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Stabilization and classification of poincare duality embeddings

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DEDICATION

This dissertation is dedicated to my mother, Janis.

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Introduction

Motivation

The object of this thesis is to develop and generalize an analog, in the Poincaré Duality setting, of the Stabilization and Classification Theorems for embeddings up to homotopy given by Connolly and Williams in [CW78]. The main results essentially give a characterization of certain Poincaré Duality embeddings in terms of the homotopy types of their complements. Throughout what follows, we wish to emphasize that the methods of homotopy theory and algebraic topology provide very tractable alternatives to their geometric ancestors in studying the topology of manifolds. The importance of developing such an underlying homotopy-theoretic theme was addressed by C.T.C Wall in the last sentence of the following quote on embeddings: ([Wa99], P.119)

> The results to be obtained are best formulated in terms of a 'triangulation or smoothing' of initial homotopy-theoretic data. It is misleading to regard this as a complete solution to the problem of embeddings: the problems raised seem to the author in some respects to be harder than the original geometrical problems. We hope to give more positive results on this point elsewhere.

The need for such "positive results" has facilitated a major part of the work of Klein ([Kl99a], [Kl99b], [Kl02a], [Kl02b], [Kl05]) and this thesis is, in many ways, a continuation of this work.

The Main Results

Let $\mathcal{T}$ denote the category of compactly-generated topological spaces with the Quillen model structure ([Qu67], Section 1) based on weak homotopy equivalences and (Serre) fibrations. In what follows, "space" will mean "cofibrant object of $\mathcal{T}$" unless otherwise stated. An object X of $\mathcal{T}$ is *homotopy finite of dimension* d, written $\mathrm{hodim}(X) = d$, if X is homotopy equivalent to a finite cell complex of dimension d, and d is the smallest such number for which this holds. Similarly, we will write $\mathrm{hodim}(X, Y) = k$ if X can be obtained from Y, up to homotopy, by attaching cells of dimension at most k.

Let $\mathcal{T}(A \xrightarrow{f} B)$ denote the category whose objects are triples (i, Y, j) such that Y is an object of $\mathcal{T}$, $i : A \to Y$ and $j : Y \to B$ are morphisms in $\mathcal{T}$, and $j \circ i = f$. A morphism $(i, Y, j) \to (i', Y', j')$ is a morphism $g : Y \to Y'$ in $\mathcal{T}$ such that $g \circ i = i'$ and $j' \circ g = j$. A morphism in $\mathcal{T}(A \xrightarrow{f} B)$ is r-connected if its underlying map of spaces is r-connected. It is well-known that $\mathcal{T}(A \xrightarrow{f} B)$ is a model category (see, e.g., [Ho99], Chapter 1) whose weak equivalences and (co)fibrations are determined by the forgetful functor to $\mathcal{T}$ which is given on objects by $(i, Y, j) \mapsto Y$. In particular, an object (i, Y, j) of $\mathcal{T}(A \xrightarrow{f} B)$ is fibrant if $j : Y \to B$ is a fibration in $\mathcal{T}$. Similarly, (i, Y, j) is cofibrant if $i : A \to Y$ is a cofibration in $\mathcal{T}$. Note that $\mathcal{T}(\emptyset \to X)$ is the category $(\mathcal{T} \downarrow X)$ of spaces over X. In particular, an object of $(\mathcal{T} \downarrow X)$ is a pair (i, Y) such that $i : Y \to X$ is a map of spaces.

Let X be a Poincaré Duality space (possibly with boundary ∂X) of formal dimension n, and fix an object $(f, K) \in (\mathcal{T} \downarrow X)$. Assume that (K, L) is a cofibration pair of connected, homotopy finite spaces with $\mathrm{hodim}(K, L) = k \leq n - 3$. Further, require that $f : (K, L) \to$

$(X, \partial X)$ be a map of pairs such that $K \to X$ is an r-connected map of spaces, $r \geq 1$, and the map $L \to \partial X$ is the underlying map of a given Poincaré Duality embedding (see Definition 1.1.2 below). In what follows, we define a space $\mathbb{E}_f(K, X \; rel \, L)$ of Poincaré Duality embeddings for f and a map

$$\sigma : \mathbb{E}_f(K, X \; rel \; L) \to \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \; rel \, \mathcal{S}_X L)$$

where $\mathcal{S}_X K$ denotes the (unreduced) fiberwise suspension of the object (f, K). We call σ the *stabilization map.*

Theorem A (Stabilization). *With the hypotheses above, the stabilization map*

$$\sigma : \mathbb{E}_f(K, X \; rel \; L) \to \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \; rel \, \mathcal{S}_X L)$$

is $(n - 2(k - r) - 3)$*-connected.*

Theorem A serves as a classification tool for Poincaré Duality embeddings for f. In particular, another space, $\mathbb{SW}_f(K, X \; rel \, L)$, is defined and can be thought of as a kind of moduli space of unstable fiberwise duals of Poincaré embeddings with underlying map $f : K \to X$. We define a *classification map*

$$\theta : \mathbb{E}_f(K, X \; rel \; L) \to \mathbb{SW}_f(K, X \; rel \; L)$$

and, as a consequence of the Theorem A, obtain

Theorem B (Classification). *With the assumptions of the previous theorem, the classification map*

$$\theta : \mathbb{E}_f(K, X \ \textit{rel}\ L) \to \mathbb{SW}_f(K, X \ \textit{rel}\ L)$$

is $(n-2(k-r)-3)$-connected.

In the case that $X = S^n$ and $L = \emptyset$, there is an evident "collapse" map

$$\mathbb{E}_{\mathcal{S}_{S^n} f}(\mathcal{S}_{S^n} K, S^n \times D^1 \ \textit{rel}\ \mathcal{S}_{S^n}\emptyset) \xrightarrow{c} \mathbb{E}_{\mathcal{S}f}(\mathcal{S}K, S^{n+1})$$

where $\mathcal{S}$ is the usual (unreduced) suspension functor. We can precompose this map with the stabilization map to get

$$\mathbb{E}_f(K, S^n) \xrightarrow{c\circ\sigma} \mathbb{E}_{\mathcal{S}f}(\mathcal{S}K, S^{n+1}).$$

In this case, the space $\mathbb{SW}_f(K, S^n)$ amounts to a space of Spanier-Whitehead n-duals for K (hence the notation "$\mathbb{SW}$"). We will show that the map c above induces a π_0 surjection and, as consequences of the Theorems A and B, obtain the following analogs, in the Poincaré Duality setting, of the "Stabilization" and "Classification" theorems of Connolly and Williams ([CW78], PP.385-386):

Theorem C. *Let K be a homotopy finite complex with* $\operatorname{hodim}(K) = k \leq n-3$. *Assume that $f : K \to S^n$ is an r-connected map of spaces, $r \geq 1$. Then the induced map*

$$\pi_0(c \circ \sigma) : \pi_0(\mathbb{E}_f(K, S^n)) \to \pi_0(\mathbb{E}_{\mathcal{S}f}(\mathcal{S}K, S^{n+1}))$$

is surjective for $n \geq 2(k-r)+3$ and injective for $n \geq 2(k-r)+4$.

Theorem D. *With the assumptions of the previous theorem, the induced map*

$$\pi_0(\theta) : \pi_0(\mathbb{E}_f(K, S^n)) \to \pi_0(\mathbb{SW}_f(K, S^n))$$

is surjective for $n \geq 2(k-r)+3$ *and injective for* $n \geq 2(k-r)+4$.

Note that the results above imply the smooth manifold embedding statements of Connolly and Williams after a suitable application of the Browder-Casson-Sullivan-Wall theorem ([Kl00], Theorem 5.3). Moreover, our theorems dispense with the extra assumptions on n found in ([CW78], PP.385-386). In contrast to the surgery-theoretic methods used in [CW78], the proofs of our theorems are purely homotopy-theoretic. We rely heavily on *fiberwise homotopy theory*, a large portion of which can be found in [Kl99a], [Kl02a], [Kl07]. The proofs of our results also rely to a great extent on the "higher homotopy excision" theorems of Goodwillie ([Go92]).

Conventions and Notation

As mentioned in the introduction, $\mathcal{T}$ will denote the category of compactly-generated topological spaces equipped with the model structure based on weak homotopy equivalences and Serre (co)fibrations. Constructions in $\mathcal{T}$, such as products and function spaces, will be understood to be topologized using the compactly-generated topology. The term "space" will mean "cofibrant object of $\mathcal{T}$" unless otherwise specified. A space X is n-connected if $\pi_i(X) = 0$ for every $i \leq n$ and for every choice of basepoint. In particular, a nonempty space is always (-1)-connected and, by convention, the empty space is (-2)-connected. A map of spaces $X \to Y$ is *n-connected* if its homotopy fiber, with respect to any basepoint of Y, is an $(n-1)$ connected space. A weak equivalence in $\mathcal{T}$ is an ∞-connected map. In the event that we specify a basepoint $*$ for a given space X, we will assume that the inclusion $* \to X$ is a cofibration. We will use the language of model categories frequently. In particular, every object in a given model category has a (co)fibrant replacement. We will assume that the reader is familiar with homotopy limits and homotopy colimits, as well as the language of homotopy (co)-cartesian diagrams. The notation $(\overline{X}, Y)$ will be used to denote the mapping cylinder $\overline{X}$ of a given a map $X \xrightarrow{f} Y$ with the inclusion of Y as $Y \times 0$. Lastly, a note about set theory. Most of the categories that we will work with are not small. To avoid set-theoretic difficulties when working with such categories, we fix a Grothendieck universe $\mathcal{U}$ and use only $\mathcal{U}$-sets to form the objects of the category. In particular, we will write $|\mathscr{C}|$ for the geometric realization of the nerve of the (possibly large) category $\mathscr{C}$. This convention is not the only feasible option (see, e.g., [GK08], P.9).

CHAPTER 1

The Stabilization and Classification Maps

1.1 Poincaré Duality Embeddings

The following definitions can be found in [Kl99a] and [Kl02b]. They are included for the sake of completeness.

Definition 1.1.1. Let X be a homotopy finite space equipped with a local coefficient system $\mathscr{L}$ which is pointwise free abelian of rank one. Let $[X]$ denote a homology class in $H_n(X;\mathscr{L})$. The data $(X, \mathscr{L}, [X])$ equip X with the structure of a *Poinacré Duality space of formal dimension n* if cap product with $[X]$ induces an isomorphism

$$\cap[X] : H^*(X, \mathscr{L}) \xrightarrow{\cong} H_{n-*}(X; \mathscr{L} \otimes \mathscr{M})$$

for every local coefficient system $\mathscr{M}$. We call $[X]$ the *fundamental class* of X and we will refer to such a space X as a *PD space of dimension n*. A cofibration pair $(X, \partial X)$ of homotopy finite spaces along with $\mathscr{L}$ and a class $[X] \in H_n(X, \partial X; \mathscr{L})$ is called a *Poincaré Duality pair of formal dimension n* if

- For all local systems $\mathscr{M}$, there is an induced isomorphism

$$\cap[X] : H^*(X; \mathscr{L}) \xrightarrow{\cong} H_{n-*}(X, \partial X; \mathscr{L} \otimes \mathscr{M})$$

- The restriction of $\mathscr{L}$ to ∂X along with the image of the fundamental class $[X]$ under the boundary homomorphism $H_n(X, \partial X; \mathscr{L}) \to H_{n-1}(\partial X; \mathscr{L})$ equips ∂X with the structure of a PD space of dimension $n-1$.

We will call such a pair $(X, \partial X)$ a *PD pair of dimension* n.

Definition 1.1.2. Let $K \xrightarrow{f} X$ denote a map from a connected, homotopy finite space K to a PD space X or PD pair $(X, \partial X)$ of dimension n. A *PD embedding for* f is specified by homotopy finite spaces A and C along with a choice of factorization $\partial X \to C \to X$ fitting into a commutative diagram

$$(\mathscr{D}) \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] & \partial X \arrow[l] \\ K \arrow[r, "f"'] & X & \end{tikzcd}$$

such that

(i) (*Stratification*) The square is ∞-cocartesian, i.e., there is a weak homotopy equivalence of spaces $K \cup_A C \simeq X$.

(ii) (*Poincaré Duality*) The image of the fundamental class $[X]$ under the composite

$$H_n(X, \partial X) \cong H_n(\overline{X}, \partial X) \to H_n(\overline{X}, C) \cong H_n(\overline{K}, A)$$

equips $(\overline{K}, A)$ with the structure of a PD pair and, similarly, the image of $[X]$ with respect to the map $H_n(X, \partial X) \to H_n(\overline{C}, \partial X \amalg A)$ equips $(\overline{C}, \partial X \amalg A)$ with the structure of a PD pair.

(iii) (*Weak Transversality*) If $\mathrm{hodim}(K) \leq k$, then the map $A \to K$ is $(n-k-1)$-connected.

We call A the *gluing space*, C the *complement*, and f the *underlying map* of the PD embedding $\mathscr{D}$.

We can relativize the above as follows: Let (K, L) be a cofibration pair, with K and L homotopy finite, and let $(X, \partial X)$ be a PD pair of dimension n. Recall that $\mathrm{hodim}(K, L) = k$ if K can be obtained from L, up to homotopy, by attaching cells of dimension k, and k is the smallest such number so that this holds. Fix a map $f = (f_K, f_L) : (K, L) \to (X, \partial X)$.

Definition 1.1.3. A *relative PD embedding for* f consists of a commutative diagram of pairs of homotopy finite spaces

$$(\mathscr{E}) \qquad \begin{array}{ccc} (A_K, A_L) & \longrightarrow & (C_K, C_L) \\ \downarrow & & \downarrow \\ (K, L) & \xrightarrow[f]{} & (X, \partial X) \end{array}$$

such that

(i) (*Stratification*) Each of the associated squares of spaces

$$(\mathscr{D}_K) \quad \begin{array}{ccc} A_K & \longrightarrow & C_K \\ \downarrow & & \downarrow \\ K & \xrightarrow[f_K]{} & X \end{array} \qquad\qquad (\mathscr{D}_L) \quad \begin{array}{ccc} A_L & \longrightarrow & C_L \\ \downarrow & & \downarrow \\ L & \xrightarrow[f_L]{} & \partial X \end{array}$$

is ∞-cocartesian, and the square $\mathscr{D}_L$ is a PD embedding for f_L.

(ii) (*Poincaré Duality*) The image of the fundamental class $[X]$ under the composite

$$H_n(X, \partial X) \cong H_n(\overline{X}, \partial X) \to H_n(\overline{X}, \partial X \cup_{C_L} C_K) \cong H_n(\overline{K}, L \cup_{A_L} A_K)$$

equips $(\overline{K}, L \cup_{A_L} A_K)$ with the structure of a PD pair and, similarly, the image of $[X]$ with respect to the map

$$H_n(X, \partial X) \to H_n(\overline{C_K}, C_L \cup_{A_L} A_K)$$

equips $(\overline{C_K}, C_L \cup_{A_L} A_K)$ with the structure of a PD pair. (Here, coefficients are given by pulling back the given local system on X).

(iii) (*Weak Transversality*) If $\operatorname{hodim}(K, L) \leq k$, then the map $A_K \to K$ is $(n-k-1)$-connected.

Remark 1.1.4. In the situation above, if a PD embedding $\mathscr{D}_L$ for f_L is given, then a PD embedding of f which coincides with $\mathscr{D}_L$ on L is said to be a *PD embedding for f relative to $\mathscr{D}_L$*. By taking $L = \emptyset$, we recover the definiton of PD embedding given in Definition 1.1.2. We may denote such a PD embeding by a diagram of pairs

$$\begin{tikzcd} (A, \emptyset) \arrow[r] \arrow[d] & (C, \partial X) \arrow[d] \\ (K, \emptyset) \arrow[r, "f"'] & (X, \partial X). \end{tikzcd}$$

Definition 1.1.5. Let $\mathscr{D}_0$ and $\mathscr{D}_1$ be PD embeddings with underlying maps f_0,$f_1 : K \to X$ and suppose that we are given a homotopy $F : K \times D^1 \to X$ from f_0 to f_1. Then we have an associated embedding with underlying map

$$f_0 + f_1 : K \amalg K \to \partial(X \times D^1).$$

Denote this embedding by $\mathscr{D}_{K \amalg K}$ and consider the associated map of pairs

$$F : (K \times D^1, K \amalg K) \to (X \times D^1, \partial(X \times D^1)).$$

A *concordance* from $\mathscr{D}_0$ to $\mathscr{D}_1$ is a PD embedding of F relative to $\mathscr{D}_{K \amalg K}$.

1.2 The Space of PD Embeddings

Fix an object $(f, K) \in (\mathcal{T} \downarrow X)$. Define a category $\mathcal{D}_f(K, X)$ as follows. An object of $\mathcal{D}_f(K, X)$ is a commutative square of spaces

$$(\mathscr{D}) \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ K \arrow[r, "f"'] & X \end{tikzcd}$$

and a morphism of $\mathcal{D}_f(K, X)$ is specified by a commutative diagram

$$\begin{tikzcd} & A' \arrow[rr] \arrow[ddl] & & C' \arrow[ddl] \\ A \arrow[ur] \arrow[rr] \arrow[d] & & C \arrow[ur] \arrow[d] & \\ K \arrow[rr, "f"'] & & X & \end{tikzcd}$$

Proposition 1.2.1. *$\mathcal{D}_f(K, X)$ is a model category.*

Proof. Let $\mathbf{n}$ denote the category associated to the ordinal n as a poset, i.e., the category with n objects $1, 2, \ldots, n$ and a morphism $i \to j$ whenever $i \leq j$. Then the functor category $\mathcal{T}^{\mathbf{2}}$

is the same as the *arrow category* of $\mathcal{T}$, whose objects are the morphisms (arrows) of $\mathcal{T}$ and whose morphisms are maps of arrows forming commutative diagrams of spaces. Moreover, the category $\mathcal{T}^{\mathbf{2}}$ carries the structure of a model category (see [Ho99], Theorem 5.1.3). The fixed object $(f, K) \in (\mathcal{T} \downarrow X)$ used to define $\mathcal{D}_f(K, X)$ determines an object $K \xrightarrow{f} X$ in the arrow category. The proposition then follows by noting that the category $D_f(K, X)$ is the same as the model category $(\mathcal{T}^{\mathbf{2}} \downarrow f)$. $\square$

In particular, it follows (for example, from [Ho99], Theorem 5.1.3) that a morphism

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ A' & \longrightarrow & C' \end{array}$$

in $\mathcal{D}_f(K, X)$ is a

- *weak equivalence* (respectively, *fibration*) if each of the maps $A \to A'$ and $C \to C'$ are weak equivalences (respectively, fibrations) in $\mathcal{T}$
- *cofibration* if both $A \to A'$ and the induced map $A' \cup_A C \to C'$ are cofibrations in $\mathcal{T}$.

Remark 1.2.2. The object $\mathscr{D}$ of $\mathcal{D}_f(K, X)$ is cofibrant (respectively, fibrant) precisely when A is cofibrant in $\mathcal{T}$ and the map $A \to C$ is a cofibration in $\mathcal{T}$ (respectively, when both of the maps $A \to K$ and $C \to X$ are fibrations in $\mathcal{T}$).

Remark 1.2.3. In the situation of Definition 1.1.3, a more general version of the model structure above exists. In this case, we have a category $D_f(K, X \textit{ rel } L)$ in which the objects are commutative squares

$$
(\mathscr{E}) \qquad \begin{tikzcd}
(A_K, A_L) \arrow[r] \arrow[d] & (C_K, C_L) \arrow[d] \\
(K, L) \arrow[r, "f"'] & (X, \partial X)
\end{tikzcd}
$$

as in Definition 1.1.3. A morphism is a square of pairs

$$
\begin{tikzcd}
(A_K, A_L) \arrow[r] \arrow[d] & (C_K, C_L) \arrow[d] \\
(A'_K, A'_L) \arrow[r] & (C'_K, C'_L)
\end{tikzcd}
$$

covering the map $(K, L) \to (X, \partial X)$ in $\mathscr{E}$. The weak equivalences in $D_f(K, X \ rel\ L)$ are morphisms as above in which both of the vertical maps are weak homotopy equivalences of pairs. The object $\mathscr{E}$ is fibrant if the vertical maps are both fibrations betweeen the larger spaces that restrict to fibrations of the subspaces. Moreover, $\mathscr{E}$ is cofibrant when A_L is cofibrant in $\mathcal{T}$, (A_K, A_L) is a cofibration pair of spaces, and the induced map $A_K \cup_{A_L} C_L \to C_K$ is a cofibration in $\mathcal{T}$. In the case that L and A_L are both empty, and $C_L = \partial X$, we recover the model structure on $D_f(K, X)$ described above.

We can now define the space of PD embeddings for a given map of homotopy finite, cofibration pairs $(K, L) \xrightarrow{f} (X, \partial X)$ such that K is connected and $(X, \partial X)$ is a PD pair of dimension n. To this end, for such a map f, let $w\mathcal{D}_f(K, X \ rel\ L)$ denote the category with the same objects as $\mathcal{D}_f(K, X \ rel\ L)$ but whose only morphisms are weak equivalences, and let $E_f(K, X \ rel\ L)$ denote the full subcategory of $w\mathcal{D}_f(K, X \ rel\ L)$ with objects given by commutative squares $\mathscr{E}$ that determine PD embeddings for f in the sense of Definition 1.1.3.

Definition 1.2.4. The *space of PD embeddings for f*, denoted by $\mathbb{E}_f(K, X \ rel\ L)$, is the geometric realization of the nerve of $E_f(K, X \text{ rel } L)$. That is,

$$\mathbb{E}_f(K, X \text{ rel } L) = |E_f(K, X \text{ rel } L)|.$$

1.3 The Space of Fiberwise Duals

We now set out to define the space $\mathbb{SW}_f(K, X \text{ rel } L)$ of fiberwise duals for a given PD embedding for f. To do so, we will need to develop some *fiberwise homotopy theory*, most of which can be found in [CK09] and [Kl07].

Let (s_Z, Z, r_Z) and (s_W, W, r_W) be objects of the category $\mathcal{T}(X \xrightarrow{id} X)$ which are both fibrant and cofibrant. When no confusion will arise, we will refer to these objects as Z and W without mention of the structure maps.

Remark 1.3.1. The category $\mathcal{T}(X \xrightarrow{id} X)$ is probably known better as the *category of retractive spaces over* X and is often denoted by $R(X)$. It carries a model structure as described in the introduction. In particular, under the assumption that Z and W are cofibrant, we have the following diagram of cofibrations in $\mathcal{T}$:

$$Z \xleftarrow{s_Z} X \xrightarrow{s_W} W$$

and the canonical map

$$\operatorname{hocolim}(Z \xleftarrow{s_Z} X \xrightarrow{s_W} W) \xrightarrow{\cong} \operatorname{colim}(Z \xleftarrow{s_Z} X \xrightarrow{s_W} W)$$

is thus a homeomorphism of spaces. We will write $Z \cup_X W$ for the common value of the spaces displayed above.

Definition 1.3.2. The *fiberwise smash product* of Z and W is the object of $\mathcal{T}(X \stackrel{id}{\to} X)$ given by

$$Z \wedge_X W := Z \times_X W \cup_{Z \cup_X W} X$$

which we will regard as an object of $(\mathcal{T} \downarrow X)$ via the forgetful functor from $\mathcal{T}(X \stackrel{id}{\to} X)$.

Here, $Z \times_X W$ is the fiber product along X via the maps r_Z and r_W. As in the previous remark, our assumption that Z and W are both fibrant ensures the homotopy invariance of the fiber product construction. That is, $Z \times_X W$ will be used to denote the common value of the spaces

$$\lim(Z \xrightarrow{r_Z} X \xleftarrow{r_W} W) \cong \operatorname{holim}(Z \xrightarrow{r_Z} X \xleftarrow{r_W} W).$$

Definition 1.3.3. Let $Y \in (\mathcal{T} \downarrow X)$ with structure map $g : Y \to X$. The *j-fold unreduced fiberwise suspension* functor

$$\mathcal{S}_X^j : (\mathcal{T} \downarrow X) \to (\mathcal{T} \downarrow X)$$

is defined by

$$\mathcal{S}_X^j Y = Y \times D^j \cup_{Y \times S^{j-1}} X \times S^{j-1}.$$

That is, $\mathcal{S}_X^j Y$ is the *fiberwise join over* X of Y with S^{j-1}, denoted by $Y *_X S^{j-1}$. In general, we define the fiberwise join of an object $Y \in (\mathcal{T} \downarrow X)$ and a space U to be the space

$$Y *_X U = \operatorname{hocolim}(Y \leftarrow Y \times U \to X \times U)$$

considered as a space over X. Note that $\mathcal{S}_X Y$ is just the double mapping cylinder of g:

$$\mathcal{S}_X Y = X \times 0 \cup_{g\times 0} Y \times [0,1] \cup_{g\times 1} X \times 1.$$

Definition 1.3.4. Given an object $Y \in \mathcal{T}(X \xrightarrow{id} X)$, its *reduced fiberwise suspension* is given by

$$\Sigma_X Y = \mathcal{S}_X Y \cup_{\mathcal{S}_X X} X.$$

Note that Σ_X is an endofunctor of $\mathcal{T}(X \xrightarrow{id} X)$.

Definition 1.3.5. A *fibered spectrum* $\mathcal{E}$ over X consists of objects $\mathcal{E}_j \in \mathcal{T}(X \xrightarrow{id} X)$ for $j \in \mathbb{N}$ together with morphisms (structure maps)

$$\Sigma_X \mathcal{E}_j \to \mathcal{E}_{j+1}$$

for each $j \geq 0$. A morphism $\mathcal{E} \to \mathcal{E}'$ of fibered spectra over X is given by a collection of morphisms $\mathcal{E}_j \to \mathcal{E}'_j$ which are compatible with the structure maps.

We say that $\mathcal{E}$ is *fibrant* if the adjoints to the structure maps are weak homotopy equivalences of underlying spaces. That is, $\mathcal{E}$ is fibrant if the spectrum of underlying spaces is an Ω-spectrum in the sense of [CK09]. Moreover, any fibered spectrum $\mathcal{E}$ has a *fibrant replacement* $\mathcal{E}^f$ in which

$$\mathcal{E}^f_j = \operatorname{hocolim}_n \Omega^n_X \mathcal{E}_{j+n}.$$

Here, the homotopy colimit is taken in the category $\mathcal{T}(X \xrightarrow{id} X)$, and Ω^n_X is the adjoint to the n-fold reduced fiberwise suspension.

Remark 1.3.6. There is a category SP_X of fibered spectra over X whose morphisms are defined as above. Furthermore, this category can be equipped with a model structure (see [MS06]) where a morphism $\mathcal{E} \to \mathcal{E}'$ is a

- *weak equivalence* if the associated morphism of fibrant replacements $\mathcal{E}^f \to (\mathcal{E}')^f$ is a levelwise weak equivalence: the map $\mathcal{E}_j^f \to (\mathcal{E}')_j^f$ is a weak equivalence in $\mathcal{T}(X \xrightarrow{id} X)$.

- *cofibration* if the maps $\mathcal{E}_0 \to \mathcal{E}_0'$ and $\mathcal{E}_j \cup_{\Sigma_X \mathcal{E}_{j-1}} \Sigma_X \mathcal{E}_{j-1}' \to \mathcal{E}_j'$ are cofibrations in $\mathcal{T}(X \xrightarrow{id} X)$.

- *fibration* if it has the right lifting property with respect the trivial cofibrations.

Definition 1.3.7. For a morphism $A \to B$ in $\mathcal{T}(X \xrightarrow{id} X)$ which is an inclusion, define the *fiberwise quotient* to be the object of $\mathcal{T}(X \xrightarrow{id} X)$ given by

$$B /\!\!/ A = B \cup_A X.$$

We may regard $B /\!\!/ A$ as an object of $(\mathcal{T} \downarrow X)$ by means of the forgetful functor. Write

$$X^+ = X /\!\!/ \partial X$$

for the *double of* X. That is, X^+ is the object whose underlying space is $X \cup_{\partial X} X$.

Remark 1.3.8. For objects (r_Z, Z) and (r_W, W) of $(\mathcal{T} \downarrow X)$, we can write their fiberwise join as

$$Z *_X W = Z \times_X C_X(W) \cup_{Z \times_X W} C_X(Z) \times_X W$$

where $C_X(W)$ is the space over X given by taking the mapping cylinder of r_W, and similarly for $C_X(Z)$. The inclusion

$$C_X(Z) \times_X W \to Z *_X W$$

allows us to form the fiberwise quotient

$$(Z *_X W) /\!\!/ (C_X(Z) \times_X W)$$

which is equivalent to

$$Z^+ \wedge_X \mathcal{S}_X W.$$

Thus, we have a quotient morphism in $(\mathcal{T} \downarrow X)$ given by

$$Z *_X W \xrightarrow{\gamma} Z^+ \wedge_X \mathcal{S}_X W.$$

Definition 1.3.9. A *fiberwise duality map* for objects objects Z and W of $\mathcal{T}(X \xrightarrow{id} X)$ is a morphism

$$d : X^+ \to Z \wedge_X W$$

such that for all (cofibrant and fibrant) fibered spectra $\mathcal{E}$ over X, the assignment $g \mapsto (g \wedge_X id_W) \circ d$ determines an isomorphism of abelian groups

$$[Z, \mathcal{E}]_X \cong [X^+, \mathcal{E} \wedge_X W]_X$$

where $[\ ,\ ]_X$ denotes fiberwise homotopy classes, and the fiberwise smash product $\mathcal{E} \wedge_X W$ is defined by $(\mathcal{E} \wedge_X W)_j = \mathcal{E}_j \wedge_X W$.

Now, fix an object $(f, K) \in (\mathcal{T} \downarrow X)$, where K is a connected, homotopy finite complex and X is a PD space, possibly with boundary ∂X. Further, assume that $L \to K$ is a cofibratoin in $\mathcal{T}$ such that $f(L) \subset \partial X$ and $f|L : L \to \partial X$ is the underlying map of a given PD embedding with complement C_L. Then Richter duality ([Kl07], Proposition 8.3) provides us with a fiberwise duality map

$$\tilde{d}_L : (\partial X)^+ \to L^+ \wedge_{\partial X} \mathcal{S}_{\partial X} C_L$$

for L^+ and $\mathcal{S}_{\partial X} C_L$. Define a category $FD_f(K, X \ \mathit{rel}\ L)$ (of fiberwise duals of f) as follows. An object of $FD_f(K, X \ \mathit{rel}\ L)$ is a pair (d_K, C_K) such that (C_K, C_L) is a cofibration pair of spaces and

$$d_K : X \to K *_X C_K$$

is a morphism of spaces over X. This morphism gives rise to a based morphism

$$\tilde{d}_K : X^+ \to K^+ \wedge_X \mathcal{S}_X C_K$$

by composing with the quotient morphism constructed in Remark 1.3.8, and sending the new copy of X given by the "+" to the base section of the target via the identity map. We require $\tilde{d}_K$ to be a fiberwise duality map for K^+ and $\mathcal{S}_X C_K$ which restricts to the given fiberwise duality map $\tilde{d}_L$. A morphism $(d_K, C_K) \to (d'_K, C'_K)$ in $FD_f(K, X \ \mathit{rel}\ L)$ is given by a map $\alpha : C_K \to C'_K$ and a cofibration $C_L \to C'_K$ such that the diagram

$$\begin{tikzcd}[column sep=small] & C_L \arrow[dl] \arrow[dr] & \\ C_K \arrow[rr, "\alpha"] \arrow[dr] & & C'_K \arrow[dl] \\ & X & \end{tikzcd}$$

commutes and such that $(id_K *_X \alpha) \circ d_K = d'_K$. Similar to the above, there is a map

$$\tilde{d'}_K : X^+ \to K^+ \wedge_X \mathcal{S}_X C'_K$$

which we again require to be a fiberwise duality map for K and $\mathcal{S}_X C'_K$ which agrees with $\tilde{d}_L$ when restricted to $(\partial X)^+$. Let $wFD_f(K, X \text{ } rel \text{ } L)$ denote the category with the same objects as $FD_f(K, X \text{ } rel \text{ } L)$ but whose only morphisms are weak equivalences $C_K \to C'_K$.

Definition 1.3.10. The *space of fiberwise duals for* f, denoted by $\mathbb{SW}_f(K, X \text{ } rel \text{ } L)$, is the geometric realization of the nerve of the category $wFD_f(K, X \text{ } rel \text{ } L)$. That is,

$$\mathbb{SW}_f(K, X \text{ } rel \text{ } L) = |wFD_f(K, X \text{ } rel \text{ } L)|.$$

1.4 The Stabilization and Classification Maps

Let (K, L) and $(X, \partial X)$ be as in Definition 1.1.3. In this section we define the *stabilization* and *classification* maps from the introduction. We first consider the stabilization map:

$$\sigma : \mathbb{E}_f(K, X \text{ rel } L) \to \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L).$$

Let

$$\begin{tikzcd} (A_K, A_L) \arrow[r] \arrow[d] & (C_K, C_L) \arrow[d] \\ (K, L) \arrow[r, "f=(f_K, f_L)"'] & (X, \partial X) \end{tikzcd} \qquad (\mathscr{D})$$

be an object of $E_f(K, X \text{ rel } L)$. That is, $\mathscr{D}$ is a PD embedding for $f = (f_K, f_L)$ relative to the PD embedding

$$\begin{tikzcd} A_L \arrow[r] \arrow[d] & C_L \arrow[d] \\ L \arrow[r, "f_L"'] & \partial X \end{tikzcd} \qquad (\mathscr{D}_L)$$

We can picture the given Poincaré stratification of the pair $(X, \partial X)$ as follows:

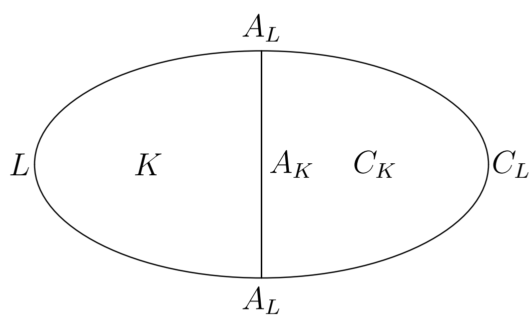


Figure 1.1: A PD Decomposition of $(X, \partial X)$

The motivation for the definition of the stabilization map below comes from "crossing the stratification above with the interval".

Definition 1.4.1. Define a functor

$$\tilde{\sigma} : E_f(K, X \text{ rel } L) \to E_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L)$$

on objects by

$$\begin{tikzcd}
(A_K, A_L) \arrow[r] \arrow[d] & (C_K, C_L) \arrow[d] \\
(K, L) \arrow[r, "f"'] & (X, \partial X)
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
(\mathcal{S}_{C_K} A_K, \mathcal{S}_{C_L} A_L) \arrow[r] \arrow[d] & (C_K \times D^1, C_L \times D^1) \arrow[d] \\
(\mathcal{S}_X K, \mathcal{S}_X L) \arrow[r, "\mathcal{S}_X f"'] & (X \times D^1, \partial(X \times D^1))
\end{tikzcd}$$

$$(\mathscr{D}) \qquad\qquad\qquad (\tilde{\sigma}\mathscr{D})$$

Note that $\mathcal{S}_X X = X \times D^1$. We will use both notations in what follows. The Poincaré stratification of $\tilde{\sigma}\mathscr{D}$ can be pictured as follows:

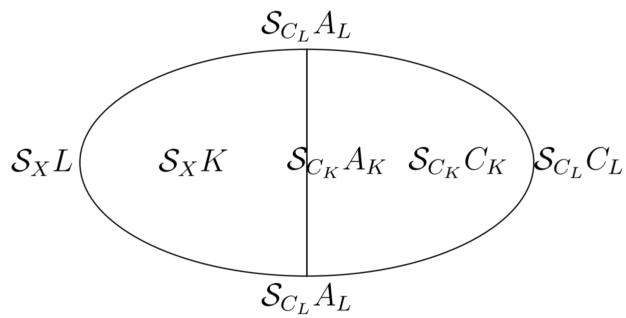


Figure 1.2: A PD Decomposition of $(X \times D^1, \partial(X \times D^1))$

Definition 1.4.2. The *stabilization map* σ is defined by applying the geometric realization functor to $\tilde{\sigma}$:

$$\sigma = |\tilde{\sigma}| : \mathbb{E}_f(K, X \text{ rel } L) \to \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L).$$

Establishing the connectivity of this map is the content of Theorem A.

We now turn to the definition of the classification map

$$\theta : \mathbb{E}_f(K, X \text{ rel } L) \to \mathbb{SW}_f(K, X \text{ rel } L).$$

To this end, let

$$(\mathscr{D}) \qquad \begin{tikzcd} (A_K, A_L) \arrow[r] \arrow[d] & (C_K, C_L) \arrow[d] \\ (K, L) \arrow[r, "f"'] & (X, \partial X) \end{tikzcd}$$

be an object of the category $E_f(K, X \text{ rel } L)$. Then $\mathscr{D}$ gives rise to a map

$$d_K : X \to K *_X C_K$$

constructed as follows. By definition, X is equivalent to $\mathrm{hocolim}(K \leftarrow A_K \to C_K)$. The square $\mathscr{D}$ provides us with a map $A_K \to K \times_X C_K$ and, thus we have an induced map

$$\mathrm{hocolim}(K \leftarrow A_K \to C_K) \to \mathrm{hocolim}(K \leftarrow K \times_X C_K \to C_K).$$

The target of this map is just the fiberwise join $K *_X C_K$. Since the source is equivalent to X, we have the desired map

$$d_K : X \to K *_X C_K.$$

Now, composing this map with the quotient map γ constructed in Remark 1.3.8 replaces the target above with $K^+ \wedge_X \mathcal{S}_X C_K$, where $K^+ = K \amalg X$. As we did when constructing the category of fiberwise duals, we can convert this composed map into a based morphism

$$\tilde{d}_K : X^+ \to K^+ \wedge_X \mathcal{S}_X C_K$$

by sending the new copy of X (provided by the "+") to the base section of the target via the identity map. This map is a fiberwise duality map, which can be seen by noting that it coincides with the following construction. There is a *fiberwise collapse* map (see [Kl07])

$$X^+ \to K /\!\!/ A_K$$

in $(\mathcal{T} \downarrow X)$ which arises as a fiberwise homotopy class by taking fiberwise quotients in the chain of fiberwise pairs

$$(X, \partial X) \leftarrow (K \cup_{A_K} C_K, \partial X) \to (K \cup_{A_K} X, X).$$

Notice that the first arrow in the chain above is a weak equivalence by the definition of PD embedding. Compose the fiberwise collapse above with the 'excision' weak equivalence

$$K /\!\!/ A_K = K \cup_{A_K} X \xrightarrow{\sim} X \cup_{C_K} X = \mathcal{S}_X C_K$$

given by the embedding diagram $\mathscr{D}$ to get a map $X^+ \to \mathcal{S}_X C_K$. Finally, compose with the *fiberwise diagonal* (see [Kl07], Section 8)

$$\mathcal{S}_X C_K \to K^+ \wedge_X \mathcal{S}_X C_K$$

to get the map

$$\tilde{d}_K : X^+ \to K^+ \wedge_X \mathcal{S}_X C_K.$$

By Richter duality ([Kl07], Proposition 8.3), this map is a fiberwise duality map. Note that by restricting the entire construction just made to the associated PD embedding with underlying map $L \to \partial X$, we obtain the given fiberwise duality map $\tilde{d}_L$. We can use this construction to define a functor

$$\tilde{\theta} : E_f(K, X \ \mathit{rel}\ L) \to wFD_f(K, X \ \mathit{rel}\ L)$$

which is given on objects by

$$\mathscr{D} \mapsto (d_K, C_K).$$

This correspondence respects the restriction to d_L and, thus, we have the classification map:

Definition 1.4.3. The *classification map* θ is defined by applying the geometric realization functor to $\tilde{\theta}$:

$$\theta = |\tilde{\theta}| : \mathbb{E}_f(K, X \ \mathit{rel}\ L) \to \mathbb{SW}_f(K, X \ \mathit{rel}\ L).$$

Establishing the connectivity of this map is the content of Theorem B.

CHAPTER 2

Classfication

2.1 Proof of Theorems B and D

We will now give proofs of Theorems B and D assuming Theorem A:

Theorem A (Stabilization). *Let $f : (K, L) \to (X, \partial X)$ be a map from a cofibration pair of homotopy finite spaces (K, L), with* $\operatorname{hodim}(K, L) = k$*, to a PD pair $(X, \partial X)$ of dimension n. Assume that $f : K \to X$ is r-connected $(r \geq 1)$ and that $k \leq n-3$. Then the stabilization map*

$$\sigma : \mathbb{E}_f(K, X \ \mathit{rel}\ L) \to \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \ \mathit{rel}\ \mathcal{S}_X L)$$

is $(n - 2(k - r) - 3)$-connected.

Define a functor

$$\tilde{\psi} : wFD_f(K, X \ \mathit{rel}\ L) \to wFD_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \ \mathit{rel}\ \mathcal{S}_X L)$$

as follows. Recall that an object of the domain category is specified by a given PD embedding with underlying map $L \to \partial X$ and complement C_L, along with a pair (d_K, C_K) such that

$$d_K : X \to K *_X C_K$$

is a morphism of spaces over X which induces a fiberwise duality map $\tilde{d}_K$ for K^+ and $\mathcal{S}_X C_K$.

The j-fold fiberwise suspension $S_X^j X$ coincides with $X \times D^j$. In particular, we can take the fiberwise suspension of the map d_K to obtain

$$\mathcal{S}_X d_K : X \times D^1 \to \mathcal{S}_X K *_X C_K.$$

This map again induces a fiberwise duality map

$$\widetilde{\mathcal{S}_X d_K} : (X \times D^1)^+ \to \mathcal{S}_X K^+ \wedge_X \mathcal{S}_X C_K$$

(see, e.g., [Kl07], Section 8). Thus, define the functor $\tilde{\psi}$ on objects by sending (d_K, C_K) to $(\mathcal{S}_X d_K, C_K)$. Applying the geometric realization functor to $\tilde{\psi}$ produces a map of spaces

$$\psi : \mathbb{SW}_f(K, X \textit{ rel } L) \to \mathbb{SW}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \textit{ rel } \mathcal{S}_X L).$$

Lemma 2.1.1. *The map ψ is $(n - 2(k - r) - 1)$-connected.*

Proof. Consider the map

$$F(X, K *_X C_K) \to F(\mathcal{S}_X X, \mathcal{S}_X K *_X C_K)$$

of underlying function spaces induced by fiberwise suspension. By the fiberwise Freudenthal suspension theorem ([Kl99a], Theorem 4.7), this map of function spaces induces a π_j-isomorphism provided that

$$\dim(\mathcal{S}_X^j X) < 2\,\mathrm{conn}(K *_X C_K) + 1$$

where the connectivity of $K *_X C_K$ is defined to be one less than the connectivity of the structure map $K *_X C_K \to X$, and $\dim(\mathcal{S}_X^j X)$ is the relative cohomological dimension of $\mathcal{S}_X^j K$. Furthermore, the map induces a surjection on π_j when the inequality above is replaced with equality. Since $C_K \to X$ is $(n-k-1)$-connected, and $K \to X$ is r-connected, we conclude that $\mathrm{conn}(K *_X C_K) = n-k+r-1$. As noted above, $\mathcal{S}^j_X X$ is just $X \times D^j$. We may regard X as having homotopy dimension n, since X is a PD space of dimension n ([Wa65]). Thus, $\dim(\mathcal{S}^j_X X) = n+j$, and the inequality above becomes

$$j < n - 2(k-r) - 1.$$

That is, the map of function spaces is $(n-2(k-r)-1)$-connected. Let

$$D(X, K *_X C_K)$$

denote the component of the space $F(X, K *_X C_K)$ consisting of those maps that induce fiberwise duality maps for K^+ and $\mathcal{S}_X C_K$. Then the induced map of components

$$D(X, K *_X C_K) \to D(\mathcal{S}_X X, \mathcal{S}_X K *_X C_K)$$

is also $(n-2(k-r)-1)$-connected. This implies that ψ is $(n-2(k-r)-1)$-connected, and finishes the proof. $\square$

Our goal is to show that the classification map (see Definition 1.4.3)

$$\mathbb{E}_f(K, X \ rel\ L) \xrightarrow{\theta} \mathbb{SW}_f(K, X \ rel\ L)$$

is $(n-2(k-r)-3)$-connected. Recall that this is the connectivity of the stabilization map, σ. Write ψ_0 for the map ψ constructed above and, in general, write

$$\psi_j : \mathbb{SW}_{\mathcal{S}^j_X f}(\mathcal{S}^j_X K, X \times D^j \ rel\ \mathcal{S}^j_X L) \to \mathbb{SW}_{\mathcal{S}^{j+1}_X f}(\mathcal{S}^{j+1}_X K, X \times D^{j+1} \ rel\ \mathcal{S}^{j+1}_X L).$$

By the previous lemma, we have $\mathrm{conn}(\psi_0) = \mathrm{conn}(\sigma) + 2$. Furthermore, it is straightforward to check that $\mathrm{conn}(\psi_j)$ increases with j. Thus, to verify that $\mathrm{conn}(\theta) = \mathrm{conn}(\sigma)$, it will suffice to show that the composed map (which we still call θ)

$$\theta : \mathbb{E}_f(K, X \ \mathit{rel}\ L) \to \mathbb{SW}^{st}(K, X \ \mathit{rel}\ L)$$

is $(n - 2(k - r) - 3)$-connected, where

$$\mathbb{SW}^{st}(K, X \ \mathit{rel}\ L) = \mathrm{colim}_j\, \mathbb{SW}_{\mathcal{S}^j_X f}(\mathcal{S}^j_X K, X \times D^j \ \mathit{rel}\ \mathcal{S}^j_X L).$$

Using the map θ, along with the stabilization map, form the square

$$(\mathscr{B}) \qquad \begin{tikzcd} \mathbb{E}_f(K, X \ \mathit{rel}\ L) \arrow[r, "\theta"] \arrow[d, "\sigma"'] & \mathbb{SW}^{st}(K, X \ \mathit{rel}\ L) \arrow[d, "="] \\ \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \ \mathit{rel}\ \mathcal{S}_X L) \arrow[r, "\theta_1"'] & \mathbb{SW}^{st}(\mathcal{S}_X K, \mathcal{S}_X X \ \mathit{rel}\ \mathcal{S}_X L) \end{tikzcd}$$

where θ_1 is a suitably adjusted version of the classification map.

Lemma 2.1.2. *The square* $\mathscr{B}$ *is* $\mathrm{conn}(\sigma) = (n - 2(k - r) - 3)$*-cartesian.*

Proof. Recall that we are assuming Theorem A. The lemma follows from ([Go92], Prop. 1.6 (ii)). □

The square $\mathscr{B}$ gives rise to the infinite ladder

$$\begin{tikzcd}
\mathbb{E}_f(K, X \ \mathit{rel}\ L) \arrow[r, "\theta"] \arrow[d] & \mathbb{SW}^{st}(K, X \ \mathit{rel}\ L) \arrow[d, "="] \\
\mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \ \mathit{rel}\ \mathcal{S}_X L) \arrow[r, "\theta_1"'] \arrow[d] & \mathbb{SW}^{st}(\mathcal{S}_X K, \mathcal{S}_X X \ \mathit{rel}\ \mathcal{S}_X L) \arrow[d, "="] \\
\vdots \arrow[d] & \vdots \arrow[d, "="] \\
\mathbb{E}_{\mathcal{S}^t_X f}(\mathcal{S}^t_X K, X \times D^t \ \mathit{rel}\ \mathcal{S}^t_X L) \arrow[r, "\theta_t"'] \arrow[d] & \mathbb{SW}^{st}(\mathcal{S}^t_X K, \mathcal{S}^t_X X \ \mathit{rel}\ \mathcal{S}^t_X L) \arrow[d, "="] \\
\vdots & \vdots
\end{tikzcd}$$

where the tower on the left is obtained by repeated application of the stabilization map.

Lemma 2.1.3. *The map θ_∞, obtained after passing to homotopy colimits in the tower above, is a weak homotopy equivalence.*

Proof. A 0-cell of $\mathbb{SW}^{st}(K, X \ \mathit{rel}\ L)$ is represented by a space C along with a stable map

$$d : X \to K *_X C$$

which induces a (stable) fiberwise duality map $\tilde{d}$ for K^+ and C. For t large, write $\mathrm{hofiber}_C(\theta_t)$ for the homotopy fiber of θ_t with respect to the space C. Suppressing the notation of pairs, we can think of a 0-cell in $\mathrm{hofiber}_C(\theta_t)$ as a PD embedding of C in $X \times D^t$:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & W \arrow[d] \\
C \arrow[r] & X \times D^t
\end{tikzcd}$$

which, after application of θ_t, gives rise to the stable fiberwise duality map

$$\tilde{d} : X^+ \to K^+ \wedge_X \mathcal{S}_X C.$$

For t large, the uniqueness of fiberwise duals then provides us with a stable identification

$$\mathcal{S}_X^t K \dashrightarrow W.$$

At the expense of (possibly) suspending further, we can assume that we have an unstable identification of $\mathcal{S}_X^t K$ with W. Hence, $\mathrm{hofiber}_C(\theta_t)$ in nonempty. Suppose, now, that we are given two embeddings in $\mathrm{hofiber}_C(\theta_t)$:

$$(\mathscr{D}_1) \quad \begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ \mathcal{S}_X^t K & \longrightarrow & X \times D^t \end{array} \qquad (\mathscr{D}_2) \quad \begin{array}{ccc} A' & \longrightarrow & W \\ \downarrow & & \downarrow \\ \mathcal{S}_X^t K & \longrightarrow & X \times D^t \end{array}$$

As above, for large t, we have an identification up to homotopy of the pairs (C, A) and (W, A'). Moreover, the duality data provides us with a commutative triangle

$$\begin{tikzcd} & K^+ \wedge_X \mathcal{S}_X C \arrow[d, "h \wedge_X id"] \\ X^+ \arrow[ur, "\tilde{d}"] \arrow[r, "\tilde{d}"'] & K^+ \wedge_X \mathcal{S}_X C \end{tikzcd}$$

which shows that we have a stable self-equivalence $h : K^+ \to K^+$. This gives rise to an associated map of pairs

$$(\mathcal{S}_X^t K \times D^1, \mathcal{S}_X^t K \times S^0) \to (X \times D^{t+1}, \partial(X \times D^{t+1}))$$

which, when combined with the identifications above, produces a concordance from $\mathscr{D}_1$ to $\mathscr{D}_2$. Thus, $\mathrm{hofiber}_C(\theta_t)$ is connected. It only remains to show that h is equivalent to the identity map of $\mathcal{S}^t_X K$. But the existence of the stable fiberwise duality map $\tilde{d}$ for K^+ and $\mathcal{S}_X C$ means precisely that we have an isomorphism of ableian groups

$$\{K^+, K^+\}_X \xrightarrow{\cong} \{X^+, K^+ \wedge_X \mathcal{S}_X C\}_X$$

where $\{,\}_X$ denotes stable fiberwise homotopy classes. Moreover, by construction, the self-equivalence h and the identity map id_{K^+} both map to $\tilde{d}$ under this isomorphism. Thus, h is stably equivalent to the identity and, as above, after suspending further, we may assume that we have the desired unstable equivalence. Recalling that we are working relatively, the uniqueness statement, along with ([Kl07], Theorems A, B, and Remark 1.1) show that $\mathrm{hofiber}_C(\theta_t)$ is highly-connected, with connectivity increasing with t.

□

We can now prove

Theorem B (Classification). *With the assumptions of Theorem A, the classification map*

$$\theta : \mathbb{E}_f(K, X \ rel\ L) \to \mathbb{SW}_f(K, X \ rel\ L)$$

is $(n - 2(k - r) - 3)$-connected.

Proof. The squares in the tower above become highly cartesian as we move downward. Thus, in light of the previous lemma, we may assume, by a downward induction, that the map θ_1 in the square $\mathscr{B}$ is $(\mathrm{conn}(\sigma) + j)$-connected, for some $j \geq 0$. Let Q denote the homotopy limit of the diagram gotten from $\mathscr{B}$ by considering only the bottom horizontal and right vertical maps. Then the canonical map

$$Q \to \mathbb{SW}_f(K, X \ rel\ L)$$

is conn(θ_1)-connected. By Lemma 2.1.2, the canonical map

$$\mathbb{E}_f(K, X \ rel\ L) \to Q$$

is $(n - 2(k - r) - 3)$-connected and, thus, we have a commutative triangle

$$\begin{tikzcd} \mathbb{E}_f(K, X \ rel\ L) \arrow[rr, "\theta"] \arrow[dr] & & \mathbb{SW}_f(K, X \ rel\ L) \\ & Q \arrow[ur] & \end{tikzcd}$$

Using this triangle and ([Go92], Prop. 1.5 (i)), we conclude that $\text{conn}(\theta) = \text{conn}(\sigma)$. This completes the proof of the theorem. $\square$

Theorem B implies the PD analog of the Connolly-Williams Classification Theorem:

Theorem D. *Let K be a homotopy finite complex with* $\text{hodim}(K) = k \leq n - 3$. *Assume that $f : K \to S^n$ is an r-connected map of spaces, $r \geq 1$. Then the induced map*

$$\pi_0(\theta) : \pi_0(\mathbb{E}_f(K, S^n)) \to \pi_0(\mathbb{SW}_f(K, S^n))$$

is surjective for $n \geq 2(k - r) + 3$ and injective for $n \geq 2(k - r) + 4$.

This establishes Theorems B and D assuming Theorem A. In the following section, we prove a technical result that will play a crucial role in the proof of Theorem A.

2.2 The 4-D Face Theorem

Recall that $\mathbf{n}$ denotes the category associated to the ordinal n as a poset, and write $P(\mathbf{n})$ for its poset of subsets, which we also regard as a category.

Theorem 2.2.1. *(4-Dimensional Face Theorem)*

Let $X : P(\mathbf{4}) \to \mathcal{T}$ be the 4-dimensional cubical diagram of spaces represented by the commutative diagram

$$\begin{tikzcd}
& X_\emptyset \arrow[rr] \arrow[dl] \arrow[ddd] & & X_3 \arrow[dl] \arrow[ddd] & & & X_4 \arrow[rr] \arrow[dl] \arrow[ddd] & & X_{34} \arrow[dl] \arrow[ddd] \\
X_1 \arrow[rr] \arrow[ddd] & & X_{13} \arrow[ddd] & & & X_{14} \arrow[rr] \arrow[ddd] & & X_{134} \arrow[ddd] & \\
& & & {} \arrow[rr] & & {} & & & \\
& X_2 \arrow[rr] \arrow[dl] & & X_{23} \arrow[dl] & & & X_{24} \arrow[rr] \arrow[dl] & & X_{234} \arrow[dl] \\
X_{12} \arrow[rr] & & X_{123} & & & X_{124} \arrow[rr] & & X_{1234} &
\end{tikzcd}$$

Assume that

- *The 4-cube X is ∞-cartesian*
- *The spaces X_S are connected for each nonempty $S \subset \mathbf{4}$*
- *Each 3-dimensional face which meets X_{1234} is strongly cocartesian*
- *Each map $X_S \to X_{S \cup \{i\}}$ is k_i-connected for S and $\{i\}$ nonempty subsets of $\mathbf{4}$, $i \notin S$.*
- *$k_i, k_j \geq 2$ for some $i \neq j$.*

Then each of the squares

$$\begin{tikzcd}
X_\emptyset \arrow[r] \arrow[d] & X_j \arrow[d] \\
X_i \arrow[r] & X_{ij}
\end{tikzcd}$$

is $\left(\sum_{i=1}^{4} k_i - 1\right)$-cocartesian for $1 \leq i < j \leq k$.

Remark 2.2.2. Let T be a nonempty subset of $\mathbf{4}$ and let $\partial_{\mathbf{4}-T}X$ denote the $|T|$-face of X terminating in X_{1234}. Suppose each of these $|T|$-faces is k_T-cartesian. One can easily check that $\min\{\sum_\alpha k_{T_\alpha}\} = \sum_{i=1}^4 k_i - 2$, where the minimum is taken over all partitions $\{T_\alpha\}_\alpha$ of $\mathbf{4}$ by nonempty subsets. By Goodwillie's generalized dual Blakers-Massey Theorem ([Go92], Theorem 2.6) X is $\left(\sum_{i=1}^4 k_i + 1\right)$-cocartesian. Write X as a map of 3-cubes $Y \to Z$. By hypothesis, Z is strongly cocartesian. Let $H_*(X)$ denote the reduced homology of the total cofiber of X, and similarly for Y and Z. Then $H_n(Z) = 0$ for all n and $H_n(X) = 0$ for $n \leq \sum_{i=1}^4 k_i + 1$. From the long exact sequence

$$\cdots \to H_n(Z) \to H_n(X) \to H_{n-1}(Y) \to H_{n-1}(Z) \to \cdots$$

we conclude that $H_n(Y) = 0$ for $n \leq \sum_{i=1}^4 k_i$. This result motivates the first claim made in the proof below.

Proof of 4D Face Theorem. As above, let X denote the 4-cube. Without loss in generality, assume that all of the maps in X are fibrations. Using Remark 2.2.2 and an argument similar to that given in the proof of ([Kl99a], Theorem 5.1), our final hypothesis in the statement of the theorem guarantees that $X_\emptyset$ is nonempty and connected.

Claim 2.2.3. *Each of the* 3*-dimensional subcubical diagrams*

$$\begin{tikzcd}
 & X_\emptyset \arrow[rr] \arrow[dl] \arrow[dd] & & X_k \arrow[dl] \arrow[dd] \\
X_i \arrow[rr] \arrow[dd] & & X_{ik} \arrow[dd] & \\
 & X_j \arrow[rr] \arrow[dl] & & X_{jk} \arrow[dl] \\
X_{ij} \arrow[rr] & & X_{ijk} &
\end{tikzcd}$$

is $\sum_{i=1}^{4} k_i$*-cocartesian for* $1 \leq i < j < k \leq 4$.

Proof of Claim: Choose one of the 3-cubes meeting $X_\emptyset$ and call it Y, say

$$\begin{tikzcd}
 & X_\emptyset \arrow[rr] \arrow[dl] \arrow[dd] & & X_3 \arrow[dl] \arrow[dd] \\
X_1 \arrow[rr] \arrow[dd] & & X_{13} \arrow[dd] & \\
 & X_2 \arrow[rr] \arrow[dl] & & X_{23} \arrow[dl] \\
X_{12} \arrow[rr] & & X_{123} &
\end{tikzcd}$$

It will be enough to prove the claim for this 3-cube. Let Z denote the 3-cube opposite Y in X. That is, Z is the 3-cube

$$\begin{tikzcd}
 & X_4 \arrow[rr] \arrow[dl] \arrow[dd] & & X_{34} \arrow[dl] \arrow[dd] \\
X_{14} \arrow[rr] \arrow[dd] & & X_{134} \arrow[dd] & \\
 & X_{24} \arrow[rr] \arrow[dl] & & X_{234} \arrow[dl] \\
X_{124} \arrow[rr] & & X_{1234} &
\end{tikzcd}$$

Let $\operatorname{hocolim}(X - X_{1234})$ denote the homotopy colimit of the restriction of X to the subposet of proper subsets of $P(\mathbf{4})$. Similarly, let $\operatorname{hocolim}(Y - X_{123})$ and $\operatorname{hocolim}(Z - X_{1234})$ denote the analogous homotopy colimits associated with Y and Z, respectively. Then there is an induced map

$$\operatorname{hocolim}(Y - X_{123}) \to \operatorname{hocolim}(Z - X_{1234}).$$

Let $\mathcal{A}$ denote the diagram

$$X_{123} \leftarrow \operatorname{hocolim}(Y - X_{123}) \to \operatorname{hocolim}(Z - X_{1234})$$

Then $\operatorname{hocolim}(\mathcal{A})$ is equivalent to $\operatorname{hocolim}(X - X_{1234})$ and we have the following commutative diagram:

$$
\begin{tikzcd}
\operatorname{hocolim}(Y - X_{123}) \arrow[rr] \arrow[dd] & & \operatorname{hocolim}(Z - X_{1234}) \arrow[dl, "f"] \arrow[dd, "\simeq"] \\
& \operatorname{hocolim}(X - X_{1234}) \arrow[dr, "g"] & \\
X_{123} \arrow[ur] \arrow[rr] & & X_{1234}
\end{tikzcd}
\tag{2.1}
$$

The right vertical map is an equivalence since Z is strongly cocartesian. The canonical map g is $\left(\sum_{i=1}^{4} k_i + 1\right)$-connected since X is $\left(\sum_{i=1}^{4} k_i + 1\right)$-cocartesian. By ([Go92], Prop 1.5(ii)) the map f is $\left(\sum_{i=1}^{4} k_i\right)$-connected. Rewrite the upper left triangle of (2.1) as the ∞-cocartesian square

$$
\begin{tikzcd}
\operatorname{hocolim}(\mathcal{Y} - X_{123}) \arrow[r] \arrow[d] & \operatorname{hocolim}(\mathcal{Z} - X_{1234}) \arrow[d, "f"] \\
X_{123} \arrow[r] & \operatorname{hocolim}(\mathcal{X} - X_{1234})
\end{tikzcd}
\tag{2.2}
$$

Our goal is to show that the left vertical map is $\left(\sum_{i=1}^{4} k_i\right)$-connected. By ([Kl99a], Lemma 5.6(2)) it will be enough to show that the top horizontal map is 2-connected. To this end, note that each of the spaces in (2.2) admits a map to X_{1234}. For any choice of basepoint in X_{1234}, the square of homotopy fibers over X_{1234} is ∞-cocartesian. The homotopy fiber of the map $\operatorname{hocolim}(Z - X_{1234}) \to X_{1234}$ is equivalent to a point since Z is ∞-cocartesian. So, we have an ∞-cocartesian square

$$
\begin{tikzcd}
\mathrm{hofiber}(\mathrm{hocolim}(Y - X_{123}) \to X_{1234}) \arrow[r] \arrow[d] & * \arrow[d] \\
\mathrm{hofiber}(X_{123} \to X_{1234}) \arrow[r] & \mathrm{hofiber}(\mathrm{hocolim}(X - X_{1234}) \to X_{1234})
\end{tikzcd}
\tag{2.3}
$$

Let s denote the connectivity of the space

$$\mathrm{hofiber}(\mathrm{hocolim}(Y - X_{123}) \to X_{1234})$$

Then the top horizontal map in (2.3) is $s+1$-connected. Hence, the homotopy cofiber of the top horizontal arrow in (2.3) is $s+1$-connected. Let C_{top} denote this homotopy cofiber.

Now, by hypothesis, the map $X_{123} \to X_{1234}$ is k_4-connected. Since the map g in (2.1) is $\left(\sum_{i=1}^{4} k_i + 1\right)$-connected we have that the map

$$X_{123} \to \mathrm{hocolim}(X - X_{1234})$$

is k_4-connected ([Go92], Prop 1.5(ii)). The five-lemma applied to the map of long exact sequences on homotopy induced by the diagram

$$
\begin{tikzcd}
\mathrm{hofiber}(X_{123} \to X_{1234}) \arrow[r] \arrow[d] & X_{123} \arrow[r] \arrow[d] & X_{1234} \arrow[d, "="] \\
\mathrm{hofiber}(\mathrm{hocolim}(X - X_{1234}) \to X_{1234}) \arrow[r] & \mathrm{hocolim}(X - X_{1234}) \arrow[r] & X_{1234}
\end{tikzcd}
$$

implies that the left vertical arrow, and hence the bottom horizontal arrow in (2.3), is k_4-connected. Thus the homotopy cofiber of the bottom horizontal arrow in (2.3) is k_4-connected. Let C_{bottom} denote this cofiber. Since (2.3) is ∞-cocartesian, we have a weak equivalence $C_{top} \xrightarrow{\sim} C_{bottom}$. This implies that $s + 1 = k_4$. But we assumed that $k_4 \geq 2$, so $s \geq 1$. That is, the connectivity of the space hofiber(hocolim$(Y - X_{123}) \to X_{1234}$) is at least 1. This implies, by definition, that the map hocolim$(Y - X_{123}) \to X_{1234}$ is at least 2-connected. But hocolim$(Z - X_{1234})$ is weakly equivalent to X_{1234}. Hence the top horizontal map in (2.2) is at least 2-connected, and the claim follows.

Now we prove the statement concerning the degree to which each 2-face meeting $X_\emptyset$ is cocartesian. Choose one of these 2-faces and call it V, say

$$\begin{tikzcd}
X_\emptyset \arrow[r] \arrow[d] & X_1 \arrow[d] \\
X_2 \arrow[r] & X_{12}
\end{tikzcd}$$

It will be enough to show that V is $\sum_{i=1}^{4} k_i - 1$-cocartesian. By hypothesis, the 3-cube

$$\begin{tikzcd}
& X_3 \arrow[rr] \arrow[dl] \arrow[dd] & & X_{34} \arrow[dl] \arrow[dd] \\
X_{13} \arrow[rr, crossing over] \arrow[dd] & & X_{134} & \\
& X_{23} \arrow[rr] \arrow[dl] & & X_{234} \arrow[dl] \\
X_{123} \arrow[rr] & & X_{1234} \arrow[from=uu, crossing over] &
\end{tikzcd}$$

is strongly cocartesian. Thus, the face

$$\begin{tikzcd}
X_3 \arrow[r] \arrow[d] & X_{13} \arrow[d] \\
X_{23} \arrow[r] & X_{123}
\end{tikzcd}$$

is ∞-cocartesian. Call this face W. As in the claim above, there is an induced map

$$\text{hocolim}\,(V - X_{12}) \to \text{hocolim}\,(W - X_{123})$$

and a commutative diagram

$$\begin{tikzcd}
\mathrm{hocolim}\,(V - X_{12}) \arrow[rr] \arrow[d] & & \mathrm{hocolim}\,(W - X_{123}) \arrow[dl, "\alpha"'] \arrow[d, "\simeq"] \\
& \mathrm{hocolim}(Y - X_{123}) \arrow[dr, "\beta"] & \\
X_{12} \arrow[ur] \arrow[rr] & & X_{123}
\end{tikzcd} \tag{2.4}$$

The right vertical arrow is an equivalence equivalence and, by the claim above, the map β is $\sum_{i=1}^{4} k_i$-connected. So, the map α is $(\sum_{i=1}^{4} k_i) - 1$-connected ([Go92], Prop 1.5(ii)). Rewrite the upper left triangle of (2.4) as the ∞-cocartesian square

$$\begin{tikzcd}
\mathrm{hocolim}\,(V - X_{12}) \arrow[r] \arrow[d] & \mathrm{hocolim}\,(W - X_{123}) \arrow[d, "\alpha"] \\
X_{12} \arrow[r] & \mathrm{hocolim}(Y - X_{123})
\end{tikzcd} \tag{2.5}$$

Our goal is to show that the left vertical map in (2.5) is $\left(\sum_{i=1}^{4} k_i - 1\right)$-connected. By ([Kl99a], Lemma 5.6(2)) it will be enough to show that the top horizontal map is 2-connected. We will make an argument similar to the one above. Map each of the spaces in the square (2.5) to X_{123}. As above, for any choice of basepoint in X_{123}, the square of homotopy fibers over X_{123} is ∞-cocartesian. The homotopy fiber of the map $\mathrm{hocolim}(W - X_{123}) \to X_{123}$ is equivalent to a point since W is ∞-cocartesian. So, we have an ∞-cocartesian square

$$\begin{tikzcd}
\mathrm{hofiber}(\mathrm{hocolim}(V - X_{12}) \to X_{123}) \arrow[r] \arrow[d] & * \arrow[d] \\
\mathrm{hofiber}(X_{12} \to X_{123}) \arrow[r] & \mathrm{hofiber}(\mathrm{hocolim}(Y - X_{123}) \to X_{123})
\end{tikzcd} \tag{2.6}$$

Let t denote the connectivity of the space

$$\text{hofiber}(\text{hocolim}(V - X_{12}) \to X_{123})$$

Then the top horizontal map in (2.6) is $t+1$-connected. Hence, the homotopy cofiber of the top horizontal arrow in (2.6) is $t+1$-connected. Let C'_{top} denote this homotopy cofiber. Now, by hypothesis, the map $X_{12} \to X_{123}$ is k_3-connected. Since the map β in (2.4) is $\left(\sum_{i=1}^{4} k_i\right)$-connected we have that the map $X_{123} \to \text{hocolim}(X - X_{1234})$ is k_3-connected ([Go92], Prop 1.5(ii)). The five-lemma applied to the map of long exact sequences on homotopy induced by the diagram

$$\begin{tikzcd}
\text{hofiber}(X_{12} \to X_{123}) \arrow[r] \arrow[d] & X_{12} \arrow[r] \arrow[d] & X_{123} \arrow[d, "="] \\
\text{hofiber}(\text{hocolim}(Y - X_{123}) \to X_{123}) \arrow[r] & \text{hocolim}(Y - X_{123}) \arrow[r] & X_{123}
\end{tikzcd}$$

implies that the left vertical arrow, and hence the bottom horizontal arrow in (2.6), is k_3-connected. Thus the homotopy cofiber of the bottom horizontal arrow in (2.6) is k_3-connected. Let C'_{bottom} denote this cofiber. Since (2.6) is ∞-cocartesian, we have a weak equivalence $C'_{top} \xrightarrow{\sim} C'_{bottom}$. This implies that $t+1 = k_3$. But we assumed that $k_3 \geq 2$, so $t \geq 1$. That is, the connectivity of the space $\text{hofiber}(\text{hocolim}(V - X_{12}) \to X_{123})$ is at least 1. This implies, by definition, that the map $\text{hocolim}(V - X_{12}) \to X_{123}$ is at least 2-connected. But $\text{hocolim}(W - X_{123})$ is weakly equivalent to X_{123}. Hence the top horizontal map in (2.5) is at least 2-connected, and the theorem follows. $\square$

CHAPTER 3

Stabilization

3.1 Decompression and Section Data for PD Embeddings

Standing Assumptions: From now on, we assume that we have a fixed object $(f, K) \in (\mathcal{T} \downarrow X)$, where $f : K \to X$ is an r-connected $(r \geq 1)$ map from a connected, homotopy finite space K to a PD space X of dimension n, possibly with boundary ∂X. Further, assume that we are given a homotopy finite space L (possibly empty) and a cofibration $L \to K$ along with a map of pairs $(K, L) \to (X, \partial X)$. Finally, assume that $\operatorname{hodim}(K, L) = k \leq n-3$.

Definition 3.1.1. Let

$$(\mathscr{D}) \qquad \begin{array}{ccc} (A_K, A_L) & \longrightarrow & (C_K, C_L) \\ \downarrow & & \downarrow \\ (K, L) & \xrightarrow[f]{} & (X, \partial X) \end{array}$$

denote an object of the category $E_f(K, X \text{ rel } L)$, which we will also think of as the corresponding 0-cell of the space $\mathbb{E}_f(K, X \text{ rel } L)$. Let f^j denote the effect of the map $f : K \to X$ followed by the inclusion $X \to X \times D^j$. Define a functor (called the *decompression* functor[1])

$$\tilde{\delta} : E_f(K, X \text{ rel } L) \to E_{f^1}(K, X \times D^1 \text{ rel } L)$$

[1]For a detailed construction of the decompression functor, see [Kl02b], Definition 2.4.

on objects by sending $\mathscr{D}$ to the relative PD embedding

$$(\tilde{\delta}\mathscr{D}) \qquad \begin{array}{ccc} (\mathcal{S}_K A_K, \mathcal{S}_L A_L) & \longrightarrow & (\mathcal{S}_X C_K, \mathcal{S}_X C_L) \\ \downarrow & & \downarrow \\ (K, L) & \xrightarrow[f^1]{} & (X \times D^1, \partial(X \times D^1)) \end{array}$$

Applying the geometric realization functor to $\tilde{\delta}$ gives the *decompression map*

$$\delta = |\tilde{\delta}| : \mathbb{E}_f(K, X \text{ rel } L) \to \mathbb{E}_{f^1}(K, X \times D^1 \text{ rel } L).$$

Remark 3.1.2. There is an alternative version of the decompression functor

$$\delta_J : E_f(K, X \ rel \ L) \to E_{f \times id}(K \times J, X \times D^1 \ rel \ L \times J)$$

where $J = [1/3, 2/3]$(see [Kl02a]). The restriction

$$E_{f\times id}(K \times J, X \times D^1 \ rel \ L \times J) \to E_{f^1}(K \times 1/2, X \times D^1 \ rel \ L \times 1/2)$$

induces an equivalence on geometric realizations.

In what follows, we will apply the stabilization map to the decompression $\tilde{\delta}\mathscr{D}$. By definition, this will land us in the space

$$(*) \qquad \mathbb{E}_{\mathcal{S}_X f^1}(\mathcal{S}_{X\times D^1} K, X \times D^2 \ rel \ \mathcal{S}_{X \times D^1} L).$$

We will also apply the decompression map to a given PD embedding with underlying map $\mathcal{S}_X f$. Using the previous remark, this will land us in the space

$$(**) \qquad \mathbb{E}_{\mathcal{S}_X f \times id}(\mathcal{S}_X K \times J, X \times D^2 \ rel \ \mathcal{S}_{X\times D^1} L \times J).$$

The restriction map displayed above, along with the weak equivalence

$$\mathcal{S}_{X\times D^1}K \xrightarrow{\sim} \mathcal{S}_X K$$

of spaces over $X \times D^2$, induces an equivalence between the realizations $(*)$ and $(**)$. Write

$$\mathbb{E}_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \ rel\ \mathcal{S}_X L)$$

for these equivalent spaces. Then the stabilization and decompression maps give rise to the following square, which commutes up to preferred weak equivalence:

$$(\mathscr{D}_{\sigma,\delta}) \qquad \begin{tikzcd} \mathbb{E}_f(K, X \ rel\ L) \arrow[r] \arrow[d] & \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \ rel\ \mathcal{S}_X L) \arrow[d] \\ \mathbb{E}_{f^1}(K, X \times D^1 \ rel\ L) \arrow[r] & \mathbb{E}_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \ rel\ \mathcal{S}_X L) \end{tikzcd}$$

Remark 3.1.3. The bulk of the proof of Theorem A lies in proving that the square $\mathscr{D}_{\sigma,\delta}$ is 0-cartesian. The rest of this section will be devoted to constructing certain maps (sections of maps in given PD embeddings) that will allow us to use the 4-D Face Theorem (Theorem 2.2.1) to prove this claim.

Lemma 3.1.4. *Let $g : X \to Y$ be a t-connected map of based spaces, and assume that* $\operatorname{conn}(X) = \operatorname{conn}(Y) = s$. *Then the square*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"'] & \Omega\Sigma X \arrow[d] \\ Y \arrow[r] & \Omega\Sigma Y \end{tikzcd}$$

is $(s+t)$-cartesian.

Proof. Recall that for a based space Z with basepoint $*$, the homotopy fiber of the inclusion

$* \hookrightarrow Z$ is the based loop space ΩZ. The square in question is then determined by taking the homotopy fibers of the horizontal maps in the ∞-cocartesian 3-cube

$$\begin{tikzcd}
& X \arrow[rr] \arrow[dl] \arrow[dd, "g"'] & & * \arrow[dl] \arrow[dd] \\
* \arrow[rr] \arrow[dd] & & \Sigma X \arrow[dd] & \\
& Y \arrow[rr] \arrow[dl] & & * \arrow[dl] \\
* \arrow[rr] & & \Sigma Y &
\end{tikzcd}$$

Denote the 3-cube by $\mathscr{X}$. For T a nonempty subset of $\{1,2,3\}$, let $k(T)$ denote the degree to which the front face $\partial^T \mathscr{X} = \{V \to \mathscr{X}(V) : V \subset T\}$ is cartesian. Labeling the "$*$"'s in the top face of $\mathscr{X}$ by 1 and 3, and labeling Y by 2, one can easily check that

$$k(\{1\}) = k(\{3\}) = s+1, \quad k(\{1,2\}) = k(\{2,3\}) = t+1, \quad k(\{2\}) = t$$

$$\text{and} \qquad k(\{1,3\}) = k(\{1,2,3\}) = \infty$$

By Goodwillie's generalized Blakers-Massey Theorem ([Go92], Theorem 2.5), $\mathscr{X}$ is $(1-3+s+t+1+1) = (s+t)$-cartesian. An application of ([Go92], Proposition 1.18) completes the proof. $\square$

The fiberwise suspension functor $\mathcal{S}_X$ can be thought of as having target category $\mathcal{T}(X \amalg X \xrightarrow{\nabla} X)$, where ∇ is the fold map. It admits a right adjoint

$$\mathcal{O}_X : \mathcal{T}(X \amalg X \xrightarrow{\nabla} X) \to (\mathcal{T} \downarrow X)$$

given on objects by

$$Y \mapsto \operatorname{holim}(X \xrightarrow{i_+} Y \xleftarrow{i_-} X)$$

where $i_\pm$ denote the restrictions of $X \coprod X \to Y$ to each summand. Recall that an object Y of $(\mathcal{T} \downarrow X)$ is *m-connected* if the structure map $Y \to X$ is an $(m+1)$-connected map of spaces.

Lemma 3.1.5. *Let $Y \in (\mathcal{T} \downarrow X)$ be a cofibrant object which is m-connected. Then there is a morphism*

$$Y \to \mathcal{O}_X \mathcal{S}_X Y$$

of $(\mathcal{T} \downarrow X)$ which is $(2m+1)$-connected.

Proof. Let $g : Y \to X$ be the structure map associated with the object Y. Then by definition, we have the following ∞-cocartesian square of spaces over X:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "g"'] & X \arrow[d, "i_1"] \\ X \arrow[r, "i_0"'] & \mathcal{S}_X Y \end{tikzcd}$$

where i_0 and i_1 are the structure maps. By the Blakers-Massey Theorem, this square is $(2m+1)$-cartesian. That is, the canonical map

$$Y \to \operatorname{holim}(X \xrightarrow{i_0} \mathcal{S}_X Y \xleftarrow{i_1} X) = \mathcal{O}_X \mathcal{S}_X Y$$

is $(2m+1)$-connected. $\square$

Let

$$
(\mathscr{A}') \qquad
\begin{tikzcd}
(A'_K, A'_L) \arrow[r] \arrow[d] & (W_K, W_L) \arrow[d] \\
(\mathcal{S}_X K, \mathcal{S}_X L) \arrow[r, "\mathcal{S}_X f"'] & (X \times D^1, \partial(X \times D^1))
\end{tikzcd}
$$

be a cofibrant and fibrant object of $E_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \ \mathit{rel}\ \mathcal{S}_X L)$, which we also think of as the corresponding 0-cell of $\mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \ \mathit{rel}\ \mathcal{S}_X L)$. The underlying PD embedding

$$
\begin{tikzcd}
A'_L \arrow[r] \arrow[d] & W_L \arrow[d] \\
\mathcal{S}_X L \arrow[r] & \partial(X \times D^1)
\end{tikzcd}
$$

is the stabilization of the PD embedding with underlying map $L \to \partial X$ determined by $\mathscr{D}$. Thus, we have an identification

$$(W_L, A'_L) \simeq (C_L \times D^1, \mathcal{S}_{C_L} A_L)$$

which allows us to write $\mathscr{A}'$ as

$$
(\mathscr{A}') \qquad
\begin{tikzcd}
(A'_K, \mathcal{S}_{C_L} A_L) \arrow[r] \arrow[d] & (W_K, C_L \times D^1) \arrow[d] \\
(\mathcal{S}_X K, \mathcal{S}_X L) \arrow[r, "\mathcal{S}_X f"'] & (X \times D^1, \partial(X \times D^1))
\end{tikzcd}
$$

Similarly, let

$$
(\mathscr{A}'') \qquad
\begin{tikzcd}
(A''_K, B''_L) \arrow[r] \arrow[d] & (W''_K, W''_L) \arrow[d] \\
(K, L) \arrow[r, "f^1"'] & (X \times D^1, \partial(X \times D^1))
\end{tikzcd}
$$

be a cofibrant and fibrant object of $E_{f^1}(K, X \times D^1 \ \mathit{rel}\ L)$, which we also think of as the corresponding 0-cell of $\mathbb{E}_{f^1}(K, X\times D^1 \ rel\, L)$. By ([Kl02b], Proposition 4.1 and the preceeding remark), there is a PD pair (C''_L, A''_L) and an identification

$$(W''_L, B''_L) \simeq (\mathcal{S}_X C''_L, \mathcal{S}_L A''_L)$$

along with an object $C''_K \in \mathcal{T}(C''_L \to X)$ which allow us to write $\mathscr{A}''$ as

$$(\mathscr{A}'') \qquad \begin{tikzcd} (A''_K, \mathcal{S}_L A''_L) \arrow[r] \arrow[d] & (\mathcal{S}_X C''_K, \mathcal{S}_X C''_L) \arrow[d] \\ (K, L) \arrow[r, "f^1"'] & (X\times D^1, \partial(X\times D^1)) \end{tikzcd}$$

Remark 3.1.6. We will suppress the notation of pairs from $\mathscr{A}'$ and $\mathscr{A}''$, keeping in mind that we are working relative to the PD embeddings of the underlying maps of subspaces that are already given (see Definition 1.1.3).

Applying the decompression map to $\mathscr{A}'$, and applying the stabilization map to $\mathscr{A}''$, we have

$$(\delta\mathscr{A}') \quad \begin{tikzcd} \mathcal{S}_{\mathcal{S}_X K} A'_K \arrow[r] \arrow[d] & \mathcal{S}_X W_K \arrow[d] \\ \mathcal{S}_X K \arrow[r] & X\times D^2 \end{tikzcd} \qquad (\sigma\mathscr{A}'') \quad \begin{tikzcd} \mathcal{S}_{\mathcal{S}_X C''_K} A''_K \arrow[r] \arrow[d] & \mathcal{S}_X C''_K \arrow[d] \\ \mathcal{S}_X K \arrow[r] & X\times D^2 \end{tikzcd}$$

where we have implicitly used the equivalence $D^1 \simeq *$ along with ([Kl99a], Lemma 2.5 (2)) to replace the spaces appearing in the lower left corner of $\delta\mathscr{A}'$ and the upper right corners of both squares with homotopy equivalent spaces. Without loss of generality, assume that $\delta\mathscr{A}'$ and $\sigma\mathscr{A}''$ are both cofibrant and fibrant objects of the category $E_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X\times D^2 \ rel\, \mathcal{S}_X L)$.

Lemma 3.1.7. *Assume that there is a path (i.e., "zig-zag" of weak equivalences) from $\delta\mathscr{A}'$*

to $\sigma\mathscr{A}''$ in the category

$$E_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \ rel \, \mathcal{S}_X L).$$

Then there is a weak equivalence

$$\mathcal{S}_X W_K \xrightarrow{\sim} \mathcal{S}_X C''_K$$

of spaces over X.

Proof. Without loss of generality, assume that W_K is cofibrant in $(\mathcal{T} \downarrow X)$. Since $\mathcal{S}_X$ preserves cofibrant objects, the object $\mathcal{S}_X W_K$ is cofibrant. Furthermore, since $\mathscr{A}''$ is fibrant, we know that the object $\mathcal{S}_X C''_K$ is fibrant. Hence, the hypothesis of the lemma provides us with a "'zig-zag" of weak equivalences in $(\mathcal{T} \downarrow X)$ from a cofibrant object to a fibrant object. This induces an isomorphism in the homotopy category which then lifts back to the desired weak equivalence. □

Remark 3.1.8. To show that the square $\mathcal{D}_{\sigma,\delta}$ is 0-cartesian, we will assume that $\delta\mathscr{A}'$ and $\sigma\mathscr{A}''$ lie in the same component of the space

$$\mathbb{E}_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \ rel \, \mathcal{S}_X L)$$

and produce a 0-cell in $\mathbb{E}_f(K, X \ rel \ L)$ (i.e., a PD embedding with underlying map f) that maps to both $\delta\mathscr{A}'$ and $\sigma\mathscr{A}''$, making the square $\mathcal{D}_{\sigma,\delta}$ commute. In fact, it will be enough to assume, as above, that there is a path (= "zig-zag" of weak equivalences) from $\delta\mathscr{A}'$ to $\sigma\mathscr{A}''$ in the category $E_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \ rel \, \mathcal{S}_X L)$ since this will produce the desired path after passing to realizations.

Lemma 3.1.9. *Similar to the previous lemma, assume that there is a path from $\delta\mathscr{A}'$ to $\sigma\mathscr{A}''$ in the category*

$$E_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \ rel\ \mathcal{S}_X L).$$

Then there are weak equivalences

(i) $\mathcal{S}_{\mathcal{S}_X K} A'_K \xrightarrow{\sim} \mathcal{S}_{\mathcal{S}_X C''_K} A''_K$ *in* $(\mathcal{T} \downarrow \mathcal{S}_X K)$

(ii) $\mathcal{S}_{\mathcal{S}_X C''_K} A''_K \xrightarrow{\sim} \mathcal{S}_{\mathcal{S}_X K} A'_K$ *in* $(\mathcal{T} \downarrow \mathcal{S}_X C''_K)$

Proof. Using the squares above, regard the spaces $\mathcal{S}_{\mathcal{S}_X K} A'_K$ and $\mathcal{S}_{\mathcal{S}_X C''_K} A''_K$ as objects of $(\mathcal{T} \downarrow \mathcal{S}_X K)$. Our assumption that $\mathscr{A}'$ is cofibrant implies that A'_K is cofibrant, so that $\mathcal{S}_{\mathcal{S}_X K} A'_K$ is cofibrant. Our assumption that $\sigma\mathscr{A}''$ is fibrant implies that $\mathcal{S}_{\mathcal{S}_X C''_K} A''$ is a fibrant object of $(\mathcal{T} \downarrow S_X K)$. The hypothesis of the lemma implies that there is a "zig-zag" of weak equivalences over $\mathcal{S}_X K$ from the cofibrant object $\mathcal{S}_{\mathcal{S}_X K} A'_K$ to the fibrant object $\mathcal{S}_{\mathcal{S}_X C''_K} A''_K$. Thus, as in the proof of the previous lemma, we have a weak equivalence

$$\mathcal{S}_{\mathcal{S}_X K} A'_K \xrightarrow{\sim} \mathcal{S}_{\mathcal{S}_X C''_K} A''_K$$

of spaces over $\mathcal{S}_X K$. Using the previous lemma, regard $\mathcal{S}_{\mathcal{S}_X K} A'$ as a space over $\mathcal{S}_{\mathcal{S}_X C''_K} A''$. Then an argument similar to the one just given produces a weak equivalence

$$\mathcal{S}_{\mathcal{S}_X C''_K} A''_K \xrightarrow{\sim} \mathcal{S}_{\mathcal{S}_X K} A'_K$$

of spaces over $\mathcal{S}_X C''_K$. $\square$

In the next two lemmas, we will use the weak equivalences above to construct sections to the maps $A'_K \to W_K$ and $A''_K \to K$ given the the diagrams $\mathscr{A}'$ and $\mathscr{A}''$. To avoid notational clutter, we will drop the subscript K.

Lemma 3.1.10. *Along with the standing assumptions given at the beginning of this section, assume that $n \geq 2(k-r)+2$. Then with the assumption of the previous lemma, there exists a map*

$$\mathcal{S}_W\emptyset = W \times S^0 \to A'$$

such that each of the restrictions $W \to A'$ is a section of the map $A' \to W$ in the diagram $\mathscr{A}'$.

Proof. The natural map $\mathcal{S}_X K \to X$ is an $(r+1) \geq 2$-connected map of spaces. Hence, every local coefficient system on $\mathcal{S}_X K$ arises by pullback from one on X. Let M_K and M_L denote the mapping cylinders of the maps $K \to X$ and $L \to X$, respectively. Then there is a relative Mayer-Vietoris sequence with coefficients in any local system given by

$$\cdots \to H^*(\mathcal{S}_X K, \mathcal{S}_X L) \to H^*(M_K, M_L) \oplus H^*(M_K, M_L) \to H^*(K, L) \to \cdots$$

Since M_K and M_L both deformation retract onto X, this sequence gives an isomorphism

$$H^*(K, L) \cong H^{*+1}(\mathcal{S}_X K, \mathcal{S}_X L) = 0 \qquad \text{for } * > k$$

Thus, $\text{hodim}(\mathcal{S}_X K, \mathcal{S}_X L) = k+1$. So, by definition, the map of spaces $A' \to \mathcal{S}_X K$ given in $\mathscr{A}'$ is $(n+1)-(k+1)-1 = (n-k-1)$-connected. Furthermore, the diagram

$$\begin{tikzcd}
\mathcal{S}_X K \arrow[rr, "\mathcal{S}_X f"] \arrow[dr] & & X \times D^1 \arrow[dl, "\simeq"] \\
& X &
\end{tikzcd}$$

implies that the map $\mathcal{S}_X f$ is an $(r+1)$-connected map of spaces. But $k \leq n-3$, so that $n-k-1 \geq 2$. Using ([Kl99a], Lemma 5.6 (2)) we infer that map $A' \to W$ in $\mathscr{A}'$ is an $(r+1)$-connected map of spaces. Thus, the Blakers-Massey Theorem implies that the square

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & W \arrow[d] \\
\mathcal{S}_X K \arrow[r] & X \times D^1
\end{tikzcd}$$

associated to $\mathscr{A}'$ is $(n-k+r-1)$-cartesian. In particular, since $\mathscr{A}'$ is fibrant, the map

$$A' \to \operatorname{holim}(\mathcal{S}_X K \to X \times D^1 \leftarrow W) = \mathcal{S}_X K \times_X W$$

is $(n-k+r-1)$-connected. Using this map and Lemma 3.1.5, form the square

$$(\mathscr{F}) \qquad \begin{tikzcd}
A' \arrow[r] \arrow[d] & \mathcal{O}_{\mathcal{S}_X K}\mathcal{S}_{\mathcal{S}_X K} A' \arrow[d] \\
\mathcal{S}_X K \times_X W \arrow[r] & \mathcal{O}_{\mathcal{S}_X K}\mathcal{S}_{\mathcal{S}_X K}(\mathcal{S}_X K \times_X W)
\end{tikzcd}$$

of spaces over $\mathcal{S}_X K$. Let

$$F = \operatorname{hofiber}(A' \to \mathcal{S}_X K)$$
$$F' = \operatorname{hofiber}(\mathcal{S}_X K \times_X W \to \mathcal{S}_X K)$$

where the homotopy fibers are taken with respect to any choice of basepoint for $\mathcal{S}_X K$. Then F and F' are both $(n-k-2)$-connected spaces, the latter being true since the connectivity of the space F' is equal to one less than the connectivity of the map of spaces $W \to X$, and this connectivity is $n-k-1$ since the squares associated to $\mathscr{A}'$ are ∞-cocartesian. Furthermore, the induced map $F \xrightarrow{h} F'$ is $(n-k+r-1)$-connected, as can be seen from the square $\mathscr{F}$. The map h fits into the square obtained by taking homotopy fibers (over any choice of basepoint for $\mathcal{S}_X K$) of the maps from $\mathscr{F}$. That is, we have a square

$$(\mathscr{F}) \qquad \begin{tikzcd} F \arrow[r] \arrow[d, "h"'] & \Omega\Sigma F \arrow[d] \\ F' \arrow[r] & \Omega\Sigma F' \end{tikzcd}$$

By Lemma 3.1.4, $\mathscr{F}$ is $(2n-2k+r-3)$-cartesian and, hence, so is $\mathscr{F}$. Using the evident map $\mathcal{S}_W\emptyset \to \mathcal{S}_X K$, as well as the weak equivalence (i) given in Lemma 3.1.9, construct the composite

$$\mathcal{S}_{\mathcal{S}_X K}(\mathcal{S}_W\emptyset) \to \mathcal{S}_X(\mathcal{S}_W\emptyset) = \mathcal{S}_{\mathcal{S}_X W}\emptyset \to \mathcal{S}_{\mathcal{S}_X W}A'' \xrightarrow{\sim} \mathcal{S}_{\mathcal{S}_X K}A'$$

of spaces over $\mathcal{S}_X K$. Apply the functor $\mathcal{O}_{\mathcal{S}_X K}$ (restricted to $(\mathcal{T} \downarrow \mathcal{S}_X K)$) to get a map

$$\mathcal{O}_{\mathcal{S}_X K}\mathcal{S}_{\mathcal{S}_X K}(\mathcal{S}_W\emptyset) \to \mathcal{O}_{\mathcal{S}_X K}\mathcal{S}_{\mathcal{S}_X K}A'.$$

Precompose with the map from $\mathcal{S}_W\emptyset$ provided by the Lemma 3.1.5 to get

$$\mathcal{S}_W\emptyset \to \mathcal{O}_{\mathcal{S}_X K}\mathcal{S}_{\mathcal{S}_X K}A'.$$

Finally, combine this map with the map $\mathcal{S}_W\emptyset = \mathcal{S}_X\emptyset \times_X W \to S_X K \times_X W$ and the square $\mathscr{F}$ to form the following diagram of spaces over $\mathcal{S}_X K$:

$$\begin{tikzcd}
\mathcal{S}_W\emptyset \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& A' \arrow[r] \arrow[d] & \mathcal{O}_{\mathcal{S}_X K}\mathcal{S}_{\mathcal{S}_X K}A' \arrow[d] \\
& \mathcal{S}_X K \times_X W \arrow[r] & \mathcal{O}_{\mathcal{S}_X K}\mathcal{S}_{\mathcal{S}_X K}(\mathcal{S}_X K \times_X W)
\end{tikzcd}$$

By obstruction theory, the dashed arrow exists provided that $\mathrm{hodim}(W) \leq 2n - 2k + r - 3$. Note that the codimension hypothesis implies that $W \to X$ is 2-connected. Hence, using duality and excision, we have the following isomorphisms for all local coefficient systems:

$$H^*(W) \cong H_{n+1-*}(\overline{W}, A') \cong H_{n+1-*}(X \times D^1, \mathcal{S}_X K).$$

Since $(X \times D^1, \mathcal{S}_X K)$ is an $(r+1)$-connected pair of spaces, the isomorphism above implies that W is cohomologically $(n-r-1)$-dimensional (i.e., its cohomology vanishes in degrees $> n-r-1$). But $r \leq k \leq n-3$, so that $n-r-1 \geq 2$. Hence, by ([GK08], Proposition 8.1), $\mathrm{hodim}(W) \leq n-r-1$. Thus, the dashed arrow exists provided that $n-r-1 \leq 2n-2k+r-3$, which is equivalent to $n \geq 2(k-r)+2$. This establishes the existence of the map

$$\mathcal{S}_W\emptyset = W \times S^0 \to A'.$$

□

We now make a similar construction associated with the square $\mathscr{A}''$:

Lemma 3.1.11. *With the assumptions of the previous lemma, there exists a map*

$$\mathcal{S}_K\emptyset = K \times S^0 \to A''$$

such that each of the restrictions $K \to A''$ is a section of the map $A'' \to K$ in the diagram $\mathscr{A}''$.

Proof. By definition, the left vertical map in $\mathscr{A}''$ is $(n-k) \geq 3$-connected, so the top horizontal map is r-connected by ([Kl99a], Proposition 5.6 (2)). By the Blakers-Massey Theorem, the square

$$\begin{array}{ccc} A'' & \longrightarrow & \mathcal{S}_X C \\ \downarrow & & \downarrow \\ K & \longrightarrow & X \times D^1 \end{array}$$

associated with $\mathscr{A}''$ is $(n-k+r-1)$-cartesian. Since $\mathscr{A}''$ was assumed fibrant, the map

$$A'' \to K \times_X \mathcal{S}_X C$$

is an $(n-k+r-1)$-connected map of spaces. Using Lemma 3.1.5, form the following commutative square of objects in $(\mathcal{T} \downarrow S_X C)$:

$$(\mathscr{G}) \qquad \begin{array}{ccc} A'' & \longrightarrow & \mathcal{O}_{\mathcal{S}_X C} \mathcal{S}_{\mathcal{S}_X C} A'' \\ \downarrow & & \downarrow \\ K \times_X \mathcal{S}_X C & \longrightarrow & \mathcal{O}_{\mathcal{S}_X C} \mathcal{S}_{\mathcal{S}_X C}(K \times_X \mathcal{S}_X C) \end{array}$$

Similar to the argument given in the previous lemma, this map gives rise to the square of homotopy fibers (for any choice of basepoint in $\mathcal{S}_X C$)

$$(\mathscr{G}') \qquad \begin{array}{ccc} G = \text{hofiber}(A'' \to \mathcal{S}_X C) & \longrightarrow & \Omega\Sigma G \\ {\scriptstyle h'}\downarrow & & \downarrow \\ G' = \text{hofiber}(K \times_X \mathcal{S}_X C \to \mathcal{S}_X C) & \longrightarrow & \Omega\Sigma G' \end{array}$$

in which $\text{conn}(G) = \text{conn}(G') = r-1$ and $\text{conn}(h') = (n-k+r-1)$. By Lemma 3.1.4, $\mathscr{G}'$ is

$(n-k+2r-2)$-cartesian and, hence, so is $\mathscr{G}$. Using the evident map $\mathcal{S}_K\emptyset \to \mathcal{S}_X C$ and the weak equivalence (ii) given in Lemma 3.1.9, construct the following composite of objects in $(\mathcal{T} \downarrow \mathcal{S}_X C)$:

$$\mathcal{S}_{\mathcal{S}_X C}(\mathcal{S}_K\emptyset) \to \mathcal{S}_X(\mathcal{S}_K\emptyset) = \mathcal{S}_{\mathcal{S}_X K}\emptyset \to \mathcal{S}_{\mathcal{S}_X K}A' \xrightarrow{\sim} \mathcal{S}_{\mathcal{S}_X C}A''.$$

As above, apply the (restricted) functor $\mathcal{O}_{\mathcal{S}_X C}$ to get a map

$$\mathcal{O}_{\mathcal{S}_X C}\mathcal{S}_{\mathcal{S}_X C}(\mathcal{S}_K\emptyset) \to \mathcal{O}_{\mathcal{S}_X C}\mathcal{S}_{\mathcal{S}_X C}A''.$$

Precompose with the map from $\mathcal{S}_K\emptyset$ provided by Lemma 3.1.5 to get

$$\mathcal{S}_K\emptyset \to \mathcal{O}_{\mathcal{S}_X C}\mathcal{S}_{\mathcal{S}_X C}A''$$

and combine this with the map $\mathcal{S}_K\emptyset = \mathcal{S}_X\emptyset \times_X K \to S_X C \times_X K$ and the square $\mathscr{G}$ to form the diagram

$$\begin{tikzcd}
\mathcal{S}_K\emptyset \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& A'' \arrow[r] \arrow[d] & \mathcal{O}_{\mathcal{S}_X C}\mathcal{S}_{\mathcal{S}_X C}A'' \arrow[d] \\
& K \times_X \mathcal{S}_X C \arrow[r] & \mathcal{O}_{\mathcal{S}_X C}\mathcal{S}_{\mathcal{S}_X C}(K \times_X \mathcal{S}_X C)
\end{tikzcd}$$

Again, by obstruction theory, the dashed arrow exists provided that $k \leq n-k+2r-2$, which is equivalent to $n \geq 2(k-r)+2$. Hence, we have the desired map

$$\mathcal{S}_K\emptyset = K \times S^0 \to A''.$$

□

3.2 The 0-Cartesian Square $\mathscr{D}_{\sigma,\delta}$

Recall the square of spaces from the previous section:

$$(\mathscr{D}_{\sigma,\delta})\quad \begin{tikzcd} \mathbb{E}_f(K, X\ \mathit{rel}\ L) \arrow[r] \arrow[d] & \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1\ \mathit{rel}\ \mathcal{S}_X L) \arrow[d] \\ \mathbb{E}_{f^1}(K, X \times D^1\ \mathit{rel}\ L) \arrow[r] & \mathbb{E}_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2\ \mathit{rel}\ \mathcal{S}_X L) \end{tikzcd}$$

Lemma 3.2.1. *Assume that $n \geq 2(k-r)+3$. Then along with the standing assumptions given at the beginning of the previous section, the square $\mathscr{D}_{\sigma,\delta}$ is 0-cartesian.*

Proof. Using the projection $X \times D^1 \to X$, write the underlying squares of $\mathscr{A}'$ and $\mathscr{A}''$ as follows:

$$(\mathscr{A}')\quad \begin{tikzcd} A'_K \arrow[r] \arrow[d] & W_K \arrow[d] \\ \mathcal{S}_X K \arrow[r] & X \end{tikzcd} \qquad (\mathscr{A}'')\quad \begin{tikzcd} A''_K \arrow[r] \arrow[d] & \mathcal{S}_X C''_K \arrow[d] \\ K \arrow[r] & X \end{tikzcd}$$

Using the fact that these squares are ∞-cocartesian, along with the maps constructed in Lemmas 3.1.10 and 3.1.11, we have the following 'excision' weak equivalences

$$(i)\quad \overline{X} \cup_{W_K} A'_K \xrightarrow{\sim} \overline{\mathcal{S}_X K} \cup_{A'_K} A'_K \xrightarrow{\sim} \mathcal{S}_X K$$
$$(ii)\quad \overline{X} \cup_K A''_K \xrightarrow{\sim} \overline{X} \cup_X \mathcal{S}_X C''_K \xrightarrow{\sim} \mathcal{S}_X C''_K$$

and, using (i) (along with an argument similar to Lemma 3.1.7 for the first weak equivalence)

$$\begin{aligned} (iii)\quad \mathcal{S}_X C''_K \cup_{\overline{X}} \mathcal{S}_X K &\xrightarrow{\sim} (\overline{X} \cup_{W_K} \overline{X}) \cup_{\overline{X}} \mathcal{S}_X K \\ &\xrightarrow{\sim} \overline{X} \cup_{W_K} \mathcal{S}_X K \\ &\xrightarrow{\sim} (\overline{X} \cup_{W_K} A'_K) \cup_{A'_K} \mathcal{S}_X K \\ &\xrightarrow{\sim} \mathcal{S}_X K \cup_{A'_K} \mathcal{S}_X K \\ &= \mathcal{S}_{\mathcal{S}_X K} A'_K \end{aligned}$$

What we have so far can be assembled to form the punctured 4-dimensional cubical diagram

$$
\begin{tikzcd}
 & & & W_K \arrow[dl] \arrow[dd] & & & W_K \arrow[rr] \arrow[dl] \arrow[dd] & & A'_K \arrow[dl] \arrow[dd] \\
K \arrow[rr] \arrow[dd] & & \overline{X} \arrow[dd] & \arrow[r] & {} & \overline{X} \arrow[rr] \arrow[dd] & & \mathcal{S}_X K \arrow[dd] & \\
 & K \arrow[rr] \arrow[dl] & & \overline{X} \arrow[dl] & & & \overline{X} \arrow[rr] \arrow[dl] & & \mathcal{S}_X K \arrow[dl] \\
A''_K \arrow[rr] & & \mathcal{S}_X C''_K & & & \mathcal{S}_X C''_K \arrow[rr] & & \mathcal{S}_{\mathcal{S}_X K} A'_K &
\end{tikzcd}
$$

Let B denote the homotopy limit of this punctured 4-cube, so that we have an ∞-cartesian 4-cube

$$
\begin{tikzcd}
 & B \arrow[rr] \arrow[dl] \arrow[dd] & & W_K \arrow[dl] \arrow[dd] & & & W_K \arrow[rr] \arrow[dl] \arrow[dd] & & A'_K \arrow[dl] \arrow[dd] \\
K \arrow[rr] \arrow[dd] & & \overline{X} \arrow[dd] & \arrow[r] & {} & \overline{X} \arrow[rr] \arrow[dd] & & \mathcal{S}_X K \arrow[dd] & \\
 & K \arrow[rr] \arrow[dl] & & \overline{X} \arrow[dl] & & & \overline{X} \arrow[rr] \arrow[dl] & & \mathcal{S}_X K \arrow[dl] \\
A''_K \arrow[rr] & & \mathcal{S}_X C''_K & & & \mathcal{S}_X C''_K \arrow[rr] & & \mathcal{S}_{\mathcal{S}_X K} A'_K &
\end{tikzcd}
$$

Remark 3.2.2. To avoid technical difficulties, we will assume that we have mapped the original punctured cube to a new punctured cube by a pointwise weak equivalence, and that the limit of the new punctured cube is the homotopy limit of the original punctured cube. The new punctured cube, together with its limit, is a strictly commutative cube. Hence, we will assume that the 4-cube above is strictly commutative, but will keep the notation as is.

We wish to apply the 4-D Face Theorem to this cube, so we check its hypotheses. As noted above, the cube is ∞-cartesian. The weak equivalences $(i), (ii)$, and (iii) above, along with the weak equivalences of Lemma 3.1.9 show that every 2-dimensional face which meets $\mathcal{S}_{\mathcal{S}_X K} A'_K$ is ∞-cocartesian. So, by ([Go92], Definition 2.5), every 3-dimensional face which meets $\mathcal{S}_{\mathcal{S}_X K} A'_K$ is strongly cocartesian. One can easily check that, in the notation of the 4-D Face Theorem, $k_1 = k_2 = n - k - 1$ and $k_3 = k_4 = r$. In particular, since $k \leq n - 3$, we have $k_1, k_2 \geq 2$. Hence, the theorem applies and as a consequence, we have that B is connected and that the square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & K \arrow[d] \\ K \arrow[r] & A''_K \end{tikzcd}$$

is $(2(n-k-1) + 2r - 1) = (2n - 2k + 2r - 3)$-cocartesian.

Claim 3.2.3. *There exists a space A and a $(2n - 2k + 2r - 5)$-connected map $A \to B$ such that the square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & K \arrow[d] \\ K \arrow[r] & A''_K \end{tikzcd}$$

(with B replaced by A) is ∞-cocartesian.

Proof of Claim 3.2.3. Our hypotheses that $r \geq 1$ and $k \leq n-3$ imply that $2(n-k+r)-3 \geq 3$. Now, choose a basepoint for B (which, in turn, bases K and A''_K) and consider the map $K \vee K \to A''_K$. Since $A''_K \to K$ is $(n-k)$-connected, we have a long exact sequence on cohomology (with respect to any local coefficient system on A''_K) given by

$$\cdots \to H^{*-1}(K \vee K) \to H^*(\overline{A''_K}, K \vee K) \to H^*(A''_K) \to \cdots$$

By definition, A''_K is a PD space of dimension n. So, since $k \leq n-3$, the sequence above implies that the relative cohomology of $K \vee K \to A''_K$ vanishes in degrees $\geq n+1$. That is, the relative cohomology of $K \vee K \to A''_K$ vanishes in degrees $> 2(n-k+r)-3$ provided that $n+1 \leq 2n-2k+2r-2$, which is equivalent to $n \geq 2(k-r)+3$. Hence, we can apply the *Cocartesian Replacement Theorem* ([Kl99a], Theorem 4.2) to obtain the desired space A. This proves the claim. $\square$

Now, consider one of the other 2-dimensional faces of the 4-cube labeled by

$$\begin{tikzcd} B \arrow[r] \arrow[d] & W_K \arrow[d] \\ K \arrow[r] & \overline{X} \end{tikzcd}$$

Replacing B with the A constructed in Claim 3.2.3, form the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & W_K \arrow[d] \\ K \arrow[r] & X \end{tikzcd}$$

By ([Kl99a], Claims 6.5 and 6.6), this square is a PD embedding for f. Writing A_K for A and C_K for W_K (and recalling that we are working relatively), we have the desired object

$$\begin{tikzcd} (A_K, A_L) \arrow[r] \arrow[d] & (C_K, C_L) \arrow[d] \\ (K, L) \arrow[r] & (X, \partial X) \end{tikzcd}$$

of $E_f(K, X \ \mathit{rel}\ L)$ and, hence, the desired 0-cell of $\mathbb{E}_f(K, X \ \mathit{rel}\ L)$. This proves that $\mathscr{D}_{\sigma,\delta}$ is 0-cartesian.

$\square$

3.3 Proof of Theorem A

Recall the statement of Theorem A:

Theorem A (Stabilization). *Let $f : (K, L) \to (X, \partial X)$ be a map from a cofibration pair of homotopy finite spaces (K, L), with* $\operatorname{hodim}(K, L) = k$*, to a PD pair $(X, \partial X)$ of dimension n. Assume that $f : K \to X$ is r-connected $(r \geq 1)$ and that $k \leq n-3$. Then the stabilization map*

$$\sigma : \mathbb{E}_f(K, X \text{ rel } L) \to \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L)$$

is $(n - 2(k - r) - 3)$-connected.

Further, recall that the object

$$\begin{tikzcd} (A_K, A_L) \arrow[r] \arrow[d] & (C_K, C_L) \arrow[d] \\ (K, L) \arrow[r] & (X, \partial X) \end{tikzcd}$$

constructed in the previous section specifies a PD embedding $\mathscr{D}_L$ with underlying map $L \to \partial X$:

$$(\mathscr{D}_L) \qquad \begin{tikzcd} A_L \arrow[r] \arrow[d] & C_L \arrow[d] \\ L \arrow[r] & \partial X \end{tikzcd}$$

After applying j-fold fiberwise suspension, $\mathscr{D}_L$ gives rise to the PD embedding

$$\begin{tikzcd}
\mathcal{S}^j_{A_K} A_L \arrow[r] \arrow[d] & \mathcal{S}^j_{C_K} C_L \arrow[d] \\
\mathcal{S}^j_K L \arrow[r] & \mathcal{S}^j_X \partial X
\end{tikzcd}$$

which, upon identifying $\mathcal{S}^j_X \partial X$ with $\partial(X \times D^j)$, is the specified "boundary PD embedding" of a relative PD embedding with underlying map $f \times id : K \times D^j \to X \times D^j$. This construction, along with Proposition A.2 (see the appendix), allows us to write

$$\pi_j(\mathbb{E}_f(K, X \ rel\ L)) = \pi_0(\mathbb{E}_{f\times id}(K \times D^j, X \times D^j \ rel\ \mathcal{S}^j_K L)).$$

Proof of Theorem A. Lemma 3.2.1 tells us that the square

$$(\mathscr{D}_{\sigma,\delta}) \qquad \begin{tikzcd}
\mathbb{E}_f(K, X \ rel\ L) \arrow[r] \arrow[d] & \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \ rel\ \mathcal{S}_X L) \arrow[d] \\
\mathbb{E}_{f^1}(K, X \times D^1 \ rel\ L) \arrow[r] & \mathbb{E}_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \ rel\ \mathcal{S}_X L)
\end{tikzcd}$$

is 0-cartesian, provided that $k \leq n-3$ and $n \geq 2(k-r)+3$. That is, under these assumptions, we have a 0-connected map $\mathbb{E}_f(K, X \ rel\ L) \to P$, where P denotes the homotopy limit of the partial diagram gotten from $\mathscr{D}_{\sigma,\delta}$ by considering only the bottom horizontal and right vertical maps. Repeated application of the decompression map on both sides of $\mathscr{D}_{\sigma,\delta}$ forms an infinite tower of embedding spaces. Further, after decompressing infinitely many times, we obtain contractible spaces on both sides of the tower. That is, we have a weak equivalence after passing to homotopy colimits. Thus, by a downward induction on codimension, we may assume that the bottom horizontal map in $\mathscr{D}_{\sigma,\delta}$ is j-connected for some $j \geq 0$. Since the square

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \ \mathit{rel}\, \mathcal{S}_X L) \arrow[d] \\
\mathbb{E}_{f^1}(K, X \times D^1 \ \mathit{rel}\ L) \arrow[r] & \mathbb{E}_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \ \mathit{rel}\, \mathcal{S}_X L)
\end{tikzcd}$$

∞-cartesian, we can then infer that the top horizontal map $P \to \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \ \mathit{rel}\, \mathcal{S}_X L)$ is j-connected. Hence, we have a commutative diagram

$$\begin{tikzcd}
\mathbb{E}_f(K, X \ \mathit{rel}\ L) \arrow[rr, "\sigma"] \arrow[dr] & & \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \ \mathit{rel}\, \mathcal{S}_X L) \\
& P \arrow[ur] &
\end{tikzcd}$$

from which it is clear that the top horizontal map (the stabilization map) is 0-connected. That is, the stabilization map induces a π_0 surjection provided that $k \leq n-3$ and $n \geq 2(k-r)+3$. Thus, by definition, the map σ induces a surjection

$$\pi_j(\mathbb{E}_f(K, X \ \mathit{rel}\ L)) \to \pi_j(\mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \ \mathit{rel}\, \mathcal{S}_X L))$$

provided that $n + j \geq 2((k+j) - r) + 3$, which is equivalent to

$$j \leq n - 2(k-r) - 3.$$

This gives the desired surjectivity statement. For the injectivity statement, assume that we are given two PD embeddings $\mathscr{D}_0$ and $\mathscr{D}_1$ with underlying maps f_0, $f_1 : K \to X$ and consider the associated embedding $\mathscr{D}_{K \amalg K}$ with underlying map $f_0 + f_1 : K \amalg K \to \partial(X \times D^1)$. Further, assume that f_0 and f_1 give rise to the same embedding with underlying map $\mathcal{S}_X K \to X \times D^1$ after applying the stabilization map. Note that this embedding is relative to the embedding $\mathscr{D}_{K \amalg K}$. Then we have an associated map of pairs

$$F : (K \times D^1, K \amalg K) \to (X \times D^1, \partial(X \times D^1)).$$

Assume that $n \geq 2(k-r)+c$ for some constant c. This is equivalent to $r \geq 2k-n+(c-r)$. According to ([Kl02b], Corollary B), $\mathscr{D}_0$ is concordant to $\mathscr{D}_1$ provided that $c-r \leq 3$. But we have assumed that $r \geq 1$, so c is at least 4. Hence the induced map

$$\pi_j(\mathbb{E}_f(K, X \;\; rel \; L)) \to \pi_j(\mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \;\; rel \; \mathcal{S}_X L))$$

is injective provided that $n+j \geq 2((k+j)-r)+4$, which is equivalent to

$$j \leq n - 2(k-r) - 4.$$

This completes the proof of Theorem A. $\square$

3.4 A Generalization of Smooth Stabilization

We now prove the PD analog of the *Stabilization Theorem* of [CW78]. To this end, fix an object $(f, K) \in (\mathcal{T} \downarrow S^n)$. Then there is a map

$$\mathbb{E}_{\mathcal{S}_{S^n} f}(\mathcal{S}_{S^n} K, S^n \times D^1 \;\; rel \; \mathcal{S}_{S^n} \emptyset) \xrightarrow{c} \mathbb{E}_{\mathcal{S} f}(\mathcal{S} K, S^{n+1})$$

given by collapsing out the copies of S^n on either end of $S^n \times D^1$.

Lemma 3.4.1. *Assume that* $\operatorname{hodim}(K) = k \leq n-3$. *Then the "collapse" map c is* 0-*connected.*

Proof. Let

$$
(\mathscr{D}_{\mathcal{S}f}) \qquad
\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[d] \\
\mathcal{S}K \arrow[r, "\mathcal{S}f"'] & S^{n+1}
\end{tikzcd}
$$

denote a vertex of the space $\mathbb{E}_{\mathcal{S}f}(\mathcal{S}K, S^{n+1})$. To lift $\mathscr{D}_{\mathcal{S}f}$ back to a vertex of $\mathbb{E}_{\mathcal{S}_{S^n}f}(\mathcal{S}_{S^n}K, S^n \times D^1 \text{ } rel \text{ } \mathcal{S}_{S^n}\emptyset)$, it will be enough to solve the lifting problem

$$
\begin{tikzcd}
 & \mathcal{S}_{S^n}K \arrow[d, "q"] \\
A \arrow[r] \arrow[ur, dashed] & \mathcal{S}K
\end{tikzcd}
$$

Obstruction theory tells us that the problem has a solution provided that $\mathrm{hodim}(A) \leq \mathrm{conn}(q)$. Now, by definition A is a PD space of dimension n and, thus, is cohomologically n-dimensional. By ([GK08], Proposition 8.1), we infer that $\mathrm{hodim}(A) \leq n$ (This uses the assumption that $k \leq n-3$, which implies that $n \geq 2$). Thus, it will be enough to show that $\mathrm{conn}(q) = n$. Notice that the quotient map q is given by

$$\mathrm{hocolim}(S^n \xleftarrow{f} K \xrightarrow{f} S^n) \xrightarrow{q} \mathrm{hocolim}(* \leftarrow K \rightarrow *)$$

which is induced by the diagram

$$
\begin{tikzcd}
S^n \arrow[d] & K \arrow[l, "f"'] \arrow[r, "f"] \arrow[d, "id"] & S^n \arrow[d] \\
* & K \arrow[l] \arrow[r] & *
\end{tikzcd}
$$

Let g denote the map $K \to *$, and assume that f and g are cofibrations by replacing their targets with the appropriate mapping cylinders. Further, let

$$S^n \xrightarrow{f'} \operatorname{hocolim}(S^n \xleftarrow{f} K \xrightarrow{f} S^n)$$
$$* \xrightarrow{g'} \operatorname{hocolim}(* \xleftarrow{g} K \xrightarrow{g} *)$$

denote the canonical maps. Then there is a commutative square

$$\begin{tikzcd}
\text{cofiber}(f) \arrow[r, "\sim"] \arrow[d] & \text{cofiber}(f') \arrow[d] \\
\text{cofiber}(g) \arrow[r, "\sim"] & \text{cofiber}(g')
\end{tikzcd}$$

where both horizontal maps are weak equivalences. Moreover, the left vertical map is n-connected since the square

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "id"'] & S^n \arrow[d] \\
K \arrow[r, "g"'] & *
\end{tikzcd}$$

is n-cocartesian. Hence, the map $\text{cofiber}(f') \to \text{cofiber}(g')$ is n-connected. Since $S^n \to *$ is n-connected, we can apply the Five Lemma to the map of long exact sequences on homology induced by the diagram

$$\begin{tikzcd}
S^n \arrow[r, "f'"] \arrow[d] & \mathcal{S}_{S^n} K \arrow[r] \arrow[d, "q"] & \text{cofiber}(f') \arrow[d] \\
* \arrow[r, "g'"'] & \mathcal{S}K \arrow[r] & \text{cofiber}(g')
\end{tikzcd}$$

to conclude that q is homologically n-connected. Since K is connected, the source and target of q are 1-connected. The Hurewicz Theorem then implies the lemma.

□

Theorem C. *Let K be a homotopy finite complex with* $\operatorname{hodim}(K) = k \leq n - 3$. *Assume* *that* $f : K \to S^n$ *is an* r*-connected map of spaces,* $r \geq 1$*. Then the induced map*

$$\pi_0(c \circ \sigma) : \pi_0(\mathbb{E}_f(K, S^n)) \to \pi_0(\mathbb{E}_{\mathcal{S}f}(\mathcal{S}K, S^{n+1}))$$

is surjective for $n \geq 2(k-r)+3$ *and injective for* $n \geq 2(k-r)+4$*.*

Proof. Invoke Theorem A and Lemma 3.4.1. □

APPENDIX

The object of this appendix is to give a construction of a semi-simplicial set [2]

$$E_f^{PD}(K, X \ rel\ L)$$

whose geometric realization coincides, up to homotopy, with the space of Poincaré Duality Embeddings $\mathbb{E}_f(K, X \ rel\ L)$. We will work under the standing assumptions given at the beginning of Chapter 3. Write Δ^m for the standard geometric m-simplex and define a j-simplex of $E_f^{PD}(K, X \ rel\ L)$ to be a commutative square of pairs of (homotopy finite, CW) $(j+2)$-ads

$$(\mathscr{D}_j) \qquad \begin{tikzcd} (A^j_\bullet, A^j_{\bullet_L}) \arrow[r] \arrow[d] & (C^j_\bullet, C^j_{\bullet_L}) \arrow[d] \\ (K \times \Delta^j, \mathcal{S}^j_K L) \arrow[r] & (X \times \Delta^j, \partial(X \times \Delta^j)) \end{tikzcd}$$

where we think of $\mathcal{S}^j_K L$ as the amalgamated union $L \times \Delta^j \cup_{L \times \partial\Delta^j} K \times \partial\Delta^j$. This gives $\mathcal{S}^j_K L$ the structure of a $(j+2)$-ad, with $j+1$ preferred subspaces given by decomposing $K \times \partial\Delta^j$ into $j+1$ copies of $K \times \Delta^{j-1}$. A similar statement applies for $\partial(X \times \Delta^j) = \mathcal{S}^j_X \partial X$. Further, the $(j+2)$-ad $A^j_\bullet$ is given by the "total" space $A^j_\bullet$ along with $j+1$ subspaces indexed by the faces of Δ^j. Let A^j_i denote the subspace of A indexd by the i^{th} face, $d_i(\Delta^j)$, of Δ^j. The $(j+2)$-ads $A^j_{\bullet_L}$, $C^j_\bullet$, and $C^j_{\bullet_L}$ are specified similarly. Moreover, $\mathscr{D}_j$ should satisfy

(i) The associated squares are ∞-cocartesian. That is, there are weak ad-homotopy equivalences

[2] By *semi-simplicial set* we mean Δ-set, i.e., simplicial set without degeneracies.

$$\operatorname{hocolim}(K \times \Delta^j \leftarrow A^j_\bullet \rightarrow C^j_\bullet) \xrightarrow{\sim} X \times \Delta^j$$
$$\operatorname{hocolim}(\mathcal{S}^j_K L \leftarrow A^j_{\bullet_L} \rightarrow C^j_{\bullet_L}) \xrightarrow{\sim} \partial(X \times \Delta^j)$$

(ii) The associated square of $(j+2)$-ads with underlying map $\mathcal{S}^j_K L \rightarrow \partial(X \times \Delta^j)$ is a PD embedding (just as in Definition 1.1.2, with minor adjustments made to suit the language of n-ads)

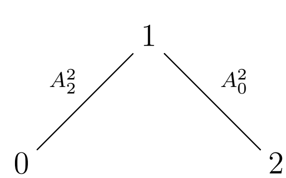
(iii) $A^j_i \rightarrow K \times d_i(\Delta^j)$ is $(n-k-1)$-connected for all i

(iv) The pairs $(\overline{K \times \Delta^j}, \mathcal{S}^j_K L \cup_{A^j_{\bullet_L}} A^j_\bullet)$ and $(\overline{C^j_\bullet}, C^j_{\bullet_L} \cup_{A^j_{\bullet_L}} A^j_\bullet)$ are $(n+j)$-dimensional PD pairs

Note that that a vertex of $E^{PD}_f(K, X \ rel\ L)$ is nothing more than a PD embedding of K in X relative to L (see Definition 1.1.3). The face maps $d^{PD}_i : E^{PD}(K,M)_j \rightarrow E^{PD}(K,M)_{j-1}$ are given by restriction. Together with these face maps, $E^{PD}_f(K, X \ rel\ L)$ has the structure of a semi-simplicial set.

Proposition A.1. $E^{PD}_f(K, X \ rel\ L)$ satisfies the Kan condition.

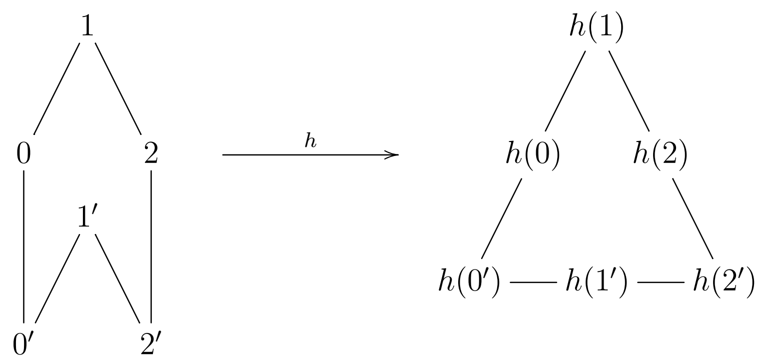
Proof. Fix j. For the sake of brevity, we will omit the notation of pairs, keeping in mind that we are working relative to the underlying map $\mathcal{S}^j_K L \rightarrow \partial(X \times \Delta^j)$. Construct a space $A^j_{\Lambda i}$ by gluing the j subspaces A^j_l, $l \neq i$, along their intersections. The following picture illustrates $A^2_{\Lambda 1}$.



Construct a space $C^j_{\Lambda i}$ similarly. Consider the space $A^j_{\Lambda i} \times [0,1]$ as the $(j+2)$-ad

$$\widehat{A^j_{\Lambda i}} = (A^j_{\Lambda i} \times [0,1]; \left\{A^j_k \times [0,1]\right\}_{k \in \left\{0,\ldots,\hat{i},\ldots,j\right\}}, A^j_{\Lambda i} \times 1).$$

and similarly for $C^j_{\Lambda i} \times [0,1]$. Let Λ^j_i denote the i^{th} horn of Δ^j. Then there is a homeomorphism $h : \Lambda^j_i \times [0,1] \to \Delta^j$ that restricts to the identity on $\Lambda^j_i \times 0$. The following picture illustrates this homeomorphism in the case $j = 2$ and $i = 1$.



Thus, we have homeomorphisms $id_K \times h : K \times \Lambda^j_i \times [0,1] \to K \times \Delta^j$ and $id_X \times h : X \times \Lambda^j_i \times [0,1] \to X \times \Delta^j$. From $\mathscr{D}_j$ above, we see that there is a commutative square of $(j+1)$-ads

$$(\mathscr{D}_{j-1}) \qquad \begin{tikzcd} A^j_{\Lambda i} \arrow[r] \arrow[d] & C^j_{\Lambda i} \arrow[d] \\ K \times \Lambda^j_i \arrow[r] & X \times \Lambda^j_i \end{tikzcd}$$

Taking the product of all of the spaces in $\mathscr{D}_{j-1}$ with $[0,1]$, we obtain the desired commutative square of $(j+2)$-ads

$$\begin{tikzcd}
\widehat{A^j_{\Lambda i}} \arrow[r] \arrow[d] & \widehat{C^j_{\Lambda i}} \arrow[d] \\
K \times \Delta^j \arrow[r] & X \times \Delta^j
\end{tikzcd}$$

□

Now, consider the nerve $NE_f(K, X \ rel \ L)$ of the category $E_f(K, X \ rel \ L)$ (see the remarks preceding Definition 1.2.4). Apply the subdivision functor ([GJ99]) to this nerve to obtain the semi-simplicial set $sdNE_f(K, X \ rel \ L)$. By definition of the subdivision, one can identify the set of components of $sdNE_f(K, X \ rel \ L)$ with the set of components of the semi-simplicial set $E_f^{PD}(K, X \ rel \ L)$ constructed above. That is,

$$\pi_0(sdNE_f(K, X \ rel \ L)) \cong \pi_0(E_f^{PD}(K, X \ rel \ L)).$$

Choose a vertex $\mathscr{D} \in E_f^{PD}(K, X \ rel \ L)_0$ and recall ([GJ99]) that

$$\pi_j(E_f^{PD}(K, X \ rel \ L), \mathscr{D}) = \pi_{j-1}(\Omega E_f^{PD}(K, X \ rel \ L), \mathscr{D})$$

where $\Omega E_f^{PD}(K, X \ rel \ L)$ is the loopspace of the semi-simplicial set $E_f^{PD}(K, X \ rel \ L)$. Using the identification of components given above, and passing to realizations, we have shown

Proposition A.2. $\pi_j(\mathbb{E}_f(K, X \ rel \ L)) \cong \pi_0(\mathbb{E}_{f\times id}(K \times D^j, X \times D^j \ rel \ \mathcal{S}^j_K L))$

□

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ABSTRACT

STABILIZATION AND CLASSIFICATION OF POINCARÉ DUALITY EMBEDDINGS

by

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A map $f : K \to X$ from a homotopy finite complex of dimension $\leq k$ to a Poincaré Duality space of dimension n is said to *Poincaré Embed* if f extends to a homotopy equivalence

$$K \cup_A C \simeq X$$

such that (K, A) and (C, A) are Poincaré pairs of dimension n. For K and X as above, we define a space $\mathbb{E}_f(K, X)$ of all such embeddings and show that there is a highly connected *stabilization map*

$$\mathbb{E}_f(K, X) \to \mathbb{E}(\mathcal{S}_X K, X \times D^1)$$

where $\mathcal{S}_X K$ denotes the unreduced fiberwise suspension of K over X. This serves as a

tool for classifying Poincaré Duality embeddings in terms of the homotopy types of their complements. In particular, a Poincaré embedding with underlying map $f : K \to X$ gives rise to a fiberwise duality map in the category of retractive spaces over X. We use this construction to obtain a highly connected *classification map*

$$\mathbb{E}_f(K, X) \to \mathbb{SW}_f(K, X)$$

where $\mathbb{SW}_f(K, X)$ is a moduli space of unstable complements for Poincaré embeddings with underlying map $f : K \to X$. As consequences, we obtain stabilization and classification results for smooth embeddings.

AUTOBIOGRAPHICAL STATEMENT

John Peter was born on September 4^{th}, 1981, in Wayne, Michigan, and has lived in Michigan all of his life. He earned a B.S. in mathematics from The University of Michigan-Dearborn in 2005, and an M.A. in mathematics from Wayne State University in 2007.