On Maximum Likelihood Estimators of the Parameters of a Modified Weibull Distribution Using Extreme Ranked Set Sampling

Amer Ibrahim Al-Omari  
*Al al-Bayt University, alomari_amer@yahoo.com*

Said Ali Al-Hadhrami  
*College of Applied Sciences, abur1972@yahoo.co.uk*

Follow this and additional works at: [http://digitalcommons.wayne.edu/jmasm](http://digitalcommons.wayne.edu/jmasm)

Part of the [Applied Statistics Commons](http://digitalcommons.wayne.edu/jmasm), [Social and Behavioral Sciences Commons](http://digitalcommons.wayne.edu/jmasm), and the [Statistical Theory Commons](http://digitalcommons.wayne.edu/jmasm)

Recommended Citation

DOI: 10.22237/jmasm/1320121020  
Available at: [http://digitalcommons.wayne.edu/jmasm/vol10/iss2/18](http://digitalcommons.wayne.edu/jmasm/vol10/iss2/18)

This Regular Article is brought to you for free and open access by the Open Access Journals at DigitalCommons@WayneState. It has been accepted for inclusion in Journal of Modern Applied Statistical Methods by an authorized editor of DigitalCommons@WayneState.
On Maximum Likelihood Estimators of the Parameters of a Modified Weibull Distribution Using Extreme Ranked Set Sampling

Amer Ibrahim Al-Omari Said Ali Al-Hadhrami
Al al-Bayt University, College of Applied Sciences,
Mafraq, Jordan Nizwa, Oman

Extreme ranked set sampling (ERSS) is considered to estimate the three parameters and population mean of the modified Weibull distribution (MWD). The maximum likelihood estimator (MLE) is investigated and compared to the corresponding one based on simple random sampling (SRS). It is found that, the MLE based on ERSS is more efficient than MLE using SRS for estimating the three parameters of the MWD. The ERSS estimator of the population mean of the MWD is also found to be more efficient than the SRS based on the same number of measured units.

Key words: Modified Weibull distribution, extreme ranked set sampling, maximum likelihood estimator, simple random sampling, information number.

Introduction

The modified Weibull distribution (MWD) was suggested by Sarhan and Zaindin (2009). The probability density function (pdf) of the MWD is given by

\[ f(x; \alpha, \beta, \gamma) = \left( \alpha + \beta \gamma x^{\gamma-1} \right) \exp \left( -\alpha x - \beta x^\gamma \right), \]

\[ x > 0, \]

and the corresponding distribution function (cdf) is

\[ F(x; \alpha, \beta, \gamma) = 1 - \exp \left( -\alpha x - \beta x^\gamma \right), x > 0, \]

where \( \gamma > 0 \) and \( \alpha, \beta \geq 0 \) such that \( \alpha + \beta > 0 \). The MWD have two shape parameters \( \gamma \) and \( \beta \), and a scale parameter \( \alpha \). The hazard function of the MWD is

\[ h(x; \alpha, \beta, \gamma) = \alpha + \beta \gamma x^{\gamma-1}, \]

which increases for \( \gamma > 1 \), decreases for \( \gamma < 1 \) and remains constant for \( \gamma = 1 \). Sarhan and Zaindin (2009) defined the \( k \)th moment, \( \mu_k \), of the MWD random variable as

\[ \mu_k = \sum_{i=0}^{\infty} \frac{(-\beta)^i}{i!} \left( \frac{\Gamma(i \gamma + k + 1)}{\alpha^{\gamma + k}} + \frac{\beta \Gamma(i \gamma + k)}{\alpha^{\gamma + k}} \right) \]

if \( \alpha, \beta > 0, \)

\[ \mu_k = \frac{\Gamma(k + 1)}{\beta^k} \]

if \( \alpha > 0, \beta = 0. \)

The moment generating function of the MWD is given by
MLEs of the parameters of a modified Weibull distribution using ERSS

Some special cases of the MWD distribution are the exponential distribution, Raleigh distribution, linear failure rate distribution and Weibull distribution. For additional details about the MWD see: Sarhan & Zaindin (2009) and Zaindin & Sarhan (2009).

The maximum likelihood estimator of the three parameters and the population mean of the modified Weibull distribution is examined, and compared to their counterparts based on simple random sampling. The MLE of the parameters based on ERSS is considered for two cases: when the set size is even and odd.

\[
M(t) = \begin{cases} 
\sum_{i=0}^\infty (-\beta)^i \frac{\alpha \Gamma(i \gamma + 1)}{(\alpha - t)^{i+1}} + \frac{\beta \gamma \Gamma(i \gamma + \gamma)}{(\alpha - t)^{i+\gamma}} & \text{if } \alpha, \beta > 0, \alpha > t, \\
\sum_{i=0}^\infty t^i \Gamma(i \gamma + 1) & \text{if } \alpha = 0, \beta > 0, \\
\frac{\alpha}{\alpha - t} & \text{if } \alpha > 0, \beta = 0, \alpha > t.
\end{cases}
\]

(5)

It should be noted that the error in ranking reduces the efficiency of the method. Extreme ranked set sampling was proposed by Samawi, et al. (1996) as a useful modification of RSS. It requires identifying the extreme units only, as opposed to all ranks as in the usual RSS. The method gives an unbiased estimate of the population mean in the case of symmetric distributions and it is more efficient than SRS.

The extreme ranked set sampling (ERSS) method can be described as follows:

Step 1: Select \( m \) random samples each of size \( m \) units from the target population.

Step 2: Rank the units within each sample with respect to a variable of interest by visual inspection or any other inexpensive method.

Step 3: For actual measurement, if the sample size \( m \) is even, from the first \( \frac{m}{2} \) sets select the lowest ranked unit of each set and from the other \( \frac{m}{2} \) sets select the largest ranked unit. If the sample size is odd, from the first \( \frac{m-1}{2} \) sets select the lowest ranked unit, from the other \( \frac{m-1}{2} \) sets select the largest ranked unit, and from the remaining set the median ranked unit is selected.

Step 4: The procedure can be repeated \( n \) times if needed to increase the sample size to \( nm \) units.

Let \( X_1, X_2, \ldots, X_m \) be a simple random sample from the probability density function \( f(x) \), with mean \( \mu \) and variance \( \sigma^2 \). Let \( X_{11}, X_{12}, \ldots, X_{1m}; X_{21}, X_{22}, \ldots, X_{2m}; \ldots; \)
\(X_{m1}, X_{m2}, \ldots, X_{mn}\) be \(m\) independent SRS each of size \(m\). Let \(X_{i(1)}, X_{i(2)}, \ldots, X_{i(m)}\) be the order statistics of the sample \(X_{i1}, X_{i2}, \ldots, X_{im}\) for \((i = 1, 2, \ldots, m)\). The pdf and cdf of the \(i^{th}\) order statistics, \(X_{i}\), respectively are

\[
f_{(i)}(x) = \frac{m!}{(i-1)!(m-i)!} [F(x)]^{i-1} [1-F(x)]^{m-i} f(x),
\]

and

\[
F_{(i)}(x) = \frac{m!}{(i-1)!(m-i)!} \int_0^{F(x)} v^{i-1} (1-v)^{m-i} dv.
\]

The mean and the variance of \(X_{(i)}\) are given by

\[
\mu_{(i)} = \int_{-\infty}^{\infty} x f_{(i)}(x) dx
\]

and

\[
\sigma_{(i)}^2 = \int_{-\infty}^{\infty} (x-\mu_{(i)})^2 f_{(i)}(x) dx,
\]

respectively (see David and Nagaraja, 2003). Takahasi and Wakimoto (1968) provided the mathematical properties of the RSS and gave the following identities

\[
f(x) = \frac{1}{m} \sum_{i=1}^{m} f_{(i)}(x), \quad \mu = \frac{1}{m} \sum_{i=1}^{m} \mu_{(i)},
\]

and

\[
\text{Var}(\hat{\mu}_{RSS}) = \frac{\sigma^2}{m} = \frac{1}{m^2} \sum_{i=1}^{m} (\mu_{(i)} - \mu)^2.
\]

They showed that the efficiency of RSS with respect to SRS is

\[
1 \leq \text{eff}(\hat{\mu}_{RSS}, \hat{\mu}_{SRS}) = \frac{\text{Var}(\hat{\mu}_{SRS})}{\text{Var}(\hat{\mu}_{RSS})} \leq \frac{m+1}{2},
\]

where \(\hat{\mu}_{SRS}\) and \(\hat{\mu}_{RSS}\) are unbiased estimators of the population mean \(\mu\) using SRS and RSS, respectively.

When \(m\) is even, the ERSS estimator of the population mean is defined as

\[
\hat{\mu}_{ERSS} = \frac{1}{rm} \sum_{j=1}^{r} \left( \sum_{i=1}^{m/2} X_{(m)i,j} + \sum_{i=m/2+1}^{m} X_{(1)i,j} \right),
\]

and when \(m\) is odd

\[
\hat{\mu}_{ERSS} = \frac{1}{rm} \sum_{j=1}^{r} \left( \sum_{i=1}^{(m-1)/2} X_{(m)i,j} + \sum_{i=(m+1)/2}^{m} X_{(1)i,j} + X_{((m+1)/2)i,j} \right),
\]

where \(X_{(k)i,j}\) denotes the \(k^{th}\) ranked from the \(i^{th}\) set at the \(j^{th}\) cycle.


Maximum Likelihood Estimation of the MWD: When \(m\) is Even

The maximum likelihood estimators (MLEs) of the three estimators \(\alpha\), \(\beta\) and \(\gamma\) when \(m\) is even are investigated based on the likelihood function \(L\) using ERSS as

\[
L = \frac{h^{\frac{\sum_{i=1}^{r} \sum_{j=1}^{p} \left\{ mf\left(x_{(m)i,j}\right) \left[ F\left(x_{(m)i,j}\right) \right]^{m-1} \right\}}{\sum_{i=1}^{r} \sum_{j=p+1}^{m} \left\{ mf\left(x_{(1)i,j}\right) \left[ 1 - F\left(x_{(1)i,j}\right) \right]^{m-1} \right\}}}}{\prod_{i=1}^{r} \prod_{j=1}^{p} \left\{ mf\left(x_{(m)i,j}\right) \left[ F\left(x_{(m)i,j}\right) \right]^{m-1} \right\} \prod_{i=1}^{r} \prod_{j=p+1}^{m} \left\{ mf\left(x_{(1)i,j}\right) \left[ 1 - F\left(x_{(1)i,j}\right) \right]^{m-1} \right\}},
\]
MLEs OF THE PARAMETERS OF A MODIFIED WEIBULL DISTRIBUTION USING ERSS

where \( p = m/2 \) and \( h \) is a constant. The variable \( X_{(k)_{i,j}} \) denotes the \( k \)th ranked unit of the \( i \)th sample at the \( j \)th cycle. The log likelihood function of (8) is

\[
L^* = C + \sum_{j=1}^{r} \sum_{i=1}^{p} \ln f(x_{(m)_{i,j}}) + (m-1) \sum_{j=1}^{r} \sum_{i=1}^{p} \ln F(x_{(m)_{i,j}}) + \sum_{j=1}^{r} \sum_{i=p+1}^{m} \ln f(x_{(1)_{i,j}}) + (m-1) \sum_{j=1}^{r} \sum_{i=p+1}^{m} \ln(1-F(x_{(1)_{i,j}})),
\]

(9)

where \( C \) is a constant. The first derivatives of \( L^* \) with respect to \( \alpha, \beta \) and \( \gamma \), respectively are

\[
\frac{\partial L^*}{\partial \alpha} = \sum_{j=1}^{r} \sum_{i=1}^{p} \left( \frac{1}{\alpha + \beta \gamma x_{(m)_{i,j}}^{\gamma-1}} - x_{(m)_{i,j}} \right) + (m-1) \sum_{j=1}^{r} \sum_{i=1}^{p} \left( \frac{x_{(m)_{i,j}} T_i}{1 - T_i} \right) - (m-1) \sum_{j=1}^{r} \sum_{i=p+1}^{m} \left( \frac{1}{\alpha + \beta \gamma x_{(1)_{i,j}}^{\gamma-1}} - x_{(1)_{i,j}} \right),
\]

(10)

\[
\frac{\partial L^*}{\partial \beta} = \sum_{j=1}^{r} \sum_{i=1}^{p} \left( \frac{\gamma x_{(m)_{i,j}}^{\gamma-1}}{\alpha + \beta \gamma x_{(m)_{i,j}}^{\gamma-1}} - x_{(m)_{i,j}}^\gamma \right) + (m-1) \sum_{j=1}^{r} \sum_{i=1}^{p} \left( \frac{x_{(m)_{i,j}} T_i}{1 - T_i} \right) - (m-1) \sum_{j=1}^{r} \sum_{i=p+1}^{m} x_{(1)_{i,j}}^\gamma + \sum_{j=1}^{r} \sum_{i=p+1}^{m} \left( \frac{\gamma x_{(1)_{i,j}}^{\gamma-1}}{\alpha + \beta \gamma x_{(1)_{i,j}}^{\gamma-1}} - x_{(1)_{i,j}}^\gamma \right),
\]

(11)

\[
\frac{\partial L^*}{\partial \gamma} = \sum_{j=1}^{r} \sum_{i=1}^{p} \left( \beta x_{(m)_{i,j}}^{\gamma-1} \left( \gamma \ln(x_{(m)_{i,j}}) + 1 \right) - \alpha \ln(x_{(m)_{i,j}}) x_{(m)_{i,j}}^\gamma \right) + \sum_{j=1}^{r} \sum_{i=p+1}^{m} \left( \beta x_{(1)_{i,j}}^{\gamma-1} \left( \gamma \ln(x_{(1)_{i,j}}) + 1 \right) - \alpha \ln(x_{(1)_{i,j}}) x_{(1)_{i,j}}^\gamma \right) + (m-1) \sum_{j=1}^{r} \sum_{i=1}^{p} \beta \ln(x_{(m)_{i,j}}) x_{(m)_{i,j}}^\gamma T_i - (m-1) \sum_{j=1}^{r} \sum_{i=p+1}^{m} \beta \ln(x_{(1)_{i,j}}) x_{(1)_{i,j}}^\gamma T_i,
\]

(12)

where

\[
T_1 = \exp(-\alpha x_{(m)_{i,j}} - \beta x_{(m)_{i,j}}^{\gamma-1}) \quad \text{and} \quad T_2 = \exp(-\alpha x_{(1)_{i,j}} - \beta x_{(1)_{i,j}}^{\gamma-1}).
\]

The MLE of the parameters \( \alpha, \beta, \) and \( \gamma \) are the solution of equations (10), (11) and (12), respectively, when set them to zero. However, the solutions are not in closed forms, in order to obtain estimates for the parameters, the three equations may be solved numerically.

Fisher information (FI) numbers describe the amount of information that a sample provides about the parameters. The FI is defined as

\[
I = -E \left( \frac{\partial^2 \log(L)}{\partial \theta^2} \right),
\]

where \( \theta \) is a parameter. The FI number from ERSS for estimating \( \alpha, \beta, \) and \( \gamma \) can be expressed as in equations, (13), (14) and (15), respectively as
\[ I_{\text{ERSS}}(\alpha) = -E\left\{ \sum_{j=1}^{r} \sum_{i=1}^{p} \left[ \frac{-1}{(\alpha + \beta \gamma x_{(m,i),j}^{(1)})} \right] \right\} + (m-1) \sum_{j=1}^{r} \sum_{i=1}^{p} \left[ \frac{-1}{1 - T_{i}} \right] \]

\[ I_{\text{ERSS}}(\beta) = -E\left\{ \sum_{j=1}^{r} \sum_{i=1}^{p} \left[ \frac{\gamma x_{(m,i),j}^{(1)}}{\alpha + \beta \gamma x_{(m,i),j}^{(1)}} \right] \right\} \]

(13)

\[ I_{\text{ERSS}}(\beta) = \]

\[ -E\left\{ \sum_{j=1}^{r} \sum_{i=1}^{p} \left[ \frac{\gamma x_{(m,i),j}^{(1)}}{\alpha + \beta \gamma x_{(m,i),j}^{(1)}} \right] \right\} \]

\[-\sum_{j=1}^{r} \sum_{i=p+1}^{m} \left[ \frac{\gamma x_{(m,i),j}^{(1)}}{\alpha + \beta \gamma x_{(m,i),j}^{(1)}} \right] \]

\[ - (m-1) \sum_{j=1}^{r} \sum_{i=1}^{p} \left[ \frac{x_{(m,i),j}^{(1)} T_{i}}{1 - T_{i}} + \left( \frac{x_{(m,i),j}^{(1)}}{1 - T_{i}} \right)^{2} \right] \}

(14)

and

\[ I_{\text{ERSS}}(\beta) = \]

\[ -E\left\{ \sum_{j=1}^{r} \sum_{i=1}^{p} \left[ \frac{\partial^{2}}{\partial \gamma^{2}} \log(f(x_{(m,i),j})) \right] \right\} \]

\[ + (m-1) \frac{\partial^{2}}{\partial \gamma^{2}} \log(F(x_{(m,i),j})) \]

\[ - \sum_{j=1}^{r} \sum_{i=1}^{p} \left[ \frac{\partial^{2}}{\partial \gamma^{2}} \log(f(x_{(i),j})) \right] \]

\[ + (m-1) \frac{\partial^{2}}{\partial \gamma^{2}} \log[1 - F(x_{(i),j})] \}

(15)

where

\[ \frac{\partial^{2}}{\partial \gamma^{2}} \log[f(x_{(i),j})] = \]

\[ \frac{\beta x_{(m,i),j}^{(1)} \ln(x_{(m,i),j})}{\alpha + \beta \gamma x_{(m,i),j}^{(1)}} \]

\[ \gamma \ln(x_{(m,i),j}) + 2 \]

\[- \left\{ \frac{\beta x_{(m,i),j}^{(1)} \ln(x_{(m,i),j}) + 1}{\alpha + \beta \gamma x_{(m,i),j}^{(1)}} \right\} \]

\[- \beta x_{(m,i),j}^{(1)} \ln^{2}(x_{(m,i),j}) \]

\[ \frac{\beta T_{i} x_{(m,i),j}^{(1)} \ln^{2}(x_{(m,i),j})(1 - \beta x_{(m,i),j}^{(1)})}{1 - T_{i}} \]

\[ - \left[ \frac{\beta \ln(x_{(m,i),j}) x_{(m,i),j}^{(1)} T_{i}}{1 - T_{i}} \right]^{2} \]

and

\[ \frac{\partial^{2}}{\partial \gamma^{2}} \log[1 - F(x_{(i),j})] = -\beta x_{(i),j}^{(1)} \ln^{2}(x_{(i),j}) \].

Maximum Likelihood Estimation of the MWD:

When \( m \) is Odd

Based on ERSS, when \( m \) is odd, the likelihood function is

\[ L = K \prod_{i=1}^{q} \left\{ \prod_{j=1}^{r} \prod_{j=q+1}^{m-1} \left[ \frac{mf(x_{(m,i),j}) F(x_{(m,i),j})^{m-1}}{m} \right] \right\} \]

\[ f(x_{(m+1)/2, j}) \left[ F(x_{(m+1)/2, j}) \right]^{m-1} \]

(16)

where \( q = (m-1)/2 \) and \( K \) is a constant. The log likelihood function of (16) is
\[ L' = \ln(L) \]
\[ = K' + \sum_{j=1}^{r} \sum_{i=1}^{q} \ln f(x_{(m)i,j}) \]
\[ + (m-1) \sum_{j=1}^{r} \sum_{i=1}^{q} \ln \left[ 1 - F(x_{(m)i,j}) \right] \]
\[ + \sum_{j=1}^{r} \sum_{i=q+1}^{m-1} \ln f(x_{(m)i,j}) \]
\[ + (m-1) \sum_{j=1}^{r} \sum_{i=q+1}^{m-1} \ln \left[ 1 - F(x_{(m)i,j}) \right] \]
\[ + \log f(x_{(m+1)i,j}) \]
\[ + \frac{m-1}{2} \left[ \log F(x_{(m+1)i,j}) + \log \left[ 1 - F(x_{(m+1)i,j}) \right] \right]. \]

(17)

Taking the first derivative of \( L' \) in (17) with respect to \( \alpha, \beta \) and \( \gamma \) results in

\[ \frac{\partial L'}{\partial \alpha} = \sum_{j=1}^{r} \sum_{i=1}^{q} \left( \frac{1}{\alpha + \beta \gamma x_{(m)i,j}^{-1}} - x_{(m)i,j} \right) \]
\[ + \sum_{j=1}^{r} \sum_{i=q+1}^{m-1} \left( \frac{1}{\alpha + \beta \gamma x_{(m)i,j}^{-1}} - x_{(m)i,j} \right) \]
\[ + (m-1) \sum_{j=1}^{r} \sum_{i=1}^{q} \frac{x_{(m)i,j}}{1 - T_1} \]
\[ - (m-1) \sum_{j=1}^{r} \sum_{i=q+1}^{m-1} x_{(m)i,j} \]
\[ + \sum_{j=1}^{r} \left( \frac{1}{\alpha + \beta \gamma x_{(m+1)i,j}^{-1}} - x_{(m+1)i,j} \right) \]
\[ + \frac{m-1}{2} \sum_{j=1}^{r} \left( \frac{x_{(m+1)i,j} T_3}{1 - T_3} - x_{(m+1)i,j} \right). \]

(18)

\[ \frac{\partial L'}{\partial \beta} = \sum_{j=1}^{r} \sum_{i=1}^{q} \left( \frac{\gamma x_{(m)i,j}^{-1}}{\alpha + \beta \gamma x_{(m)i,j}^{-1}} - x_{(m)i,j} \right) \]
\[ + \sum_{j=1}^{r} \sum_{i=q+1}^{m-1} \left( \frac{\gamma x_{(m)i,j}^{-1}}{\alpha + \beta \gamma x_{(m)i,j}^{-1}} - x_{(m)i,j} \right) \]
\[ + (m-1) \sum_{j=1}^{r} \sum_{i=1}^{q} \frac{x_{(m)i,j} T_1}{1 - T_1} \]
\[ + (m-1) \sum_{j=1}^{r} \sum_{i=q+1}^{m-1} x_{(m)i,j} \]
\[ + \frac{m-1}{2} \sum_{j=1}^{r} \left( \frac{x_{(m+1)i,j} T_3}{1 - T_3} - x_{(m+1)i,j} \right). \]

(19)

and

\[ \frac{\partial L'}{\partial \gamma} = \sum_{j=1}^{r} \sum_{i=1}^{q} \left( \frac{\beta x_{(m)i,j}^{-1}}{\alpha + \beta \gamma x_{(m)i,j}^{-1}} - x_{(m)i,j} \right) \]
\[ + \sum_{j=1}^{r} \sum_{i=q+1}^{m-1} \left( \frac{\beta x_{(m)i,j}^{-1}}{\alpha + \beta \gamma x_{(m)i,j}^{-1}} - x_{(m)i,j} \right) \]
\[ + (m-1) \sum_{j=1}^{r} \sum_{i=1}^{q} \frac{x_{(m)i,j} T_1}{1 - T_1} \]
\[ - (m-1) \sum_{j=1}^{r} \sum_{i=q+1}^{m-1} x_{(m)i,j} \]
\[ + \frac{m-1}{2} \sum_{j=1}^{r} \left( \frac{x_{(m+1)i,j} T_3}{1 - T_3} - x_{(m+1)i,j} \right). \]
respectively, where
\[ T_1 = \exp\left( -\alpha x_{(m)i,j} - \beta x_{(m)i,j}^{-1} \right), \]
\[ T_2 = \exp\left( -\alpha x_{(1)i,j} - \beta x_{(1)i,j}^T \right) \]
and
\[ T_3 = \exp\left( -\alpha x_{((m+1)/2)i,j} - \beta x_{((m+1)/2)i,j}^T \right). \]

The Fisher Information number of \( \alpha, \beta \) and \( \gamma \) from the samples, respectively are
\[
I_{E_{RSS}}(\alpha) = -E \left\{ \sum_{j=1}^{r} \sum_{i=1}^{q} \left[ \frac{-1}{\alpha + \beta \gamma x_{(m)i,j}^{-1}} \right]^2 \right. \\
- (m-1) \sum_{j=1}^{r} \sum_{i=1}^{q} \left[ \frac{x_{(m)i,j} T_1 + \left( x_{(m)i,j} T_1 \right)^2}{1-T_1} \right]^2 \\
+ \sum_{j=1}^{r} \sum_{i=q+1}^{m-1} \left[ \frac{-1}{\alpha + \beta \gamma x_{(1)i,j}^{-1}} \right]^2 \\
+ \sum_{j=1}^{r} \left[ \frac{-1}{\alpha + \beta \gamma x_{((m+1)/2)i,j}^{-1}} \right]^2 \\
- m-1 \sum_{j=1}^{r} \left[ \frac{x_{((m+1)/2)i,j} T_3 + \left( x_{((m+1)/2)i,j} T_3 \right)^2}{1-T_3} \right] \right\}, \tag{21}
\]
\[
I_{E_{RSS}}(\beta) = -E \left\{ \sum_{j=1}^{r} \sum_{i=1}^{q} \left[ \frac{- \gamma x_{(m)i,j}}{\alpha + \beta \gamma x_{(m)i,j}^{-1}} \right]^2 \right. \\
+ (m-1) \sum_{j=1}^{r} \sum_{i=q+1}^{m-1} \left[ \frac{- \gamma x_{(1)i,j}}{\alpha + \beta \gamma x_{(1)i,j}^{-1}} \right]^2 \\
- (m-1) \sum_{j=1}^{r} \sum_{i=1}^{q} \left[ \frac{x_{(m)i,j}^2 T_1 + \left( x_{(m)i,j}^2 T_1 \right)^2}{1-T_1} \right] \\
+ \sum_{j=1}^{r} \sum_{i=q+1}^{m-1} \left[ \frac{- \gamma x_{((m+1)/2)i,j}}{\alpha + \beta \gamma x_{((m+1)/2)i,j}^{-1}} \right]^2 \\
- m-1 \sum_{j=1}^{r} \left[ \frac{x_{((m+1)/2)i,j}^2 T_3 + \left( x_{((m+1)/2)i,j}^2 T_3 \right)^2}{1-T_3} \right] \right\}, \tag{22}
\]
\[
I_{E_{RSS}}(\gamma) = \\
- E \left\{ \sum_{j=1}^{r} \sum_{i=1}^{q} \frac{\partial^2}{\partial \gamma^2} \log\left[ f\left(x_{(m)i,j}\right)\right] \\
+(m-1) \frac{\partial^2}{\partial \gamma^2} \log\left[ F\left(x_{(m)i,j}\right)\right] \\
\right\} + \sum_{j=1}^{r} \sum_{i=1}^{q} \left[ \frac{\partial^2}{\partial \gamma^2} \log\left[ f\left(x_{(1)i,j}\right)\right] \\
+(m-1) \frac{\partial^2}{\partial \gamma^2} \log\left[ 1 - F\left(x_{(1)i,j}\right)\right] \right\} + \sum_{j=1}^{r} \left[ \frac{\partial^2}{\partial \gamma^2} \log\left[ f\left(x_{((m+1)/2)i,j}\right)\right] \\
+ \frac{m-1}{2} \sum_{j=1}^{r} \frac{\partial^2}{\partial \gamma^2} \log\left[ F\left(x_{((m+1)/2)i,j}\right)\right] \\
+ \frac{\partial^2}{\partial \gamma^2} \log\left[ 1 - F\left(x_{((m+1)/2)i,j}\right)\right] \right\}, \tag{23}
\]
where
MLEs OF THE PARAMETERS OF A MODIFIED WEIBULL DISTRIBUTION USING ERSS

\[ \frac{\partial^2}{\partial \gamma^2} \log \left[ f \left( x_{(i,j)} \right) \right] = \]
\[ \left[ \beta \alpha x_{(i,j)}^{\gamma-1} \ln \left( x_{(i,j)} \right) \right] \gamma \ln \left( x_{(i,j)} \right) + 2 \]
\[ \alpha + \beta \gamma x_{(i,j)}^{\gamma-1} \]
\[ \left[ \beta \alpha x_{(i,j)}^{\gamma-1} \left( \gamma \ln \left( x_{(i,j)} \right) + 1 \right) \right] - \beta x_{i,j}^{\gamma} \ln^2 \left( x_{(i,j)} \right) , \]

\[ \frac{\partial^2}{\partial \gamma^2} \log \left[ F \left( x_{(i,j)} \right) \right] = \]
\[ \beta T_2 \left( x_{(i,j)}^{\gamma} \right) \ln^2 \left( x_{(i,j)} \right) \left( 1 - \beta x_{i,j}^{\gamma} \right) \]
\[ 1 - T_2 \]
\[ - \left[ \beta \ln \left( x_{(i,j)} \right) x_{i,j}^{\gamma} T_1 \right] ^2 , \]

and

\[ \frac{\partial^2}{\partial \gamma^2} \left[ 1 - F \left( x_{(i,j)} \right) \right] = - \beta x_{i,j}^{\gamma} \ln^2 \left( x_{(i,j)} \right) , \]

where

\[ T_1 = \exp \left( -\alpha x_{(m)i,j} - \beta x_{(m)i,j}^{\gamma-1} \right) , \]
\[ T_2 = \exp \left( -\alpha x_{(i)i,j} - \beta x_{(i)i,j}^{\gamma} \right) , \]
and

\[ T_3 = \exp \left( -\alpha x_{(m+1/2)i,j} - \beta x_{(m+1/2)i,j}^{\gamma} \right) . \]

Methodology

Simulation Study

To investigate the properties of the MLEs of the three parameters of the MWD a simulation was conducted. The inverse transform method was used to generate samples from MWD (see Ros, 1997). The inverse transform algorithm can be described as: generate $U$ from the uniform $(0, 1)$, initiate $X_1$, and then find a new $X_1$ using

\[ X_1 = -\frac{\beta}{\alpha} X_1^{\gamma} - \frac{1}{\alpha} \ln \left( 1 - U \right) ; \]

repeat until stability of $X_1$ is reached, which eventually

represents a random number from MWD. The samples generated are then used to obtain the Fisher Information numbers, $I_{ERSS}$ and $I_{SRS}$, when using ERSS and SRS. The asymptotic relative efficiency (RP) is found as the ratio $I_{ERSS} / I_{SRS}$.

Results

For $\alpha = 3$, $\beta = 1.2$, and $\gamma = 1.3$, the results are presented in Tables 1, 2 and 3, respectively.

Table 1: Information Numbers and Asymptotic RP of the MLE of $\alpha$ Based on ERSS with respect to SRS

<table>
<thead>
<tr>
<th>$m$</th>
<th>$I_{ERSS}$</th>
<th>$I_{SRS}$</th>
<th>Asymptotic RP</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.3613</td>
<td>0.1854</td>
<td>1.9490</td>
</tr>
<tr>
<td>4</td>
<td>0.6069</td>
<td>0.2392</td>
<td>2.5372</td>
</tr>
<tr>
<td>5</td>
<td>0.7933</td>
<td>0.2849</td>
<td>2.7845</td>
</tr>
<tr>
<td>6</td>
<td>0.9818</td>
<td>0.3478</td>
<td>2.8229</td>
</tr>
<tr>
<td>7</td>
<td>1.3030</td>
<td>0.4057</td>
<td>3.2119</td>
</tr>
</tbody>
</table>

Table (2): Information Numbers and Asymptotic RP of the MLE of $\beta$ Based on ERSS with respect to SRS

<table>
<thead>
<tr>
<th>$m$</th>
<th>$I_{ERSS}$</th>
<th>$I_{SRS}$</th>
<th>Asymptotic RP</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.1401</td>
<td>0.0542</td>
<td>2.5849</td>
</tr>
<tr>
<td>4</td>
<td>0.2894</td>
<td>0.1014</td>
<td>2.8554</td>
</tr>
<tr>
<td>5</td>
<td>0.4627</td>
<td>0.1551</td>
<td>2.9832</td>
</tr>
<tr>
<td>6</td>
<td>0.6335</td>
<td>0.1956</td>
<td>3.2382</td>
</tr>
<tr>
<td>7</td>
<td>0.8606</td>
<td>0.2314</td>
<td>3.7191</td>
</tr>
</tbody>
</table>

Table (3): Information Numbers and Asymptotic RP of the MLE of $\gamma$ based on ERSS with respect to SRS

<table>
<thead>
<tr>
<th>$m$</th>
<th>$I_{ERSS}$</th>
<th>$I_{SRS}$</th>
<th>Asymptotic RP</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.6451</td>
<td>0.5348</td>
<td>1.2062</td>
</tr>
<tr>
<td>4</td>
<td>1.1279</td>
<td>0.8684</td>
<td>1.2987</td>
</tr>
<tr>
<td>5</td>
<td>1.2494</td>
<td>0.7336</td>
<td>1.7032</td>
</tr>
<tr>
<td>6</td>
<td>1.7856</td>
<td>0.9847</td>
<td>1.8133</td>
</tr>
<tr>
<td>7</td>
<td>1.9196</td>
<td>0.8459</td>
<td>2.2693</td>
</tr>
</tbody>
</table>

614
For $\alpha = 2.3$, $\beta = 1.3$ and $\gamma = 1.6$, results are summarized in Tables 4, 5 and 6 respectively.

Table (4): Information Numbers and Asymptotic RP of the MLE of $\alpha$ Based on ERSS with respect to SRS

<table>
<thead>
<tr>
<th>$m$</th>
<th>$I_{ERSS}$</th>
<th>$I_{SRS}$</th>
<th>Asymptotic RP</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.2523</td>
<td>0.1201</td>
<td>2.1010</td>
</tr>
<tr>
<td>4</td>
<td>0.3739</td>
<td>0.1563</td>
<td>2.3922</td>
</tr>
<tr>
<td>5</td>
<td>0.5461</td>
<td>0.1987</td>
<td>2.7483</td>
</tr>
<tr>
<td>6</td>
<td>0.6521</td>
<td>0.2254</td>
<td>2.8931</td>
</tr>
<tr>
<td>7</td>
<td>0.8796</td>
<td>0.2695</td>
<td>3.2632</td>
</tr>
</tbody>
</table>

Table (5): Information Numbers and Asymptotic RP of the MLE of $\beta$ Based on ERSS with respect to SRS

<table>
<thead>
<tr>
<th>$m$</th>
<th>$I_{ERSS}$</th>
<th>$I_{SRS}$</th>
<th>Asymptotic RP</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.1195</td>
<td>0.0481</td>
<td>2.4871</td>
</tr>
<tr>
<td>4</td>
<td>0.1681</td>
<td>0.0595</td>
<td>2.8263</td>
</tr>
<tr>
<td>5</td>
<td>0.2383</td>
<td>0.0766</td>
<td>3.1110</td>
</tr>
<tr>
<td>6</td>
<td>0.2913</td>
<td>0.0814</td>
<td>3.5787</td>
</tr>
<tr>
<td>7</td>
<td>0.3852</td>
<td>0.0919</td>
<td>4.1902</td>
</tr>
</tbody>
</table>

Table (6): Information Numbers and Asymptotic RP of the MLE of $\gamma$ Based on ERSS with respect to SRS

<table>
<thead>
<tr>
<th>$m$</th>
<th>$I_{ERSS}$</th>
<th>$I_{SRS}$</th>
<th>Asymptotic RP</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.2459</td>
<td>0.7885</td>
<td>1.5801</td>
</tr>
<tr>
<td>4</td>
<td>1.9567</td>
<td>1.0232</td>
<td>1.9123</td>
</tr>
<tr>
<td>5</td>
<td>2.9202</td>
<td>1.3510</td>
<td>2.1615</td>
</tr>
<tr>
<td>6</td>
<td>3.8283</td>
<td>1.5697</td>
<td>2.4388</td>
</tr>
<tr>
<td>7</td>
<td>5.0158</td>
<td>1.7128</td>
<td>2.9284</td>
</tr>
</tbody>
</table>

Simulation results are summarized in Tables 7-9 for some values of the population parameters.

From results shows in Tables 7-9, it may be concluded that the ERSS estimators are biased and more efficient than the SRS estimator for all cases considered in this study. However, as demonstrated by Samawi, et al. (1996) it is better to use ERSS with small sample size. Also note that the efficiency of the mean estimation depends on the values of $\alpha, \beta, \gamma$, as well as the sample size.

Tables 1-3 show that:

- The ERSS estimators dominate the estimators based on SRS.
- The information numbers from ERSS are greater than those of SRS.

Estimation of the Population Mean of the MWD

The problem of estimating the population mean of the MWD is now considered and compared with the SRS estimator of the population mean $\hat{\mu}_{SRS} = \frac{\sum_{i=1}^{m} x_i}{m}$, which has variance $\sigma^2 / m$. The efficiency of $\hat{\mu}_{ERSS1}$ and $\hat{\mu}_{ERSS2}$ respectively with respect to $\hat{\mu}_{SRS}$ are defined as

$$\text{eff} (\hat{\mu}_{ERSSi}, \hat{\mu}_{SRS}) = \frac{\text{MSE}(\hat{\mu}_{SRS})}{\text{MSE}(\hat{\mu}_{ERSSi})}, \ i = 1,2.$$

Conclusion

Maximum likelihood estimators for the three parameters of the modified Weibull distribution were studied based on extreme ranked set sampling. These MLEs are not in closed forms, so numerical method is used. Results show that the Fisher information numbers obtained from ERSS are greater than that from SRS. Also, it was shown that ERSS is more efficient than SRS in estimating the population mean and it has a small bias. However, the ERSS estimators dominate the corresponding estimators based on SRS for estimating the population mean of the MWD.
MLEs OF THE PARAMETERS OF A MODIFIED WEIBULL DISTRIBUTION USING ERSS

Table 7: Efficiency and Bias Values of Estimating the Population Mean of the MWD Using ERSS with respect to SRS for $\alpha = 2$, $\beta = 1.2$ and $\gamma = 1.3$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\text{Bias(ERSS)}$</th>
<th>$\text{MSE(SRS)}$</th>
<th>$\text{MSE(ERSS)}$</th>
<th>Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.0015</td>
<td>0.0173</td>
<td>0.0097</td>
<td>1.7835</td>
</tr>
<tr>
<td>4</td>
<td>0.0284</td>
<td>0.0140</td>
<td>0.0075</td>
<td>1.8667</td>
</tr>
<tr>
<td>5</td>
<td>0.0256</td>
<td>0.0099</td>
<td>0.0053</td>
<td>1.8679</td>
</tr>
<tr>
<td>6</td>
<td>0.0544</td>
<td>0.0088</td>
<td>0.0064</td>
<td>1.3750</td>
</tr>
<tr>
<td>7</td>
<td>0.0486</td>
<td>0.0079</td>
<td>0.0049</td>
<td>1.6122</td>
</tr>
</tbody>
</table>

Table 8: Efficiency and Bias Values of Estimating the Population Mean of the MWD Using ERSS with respect to SRS for $\alpha = 4$, $\beta = 2$ and $\gamma = 3$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\text{Bias(ERSS)}$</th>
<th>$\text{MSE(SRS)}$</th>
<th>$\text{MSE(ERSS)}$</th>
<th>Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.0028</td>
<td>0.0088</td>
<td>0.0047</td>
<td>1.8723</td>
</tr>
<tr>
<td>4</td>
<td>0.0165</td>
<td>0.0071</td>
<td>0.0038</td>
<td>1.8684</td>
</tr>
<tr>
<td>5</td>
<td>0.0064</td>
<td>0.0051</td>
<td>0.0020</td>
<td>2.5500</td>
</tr>
<tr>
<td>6</td>
<td>0.0362</td>
<td>0.0047</td>
<td>0.0032</td>
<td>1.4688</td>
</tr>
<tr>
<td>7</td>
<td>0.0221</td>
<td>0.0041</td>
<td>0.0016</td>
<td>2.5625</td>
</tr>
</tbody>
</table>

Table 9: Efficiency and Bias Values of Estimating the Population Mean of the MWD Using ERSS with respect to SRS for $\alpha = 3.5$, $\beta = 2$ and $\gamma = 1.5$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\text{Bias(ERSS)}$</th>
<th>$\text{MSE(SRS)}$</th>
<th>$\text{MSE(ERSS)}$</th>
<th>Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.0285</td>
<td>0.0067</td>
<td>0.0036</td>
<td>1.8611</td>
</tr>
<tr>
<td>4</td>
<td>0.0029</td>
<td>0.0076</td>
<td>0.0041</td>
<td>1.8537</td>
</tr>
<tr>
<td>5</td>
<td>0.0071</td>
<td>0.0040</td>
<td>0.0014</td>
<td>2.8570</td>
</tr>
<tr>
<td>6</td>
<td>0.0122</td>
<td>0.0055</td>
<td>0.0027</td>
<td>1.8519</td>
</tr>
<tr>
<td>7</td>
<td>0.0025</td>
<td>0.0029</td>
<td>0.0008</td>
<td>3.6250</td>
</tr>
</tbody>
</table>

References


