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Journal of Modern Applied Statistical Methods

Volume 9 | Issue 1 Article 9

5-1-2010

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Recommended Citation

Holgersson, H.E.T. and Karlsson, Peter S. (2010) "Model Based vs. Model Independent Tests for Cross-Correlation," *Journal of Modern Applied Statistical Methods*: Vol. 9 : Iss. 1 , Article 9.
DOI: 10.22237/jmasm/1272686880
Available at: http://digitalcommons.wayne.edu/jmasm/vol9/iss1/9

Journal of Modern Applied Statistical Methods
May 2010, Vol. 9, No. 1, 75-89

1538 – 9472/10/$95.00

Model Based vs. Model Independent Tests for Cross-Correlation

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This article discusses the issue of whether cross correlation should be tested by model dependent or model independent methods. Several different tests are proposed and their main properties are investigated analytically and with simulations. It is argued that model independent tests should be used in applied work.

Key words: Cross correlation, residuals, lag window, hypothesis tests.

Introduction

Statistical analysis frequently involves the problem of whether two variables are related to each other. One of the most popular approaches is correlation analysis, initially proposed by Galton (1888) and refined by Fisher (1915, 1921). Later on correlations became popular also in time series contexts. When estimated correlation coefficients are used to test formal hypotheses, a test statistic with a (asymptotically) known null distribution is needed. In the case of independently distributed data (i.i.d.) there are several known standard error formulas for the correlation coefficient (Stuart & Ord, 1994). If autocorrelation exists in the data, however, these null distributions are not valid because the variance of the test statistic will depend on the unknown autocorrelation. It is therefore important to develop tests that take this aspect into account.

Through the last three decades a number of articles have been concerned with this issue. Some important works include Haugh (1976) and McLeod (1979) both of whom dealt with the distributional properties of residual based cross-correlation coefficients, Koch and Yang (1986) extended these methods to include pattern in the cross-correlation function, and Hallin and Saidi (2001) extended these two methods to the general multivariate case. Hong (1996) proposed a different approach of using an AR(p) model where p is allowed to grow asymptotically with the sample size T, and Bouhaddioui and Roy (2006) further developed this idea in a more general VAR(p) context.

All of these studies share the property that they involve residual based tests, constructed by first pre-whitening the data. The rationale behind this method is that the variance of the cross-correlation coefficient is somewhat complicated for autocorrelated data, and becomes much easier to handle for variables without autocorrelation. Thus, as residuals are asymptotically uncorrelated and the main interest is in the possible cross-correlation - not in the autocorrelation - this approach is reasonable. However, there is also an option to use some linear function of the sample cross-correlations and to construct a model independent test.

Model based tests have the disadvantage that a misspecified model may lead to an inconsistent procedure but also have the potential of being more efficient than model independent tests because they are more parsimonious regarding the number of parameters. It may be questioned how model

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dependent tests perform relative to model independent tests, or is the potential efficiency gain of model based methods worth the risk of using a misspecified model? The aim of this article is to examine the properties of five different, simple tests of cross-correlation of weakly stationary bivariate processes. These involve a test dependent on a known model plus known parameters, two tests dependent on a known model but not of known parameters and two model independent tests. The asymptotic properties of the tests are established analytically and the small sample properties are examined by Monte Carlo simulations.

Methodology

Some properties of the sample correlation coefficient calculated from two possibly autocorrelated variables are considered; in particular, the focus is on the variance of the correlation coefficient. A few relevant measures must first be defined. Let X_t and Y_t be two random sequences such that

$$X_t = \mu_X + \sum_{i=0}^{\infty} \psi_{x,i} \varepsilon_{x,t-i}$$

and

$$Y_t = \mu_Y + \sum_{i=0}^{\infty} \psi_{y,i} \varepsilon_{y,t-i},$$

where $\{\varepsilon_{x,t}\}_{t=1}^{\infty}$ and $\{\varepsilon_{y,t}\}_{t=1}^{\infty}$ are two sequences of zero mean i.i.d. random variables and ψ_x and ψ_y are absolutely summable, that is, $\sum_{i=0}^{\infty} |\psi_{x,i}| < \infty$ and $\sum_{i=0}^{\infty} |\psi_{y,i}| < \infty$, $V[\varepsilon_X] < \infty$ and $V[\varepsilon_Y] < \infty$. Letting σ_X^2 and σ_Y^2 be the variance of X_t and Y_t respectively, the cross correlation coefficient is defined by

$$\rho_{XY}(k) = \frac{E[X_t - \mu_x] E\left[Y_{t-k} - \mu_y\right]}{\sqrt{\sigma_X^2}\sqrt{\sigma_Y^2}} = \rho_{YX}(-k) \quad . \tag{1}$$

The main thrust of this article is the following hypothesis:

$$\begin{aligned} &\mathrm{H}_0 : \rho_{XY}(k) = 0 \ \forall k \in \mathbb{Z} \\ &\mathrm{H}_\mathrm{A} : \rho_{XY}(k) \neq 0 \ \exists k \in \mathbb{Z} \end{aligned} \tag{2}$$

where X and Y are covariance stationary but possibly autocorrelated. In order to test this hypothesis, a proper test statistic with an asymptotically known null distribution is needed. The population correlation $\rho_{xy}(k)$ may be estimated by

$$\hat{\rho}_{XY}(k) = \frac{\hat{\sigma}_{XY}(k)}{\sqrt{\hat{\sigma}_X^2}\sqrt{\hat{\sigma}_Y^2}},$$

where

$$\hat{\sigma}_y^2 = (1/T)\sum_{t=1}^{T}\left(Y_t - \bar{Y}\right)^2,$$

$$\hat{\sigma}_{xy}(k) = (1/T)\sum_{t=1}^{T-k}\left(X_t - \bar{X}\right)\left(Y_{t+k} - \bar{Y}\right)$$

and T is the number of observations. For identically independently distributed data it is well known that, if $\rho_{XY}(k) = 0$, then

$$\hat{\rho}_{XY}(k) \xrightarrow{\ell} N(0, 1/T) \tag{3}$$

where ℓ denotes convergence in law. An improvement of (2.3) is given by Fisher's z-transformation (Fisher, 1921; Stuart & Ord, 1994). In cases when the data is not independent this variance is no longer valid. Using the well-known Bartlett approximation (for example, see Box, et al., 1994) the variance of the sample cross correlation is given by

$$Var\left(\hat{\rho}_{XY}(k)\right) \approx T^{-1}\sum_{\tau=-\infty}^{\infty}\left[\begin{aligned} &\rho_{XX}(\tau)\rho_{YY}(\tau) \\ &+\rho_{XY}(k+\tau)\rho_{YX}(k-\tau) \\ &-2\rho_{XY}(k)\begin{pmatrix}\rho_{XX}(\tau)\rho_{XY}(\tau+k) \\ +\rho_{YY}(\tau+k)\rho_{XY}(-\tau)\end{pmatrix} \\ &+\rho_{XY}^2(k)\begin{pmatrix}\rho_{XY}^2(\tau)+0,5\rho_{XX}^2(\tau) \\ +0,5\rho_{YY}^2(\tau)\end{pmatrix}\end{aligned}\right] \tag{4}$$

where $\rho_{XY}(\tau)$ is the correlation between X_t and $Y_{t-\tau}$. Equation (4) gives the variance of the sample cross-correlation coefficient between X and Y with a lag shift of k steps. Hence, under the simple null hypothesis that $\rho_{xy}(k)=0$ the equation (4) reduces to

$$Var\left[\hat{\rho}_{XY}(k)\middle|\rho_{XY}(k)=0\right]\approx T^{-1}\sum_{\tau=-\infty}^{\infty}\begin{bmatrix}\rho_{XX}(\tau)\rho_{YY}(\tau)\\ +\rho_{xy}(\tau+k)\rho_{yx}(k-\tau)\end{bmatrix}. \quad (5)$$

Furthermore, under the null hypothesis that all cross covariances are zero (as in (2)), results in

$$\lambda := Var\left[\hat{\rho}_{XY}(k)\middle|\rho_{XY}(k)=0\ \forall k\in\mathbb{Z}\right]\approx T^{-1}\sum_{\tau=-\infty}^{\infty}\left[\rho_{XX}(\tau)\rho_{YY}(\tau)\right]. \quad (6)$$

Accordingly, if a consistent estimate of λ can be obtained (for example, $\hat{\lambda}$), it follows that

$$\frac{\left(\hat{\rho}_{XY}(k)\middle|\rho_{XY}(k)=0\ \forall k\in\mathbb{Z}\right)}{\sqrt{\hat{\lambda}}}\xrightarrow{\ell}N(0,1). \quad (7)$$

From these formulas it is apparent that several possible ways exist with which to test for zero cross-correlation.

Firstly, one may test if a particular cross-correlation at lag k is zero while allowing for non-zero cross-correlations at other lags; then an estimate of (5) is sufficient to form a proper test statistic. Secondly, one might like to test whether there are any non-zero cross-correlations above a certain lag. Thirdly, one may test whether there are any non-zero cross-correlations at all. This is the hypothesis expressed in (2) and is the main issue here. The question is how to construct a test that is both consistent and also reasonably simple to perform. Observably, equation (7) can be used to form a consistent test if $\rho_{XY}(k)\neq 0$. However, the null hypothesis states that the cross-correlations are zero at all lags. The question is then what will happen if the cross-correlation at lag k is zero but there is at least one non-zero coefficient at some other lag, e.g. if $\rho_{XY}(k)=0$ but $\rho_{XY}(k+l)\neq 0$ for some $l\neq 0$.

To address this question, two things should be noted. First, the cross-correlation function is, in most cases, exponentially decaying so that even if the value of k corresponding to the largest cross–correlation is not specified there will still be a non-zero cross-correlation at k. Thus, it is not likely that an inappropriately chosen k is specified such that $\rho_{XY}(k)=0$ under the alternative hypothesis. Second, in a comparison of equations (5) and (6), there will still be a sense in which the test is consistent as the test statistic will diverge from its null distribution. In other words, specifying a value k that does not correspond exactly to the largest cross-correlation is merely a matter of optimality rather than consistency. There also exists a possibility to involve several $\rho_{xy}(k)$ explicitly in the test: one might use the sum of squared cross-correlations within a certain interval, for example, $\hat{\rho}^2_{XY}(-h)+\ldots+\hat{\rho}^2_{XY}(h)$.

Unfortunately such an approach will introduce additional complications as the sample cross-correlations will not be uncorrelated even under the null hypothesis (apart from the unlikely special case of independent data). Therefore, several authors, including Haugh (1976), McLeod (1979), Koch and Yang (1986) and Hallin and Saidi (2001) proposed model dependent tests and then applied this kind of test on the asymptotically uncorrelated residuals. For example, if $\hat{u}$, and $\hat{v}$ are residuals from ARMA models, a test may be defined by $Q(h)=\hat{\rho}^2_{\hat{u}\hat{v}}(-h)+\ldots+\hat{\rho}^2_{\hat{u}\hat{v}}(h)$.

A slightly different situation arises in cases where there is some sort of a priori knowledge of which lag the largest cross-correlation might be (if any), the null hypothesis (2) can be tested by the asymptotic null distribution of (7); then one is left with the issue of how to estimate the variance λ of equation (6). This approach is followed here because the other is fairly well investigated in the literature. In particular, two different approaches are investigated: (i) tests dependent upon a model, and (ii) tests independent of model assumptions. Case (i) may be dealt with as follows: if X_t and Y_t are known to follow a finite-order ARMA process, then the autocorrelations ρ_{XX} and ρ_{YY} may be expressed as functions of the autoregressive parameters. For example, if X_t and Y_t are given by two ARMA(1,1) processes, that is, if

$$X_t - \phi_X X_{t-1} = \varepsilon_{X,t} - \theta_X \varepsilon_{X,t-1}$$

and

$$Y_t - \phi_Y Y_{t-1} = \varepsilon_{Y,t} - \theta_Y \varepsilon_{Y,t-1}$$

then the autocorrelations of X_t are known to be given by

$$\rho_{XX}(1) = \frac{(1-\phi_X\theta_X)(\phi_X - \theta_X)}{1+\theta_X^2 - 2\phi_X\theta_X},$$

$$\rho_{XX}(\tau) = \phi_X \rho_{XX}(1),\ \tau > 1.$$

Hence, using obvious notation,

$$\rho_{XX}(1)\rho_{YY}(1) = \frac{(1-\varphi_X\theta_X)(\varphi_X - \theta_X)}{(1+\theta_X^2 - 2\varphi_X\theta_X)} \frac{(1-\varphi_Y\theta_Y)(\varphi_Y - \theta_Y)}{(1+\theta_Y^2 - 2\varphi_Y\theta_Y)}$$

$$\rho_{XX}(\tau)\rho_{YY}(\tau) = \phi_X^{\tau-1}\rho_{XX}(1)\phi_Y^{\tau-1}\rho_{YY}(1)$$

results in

$$\sum_{\tau=1}^{\infty} \rho_{XX}(\tau)\rho_{YY}(\tau) = \sum_{\tau=1}^{\infty} \phi_X^{\tau-1}\rho_{XX}(1)\phi_Y^{\tau-1}\rho_{YY}(1) = \rho_{XX}(1)\rho_{YY}(1)\left(1/(1-\phi_X\phi_Y)\right).$$

Thus, if X_t and Y_t are two ARMA(1,1) processes it follows that

$$Var\left[\hat{\rho}_{XY}(k)\right] \approx T^{-1} \frac{(1-\phi_X\theta_X)(\phi_X - \theta_X)}{(1+\theta_X^2 - 2\phi_X\theta_X)} \frac{(1-\phi_Y\theta_Y)(\phi_Y - \theta_Y)}{(1+\theta_Y^2 - 2\phi_Y\theta_Y)} \left(\frac{1}{(1-\phi_X\phi_Y)}\right). \tag{8}$$

From (2.8) the variance for AR(1) or MA(1) processes are immediately obtained by setting the irrelevant parameter to zero. This estimator can easily be generalised to ARMA(*p, q*) processes of arbitrary orders by substituting $\rho_{XX}(\tau)$ and $\rho_{YY}(\tau)$ with the model-based autocorrelations. These are acquired by the autocorrelation generating function which can be found in the time series literature (see, for example, Hamilton, 1994). The unknown parameters of (8) should be replaced by any consistent estimates such as maximum likelihood estimates or non-linear least squares (see Brockwell & Davis, 1991; Box, et al., 1994 for further details on estimations of ARMA parameters).

An alternative way to use model based tests is to use the asymptotically independent residuals: If the parameters of the ARMA model were actually known, then the two marginal models X_t and Y_t could be reformulated according to

$$\varepsilon_{X,t} = X_t - \phi_X X_{t-1} + \theta_X \varepsilon_{X,t-1}$$

and

$$\varepsilon_{Y,t} = Y_t - \phi_Y Y_{t-1} + \theta_Y \varepsilon_{Y,t-1} \tag{9}$$

Thus, by replacing the true ARMA parameter by consistent estimates the resulting asymptotically white noise residuals, $\hat{\varepsilon}_{X,t}$ and $\hat{\varepsilon}_{Y,t}$, can be used to test for cross-correlations because the variance of the cross-correlation may be approximated by $1/T$, according to equation (3).

Residual based tests have been proposed earlier in the literature, including the citations above, but will still be considered for comparison. The advantage of ARMA based tests is that they are parsimonious, although the disadvantage is that they are model-dependent and a rough approximation to the true unknown functional form may lead to an inconsistent variance estimate. Hence, it is of interest to also consider a variance estimate that does not rely on any model assumptions. In particular, the cross correlations $\hat{\rho}_{XY}(\tau)$ of equation (6) could be substituted directly with the sample autocorrelations:

$$\hat{\rho}_{XX}(\tau) = \frac{\sum_{t=1}^{T}\left(X_t - \overline{X}\right)\left(X_{t-\tau} - \overline{X}\right)}{\sum_{t=1}^{T}\left(X_t - \overline{X}\right)^2}.$$

However, as (6) is a sum of infinitely many parameters some care needs to be taken: If a stochastic process is absolutely summable with finite fourth-order moments, then $\hat{\rho}_{XX}(\tau) \xrightarrow{\ell} N\left(\rho_{XX}(\tau),\, W_{\tau\tau}\right)$ for some $W_{\tau\tau} < \infty$ (see Brockwell & Davis, 1991). Hence the variance and the bias of $\hat{\rho}_{\tau\tau}$ are of the order $o\left(T^{-\alpha}\right)$ for any $\alpha \in (0,1/2)$, and $\hat{\rho}_{\tau\tau}(\tau)$ converges in mean square to $\rho_{\tau\tau}(\tau)$ at the rate $o\left(T^{-\alpha}\right)$, that is, $\hat{\rho}_{XX}(\tau) = \rho_{XX}(\tau) + o_p\left(T^{-\alpha}\right)$. Accordingly it follows that $\hat{\rho}_{XX}(\tau)\hat{\rho}_{YY}(\tau) = \rho_{XX}(\tau)\rho_{YY}(\tau) + o_p\left(T^{-\alpha}\right)$, and an estimate of (6) can be formulated. In particular, absolute summability of the original variables X and Y implies absolute summability of the sequence $\left\{\rho_{XX}(\tau)\rho_{YY}(\tau)\right\}_{\tau=0}^{\infty}$. Thus, for some monotonically increasing function $q = q(T)$,

$$\begin{aligned}&\delta(q)\\ &= \sum_{\tau=1}^{q}\hat{\rho}_{XX}(\tau)\hat{\rho}_{YY}(\tau) - \sum_{\tau=1}^{\infty}\rho_{XX}(\tau)\rho_{YY}(\tau)\\ &= \sum_{\tau=1}^{q}\left[\hat{\rho}_{XX}(\tau)\hat{\rho}_{YY}(\tau) - \rho_{XX}(\tau)\rho_{YY}(\tau)\right] - o(1)\end{aligned} \tag{10}$$

The literature concerning the convergence of sequences of the type $\delta(q)$ is extensive, one of the most cited being Newey and West (1987). If $\hat{\rho}_{XX}(\tau) = \rho_{XX}(\tau) + o_p\left(T^{-1/2+\varepsilon}\right)$ for all $\varepsilon > 0$ (which is the convergence rate met in most linear estimates) but the convergence of $\hat{\rho}_{XX}(\tau)$ cannot be assumed to hold uniformly in τ, then q must be restricted to values below $T^{1/4}$ in order to ensure that $\delta(q) = o_p(1)$.

However, for linear processes with finite fourth moments, i.i.d. innovations and absolute summable coefficients, q may be relaxed to values below $T^{1/2}$, $\delta\left(T^{-1/2+\varepsilon}\right) = o_p(1)$. Moreover, recent results (e.g., Lobato & Velasco, 2004; Robinson, 1998) have shown that, in many cases, sequences of the above type may converge for values up to T, $\delta\left(T^{-1+\varepsilon}\right) = o_p(1)$. This is mainly a consequence of operating with sums containing stochastic down weighting such as $\sum_{\tau}\hat{\rho}_{XX}(\tau)\hat{\rho}_{YY}(\tau)$ herein; $\hat{\rho}_{XX}(\tau)$ down weights $\hat{\rho}_{YY}(\tau)$ and vice versa and both decrease individually in τ. These properties indicate that restricting q to values below $T^{1/4}$ might be unnecessarily stringent; therefore the compromise $\delta\left(T^{-1/2+\varepsilon}\right)$ is used in this article so that the proposed model-free estimate of (6) takes the form

$$\begin{aligned}&\hat{\lambda} = \sum_{\tau=1}^{q}\hat{\rho}_{XX}(\tau)\hat{\rho}_{YY}(\tau),\\ &\text{where } q \le \operatorname{int}\left(T^{-1/2+\varepsilon}\right)\end{aligned}. \tag{11}$$

Hence, the variance estimate of (11) consistently estimates the variance component of (6). But, this estimate is not guaranteed to be positive in small samples, for this reason another variance estimate which is strictly non-negative is also considered:

$$\tilde{\lambda} = \sum_{\tau=0}^{q}\left(1 - \tau/(q+1)\right)\hat{\rho}_{XX}(\tau)\hat{\rho}_{YY}(\tau). \tag{12}$$

The non-negativeness of (12) is easily established: When X_t and Y_t are two absolutely summable stochastic processes with finite fourth moments and $\hat{\mathbf{\Gamma}}_{X,T}$ and $\hat{\mathbf{\Gamma}}_{Y,T}$ are the matrices of the sample autocorrelations, it is well known that $\hat{\mathbf{\Gamma}}_{X,T}$ and $\hat{\mathbf{\Gamma}}_{Y,T}$ are both non-negative matrices (Brockwell and Davis, 1991). Moreover, because direct products (symbolized by $\odot$) of non-negative matrices are also non-negative (Schott, 1997), it follows that $\hat{\mathbf{\Gamma}}_{X,T} \odot \hat{\mathbf{\Gamma}}_{Y,T}$ is non-negative as well. Hence there exists an $\mathbf{L}$ such that $\hat{\mathbf{\Gamma}}_{X,T} \odot \hat{\mathbf{\Gamma}}_{Y,T} = \mathbf{L}'\mathbf{L} \geq 0$, then if $\mathbf{1}_q' = (111...100..0)$ such that $\mathbf{1}_q'\mathbf{1}_q = q$, it follows that:

$$\begin{aligned}\tilde{\lambda}_T &= \sum_{\tau=0}^{q}\left(1-\tau/(q+1)\right)\hat{\rho}_{XX}(\tau)\,\hat{\rho}_{YY}(\tau) \\ &= \mathbf{1}_q'\hat{\mathbf{\Gamma}}_{X,T} \odot \hat{\mathbf{\Gamma}}_{Y,T}\mathbf{1}_q \\ &= \mathbf{1}_q'\mathbf{L}'\mathbf{L}\mathbf{1}_q \\ &= \left(\mathbf{L}\mathbf{1}_q\right)'\left(\mathbf{L}\mathbf{1}_q\right) \geq 0\end{aligned} .$$

In other words, if X, Y are two linear processes with finite fourth order moments and absolute summable coefficients and $q \leq \operatorname{int}\left(T^{-1/2+\varepsilon}\right)$, then $\tilde{\lambda}$ is a non-negative and consistent estimate of (6). Truncating the sample autocorrelation function at a certain point, as in (11), is sometimes referred to as a rectangular lag window, and estimates of the kind in (12) are referred to as a triangular window. That terminology is adopted later, even though here work with products of correlations is employed as opposed to individual correlations (which is the usual case).

To sum up, four estimates of the variance of equation (6) have been proposed, two model-independent and two model-based estimates. The first two use the same information set, namely the ARMA model and its parameter estimates; the other two depend only upon the truncation point and the choice of lag window. Of particular interest is the potential difference between the model based and the model independent tests; how much gain is there in knowing the true model? It is also of interest to investigate the possible difference within each type of test, asking the questions: Does it matter how one makes use of the known model and does the choice of lag window make a difference?

Results

When investigating the properties of a test procedure, two aspects are of prime importance. First it is necessary to determine whether the actual size of the test - the probability of rejecting the null hypothesis when it is true - is close to the nominal size. Given that the actual size is a reasonable approximation to the nominal size, it is then necessary to investigate the actual power of the test - the probability of rejecting the null hypothesis when it is false - for a number of different parameter settings. The number of replicates in the computer simulations is 100,000 for each size and power simulation.

In this study the relevant factor is first and foremost the choice of test. Five different tests are considered based on the statistic (7) but with different estimates of the standard error λ, namely (i) the ARMA based test using the asymptotically white noise residuals (so that $\lambda = 1/T$), (ii) standard error obtained from (8) using the true ARMA parameters, (iii) standard error obtained by (8) using maximum likelihood estimate of the ARMA parameters, (iv) standard error using the rectangular lag window (11) with truncation point $q = \operatorname{int}\left[T^{-0.45}\right]$, and finally (v) the test based on the standard error using the triangular lag window (12), again with truncation point $q = \operatorname{int}\left[T^{-0.45}\right]$.

It is critical to identify possible differences between these five tests, and in order to do so some different autocorrelation patterns must be considered. For that purpose the AR(1) process, MA(1) process and ARMA(1,1) processes are used with different values of autoregressive parameters, ranging from white noise (independent data) up to high autocorrelation. Moreover, two different sample sizes are used: 30 observations (which is usually considered as a small sample in time series analysis) and 200 observations (medium-sized sample). Finally, in order to investigate the tests' power to detect correlation, cross-correlations ranging from 0 (no correlation) up to 0.9 (very strong correlation) are considered. The significance level is set to the 0.05 level in all models so that the critical values are −1.96 and 1.96 in all tests.

By counting the number of rejections the empirical significance level is identified for each test conducted. The results are presented in Tables 1-8. According to Table 1, which deals with the special case of two independent white noise processes, it is observed that all tests have an almost perfect size relative to their nominal sizes, except perhaps the residual test for the smallest samples. Although this is not an unexpected result (because the sample autocorrelations converge rapidly for white noise) it is still interesting because it reveals that the choice of test is almost irrelevant for white noise data. Unfortunately, the choice of test becomes less obvious when considering the size properties of autocorrelated data.

As shown in Table 2, there are some notable differences between the various tests. In particular, the rejection frequencies of the model-based tests (as functions of true respectively estimated parameters) reveal that there is no obvious gain in knowing the true ARMA parameters. Even though the under-rejection of both these tests seems to worsen for larger values of the autocorrelation parameter, the test of estimated ARMA parameters underestimates less when compared with the corresponding test of the true parameters. Moreover, there is also a somewhat drastic difference between the two model-independent tests. In fact, the test of the rectangular lag window seems to uniformly outperform that of the triangular lag window. Although the test of the rectangular lag window slightly over rejects for high autocorrelation, the effect is not that serious in contrast to that of the triangular lag window which shows a rejection frequency of 0.11 at high autocorrelation and small T. It is noteworthy that the residual-based test behaves satisfactorily at all sample sizes and autocorrelations.

Table 3 shows some interesting differences compared to Table 2. The residual-based test no longer maintains its good size properties, no difference exists between the two model-based tests and, additionally, the difference between the two model-independent tests is now very small (they both stay fairly close to the nominal size though the rectangular window is slightly closer).

Not unexpectedly, the rejection frequencies shown in Table 4 are a mixture of the results shown in Tables 2 and 3. Hence it is not easy to select a test that is generally better than another when it comes to size properties, though the residual-based test and the model-free test using the rectangular lag window may be said to have good overall properties.

The power simulations in Tables 5 and 6 present rejection frequencies for AR(1) properties at two sample sizes, 30 and 200 observations respectively. It is striking that the differences of the various tests are negligible for white noise, irrespective of whether the sample size is 30 or 200. Conversely, there appears to be a difference when the autoregressive parameter is 0.7.

The general pattern is that the model-based tests have surprisingly low power although the residual-based test has higher power than any other test. In fact, the difference is even more accentuated for the large sample size. The two model-independent tests have power properties between the model-based test and the residual-based one. The residual-based test maintains its superior power for the MA process (Tables 7 and 8) even if the difference to the other tests is now less drastic.

For most parameter values and sample sizes the model-free tests are not far behind those of the residual test. If one or two winners of the 5 tests are to be selected, one should start by considering tests that have fairly acceptable size properties - even for strong autocorrelation. This rules out the model-based tests (i) and (ii) as well as the model-independent test using a triangular lag window. The remaining two tests both have their own pros and cons; the residual-based test uniformly outperforms the model-independent test, but at the same time it should be noted that it is somewhat difficult to assume the model to be known. For this reason, and because the model-independent test is clearly consistent and not much weaker in power than the residual-based test, one might want to recommend the test of the rectangular window for an applied situation unless the true model is known.

Table 1: Estimated Size for White Noise Process

Sample Size T	Residual-Based Test	Model-Based, True Parameters	Model-Based, Estimated Parameters	Model-Free, Rectangle Window	Model-Free, Triangle Window
20	0.060	0.052	0.052	0.050	0.052
30	0.058	0.049	0.049	0.048	0.048
40	0.056	0.050	0.050	0.048	0.049
50	0.054	0.050	0.050	0.048	0.049
70	0.054	0.051	0.051	0.049	0.050
100	0.052	0.052	0.052	0.051	0.051
200	0.052	0.052	0.052	0.053	0.052
500	0.050	0.051	0.051	0.052	0.051

Table 2: Estimated Size for AR(1) Process

Sample Size T	Residual-Based Test	Model-Based, True Parameters	Model-Based, Estimated Parameters	Model-Free, Rectangle Window	Model-Free, Triangle Window
Phi = 0.2					
20	0.057	0.050	0.051	0.052	0.055
30	0.057	0.051	0.051	0.054	0.055
40	0.055	0.049	0.051	0.053	0.055
50	0.057	0.051	0.051	0.054	0.055
70	0.054	0.050	0.049	0.050	0.053
100	0.056	0.050	0.049	0.051	0.052
200	0.053	0.053	0.053	0.055	0.056
500	0.050	0.054	0.054	0.054	0.055
Phi = 0.5					
20	0.056	0.028	0.034	0.057	0.069
30	0.056	0.035	0.039	0.057	0.066
40	0.056	0.038	0.039	0.056	0.065
50	0.057	0.041	0.042	0.055	0.062
70	0.054	0.043	0.045	0.055	0.063
100	0.056	0.046	0.047	0.055	0.061
200	0.052	0.050	0.050	0.054	0.060
500	0.050	0.052	0.052	0.053	0.057
Phi = 0.8					
20	0.057	0.001	0.006	0.071	0.114
30	0.056	0.003	0.011	0.070	0.111
40	0.055	0.009	0.015	0.068	0.110
50	0.059	0.016	0.020	0.065	0.104
70	0.053	0.027	0.031	0.064	0.097
100	0.056	0.033	0.037	0.065	0.093
200	0.053	0.041	0.043	0.059	0.081
500	0.050	0.049	0.049	0.056	0.072

Table 3: Estimated Size for MA(1) Process

Sample Size T	Residual-Based Test	Model-Based True Parameters	Model-Based, Estimated Parameters	Model-Free, Rectangle Window	Model-Free, Triangle Window
Theta = 0.2					
20	0.056	0.054	0.054	0.052	0.055
30	0.054	0.052	0.052	0.052	0.055
40	0.054	0.054	0.053	0.054	0.057
50	0.055	0.057	0.056	0.058	0.058
70	0.057	0.058	0.058	0.059	0.061
100	0.057	0.057	0.057	0.059	0.059
200	0.053	0.054	0.054	0.055	0.056
500	0.051	0.052	0.052	0.052	0.053
Theta = 0.5					
20	0.058	0.052	0.051	0.053	0.059
30	0.054	0.051	0.051	0.054	0.061
40	0.054	0.054	0.054	0.057	0.063
50	0.054	0.058	0.058	0.059	0.064
70	0.058	0.057	0.058	0.061	0.064
100	0.058	0.058	0.057	0.058	0.061
200	0.053	0.055	0.055	0.056	0.060
500	0.051	0.053	0.053	0.053	0.055
Theta = 0.8					
20	0.105	0.051	0.049	0.054	0.066
30	0.091	0.051	0.051	0.055	0.063
40	0.080	0.053	0.052	0.058	0.065
50	0.072	0.056	0.054	0.059	0.067
70	0.065	0.058	0.057	0.063	0.067
100	0.061	0.058	0.057	0.059	0.064
200	0.054	0.056	0.056	0.057	0.061
500	0.051	0.052	0.052	0.052	0.054

Table 4: Estimated Size for ARMA(1,1) Process

Sample Size T	Residual-Based Test	Model-Based, True Parameters	Model-Based, Estimated Parameters	Model-Free, Rectangle Window	Model-Free, Triangle Window
Phi = 0.1, Theta = 0.1					
20	0.065	0.043	0.050	0.051	0.054
30	0.062	0.042	0.048	0.050	0.052
40	0.059	0.039	0.045	0.048	0.049
50	0.055	0.040	0.048	0.050	0.051
70	0.052	0.041	0.051	0.051	0.054
100	0.052	0.043	0.051	0.054	0.053
200	0.051	0.047	0.051	0.051	0.052
500	0.049	0.046	0.047	0.048	0.049
Phi = 0.25, Theta = 0.25					
20	0.068	0.035	0.041	0.053	0.059
30	0.057	0.037	0.044	0.054	0.061
40	0.057	0.041	0.049	0.056	0.063
50	0.051	0.037	0.045	0.052	0.056
70	0.055	0.044	0.050	0.053	0.058
100	0.051	0.044	0.048	0.050	0.054
200	0.052	0.049	0.050	0.052	0.056
500	0.054	0.052	0.053	0.053	0.056
Phi = 0.5, Theta = 0.5					
20	0.075	0.029	0.026	0.061	0.084
30	0.061	0.037	0.037	0.060	0.076
40	0.055	0.042	0.042	0.061	0.081
50	0.055	0.042	0.043	0.060	0.072
70	0.053	0.043	0.045	0.055	0.067
100	0.049	0.045	0.047	0.054	0.064
200	0.046	0.042	0.042	0.047	0.055
500	0.052	0.049	0.049	0.050	0.056

Table 5: Estimated Power for AR(1) Process
Sample Size = 30

Underlying Cross-Correlation	Residual-Based Test	Model-Based, True Parameters	Model-Based, Estimated Parameters	Model-Free, Rectangle Window	Model-Free, Triangle Window
Phi = 0					
0	0.058	0.054	0.054	0.052	0.054
0.1	0.093	0.090	0.088	0.086	0.087
0.2	0.191	0.194	0.193	0.188	0.192
0.3	0.365	0.382	0.380	0.369	0.376
0.4	0.589	0.619	0.616	0.606	0.613
0.5	0.802	0.833	0.829	0.822	0.828
0.6	0.945	0.959	0.957	0.954	0.957
0.7	0.992	0.994	0.995	0.994	0.994
0.8	1	1	1	1	1
0.9	1	1	1	1	1
Phi = 0.3					
0	0.058	0.050	0.047	0.053	0.057
0.1	0.091	0.076	0.076	0.083	0.088
0.2	0.192	0.160	0.162	0.175	0.183
0.3	0.366	0.323	0.326	0.346	0.357
0.4	0.590	0.542	0.548	0.568	0.583
0.5	0.804	0.765	0.767	0.780	0.792
0.6	0.946	0.922	0.923	0.929	0.934
0.7	0.992	0.988	0.988	0.989	0.990
0.8	1	0.999	0.999	0.999	0.999
0.9	1	1	1	1	1
Phi = 0.7					
0	0.058	0.017	0.023	0.063	0.087
0.1	0.091	0.025	0.032	0.086	0.112
0.2	0.192	0.049	0.063	0.144	0.180
0.3	0.367	0.100	0.124	0.248	0.288
0.4	0.596	0.196	0.223	0.387	0.436
0.5	0.810	0.336	0.363	0.563	0.614
0.6	0.949	0.541	0.549	0.756	0.791
0.7	0.992	0.758	0.753	0.906	0.923
0.8	1	0.928	0.898	0.983	0.987
0.9	1	0.994	0.968	1	1

Table 6: Estimated Power for AR(1) Processes
Sample Size = 200

Underlying Cross-Correlation	Residual-Based Test	Model-Based, True Parameters	Model-Based, Estimated Parameters	Model-Free, Rectangle Window	Model-Free, Triangle Window
Phi = 0					
0	0.052	0.051	0.051	0.051	0.051
0.1	0.292	0.290	0.291	0.289	0.290
0.2	0.811	0.815	0.816	0.815	0.816
0.3	0.990	0.991	0.990	0.991	0.991
0.4	1	1	1	1	1
0.5	1	1	1	1	1
0.6	1	1	1	1	1
0.7	1	1	1	1	1
0.8	1	1	1	1	1
0.9	1	1	1	1	1
Phi = 0.3					
0	0.053	0.051	0.051	0.053	0.054
0.1	0.291	0.252	0.254	0.257	0.260
0.2	0.810	0.744	0.745	0.749	0.753
0.3	0.990	0.979	0.980	0.981	0.982
0.4	1	1	1	1	1
0.5	1	1	1	1	1
0.6	1	1	1	1	1
0.7	1	1	1	1	1
0.8	1	1	1	1	1
0.9	1	1	1	1	1
Phi = 0.7					
0	0.053	0.045	0.047	0.056	0.069
0.1	0.291	0.123	0.126	0.143	0.166
0.2	0.810	0.383	0.384	0.417	0.454
0.3	0.990	0.732	0.737	0.760	0.785
0.4	1	0.943	0.946	0.953	0.962
0.5	1	0.996	0.996	0.997	0.998
0.6	1	1	1	1	1
0.7	1	1	1	1	1
0.8	1	1	1	1	1
0.9	1	1	1	1	1

Table 7: Estimated Power for MA(1) Process
Sample Size = 30

Underlying Cross-Correlation	Residual-Based Test	Model-Based, True Parameters	Model-Based, Estimated Parameters	Model-Free, Rectangle Window	Model-Free, Triangle Window
Theta =0					
0	0.055	0.055	0.055	0.051	0.052
0.1	0.081	0.083	0.083	0.080	0.082
0.2	0.179	0.187	0.186	0.179	0.182
0.3	0.346	0.374	0.371	0.361	0.366
0.4	0.566	0.607	0.604	0.590	0.598
0.5	0.788	0.827	0.824	0.812	0.821
0.6	0.932	0.954	0.953	0.947	0.952
0.7	0.987	0.995	0.994	0.994	0.995
0.8	0.998	1	1	1	1
0.9	1	1	1	1	1
Theta =0.3					
0	0.053	0.052	0.053	0.055	0.057
0.1	0.082	0.079	0.079	0.081	0.087
0.2	0.177	0.166	0.169	0.172	0.178
0.3	0.347	0.330	0.336	0.336	0.349
0.4	0.562	0.548	0.551	0.552	0.567
0.5	0.780	0.769	0.775	0.772	0.787
0.6	0.925	0.926	0.929	0.928	0.935
0.7	0.985	0.990	0.990	0.990	0.991
0.8	0.996	0.999	1	1	1
0.9	0.999	1.000	1	1	1
Theta =0.7					
0	0.066	0.053	0.051	0.056	0.063
0.1	0.094	0.075	0.074	0.079	0.088
0.2	0.177	0.143	0.139	0.149	0.164
0.3	0.333	0.271	0.271	0.281	0.307
0.4	0.532	0.450	0.451	0.465	0.493
0.5	0.740	0.672	0.673	0.687	0.712
0.6	0.882	0.853	0.862	0.869	0.883
0.7	0.956	0.964	0.968	0.971	0.975
0.8	0.986	0.997	0.998	0.998	0.998
0.9	0.996	1	1	1	1

Table 8: Estimated Power for MA(1) Processes
Sample Size = 200

Underlying Cross-Correlation	Residual-Based Test	Model-Based, True Parameters	Model-Based, Estimated Parameters	Model-Free, Rectangle Window	Model-Free, Triangle Window
Theta = 0					
0	0.049	0.049	0.049	0.050	0.049
0.1	0.282	0.284	0.282	0.283	0.283
0.2	0.809	0.813	0.814	0.813	0.813
0.3	0.990	0.991	0.991	0.990	0.990
0.4	1	1	1	1	1
0.5	1	1	1	1	1
0.6	1	1	1	1	1
0.7	1	1	1	1	1
0.8	1	1	1	1	1
0.9	1	1	1	1	1
Theta = 0.3					
0	0.049	0.051	0.051	0.052	0.052
0.1	0.282	0.260	0.260	0.262	0.265
0.2	0.809	0.755	0.755	0.757	0.761
0.3	0.990	0.982	0.982	0.982	0.983
0.4	1	1	1	1	1
0.5	1	1	1	1	1
0.6	1	1	1	1	1
0.7	1	1	1	1	1
0.8	1	1	1	1	1
0.9	1	1	1	1	1
Theta = 0.7					
0	0.049	0.051	0.052	0.051	0.054
0.1	0.283	0.216	0.218	0.219	0.227
0.2	0.811	0.657	0.654	0.660	0.669
0.3	0.989	0.949	0.950	0.952	0.955
0.4	1	0.999	0.999	0.999	0.999
0.5	1	1	1	1	1
0.6	1	1	1	1	1
0.7	1	1	1	1	1
0.8	1	1	1	1	1
0.9	1	1	1	1	1

Conclusion

This study used five tests for cross-correlation with the purpose of investing the possible gain of knowing the true model, or the true parameters, relative to model independent tests. The size and power properties of five tests, each relying on different amounts of information, were investigated via the use of Monte Carlo simulations. It was observed that the size properties are essentially the same for all tests in case of white noise data. For autocorrelated data the size properties diverge; for slowly decaying autocorrelations the residual based test is markedly better than the others, although for rapidly decaying autocorrelations the residual based test is inferior to the others in that it over rejects, thus, none of the tests has uniformly best size properties.

The power properties of the tests are the same for white noise data, but in the case of autocorrelation there are some apparent differences. For slowly decaying autocorrelations the residual based test is markedly better than the others, but for rapidly decaying autocorrelations the power properties are about the same for all tests. It was also observed that the choice of lag window for the model independent estimates is of some importance. The size properties are uniformly better for the rectangular lag window but the power properties are about the same. In general, the residual based test dominates the model independent test in terms of power, but the potency of the residual based test should be weighed against the risk of using a misspecified model.

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