds+\int_{S_1}^t \beta(Z_s)dW_s$$

$$= -\varepsilon + c(s-S_1)$$

$$\geq -\varepsilon.$$

This contradicts the definition of infimum in the definition of the stopping time S_1.

Since ε is arbitrary, $Z_t \geq 0$ a.s. for all t. By the continuity of paths of the solution, Z_t is non-negative valued for all t almost surely. Therefore, the processes $\{Z_t\}$ and $\{Y_t\}$ coincide.

STEP 3: Let us consider the solution Y_t over the time interval $[0,T]$ for any fixed, finite time T. Since

$$EY_t \leq E\xi + b\int_0^t EY_s ds + ct$$

it follows that

$$EY_t \leq (E\xi + cT)e^{bt}$$

by the Gronwall Lemma.

Likewise,

$$E(Y_t^{\,2}) \le E(\xi^2) + 2b\int_0^t E(Y^2)ds + 2c\int_0^t EY_s ds + \sigma^2\int_0^t E(Y_s^{\,2})ds$$

By the Gronwall inequality, for all $0 \le t \le T$,

$$E(Y_t^{\,2}) \le K_T e^{(2b+\sigma^2)t}$$

where $K_T = E(\xi^2) + 2c\left\{E\xi + cT\dfrac{e^{bT}}{b}\right\}$. One can also bound $E(Y_t^{\,4})$ in a similar manner by using the Itô Lemma and the estimate (3.4). This is possible since we have assumed that $E(\xi^4) < \infty$.

STEP 4: We will now estimate $E\left[\sup_{0\le t\le T} Y_t^{\,2}\right]$. By the Itô Lemma,

$$Y_t^{\,2} = \xi^2 + 2\int_0^t Y_s a_n(Y_s)ds + 2\int_0^t Y_s\sigma_n(Y_s)dW_s + \int_0^t \sigma^2(Y_s)ds$$

$$\le \xi^2 + 2\int_0^t (bY_s^{\,2} + cY_s)ds + 2\int_0^t \sigma(Y_s \wedge n)^2 dW_s + \sigma^2\int_0^t Y_s^{\,2} ds$$

$$= \xi^2 + (2b+\sigma^2)\int_0^t Y_s^{\,2} ds + 2c\int_0^t Y_s ds + 2\sigma + \int_0^t (Y_s \wedge n)^2 dWs$$

Therefore,

$$\sup_{0\le t\le u}(Y_t^{\,2}+1) \le (1+\xi^2) + (2b+\sigma^2+2c)\int_0^u \sup_{0\le r\le s}(Y_r^{\,2}+1)ds$$

$$+\ 2\sigma\sup_{0\le t\le u}\left|\int_0^t (Y_n \wedge n)^2 dW_s\right|$$

Upon taking expectation, and setting $A_u := \sup_{0\le t\le u}(Y_t^{\,2}+1)$,

$$E(A_u) \le E(1+\xi^2) + (2b+\sigma^2+2c)\int_0^u E(A_s)ds + 2\sigma\sqrt{2}E\left(\int_0^T Y_s^{\,4} ds\right)^{1/2}$$

by the Burkholder-Davis-Gundy inequality. Using the bound for $E(Y_s^4)$, the last term on the right side can be bounded by a constant K. By the Gronwall inequality,

$$EA_T \le E(\xi^2 + K + 1)e^{(2b+\sigma^2+2c)T}$$

so that $E(\sup_{0\le s\le T} Y_s^2)$ is finite. Let us call it as C.

STEP 5: Consider

$$P\{\tau_n < T\} = P\left\{\sup_{0\le s\le T} Y_s \ge n\right\}$$

$$= \frac{1}{n^2} E(\sup_{0\le s\le T} Y_s^2) = \frac{C}{n^2}$$

Hence $\sum_n P\{\tau_n < T\} < \infty$, and by the Borel-Cantelli Lemma, we infer that $P\{\tau_n < T$ infinitely often $\}$ = 0. Therefore, $\tau_n \ge T$ eventually with probability one. If we denote the limit of τ_n by τ_∞, then

$$P\{T_\infty \ge T\} = 1.$$

Since T is arbitrary, we can conclude that $T_\infty = \infty$ a.s. Therefore, the solution doesn't explode, and we are guaranteed the existence of a unique, non-negative solution without explosion.

3.3 Implications for the Reference Pressure

3.3.1 Constraints of existence, positivity and non-explosion

The proof of existence depends upon the Lipschitz growth conditions, in which the drift and infinitesimal standard deviation enter as differences between distinct points in the state space. Since the inhomogeneous parameter 'c' drops out in forming the difference, it can have any value-positive or negative-without compromising existence. Because the growth conditions involve absolute values, the parameters a, b, and σ can have any values—subject to the obvious requirement that $\sigma > 0$ by definition-without jeopardizing existence. The proof by contradiction argument to establish positivity of the process depends upon the assumption that $c \geq 0$. Thus, if $c < 0$, the process value could become negative. This may be legitimate for some applications-for example if the state variable denotes the voltage in a circuit-but inappropriate for problems in which only positive values of the state variable are meaningful, such as, for example, if the state variable denotes the power in an electrical circuit, or the size of a population. In establishing non-explosion, we capitalized on the Gronwall inequality by showing that the first and second moments of the process are bounded above at every point in time, and that required the condition $a < 0$. Thus, for $a > 0$, the process could explode.

3.3.2 The case for zero reference pressure

Our analysis has provocative implications for settling the controversy regarding the reference pressure parameter p_0 in the celebrated Marmarou [32] model. That model is a nonlinear differential equation based on physical analogies between cerebrospinal fluid dynamics and an electrical circuit, and the drift term in the stochastic extension of the Marmarou model has the form $ax^2 + bx + c$. As discussed earlier, the parameter c is taken to be zero by many brain physics and neuroscience researchers, including Marmarou himself. However, the reason that c is assumed to be zero is because it is proportional to the reference pressure parameter p_0 which is “a parameter of uncertain significance”[29].

On the grounds that the clinical and neurological meaning of p0 are presently unclear, researchers set $p_0 = 0$ which results in $c = 0$. However, is it not somewhat capricious to set $p_0 = 0$ merely because its clinical and neurological significance are not currently well-understood? Would better understanding in the future then decree a non-zero value for p_0?

Our analysis substitutes a rigorous mathematical rationale for the arbitrariness inherent in the current approach driven by subjectivity: p_0 should equal zero not because its meaning is unclear, but because a non-zero value for p_0 would create an explosive neurological process. That’s because $c = -(I + p_b/R)Ep_0/R$, and, because all the quantities on the right hand side are positive, $c< 0$. But we proved that explosions can be conclusively ruled out only when $c \geq 0$. And that mathematical fact determines whether p_0 should be zero or non-zero. p_0 should be zero because that is the only way to assure that $c = 0$, thereby forbidding explosions. It is pleasing that mathematical rigor resolves a practical matter so elegantly. Normally, advanced mathematics offers little commentary on practice, but here the practical issue is enlightened and aided by advanced mathematical reasoning. While the mathematical considerations favor a zero value for ’p_0,’ we certainly do not suggest that the mathematics should supplant neuroscientific knowledge generated by experimentation; rather we urge that the mathematical insights should augment neuroscience experiments and guide the research inquiry in the most promising directions. The mathematical logic presents one possible rationale for a zero value for ’p_0,’ based on the theoretical desirability of ruling out unbounded solutions—however, more empirical work is needed to conclusively settle the issue.

3.4 Conclusions

Our research contributes by establishing the precise conditions that will guarantee existence, positivity and non-explosion of the solutions for a class of SDEs that are ubiquitous in neuroscience and engineering. These theoretical issues are not infrequently ignored or relegated to the realm of speculation, often bolstered by the comforting thought that no harm is done as long as an approximate solution works. But here it is established that mathematical rigor is accompanied by unheralded benefits. While our results about global behavior enrich existing theory, we demonstrate that, in this case, the theoretical results also illuminate a pragmatic issue and bestow mathematical rigor upon an ad hoc rule used in a fundamental model in the neurosciences. The results facilitate the engineer's search for insights through simulations for nonlinear control problems in which the drift is quadratic. In Chapter 4, we develop an algorithm to quasi-analytically solve higher order polynomial systems—a result that will permit solving the stochastic Marmarou model for the non-zero p_0 case in future research.

CHAPTER 4

STOCHASTIC DIFFERENTIAL EQUATIONS WITH POLYNOMIAL DRIFT

Signals described by stochastic differential equations in which the drift is a polynomial function of the state variable and the infinitesimal standard deviation is proportional to the state variable are pervasive in electrical engineering and the physical sciences. Most of these equations cannot be solved in closed-form. The innovation of this research is a two-stage algorithm that first transforms the SDE into an ordinary differential equation with time-varying and random coefficients, and then converts that into a differential equation in which every coefficient except one is constant. The contributions are an algorithm that solves polynomial SDEs with linear noise, and that harnesses noise to influence the equilibria and characteristic response time of the system.

4.1 Background

4.1.1 Relationship to general Langevin equation

The general Langevin equation was introduced in equation (1.10) in Chapter 1 of this dissertation. It is reproduced here:

$$dX = f(X,u)dt + \sigma(X,u)\, dW(t) \tag{4.1}$$

Equation (4.1) covers all the cases in this research and most cases of practical interest in engineering, neuroscience and operations research. In the absence of control variables 'u,' the signals considered in this chapter correspond to the specification that f(X) is a polynomial in X and σ(X) is proportional to X:

$$dX = \left(\sum_{i=0}^{i=n} a_i X^i \right) dt + \sigma X \, dW \qquad (4.2)$$

4.1.2 Relevance to cerebrospinal fluid dynamics

The classic Marmarou [32] model that was extended in Chapter 2 to accommodate noise in CSF flow dynamics corresponds to a homogeneous ($a_0 = 0$) quadratic specification of the drift in equation (4.2):

$$dX = (a_2 X^2 + a_1 X) dt + \sigma X dW \qquad (4.3)$$

The non-zero reference pressure case discussed in Chapter 3 corresponds to an inhomogeneous ($a_0 \neq 0$) quadratic specification of the drift in equation (4.2):

$$dX = (a_2 X^2 + a_1 X + a_0) dt + \sigma X dW \qquad (4.4)$$

Thus, the algorithm developed in this chapter can solve the stochastic Marmarou model for the general case of non-zero reference pressure.

4.1.3 Relevance to electrical engineering

Many signals in electrical engineering are described by stochastic differential equations (SDEs) with infinitesimal drift coefficients that are polynomial in the state variable. For example, Rebolledo, Rios, Trigo and Matus [37] show that the current and power in gas discharge lamps are both well-described by a SDE with infinitesimal drift and standard deviation terms proportional to the state—a simple linear example of polynomial systems. The post-breakdown current-time characteristics of constant voltage-stressed metal-oxide-semiconductor (MOS) capacitors with ultra-thin oxides is well-captured by the logistic model [38, 39], whose natural stochastic extension to incorporate the pervasive noise in the current is the stochastic logistic model [39]—a quadratic example of polynomial systems. SDEs with polynomial drift arise in the analysis of sampling mixers [60], as models of noisy signals in electrical circuits, and as special cases of the Langevin equation [35, 10]. SDEs with polynomial drift are solvable in the simplest linear case but remain tractable only in a handful of nonlinear models. Because SDEs with polynomial drift and linear noise are ubiquitous in engineering and science, there is strong interest in solving these systems.

4.2 A quasi analytical algorithm to solve polynomial SDEs

4.2.1 The reducibility condition

The key player in this chapter is equation (4.2). Some models in the class of SDEs represented by equation (4.2) satisfy the reducibility condition [61] which states that, if the signal X(t) satisfies the SDE,

$$dX = \mu(X,t)dt + \sigma(X,t)dW \tag{4.5}$$

and if the coefficients $\mu(.)$ and $\sigma(.)$ obey the following condition,

$$\frac{\partial}{\partial x}\left[\frac{1}{\sigma^2}\frac{\partial\sigma}{\partial t} - \frac{\partial}{\partial x}\left[\frac{\mu}{\sigma}\right] + \frac{1}{2}\frac{\partial^2\sigma}{\partial x^2}\right] = 0 \tag{4.6}$$

then, it may be shown [62], that under a suitable transformation Z(t) = F[X(t)], the transformed signal Z(t) will satisfy an SDE in which the drift and variance terms are independent of Z,

$$dZ = p(t)dt + q(t)dW \tag{4.7}$$

for which the solution leaps out at a glance because both the infinitesimal drift and standard deviation terms are free of ‘X.’

If the majority of SDEs arising in applications respected the reducibility condition, the case for more sophisticated solution methods would be less compelling. But many SDEs of practical interest unabashedly violate the reducibility condition. And the slightest change in the drift or variance term of the SDE can transform a reducible equation into one that staunchly resists reducibility. For example, the stochastic logistic model with homogeneous quadratic drift (equation (4.3)) satisfies the reducibility condition and can be explicitly solved, as shown in Chapter 2. However, the stochastic logistic model with inhomogeneous quadratic drift (equation (4.4)) violates the reducibility condition, turning the quest for its solution into a non-trivial and arduous task, as mentioned in Chapter 3.

Consequently, an algorithm that can handle the irreducible case is needed. And although lower degree polynomials such as linear, quadratic or cubic may suffice for some applications, the methodology should ideally be applicable to polynomials of arbitrarily high degree. The quest for such generality is prompted not by idle mathematical curiosity but by pragmatic sentiments. For a methodology that can solve SDEs containing a polynomial drift with arbitrarily high degrees can be harnessed to solve more general autonomous SDEs with arbitrarily small error. Such a system has the form:

$$dX = f(X)dt + \sigma X dW \tag{4.8}$$

Provided that the infinitesimal standard deviation term is proportional to the state variable, the algorithm developed in this chapter will approximately solve equation (4.8), as will be shown in section 4.3.3.

4.2.2 A stochastic exponential martingale transform

Consider a probability space consisting of a set Ω of all possible outcomes, a sigma algebra of events $\mathfrak{F}$, and a probability measure P defined on $\mathfrak{F}$. As usual, the probability space (Ω, $\mathfrak{F}$, P) is assumed to be complete, so that $\mathfrak{F}$ contains all the P-null events, a technical assumption that is standard in the literature (Wong and Hajek 1985, p. 3). Furthermore, the probability space (Ω, $\mathfrak{F}$, P) is equipped with a filtration $\{\mathfrak{F}_t\}$ defined on the sigma algebra $\mathfrak{F}$. A filtration $\mathfrak{F}_t$ is a σ field of events satisfying the condition that $t \geq s$ implies that $\mathfrak{F}_t \supseteq \mathfrak{F}_s$, a definition that intuitively captures the notion of increasing information patterns over time [64]. A process X_t is defined to be a martingale with respect to the filtration $\mathfrak{F}_t$ if it satisfies the property, $E[X_t | \mathfrak{F}_s] = X_s$, for any $s < t$.

Let W(t) be a standard Brownian Motion process. Then the process $Y(t) = \exp\left[\sigma W(t) - \frac{\sigma^2 t}{2}\right]$ is a martingale [10]. Consequently, the process $Y(t) = \exp\left[-\sigma W(t) + \frac{\sigma^2 t}{2}\right]$ is also a martingale. This latter process is called a stochastic exponential martingale and will be a key player in the development of our algorithm.

4.2.3 Statement of the algorithm

The central problem is to solve the SDE:

$$dX = \left(\sum_{i=0}^{i=n} a_i X^i\right) dt + \sigma X\, dW \qquad (4.9)$$

Because the system is a SDE, its solution is a stochastic process, specifically a Markov process, whose temporal behavior will depend upon the realization of the Brownian Motion process W(t). For each possible realization, the solution can be derived on a path-by-path basis as specified in the algorithm below.

Algorithm to solve SDEs with Polynomial Drift

The first step in the solution strategy is to convert the SDE into a nonlinear ODE in which *every* coefficient *except one* is time-varying and random. The next step transforms the nonlinear ODE with time-varying and random coefficients into a significantly simpler ODE in which *only one* coefficient is time-varying and random. The Euler algorithm then solves the ODE for that specific realization of the process. The following steps implement the algorithm.

(1) Given the signal X(t) described by equation (4.9),), consider the transformed signal $Y(X, W, t) = X\, e^{-\sigma W + \sigma^2 t/2}$.

(2) The signal Y(t) satisfies the ODE

$$\frac{dY}{dt} = a_0 g + a_1 Y + \frac{a_2}{g} Y^2 + \frac{a_3}{g^2} Y^3 + \cdots \frac{a_n}{g^{n-1}} Y^{n-1} \tag{4.10}$$

where $g = e^{-\sigma W + \sigma^2 t/2}$, in which all the coefficients except a_1 are random and time-varying.

(3) Consider a specific realization of the Brownian Motion process W(t), say $W(t_k) = w_k$, for $t = t_k, 0 \le k \le N$. Given $W(t) = w_{k-1}$ for $t_{k-1} \le t < t_k$, the evolution of the system is deterministic over the interval $[t_{k-1}, t_k]$. Consider the deterministic process U(t) associated, through a series of transformations described in the

next section, with the ODE shown in step 2, such that U(t) satisfies

$$\frac{dU}{dt} = a_0 e^{\sigma^2 t} + \left(a_1 + \frac{\sigma^2}{2}\right)U + a_2 U^2 + a_3 U^3 + \ldots + a_{n-1}U^{n-1} + a_n U^n$$

(4.11)

Solve the ODE for the function U(t). This can be done in closed-form for the linear and quadratic case, and numerically for higher-degree systems. Note that the *only* time-varying term in the ODE satisfied by U(t) is the term that is free of U.

(4) Compute the quantity $\exp\left[-\left(a_1 - \sigma + \frac{\sigma^2}{2}\right)w_{k-1}\right]\exp\left[-\left(2a_1 + \sigma^2\right)t\right]$.

(5) Given the specific value $W(t) = w_{k-1}$, the solution to the SDE over the interval $[t_{k-1}, t_k]$ is

$$X_t = \exp\left[-\left(a_1 - \sigma + \frac{\sigma^2}{2}\right)w_{k-1}\right]\exp\left[-\left(2a_1 + \sigma^2\right)t\right]U_t$$

(4.12)

By sampling the process more frequently, the accuracy of the solution can be increased as desired. Between sampling points, the evolution of the system is described by the above solution.

4.2.4 Derivation of the algorithm

The algorithm will be proved in two stages. In the first stage, a stochastic exponential martingale is employed to transform the SDE into an ODE with random and time-varying coefficients. Next that system is reduced to an ODE in which only *one* coefficient is time-varying, over each interval $[t_{k-1}, t_k]$, $1 \le k \le N$, from which the solution to the original system is recovered through a series of simple transformations.

Given the SDE (4.9), consider the function $f(x, w, t) = x\, e^{-\sigma w + \sigma^2 t/2} = x\, g(w, t)$ and let Y_t denote the transformed process $X_t\, e^{-\sigma w + \sigma^2 t/2}$, so that $Y_t = f(X_t, W_t, t)$.

By the Ito Lemma,

$$Y_t = y_0 + \int_0^t f_x(x_s, w_s, s)\, dx_s + \int_0^t f_w(x_s, w_s, s)\, dw_s + \int_0^t f_s(x_s, w_s, s)\, ds + \frac{1}{2}\int_0^t f_{xx}(x_s, w_s, s)\, d < X >_s + \frac{1}{2}\int_0^t f_{ww}(x_s, w_s, s)\, d < W >_s + \int_0^t f_{xw}(x_s, w_s, s)\, d < X, W >_s \tag{4.13}$$

In the above, the processes $<X>_t$,$<W>_t$ and $<X, W>_t$ denote, respectively, the quadratic variation process of X, W and the cross-variation process between X and W, all at time ‘t.’

Then:

$f_x = g$, $f_{xx} = 0$, $f_w = -\sigma f$, $f_{ww} = \sigma^2 f$, $f_{wx} = f_{xw} = -\sigma g$, $f_t = \sigma^2 f/2$

Next, the quadratic and cross variation processes are computed as follows:

$d<X>_t = \sigma^2 X^2\, dt$, $d<W>_t = dt$, $d<X, W>_t = \sigma X_t\, dt$

Using the above computed quantities in the Ito Lemma, it is found:

$$Y_t = y_0 + \int_0^t f_x(x_s, w_s, s)\,dx_s + \int_0^t f_w(x_s, w_s, s)\,dw_s + \int_0^t f_s(x_s, w_s, s)\,ds +$$
$$\frac{1}{2}\int_0^t f_{ww}(x_s, w_s, s)\,ds + \int_0^t f_{xw}(x_s, w_s, s)\,\sigma x_s\,ds$$

(4.14)

The 3rd term of (4.14) is

$$\int_0^t f_s(x_s, w_s, s)\,ds = \int_0^t \frac{\sigma^2}{2} f(x_s, w_{s,} s)\,ds = \frac{\sigma^2}{2}\int_0^t y_s\,ds$$

The 4th term of (4.14) is

$$\int_0^t f_s(x_s, w_s, s)\,ds = \int_0^t \frac{\sigma^2}{2} f(x_s, w_{s,} s)\,ds = \frac{\sigma^2}{2}\int_0^t y_s\,ds$$

The 5th term of (4.14) is

$$\frac{1}{2}\int_0^t f_{ww}(x_s, w_s, s)\,d<W>_s = \frac{\sigma^2}{2}\int_0^t y_s ds$$

The last term of (4.14) is

$$\int_0^t f_{xw}(x_s, w_s, s)\,d<X,W>_s = -\sigma\int_0^t g(w_s, s)\,\sigma x_s ds = -\sigma^2\int_0^t y_s ds$$

Addition of terms 2 through 6 sparks a sequence of spirited cancellations, leaving the delightfully frugal representation:

$$Y_t = y_0 + \int_0^t \left(\sum_{i=0}^{i=n} a_i X^i\right) f_x ds \qquad (4.15)$$

And using $f_x = g$ in the above:

$$Y_t = y_0 + \int_0^t \left(\sum_{i=0}^{i=n} a_i X^i\right) g ds \qquad (4.16)$$

Therefore

$$Y_t = y_0 + \int_0^t g\left\{a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n\right\} ds = y_0 + \int_0^t \left(a_0 g + a_1 g X + \cdots + a_n g X^n\right) ds$$

(4.17)

Next, $Y_t = gX_t \Rightarrow X_t = \frac{Y_t}{g}$, and therefore

$$Y_t = y_0 + \int_0^t \left(a_0 g + a_1 Y + \frac{a_2}{g} Y^2 + \frac{a_3}{g^2} Y^3 + \cdots \frac{a_n}{g^{n-1}} Y^{n-1}\right) ds$$

(4.18)

It follows that the process Y_t satisfies the following ODE in which *all* the coefficients *except* a_1 are random, time-varying parameters:

$$\frac{dY}{dt} = a_0 g + a_1 Y + \frac{a_2}{g} Y^2 + \frac{a_3}{g^2} Y^3 + \cdots \frac{a_n}{g^{n-1}} Y^{n-1} \qquad (4.19)$$

Thus, solving the original SDE is equivalent to solving this random ODE: for each ω Є Ω, solve the above ODE. Ω can be taken as $C(0,\infty)$. And $\omega_t(\omega) = \omega(t)$ and P, the Wiener measure. For each ωЄ Ω, the above ODE is a polynomial in which *all but one* of the coefficients are time-varying. While the ODE is significantly more tractable than the SDE, considerable simplifications reward additional work. The above will be reduces to an algebraically simpler and computationally more efficient system, in which the coefficients associated with powers of Y *are all constant*, and only a *single* term is time-varying.

Consider a specific realization of W(t), say $W(t_k) = w_k$, for $t = t_k, 1 \le k \le N$. For notational convenience, denote the process value by w for t between t_{k-1} and t_k, and replace w by w_{k-1} later. Then, starting from t = t_{k-1}, the process evolves according to the ODE:

$$\frac{dY}{dt} = a_n \frac{Y^n}{g^{n-1}} + a_{n-1}\frac{Y^{n-1}}{g^{n-2}} + \cdots + a_2 \frac{Y^2}{g} + a_1 Y + a_0 g \Rightarrow$$

$$\frac{dY}{dt} - a_1 Y = a_n \frac{Y^n}{g^{n-1}} + a_{n-1}\frac{Y^{n-1}}{g^{n-2}} + \cdots + a_2 \frac{Y^2}{g} + a_0 g \Rightarrow$$

$$\frac{d}{dt}\left[Ye^{-a_1 t}\right] = e^{-a_1 t}\left[a_n \frac{Y^n}{g^{n-1}} + a_{n-1}\frac{Y^{n-1}}{g^{n-2}} + \cdots + a_2 \frac{Y^2}{g} + a_0 g\right] \Rightarrow$$

$$\frac{d}{dt}\left[Ye^{-a_1 t}\right] = \left[a_n e^{(n-1)a_1 t}\frac{\left(Ye^{-a_1 t}\right)^n}{g^{n-1}} + a_{n-1}e^{(n-2)a_1 t}\frac{\left(Ye^{-a_1 t}\right)^{n-1}}{g^{n-2}} + \cdots + a_2 e^{a_1 t}\frac{\left(Ye^{-a_1 t}\right)^2}{g} + a_0 g e^{-a_1 t}\right]$$

Let

$$Z_t = Y_t e^{-a_1 t} \tag{4.20}$$

Then

$$\frac{dZ}{dt} = \left[a_n e^{(n-1)a_1 t}\frac{Z^n}{g^{n-1}} + a_{n-1}e^{(n-2)a_1 t}\frac{Z^{n-1}}{g^{n-2}} + \cdots + a_2 e^{a_1 t}\frac{Z^2}{g} + a_0 g e^{-a_1 t}\right]$$

Therefore

$$\frac{dZ}{dt} = \left[\begin{array}{l} a_n e^{(n-1)a_1 t} e^{(n-1)\sigma w + \frac{(n-1)\sigma^2 t}{2}} Z^n + a_{n-1} e^{(n-2)a_1 t} e^{(n-2)\sigma w + \frac{(n-2)\sigma^2 t}{2}} Z^{n-1} + \cdots \\ + a_2 e^{a_1 t} e^{\sigma w + \frac{\sigma^2 t}{2}} Z^2 + a_0 e^{-\sigma w + \frac{\sigma^2 t}{2} - a_1 t} \end{array}\right]$$

Let

$$\theta = a_1 + \frac{\sigma^2}{2} \tag{4.21}$$

Then

$$\frac{dZ}{dt}=\begin{bmatrix} a_n e^{(n-1)\sigma w}\left(e^{\theta t}\right)^{n-1} Z^n + a_{n-1}e^{(n-2)\sigma w}\left(e^{\theta t}\right)^{n-2} Z^{n-1}+\cdots \\ +a_2 e^{\sigma w}e^{\theta t}Z^2 + a_0 e^{-\sigma w+\frac{\sigma^2 t}{2}-a_1 t} \end{bmatrix}\Rightarrow$$

$$\frac{dZ}{dt}=\begin{bmatrix} a_n e^{(n-1)\sigma w}e^{-\theta t}\left(Ze^{\theta t}\right)^{n} + a_{n-1}e^{(n-2)\sigma w}e^{-\theta t}\left(Ze^{\theta t}\right)^{n-1}+\cdots \\ +a_2 e^{\sigma w}e^{-\theta t}\left(Ze^{\theta t}\right)^2 + a_0 e^{-\sigma w+\frac{\sigma^2 t}{2}-a_1 t} \end{bmatrix}\Rightarrow$$

$$e^{\theta t}\frac{dZ}{dt}=\begin{bmatrix} a_n e^{(n-1)a_1 t}e^{(n-1)\sigma w+\frac{(n-1)\sigma^2 t}{2}} Z^n + a_{n-1}e^{(n-2)a_1 t}e^{(n-2)\sigma w+\frac{(n-2)\sigma^2 t}{2}} Z^{n-1}+\cdots \\ +a_2 e^{a_1 t}e^{\sigma w+\frac{\sigma^2 t}{2}}Z^2 + a_0 e^{-\sigma w+\frac{\sigma^2 t}{2}-a_1 t+a_1 t+\frac{\sigma^2 t}{2}} \end{bmatrix}$$

Therefore

$$\frac{d\left[Ze^{\theta t}\right]}{dt}-\theta Ze^{\theta t}=\begin{bmatrix} a_n e^{(n-1)\sigma w}\left(Ze^{\theta t}\right)^{n} + a_{n-1}e^{(n-2)\sigma w}\left(Ze^{\theta t}\right)^{n-1}+\cdots \\ +a_2 e^{\sigma w}\left(Ze^{\theta t}\right)^2 + a_0 e^{-\sigma w+\sigma^2 t} \end{bmatrix}$$

Let

$$V_t = Z_t e^{\theta t} \tag{4.22}$$

Then

$$\frac{dV}{dt}-\theta V=\begin{bmatrix} a_n e^{(n-1)\sigma w}V^n + a_{n-1}e^{(n-2)\sigma w}V^{n-1}+\cdots \\ +a_2 e^{\sigma w}V^2 + a_0 e^{-\sigma w+\sigma^2 t} \end{bmatrix}\Rightarrow$$

$$\frac{dV}{dt}=a_0 e^{-\sigma w+\sigma^2 t}+\theta V + a_2 e^{\sigma w}V^2 + a_3 e^{2\sigma w}V^3+\ldots+a_{n-1}e^{(n-2)\sigma w}V^{n-1}$$
$$+a_n e^{(n-1)\sigma w}V^n \Rightarrow$$

$$e^{\sigma w}\frac{dV}{dt}=a_0 e^{\sigma^2 t}+\theta e^{\sigma w}V + a_2\left(e^{\sigma w}V\right)^2 + a_3\left(e^{\sigma w}V\right)^3+\ldots$$
$$+a_{n-1}\left(e^{\sigma w}V\right)^{n-1}+a_n\left(e^{\sigma w}V\right)^n \Rightarrow$$

$$\frac{d\left[e^{\sigma w}V\right]}{dt}=a_0 e^{\sigma^2 t}+\theta e^{\sigma w}V + a_2\left(e^{\sigma w}V\right)^2 + a_3\left(e^{\sigma w}V\right)^3+$$
$$\ldots+a_{n-1}\left(e^{\sigma w}V\right)^{n-1}+a_n\left(e^{\sigma w}V\right)^n$$

Let

$$U_t = e^{\theta w} V_t \tag{4.23}$$

Then

$$\frac{dU}{dt} = a_0 e^{\sigma^2 t} + \theta U + a_2 U^2 + a_3 U^3 + \ldots + a_{n-1} U^{n-1} + a_n U^n$$

Finally,

$$\frac{dU}{dt} = a_0 e^{\sigma^2 t} + \left(a_1 + \frac{\sigma^2}{2} \right) U + a_2 U^2 + a_3 U^3 + \ldots + a_{n-1} U^{n-1} + a_n U^n \tag{4.24}$$

All the coefficients in the above system are constant except for the term that is free of U, and the equation is solvable by numerical algorithms, for example by using the Runge-Kutta algorithm, over each interval [t_{k-1}, t_k], using w_{k-1} as the constant value of w in the above computations. Recover V from U, Z from V, Y from Z, and finally X from Y. Upon working one's way through this chain of transformations, the solution to the original SDE is discovered, as exhibited in the algorithm.

4.3 Applications of algorithm

Despite its historical plight of being targeted for elimination, the salubrious effects of noise are now increasingly acknowledged in a growing body of work [65, 66, 67, 68]. Far from being an embarrassment to be expunged, research shows that noise plays a constructive role in stochastic resonance phenomena [69, 70] and in molecular motors featuring in nano-engineering [71, 72]. In the next two subsections, it will be shown that the steady-state behavior of a system can be palpably altered by using noise judiciously.

4.3.1 Utilizing noise to influence system behavior at steady-state

The idea of harnessing noise to influence system behaviour is not new, as evidenced by the abundant literature on stochastic resonance [65, 66, 67, 68]. The novelty here is that we suggest ways of using noise, not to improve the signal-to-noise ratio as in stochastic resonance, but to alter the structural properties of the system. Furthermore, unlike some forms of stochastic resonance, no threshold effects are needed. A closely related concept is the notion of vibrational control, extensively developed in a series of papers by Meerkov [73], Bellman et al. [74], and Bellman et al. [75]. The idea motivating vibrational control is to stabilize certain types of systems by adding an oscillatory function into the differential equation, and after employing a suitable averaging technique described in [76], transform it into one for which a previously unstable point becomes stable. But noise is not involved in vibrational control. It has been applied only to deterministic systems. Furthermore, whether or not vibrational control is successful for a particular application is a structural property of that system [73]; here, we demonstrate how noise can be used to *alter* the structural property of the system.

Two significant properties of a dynamical system are its set of fixed points, and its typical response time, called the characteristic time [77]. By regarding noise intensity as a control parameter, noise can be utilized to influence both these dynamical system properties. It is clear from the solution

$$X_t = \exp\left[-\left(a_1 - \sigma + \frac{\sigma^2}{2}\right) w_{k-1}\right] \exp\left[-\left(2a_1 + \sigma^2\right) t\right] U_t$$

that the random variable w_{k-1} observed at $t = t_{k-1}$ influences the solution U_t

merely by a multiplicative scaling factor. Consequently, when U_t inhabits one of its fixed points, it will stay there provided that the fixed point is a stable equilibrium. This simple observation paves the way to using noise to influence the fixed points of the system, as demonstrated explicitly for the class of polynomial SDEs with homogeneous drift.

The class of SDEs with homogeneous drift is defined by $a_0 = 0$, and for that class, the corresponding ODE has *no* time-varying parameter. With $a_0 = 0$, the non-autonomous ODE satisfied by the function U(t) reduces to the autonomous constant-parameter system.

$$\frac{dU}{dt} = \left(a_1 + \frac{\sigma^2}{2}\right)U + a_2U^2 + a_3U^3 + \ldots + a_{n-1}U^{n-1} + a_nU^n \qquad (4.25)$$

Steady-state solutions are found by setting $\frac{dU}{dt} = 0$, which results in a polynomial of degree ‘n’ in U. Steady-state solutions are guaranteed to exist by the fundamental theorem of algebra [78]. Consequently, after the transients have died out, the system will settle down into one of its equilibria (fixed points) at steady-state. Assume that a sufficiently long time has elapsed for the condition at steady-state to be satisfied so that the fixed points of the system satisfy:

$$\left(a_1 + \frac{\sigma^2}{2}\right)U + a_2U^2 + a_3U^3 + \ldots + a_{n-1}U^{n-1} + a_nU^n = 0 \qquad (4.26)$$

One of the roots is clearly U = 0, showing that the origin is an equilibrium. But other equilibria exist, some of which may portend unwelcome outcomes to the control engineer. The fundamental theorem of algebra guarantees the existence of the remaining (n-1) roots to the above equation—not necessarily all distinct or all real-valued. The variance parameter σ^2 is seen to play a fundamental role in determining the nature of these solutions—a fact that can be judiciously exploited to control the system behavior in desirable ways.

4.3.2 Three examples

Example 1. As an example, consider the homogeneous quadratic system:

$dX = \left(a_1 X + a_2 X^2\right)dt + \sigma X dW$, for which the corresponding ODE is

$$f(U) = \left(a_1 + \frac{\sigma^2}{2}\right)U + a_2 U^2 = 0$$

Assume that $a_1 < 0$. The two equilibria are $U^* = 0$ and

$$U^* = -\frac{\left(a_1 + \frac{\sigma^2}{2}\right)}{a_2}$$

In a completely deterministic system, $\sigma = 0$, and the only non-zero equilibrium is $U^* = -a_1/a_2$, which is fixed and unalterable for a given system defined by the parameters a_1 and a_2. In many engineering applications, performance criteria mandate stable equilibria. The origin is a stable equilibrium because $f'(0) = a_1 < 0$. So if the system settles down at the origin, it will be stable. But if the system ends up in the non-zero equilibrium $U^* = -a_1/a_2$, it will be unstable because $f'(-a_1/a_2) = -a_1 > 0$.

In the presence of noise, the non-zero equilibrium can be moved around in desired directions by manipulating the noise intensity σ^2. In particular, noise can be exploited to either keep all fixed points at the origin or to stabilize the non-zero equilibrium. The first objective is achieved by driving the only non-zero equilibrium to the origin—by increasing or reducing the noise till $\sigma^2 = -2a_1$. Doing so guarantees that no matter which of its fixed points the system inhabits at steady-state, it will remain at the origin. In fact, with the choice of $\sigma^2 = -2a_1$, the system is transformed to $\frac{dU}{dt} = 0$, a situation that results in a “whole line of fixed points,” in which perturbations neither grow nor decay [77]. Another possibility is to choose $\sigma^2 > -2a_1$, and under this choice,

$f'(0) = \left(a_1 + \frac{\sigma^2}{2}\right) < 0$ and $f'\left(-\frac{\left(a_1 + \frac{\sigma^2}{2}\right)}{a_2}\right) = -\left(a_1 + \frac{\sigma^2}{2}\right) < 0$. This choice leaves both the origin and the non-zero equilibrium stable. Thus, while noise is generally shunned as an annoyance that hinders the operation of the system, here it is shown that noise helps the control engineer *improve* the performance and stability of the system.

Example 2. Our second example is the homogeneous cubic system:

$dX = \left(a_1 X + a_2 X^2 + a_3 X^3\right)dt + \sigma X dW$, for which the corresponding ODE is

$$f(U) = \left(a_1 + \frac{\sigma^2}{2}\right)U + a_2 U^2 + a_3 U^3 = 0.$$

Assume that $a_3 < 0$, $a_1 < 0$, $a_2 > 0$, and $a_2^2 > 4a_3a_1$. The three equilibria are $U^* = 0$ and $U^* = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_3\left(a_1 + \frac{\sigma^2}{2}\right)}}{2a_3}$. In a completely deterministic system, $\sigma = 0$, and there are two non-zero equilibria defined by

$U_1{*}=\dfrac{-a_2+\sqrt{a_2{}^2-4a_3a_1}}{2a_3}$, and $U_2{*}=\dfrac{-a_2-\sqrt{a_2{}^2-4a_3a_1}}{2a_3}$ which are fixed and unalterable for a given system defined by the parameters a_1, a_2, and a_3. The origin is a stable equilibrium because f'(0) = $a_1 < 0$. To test the stability of U_1* and U_2*, note that

$f\,'(U_2{*})=\dfrac{a_2{}^2-4a_3a_1+a_2\sqrt{a_2{}^2-4a_3a_1}}{2a_3}<0$, so U_2* is a stable equilibrium, but

$f\,'(U_1{*})=\dfrac{a_2{}^2-4a_3a_1-a_2\sqrt{a_2{}^2-4a_3a_1}}{2a_3}<0$ if and only if $a_2{}^2-4a_3a_1>a_2\sqrt{a_2{}^2-4a_3a_1}$.

Next consider the stochastic system. The roots are

$U_1{*}=\dfrac{-a_2+\sqrt{a_2{}^2-4a_3\left(a_1+\dfrac{\sigma^2}{2}\right)}}{2a_3}$, and $U_2{*}=\dfrac{-a_2-\sqrt{a_2{}^2-4a_3\left(a_1+\dfrac{\sigma^2}{2}\right)}}{2a_3}$.

Corresponding to these roots,

$f\,'(U_2{*})=\dfrac{a_2{}^2-4a_3a_1-2a_3\sigma^2+a_2\sqrt{a_2{}^2-4a_3a_1-2a_3\sigma^2}}{2a_3}<0$, but

$f\,'(U_1{*})=\dfrac{a_2{}^2-4a_3a_1-2a_3\sigma^2-a_2\sqrt{a_2{}^2-4a_3a_1-2a_3\sigma^2}}{2a_3}<0$ if and only if

$a_2{}^2-4a_3a_1-2a_3\sigma^2>a_2\sqrt{a_2{}^2-4a_3a_1-2a_3\sigma^2}$, which can be guaranteed for sufficiently large σ^2 because the left hand side increases faster with σ^2 than the right hand side. This intuition can be rigorously substantiated. The proof is by contradiction. The assumption $a_2^2 > 4a_3a_1$ guarantees that $a_2{}^2-4a_3a_1>0$. Denote $A^2=a_2{}^2-4a_3a_1$ to reinforce its positivity. The assumption $a_3 < 0$

guarantees that $-2a_3$ is always positive. Let $B^2 = -2a_3$. Then, if there exists no σ^2 such that the above inequality is true, then for all σ^2, $A^2 + B^2\sigma^2 \le a_2\sqrt{A^2 + B^2\sigma^2}$.

Consider the inequality for $0 < a_2 < 1$, and choose $\sigma^2 = \frac{A^2}{B^2}\left(\frac{1-a_2{}^2}{a_2{}^2}\right)$. For that choice of σ^2, the above inequality reads $B^2\sigma^2 \le 0$, which is impossible. Therefore, $f'(U_1^*) < 0$ for sufficiently large σ^2 and the equilibrium U_1^* can be made stable by adjusting the noise intensity.

Example 3. Our third example shows that noise can be exploited to change the characteristic time of a system.

Strogatz [77] defines the characteristic time of a system as the time required for the system to vary significantly in the neighborhood of its fixed points. Noise can be used to *change* the characteristic time of the system. The characteristic time of the dynamic process U(t) is defined as follows:

$$\text{Characteristic Time} = \frac{1}{|f'(U^*)|}, \text{ where}$$

$$f(U) = \left(a_1 + \frac{\sigma^2}{2}\right)U + a_2U^2 + a_3U^3 + \ldots + a_{n-1}U^{n-1} + a_nU^n$$

U^* denotes a fixed point or equilibrium of the system. For the quadratic system, at both the equilibria $U^* = 0$ and $U^* = -\frac{\left(a_1+\frac{\sigma^2}{2}\right)}{a_2}$, the characteristic time is given by $\frac{1}{\left(a_1+\frac{\sigma^2}{2}\right)}$. Consequently, the characteristic time can be controlled by adding or reducing the noise intensity. For higher noise intensity (larger σ^2), the characteristic time of the system becomes smaller.

4.3.3 Solving a general Langevin equation

The general Langevin equation is of the form:

$$dX = f(X,u)dt + \sigma(X,u)\, dW(t) \tag{4.27}$$

Within the class of SDEs represented by equation (4.27), consider the class of SDEs in which the infinitesimal standard deviation term is proportional to the state variable, and there is no control variable ‘u’:

$$dX = f(X)dt + \sigma X dW \tag{4.28}$$

Equation (4.28) can be solved approximately by the algorithm developed in this chapter by using a polynomial approximation for the general drift function f(X). The link between autonomous SDEs and the class of SDEs with polynomial drift is spawned by the Weierstrass Approximation Theorem. Given a continuous but otherwise perfectly general autonomous drift function f(.), the Weierstrass Approximation Theorem [79] guarantees a polynomial to approximate f(.) within any desired degree of accuracy. Thus, for an arbitrarily small approximation error ε, there exists a polynomial of degree $n(\varepsilon)$ that will mimic the function f(.) within the tolerable error ε. Consequently, a mathematical algorithm that solves SDEs with polynomial drift will also solve general autonomous systems with arbitrarily small approximation error, thereby considerably expanding the range of engineering phenomena accommodated by polynomial SDEs.

4.4 Conclusions

4.4.1 Implications for CSF dynamics

The stochastic Marmarou model with reference pressure $p_0 \neq 0$ cannot be solved in closed-form because it is an inhomogeneous stochastic logistic SDE which was shown to violate the reducibility condition in section 4.2.1. The algorithm developed in this chapter can be used to obtain a quasi-analytical solution for the non-zero reference pressure case. Solving the stochastic Marmarou model for $p_0 \neq 0$ would be an important contribution to the neuroscience literature because it will provide another perspective on the controversy over whether $p_0 \neq 0$ or $p_0 = 0$. By comparing the predictions from the solution in Chapter 2 for the $p_0 = 0$ case with those from the quasi-analytical solution for $p_0 \neq 0$, these two possibilities can be tested to discover the one that provides superior predictions. This is on the future research agenda.

4.4.2 Implications for electrical engineering

The algorithm to solve SDEs with polynomial drift and linear noise is relevant to signal processing and several areas of electrical engineering in which SDEs are pervasive, as discussed in section 4.1.3. The algorithm illuminates the counter-intuitive role of noise. By analyzing the special case of SDEs with homogeneous drift, insights were obtained into the effect of noise on the system's steady-state behavior. Contrary to the intuitive view of noise as an unwelcome interference that degrades system performance, these results show that noise can be utilized constructively to achieve desirable system behavior.

CHAPTER 5

STOCHASTIC OPTIMAL CONTROL IN OPERATIONS RESEARCH

The determination of optimal advertising spending occupies a key position in operations research literature. Advertisers spread their budget out over time because memory effects cause the influence of advertising to decay over time. In response to advertising decay, marketers have developed different temporal scheduling patterns for advertising. But these temporal patterns do not maximize profits because they driven by managerial judgment rather than rigorous mathematical reasoning. What is the profit-maximizing way to allocate an advertising budget over time? Is it better to spread the budget evenly over time, to decrease spending over time, to increase spending over time, or to do something more elaborate? Using stochastic optimal control, this research shows that all these spending patterns emerge at optimality for the same response function dynamics, due to differences in salvage value assumptions. We use these results to develop a methodology for determining the optimal planning horizon length for each pattern of spending.

5.1 Background

Past research has identified conditions favoring one or the other of these options. The best option is determined by the interplay of at least six different factors: the dynamics of demand, the dynamics of production cost, the dynamics of temporal preference for money, the forces of competition, uncertainty, and salvage value [80, 81, 82, 83, 84]. Sasieni [85] established that it is dynamically optimal to spread advertising expenses evenly over an infinite planning horizon for a large class of response models. But the influences of salvage value and uncertainty have received scant attention in the extant literature. The salvage value of a dynamic process is the value of the final level of the state variable at the end of the planning horizon. Salvage value constraints are the terminal time or boundary conditions in a finite-horizon stochastic control problem.

5.1.1 Noise in marketing communications

The marketing communications mix consists of advertising, promotions, personal selling and all the media through which a firm communicates with its customers. Market response functions relate the marketing communications expenditures to market outcomes, typically sales. Sales are however, influenced by many factors other than the marketing communications mix alone—such as price, product quality, distribution channel and uncontrollable elements such as competition and the economy. To the extent that these factors vary, they will affect market response. Consequently, market response functions will contain some unexplained error variance due to the omission of factors influencing sales. Therefore, market response functions are stochastic, the source of noise being factors that affect sales but have been omitted from the model [86, 87].

5.1.2 Finite horizon decision-making

Salvage value assumptions play a central role in finite horizon decision making. Salvage values reflect the decision-maker's assumptions about the nature of the market and product. High tech markets are characterized by rapid product obsolescence and short product life cycles; under such conditions, the salvage value at the end of the planning horizon would be zero. For some products, the availability of a secondary market imparts non-zero value to the decision maker at the end of the planning horizon. Leased cars are an example of secondary markets—they can be sold in the market for used cars at the end of the leasing period.

Questions remain about the influence of salvage value on optimal advertising because finite horizon problems are mathematically more challenging than infinite horizon problems, and thus numerical analysis is the norm in previous work [80]. Yet it is important to understand salvage value effects analytically because they are a fundamental determinant of the temporal pattern of advertising spending [83]. Bass et al. [80] solves finite-horizon problems but the influence of salvage value constraints on advertising policies remains significantly under researched.

Salvage values influence the choice of best horizon in dynamic decision-making. Sethi and Chand [89], Chand, Sethi and Sorger [90] and Sethi and Sorger [91] have made some contributions in this area but they do not address the following questions. Is it better (in an expected profit-maximizing sense) to use a short or long planning horizon? Should the planning horizon be larger or smaller when advertising effectiveness (decay) is larger? A short planning horizon is inconsistent with dynamic optimization, but a long planning horizon is undesirable in a rapidly changing industry or, as Starr [92] notes, in an uncertain environment.

5.2 A stochastic model of communications response

5.2.1 Model formulation

Nerlove and Arrow [93] conceptualized the long-term effect of advertising through the construct of goodwill. Following Rao [94], we postulate the following SDE for the goodwill process G(t), driven by advertising u(t):

$$dG = (-\delta\, G + \beta u)\,dt \; + \; \sigma\, dW \tag{5.1}$$

where σ is the infinitesimal standard deviation and W(t) is a standard Brownian motion process. In equation (5.1), 'u(t)' is the control variable. The implication of equation (5.1) is that goodwill evolves as a controlled Markov process—for every possible advertising spending trajectory u(t), the goodwill process G(t) is a Markov process.

5.2.2 Stochastic optimal control problem

The decision-maker's objective is find a trajectory u(t) to achieve the following maximization:

$$\underset{u(t)}{\text{Max}}\; E_{G_o}\; \{\int_0^T e^{-\rho s}\, \pi(s)\, ds + \phi(G(T), T)\} \tag{5.2}$$

subject to the evolution of the SDE (5.1), where $\pi(s) = mG(s) - u^2(s)$ is the instantaneous profit at time "s." We consider the following family of salvage values, multiplicatively separable in g and T: $\varphi(G(T),T) = e^{-\rho T} mg\theta$, for any G(T) = g at t=T. The parameter $\theta > 0$ captures a number of substantively interesting scenarios.

5.2.3 Salvage value specifications

Specification I: Natural salvage value, $\theta = 1/(\delta + \rho)$

This boundary specification is a natural consequence of Nerlove-Arrow dynamics because the accumulated goodwill at the end of the planning horizon will decay in the absence of advertising thereafter. Therefore for any arbitrary level G(T) = g, the discounted value of the profit stream over [T, ∞) with u(t) = 0, for t>T is ($e^{-\rho T}$ mg)/(δ+ρ).

Proof

Let the process X_t be an Ito diffusion $dX = f(x)dt + \sigma(X)dW$. For $\alpha > 0$, and g(.) a bounded continuous function on R^n, define the resolvent operator R_α by $R_\alpha g(x) = E_x\left[\int_{t=0}^{t=\infty} e^{-\alpha t} g(X_t)dt\right]$, where E_x is the expectation operator given $X(0) = x$; then Oksendal [10] shows that $R_\alpha g(x) = \int_{t=0}^{t=\infty} e^{-\alpha t} E_x\left[g(X_t)\right]dt$.

Apply Oksendal's [10] result to evaluate $E_x(\int_{t=T}^{t=\infty} \pi(t)dt)$ with over $\left[T, \infty\right)$. To evaluate $E_g\left[G(t)\right]$, set $u(t) = 0$ in the stochastic differential equation (SDE) satisfied by G(t) to get $dG = -\delta G dt + \sigma dW$; apply E_g to both sides of the SDE for G(t), interchange d and E_g operators (allowed by Fubini's Thereom) and solve the resulting ordinary differential equation for $E_g\left[G(t)\right]$ using the condition at the boundary $t = T$ for $G(t)$ to get $E_g\left[G(t)\right] = G(T)e^{-\delta(t-T)}$; finally $\int_{t=T}^{t=\infty} e^{-\rho t} E_x\left[g(X_t)\right]dt = \int_{t=T}^{t=\infty} e^{-\rho t} G(T)e^{-\delta(t-T)}dt \Rightarrow \varphi(G(T),T) = \left(e^{-\rho T}mg/(\delta+\rho)\right)$ for any G(t) = g.

Specification II: Zero salvage value, θ = 0

A zero salvage value specification would be appropriate for an industry characterized by rapid product obsolescence or short product life cycles (the latter is typically though not always, a *consequence* of the former), so that the residual goodwill at the end of the horizon is worth nothing to the firm.

Specification III: Secondary market salvage value, $\theta = 1$

This boundary specification attaches no value whatsoever to any level of goodwill G(t) *after* t > T but recognizes that goodwill accumulated *until* time T may have value in a secondary market. The firm can dispose of its accumulated goodwill G(T) at \$m/unit in a secondary market but since it has to wait till t = T, it discounts the value of G(T) back to time t = 0.

Specification IV: High equity salvage value, $\theta > 1$

This specification is appropriate for a firm with high brand equity such as Coca-Cola in consumer non-durables or Microsoft Windows in consumer durables. The goodwill enjoyed by such brands could arise from strong brand loyalty (as for Coke) or a captive customer base due to high switching costs (as for Microsoft Windows).

5.3 Derivation of stochastic optimal control

5.3.1 Solution strategy

The value function V(g, t, T, ϕ) denotes the optimal expected performance over the remaining time horizon [t, T] when using an optimal policy and is defined as:

$$V(g, t, T, \phi) = \underset{u(t)}{\mathrm{Max}}\, E_g \, \{\int_t^T e^{-\rho s}\, \pi(s)\, ds + \varphi(G(T), T)\} \tag{5.3}$$

where E_g denotes the expectation operator, given G(t) = g, and $\phi = (\beta,\delta,\rho,\theta,m,c)$. The boundary condition is $V(g,t,T,\phi) = \varphi(G(T),T)$ where $\varphi(G(T),T)$ is the salvage value of terminal goodwill. $V(g,t,T,\phi)$ satisfies the Hamilton-Jacobi-Bellman (HJB) partial differential equation:

$$-V_t = \max_{u(t)} \left[e^{-\rho t}\pi(t) + f(g,u)\,V_g + \frac{\sigma^2 V_{gg}}{2} \right] \tag{5.4}$$

where $V_t = \partial V/\partial t$, $V_g = \partial V/\partial g$ and $V_{gg} = \partial^2 V/\partial g^2$. The above non-linear partial differential equation is solved subject to $G(t) = G_0$, and boundary conditions $\varphi(G(T),T) = e^{-\rho T}\, mg\theta$.

Given (5.1) and (5.3), the HJB equation is:

$$V_t \; - g\delta V_g \; + \frac{e^{\rho t}\beta^2 V_g^{\;2}}{4} \; + \frac{\sigma^2 V_{gg}}{2} + e^{-\rho t}\, gm \; = 0 \tag{5.5}$$

The stochastic optimal control is derived by solving equation (5.5) subject to the boundary condition $\varphi(G(T),T) = e^{-\rho T}\, mg\theta$. The Riccati structure of (5.5) suggests the following functional form for $V(g,t,T,\phi)$:

$$V(g,t,T,\phi) = e^{-\rho t}\{k_1(t)g^2 + k_2(t)g + k_3(t)\} \tag{5.6}$$

Substitution of (5.6) into the HJB equation generates three coupled non-linear differential equations satisfied simultaneously by the functions $k_1(t)$, $k_2(t)$, and $k_3(t)$. These are solved subject to the boundary conditions given by $\varphi(g,t)$ at $t=T$

5.3.2 Optimal solution

Substituting $V(g,t,T,\phi) = [e^{-\rho t}\{k_1(t)g^2 + k_2(t)g + k_3(t)]$ into the HJB, we obtain:

$$\begin{aligned}&\sigma^2 k_1(t) + \frac{1}{4}\beta^2 k_2(t)^2 - \rho k_3(t) + k'_3(t) \; + \\ &g\,(m - (\delta+\rho)\,k_2(t) + \beta^2 k_1(t)\,k_2(t) + k'_2(t)) + \\ &g^2(k'_1(t) + \beta^2 k_1(t)^2 - (2\,\delta+\rho)k_1(t)) \; = \; 0\end{aligned} \tag{5.7}$$

where $k_i^{'}(t) = \frac{dk_i}{dt}$. The $k_i'(t)$ satisfy three coupled nonlinear differential equations:

$$k'_1(t) + \beta^2 k_1(t)^2 - (2\,\delta+\rho)k_1(t) = 0 \tag{5.8}$$

$$m - (\delta+\rho)\,k_2(t) + \beta^2 k_1(t)\,k_2(t) + k'_2(t) = 0 \tag{5.9}$$

$$\sigma^2 k_1(t) + \frac{1}{4}\beta^2 k_2(t)^2 - \rho k_3(t) + k'_3(t) = 0 \tag{5.10}$$

Solving equations (5.8) – (5.10) yields the value function $V(g,t,T,\phi)$:

$$V(g,t,T,\phi) = e^{-\rho t}\left(\frac{e^{\rho(t-T)} m^2\beta^2(-2+2\theta\rho+\delta\theta^2\rho)}{4\delta\rho(2\delta+\rho)} + \frac{gm(1+e^{(\delta+\rho)(t-T)}(-1+(\delta+\rho)\theta))}{\delta+\rho} - \frac{m^2\beta^2\left(-\frac{1}{\rho} + \frac{2e^{(\delta+\rho)(t-T)}(-1+(\delta+\rho)\theta)}{\delta} + \frac{e^{2(t-T)(\delta+\rho)}(-1+(\delta+\rho)\theta)^2}{2\delta+\rho}\right)}{4(\delta+\rho)^2}\right) \tag{5.11}$$

Next, we substitute $V_g = \frac{\partial V}{\partial g}$ into $u(g,t,T,\theta) = \frac{1}{2}\{e^{\rho t}\,\beta\, V_g)$ to obtain:

$$u(g,t,T,\theta) = \frac{m\,\beta}{2\,(\delta+\rho)} + \frac{e^{(\delta+\rho)(t-T)}\, m\,\beta\,(-1+(\delta+\rho)\theta)}{2\,(\delta+\rho)} \tag{5.12}$$

Finally, $V(g, t, T, \phi) = e^{-\rho T} mg\theta$ at $t=T$, and so the boundary condition is indeed satisfied. Since the optimal policy does not depend upon the state variable for any θ, it is an open-loop policy.

Define:

$$\text{EvenPolicy} = \frac{m\beta}{2(\delta+\rho)} \tag{5.13}$$

Equation (5.12) is the optimal policy for the general family of salvage values $\varphi(G(T),T) = e^{-\rho T} mg\theta$, for any G(T) = g at t=T. The optimal policy for each of the four market conditions corresponds to the specific value of θ characterizing that market condition.

Natural salvage value, $\theta = 1/(\delta + \rho)$

$$u(t,T) = \frac{m\,\beta}{2\,(\delta+\rho)} \qquad (5.14)$$

Thus it is optimal to maintain a constant advertising level, called an *Even* policy [95, 96] defined in equation (5.13). The optimal policy increases with margin (m), effectiveness (β) and decreases with decay rate (δ) and the discount rate (ρ). Since the finite horizon problem with natural boundary specification yields the same optimal policy as the infinite horizon problem, this result means that a decision-maker with a *long-term* perspective should keep her advertising level constant in markets described by Nerlove-Arrow dynamics.

Specification II: Zero salvage value, $\theta = 0$

$$u(t,T) = \frac{m\,\beta}{2\,(\delta+\rho)} - \frac{e^{(\delta+\rho)(t-T)}\,m\,\beta}{2\,(\delta+\rho)} \qquad (5.15)$$

From equation (5.13) and (5.15),

$$\text{ZeroSalvagePolicy} = \text{EvenPolicy}\left(1 - e^{(\delta+\rho)(t-T)}\right)$$

Specification III: Secondary market salvage value, $\theta = 1$

$$u(t,T) = \frac{m\,\beta}{2\,(\delta+\rho)} + \frac{e^{(\delta+\rho)(t-T)}\,m\,\beta\,(-1+\delta+\rho)}{2\,(\delta+\rho)} \qquad (5.16)$$

From equation (5.13) and (5.16),

$$\text{SecondaryMarketPolicy} = \text{EvenPolicy}\left(1 - (1 - (\delta+\rho))\ e^{(\delta+\rho)(t-T)}\right)$$

Specification IV: High equity salvage value, θ > 1

$$u(t,T) = \frac{m\,\beta}{2\,(\delta+\rho)} + \frac{e^{(\delta+\rho)(t-T)}\ m\,\beta\,(-1+(\delta+\rho)\theta)}{2\,(\delta+\rho)} \tag{5.17}$$

From equation (5.13) and (5.17),

$$\text{HighEquityPolicy} = \text{EvenPolicy}\left(1 + ((\delta+\rho)\theta-1)\ e^{(\delta+\rho)(t-T)}\right)$$

Summing up, it is optimal to spend less than the even policy in both the zero salvage and secondary market conditions. Thus a decision-maker ignoring the zero salvage or secondary market constraint will *overspend* relative to an optimal decision maker. Under zero salvage and secondary market conditions, the qualitative nature of the policy is *time-varying* rather than even.

5.3.3 Properties of the optimal solution

Asymptotic behavior of the optimal policy

From equation (5.12), the optimal policy for the general optimal policy is asymptotically even because

$$\lim_{t->\infty}[u(t,T)] = \text{EvenPolicy} \tag{5.18}$$

Temporal behavior of the optimal policy

Differentiating u(t, T) with respect to θ:

$$\frac{\partial u(t,T)}{\partial t} = \frac{1}{2}\, e^{(\delta+\rho)(t-T)}\ m\,\beta\,(-1+(\delta+\rho)\theta) \tag{5.19}$$

Define $\theta_{crit} = 1/(\delta+\rho)$ and derive:

$$\frac{\partial u(t,T)}{\partial t}\begin{cases}<0 \text{ for } 0\leq\theta<\theta_{crit}\\ =0 \text{ for } \theta=\theta_{crit}\\ >0 \text{ for } \theta>\theta_{crit}\end{cases} \tag{5.20}$$

Terminal value of the optimal policy

$$u(T,T,\theta)=\begin{cases}0 \text{ for } \theta=0\\ \dfrac{m\beta}{2(\rho+\delta)} \text{ for } \theta=\theta_{crit}\\ \dfrac{m\beta}{2} \text{ for } \theta=1\\ \dfrac{m\beta\theta}{2} \text{ for } \theta>\theta_{crit}\end{cases} \tag{5.21}$$

5.4 Optimizing the length of the planning horizon

5.4.1 Impact of planning horizon on performance

Many decision-makers use planning horizons of three, five or ten years, but Starr (1966) points out that the practice is based on executive judgment rather than quantitative analysis. What is the best planning horizon length for advertising spending decisions and how is the answer influenced by market parameters? Determining the planning horizon length to maximize the expected profit makes the issue unambiguous.

5.4.2 Optimal planning horizon length in general

Given an exogeneously predetermined T, the value function V(g, t, T, ϕ) evaluated at t = 0 gives the maximum expected profit over [0,T]. Let C(T) be the cost associated with a horizon of length T, such that C(T) = cT [59]. The problem is to endogeneously determine T to maximize the value, net of the cost associated with the planning horizon length. Thus the problem is to

$\arg\max_{T}\left(V(g,0,T,\varphi) - cT\right)$.

The value function V(g, t, T, ϕ) is shown in equation (5.11). Evaluating V(g, t, T, ϕ) at t = 0, we obtain the optimal expected value over [0,T] for fixed T and any arbitrary initial level g.

$$V(g,0,T,\phi) = \left(\begin{array}{l} \dfrac{e^{-\rho T}\, m^2\beta^2\,(-2+2\,\theta\,\rho+\delta\,\theta^2\rho)}{4\,\delta\,\rho\,(2\delta+\rho)} + \\ \dfrac{g\,m\,(1+e^{-(\delta+\rho)T}(-1+(\delta+\rho)\theta))}{\delta+\rho} - \\ \dfrac{m^2\beta^2\left(-\dfrac{1}{\rho}+\dfrac{2\,e^{-(\delta+\rho)T}(-1+(\delta+\rho)\theta)}{\delta}+\dfrac{e^{-2(\delta+\rho)T}\,(-1+(\delta+\rho)\theta)^2}{2\,\delta+\rho}\right)}{4\,(\delta+\rho)^2} \end{array} \right) \tag{5.22}$$

The expected profit function is the difference between the expected value over [0, T] and the cost associated with a planning horizon of length T.

$$\Pi(t) = V(g,0,T,\phi) - cT \tag{5.23}$$

Setting $\dfrac{\partial\Pi}{\partial T} = 0$ gives the optimality equation for the expected profit-maximizing T, and its solution yields the optimal T provided that the sufficiency condition $\dfrac{\partial^2\Pi}{\partial T^2} < 0$ holds at the root of the optimality equation. Successively setting θ = 1/(ρ+δ), 0, 1, we get the optimality equations for T for the natural, secondary market and high equity cases respectively. A closed-form solution can be found for T in the case of natural boundary condition. For the other three market conditions, no closed-form solution is available.

5.4.3 Optimal planning horizon length for natural boundary

condition

Let T* denote the optimal T.

$$\frac{\partial \Pi}{\partial T} = \frac{e^{-\rho T}\,(m^2\,\beta^2 - 4\,c\,e^{-\rho T}\,(\delta+\rho)^{\,2}\,)}{4\,(\delta+\rho)^2} = 0$$

$$\frac{\partial^2 \Pi}{\partial T^2} = -\,\frac{e^{-T\rho} m^2 \beta^2 \rho}{4\,(\delta+\rho)^2} < \,0$$

Consequently, the sufficiency condition is met and from $\frac{\partial \Pi}{\partial T} = 0$, we obtain T*:

$$T^* = \frac{\mathrm{Log}\left[\dfrac{m^2\beta^2}{4c(\delta+\rho)^2}\right]}{\rho} \tag{5.24}$$

5.5 Conclusions

The central issues in this research were the optimality of different temporal patterns of advertising spending and their implications for the optimal planning horizon. The focal questions were: is it better to spend evenly over time, or use more elaborate spending patterns, and how is the optimal planning horizon related to the spending pattern? Although an even policy is optimal under a specific boundary constraint, other boundary constraints generate patterns of optimal spending quite different from the even policy. Across different market conditions (as captured in the salvage values), longer planning horizons are optimal when the advertising effectiveness increases or when the decay rate decreases.

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

No field has contributed more to a rigorous and systematic study of noise than electrical engineering—methodologically, conceptually and pragmatically. By its very nature and because of its ubiquity across disciplines, noise is an interdisciplinary topic—thus electrical engineering technologies for studying noise solve fundamental problems in other disciplines, whose solutions spawn new problems leading to new methodologies and algorithms for the analysis of noise, thereby driving a closed-loop research process in which interdisciplinary applications enrich electrical engineering methodology. Such has indeed been the case in this research, as discussed below.

This research makes interdisciplinary contributions by mathematically modeling the impact of noise on dynamic phenomena in neuroscience, electrical engineering and operations research. The model for the dynamics of cerebrospinal fluid (CSF) flow in neuroscience is inspired by an electrical circuit analogy, the extended CSF flow model accommodates noise through SDE technology, both the neuroscience and operations research applications exploit the power of SDEs, the extended SDE model of CSF flow dynamics developed in chapter 2 derives results of clinical significance and offers a novel perspective on an ongoing neuroscience controversy in hydrocephalus research in chapter 3. Additionally, the new stochastic model of cerebrospinal fluid flow dynamics provides the mathematical basis for an automatic nonlinear regulator to keep the intracranial pressure (ICP) within safe limits in patients suffering from hydrocephalus—this is a contribution to the biomedical engineering literature. A natural generalization of the SDE for CSF flow dynamics leads to SDEs with polynomial drift of arbitrarily high order, which are relevant to both neuroscience and electrical engineering applications. Responding to this relevance, a new algorithm to solve SDEs with polynomial drift is developed in chapter 4, applicable to a large class of signal processing models in electrical engineering and neuroscience. Thus the new algorithm is a methodological contribution to both the electrical engineering and applied mathematics literature. Not only does the new algorithm solve a large class of SDEs; it also suggests ways in which noise can be harnessed constructively to improve the performance of engineering systems, as shown in chapter 4. In chapter 5, the technology of SDEs and stochastic optimal control solve a fundamental puzzle in operations research on marketing communications, specifically advertising. Temporal patterns observed for advertising include constant spending over time, decreasing spending over time and increasing spending over time. Rigorous mathematical analysis has previously established that constant spending is dynamically optimal for a large class of models. Past research also shows that a number of demand, cost and temporal dynamics dictate dynamically optimal time-varying spending rather than constant spending. The joint impact of terminal time constraints at the end of a finite planning horizon and response uncertainty have been neglected in past work, as has the issue of the best planning horizon length. The research reported here shows that different spending patterns emerge at optimality for the same response function dynamics, due to differences in salvage value assumptions at the terminal time. Modeling response dynamics by a special case of the Langevin model, we show that constant and time-varying policies are each optimal under different terminal value constraints. The general Langevin equation provides the unifying mathematical framework that ties together all the chapters in this dissertation.

This dissertation has already spawned several new projects for future work, many of which are now at various stages of completion. Thermal noise affects the operation of gas discharge lamps, and while there is an existing stochastic model of this phenomenon, several unaddressed issues remain that have been resolved by using methods developed in this research. The existing results on gas discharge lamps are generalized and new probabilistic results that are useful in assessing their performance are derived. By doing so, we expand the scope of previous results on gas discharge lamp dynamics and introduce sophisticated martingale techniques in power engineering to compute probabilities of engineering relevance for gas discharge lamps. Noise affects the degradation dynamics of ultra-thin metal oxides in MOS capacitors, and, although an existing stochastic model describes this phenomenon, the statistical quality of the estimators of that model leave room for improvement, and a generalization of the methodology of this dissertation leads to continuous-time estimators that will inform future research in that area. These new continuous time maximum likelihood estimators are expected to improve theory and practice in studies of the degradation dynamics of ultra-thin metal oxides in MOS capacitors.

The methods developed to address these issues are applicable with only slight modification to future research on Brownian motors in nanotechnology. Almost every function of life involves motion. Furthermore, every biological process involving motion arises from the action of biological 'molecular motors.' The world in which molecular motors and nanomachines operate is the world of Brownian Motion. Any description of molecular motors has to be stochastic in nature because molecular motors operate in a Brownian environment. Consequently, researchers in the nanosciences are now increasingly interested in the development and applications of Brownian Motion models. These molecular motors are the essential agents of movement at the molecular level. Understanding how they work is a major challenge that requires the blending of ideas from multiple disciplines. At the macroscopic level, the physics of movement can be understood by applying Newtonian mechanics, but movement at the microscopic level is not purely deterministic because molecules are in unceasing thermal (random) motion. Indeed, all micron-sized and smaller particles are in constant motion due to thermal fluctuations. The puzzle is: how do the molecular motors inside a living cell overcome this randomness to produce orderly macroscopic motion? In other words, how do molecular motors produce useful mechanical work? The key to modeling molecular motors is known to be the Langevin approach and the methodology developed in this dissertation for solving a large class of Langevin SDEs provides the analytical apparatus for future contributions to nanotechnology.

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ABSTRACT

PROCESSING RANDOM SIGNALS IN NEUROSCIENCE, ELECTRICAL ENGINEERING AND OPERATIONS RESEARCH

by

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Major: Electrical Engineering

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The topic of this dissertation is the study of noise in electrical engineering, neuroscience, biomedical engineering, and operations research through mathematical models that describe, explain, predict and control dynamic phenomena. Noise is modeled through Brownian Motion and the research problems are mathematically addressed by different versions of a generalized Langevin equation. Our mathematical models utilize stochastic differential equations (SDEs) and stochastic optimal control, both of which were born in the soil of electrical engineering. Central to this dissertation is a brain-physics based model of cerebrospinal fluid (CSF) dynamics, whose structure is fundamentally determined by an electrical circuit analogy. Our general Langevin framework encompasses many of the existing equations used in electrical engineering, neuroscience, biomedical engineering and operations research.

The generalized SDE for CSF dynamics extends a fundamental model in the field to discover new clinical insights and tools, provides the basis for a

nonlinear controller, and suggests a new way to resolve an ongoing controversy regarding CSF dynamics in neuroscience. The natural generalization of the SDE for CSF dynamics is a SDE with polynomial drift. We develop a new analytical algorithm to solve SDEs with polynomial drift, thereby contributing to the electrical engineering literature on signal processing models, many of which are special cases of SDEs with polynomial drift. We make new contributions to the operations research literature on marketing communication models by unifying different types of dynamically optimal trajectories of spending in the framework of a classic model of market response, in which these different temporal patterns arise as a consequence of different boundary conditions.

The methodologies developed in this dissertation provide an analytical foundation for the solution of fundamental problems in gas discharge lamp dynamics in power engineering, degradation dynamics of ultra-thin metal oxides in MOS capacitors, and molecular motors in nanotechnology, thereby establishing a rich agenda for future research.

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Publications and Presentations Related to the Dissertation Work:

1. Raman, Kalyan, "A Stochastic Differential Equation Analysis of Cerebrospinal Fluid Dynamics," *Fluids and Barriers of the CNS*, January 2011, 8:9, 1-10. (CNS stands for Central Nervous System).

2. Raman, Kalyan and P. Sundar, "Towards a Resolution of the Reference Pressure Controversy in Cerebrospinal Fluid Dynamics," *Cerebrospinal Fluid—Functions, Composition and Disorders*, Ed., Nadya S. Gotsiridze-Columbus, Nova Science Publishers Inc., New York, In Press.

3. Raman, Kalyan, "Boundary Value Problems in Stochastic Optimal Control of Advertising," *Automatica* 2006, Volume 42, Issue 8, August, 1357-1362.

4. Raman, Kalyan, "Modeling, Estimation and Optimal Control Issues in Cerebrospinal Fluid Dynamics," *Cerebrospinal Fluid Research* 2009, 6 (Suppl 2), S23, 27th November.

Intellectual Property & Provisional Patent Related to the Dissertation Work

5. Raman, Kalyan, 'A Non-Linear Controller for Intracranial Pressure Maintenance." New Invention Disclosure for a U.S. Patent, Technology Transfer Program, *Northwestern University* Reference NU29140. Successfully passed stage 1 of consideration for a US Patent, called the provisional stage lasting one year. Stage 1 establishes the novelty, originality and uniqueness of the invention. Now the invention is in stage 2: U.S. Serial No. 61/276,672 filed unprovisionally in September 2010.