1-1-2011

Moduli spaces and cw structures arising from morse theory

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MODULI SPACES AND CW STRUCTURES ARISING FROM MORSE THEORY

by

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DISSERTATION

Submitted to the Graduate School

of Wayne State University

Detroit, Michigan

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

2011

MAJOR: MATHEMATICS

Approved by:

Advisor Date

DEDICATION

This dissertation is dedicated to my family.

ACKNOWLEDGEMENTS

I am very grateful to my Ph.D. advisor Professor John Klein. He has given me direction, patient instruction and constant encouragement during my progress on completing this Ph.D. program. He has also made a great effort in the development of my career. Moreover, his sense of humor made our discussions enjoyable.

I wish to thank Professors Dan Burghelea, Ralph Cohen, Octav Cornea, Paul Kirk and John Milnor, who have supported me in finishing this Ph.D. program. I wish to thank Professor Bruce Williams for his help in starting my career.

I wish to thank Professors Robert Bruner, Vladimir Chernyak, Daniel Frohardt, Peter Malcolmson, Claude Schochet and Peiyong Wang, from whom I have learned a great deal of mathematics. I wish to express my gratitude to Professor Zhimin Zhang for his help in starting my Ph.D. program. I wish to thank Ms. Mary Klamo for her support.

I wish to thank the topology group of graduate students, especially Matthew Buckman, John Peter and Sean Tilson, for valuable mathematics discussions.

I wish to thank my friends in the mathematics department, especially Zhuo Jin, James Veneri, Changhong Wen and Huiqing Zhu, who have helped me so much.

Most importantly, I am deeply indebted to my family. The warmth of my family always gives me power to make progress. This dissertation is dedicated to them.

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1 Introduction

Invented in the 1920s (see [40] and [41]), Morse theory has been a crucial tool in the study of smooth manifolds. In the past two decades, largely due to the influence of Floer, there has been a resurgence in activity in Morse theory in its geometrical and dynamical aspects, especially in infinite dimensional situations. An explosion of new ideas produced many *folklore theorems* which were apparently widely acknowledged, highly anticipated or even frequently used. Unfortunately, the literature has not kept pace with the folklore. Some previously asserted results are still stated without proof and, having asked various experts in the field, the author could not ascertain what is sufficiently proved or what is even regarded as true. The purpose of this dissertation is to give a self-contained and detailed treatment proving some of these claims.

In order to develop his homology, Floer invented two techniques in Morse theory (see e.g. [25] and [26]). One is the compactification of the moduli spaces of negative gradient flow lines. The other one is the gluing of broken flow lines. These two arguments have continuously impacted on Morse theory since then.

In this dissertation, we shall study the manifold structures of compactified moduli spaces, the orientation of compactified moduli spaces, the CW structure resulting from the negative gradient vector fields and its relation with moduli spaces, and the associativity of gluing of moduli spaces.

In the simplest instance, suppose one is given a Morse function on a finite dimensional closed smooth manifold. By choosing a Riemannian metric, one obtains a negative gradient flow. This determines a stratification in which two points lie in the same stratum if they lie on the same unstable manifold. Now each such unstable manifold (or descending manifold) is homeomorphic to an open cell, and it is desirable to know whether this open cell can be compactified in such a way that it becomes the image of a closed cell arising from a CW structure on the manifold. This is one of the problems we will be addressing. Another related problem is to consider moduli spaces of flow lines between any pair of critical points. Using piecewise flow lines, one obtains a compactification of these moduli spaces. The question in this case to decide when one obtains a manifold with corner structure from this compactification.

In addition to the finite dimensional case, our results will generalize in two ways. Firstly, there will be an infinite dimensional version in which the underlying manifold is a complete Hilbert manifold and the Morse function satisfies Condition (C) and has finite index at each critical point. This situation will be called the *CF case*. Secondly, we will also strengthen some results in the finite dimensional case. For example, we will obtain a certain result about simple homotopy type in Theorem 6.2.

The following is a brief description of our main results.

Chapter 3 studies the compactness. Roughly speaking, compactness means the space of unbroken flow lines can be compactified by adding broken flow lines. When the underlying manifold M is finite dimensional, similar results are well-known, for example, [58, thm. 2.3, p. 798], [11, prop. 3] and [55, prop. 2.35]. For the infinite dimensional Floer case, there are results in [25], [26] and [54]. Even in the finite dimensional case, some assumptions on M (e.g. compactness or Condition (c)) are needed in order to prove such results.

Chapters 4 and 8 deal with certain compactified spaces. Some spaces arise naturally from the study of negative gradient dynamical systems. Let $\mathcal{D}(p)$ and $\mathcal{A}(p)$ be the descending (unstable) and ascending (stable) manifolds of a critical point p respectively. We say the dynamical system satisfies transversality if each $\mathcal{D}(p)$ is transverse to each $\mathcal{A}(q)$. Assuming transversality of the dynamical system, let $\mathcal{W}(p,q)$ be the intersection manifold of $\mathcal{D}(p)$ and $\mathcal{A}(q)$. Since points can travel along flows, there is a flow action (or R-action) on $\mathcal{W}(p,q)$. Let $\mathcal{M}(p,q)$ be the orbit space of $\mathcal{W}(p,q)$ with respect to the action of the flow. It's well-known that these manifolds can be compactified in a standard way (see e.g. [17], [18], [35] and [11]). These two chapters consider the manifold structures of the compactified spaces of $\mathcal{M}(p,q)$, $\mathcal{D}(p)$ and $\mathcal{W}(p,q)$. Denote the compactified spaces by $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$. A central problem is to equip them with smooth structures in such a way that they are manifolds with corners that are compatible with the given stratifications.

The space $\overline{\mathcal{M}(p,q)}$ has extensive applications in geometry, topology and dynamical systems such as finding closed orbits, thickening of CW complexes, examples of Poincaré duality spaces and geometric realizations of the Floer complex (see e.g. [28], [35], [11], [4], [5], [19]-[23], [15], [16], [17], and [18]). The smooth structure of $\overline{\mathcal{M}(p,q)}$ is important for the geometric constructions in those applications. The smooth structure of $\overline{\mathcal{D}(p)}$ is useful for Witten Deformations, for example, see [35] and [11]. The papers [12] and [36] (see also [63, sec. 6.4]) use the "smooth structure" of $e(\overline{\mathcal{D}(p)})$, where e is the evaluation map defined in (3) of Theorem 4.5. The smooth structure of $\overline{\mathcal{W}(p,q)}$ is useful for computing the cup product of $H^*(M;R)$ via Morse Theory (see [3, sec. 2.4] and [60]). To the best of my knowledge, when M is finite dimensional, and the metric is locally trivial (we shall explain this later), the cases of $\overline{\mathcal{M}(p,q)}$ and $\overline{\mathcal{D}(p)}$ are solved by [35] and [11]. (Actually, these two papers consider closed 1-forms which are more general than Morse functions.) However, this problem still remains open in the general case, in particular, when the metric is nontrivial near the critical points.

This problem is closely related to the associativity of gluing of broken flow lines which is also a well-known open problem. (We shall study the associativity of gluing in Chapter 11.) In addition, few papers in the literature study $\overline{\mathcal{W}(p,q)}$.

Chapters 5 and 9 discusses the orientations of $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$. Since the descending manifolds $\mathcal{D}(p)$ are finite dimensional, we can assign orientations to them arbitrarily. This determines naturally the orientations of $\mathcal{M}(p,q)$, $\mathcal{W}(p,q)$ and the compactified manifolds $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$. The codimension 1 faces of these compactified manifolds have two types of orientations, boundary orientations and product orientations. The orientation formulas Theorem 5.1 and 9.7 show the relation between these two. Actually, they are folklore theorems. Some claims on the finite dimensional case can be found, for example, in [3] and [35].

These orientation formulas have some applications. As pointed out in [3, prop. 2.8], the formula for $\overline{\mathcal{M}(p,q)}$ gives an immediate proof of $\partial^2 = 0$ for the Thom-Smale complex in Morse homology. Actually, the formula for $\overline{\mathcal{M}(p,q)}$ is a strong extension of the notion of *coherent orientations* which is crucial to all chain complexes defined by Morse theory such as the Floer homology (see [27] and [55, sec. 3.2]). The formula for $\overline{\mathcal{D}(p)}$ tells us how to apply Stokes' theorem correctly when a differential form is integrated on $\overline{\mathcal{D}(p)}$ (compare [36, prop. 6]). In this dissertation, it also straightforwardly describes the boundary operator of the cellular chain complex associated with the CW decomposition of the underlying manifold. As mentioned above, the papers [3] and [60] compute the cup product of $H^*(M;R)$ via Morse Theory. Both [3, (2.2)] and [60, lem. 2 and 3] neglect signs. If we do care about the signs in their formulas, the formula for $\overline{\mathcal{W}(p,q)}$ can tell us the answer.

Chapters 6 and 10 consider the problem of constructing a CW structure from the descending manifolds of the Morse function. Suppose a Morse function f on M is bounded below. It's well known that the descending manifolds $\mathcal{D}(p)$ for the critical points p are disjoint. Let $K^a = \bigsqcup_{f(p)\leq a} \mathcal{D}(p)$. A natural question is whether or not K^a is a CW complex with open cells $\mathcal{D}(p)$. This has been considered by Thom ([59]), Bott ([8, p. 104]) and Smale ([57, p. 197]). If the answer is positive, then Morse theory will give a compact manifold a bona fide CW decomposition. It's necessary to point out that Milnor's book [37, thm. 3.5] gives a CW complex resulting from a Morse function. But that CW complex is only homotopy equivalent to M. This bona fide CW decomposition is homeomorphic to M. Thus it is stronger.

In order to give a positive answer to the above question, we have to construct a characteristic map $e : D \longrightarrow M$ such that e maps the interior D° homeomorphically onto $\mathcal{D}(p)$ for each p, where D is a closed disk. This has been solved by [34, thm. 1] and [36, rem. 3] when M is finite dimensional and the metric is locally trivial. In this dissertation, these results will be further improved as follows.

In fact, the papers [34] and [36] show that there exists such a characteristic map. Theorem 4.5 shows that, even in the infinite dimensional CF case, $\mathcal{D}(p)$ can be compactified to be $\overline{\mathcal{D}(p)}$ and there is the map $e : \overline{\mathcal{D}(p)} \longrightarrow M$ which is explicitly constructed. If $\overline{\mathcal{D}(p)}$ is homeomorphic to a closed disk (this is Theorem 6.1), then K^a is a CW complex, and what's more, the characteristic maps $e : \overline{\mathcal{D}(p)} \longrightarrow M$ are explicit. In order to get an elementary proof of Theorem 6.1, I asked Prof. John Milnor for help. (Actually, there is a quick but non-elementary proof based on the Poincaré Conjecture in all dimensions. It is given in Section 10.2.) I had not known the existence of characteristic maps had been proved by [34] and [36] at that time. Prof. Milnor helped me greatly. First, he referred me to [34]. Second, he suggested that we may add a vector field to $-\nabla f$ on $\mathcal{D}(p)$ to control the limit behavior of $-\nabla f$. Motivated by his suggestion and [34], I found the desired proof. In particular, the key Lemma 6.10 fulfills his suggestion.

In addition, Theorem 6.1 and Lemma 6.10 help us prove more results. Let $M^a = f^{-1}((-\infty, a])$, the paper [33, cor., p. 543] (see also [34, sec. 4.5]) shows that K^a is a strong deformation retract of M^a when f is bounded below and proper and a is regular. Theorems 6.2 and 10.1 shows that, in this case, M^a even has a CW decomposition such that K^a expands to M^a by elementary expansions.

Theorems 6.3 and 10.3 compute the boundary operator of the CW chain complex associated with K^a. This relates Morse homology to a cellular chain complex. The proofs of Theorems 6.2 and 6.3 reflect the advantage of Theorem 6.1 and Lemma 6.10.

The results in Chapters 4, 5 and 6 are based on the assumption of that the Riemmannian metric (or the negative gradient vector field) is locally trivial. This means the metric is Euclidean and fits with the function value well near each critical point. Thus the vector field has the simplest form near each critical point.

Chapters 8, 9 and 10 extend those results by dropping the above assumption provided that the Morse function is proper. Here the underlying manifold has to be finite dimensional but not necessarily compact.

The following is the reason for making such an extension. There are at least two disadvantages of the locally trivial metric. Firstly, local triviality is not a generic property. Sometimes, especially in the infinite dimensional setting such as in Floer theory, it is not usually the case that one can find a metric satisfying both the local triviality and transversality conditions. Secondly, the assumption of local triviality of the metric contradicts symmetry. Take for example a homogeneous Riemannian manifold. If the metric is locally trivial, then the curvature tensor must vanish near each critical point. Since the metric is homogeneous, the curvature tensor must vanish globally. Thus only a tiny class of homogeneous Riemannian manifolds have this type of metric.

Actually, the local triviality assumption on the metric is made in Chapters 4, 5 and 6 exclusively because of the techniques employed there. The statements of (3) of Theorem 4.4, (3) and (4) of Theorem 4.5, and (3) of Theorem 4.6 show that, under the assumption of a locally trivial metric, the compactified moduli spaces have smooth structures compatible with that of the underlying manifold. However, as we will see it later, if the metric is not locally trivial, there is no such compatiblity. Thus the case of a locally trivial metric has several distinct features from the general case. In fact, the proofs of Theorems 6.1 and 6.2 rely heavily on the compatibility.

In this situation, it's natural to pose the following strategy for obtaining results about Morse moduli spaces in the case of a general metric. As a first step, we implement the subtle and technical arguments in the special case. In the second and final step, we try to convert the general case to the special case. Chapters 4, 5 and 6 complete the first step. Chapters 8, 9 and 10 achieve the second one.

Franks' paper [28, prop. 1.6] proposes an excellent idea to reduce the general case to the special case as follows. The proof of [42, lem. 2] claims that there exists a regular path connecting a general negative gradient vector field X with a special Y. Here X and Y satisfy transversality. More importantly, Y is locally trivial. By a regular path, we mean a continuous path of negative gradient fields in which each single vector field satisfies transversality.

Since a negative gradient field satisfying transversality is structurally stable, we get X is topologically equivalent to Y, which converts the general vector field X to the locally trivial Y.

However, there is a serious issue in the proof in [42]. It's well known that, for negative gradient fields, transversality is preserved under small C^1 perturbations. However, the vector fields certainly change largely in the C^1 topology along the above path. *How can we guarantee the transversality?* Franks' paper [28] refers the proof to [42], and the latter outlines the construction of the path. Both [28] and [42] indicate that the λ-Lemma in [49] verifies the transversality. Unfortunately, none of them explain *why* the λ-Lemma works in this setting.

Chapter 7 supports the idea in [42]. Precisely, following this idea, we shall give a self-contained and detailed proof of Theorem 7.7. However, the statement of Theorem 7.7 is slightly different from that in [42] such that it becomes better in the setting of Morse theory. (Actually, the papers [42] and [28] emphasize the setting of dynamical systems. However, our argument also proves the result in [42].)

It's necessary to point out that Chapter 8 has to be a partial extension of Chapter 4. As mentioned before, the spaces $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$ have perfect relations with the underlying manifold if the metric is locally trivial. However, Example 8.1 shows that these relations have to be bad if the metric is not locally trivial. This is a remarkable difference between these two types of metrics. Therefore, the statements of theorems in Chapter 8 are weaker than their counterparts in Chapter 4.

Finally, Chapter 11 deals with the associativity of gluing. It shows that the associativity of gluing exclusively follows from the compatible manifold structures of the compactified moduli spaces. This does *not* rely on any speciality of Morse theory.

Suppose p_1, p_2 and p_3 are critical points, γ_1 is a flow line from p_1 to p_2 and γ_2 is a flow line from p_2 to p_3. In a strict sense, the pair (γ_1, γ_2) of consecutive flow lines is not a flow line. We consider (γ_1, γ_2) as a broken flow line from p_1 to p_3. A gluing of (γ_1, γ_2) is a construction of $\gamma_1 \#_\lambda \gamma_2$, where $\lambda \in [0, \epsilon)$ is the parameter for the gluing. We have $\gamma_1 \#_\lambda \gamma_2$ is an unbroken flow line from p_1 to p_3 when $\lambda \neq 0$, and $\gamma_1 \#_0 \gamma_2 = (\gamma_1, \gamma_2)$. Furthermore, $\gamma_1 \#_\lambda \gamma_2$ smoothly depends on γ_1, γ_2 and λ in a certain sense.

Suppose γ_1, γ_2 and γ_3 are three consecutive flow lines. We glue them inductively by pairs. There are two such gluings $(\gamma_1 \#_{\lambda_1} \gamma_2) \#_{\lambda_2} \gamma_3$ and $\gamma_1 \#_{\lambda_1} (\gamma_2 \#_{\lambda_2} \gamma_3)$ up to their orders. If we have

$$(\gamma_1 \#_{\lambda_1} \gamma_2) \#_{\lambda_2} \gamma_3 = \gamma_1 \#_{\lambda_1} (\gamma_2 \#_{\lambda_2} \gamma_3),$$

then this gluing satisfies associativity.

As mentioned before, the manifold structure of a compactified moduli space is actually related to the associativity of gluing. One can derive the manifold structure from the associativity of gluing because the latter provides nice coordinate charts for the former. However, this is *not* the unique way to get the manifold structure. The papers [35] and [11], and Chapters 4 and 8 in this dissertation obtain the manifold structures without using any gluing arguments.

In Chapter 11, we shall strengthen the above relation by working in the inverse direction. Theorems 11.2 and 11.3 show that, if the manifold structures satisfy Assumption 11.1, then one will get the associativity of gluing for free. In fact, we reformulate a gluing of broken flow lines as parametrizations of collar neighborhoods of the strata of the compactified moduli spaces. The associativity of gluing is equivalent to a compatible collar structure. Thus the above theorems can be even generalized to be Theorem 11.6 which is purely on the compatible collar structures of manifolds with faces.

In short, these theorems convert the problem of the associativity of gluing to the problem of manifold structures. By the results on manifold structures in Chapters 4 and 8, we get Propositions 11.4 and 11.5. They establish the associativity of gluing.

A byproduct of our work is Proposition 11.19 which is also about compatible collar structures. Theorem 11.6 is on a family of manifolds with faces, while Proposition 11.19 is on a single one. However, the assumption of Proposition 11.19 is more general.

The outline of this dissertation is as follows.

Chapter 2 gives some definitions, notation and elementary results mostly used in this dissertation. The subsequent chapters can be divided into four parts.

The first part is Chapter 3 which studies the compactness of flow lines. The hypotheses of theorems in this part is really weak. Transversality of the vector field is even not required. After Chapter 3, transversality is subsequently required throughout.

The second part consists of Chapters 4, 5 and 6. It discusses the manifold structures, orientations and CW structures under the assumption of the local triviality of the Riemannian metric.

The third part is from Chapters 7 to 10. It extends the results in the second part by dropping the local triviality of the metric. Unlike other parts, this part exclusively deals with finite dimensional manifolds.

The last part is Chapter 11 which considers the associativity of gluing.

2 Preliminaries

In this chapter, we give some definitions, notation and elementary results mostly used in this dissertation.

2.1 Condition (C)

Suppose M is a Hilbert manifold with a *complete* Riemannian metric. The completeness of the metric is necessary for Theorem 2.2 (compare [37, rem., p. 13]). Let f be a Morse function on M. Denote the index of a critical point p by $\mathrm{ind}(p)$. Denote $f^{-1}([a,b])$ by $M^{a,b}$. Denote $f^{-1}((-\infty,a])$ by M^a.

We need the well-known Condition (C) or Palais-Smale Condition (see [46]).

Condition (C): *If S is a subset of M on which f is bounded but on which $\|\nabla f\|$ is not bounded away from 0, then there is a critical point of f in the closure of S.*

Assuming this condition, its easy to prove the following results. Good references are [46, thm. 1 and 2], [44] and [47, sec. 9.1].

Theorem 2.1. *If (M,f) satisfies Condition (C), then for all a,b such that $-\infty < a < b < +\infty$, $M^{a,b}$ contains only finite many critical points.*

We cite [44, thm. (3), p. 333] as follows.

Theorem 2.2. *If (M,f) satisfies Condition (C), $x \in M$, and $\phi_t(x)$ is the maximal flow of $-\nabla f$ with initial value x, then $\phi_t(x)$ satisfies one of the following two conditions:*

(1) $f(\phi_t(x))$ has no lower (upper) bound; or

(2) $f(\phi_t(x))$ has a lower (upper) bound, $\phi_t(x)$ can be defined as a function of t on $[0,+\infty)$ $((-\infty,0])$, $\lim_{t\to+\infty}\phi_t(x)$ $(\lim_{t\to-\infty}\phi_t(x))$ exists and is a critical point of f.

By Theorem 2.2, we get an immediate corollary.

Corollary 2.3. *Suppose (M,f) satisfies Condition (C) and $-\infty<a<b<+\infty$. Then all flow lines in $M^{a,b}$ are from $f^{-1}(b)$ or a critical point in $M^{a,b}$ to $f^{-1}(a)$ or a critical point in $M^{a,b}$.*

Definition 2.4. *Let $\phi_t(x)$ be the flow generated by $-\nabla f$ with initial value x. Suppose p is a critical point. Define the descending manifold of p to be $\mathcal{D}(p)=\{x\in M \mid \lim_{t\to-\infty}\phi_t(x)=p\}$. Define the ascending manifold of p to be $\mathcal{A}(p)=\{x\in M \mid \lim_{t\to+\infty}\phi_t(x)=p\}$. We call $\mathcal{D}(p)$ and $\mathcal{A}(p)$ the invariant manifolds of p. If the manifolds come from a vector field X, we also denote $\mathcal{D}(p)$ by $\mathcal{D}(p;X)$ and denote $\mathcal{A}(p)$ by $\mathcal{A}(p;X)$ in order to indicate the vector field X.*

Both $\mathcal{D}(p)$ and $\mathcal{A}(p)$ are embedded submanifolds diffeomorphic to (maybe infinite dimensional) open disks. (By [30] and [31], we know they are immersed open disks. Since they come from a Morse function, it's easy to show that they are actually embedded.) By Theorem 2.1 and Corollary 2.3, we get the following.

Corollary 2.5. *Suppose (M,f) satisfies Condition (C) and $-\infty<a<b<+\infty$. Suppose $\{p_1,\cdots,p_n\}$ consists of all critical points in $M^{a,b}$. Denote $\mathcal{A}(p_i)\cap f^{-1}(b)$ by S_i^+, and $\mathcal{D}(p_i)\cap f^{-1}(a)$ by S_i^-. Then the flow map can be defined and gives a diffeomorphism:*

$$\psi: f^{-1}(b)-\bigcup_{i=1}^{n}S_i^+ \longrightarrow f^{-1}(a)-\bigcup_{i=1}^{n}S_i^-.$$

In particular, if there is no critical point in $M^{a,b}$, we have the following diffeomorphism:

$$\psi : f^{-1}(b) \longrightarrow f^{-1}(a).$$

Here, if $x \in f^{-1}(b)$, $\phi_t(x) = y \in f^{-1}(a)$ for some t, the flow map is defined by $\psi(x) = y$.

Remark 2.1. *Although we use the notation $S_i^{\pm}$ in Corollary 2.5, $S_i^{\pm}$ are not necessarily homeomorphic to spheres.*

Definition 2.6. *If (M, f) satisfies Condition (C) and ind$(p) < +\infty$ for all critical points p, then we call (M, f) a CF pair.*

Condition (C) is a generalization of the proper condition to the infinitely dimensional setting.

Definition 2.7. *We call a Morse function f is proper if $f^{-1}(K)$ is compact for any compact subset K in R.*

Obviously, if f is proper, then the underlying manifold M has to be finitely dimensional. In this case, (M, f) is certainly a CF pair.

2.2 Flows and Moduli Spaces

Definition 2.8. *If the descending manifold $\mathcal{D}(p)$ and the ascending manifold $\mathcal{A}(q)$ are transversal for all critical points p and q, then we say $-\nabla f$ satisfies transversality.*

Remark 2.2. *Some papers in the literature call Definition 2.8 Morse-Smale Condition.*

If $-\nabla f$ satisfies transversality, then $\mathcal{D}(p) \cap \mathcal{A}(q)$ is an embedded submanifold which consists of points on flow lines from p to q. Since a flow line has an R-action, we may take the quotient of $\mathcal{D}(p) \cap \mathcal{A}(q)$ by this R-action, i.e. consider its orbit space acted upon by the flow. This leads to the following definition. (See also [11, observation 4], [17, p. 3], [55, defn. 2.32] and [9, p. 158].)

Definition 2.9. *Suppose $-\nabla f$ satisfies transversality. Define $\mathcal{W}(p,q) = \mathcal{D}(p) \cap \mathcal{A}(q)$. Define the moduli space $\mathcal{M}(p,q)$ to be the orbit space $\mathcal{W}(p,q)/R$.*

Clearly, both $\mathcal{W}(p,q)$ and $\mathcal{M}(p,q)$ are smooth manifolds. Suppose γ_1 and γ_2 are two flow lines such that $\gamma_1(-\infty) = \gamma_2(-\infty) = p$, $\gamma_1(+\infty) = \gamma_2(+\infty) = q$ and $\gamma_1(0) = \gamma_2(t_0)$ for some $t_0 \neq 0$. Then γ_1 and γ_2 are two distinct flow lines which represent the same point of $\mathcal{M}(p,q)$. For convenience and briefness, we identify them as the same flow line. Then $\mathcal{M}(p,q) = \{\gamma \mid \gamma \text{ is a flow line, } \gamma(-\infty) = p \text{ and } \gamma(+\infty) = q.\}$. Suppose $a \in (f(q), f(p))$ is a regular value. For all $\gamma \in \mathcal{M}(p,q)$, it intersects with $f^{-1}(a)$ at a unique point. This gives $\mathcal{M}(p,q)$ a natural identification with $\mathcal{W}(p,q) \cap f^{-1}(a)$ which is a diffeomorphism.

We generalize the concept of flow lines. Suppose γ is a flow line. If it passes through a singularity, it is a constant flow line. Otherwise, it is nonconstant. The following definition is slightly different from the "broken trajectories" in [11, defn. 4].

Definition 2.10. *An ordered sequence of flow lines $\Gamma = (\gamma_1, \cdots, \gamma_n)$, $n \geq 1$, is a generalized flow line if $\gamma_i(+\infty) = \gamma_{i+1}(-\infty)$ and γ_i are constant or nonconstant alternatively according the order of their places in the sequence. γ_i is a component of Γ. Γ is a unbroken generalized flow line if $n = 1$ and a broken generalized flow line if $n > 1$.*

Example 2.1. *Suppose p is a singularity. Assume γ_1, γ_2 and γ_3 are flow lines in which*

γ_1 and γ_3 are nonconstant and $\gamma_1(+\infty) = \gamma_3(-\infty) = p$, $\gamma_2(t) \equiv p$. Then (γ_1), (γ_1, γ_2), (γ_2, γ_3) and $(\gamma_1, \gamma_2, \gamma_3)$ are generalized flow lines, (γ_1) is unbroken, and others are broken. Furthermore, (γ_1, γ_3) is not a generalized flow line.

For convenience, we may identify a flow line γ with the generalized flow line (γ). Definition 2.10 is a generalization of flow lines.

Definition 2.11. *Suppose x and y are two points in M. A generalized flow line $(\gamma_1, \cdots, \gamma_n)$ connects x and y if there exist $t_1, t_2 \in (-\infty, +\infty)$ such that $\gamma_1(t_1) = x$ and $\gamma_n(t_2) = y$. A point z is a point on $(\gamma_1, \cdots, \gamma_n)$ if there exists γ_i and $t \in (-\infty, +\infty)$ such that $\gamma_i(t) = z$.*

Example 2.2. *Suppose p and q are two critical points. Let γ_1, γ_2 and γ_3 be flow lines such that $\gamma_1(t) \equiv p$, $\gamma_3(t) \equiv q$, $\gamma_2(-\infty) = p$ and $\gamma_2(+\infty) = q$. Then $(\gamma_1, \gamma_2, \gamma_3)$ is a generalized flow line connecting p and q, while γ_2 is not.*

We need to consider the relations between two critical points.

Definition 2.12. *Suppose p and q are two critical points. We define the relation $p \succeq q$ if there is a flow line from p to q. We define the relation $p \succ q$ if $p \succeq q$ and $p \neq q$.*

If $-\nabla f$ satisfies transversality, then "$\succeq$" is a partial order on the set consisting of all critical points (see [50, p. 85, cor. 1]).

Definition 2.13. *An ordered set $I = \{r_0, r_1, \cdots, r_{k+1}\}$ is a critical sequence if r_i ($i = 0, \cdots, k+1$) are critical points and $r_0 \succ r_1 \succ \cdots \succ r_{k+1}$. We call r_0 the head of I, and r_{k+1} the tail of I. The length of I is $|I| = k$.*

Suppose $I = \{r_0, r_1, \cdots, r_{k+1}\}$ is a critical sequence. We denote the following product manifolds by $\mathcal{M}_I$ and $\mathcal{D}_I$.

$$\mathcal{M}_I = \prod_{i=0}^{k} \mathcal{M}(r_i, r_{i+1}), \qquad \mathcal{D}_I = \prod_{i=0}^{k} \mathcal{M}(r_i, r_{i+1}) \times \mathcal{D}(r_{k+1}). \tag{2.1}$$

We shall also use negative gradient-like flows.

Definition 2.14. *Suppose f is a Morse function on a Hilbert manifold. A vector field X is a gradient-like vector field of f if $X = \nabla f$ near each critical point of f and $Xf > 0$ at each regular point of f.*

Remark 2.3. *Some papers in the literature include the local triviality of X into the definition of a gradient-like vector field. We follow the style of [56] and exclude it.*

By Definition 2.14, every gradient vector field is obviously a gradient-like vector field. On the other hand, Smale [56, remark after thm. B] points out that all gradient-like fields are actually gradient fields on a finitely dimensional manifold. This is even true for a Hilbert manifold.

Lemma 2.15. *If X is a gradient-like vector field of a Morse function f on a Hilbert manifold M, then there is a metric on M such that this metric equals the original one associated with X near each critical point of f and $\nabla f = X$ for this metric.*

Proof. Define a matrix

$$A_1 = \begin{pmatrix} \cos\theta & \sin\theta \\ \\ \sin\theta & \frac{1}{\cos\theta} \end{pmatrix}$$

for $\theta \in [0, \frac{\pi}{2})$. Then A_1 is a symmetric positive matrix and $A_1(1,0)^T = (\cos\theta, \sin\theta)^T$, where $(*,*)^T$ is the transpose of $(*,*)$.

Suppose V_1 and V_2 are two vectors in a Hilbert space such that $\langle V_1, V_2\rangle > 0$, i.e., the angle between V_1 and V_2 is less than $\frac{\pi}{2}$. We define a symmetric positive operator $A(V_1, V_2)$ such that $A(V_1, V_2)V_1 = V_2$ as follows.

If V_1 and V_2 are colinear, then define $A(V_1, V_2) = \frac{\|V_2\|}{\|V_1\|}\mathrm{Id}$. If V_1 and V_2 are not colinear, then they span a plane $V_1 \wedge V_2$. First, we define an operator $A_2(e_1, e_2)$ for $e_1 = \frac{V_1}{\|V_1\|}$ and $e_2 = \frac{V_2}{\|V_2\|}$. In $(V_1 \wedge V_2)^\perp$, $A_2(e_1, e_2)$ is the identity. In $V_1 \wedge V_2$, $A_2(e_1, e_2)$ is the above A_1 mapping e_1 to e_2. Define $A(V_1, V_2) = \frac{\|V_2\|}{\|V_1\|}A_2(e_1, e_2)$.

Thus, in general, $A(V_1, V_2) = \frac{\|V_2\|}{\|V_1\|}A_2(\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|})$ for $\langle V_1, V_2\rangle > 0$. Here, for $\|e_1\| = \|e_2\| = 1$ and $\langle e_1, e_2\rangle > 0$, we have

$$\begin{aligned} A_2(e_1, e_2)Y &= Y + \frac{\langle e_1, e_2\rangle\langle e_2, Y\rangle - (1 + \langle e_1, e_2\rangle + \langle e_1, e_2\rangle^2)\langle e_1, Y\rangle}{1 + \langle e_1, e_2\rangle}e_1 \\ &\quad + \frac{\langle e_1, e_2\rangle^2\langle e_1, Y\rangle + \langle e_2, Y\rangle}{\langle e_1, e_2\rangle(1 + \langle e_1, e_2\rangle)}e_2. \end{aligned}$$

Then $A(V_1, V_2)$ smoothly depends on V_1 and V_2, and $A(V_1, V_1) = \mathrm{Id}$.

Let G_1 be the metric associated with X. Denote the gradient vector field of f with respect to G_1 by $\nabla_{G_1}f$. Then $\nabla_{G_1}f$ equals X near each critical point, and $\langle \nabla_{G_1}f, X\rangle = Xf > 0$ at each regular point. Define the operator $A(X, \nabla_{G_1}f)$ as above at each regular point. Define $A(X, \nabla_{G_1}f) = \mathrm{Id}$ at each critical point. Then $A(X, \nabla_{G_1}f)$ is smooth on M. Define a new metric G_2 such that $\langle *, *\rangle_{G_2} = \langle A(X, \nabla_{G_1}f)*, *\rangle_{G_1}$. Then $\nabla_{G_2}f = X$. $\square$

Suppose p is critical point. By the Morse Lemma, there exist $\epsilon > 0$ and a diffeomorphism

$$h : B(\epsilon) \longrightarrow U \tag{2.2}$$

such that

$$f \circ h(v_1, v_2) = f(p) - \frac{1}{2}\langle v_1, v_1 \rangle + \frac{1}{2}\langle v_2, v_2 \rangle. \tag{2.3}$$

Here $B(\epsilon) = \{(v_1, v_2) \in T_pM \mid v_1 \in V_-, v_2 \in V_+, \|v_1\|^2 < 2\epsilon \text{ and } \|v_2\|^2 < 2\epsilon\}$, $V_- \times \{0\}$ is the negative spectrum space of $\nabla^2 f$ and $\{0\} \times V_+$ is the positive spectrum space of $\nabla^2 f$, U is a neighborhood of p and $h(0,0) = p$.

Definition 2.16. *If the map h in (2.2) also preserves the metric, then we say that the metric of M is locally trivial at p. If it is locally trivial at each critical point, then we say that the metric on M is locally trivial.*

If the metric is locally trivial at p, then we have

$$-\nabla f|_U = dh \cdot (v_1, -v_2). \tag{2.4}$$

When the metric is locally trivial, Figure 1 shows the standard model of the neighborhood U, where U is identified with $B(\epsilon)$. Here, a and b are regular values such that $b < f(p) < a$, and $f^{-1}(a)$ and $f^{-1}(b)$ are two level surfaces. The arrows indicate the directions of the flow. The points (v_1, v_2), (v_3, v_4) and (v_5, v_6) are on the same flow line, whereas (v_7, v_8), (v_9, v_{10}), (v_{11}, v_{12}) and $(0, 0)$ are on the same broken generalized flow line. Figure 1 will provide geometric intuition for the arguments in this paper.

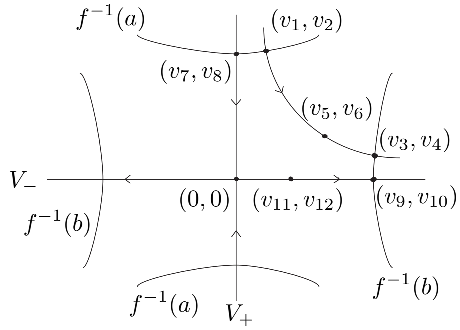

Figure 1: Standard Model

If U is a local coordinate chart near p such that p has the coordinate $(0,0)$ and f has the form as (2.3), then we call U a Morse chart. We also say a negative gradient-like field X is trivial near p if it has the form as (2.4) in a Morse chart. We say X is locally trivial if it is trivial near each critical point.

2.3 Manifolds with Corners

We shall consider the manifold structures of compactifications of the spaces $\mathcal{M}(p,q)$, $\mathcal{D}(p)$ and $\mathcal{W}(p,q)$. They usually have corners. For the definition of manifold with corners, we follow [24, p. 2] and [32, sec. 1.1].

Definition 2.17. *An n dimensional smooth manifold with corners is a space defined in the same way as a smooth manifold except that its atlases are open subsets of $[0,+\infty)^n$.*

If L is a smooth manifold with corners, $x \in L$, a neighborhood of x is differomorphic to $(0,\epsilon)^{n-k} \times [0,\epsilon)^k$, then define $c(x) = k$. Clearly, $c(x)$ does not depend on the choice of atlas.

Definition 2.18. *Suppose L is a smooth manifold. We call $\{x \in L \mid c(x) = k\}$ the k-stratum of L. Denote it by $\partial^k L$.*

Clearly, $\partial^k L$ is a submanifold *without* corners inside L, its codimension is k.

Definition 2.19. *A smooth manifold L with faces is a smooth manifold with corners such that each x belongs to the closures of $c(x)$ different components of $\partial^1 L$.*

Suppose L is manifold with faces. The closure of a component of $\partial^1 L$ (see Definition 2.18) is still connected. Following the terminology of [32], we have the following definition.

Definition 2.20. *We call the closure of a component of $\partial^1 L$ a connected (closed) face of L. We call any union of pair-wisely disjoint connected faces a face of L.*

Definition 2.21. *Suppose L is a manifold with corners. For all $x \in L$,*

$$A_x L = \{v \in T_x L \mid v = \gamma'(0) \text{ for some smooth curve } \gamma : [0, \epsilon) \longrightarrow L.\}$$

is the tangent sector of L at x.

Definition 2.21 is equivalent to the *secteur tangent* in [24, p. 3].

3 Compactness

In this chapter, we study the compactness properties of the CF case.

3.1 Main Theorems

We shall prove the following two theorems on compactness in this chapter. They are essential for proving the compactness of certain compactified spaces later. Theorem 3.1 shows that the closure of the space of unbroken flow lines is compact and is contained in the (maybe broken) generalized flow lines (see Definition 2.10). Theorem 3.2 considers the set consisting of points on the generalized flow lines with a fixed head and tail. They are essential for proving the compactness of certain compactified spaces later.

Theorem 3.1 (Compactness of Flows)**.** *Suppose (M, f) is a CF pair. Suppose p and q are two distinct critical points and $\{\gamma_n\}_{n=1}^{\infty}$ are flow lines such that $\gamma_n(-\infty) = p$ and $\gamma_n(+\infty) = q$. Then there exist finite many distinct critical points r_i $(i = 0, \cdots, l+1)$ and flow lines $\hat{\gamma}_i$ $(i = 0, \cdots, l)$ such that $\hat{\gamma}_i(-\infty) = r_i$, $\hat{\gamma}_i(+\infty) = r_{i+1}$, $r_0 = p$ and $r_{l+1} = q$. There exist a subsequence $\{\gamma_{n_k}\} \subseteq \{\gamma_n\}$ and time $s^0_{n_k} < \cdots < s^l_{n_k}$ such that $\lim\limits_{k\to\infty} \gamma_{n_k}(s^i_{n_k}) = \hat{\gamma}_i(0)$.*

Remark 3.1. *The papers [1] and [2] prove results similar to Theorem 3.1. The proof of [1] relies on the study of differential operators on vector fields, which is a very different approach from that of this dissertation. However, the proof of [2, prop. 2.4, 1.17 and 2.2] is essentially the same as that of Theorem 3.1. Thus, theoretically, it's unnecessary to include the proof here. Nevertheless, for the sake of completeness, we still keep it.*

Theorem 3.2 (Compactness of Points). *Suppose, for any real numbers $a < b$, $M^{a,b}$ only contains finite many critical points. Suppose, for any two critical points p and q, the conclusion of Theorem 3.1 holds. Let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of generalized flow lines connecting p and q. Then we have the following results.*

(1). Suppose x_n is on Γ_n. Then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\lim\limits_{k\to\infty} x_{n_k}$ exists and is on a generalized flow line connecting p and q.

(2). Suppose x_n^i are on Γ_n and $\lim\limits_{n\to\infty} x_n^i$ exist ($i = 1, \cdots, k$). Then these limit points are on a same generalized flow line connecting p and q.

In particular, if (M, f) is a CF pair, then the above (1) and (2) hold.

Remark 3.2. *Essentially, the compactness of points follows from the compactness of flows. For a more precise description, see Proposition 3.6.*

3.2 Proof of Theorem 3.1

In order to prove Theorem 3.1, we need the classical Grobman-Hartman Theorem in Banach spaces.

Suppose U_i ($i = 1, 2$) are two open subsets in two Banach spaces E_i. Let X_i be a smooth vector field on U_i. Let $\phi_t^i(x_i)$ be the associated flow on U_i with initial value x_i. We say ϕ_t^i ($i = 1, 2$) are topologically conjugate if there exists a homeomorphism $h : U_1 \longrightarrow U_2$ such that $h(\phi_t^1(x_1)) = \phi_t^2(h(x_1))$ (see also (7.3)). The Grobman-Hartman Theorem states that, if p is a hyperbolic singularity of X on an open subset U of a Banach space E, then the flow generated by X is locally topologically conjugate to that generated by the linear vector field $\nabla X(p)v$ near 0 on T_pU (see [53, sec. 4], [48, sec. 5] and [50, thm. 4.10, p. 66]. Although the statements in [48] and [50] are only up to topological equivalence, they actually construct the conjugate.)

In our case, $\nabla^2 f(p)$ splits T_pM into two subspace $T_pM = V_- \times V_+$, where $\{0\} \times V_+$ ($V_- \times \{0\}$) is the positive (negative) spectrum space of $\nabla^2 f(p)$. Thus the flow of $-\nabla f$ is topologically conjugate to the flow of $(-\nabla^2 f(p)v_1, -\nabla^2 f(p)v_2)$ on T_pM. Furthermore, $-\nabla^2 f(p)$ is symmetric and negative (positive) definite in $\{0\}\times V_+$ ($V_-\times\{0\}$), thus $-\nabla^2 f(p)v_i$ is transversal to the unit sphere in $V_\pm$. By the method of the proof of [50, prop. 2.15, p. 52], we have the flow of $(-\nabla^2 f(p)v_1, -\nabla^2 f(p)v_2)$ is topologically conjugate to the flow of $(v_1, -v_2)$. Thus we get the flowing lemma (compare (2.4)).

Lemma 3.3. *The flow generated by $-\nabla f$ near a critical point p is locally topologically conjugate to the flow generated by $(v_1, -v_2)$ near 0 on $T_pM = V_- \times V_+$.*

Lemma 3.4. *Suppose $\{\gamma_n(t)\}_{n=1}^{\infty}$ and $\hat{\gamma}_1(t)$ are flow lines such that $\lim_{n\to\infty} \gamma_n(0) = \hat{\gamma}_1(0)$, $\hat{\gamma}_1(+\infty) = p$ with ind$(p) < +\infty$, and, for all n, $\gamma_n(+\infty) \neq p$. Then there exist a subsequence $\{\gamma_{n_k}\} \subseteq \{\gamma_n\}$, time $s_{n_k} > 0$ and a nonconstant flow line $\hat{\gamma}_2(t)$ such that $\hat{\gamma}_2(-\infty) = p$ and $\lim_{k\to\infty} \gamma_{n_k}(s_{n_k}) = \hat{\gamma}_2(0)$.*

Proof. By Lemma 3.3, there exist a neighborhood U_2 of p in M, a neighborhood U_1 of 0 in T_pM and a homeomorphism $h : U_1 \longrightarrow U_2$ such that $h(0) = p$ and h conjugates between the flow generated by $(v_1, -v_2)$ in U_1 and the flow generated by $-\nabla f$ in U_2 (see Figure 2).

Choosing an open subset if necessary, we may assume $U_1 = D_1(\epsilon) \times D_2(\epsilon)$ for some ϵ, where $D_1(\epsilon) = \{v_1 \in V_- \mid \|v_1\| < \epsilon\}$ and $D_2(\epsilon) = \{v_2 \in V_+ \mid \|v_2\| < \epsilon\}$. In U_1, $(V_-\times\{0\})\cap U_1$ is the unstable submanifold, $(\{0\} \times V_+) \cap U_1$ is the stable submanifold. Thus $h(V_- \times \{0\})$ and $h(\{0\} \times V_+)$ are locally unstable and locally stable submanifolds respectively in U_2.

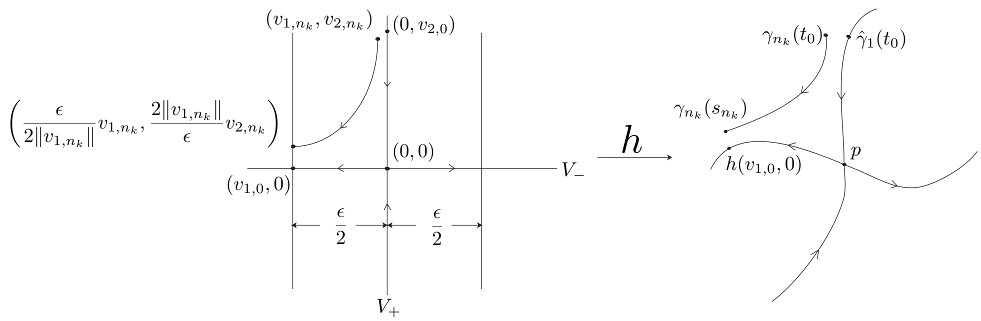

Figure 2: Topological Conjugate

Since $\hat{\gamma}_1(+\infty) = p$, $\exists t_0$ such that $\forall t \geq t_0$, $\hat{\gamma}_1(t) \in h(\{0\} \times V_+)$. Suppose $h^{-1}(\hat{\gamma}_1(t_0)) = (0, v_{2,0})$. Since $\gamma_n(0) \to \hat{\gamma}_1(t_0)$, we have $\gamma_n(t_0) \in U_2$, $h^{-1}(\gamma_n(t_0)) = (v_{1,n}, v_{2,n})$ and $\|v_{1,n}\| < \frac{\epsilon}{2}$ when n is large enough. Since $\gamma_n(+\infty) \neq p$, we have $v_{1,n} \neq 0$. As a result, in U_1, the flow line passing through $(v_{1,n}, v_{2,n})$ intersects $S_1(\frac{\epsilon}{2}) \times D_2(\epsilon)$ at $\left(\frac{\epsilon}{2\|v_{1,n}\|}v_{1,n}, \frac{2\|v_{1,n}\|}{\epsilon}v_{2,n}\right)$, where $S_1(\frac{\epsilon}{2}) = \{v_1 \in V_- \mid \|v_1\| = \frac{\epsilon}{2}\}$. When $n \to \infty$, we have $(v_{1,n}, v_{2,n}) \to (0, v_{2,0})$. Thus $\frac{2\|v_{1,n}\|}{\epsilon}v_{2,n} \to 0$. Since it is a $\operatorname{ind}(p) - 1$ dimensional sphere and $\operatorname{ind}(p) < +\infty$, we have $S_1\left(\frac{\epsilon}{2}\right)$ is compact. So there exists a subsequence $\left\{\frac{\epsilon v_{1,n_k}}{2\|v_{1,n_k}\|}\right\}$ of $\left\{\frac{\epsilon v_{1,n}}{2\|v_{1,n}\|}\right\}$ such that $\lim_{k\to\infty} \frac{\epsilon}{2\|v_{1,n_k}\|}v_{1,n_k} = v_{1,0}$. Clearly, there exists $s_{n_k} > 0$ such that

$$\gamma_{n_k}(s_{n_k}) = h\left(\frac{\epsilon}{2\|v_{1,n_k}\|}v_{1,n_k}, \frac{2\|v_{1,n_k}\|}{\epsilon}v_{2,n_k}\right).$$

Thus

$$\lim_{k\to\infty} \gamma_{n_k}(s_{n_k}) = h(v_{1,0}, 0).$$

Denote the flow line with initial value $h(v_{1,0}, 0)$ by $\hat{\gamma}_2(t)$. Then $\lim_{k\to\infty} \gamma_{n_k}(s_{n_k}) = \hat{\gamma}_2(0)$. Since $h^{-1}(\hat{\gamma}_2(0)) = (v_{1,0}, 0) \in V_- \times \{0\}$, we know that $h^{-1}(\hat{\gamma}_2(-\infty)) = (0, 0)$ or $\hat{\gamma}_2(-\infty) = p$.

Since $\hat{\gamma}_2(0) \neq p$, $\hat{\gamma}_2$ is nonconstant. □

Proof of Theorem 3.1. Let a be a regular value such that $a < f(p)$ and there is no critical value in $(a, f(p))$. Let $S_p^- = \mathcal{D}(p) \cap f^{-1}(q)$. Then S_p^- is a sphere with dimension $\operatorname{ind}(p) < +\infty$, and it is compact. Suppose $\gamma_n(s_n^0) \in S_p^-$. Then there exists a subsequence of $\{\gamma_n(s_n^0)\}$, we may still denote it by $\{\gamma_n(s_n^0)\}$, which converges. Suppose $\lim_{n\to\infty} \gamma_n(s_n^0) = x_0$. Then $x_0 \in S_p^-$. Denote the flow line with initial value x_0 by $\hat{\gamma}_0$. Then $\hat{\gamma}_0(-\infty) = p$ because $\hat{\gamma}_0(0) = x_0 \in S_p^- \subseteq \mathcal{D}(p)$. Since $\gamma_n(+\infty) = q$, we have, for all t, $f(\gamma_n(s_n^0 + t)) \geq f(q)$. Thus, for all t,

$$f(\hat{\gamma}_0(t)) = \lim_{n\to\infty} f(\gamma_n(s_n^0 + t)) \geq f(q),$$

i.e., $f(\hat{\gamma}_0(t))$ has a lower bound $f(q)$. By Theorem 2.2, $\lim_{t\to+\infty} \hat{\gamma}_0(t)$ exists and $\hat{\gamma}_0(+\infty) = r_1$ is a critical point in $M^{f(q),f(p)}$. Clearly, $\hat{\gamma}_0$ is nonconstant. Thus $r_1 \neq p$. There are exactly the following two cases.

Case (1): $r_1 = q$. In this case, the proof is finished.

Case (2): $r_1 \neq q$. Since $\gamma_n(+\infty) = q \neq r_1$ and $\operatorname{ind}(r_1) < +\infty$, by Lemma 3.4, there exists a nonconstant flow line $\hat{\gamma}_1$ such that $\hat{\gamma}_1(-\infty) = r_1$. Furthermore, there exists a subsequence of $\{\gamma_n\}$, which we still denote by $\{\gamma_n\}$, and time $s_n^1 > s_n^0$ such that $\lim_{n\to\infty} \gamma_n(s_n^1) = \hat{\gamma}_1(0)$. Similar to the case of $\hat{\gamma}_0$, we have $\lim_{t\to+\infty} \hat{\gamma}_1(t)$ exists and $\hat{\gamma}_1(+\infty) = r_2$ is also a critical point in $M^{f(q),f(p)}$. Since $\hat{\gamma}_1$ is nonconstant, p, r_1 and r_2 are distinct. If $r_2 = q$, the proof is finished. Otherwise, repeat the argument of Case (2).

By Theorem 2.1, there are only finitely many critical points in $M^{f(q),f(p)}$, the process of the above argument terminates in finitely many steps. □

3.3 Proof of Theorem 3.2

We first give two results needed for the proof of Theorem 3.2.

Lemma 3.5. *Suppose* $\{\gamma_n\}_{n=1}^{\infty}$ *and* $\hat{\gamma}$ *are flow lines such that* $\hat{\gamma}(-\infty) = p$, $\hat{\gamma}(+\infty) = q$ *and* $\lim_{n\to\infty} \gamma_n(s_n) = \hat{\gamma}(0)$. *If* $\lim_{n\to\infty}(t_n - s_n) = +\infty$ *(*$\lim_{n\to\infty}(t_n - s_n) = -\infty$*), then* $\limsup_{n\to\infty} f(\gamma_n(t_n)) \leq f(q)$ *(*$\liminf_{n\to\infty} f(\gamma_n(t_n)) \geq f(p)$*).*

Proof. It suffices to prove the case $\lim_{n\to\infty}(t_n - s_n) = +\infty$.

Since $\hat{\gamma}(+\infty) = q$, then $\forall \epsilon > 0$, $\exists T$, such that $\forall t \geq T$, we have $f(\hat{\gamma}(t)) < f(q) + \epsilon$. By that $\lim_{n\to\infty}(t_n - s_n) = +\infty$, we have $t_n > s_n + T$ and $f(\gamma_n(t_n)) < f(\gamma_n(s_n + T))$ when n is large enough. Since $\lim_{n\to\infty} \gamma_n(s_n) = \hat{\gamma}(0)$, we infer

$$\lim_{n\to\infty} f(\gamma_n(s_n + T)) = f(\hat{\gamma}(T)) < f(q) + \epsilon.$$

Thus

$$\limsup_{n\to\infty} f(\gamma_n(t_n)) \leq \lim_{n\to\infty} f(\gamma_n(s_n + T)) < f(q) + \epsilon.$$

Now let $\epsilon \to 0$. Then we get $\limsup_{n\to\infty} f(\gamma_n(t_n)) \leq f(q)$. □

The following proposition requires neither Condition (C) nor finite indices.

Proposition 3.6. *Suppose* p *and* q *are two critical points,* $\{\gamma_n\}_{n=1}^{\infty}$ *are flow lines such that* $\gamma_n(-\infty) = p$ *and* $\gamma_n(+\infty) = q$*, and there exist* $s_n^0 < \cdots < s_n^l$ *such that* $\lim_{n\to\infty} \gamma_n(s_n^i) = \hat{\gamma}_i(0)$. *Here* $\hat{\gamma}_i$ *are flow lines such that* $\hat{\gamma}_i(-\infty) = r_i$, $\hat{\gamma}_i(+\infty) = r_{i+1}$*, and* $r_0 = p$, $r_{l+1} = q$. *Then we have the following convergence result.*

(1). If $\lim_{n\to\infty}(t_n - s_n^i) = \tau$, $|\tau| < +\infty$*, then* $\lim_{n\to\infty} \gamma_n(t_n) = \hat{\gamma}_i(\tau)$*;*

(2). If $s_n^i < t_n < s_n^{i+1}$, and $\lim_{n\to\infty}(t_n - s_n^i) = \lim_{n\to\infty}(s_n^{i+1} - t_n) = +\infty$, then $\lim_{n\to\infty}\gamma_n(t_n) = r_{i+1}$, where $s_n^{-1} = -\infty$ and $s_n^{l+1} = +\infty$.

Proof. Case (1) is obvious. We only need to prove Case (2).

We may assume $s_n^i < t_n < s_n^{i+1}$ and $i \geq 0$ because the subcase of $i = -1$ will be converted to the subcase of $i = l$ if f is replaced by $-f$.

We shall prove $\lim_{n\to\infty}\gamma_n(t_n) = r_{i+1}$ by contradiction.

Suppose it doesn't hold, then there exist a subsequence of $\{\gamma_n(t_n)\}$, which we still denote by $\{\gamma_n(t_n)\}$, and a neighborhood U of r_{i+1} such that $\gamma_n(t_n) \notin U$. Choose an open geodesic disk $D(r_{i+1}, \epsilon)$ with center r_{i+1} and radius ϵ such that $\overline{D(r_{i+1}, \epsilon)} \subseteq U$. Since r_{i+1} is a nondegenerate critical point, by the Taylor expansion, we may choose ϵ small enough such that, there exist constants C_1 and C_2, and $0 < C_1 \leq \|\nabla f\| \leq C_2$ in $\overline{D(r_{i+1}, \epsilon)} - D(r_{i+1}, \frac{\epsilon}{2})$ for a fixed ϵ.

Suppose $\gamma(t)$ is a flow line, $\tau_1 < \tau_2$, such that $\gamma(\tau_1) \in D(r_{i+1}, \frac{\epsilon}{2})$ and $\gamma(\tau_2) \notin \overline{D(r_{i+1}, \epsilon)}$. Thus there exist τ_1', τ_2' such that $\tau_1 < \tau_1' < \tau_2' < \tau_2$, $\gamma([\tau_1', \tau_2']) \subseteq \overline{D(r_{i+1}, \epsilon)} - D(r_{i+1}, \frac{\epsilon}{2})$, $\gamma(\tau_1') \in \partial D(r_{i+1}, \frac{\epsilon}{2})$ and $\gamma(\tau_2') \in \partial D(r_{i+1}, \epsilon)$.

Consider the distance $d(\gamma(\tau_1'), \gamma(\tau_2'))$ between $\gamma(\tau_1')$ and $\gamma(\tau_2')$. Clearly, $d(\gamma(\tau_1'), \gamma(\tau_2')) \geq \frac{\epsilon}{2}$. Thus

$$
\begin{aligned}
\frac{\epsilon}{2} \leq d(\gamma(\tau_1'), \gamma(\tau_2')) &\leq \int_{\tau_1'}^{\tau_2'} \left\| \frac{d}{dt}\gamma(t) \right\| dt \\
= \int_{\tau_1'}^{\tau_2'} \|\nabla f(\gamma(t))\| dt &\leq \int_{\tau_1'}^{\tau_2'} C_2 dt = C_2(\tau_2' - \tau_1').
\end{aligned}
$$

We have $\tau_2' - \tau_1' \geq \frac{\epsilon}{2C_2}$. Then

$$\int_{\tau_1'}^{\tau_2'} \|\nabla f\|^2 \geq \int_{\tau_1'}^{\tau_2'} C_1^2 \geq \frac{C_1^2 \epsilon}{2C_2}.$$

Thus we get

$$f(\gamma(\tau_1)) - f(\gamma(\tau_2)) = \int_{\tau_1}^{\tau_2} \|\nabla f\|^2 \geq \int_{\tau_1'}^{\tau_2'} \|\nabla f\|^2 \geq \frac{C_1^2 \epsilon}{2C_2} > 0.$$

Denoting $\frac{C_1^2 \epsilon}{2C_2}$ by K, we get

$$f(\gamma(\tau_1)) - f(\gamma(\tau_2)) \geq K > 0. \tag{3.1}$$

Since $\hat{\gamma}_i(+\infty) = r_{i+1}$, then there exists t_∞ such that $\hat{\gamma}_i(t_\infty) \in B(r_{i+1}, \frac{\epsilon}{2})$ and $f(\hat{\gamma}_i(t_\infty)) < f(r_{i+1}) + \frac{K}{2}$. Since $\gamma_n(s_n^i) \to \hat{\gamma}_i(0)$, we have $\gamma_n(s_n^i + t_\infty) \in B(r_{i+1}, \frac{\epsilon}{2})$ and $f(\gamma_n(s_n^i + t_\infty)) < f(r_{i+1}) + \frac{K}{2}$ when n is large enough. Also since $(t_n - s_n^i) \to +\infty$, we get $t_n > s_n^i + t_\infty$ when n is large enough. Now we can replace $\gamma(\tau_1)$ and $\gamma(\tau_2)$ in (3.1) by $\gamma_n(s_n^i + t_\infty)$ and $\gamma_n(t_n)$, then $f(\gamma_n(s_n^i + t_\infty)) - f(\gamma_n(t_n)) \geq K$. Furthermore,

$$f(\gamma_n(t_n)) \leq f(\gamma_n(s_n^i + t_\infty)) - K < f(r_{i+1}) - \frac{K}{2}.$$

Thus

$$\limsup_{n\to\infty} f(\gamma_n(t_n)) \leq f(r_{i+1}) - \frac{K}{2} < f(r_{i+1}). \tag{3.2}$$

However, since $\hat{\gamma}_{i+1}(-\infty) = r_{i+1}$, and $(t_n - s_n^{i+1}) \to -\infty$, by Lemma 3.5, we have

$$\liminf_{n\to\infty} f(\gamma_n(t_n)) \geq f(r_{i+1}), \tag{3.3}$$

which is a contradiction. □

Proof of Theorem 3.2. (1). By assumption, there are only finite many critical points in $M^{f(q),f(p)}$. We can find two critical points p' and q', a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that x_{n_k} is on γ_{n_k}, $\gamma_{n_k}(-\infty) = p'$, $\gamma_{n_k}(+\infty) = q'$ and γ_{n_k} is a component of Γ_{n_k}. Clearly, a generalized flow line connecting p' and q' can be extended to one connecting p and q. If there is a cluster point of $\{x_{n_k}\}_{k=1}^{\infty}$ on a generalized flow line connecting p' and q', this cluster point is also on one connecting p and q. So we may assume that x_n is on γ_n, $\gamma_n(-\infty) = p$ and $\gamma_n(+\infty) = q$.

If $p = q$, this is obviously true. Now we assume $p \neq q$. Suppose $\gamma_n(t_n) = x_n$. Since the conclusion of Theorem 3.1 holds, choosing a subsequence if necessary, we can find $s_n^0 < \cdots < s_n^l$ such that $\lim_{n\to\infty} \gamma_n(s_n^i) = \hat{\gamma}_i(0)$, where $\hat{\gamma}_i(-\infty) = r_i$, $\hat{\gamma}_i(+\infty) = r_{i+1}$, and $r_0 = p$, $r_{l+1} = q$.

Choosing a subsequence again if necessary, we can find a fixed i such that, for all n, we have $t_n \in [s_n^i, s_n^{i+1}]$, where $s_n^{-1} = -\infty$ and $s_n^{l+1} = +\infty$. In addition, we may assume there are exactly the following three cases when $n \to \infty$. By Proposition 3.6, we have:

Case (a): $\lim_{n\to\infty} (t_n - s_n^i) = \tau < +\infty$. Then x_n converges to a point on $\hat{\gamma}_i$;

Case (b): $\lim_{n\to\infty} (s_n^{i+1} - t_n) = \tau < +\infty$. Then x_n converges to a point on $\hat{\gamma}_{i+1}$;

Case (c): $\lim_{n\to\infty} (t_n - s_n^i) = \lim_{n\to\infty} (s_n^{i+1} - t_n) = +\infty$. Then x_n converges to $r_{i+1} = \hat{\gamma}_i(+\infty) = \hat{\gamma}_{i+1}(-\infty)$.

This completes the proof of the first result.

(2). Since the limit of $\{x_n^i\}$ exists, its subsequences share the same limit with it. So we only need to check the limit of a subsequence of $\{x_n^i\}$. Since there are only finitely many critical points in $M^{f(q),f(p)}$, we may argue as in (1): choosing a subsequence if necessary, we may assume $\Gamma_n = (\gamma_{n,1}, \cdots, \gamma_{n,m})$, $\gamma_{n,j}(-\infty) = r_j$ and $\gamma_{n,j}(+\infty) = r_{j+1}$ are fixed and independent of n. In addition, $\forall i$, there is a fixed j such that for all n, x_n^i is on $\gamma_{n,j}$. If $r_j = r_{j+1}$, then $\gamma_{n,j}$ converges to the constant flow connecting r_j and r_j. Otherwise, choosing a subsequence again if necessary, $\{\gamma_{n,j}\}_{n=1}^{\infty}$ converges to a generalized flow line connecting r_j and r_{j+1}. The combination of the limits of $\{\gamma_{n,j}\}_{n=1}^{\infty}$ for $j = 1, \cdots, m$ yields a generalized flow line, Γ, connecting p and q. By an argument similar to that of (1), the limits of all $\{x_n^i\}$ are on Γ. $\square$

4 Manifold Structures (I)

In this chapter, under the assumption of the local triviality of the metric, we consider the compactification of $\mathcal{M}(p,q)$, $\mathcal{D}(p)$ and $\mathcal{W}(p,q)$, and the manifold structures of these compactified spaces.

4.1 Preparation Lemmas

The following two lemmas, Lemmas 4.1 and 4.2 are crucial for us to construct manifold structures of compactified spaces. Example 8.1 shows that they necessarily depend on the local triviality of the metric. These two lemmas are announced in [11, observations 8 and 9]. A proof for them in the finite dimensional case is given in [10]. For the importance of them, we present a proof which follows that in [10].

Figure 1 gives an illustration for the following argument. Suppose c is a critical value of f. The critical points with function value c are exactly $p_1, \cdots, p_n$. Just as (2.2), we have diffeomorphisms $h_i : B_i(\epsilon) \longrightarrow U_i$ such that (2.3) and (2.4) hold, where $B_i(\epsilon)$ is the open subset of $T_{p_i}M$ and U_i is the neighborhood of p_i. Choose ϵ small enough such that there is no critical value in $[c-\epsilon, c+\epsilon]$ other than c. Let $M_c^+ = \{x \in M \mid f(x) = c + \frac{1}{2}\epsilon\}$ and $M_c^- = \{x \in M \mid f(x) = c - \frac{1}{2}\epsilon\}$. Let

$$P_c = \{(x^+, x^-) \in M_c^+ \times M_c^- \mid x^+ \text{ and } x^- \text{ are connected by a generalized flow line}\}.$$

Clearly, x^+ and x^- are connected by broken generalized flow lines if and only if $(x^+, x^-) \in \bigsqcup_{p_i} S_{p_i}^+ \times S_{p_i}^-$, where $S_{p_i}^+$ and $S_{p_i}^-$ are $\mathcal{A}(p_i) \cap M_c^+$ and $\mathcal{D}(p_i) \cap M_c^-$ respectively. Suppose the smallest (largest) critical value greater (smaller) than c is c_+ (c_-). Here $c_\pm$ may be $\pm\infty$. Define $M(c) = f^{-1}((c_-, c_+))$. Let

$$Q_c^+ = \{(x^+, z) \in M_c^+ \times M(c) \mid x^+ \text{ and } z \text{ are connected by a generalized flow line}\}.$$

$$Q_c^- = \{(z, x^-) \in M(c) \times M_c^- \mid x^- \text{ and } z \text{ are connected by a generalized flow line}\}.$$

Then $x^\pm$ and z are connected by broken generalized flow lines if and only if $(x^+, z) \in \bigsqcup_{p_i} S_{p_i}^+ \times D_{p_i}$ and $(z, x^-) \in \bigsqcup_{p_i} A_{p_i} \times S_{p_i}^-$ respectively, where $D_{p_i} = \mathcal{D}(p_i) \cap M(c)$ and $A_{p_i} = \mathcal{A}(p_i) \cap M(c)$.

Lemma 4.1. *Suppose the metric is locally trivial. Then P_c is a smoothly embedded submanifold with boundary $\bigsqcup_{p_i} S_{p_i}^+ \times S_{p_i}^-$ of $M_c^+ \times M_c^-$.*

Proof. There is no essential difference in the proof between the case of one critical point and that of several critical points. For convenience, we may assume there is only one critical point p in $M^{c-\epsilon, c+\epsilon}$. We shall prove that P_c is a smooth embedding submanifold with boundary $S_p^+ \times S_p^-$ of $M_c^+ \times M_c^-$.

Firstly, we shall prove $P_c - S_p^+ \times S_p^-$ is a smoothly embedded submanifold of $M_c^+ \times M_c^-$.

Since it is an open subset of M_c^+, $M_c^+ - S_p^+$ is a smooth submanifold of M_c^+. By Corollary 2.5, we can define the flow map $\psi : M_c^+ - S_p^+ \longrightarrow M_c^- - S_p^-$. Define $\varphi : M_c^+ - S_p^+ \longrightarrow M_c^+ \times M_c^-$ by $\varphi(x_+) = (x_+, \psi(x_+))$. Clearly, φ is smooth and $\mathrm{Im}(\varphi) = P_c - S_p^+ \times S_p^-$. Define $\pi_+ : M_c^+ \times M_c^- \longrightarrow M_c^+$ to be the natural projection. We have π_+ is smooth and $\pi_+ \varphi = \mathrm{Id}$, so φ is a homeomorphism to its image. Since $d\pi_+ d\varphi = \mathrm{Id}$, $d\varphi$ is an isomorphism to its image. Thus $P_c - S_p^+ \times S_p^- = \mathrm{Im}(\varphi)$ is a smooth manifold of $M_c^+ \times M_c^-$.

Secondly, we shall prove that there is an open neighborhood W of $S_p^+ \times S_p^-$ in $M_c^+ \times M_c^-$ such that $W \cap P_c$ is a smooth submanifold with boundary $S_p^+ \times S_p^-$ in W.

By local triviality of the metric, there is a diffeomorphism $h : B \longrightarrow U$ given by (2.2) which satisfies (2.3) and (2.4). For convenience, we identify B with U. Then $S_p^+ = \{(0, v_2) \mid \|v_2\|^2 = \epsilon\}$, $S_p^- = \{(v_1, 0) \mid \|v_1\|^2 = \epsilon\}$, $M_c^+ \cap U = \{(v_1, v_2) \mid \|v_1\|^2 < 2\epsilon$, and $\|v_2\|^2 < 2\epsilon, -\|v_1\|^2 + \|v_2\|^2 = \epsilon\}$ and $M_c^- \cap U = \{(v_1, v_2) \mid \|v_1\|^2 < 2\epsilon$, and $\|v_2\|^2 < 2\epsilon, -\|v_1\|^2 + \|v_2\|^2 = -\epsilon\}$. Let $U_+ = \{(v_1, v_2) \mid \|v_1\|^2 < \epsilon$ and $\frac{\epsilon}{2} < \|v_2\|^2 < 2\epsilon.\}$ and $U_- = \{(v_1, v_2) \mid \|v_2\|^2 < \epsilon$ and $\frac{\epsilon}{2} < \|v_1\|^2 < 2\epsilon.\}$. Then $U_+ \times U_-$ is an open neighborhood of $S_p^+ \times S_p^-$ in $M^{c-\epsilon, c+\epsilon} \times M^{c-\epsilon, c+\epsilon}$. For convenience, we identify S_p^- with $\{v_1 \mid (v_1, 0) \in S_p^-\}$ and S_p^+ with $\{v_2 \mid (0, v_2) \in S_p^+\}$.

Consider the map $\varphi : S_p^+ \times S_p^- \times [0, 1) \longrightarrow U_+ \times U_-$ satisfying

$$\varphi(v_2, v_1, s) = \left((sv_1, (1+s^2)^{\frac{1}{2}} v_2), ((1+s^2)^{\frac{1}{2}} v_1, sv_2)\right).$$

Clearly, φ is smooth, $\mathrm{Im}(\varphi) = P_c \cap (U_+ \times U_-)$ and $\varphi|_{S_p^+ \times S_p^-} = \mathrm{Id}$. On the other hand, consider the map $\alpha : U_+ \times U_- \longrightarrow S_p^+ \times S_p^- \times [0, 1)$ satisfying

$$\alpha((z_1, z_2), (z_3, z_4)) = \left(\epsilon^{\frac{1}{2}} \frac{z_2}{\|z_2\|}, \epsilon^{\frac{1}{2}} \frac{z_3}{\|z_3\|}, \epsilon^{-\frac{1}{2}} \|z_1\|\right).$$

Then α is continuous and $\alpha\varphi = \mathrm{Id}$. In addition, α is smooth when $z_1 \neq 0$. Then φ is a homeomorphism to its image, and $d\varphi$ is an isomorphism onto its image when $s \neq 0$.

Now we consider the case of $s = 0$. We shall prove that $d\varphi|_{s=0}$ is an isomorphism onto its image. It suffices to prove that there exists $\lambda > 0$, for all $v \in T(S_p^+ \times S_p^- \times [0, 1))$, such that

$$\|d\varphi \cdot v\| \geq \lambda \|v\|. \tag{4.1}$$

Let $\frac{\partial}{\partial s}$ be the positive unit tangent vector of $[0, 1)$, e_2 and e_1 are tangent vectors of S_p^+ and S_p^-. Then

$$d\varphi|_{s=0}\left(\frac{\partial}{\partial s}\right) = (v_1, 0, 0, v_2), \quad d\varphi|_{s=0}(e_1) = (0, 0, e_1, 0), \quad d\varphi|_{s=0}(e_2) = (0, e_2, 0, 0).$$

It's easy to see (4.1) holds.

Thus φ is a smooth embedding into $U_+ \times U_-$. Let $W = (U_+ \times U_-) \cap (M_c^+ \times M_c^-)$. Then W is an open neighborhood of $S_p^+ \times S_p^-$ in $M_c^+ \times M_c^-$, $P_c \cap W = \mathrm{Im}(\varphi)$ and $P_c \cap W$ is a smoothly embedded submanifold with boundary $S_p^+ \times S_p^-$. $\square$

Lemma 4.2. *Suppose the metric is locally trivial. Then Q_c^+ (Q_c^-) is a smoothly embedded submanifold with boundary $\bigsqcup_{p_i} S_{p_i}^+ \times D_{p_i}$ ($\bigsqcup_{p_i} A_{p_i} \times S_{p_i}^-$) of $M_c^+ \times M(c)$ ($M(c) \times M_c^-$).*

Proof. We only need to prove the case of Q_c^+.

Let $\tilde{Q}_c^+ = \{(x^+, z) \in Q_c^+ \mid f(z) \in (c_i - \frac{\epsilon}{2}, c_i + \frac{\epsilon}{2})\}$. If we shrink $M(c)$ by an isotopy along flow lines, we get a diffeomorphism from $M(c)$ to $f^{-1}((c_i - \frac{\epsilon}{2}, c_i + \frac{\epsilon}{2}))$. This diffeomorphism preserves flow lines. Thus it induces a diffeomorphism from Q_c^+ to $\tilde{Q}_c^+$. Then we only need to prove that $\tilde{Q}_c^+$ is a submanifold of $M_c^+ \times f^{-1}((c_i - \frac{\epsilon}{2}, c_i + \frac{\epsilon}{2}))$. We can therefore assume $M(c) = f^{-1}((c_i - \frac{\epsilon}{2}, c_i + \frac{\epsilon}{2}))$.

The proof is very similar to that of Lemma 4.1. We assume there is only one critical point in $M(c)$.

Firstly, we prove $Q_c^+ - S_p^+ \times D_p$ is a smooth embedding submanifold of $M_c^+ \times M(c)$. There is a smooth map $\varphi : M(c) - D_p \longrightarrow M_c^+ \times M(c)$ such that $\varphi(z) = (\psi(z), z)$, where ψ is the flow map from $M(c) - D_p$ to M_c^+. Similarly to Lemma 4.1, φ is also a smooth embedding. This gives the proof.

Secondly, we shall find an open neighborhood W of $S_p^+ \times D_p$ such that $Q_c^+ \cap W$ is a smoothly embedded submanifold with boundary $S_p^+ \times D_p$ of $M_c^+ \times M(c)$.

Just as the proof of Lemma 4.1, we use the same notation of h, B, U and U_+, identify U with B, and we define $\widetilde{U}_- = \{(v_1, v_2) \mid \|v_2\|^2 < \epsilon \text{ and } \|v_1\|^2 < 2\epsilon.\}$. Define $\varphi : S_p^+ \times D_p \times [0,1) \longrightarrow U_+ \times \widetilde{U}_-$ by

$$\varphi(v_2, v_1, s) = \left((sv_1, (s^2\|v_1\|^2 + \epsilon)^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}v_2), (v_1, s(s^2\|v_1\|^2 + \epsilon)^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}v_2)\right).$$

Define $\alpha : U_+ \times \widetilde{U}_- \longrightarrow S_p^+ \times D_p \times [0,1)$ by

$$\alpha((z_1, z_2), (z_3, z_4)) = \left(\epsilon^{\frac{1}{2}}\frac{z_2}{\|z_2\|}, z_3, \frac{\|z_4\|}{\|z_2\|}\right).$$

Then $\alpha\varphi = \mathrm{Id}$. Similar to the proof of Lemma 4.1, φ is a homeomorphism. And $d\varphi$ is an isomorphism to its image when $s \neq 0$. When $s = 0$,

$$d\varphi|_{s=0}\left(\frac{\partial}{\partial s}\right) = (v_1, 0, 0, v_2), \tag{4.2}$$

$$d\varphi|_{s=0}(e_1) = (0, 0, e_1, 0), \qquad d\varphi|_{s=0}(e_2) = (0, e_2, 0, 0),$$

and $d\varphi$ is also an isomorphism to its image. Thus φ is a smooth embedding. Let $W = (U_+ \times \widetilde{U}_-) \cap (M_c^+ \times M(c))$. This finishes the proof. □

We shall cut out a submanifold with corners from a manifold with corners. This requires a result about transversality. (See [45, II. E] for more details about transversality on Hilbert manifolds.) First we recall a classical result about manifold with boundary. Suppose L is a Hilbert manifold with boundary, and N_1 and N_2 are Hilbert manifolds. Assume N_2 is an embedded submanifold of N_1. Suppose $g : L \longrightarrow N_1$ is a smooth manifold transversal to N_2 both in $L^\circ = L - \partial L$ and in ∂L. Then $g^{-1}(N_2)$ is an embedded submanifold with boundary inside L, and $\partial g^{-1}(N_2) = g^{-1}(N_2) \cap \partial L$. Now we extend this result to the product of manifolds with boundary. Suppose L_i $(i = 1, \cdots, n)$ are Hilbert manifolds with boundary. Then $\prod_{i=1}^n L_i$ is a Hilbert manifold with corners. Its k-stratum is just $\partial^k \prod_{i=1}^n L_i = \bigsqcup_{|\Lambda|=k} (\prod_{i\in\Lambda} \partial L_i \times \prod_{i\notin\Lambda} L_i^\circ)$, where Λ is a subset of $\{1, \cdots, n\}$. The above extends Definitions 2.17 and 2.18. We have the following result, whose proof is a straightforward extension of that in the case of a manifold with boundary.

Lemma 4.3. *If $g : \prod_{i=1}^n L_i \longrightarrow N_1$ is transversal to N_2 in each stratum of $\prod_{i=1}^n L_i$, then $g^{-1}(N_2)$ is a smoothly embedded submanifold with corners of $\prod_{i=1}^n L_i$ such that $\partial^k g^{-1}(N_2) = g^{-1}(N_2) \cap \partial^k \prod_{i=1}^n L_i$.*

4.2 Compactfied Spaces of $\mathcal{M}(p, q)$

The compactification of $\mathcal{M}(p, q)$ is standard. Define the compactified space of $\mathcal{M}(p, q)$ as

$$\overline{\mathcal{M}(p,q)} = \bigsqcup_I \mathcal{M}_I, \tag{4.3}$$

where the disjoint union is over all critical sequences with head p and tail q (see Definition 2.13).

We can give $\overline{\mathcal{M}(p,q)}$ another equivalent definition which is sometimes more convenient. If $\alpha \in \mathcal{M}_I \subseteq \overline{\mathcal{M}(p,q)}$, then $\alpha = (\gamma_0, \cdots, \gamma_k)$, where $\gamma_i \in \mathcal{M}(r_i, r_{i+1})$, $r_0 = p$ and $r_{k+1} = q$. Denote the constant flow line passing through r_i by $\beta(r_i)$. We can identify α with the generalized flow line $(\beta(r_0), \gamma_0, \beta(r_1), \cdots, \gamma_k, \beta(r_{k+1}))$ connecting p with q. Thus we get

$$\overline{\mathcal{M}(p,q)} = \{\Gamma \mid \Gamma \text{ is a generalized flow line connecting } p \text{ with } q\}.$$

Suppose $\alpha \in \mathcal{M}_I \subseteq \overline{\mathcal{M}(p,q)}$. Then $\alpha = (\gamma_0, \cdots, \gamma_k)$, where $\gamma_i \in \mathcal{M}(r_i, r_{i+1})$, $r_0 = p$ and $r_{k+1} = q$. By Condition (C), there are only finitely many critical values in $[f(q), f(p)]$. Suppose the critical values of f divide $[f(q), f(p)]$ into $l+1$ intervals $[c_{i+1}, c_i]$ $(i = 0, \cdots, l)$, where $c_0 = f(p)$ and $c_{l+1} = f(q)$. For all $a_i \in (c_{i+1}, c_i)$, they are regular. The union of the components of α intersects with $f^{-1}(a_i)$ at exactly one point $x_i(\alpha)$. There is an evaluation map $E : \overline{\mathcal{M}(p,q)} \longrightarrow \prod_{i=0}^{l} f^{-1}(a_i)$ such that

$$E(\alpha) = (x_0(\alpha), \cdots, x_l(\alpha)). \tag{4.4}$$

If $\alpha_1 \in \prod_{i=0}^{j-1} \mathcal{M}(r_i, r_{i+1}) \subseteq \overline{\mathcal{M}(r_0, r_j)}$ and $\alpha_2 \in \prod_{i=j}^{k} \mathcal{M}(r_i, r_{i+1}) \subseteq \overline{\mathcal{M}(r_j, r_k)}$, then $(\alpha_1, \alpha_2) \in \prod_{i=0}^{k} \mathcal{M}(r_i, r_{i+1}) \subseteq \overline{\mathcal{M}(r_0, r_k)}$. This gives a map $i_{(p,r,q)} : \overline{\mathcal{M}(p,r)} \times \overline{\mathcal{M}(r,q)} \longrightarrow \overline{\mathcal{M}(p,q)}$. We shall prove the following theorem.

Theorem 4.4 (Smooth Structure of $\overline{\mathcal{M}(p,q)}$)**.** *Let (M, f) be a CF pair satisfying transversality and having a locally trivial metric. Then, for each pair of critical points (p,q), there is a smooth structure on $\overline{\mathcal{M}(p,q)}$ which satisfies the following properties.*

(1). It is a compact manifold with faces whose k-stratum is exactly $\bigsqcup_{|I|=k} \mathcal{M}_I$, where the

disjoint union is over all critical sequences I with head p and tail q.

(2). The smooth structure is compatible with that of $\mathcal{M}_I$ in each stratum.

(3). The evaluation map $E : \overline{\mathcal{M}(p,q)} \longrightarrow \prod_{i=0}^{l} f^{-1}(a_i)$ is a smooth embedding, where E is defined by (4.4).

(4). The smooth structures are compatible with critical pairs, i.e., $i_{(p,r,q)} : \overline{\mathcal{M}(p,r)} \times \overline{\mathcal{M}(r,q)} \longrightarrow \overline{\mathcal{M}(p,q)}$ is a smooth embedding.

Proof. (1) & (2). We only prove the corner structure now. The face structure will follow from (4). Suppose the critical values in $[f(q), f(p)]$ are exactly $c_{l+1} < \cdots < c_1 < c_0$, where $c_0 = f(p)$ and $c_{l+1} = f(q)$. Define

$$P = \prod_{i=1}^{l} P_i, \quad R = S_p^- \times \prod_{i=1}^{l-1} M_i^- \times S_q^+, \quad O = \prod_{i=1}^{l} (M_i^+ \times M_i^-).$$

Here $P_i = P_{c_i}$, $M_i^+ = M_{c_i}^+$, $M_i^- = M_{c_i}^-$, $S_p^- = \mathcal{D}(p) \cap M_0^-$ and $S_q^+ = \mathcal{A}(q) \cap M_{l+1}^+$ are defined as before Lemma 4.1. By Lemma 4.1, P is a manifold with corners whose k-stratum is exactly the disjoint union of $\prod_{i=1}^{k} (S_{r_i}^+ \times S_{r_i}^-) \times \prod_{j \notin \Lambda_I} P_j^\circ$, where $I = \{p, r_1, \cdots, r_k, q\}$ is a critical sequence and $\Lambda_I = \{j \mid c_j = f(r_i), i = 1, \cdots, k\}$. Clearly, P is a submanifold of O, so there is an inclusion $\iota : P \longrightarrow O$. On the other hand, define a smooth embedding $\Delta : R \longrightarrow O$ as follows. Since there is no critical point in $M^{c_{i+1}+\frac{\epsilon}{2}, c_i - \frac{\epsilon}{2}}$, by Corollary 2.5, we have a flow map $\psi_i : M_i^- \longrightarrow M_{i+1}^+$. Define

$$\Delta(y_0^-, y_1^-, \cdots, y_{l-1}^-, y_{l+1}^+) = (\psi_0 y_0^-, y_1^-, \psi_1 y_1^-, \cdots, y_{l-1}^-, \psi_{l-1} y_{l-1}^-, \psi_{l+1}^{-1} y_{l+1}^+).$$

Now we point out that ι is transversal to Δ in each stratum of P. When M is compact, transversality is proved by [11, thm. 1]. (The paper [11] uses different notations from ours. Its $\mathcal{P}$, $\mathcal{S}$ and $\mathcal{O}$ are our P, R and O respectively. Its maps p and s are our ι and Δ respectively.) The proof needs Corollary 2.5 which is trivial in the compact case. Our proof of the transverality duplicates that in [11], so we omit it.

Denote $K = \iota^{-1}(\mathrm{Im}(\Delta))$. By Lemma 4.3, K is a smoothly embedded submanifold of P whose k-stratum is exactly the intersection of K with the k-stratum of P.

Now we identify the strata of K with the disjoint unions of $\mathcal{M}(p, r_1) \times \mathcal{M}(r_1, r_2) \times \cdots \times \mathcal{M}(r_k, q)$ as smooth manifolds.

It's easy to see that

$$\begin{aligned} K &= \{(x_1^+, x_1^-, \cdots, x_l^+, x_l^-) \in O \mid x_i^\pm \ (1 \leq i \leq l) \\ &\qquad \text{are on a same generalized flow line connecting } p \text{ and } q\}. \end{aligned} \tag{4.5}$$

Let $I = \{p, r_1, \cdots, r_k, q\}$ be a critical sequence. For all $\Gamma \in \mathcal{M}_I$ (see (**??**)), Γ intersects $M_i^\pm$ at exactly one point $x_i^\pm(\Gamma)$. Thus there is also an evaluation map $\tilde{E}_I : \mathcal{M}_I \longrightarrow O$ such that $\tilde{E}_I(\Gamma) = (x_1^+(\Gamma), \cdots, x_l^-(\Gamma))$. Clearly, $\tilde{E}_I$ is a smooth embedding, and $\mathrm{Im}(\tilde{E}_I)$ is exactly $K \cap (\prod_{i=1}^k (S_{r_i}^+ \times S_{r_i}^-) \times \prod_{j \notin \Lambda_I} P_j^\circ)$ which is an open subset of the k-stratum of K. This gives an identification preserving smooth structures.

As a result, identifying $\overline{\mathcal{M}(p,q)}$ with K, we give $\overline{\mathcal{M}(p,q)}$ a smooth structure which is compatible with the smooth structure of $\prod_{i=0}^k \mathcal{M}(r_i, r_{i+1})$ for all critical sequences and its k-stratum is exactly $\bigsqcup_{|I|=k} \mathcal{M}_I$.

Now we prove the compactness of $\overline{\mathcal{M}(p,q)}$.

By (4.5), for all $\{x_n\}_{n=1}^\infty \subseteq K$, $x_n = (x_{n,1}^+, x_{n,1}^-, \cdots, x_{n,l}^+, x_{n,l}^-)$, $x_{n,i}^\pm \in M_i^\pm$ and are on a same generalized flow line connecting p and q. By Theorem 3.2, $\{x_n\}$ has a cluster point $x_0 = (x_1^+, x_1^-, \cdots, x_l^+, x_l^-)$, and $x_i^\pm$ are on a same generalized flow line connecting p and q. Since $M_i^\pm$ is closed, $x_i^\pm \in M_i^\pm$ or $x_0 \in K$. So K and then $\overline{\mathcal{M}(p,q)}$ are compact.

(3). Since a_i, $c_i - \frac{\epsilon}{2}$ and $c_{i+1} + \frac{\epsilon}{2}$ are in (c_{i+1}, c_i) and there is no critical value in (c_{i+1}, c_i), by Corollary 2.5, the flow map gives a smooth map from $f^{-1}(a_i)$ to $M_i^- \times M_{i+1}^+$. This induces a map $\varphi : \prod_{i=0}^{l} f^{-1}(a_i) \longrightarrow O$. Clearly, $\varphi \circ E : \overline{\mathcal{M}(p,q)} \longrightarrow O$ is exactly the inclusion if we identify $\overline{\mathcal{M}(p,q)}$ with K. So $\varphi \circ E$ and then E are smooth embeddings.

(4). Suppose $f(r) = c_k$. By (3), we have the following commutative diagram. Here $E_{p,q}$, $E_{p,r} : \overline{\mathcal{M}(p,r)} \longrightarrow \prod_{i=0}^{k-1} f^{-1}(a_i)$, and $E_{r,q} : \overline{\mathcal{M}(r,q)} \longrightarrow \prod_{i=k}^{l} f^{-1}(a_i)$ are evaluation maps.

$$\begin{tikzcd}
\overline{\mathcal{M}(p,r)} \times \overline{\mathcal{M}(r,q)} \arrow[r, "E_{p,r} \times E_{r,q}"] \arrow[d, "i_{(p,r,q)}"'] & \prod_{i=0}^{l} f^{-1}(a_i) \\
\overline{\mathcal{M}(p,q)} \arrow[ur, "E_{p,q}"'] &
\end{tikzcd}$$

Also by (3), the above three evaluation maps are smooth embeddings. Then so is $i_{(p,r,q)}$. This completes the proof of (4).

Finally, we establish the face structure of $\overline{\mathcal{M}(p,q)}$. Suppose x is in the k-stratum. Then $x \in \mathcal{M}_I$ for some $I = \{p, r_1, \cdots, r_k, q\}$. Thus $x \in \overline{\mathcal{M}(p,r_i)} \times \overline{\mathcal{M}(r_i,q)}$ for $i = 1, \cdots, k$. Clearly, $\mathcal{M}(p,r_i) \times \mathcal{M}(r_i,q)$ are k pairwise disjoint open subsets of the 1-stratum. We only need to prove that their closures are $\overline{\mathcal{M}(p,r_i)} \times \overline{\mathcal{M}(r_i,q)}$ respectively. On the one hand, since it is compact, $\overline{\mathcal{M}(p,r_i)} \times \overline{\mathcal{M}(r_i,q)}$ contains the closure of $\mathcal{M}(p,r_i) \times \mathcal{M}(r_i,q)$ in $\overline{\mathcal{M}(p,q)}$. On the other hand, as $\mathcal{M}(p,r_i) \times \mathcal{M}(r_i,q)$ is the 0-stratum (the interior) of $\overline{\mathcal{M}(p,r_i)} \times \overline{\mathcal{M}(r_i,q)}$, we infer that the closure of $\mathcal{M}(p,r_i) \times \mathcal{M}(r_i,q)$ contains $\overline{\mathcal{M}(p,r_i)} \times \overline{\mathcal{M}(r_i,q)}$. Thus the closure is exactly $\overline{\mathcal{M}(p,r_i)} \times \overline{\mathcal{M}(r_i,q)}$. $\square$

4.3 Compactfied Spaces of $\mathcal{D}(p)$

Define the compactified space of $\mathcal{D}(p)$ as

$$\overline{\mathcal{D}(p)} = \bigsqcup_I \mathcal{D}_I, \tag{4.6}$$

where the disjoint union is over all critical sequences with head p.

We can also give $\overline{\mathcal{D}(p)}$ another equivalent definition. Suppose $(\alpha, x) \in \mathcal{M}_I \times \mathcal{D}(r_k) \subseteq \overline{\mathcal{D}(p)}$. We can identify α with a generalized flow line connecting p and r_k. Adding the flow line passing through x to the above generalized flow line, we get a generalized flow line connecting p and x. The latter generalized flow line is uniquely determined by (α, x). Thus we get

$$\overline{\mathcal{D}(p)} = \{(\Gamma, x) \mid \Gamma \text{ is a generalized flow line connecting } p \text{ and } x\}. \tag{4.7}$$

We define the evaluation map $e : \overline{\mathcal{D}(p)} \longrightarrow M$ as follows. The restriction of e on $\mathcal{D}_I = \mathcal{M}_I \times \mathcal{D}(r_k)$ is just the coordinate projection $\mathcal{M}_I \times \mathcal{D}(r_k) \longrightarrow \mathcal{D}(r_k)$. This defines the map since $\mathcal{D}(r_k) \subseteq M$.

Figure 3 shows a standard example on a torus $T^2 = S^1 \times S^1$. Consider S^1 as the unit circle on the complex plane. Define a Morse function on T^2 by $f(z_1, z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$. Then f has 4 critical points p, r, s and q. Their indices are 2, 1, 1 and 0 respectively. Equip T^2 with the standard metric. The left part of Figure 3 shows the flow on T^2, where the opposite sides of the square are identified with each other. The right part is $\overline{\mathcal{D}(p)}$ which is an octagon. Here $\mathcal{M}(p,r) \times \mathcal{D}(r)$ (or $\mathcal{M}(p,s) \times \mathcal{D}(s)$) consists of open edges containing r_i (or s_i), where $i = 1, 2$. In addition, $\mathcal{M}(p,q) \times \mathcal{D}(q)$ consists of the other 4 open edges,

$(\mathcal{M}(p,r) \times \mathcal{M}(r,q) \times \mathcal{D}(q)) \cup (\mathcal{M}(p,s) \times \mathcal{M}(s,q) \times \mathcal{D}(q))$ consists of the 8 vertices, and e maps r_i (or s_i) to r (or s).

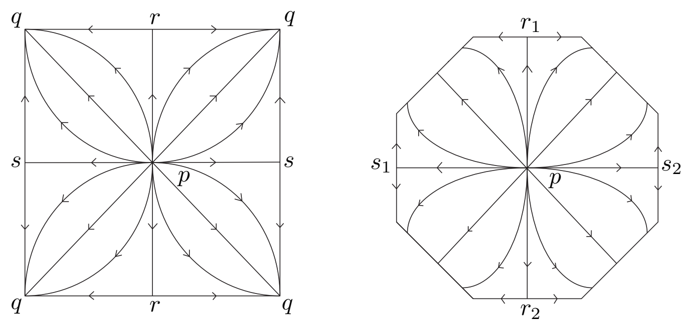

Figure 3: Compactification of the Descending Manifolds

If $\alpha_1 \in \prod_{i=0}^{j-1} \mathcal{M}(r_i, r_{i+1}) \subseteq \overline{\mathcal{M}(r_0, r_j)}$ and $(\alpha_2, x) \in \prod_{i=j}^{k} \mathcal{M}(r_i, r_{i+1}) \times \mathcal{D}(r_k) \subseteq \overline{\mathcal{D}(r_j)}$, then $(\alpha_1, \alpha_2, x) \in \prod_{i=0}^{k} \mathcal{M}(r_i, r_{i+1}) \times \mathcal{D}(r_k) \subseteq \overline{\mathcal{D}(r_0)}$. This gives a map $i_{(p,r)} : \overline{\mathcal{M}(p,r)} \times \overline{\mathcal{D}(r)} \longrightarrow \overline{\mathcal{D}(p)}$.

If we assume f is bounded below, then, by Theorem 2.1, the critical points in $(-\infty, f(p)]$ are finitely many. Suppose they are $f(p) = c_0 > c_1 > \cdots > c_l$. Choose $a_i \in (c_{i+1}, c_i)$. Define $U(i) \subseteq \mathcal{D}(p)$ as

$$U(i) = \{(\Gamma, x) \mid c_{i+1} < f(x) < c_{i-1}\}, \tag{4.8}$$

where $c_{-1} = +\infty$ and $c_{l+1} = -\infty$. Define the intersection of Γ with $f^{-1}(a_j)$ as $x_j(\Gamma)$. Define $E(i) : U(i) \to \prod_{j=0}^{i-1} f^{-1}(a_j) \times M$ as

$$E(i)(\Gamma, x) = (x_0(\Gamma), x_1(\Gamma), \cdots, x_{i-1}(\Gamma), x). \tag{4.9}$$

Theorem 4.5 (Smooth Structure of $\overline{\mathcal{D}(p)}$)**.** *Under the assumptions of Theorem 4.4, suppose f has a lower bound. Then, for each critical point p, there is a smooth structure on $\overline{\mathcal{D}(p)}$*

satisfying the following properties.

(1). It is a compact manifold with faces whose k-stratum is exactly $\bigsqcup_{|I|=k-1} \mathcal{D}_I$ where the disjoint union is over all critical sequences with head p.

(2). The smooth structure is compatible with that of $\mathcal{D}_I$ in each stratum.

(3). The evaluation map $e : \overline{\mathcal{D}(p)} \longrightarrow M$ is smooth, where the restriction of e on $\mathcal{D}_I = \mathcal{M}_I \times \mathcal{D}_{r_k}$ is the coordinate projection onto $\mathcal{D}_{r_k} \subseteq M$.

(4). Each $U(i)$ is an open subset of $\overline{\mathcal{D}(p)}$ and $E(i) : U(i) \to \prod_{j=0}^{i-1} f^{-1}(a_j) \times M$ is a smooth embedding.

(5). The smooth structures are compatible with critical pairs, i.e., $i_{(p,r)} : \overline{\mathcal{M}(p,r)} \times \overline{\mathcal{D}(r)} \longrightarrow \overline{\mathcal{D}(p)}$ is a smooth embedding, where the smooth structure of $\overline{\mathcal{M}(p,r)}$ is defined in Theorem 4.4.

Remark 4.1. *It's easy to see that Theorem 4.5 will not be true if we don't assume that f is bounded below.*

Proof. Denote $M(c_i)$ by $M(i)$, P_{c_i} by P_i and $Q_{c_i}^+$ by Q_i^+, where $M(c_i)$, P_{c_i} and $Q_{c_i}^+$ are as defined before Lemma 4.1. Clearly $U(i) = e^{-1}(M(i))$.

(1), (2) & (3). We shall give each $U(i)$ a smooth structure, and show that $U(i) \cap U(j)$ is open in both $U(i)$ and $U(j)$ and smooth structures are compatible in $U(i) \cap U(j)$.

Firstly, when $i = 0$, $U(0)$ is identified with $\mathcal{D}(p) \cap M(0)$. $\mathcal{D}(p) \cap M(0)$ is a smooth embedded submanifold of M. Thus $U(0)$ has a smooth structure by this identification.

Secondly, when $i > 0$, let $Q(i) = \prod_{j=1}^{i-1} P_j \times Q_i^+$, $O(i) = \prod_{j=1}^{i-1}(M_j^+ \times M_j^-) \times M_i^+$ and $R(i) = S_p^- \times \prod_{j=1}^{i-1} M_j^-$. We know that, $\forall x \in Q(i)$, $x = (x_1^+, x_1^-, \cdots, x_{i-1}^-, x_i^+, z_i)$, where

$x_j^{\pm} \in M_j^{\pm}$ and $z_i \in M(i)$. Define a smooth map $\iota_i : Q(i) \longrightarrow O(i)$ by

$$\iota_i(x_1^+, x_1^-, \cdots, x_{i-1}^-, x_i^+, z_i) = (x_1^+, x_1^-, \cdots, x_{i-1}^-, x_i^+).$$

Define a smooth embedding $\Delta_i : R(i) \longrightarrow O(i)$ by

$$\Delta_i(y_0^-, y_1^-, \cdots, y_{i-1}^-) = (\psi_0 y_0^-, y_1^-, \psi_1 y_1^-, \cdots, y_{i-1}^-, \psi_{i-1} y_{i-1}^-),$$

where ψ_j is the flow map from M_j^- to M_{j+1}^+.

As in the proof of Theorem 4.4, we point out that ι_i is transversal to Δ_i in each stratum of $Q(i)$. The proof is similar to that of Theorem 4.4.

Thus $\tilde{U}(i) = \iota_i^{-1}(\mathrm{Im}(\Delta_i))$ is a smooth embedding submanifold of $Q(i)$ whose k-stratum is exactly the intersection of $\tilde{U}(i)$ with the k-stratum of $Q(i)$.

Now we identify $U(i)$ with $\tilde{U}(i)$. It's easy to see that

$$\begin{aligned} \tilde{U}(i) \;=\; & \{(x_1^+, x_1^-, \cdots, x_{i-1}^-, x_i^+, z_i) \in O(i) \times M(i) \mid x_j^{\pm} \\ & \text{are on a same generalized flow line connecting } p \text{ and } z_i.\}. \end{aligned} \tag{4.10}$$

Let $I = (p, r_1, \cdots, r_k)$ be a critical sequence. For any element $(\Gamma, x) \in \mathcal{D}_I \cap U(i)$ (see (4.7)), Γ intersects $M_j^{\pm}$ at exactly one point $x_j^{\pm}(\Gamma)$. Thus there is an evaluation map $\tilde{E}_I : \mathcal{D}_I \cap U(i) \longrightarrow O(i) \times M(i)$ such that $\tilde{E}_I(\Gamma, x) = (x_1^+(\Gamma), x_1^-(\Gamma), \cdots, x_i^+(\Gamma), x)$. Similar to the identification of K with $\overline{\mathcal{M}(p,q)}$ in the proof of Theorem 4.4, this also identifies $U(i)$ with $\tilde{U}(i)$ and preserves the smooth structure of the strata. So we get a desired smooth structure on U_i.

In each $\tilde{U}(i)$, define $\tilde{e}_i : \tilde{U}(i) \longrightarrow M$ by $\tilde{e}_i(x_1^+, \cdots, x_i^+, z_i) = z_i$, then $\tilde{e}_i$ is smooth. When we identify $U(i)$ with $\tilde{U}(i)$, we have $e|_{U(i)} = \tilde{e}_i$. Thus $e|_{U(i)}$ is smooth.

Now we check the compatibility of smooth structures for all $U(i)$ $(0 \le i \le l)$. Clearly, if $|i-j| > 1$, then $U(i) \cap U(j) = \emptyset$. We only need to check the compatibility of $U(i)$ and $U(i+1)$.

Denote $M(i)^- = f^{-1}((c_{i+1}, c_i))$. For clarity, when we consider $U(i) \cap U(i+1)$ as a topological subspace of $U(i)$ (or $U(i+1)$), we denote it by $U(i, i+1)$ (or $U(i+1, i)$). Since $U(i, i+1) = e|_{U(i)}^{-1}(M(i)^-)$, it is an open subset of $U(i)$. Furthermore, $U(i+1, i)$ is an open subset of $U(i+1)$. When $i \ge 1$, $U(i, i+1) \subseteq \prod_{j=1}^{i-1}(M_j^+ \times M_j^-) \times M_i^+ \times M(i)^-$ and $U(i+1, i) \subseteq \prod_{j=1}^{i}(M_j^+ \times M_j^-) \times M_{i+1}^+ \times M(i)^-$. Define $\pi : \prod_{j=1}^{i}(M_j^+ \times M_j^-) \times M_{i+1}^+ \times M(i)^- \longrightarrow \prod_{j=1}^{i-1}(M_j^+ \times M_j^-) \times M_i^+ \times M(i)^-$ be the natural projection. Define $\varphi : \prod_{j=1}^{i-1}(M_j^+ \times M_j^-) \times M_i^+ \times M(i)^- \longrightarrow \prod_{j=1}^{i}(M_j^+ \times M_j^-) \times M_{i+1}^+ \times M(i)^-$ such that

$$\varphi(x_1^+, x_1^-, \cdots, x_{i-1}^-, x_i^+, z_i) = (x_1^+, x_1^-, \cdots, x_{i-1}^-, x_i^+, \psi_-(z_i), \psi_+(z_i), z_i),$$

where ψ_- and ψ_+ are flow maps from $M(i)^-$ to M_i^- and M_{i+1}^+ respectively. Then $\pi(U(i+1, i)) = U(i, i+1)$, $\varphi(U(i, i+1)) = U(i+1, i)$, $\pi\varphi|_{U(i,i+1)} = \mathrm{Id}$, and $\varphi\pi|_{U(i+1,i)} = \mathrm{Id}$. Thus π and φ are diffeomorphisms between $U(i, i+1)$ and $U(i+1, i)$, and they are the identity on the set $U(i) \cap U(i+1)$. Thus $U(i)$ and $U(i+1)$ have compatible smooth structures when $i \ge 1$.

Similarly, $U(0, 1) \subseteq M(0)^-$ and $U(1, 0) \subseteq M_1^+ \times M(0)^-$, and $U(0, 1)$ and $U(1, 0)$ also have compatible smooth structures.

As a result, we can patch the smooth structures on all $U(i)$ together to give a smooth structure on $\overline{\mathcal{D}(p)}$ satisfying all properties of (1) and (2) but the face structure and compactness. Similar to Theorem 4.4, the face structure will follow from (4). Also e is smooth since $e|_{U(i)}$ is smooth. This proves (3).

Finally, we prove compactness.

Let $K(i) = e^{-1}(L(i))$, where $L(i) = f^{-1}([\frac{c_{i+1}+c_i}{2}, \frac{c_i+c_{i-1}}{2}])$. Then $L(i)$ is closed. Similar to proving the compactness of K in the proof of Theorem 4.7, we get $K(i)$ is compact. Thus $\overline{\mathcal{D}(p)}$ is compact because $\overline{\mathcal{D}(p)} = \bigcup_{i=0}^{l} K(i)$.

This completes the proof of (1), (2) and (3).

(4). Similar to the proof of (3) of Theorem 4.4, we can prove (4).

(5). Clearly, $i_{(p,r)}$ is one to one. Suppose $f(r) = c_m$. For clarity, denote the evaluation map from $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{D}(r)}$ to M by e_p and e_r respectively. Let $U_p(k) = e_p^{-1}(M(k))$ and $U_r(k) = e_r^{-1}(M(k))$. Since $\overline{\mathcal{M}(p,r)} \times \overline{\mathcal{D}(r)}$ is compact, we only need to prove that $i_{(p,r)}(k) : \overline{\mathcal{M}(p,r)} \times U_r(k) \longrightarrow \overline{\mathcal{D}(p)}$ is a smooth embedding for $k \geq m$.

By (4), $E_r(k) : U_r(k) \longrightarrow \prod_{j=m}^{k-1} f^{-1}(a_j) \times M$ and $E_p(k) : U_p(k) \longrightarrow \prod_{j=0}^{k-1} f^{-1}(a_j) \times M$ are smooth embeddings. By (3) of Theorem 4.4, we know that $E_{p,r} : \overline{\mathcal{M}(p,r)} \longrightarrow \prod_{j=0}^{m-1} f^{-1}(a_j)$ is also a smooth embedding. Thus $E_{p,r} \times E_r(k) : \overline{\mathcal{M}(p,r)} \times U_r(k) \longrightarrow \prod_{i=0}^{k-1} f^{-1}(a_i) \times M$ is a smooth embedding. In addition, $E_{p,r} \times E_r(k) = E_p(k) \circ i_{(p,r)}(k)$. Thus $i_{(p,r)}(k)$ is a smooth embedding.

Finally, $\overline{\mathcal{D}(p)}$ has $\mathcal{M}(p,r) \times \mathcal{D}(r)$ which are disjoint open subsets of 1-stratum. Their closures are $\overline{\mathcal{M}(p,r)} \times \overline{\mathcal{D}(r)}$. This gives the face structure of (1). $\square$

4.4 Compactfied Spaces of $\mathcal{W}(p,q)$

First, we introduce some notation. Suppose $I_1 = (p, r_1, \cdots, r_s)$ and $I_2 = (r_{s+1}, \cdots, r_k,\ q)$ are critical sequences (see Definition 2.13) and $r_s \succeq r_{s+1}$. Let $(I,s) = (p, r_1, \cdots, q)$. It is not necessarily a critical sequence since r_s may equal r_{s+1}. Denote the following product manifold by $\mathcal{W}_{I,s}$.

$$\mathcal{W}_{I,s} = \mathcal{M}_{I_1} \times \mathcal{W}(r_s, r_{s+1}) \times \mathcal{M}_{I_2}. \tag{4.11}$$

Define the compactified space of $\mathcal{W}(p,q)$ as

$$\overline{\mathcal{W}(p,q)} = \bigsqcup_{(I,s)} \mathcal{W}_{I,s}, \tag{4.12}$$

where the disjoint union is over all $(I,s) = (p, r_1, \cdots, r_k, q)$ such that $p \succ r_1 \succ \cdots \succ r_s \succeq r_{s+1} \succ \cdots \succ r_k \succ q$ for all k.

We can also give $\overline{\mathcal{W}(p,q)}$ another equivalent definition which is

$$\overline{\mathcal{W}(p,q)} = \{(\Gamma, x) \mid \Gamma \in \overline{\mathcal{M}(p,q)},\ x \text{ is on } \Gamma\}. \tag{4.13}$$

If $(\Gamma_1, x) \in \overline{\mathcal{W}(p,r)}$ and $\Gamma_2 \in \overline{\mathcal{M}(r,q)}$, then the combination of Γ_1 and Γ_2 gives an element in $\overline{\mathcal{M}(p,q)}$ and x is on it. This defines the map $i^1_{(p,r,q)}$ in (4) of Theorem 4.6. $i^2_{(p,r,q)}$ is defined in a similar way.

Suppose the critical values of f divide $[f(q), f(p)]$ into $l+1$ intervals $[c_{i+1}, c_i]$ $(i = 0, \cdots, l)$, where $c_0 = f(p)$ and $c_{l+1} = f(q)$. choose $a_i \in (c_{i+1}, c_i)$.

Define the map $\widetilde{E} : \overline{\mathcal{W}(p,q)} \to \prod_{i=0}^{l} f^{-1}(a_i) \times M$ as

$$\widetilde{E}(\Gamma, x) = (E(\Gamma), x), \tag{4.14}$$

where E is defined in (3) of Theorem 4.4.

Define $i : \overline{\mathcal{W}(p,q)} \to \overline{\mathcal{M}(p,q)} \times M$ as the natural inclusion.

Theorem 4.6 (Smooth Structure of $\overline{\mathcal{W}(p,q)}$)**.** *Under the assumptions of Theorem 4.4, for each pair of critical points (p,q), there is a smooth structure on $\overline{\mathcal{W}(p,q)}$ satisfying the following properties.*

(1). It is a compact manifold with faces whose k-stratum is exactly $\bigsqcup_{(I,s)} \mathcal{W}_{I,s}$. Here $(I,s) = (p, r_1, \cdots, r_k, q)$ such that $p \succ r_1 \succ \cdots \succ r_s \succeq r_{s+1} \succ \cdots \succ r_k \succ q$. The disjoint union is over all (I,s) which contain $k+2$ components.

(2). The smooth structure is compatible with that of $\mathcal{W}_{I,s}$ in each stratum.

(3). The maps $i : \overline{\mathcal{W}(p,q)} \to \overline{\mathcal{M}(p,q)} \times M$ and $\widetilde{E} : \overline{\mathcal{W}(p,q)} \to \prod_{i=0}^{l} f^{-1}(a_i) \times M$ are smooth embeddings.

(4). The smooth structures are compatible with critical pairs, i.e., $i^1_{(p,r,q)} : \overline{\mathcal{W}(p,r)} \times \overline{\mathcal{M}(r,q)} \longrightarrow \overline{\mathcal{W}(p,q)}$ and $i^2_{(p,r,q)} : \overline{\mathcal{M}(p,r)} \times \overline{\mathcal{W}(r,q)} \longrightarrow \overline{\mathcal{W}(p,q)}$ are smooth embeddings.

Here the smooth structure of $\overline{\mathcal{M}(*,*)}$ is defined in Theorem 4.4.

The proof of Theorem 4.6 is a mixture of the proofs of Theorems 4.4 and 4.5. Thus we only need to give the key constructions in the proof. Just as the proofs of the previous two theorems, we still use the notation $M(k)$, P_k and $Q_k^{\pm}$. Suppose the critical values in $[f(q), f(p)]$ are exactly $f(q) = c_{l+1} < \cdots < c_0 = f(p)$. Define $U(i) \subseteq \overline{\mathcal{W}(p,q)}$ as

$U(k) = e^{-1}(M(k))$. Use the notation of $S_p^{\pm}$, D_p and A_p as those appearing before Lemma 4.1.

Proof. Similarly to the proof of Theorem 4.5, we shall give each $U(k)$ a smooth structure and then patch them together.

Define $Q(0) = Q_0^- \times \prod_{j=1}^{l} P_j$, $R(0) = D_p \times \prod_{j=0}^{l-1} M_j^- \times S_q^+$ and $O(0) = M(0) \times M_0^- \times \prod_{j=1}^{l}(M_j^+ \times M_j^-)$. Define $\Delta_0 : R(0) \longrightarrow O(0)$ by

$$\Delta_0(z_0, y_0^-, \cdots, y_{l-1}^-, y_{l+1}^+) = (z_0, y_0^-, \psi_0 y_0^-, \cdots, y_{l-1}^-, \psi_{l-1} y_{l-1}^-, \psi_l^{-1} y_{l+1}^+).$$

Define $Q(l+1) = \prod_{j=1}^{l} P_j \times Q_{l+1}^+$, $R(l+1) = S_p^- \times \prod_{j=1}^{l} M_j^- \times A_q$, and $O(l+1) = \prod_{j=1}^{l}(M_j^+ \times M_j^-) \times M_{l+1}^+ \times M(l+1)$. Define $\Delta_{l+1} : R(l+1) \longrightarrow O(l+1)$ by

$$\Delta_{l+1}(y_0^-, \cdots, y_l^-, z_{l+1}) = (\psi_0 y_0^-, y_1^-, \psi_1 y_1^-, \cdots, y_l^-, \psi_l y_l^-, z_{l+1}).$$

When $1 \leq k \leq l$, define $Q(k) = \prod_{j=1}^{k-1} P_j \times Q_k^+ \times Q_k^- \times \prod_{j=k+1}^{l} P_j$, $R(k) = S_p^- \times \prod_{j=1}^{k-1} M_j^- \times M(k) \times \prod_{j=k}^{l-1} M_j^- \times S_q^+$ and $O(k) = \prod_{j=1}^{k-1}(M_j^+ \times M_j^-) \times M_k^+ \times M(k) \times M(k) \times M_k^- \times \prod_{j=k+1}^{l}(M_j^+ \times M_j^-)$. Define $\Delta_k : R(k) \longrightarrow O(k)$ by

$$\begin{aligned} & \Delta_k(y_0^-, y_1^-, \cdots, y_{k-1}^-, z_k, y_k^-, \cdots, y_{l-1}^-, y_{l+1}^+) \\ = \; & (\psi_0 y_0^-, y_1^-, \psi_1 y_1^-, \cdots, y_{k-1}^-, \psi_{k-1} y_{k-1}^-, z_k, z_k, y_k^-, \psi_k y_k^-, \cdots, \psi_{l-1} y_{l-1}^-, \psi_l^{-1} y_{l+1}^+). \end{aligned}$$

In the above, ψ_k are flow maps from M_k^- to M_{k+1}^+.

Define $\iota_k : Q(k) \longrightarrow O(k)$ to be the inclusion for all $k = 0, \cdots, l+1$.

Similar to the proof of Theorems 4.4 and 4.5, ι_k is transversal to Δ_k in each stratum of $Q(k)$. Thus $\tilde{U}(k) = \iota_k^{-1}(\mathrm{Im}(\Delta_k))$ is a smooth manifold with corners. $U(k)$ can be identified with $\tilde{U}(k)$ and the smooth structures are preserved. This gives a smooth structure to each $U(k)$.

Clearly, $\widetilde{E}|_{U(k)}$ is a smooth embedding, and $U(k)$ and $U(j)$ have compatible smooth structures. Thus $\widetilde{E}$ is a smooth embedding. By (3) of Theorem 4.4, we get the map $i : \overline{\mathcal{W}(p,q)} \to \overline{\mathcal{M}(p,q)} \times M$ is also an smooth embedding.

The face structures will follow from (4).

Let $L(k) = f^{-1}([\frac{c_{k+1}+c_k}{2}, \frac{c_k+c_{k-1}}{2}])$, then $e^{-1}(L(k))$ is compact. Thus $\mathcal{W}(p,q)$ is compact. This finishes the proof of (1), (2) and (3).

Finally, (4) is proved by an argument similar to that in (4) of Theorem 4.5.

This completes the proof. $\square$

4.5 Additional Results

We prove two results which are needed later.

First, we have the following result which follows straightforwardly from the face structure of $\overline{\mathcal{D}(p)}$ (see Definition 2.19).

Lemma 4.7. *Suppose $I = \{p, r_1, \cdots, r_k\}$ is a critical sequence and $x \in \mathcal{D}_I \subseteq \overline{\mathcal{D}(p)}$. Then there exist an open neighborhood W of x in $\mathcal{D}_I$ and a smooth map $\varphi : W \times [0,\epsilon)^k \longrightarrow \overline{\mathcal{D}(p)}$, where φ is a diffeomorphism onto an open neighborhood of x in $\overline{\mathcal{D}(p)}$ satisfying the following stratum condition. For all $y \in W$, $\rho_I = (\rho_1, \cdots, \rho_k) \in [0,\epsilon)^k$ and $J = \{p, r_{i_1}, \cdots, r_{i_s}\}$, we have $\varphi(x, \rho_I) \in \mathcal{D}_J$ if and only if $\rho_j > 0$ when $r_j \notin J$ and $\rho_j = 0$ when $r_j \in J$.*

Proof. Since $\mathcal{D}_I$ is an open subset of the k-stratum of $\overline{\mathcal{D}(p)}$, there is a smooth map $\varphi : W \times [0,\epsilon)^k \longrightarrow \overline{\mathcal{D}(p)}$ which is a diffeomorphism onto an open neighborhood of x in $\overline{\mathcal{D}(p)}$. As mentioned at the end of the proof of Theorem 4.5, $\mathcal{D}_I$ is contained in $\overline{F_i} = \overline{\mathcal{M}(p,r_i)} \times \overline{\mathcal{D}(r_i)}$, the closure of k disjoint faces $F_i = \mathcal{M}(p,r_i) \times \mathcal{D}(r_i)$ $(i = 1, \cdots, k)$. Furthermore, $W \times [0,\epsilon)^k$ also has k disjoint faces $G_i = W \times (0,\epsilon)^{i-1} \times \{0\} \times (0,\epsilon)^{k-i}$. The closure of G_i is $\overline{G_i} = W \times [0,\epsilon)^{i-1} \times \{0\} \times [0,\epsilon)^{k-i}$. Since it is a diffeomorphism, φ maps a face into a face. Choose W to be connected, then permutating the coordinates of $[0,\epsilon)^k$ if necessary, we have $\varphi(G_i) \subseteq F_i$. Thus $\varphi(\overline{G_i}) \subseteq \overline{F_i}$. Moreover, using the fact that φ is a diffeomorphism again, x is in the i-stratum if and only if $\varphi(x)$ is in the i-stratum. $\square$

Lemma 4.8. *Let $e : \overline{\mathcal{D}(p)} \longrightarrow M$ be the map in (3) of Theorem 4.5, and let $I = \{p, r_1, \cdots, r_k\}$ and $J = \{p, r_1, \cdots, r_{k-1}\}$ be critical sequences. Suppose $(\alpha, r_k) \in \mathcal{M}_I \times \mathcal{D}(r_k) = \mathcal{D}_I$. Let $\mathcal{N} \in T_{(\alpha,r_k)}(\mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})})$ represent an inward normal vector in $N_{(\alpha,r_k)}(\mathcal{D}_I, \mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})})$, and $de(\mathcal{N}) = (\mathcal{N}_1, \mathcal{N}_2) \in V_- \times V_+ = T_{r_k}M$. Then $\mathcal{N}_2 \neq 0$. (Here $N_{(\alpha,r_k)}(\mathcal{D}_I, \mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})}) = \dfrac{T_{(\alpha,r_k)}\mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})}}{T_{(\alpha,r_k)}\mathcal{D}_I}$ is the normal space of $\mathcal{D}_I$ in $\mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})}$, and de is the derivative of e.)*

Proof. Suppose the critical values in $(-\infty, f(p)]$ are exactly $c_0 > c_1 > \cdots > c_l$. Let $c_{-1} = +\infty$ and $c_{l+1} = -\infty$. Suppose $f(r_i) = c_{t_i}$ $(i = 1, \cdots, k)$. Recall the evaluation map e in Theorem 4.5. Let $U(t_k) = e^{-1} \circ f^{-1}((c_{t_k+1}, c_{t_k-1}))$. Then $(\alpha, r_k) \in \mathcal{D}_I \cap U(t_k)$.

Returning to the proof of Theorem 4.5, we have $U(t_k)$ is an embedded submanifold of $\prod_{i=1}^{t_k-1} P_i \times Q_{t_k}^+$. We may assume r_i is the unique critical point with function value c_{t_i}. Otherwise, replace P_{t_i} by its open subset $\{(x,y) \in P_{t_i} \mid \forall r \neq r_{t_i},\ x \notin \mathcal{A}(r) \cap M_{t_i}^+\}$ and replace $Q_{t_k}^+$ by its open subset $\{(x,y) \in Q_{t_k}^+ \mid \forall r \neq r_{t_k},\ x \notin \mathcal{A}(r) \cap M_{t_i}^+\}$ in this proof.

Denote $\mathcal{D}_I \cap U(t_k)$ by D_I, and $(\mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})}) \cap U(t_k)$ by D_J. Then $T_{(\alpha,r_k)}\mathcal{D}_I = T_{(\alpha,r_k)}D_I$ and $T_{(\alpha,r_k)}(\mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})}) = T_{(\alpha,r_k)}D_J$. Denote $\prod_{j\neq t_s} P_j^\circ \times \prod_{j<k} \partial P_{t_j} \times Q_{t_k}^+$ by H. Then $\partial H = \prod_{j\neq t_s} P_j^\circ \times \prod_{j<k} \partial P_{t_j} \times \partial Q_{t_k}^+$. Here $P_j^\circ = P_j - \partial P_j$.

Clearly, $D_I = \partial H \cap \iota_{t_k}^{-1}(\mathrm{Im}\Delta_{t_k})$ and $D_J = H \cap \iota_{t_k}^{-1}(\mathrm{Im}\Delta_{t_k})$. We have the following inclusion of pairs

$$(T_{(\alpha,r_k)}D_I, T_{(\alpha,r_k)}D_J) \longrightarrow (T_{(\alpha,r_k)}\partial H, T_{(\alpha,r_k)}H).$$

Since ι_{t_k} is transversal to Δ_{t_k} in ∂H, the above inclusion induce an isomorphism

$$N_{(\alpha,r_k)}(D_I, D_J) \longrightarrow N_{(\alpha,r_k)}(\partial H, H)$$

Thus $\mathcal{N}$ also represents an inward normal vector in $N_{(\alpha,r_k)}(\partial H, H)$.

By the proof of Lemma 4.2 (see (4.2)), another such representative element is

$$\widetilde{\mathcal{N}} = (0, \cdots, 0, (v_1, 0), (0, v_2)) \in T_{(\alpha,r_k)}\left(\prod_{j\neq t_i} P_j^\circ \times \prod_{j=1}^{k-1} \partial P_{t_j} \times Q_{t_k}^+\right) = T_{(\alpha,r_k)}H,$$

where $((v_1, 0), (0, v_2)) \in TQ_{t_k}^+ \subseteq TM_{t_k}^+ \times TM(t_k)$, and $0 \neq (0, v_2) \in V_- \times V_+ = T_{r_k}M$.

Since both $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are inward normal vectors, we have $\mathcal{N} = a\widetilde{\mathcal{N}} + w$ for some $a > 0$ and $w \in T_{(\alpha,r_k)}(\prod_{j\neq t_i} P_j^\circ \times \prod_{j=1}^{k-1} \partial P_{t_j} \times \partial Q_{t_k}^+) = T_{(\alpha,r_k)}\partial H$. Clearly,

$$w = (w_1, \cdots, w_{t_k - 1}, (0, \tilde{v}_2), (\tilde{v}_1, 0)),$$

where $(0, \tilde{v}_2) \in TS_{t_k}^+$ and $(\tilde{v}_1, 0) \in V_- \times \{0\} = T_{r_k}\mathcal{D}(r_k)$.

Since the evaluation map e on $U(t_k)$ is just the projection $\prod_{j=1}^{t_k-1} P_j \times Q_{t_k}^+ \longrightarrow Q_{t_k}^+ \subseteq$

$M(t_k)$, we have $de(\mathcal{N}) = (\tilde{v}_1, av_2)$. Thus $\mathcal{N}_2 = av_2 \neq 0$. $\square$

5 Orientations (I)

In this chapter, under the assumption of the local triviality of the metric, we discuss the orientations of the codimension 1 stratum of $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$.

5.1 Orientation Formulas

Before defining the orientations of $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$, we give a general way to get an orientation by transversality.

Suppose M_1, M_2 and M_3 are three Hilbert manifolds such that M_2 is embedded in M_3. The normal bundle of M_2 with respect to M_3 is defined as $N(M_2, M_3) = \dfrac{T_{M_2}M_3}{TM_2}$. Here $T_{M_2}M_3$ is the restriction of TM_3 on M_2. If $\varphi : M_1 \longrightarrow M_3$ is transversal to M_2, then $M_0 = \varphi^{-1}(M_2)$ is an embedded submanifold of M_1, and $d\varphi$ induces a bundle map $d\varphi : N(M_0, M_1) \longrightarrow N(M_2, M_3)$, i.e., $d\varphi$ is an isomorphism in each fiber. If M_1 is finite dimensional and oriented and $N(M_2, M_3)$ is a finite dimensional (i.e., the fiber is finite dimensional) and oriented bundle, then we can give an orientation of M_0 as follows. The orientation of $N(M_2, M_3)$ gives an orientation to $N(M_0, M_1)$ via $d\varphi$. Let $\pi : T_{M_0}M_1 \longrightarrow N(M_0, M_1)$ be the natural projection. For all $x \in M_0$, choose $\{e_{k+1}, \cdots, e_n\} \subseteq T_xM_1$ such that $\{\pi(e_{k+1}), \cdots, \pi(e_n)\}$ is a positive base of $N_x(M_0, M_1)$. Choose $\{e_1, \cdots, e_k\} \subseteq T_xM_0$ such that $\{e_1, \cdots, e_k, e_{k+1}, \cdots, e_n\}$ is a positive base of T_xM_1, then $\{e_1, \cdots, e_k\}$ gives M_0 an orientation. Clearly, this is well defined and only depends on the orientations of M_1 and $N(M_2, M_3)$.

Since $\dim(\mathcal{D}(p)) = \mathrm{ind}(p) < +\infty$, we can assign $\mathcal{D}(p)$ an orientation arbitrarily. By the above method, we can derive the orientations of $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$ provided that the orientations of $\mathcal{D}(p)$ and $\mathcal{D}(q)$ have been assigned arbitrarily.

Firstly, we can give $\mathcal{W}(p,q)$ an orientation.

Since $\mathcal{A}(q)$ is transversal to $\mathcal{D}(q)$ at q, then the orientation of $T_q\mathcal{D}(q)$ induces an orientation of $N(\mathcal{A}(q),M)$. Let $i : \mathcal{D}(p) \longrightarrow M$ be the inclusion. i is transversal to $\mathcal{A}(q)$ and $i^{-1}(\mathcal{A}(q)) = \mathcal{W}(p,q)$. The orientations of $\mathcal{D}(p)$ and $N(\mathcal{A}(q),M)$ determine an orientation of $\mathcal{W}(p,q)$.

Secondly, we can give $\mathcal{M}(p,q)$ an orientation.

Choose a regular value $a \in (-\infty, f(p))$. We give $S_p^- = \mathcal{D}(p) \cap f^{-1}(a)$ the **induced orientation** from $\mathcal{D}(p)$ as follows. For all $x \in S_p^-$, $\{e_1, \cdots, e_n\}$ is a positive base of $T_xS_p^-$ if and only if $\{-\nabla f, e_1, \cdots, e_n\}$ is a positive base of $T_x\mathcal{D}(p)$. Suppose $a \in (f(q), f(p))$. Denote $\mathcal{A}(q) \cap f^{-1}(a)$ by S_q^+. Then both S_p^- and S_q^+ are embedded submanifolds of $f^{-1}(a)$ which are transversal to each other. S_p^- has its induced orientation from $\mathcal{D}(p)$ as above. There is a natural bundle map from $N(S_q^+, f^{-1}(a))$ to $N(\mathcal{A}(q),M)$. Thus $N(S_q^+, f^{-1}(a))$ is an oriented bundle. The orientations of S_p^- and $N(S_q^+, f^{-1}(a))$ give $S_p^- \cap S_q^+ = \mathcal{W}(p,q) \cap f^{-1}(a)$ an orientation. The natural identification between $\mathcal{M}(p,q)$ and $\mathcal{W}(p,q) \cap f^{-1}(a)$ (see the comment below Definition 2.9) moves the orientation of $\mathcal{W}(p,q) \cap f^{-1}(a)$ to an orientation of $\mathcal{M}(p,q)$. Clearly, this orientation only depends on those of $\mathcal{D}(p)$ and $\mathcal{D}(q)$.

Thirdly, since $\mathcal{M}(p,q)$, $\mathcal{D}(p)$ and $\mathcal{W}(p,q)$ are the interiors of $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$ respectively, the orientation of each interior determines a unique orientation of each compactified space.

Assign orientations to descending manifolds of all critical points arbitrarily. We can consider the orientations of the 1-strata $\partial^1\overline{\mathcal{M}(p,q)}$, $\partial^1\overline{\mathcal{D}(p)}$ and $\partial^1\overline{\mathcal{W}(p,q)}$ of $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$. As unoriented manifolds, $\partial^1\overline{\mathcal{M}(p,q)} = \bigsqcup_{p \succ r \succ q} \mathcal{M}(p,r) \times \mathcal{M}(r,q)$. There are two orientations of it. First, since $\overline{\mathcal{M}(p,q)}$ has an orientation, $\mathcal{M}(p,q) \sqcup \partial^1\overline{\mathcal{M}(p,q)}$ is an oriented manifold with boundary $\partial^1\overline{\mathcal{M}(p,q)}$. For all $x \in \partial^1\overline{\mathcal{M}(p,q)}$, let $\mathcal{N}$ be an outward normal vector at x. We define an oriented base $\{e_1, \cdots, e_k\}$ of $T_x\partial^1\overline{\mathcal{M}(p,q)}$ to be positive if and only if $\{\mathcal{N}, e_1, \cdots, e_k\}$ is a positive base of $T_x(\mathcal{M}(p,q) \sqcup \partial^1\overline{\mathcal{M}(p,q)})$. We call this the **boundary orientation** of $\partial^1\overline{\mathcal{M}(p,q)}$. Second, since both $\mathcal{M}(p,r)$ and $\mathcal{M}(r,q)$ have orientations, $\mathcal{M}(p,r) \times \mathcal{M}(r,q)$ has the product orientation of these two orientations. This gives $\partial^1\overline{\mathcal{M}(p,q)}$ the **product orientation**. Similarly, we can also define the boundary orientations and the product orientations for $\partial^1\overline{\mathcal{D}(p)}$ and $\partial^1\overline{\mathcal{W}(p,q)}$.

Theorem 5.1 answers the relations between the boundary orientations and the product orientations of the above 1-strata. The proof of this theorem occupies the rest of this chapter.

Theorem 5.1 (Orientation Formulas)**.** *Under the assumption of Theorem 4.4, as oriented manifolds, we have*

(1). $\partial^1\overline{\mathcal{M}(p,q)} = \bigsqcup_{p \succ r \succ q} (-1)^{ind(p)-ind(r)}\mathcal{M}(p,r) \times \mathcal{M}(r,q)$;

(2). $\partial^1\overline{\mathcal{D}(p)} = \bigsqcup_{p \succ r} \mathcal{M}(p,r) \times \mathcal{D}(r)$, *where* f *is bounded below;*

(3). $\partial^1\overline{\mathcal{W}(p,q)} = \bigsqcup_{p \succeq r \succ q} (-1)^{ind(p)-ind(r)+1}\mathcal{W}(p,r) \times \mathcal{M}(r,q) \sqcup \bigsqcup_{p \succ r \succeq q} \mathcal{M}(p,r) \times \mathcal{W}(r,q)$.

In the above, $\partial^1\Box$ *are equipped with boundary orientations,* $\Box \times \Box$ *are equipped with product orientations.*

Remark 5.1. *The papers [3, lem. 3.4] and [35, sec. 2.14 and 2.15] announce formulas similar to (1) and (2) of Theorem 5.1 in finite dimensional case ([3] even does the Morse-Bott case). Our method to define orientations is different from theirs. Thus our formulas are different from theirs. By our definition of orientations, there is no sign in (2) of Theorem 5.1.*

5.2 Proof of (1) of Theorem 5.1

Proof. We only need to prove that, for all r,

$$\partial(\mathcal{M}(p,q)\sqcup\mathcal{M}(p,r)\times\mathcal{M}(r,q))=(-1)^{\mathrm{ind}(p)-\mathrm{ind}(r)}\mathcal{M}(p,r)\times\mathcal{M}(r,q).$$

Denote $\mathcal{M}(p,q)\sqcup\mathcal{M}(p,r)\times\mathcal{M}(r,q)$ by $\widehat{\mathcal{M}(p,q)}$. By local triviality of the metric, we have the diffeomorphism h in (2.2) such that (2.3) and (2.4) hold. In addition, choose ϵ small enough such that $f(r)$ is the only critical value in $[f(r)-\epsilon,f(r)+\epsilon]$. For now on, we identify U with B without any difference. Let $M^+=f^{-1}(f(r)+\frac{1}{2}\epsilon)$ and $M^-=f^{-1}(f(r)-\frac{1}{2}\epsilon)$. Let $S_p^-=\mathcal{D}(p)\cap M^+$, $S_q^+=\mathcal{A}(q)\cap M^-$, $S_r^+=\mathcal{A}(r)\cap M^+$ and $S_r^-=\mathcal{D}(r)\cap M^-$. Then $S_r^+=\{(0,v_2)\in V_-\times V_+\mid \|v_2\|^2=\epsilon\}$ and $S_r^-=\{(v_1,0)\in V_-\times V_+\mid \|v_1\|^2=\epsilon\}$.

Define

$$L=\{(x,y)\in S_p^-\times M^-\mid x \text{ and } y \text{ are connected by a generalized flow line.}\}.$$

We may assume there is only one critical point r in $f^{-1}([f(r)-\epsilon,f(r)+\epsilon])$. Otherwise, define L to be

$$\{(x,y)\in(S_p^- - \bigcup_{r_i\neq r}S_{r_i}^+)\times M^-\mid x \text{ and } y \text{ are connected by a generalized flow line.}\}$$

in this argument. Consider the projection $\pi_+:M^+\times M^-\longrightarrow M^+$, then $L=\pi_+^{-1}(S_p^-)\cap P_c$, where P_c is defined in Lemma 4.1 and $c=f(r)$. By transversality, L is an smoothly embedded submanifold with boundary of $M^+ \times M^-$. The interior of L is

$$L^\circ = \{(x, y) \in L \mid x \text{ and } y \text{ are connected by a unbroken flow line.}\},$$

and $\partial L = (S_p^- \cap S_r^+) \times S_r^-$. Clearly, $S_p^- \cap S_r^+$ can be identified with $\mathcal{M}(p, r)$. We consider it as $\mathcal{M}(p, r)$. Then $\partial L = \mathcal{M}(p, r) \times S_r^-$.

Consider the projection $\pi_\pm : M^+ \times M^- \longrightarrow M^\pm$. We have $\pi_+(L^\circ) = S_p^- - S_r^+$, $\pi_-(L^\circ) = \mathcal{D}(p) \cap M^-$, and $\pi_\pm$ give diffeomorphisms from L° to its images. Give $S_p^- - S_r^+$ and $\mathcal{D}(p) \cap M^-$ the induced orientations from $\mathcal{D}(p)$ (see Section 5.1). Then π_+ and π_- move the above two orientations to L°. These orientations on L° are the same. Thus L° has a preferred orientation.

Clearly, $\pi_- : L \longrightarrow M^-$ is transversal to S_q^+ in L° and ∂L. Just as in (3) of Theorem 4.4, $\pi_-^{-1}(S_q^+)$ can be identified with $\widehat{\mathcal{M}(p, q)}$ because $(x, y) \in \pi_-^{-1}(S_q^+)$ is a pair of points on a generalized flow line $\Gamma \in \widehat{\mathcal{M}(p, q)}$. Likewise $(\pi_-|_{\partial L})^{-1}(S_q^+)$ can be identified with $\partial\widehat{\mathcal{M}(p, q)}$. The boundary of $\pi_-^{-1}(S_q^+)$ is exactly $(\pi_-|_{\partial L})^{-1}(S_q^+)$. We consider the orientation of L first in order to study the one of $\widehat{\mathcal{M}(p, q)}$.

Similarly to $\partial\widehat{\mathcal{M}(p, q)}$, there are two orientations of ∂L. First, the orientation of L gives it a boundary orientation. Second, the orientations of $\mathcal{M}(p, r)$ and S_r^- give $\partial L = \mathcal{M}(p, r) \times S_r^-$ a product orientation, where the orientation of S_r^- is induced from that of $\mathcal{D}(r)$ (see Section 5.1). The following key lemma shows the difference between these two orientations of ∂L.

Lemma 5.2. *$\partial L = (-1)^{ind(p)-ind(r)}\mathcal{M}(p, r) \times S_r^-$. Here, ∂L is given the boundary orientation and $\mathcal{M}(p, r) \times S_r^-$ is given the product orientation.*

The proof of Lemma 5.2 is based on a good local collar embedding of ∂L into L and a subtle computation of orientations. The collar embedding is provided by the following two lemmas.

Fix a point$(0, x_2) \in \mathcal{M}(p, r)$. We know $\mathcal{M}(p, r) = S_p^- \cap S_r^+ \subseteq \{0\} \times V_+$. Define $\widetilde{\mathcal{M}}(p, r) = \{v_2 \in V_+ \mid (0, v_2) \in \mathcal{M}(p, r)\}$.

Lemma 5.3. *There exist an open neighborhood Ω of x_2 in V_+, a $\delta > 0$, and a map $\tilde{\theta}$: $B_1(\delta) \times (\Omega \cap \widetilde{\mathcal{M}}(p, r)) \longrightarrow V_- \times V_+$ such that $\tilde{\theta}(v_1, v_2) = (v_1, \theta(v_1, v_2))$, $\theta(0, v_2) = v_2$ and $\tilde{\theta}$ is a diffeomorphism from $B_1(\delta) \times (\Omega \cap \widetilde{\mathcal{M}}(p, r))$ to $S_p^- \cap (B_1(\delta) \times \Omega)$. Here $B_1(\delta) = \{v_1 \in V_- \mid \|v_1\|^2 < \delta\}$.*

Let $\tilde{S}_r^+ = \{v_2 \in V_+ \mid (0, v_2) \in S_r^+\}$ and $\tilde{S}_r^- = \{v_1 \in V_- \mid (v_1, 0) \in S_r^-\}$. We can identify $\mathcal{M}(p, r)$ with $\widetilde{\mathcal{M}}(p, r)$ and $\tilde{S}_r^{\pm}$ with $S_r^{\pm}$ naturally. Fix a point $(x_1, 0) \in S_r^-$.

Lemma 5.4. *There exist $\delta > 0$, a neighborhood Ω_2 of x_2 in V_+ and a neighborhood Ω_1 of x_1 in V_+ such that $\varphi : [0, \delta) \times (\Omega_2 \cap \widetilde{\mathcal{M}}(p, r)) \times (\Omega_1 \cap \tilde{S}_r^-) \longrightarrow V_- \times V_+ \times V_- \times V_+$ is a local collar neighborhood embedding of ∂L into L near $((0, x_2), (x_1, 0))$. Here*

$$\varphi(s, v_2, v_1) = (sv_1, \theta(sv_1, v_2), \epsilon^{-\frac{1}{2}}\|\theta(sv_1, v_2)\|v_1, s\epsilon^{\frac{1}{2}}\|\theta(sv_1, v_2)\|^{-1}\theta(sv_1, v_2)),$$

and θ is defined in Lemma 5.3.

The proof of these three lemmas will be given later.

Since L and $N(S_q^+, M^-)$ have orientations, $\pi_-^{-1}(S_q^+)$ has an orientation. By the definitions of the orientations of L and $\widehat{\mathcal{M}(p, q)}$, the orientations of $\pi_-^{-1}(S_q^+)$ and $\widehat{\mathcal{M}(p, q)}$ are the same under this identification. The boundary orientation of ∂L and the orientation of

$N(S_q^+, M^-)$ also give $(\pi_-|_{\partial L})^{-1}(S_q^+)$ an orientation. This orientation of $(\pi_-|_{\partial L})^{-1}(S_q^+)$ coincides with the boundary orientation induced from $\pi_-^{-1}(S_q^+)$. The reason is as follows. At $((0, x_2), (x_1, 0)) \in (\pi_-|_{\partial L})^{-1}(S_q^+)$, let $\{e_1, \cdots, e_k\}$ be a base of $T(\pi_-|_{\partial L})^{-1}(S_q^+)$ and $\{e_{k+1}, \cdots, e_n\} \subseteq T(\partial L)$ represent a base of $N((\pi_-|_{\partial L})^{-1}(S_q^+), \partial L)$. Let $\mathcal{N}$ be an outward normal vector of $(\pi_-|_{\partial L})^{-1}(S_q^+)$ with respect to $\pi_-^{-1}(S_q^+)$. Then $\{\mathcal{N}, e_1, \cdots, e_k\}$ gives an orientation of $\pi_-^{-1}(S_q^+)$, $\{e_1, \cdots, e_k\}$ gives an orientation of $(\pi_-|_{\partial L})^{-1}(S_q^+)$, $\{\mathcal{N}, e_1, \cdots, e_n\}$ gives an orientation of L, and $\{e_1, \cdots, e_n\}$ gives an orientation of ∂L. When $\{e_{k+1}, \cdots, e_n\}$ is positively oriented, $\{e_1, \cdots, e_k\}$ gives the boundary orientation if and only if $\{e_1, \cdots, e_n\}$ gives the boundary orientation. This is the reason.

Thus $(\pi_-|_{\partial L})^{-1}(S_q^+)$ has the boundary orientation of $\partial\widehat{\mathcal{M}(p, q)}$ under this identification if ∂L is equipped with the boundary orientation.

On the other hand, if we give ∂L the product orientation, i.e., we consider it as $\mathcal{M}(p, r) \times S_r^-$, then $(\pi_-|_{\partial L})^{-1}(S_q^+)$ will have the product orientation of $\mathcal{M}(p, r) \times \mathcal{M}(r, q)$ under this identification.

By Lemma 5.2, we have completed the proof of (1) of Theorem 5.1. $\square$

Proof of Lemma 5.3. Since $\widetilde{\mathcal{M}}(p, r)$ is an embedded submanifold of V_+, there exist a neighborhood Ω of x_2 and a diffeomorphism $\alpha : \Omega \longrightarrow V_+$ such that $V_+ = K_1 \times K_2$, $\alpha(\Omega \cap \widetilde{\mathcal{M}}(p, r)) = K_1 \times \{0\}$ and $\alpha(x_2) = (0, 0)$. Here K_1 and K_2 are two Hilbert spaces. Define $\beta : B_1(\delta) \times \Omega \longrightarrow B_1(\delta) \times V_+$ by $\beta(v_1, v_2) = (v_1, \alpha(v_2))$. Then β is also a diffeomorphism. $\beta(B_1(\delta) \times (\Omega \cap \widetilde{\mathcal{M}}(p, r))) = B_1(\delta) \times K_1 \times \{0\}$, $\beta(\{0\} \times \Omega) = \{0\} \times V_+$ and $\beta(0, x_2) = (0, 0, 0)$.

Since S_p^- is transversal to $\{0\} \times \Omega$, then $\beta(S_p^- \cap (B_1(\delta) \times \Omega))$ is also transversal to

$\{0\} \times V_+ = \beta(\{0\} \times \Omega)$. Denote $\beta(S_p^- \cap (B_1(\delta) \times \Omega))$ by S. Then

$$T_{(0,0,0)}S + T_{(0,0,0)}(\{0\} \times V_+) = T_{(0,0,0)}(B_1(\delta) \times V_+) = V_- \times V_+. \tag{5.1}$$

Consider the map $\pi_1 : B_1(\delta) \times V_+ \longrightarrow B_1(\delta) \times \{(0,0)\}$, where $\pi_1(v_1, k_1, k_2) = (v_1, 0, 0)$. By (5.1), we get

$$d\pi_1 : T_{(0,0,0)}S \longrightarrow T_{(0,0,0)}(B_1(\delta) \times \{(0,0)\}) = V_- \times \{(0,0)\}$$

is surjective. In addition, since $\{0\} \times (\Omega \cap \widetilde{\mathcal{M}}(p,r)) \subseteq S_p^- \cap (B_1(\delta) \times \Omega)$, we have

$$\{0\} \times K_1 \times \{0\} = \beta(\{0\} \times (\Omega \cap \widetilde{\mathcal{M}}(p,r))) \subseteq S.$$

Thus

$$\{0\} \times K_1 \times \{0\} = T_{(0,0,0)}(\{0\} \times K_1 \times \{0\}) \subseteq T_{(0,0,0)}S. \tag{5.2}$$

Consider the map $\pi_2 : B_1(\delta) \times V_+ \longrightarrow B_1(\delta) \times K_1 \times \{0\}$, where $\pi_2(v_1, k_1, k_2) = (v_1, k_1, 0)$. By the surjectivity of $d\pi_1$ on S and (5.2), we know that

$$d\pi_2 : T_{(0,0,0)}S \longrightarrow T_{(0,0,0)}(B_1(\delta) \times K_1 \times \{0\}) = V_- \times K_1 \times \{0\}$$

is surjective.

Now we count the dimensions of S and $B_1(\delta) \times K_1 \times \{0\}$.

$$\begin{aligned} \dim(S) &= \dim(S_p^-) = \mathrm{ind}(p) - 1 = \mathrm{ind}(p) - \mathrm{ind}(r) - 1 + \mathrm{ind}(r) \\ &= \dim(\mathcal{M}(p,r)) + \dim(V_-) = \dim(K_1 \times \{0\}) + \dim(B_1(\delta)) \\ &= \dim(B_1(\delta) \times K_1 \times \{0\}) \end{aligned}$$

By the Inverse Function Theorem, shrinking δ and Ω if necessary, we have that π_2 gives a diffeomorphism from $S = \beta(S_p^- \cap (B_1(\delta) \times \Omega))$ to $B_1(\delta) \times K_1 \times \{0\} = \beta(B_1(\delta) \times (\Omega \cap \widetilde{\mathcal{M}}(p,r)))$. Also, $(\pi_2|_S)^{-1}(v_1, k_1, 0) = (v_1, \hat{\theta}(v_1, k_1, 0))$ for some $\hat{\theta}$. It's easy to see that $S \cap (\{0\} \times V_+) = \{0\} \times K_1 \times \{0\}$. Then $\hat{\theta}(0, k_1, 0) = (k_1, 0)$.

Defining $\tilde{\theta} = \beta^{-1} \circ (\pi_2|_S)^{-1} \circ \beta$ on $B_1(\delta) \times (\Omega \cap \widetilde{\mathcal{M}}(p,r))$, completes the proof. □

Proof of Lemma 5.4. We may assume $\epsilon \leq 1$. Choose δ as in Lemma 5.3. Choose Ω_2 to be Ω in Lemma 5.3. Consider φ as a map defined in $(-\delta, \delta) \times (\Omega_2 \cap \widetilde{\mathcal{M}}(p,r)) \times \tilde{S}_r^-$. By Lemma 5.3, we have $\mathrm{Im}(\varphi) \subseteq L^\circ$ when $s > 0$, $\mathrm{Im}(\varphi) \cap L = \emptyset$ when $s < 0$ and $\varphi(0, v_2, v_1) = ((0, v_2), (v_1, 0)) \in \partial L$.

Now we compute $d\varphi$. First, we introduce some notation. Let $\frac{\partial}{\partial s}$ be the positive unit tangent vector of $(-\delta, \delta)$. Let $\frac{\partial}{\partial x_1}$ be a base of $T_{x_1}\tilde{S}_r^- \subseteq V_-$, i.e.

$$\frac{\partial}{\partial x_1} = \{e_1, \cdots, e_{\mathrm{ind}(r)-1}.\}$$

Let $\frac{\partial}{\partial x_2}$ be a base of $T_{x_2}\widetilde{\mathcal{M}}(p,r)) \subseteq V_+$. The notation $(d\varphi)\frac{\partial}{\partial x_1}$ means

$$(d\varphi)\frac{\partial}{\partial x_1} = \{(d\varphi)e_1, \cdots, (d\varphi)e_{\mathrm{ind}(r)-1}\}.$$

In the following calculation, omit $d\varphi\frac{\partial}{\partial x_2}$ if $\dim(\widetilde{\mathcal{M}}(p,r))=0$ and omit $d\varphi\frac{\partial}{\partial x_1}$ if $\dim(\tilde{S}_r^-)=0$. At (s,x_1,x_2)

$$\begin{aligned}(d\varphi)\frac{\partial}{\partial s} &= (x_1,(d\theta)x_1,\epsilon^{-\frac{1}{2}}\|\theta\|^{-1}\langle\theta,(d\theta)x_1\rangle x_1,\epsilon^{\frac{1}{2}}\|\theta\|^{-1}h+s*) \\ (d\varphi)\frac{\partial}{\partial x_2} &= \left(0,(d\theta)\frac{\partial}{\partial x_2},\epsilon^{-\frac{1}{2}}\|\theta\|^{-1}\left\langle\theta,(d\theta)\frac{\partial}{\partial x_2}\right\rangle x_1,s*\right), \\ (d\varphi)\frac{\partial}{\partial x_1} &= \left(s\frac{\partial}{\partial x_1},s(d\theta)\frac{\partial}{\partial x_1},\epsilon^{-\frac{1}{2}}\|\theta\|\frac{\partial}{\partial x_1}+s*,s^2*\right).\end{aligned} \tag{5.3}$$

Here $*$ stands for some smooth functions which are not important. Since $\theta(0,v_2)\equiv v_2$, we have $(d\theta)(0,x_2)\frac{\partial}{\partial x_2}=\frac{\partial}{\partial x_2}$. And since $\frac{\partial}{\partial x_2}$ is contained in $T_{x_2}\tilde{S}_r^+$ and is orthogonal to x_2, we have $\langle\theta(0,x_2),(d\theta)(0,x_2)\frac{\partial}{\partial x_2}\rangle=0$. In addition, $\|x_2\|=\epsilon^{\frac{1}{2}}$. Thus

$$(d\varphi)(0,x_2,x_1)\frac{\partial}{\partial s}=(x_1,d\theta(0,x_2)x_1,\epsilon^{-1}\langle x_2,d\theta(0,x_2)x_1\rangle x_1,x_2),$$

$$(d\varphi)(0,x_2,x_1)\frac{\partial}{\partial x_2}=\left(0,\frac{\partial}{\partial x_2},0,0\right),\quad (d\varphi)(0,x_2,x_1)\frac{\partial}{\partial x_1}=\left(0,0,\frac{\partial}{\partial x_1},0\right). \tag{5.4}$$

Clearly, $d\varphi(0,x_2,x_1)\{\frac{\partial}{\partial s},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_1}\}$ is linear independent. Since $\dim(L)=\dim([0,\,\delta)\times(\Omega_2\cap\widetilde{\mathcal{M}}(p,r))\times\tilde{S}_r^-)$, by the Inverse Function Theorem, we have that this lemma is true. $\square$

Proof of Lemma 5.2. Let $((0,x_2),(x_1,0))$ be an arbitrary point in ∂L. We only need to prove the orientation difference is $(-1)^{\mathrm{ind}(p)-\mathrm{ind}(r)}$ at this point.

Suppose $(0,\frac{\partial}{\partial x_2})$ and $(\frac{\partial}{\partial x_1},0)$ are a positive basis of $T_{(0,x_2)}\mathcal{M}(p,r)$ and $T_{(x_1,0)}S_r^-$ respectively. We use the locally collar embedding φ in Lemma 5.4. Fix x_2 and x_1, change s. By (5.4), $d\varphi(0,x_2,x_1)\frac{\partial}{\partial x_2}=(0,\frac{\partial}{\partial x_2},0,0)$ and $d\varphi(0,x_2,x_1)\frac{\partial}{\partial x_1}=(0,0,\frac{\partial}{\partial x_1},0)$. So $\{d\varphi(0,x_2,x_1)\frac{\partial}{\partial x_2},d\varphi(0,x_2,x_1)\frac{\partial}{\partial x_1}\}$ is a positive basis of $\mathcal{M}(p,r)\times S_r^-$. When $\dim(\mathcal{M}(p,r))=0$ or $\dim(S_r^-)=0$, the orientation of $T_{(0,x_2)}\mathcal{M}(p,r)$ or $T_{(x_1,0)}S_r^-$ is a sign ± 1, and $d\varphi(s,x_2,x_1)\frac{\partial}{\partial x_2}$ or $d\varphi(s,x_2,x_1)\frac{\partial}{\partial x_1}$ is replaced by this sign.

Now, $-(d\varphi)(0,x_2,x_1)\frac{\partial}{\partial s}$ is an outward normal vector of ∂L. Thus, when $s=0$, $\{d\varphi\frac{\partial}{\partial x_2}, d\varphi\frac{\partial}{\partial x_1}\}$ is a positive base of ∂L if and only if $\{-d\varphi\frac{\partial}{\partial s}, d\varphi\frac{\partial}{\partial x_2}, d\varphi\frac{\partial}{\partial x_1}\}$ is a positive base of L. This is also equivalent to the statement that, when $s \neq 0$, $\{-d\varphi\frac{\partial}{\partial s}, d\varphi\frac{\partial}{\partial x_2}, d\varphi\frac{\partial}{\partial x_1}\}$ is a positive base of L.

When $s \neq 0$, $\varphi(s,x_2,x_1) \in L^\circ$, and $\pi_+ : L^\circ \longrightarrow S_p^-$ preserves orientation. Thus, by (5.3), the above consideration is equivalent to the statement that,

$$\begin{aligned} & \left\{-d\pi_+ \cdot d\varphi\frac{\partial}{\partial s}, d\pi_+ \cdot d\varphi\frac{\partial}{\partial x_2}, d\pi_+ \cdot d\varphi\frac{\partial}{\partial x_1}\right\} \\ = & \left\{-(x_1, d\theta\cdot x_1), \left(0, d\theta\frac{\partial}{\partial x_2}\right), \left(s\frac{\partial}{\partial x_1}, s\cdot d\theta\frac{\partial}{\partial x_1}\right)\right\} \end{aligned}$$

is a positive base of S_p^-. We change this base to another base

$$\left\{-(x_1, d\theta\cdot x_1), \left(0, d\theta\frac{\partial}{\partial x_2}\right), \left(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1}\right)\right\}. \tag{5.5}$$

The new base (5.5) has the same orientation as the old one. Its advantage is that, when $s=0$, (5.5) is still a base of S_p^-. The reason is as follows. When $s \neq 0$, (5.5) is in TS_p^-. Thus, by continuity, it is still in TS_p^- when $s=0$. In addition, when $s=0$, $(0, d\theta\frac{\partial}{\partial x_2}) = (0, \frac{\partial}{\partial x_2})$. As a base of $T_{x_1}V_-$ and $T_{x_2}\widetilde{\mathcal{M}}(p,r)$ respectively, both $\{-x_1, \frac{\partial}{\partial x_1}\}$ and $\frac{\partial}{\partial x_2}$ are linearly independent. So (5.5) remains linearly independent when $s=0$.

When s varies in $[0,\delta)$, the orientation difference between $\{-(x_1, d\theta\cdot x_1), (0, d\theta\frac{\partial}{\partial x_2}), (\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1})\}$ and S_p^- is fixed. So we only need to check the difference when $s=0$. As a base of $T_{x_2}\mathcal{M}(p,r)$, $(0,\frac{\partial}{\partial x_2})$ contains $\mathrm{ind}(p)-\mathrm{ind}(r)-1$ vectors. Denote the orientation of a base $\{*\}$ by $\mathrm{Or}\{*\}$. Then, when $s=0$,

$$\begin{aligned} & \mathrm{Or}\left\{-(x_1, d\theta\cdot x_1), \left(0, d\theta\frac{\partial}{\partial x_2}\right), \left(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1}\right)\right\} \\ = \; & \mathrm{Or}\left\{-(x_1, d\theta\cdot x_1), \left(0, \frac{\partial}{\partial x_2}\right), \left(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1}\right)\right\} \\ = \; & (-1)^{\mathrm{ind}(p)-\mathrm{ind}(r)}\mathrm{Or}\left\{\left(0, \frac{\partial}{\partial x_2}\right), (x_1, d\theta\cdot x_1), \left(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1}\right)\right\}. \end{aligned}$$

Since $(x_1,0)=-\nabla f(x_1,0)$, $(\frac{\partial}{\partial x_1},0)$ is a positive base of $T_{(x_1,0)}S_r^+$, then $\{(x_1,0),(\frac{\partial}{\partial x_1},0)\}$ is a positive base of $T_{(x_1,0)}(V_-\times\{0\})=V_-\times\{0\}=T_r\mathcal{D}(r)$. Thus $\{(x_1,d\theta\cdot x_1),(\frac{\partial}{\partial x_1},d\theta\frac{\partial}{\partial x_1})\}$ represents a positive base of the normal space $N_{(0,x_2)}(\mathcal{M}(p,r),S_p^-)$. Since $(0,\frac{\partial}{\partial x_2})$ is a positive base of $T_{(0,x_2)}\mathcal{M}(p,r)$, we infer that $\{(0,\frac{\partial}{\partial x_2}),(x_1,d\theta\cdot x_1),(\frac{\partial}{\partial x_1},d\theta\frac{\partial}{\partial x_1})\}$ is a positive base of $T_{(0,x_2)}S_p^-$.

As a result, $(-1)^{\mathrm{ind}(p)-\mathrm{ind}(r)}\mathrm{Or}\{d\varphi(0,x_2,x_1)\frac{\partial}{\partial x_2}, d\varphi(0,x_2,x_1)\frac{\partial}{\partial x_1}\}$ represents the orientation of ∂L. This completes the proof. □

5.3 Proof of (2) of Theorem 5.1

The proof of (2) is similar to that of (1). In particular, they share many details. We shall only give the outline and the key calculation of this proof.

Proof. We only need to prove that $\partial(\mathcal{D}(p)\sqcup\mathcal{M}(p,r)\times\mathcal{D}(r))=\mathcal{M}(p,r)\times\mathcal{D}(r)$ as oriented manifolds. Actually, we only need to argue this in an open subset containing $\mathcal{M}(p,r)\times\mathcal{D}(r)$ of $\mathcal{D}(p)\sqcup\mathcal{M}(p,r)\times\mathcal{D}(r)$. Recall the evaluation map $e:\mathcal{D}(p)\sqcup\mathcal{M}(p,r)\times\mathcal{D}(r)\longrightarrow M$ in (3) of Theorem 4.5. We have $e^{-1}\circ f^{-1}((-\infty,f(r)+\epsilon))$ is such an open subset. Moreover, we can simplify this problem again. Let $M(r) = f^{-1}((f(r)-\epsilon, f(r)+\epsilon))$. Consider the open subset $e^{-1}(M(r))$. For all $x \in e^{-1} \circ f^{-1}((-\infty, f(r)+\epsilon)) \cap \mathcal{M}(p,r) \times \mathcal{D}(r)$, there exist $y \in e^{-1}(M(r)) \cap \mathcal{M}(p,r) \times \mathcal{D}(r)$ and a flow map ψ in $\overline{\mathcal{D}(p)}$, such that $\psi(y) = x$ (see Lemma 6.4). From y to x, $d\psi$ preserves the orientations of $\mathcal{D}(p)$ and $\mathcal{M}(p,r) \times \mathcal{D}(r)$ and the outward normal direction. Then $d\psi$ preserves the orientation difference between $\partial(\mathcal{D}(p) \sqcup \mathcal{M}(p,r) \times \mathcal{D}(r))$ and $\mathcal{M}(p,r) \times \mathcal{D}(r)$. Thus we only need to show this is true in $e^{-1}(M(r))$. Now denote $\mathcal{D}(p) \cap M(r)$ by D_p, $\mathcal{D}(r) \cap M(r)$ by D_r and $e^{-1}(M(r))$ by $\widehat{D_p}$. Then $\widehat{D_p} = D_p \sqcup \mathcal{M}(p,r) \times D_r$. We only need to show that $\partial\widehat{D_p} = \mathcal{M}(p,r) \times D_r$ as oriented manifolds.

We use the same notation of $M^{\pm}$, S_p^- and S_r^+ as in the proof of (1). Also identify $S_p^- \cap S_r^+$ with $\mathcal{M}(p,r)$ and define $\widetilde{\mathcal{M}}(p,r)$ as in the proof of (1). Define $\widetilde{D}_r = \{v_2 \mid (0, v_2) \in D_r\}$. We also assume that there is only one critical point r in $M(r)$.

Define

$$L = \{(x,y) \in S_p^- \times M(r) \mid x \text{ and } y \text{ are connected by a generalized flow line.}\}.$$

Then $\partial L = \mathcal{M}(p,r) \times D_r$. And

$$L^\circ = \{(x,y) \in L \mid x \text{ and } y \text{ are connected by a unbroken flow line.}\},$$

L is identified with $\widehat{D_p}$ because $(x,y) \in L$ is a pair of points on a generalized flow line connecting p and y. Since $L \subseteq S_p^- \times M(r)$, we may consider the natural projection $\pi : L \longrightarrow M(r)$. Moreover, π identifies L° with D_p, and π coincides with the above identification between L and $\widehat{D_p}$. The orientation of $\widehat{D_p}$ gives L an orientation, and L gives ∂L a boundary orientation. We only need to check the difference between the boundary orientation and the product orientation of ∂L.

Fix $((0,x_2),(x_1,0)) \in \mathcal{M}(p,r) \times D_r$. Just as Lemma 5.4, we give a locally collar neighborhood parametrization $\varphi : [0,\delta) \times (\Omega_2 \cap \widetilde{\mathcal{M}}(p,r)) \times \widetilde{D}_r \longrightarrow V_- \times V_+ \times V_- \times V_+$ such that

$$\varphi(s, v_2, v_1) = (sv_1, \theta(sv_1, v_2), v_1, s\theta(sv_1, v_2)), \tag{5.6}$$

where θ is defined in Lemma 5.3. It's necessary to point out that this argument includes the special case of $\operatorname{ind}(r) = 0$. In this case, $\widetilde{D}_r = \{0\}$, $\varphi(s, v_2, v_1) = (0, v_2, 0, sv_2)$ and $d\varphi\frac{\partial}{\partial x_1}$ is the sign ± 1 assigned to D_r.

Suppose $(0, \frac{\partial}{\partial x_2})$ and $(\frac{\partial}{\partial x_1}, 0)$ are positive basis of $T_{(0,x_2)}\mathcal{M}(p,r)$ and $T_{(x_1,0)}D_r$ respectively. At (s, x_2, x_1), we have

$$d\varphi\frac{\partial}{\partial s} = (x_1, d\theta \cdot x_1, 0, \theta + s \cdot d\theta \cdot x_1), \tag{5.7}$$

$$d\varphi\frac{\partial}{\partial x_2} = \left(0, d\theta\frac{\partial}{\partial x_2}, 0, s \cdot d\theta\frac{\partial}{\partial x_2}\right), \; d\varphi\frac{\partial}{\partial x_1} = \left(s\frac{\partial}{\partial x_1}, s \cdot d\theta\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}, s^2 \cdot d\theta\frac{\partial}{\partial x_1}\right).$$

We shall check that, when $s \in [0,\delta)$, $\{-d\varphi\frac{\partial}{\partial s}, d\varphi\frac{\partial}{\partial x_2}, d\varphi\frac{\partial}{\partial x_1}\}$ coincides with the orientation of L.

When $s \neq 0$, $\varphi(s, x_2, x_1) \in L^\circ$. By the definition of the orientation of L°, $\pi : L^\circ \longrightarrow D_p$ preserves its orientation. Thus, we only need to show that, when $s \neq 0$,

$$\begin{aligned} & \left\{-d\pi \cdot d\varphi\frac{\partial}{\partial s}, d\pi \cdot d\varphi\frac{\partial}{\partial x_2}, d\pi \cdot d\varphi\frac{\partial}{\partial x_1}\right\} \\ = & \left\{(0, -\theta - s \cdot d\theta \cdot x_1), \left(0, s \cdot d\theta\frac{\partial}{\partial x_2}\right), \left(\frac{\partial}{\partial x_1}, s^2 \cdot d\theta\frac{\partial}{\partial x_1}\right)\right\} \end{aligned}$$

gives the orientation of D_p at $\pi\varphi(s, x_2, x_1)$. By (5.6), we know that $\pi\varphi(s, x_2, x_1) = (x_1, s\theta(sx_1, x_2))$ is connected with $(sx_1, \theta(sx_1, x_2)) \in S_p^-$ by an unbroken flow line. Consider the flow map ψ in U such that $\psi(v_1, v_2) = (s^{-1}v_1, sv_2)$. Then $\psi(sx_1, \theta(sx_1, x_2)) = (x_1, s\theta(sx_1, x_2))$ and ψ preserves the orientation of D_p. Thus we only need to check that

$$\begin{aligned} & \left\{-d\psi^{-1} \cdot d\pi \cdot d\varphi\frac{\partial}{\partial s}, d\psi^{-1} \cdot d\pi \cdot d\varphi\frac{\partial}{\partial x_2}, d\psi^{-1} \cdot d\pi \cdot d\varphi\frac{\partial}{\partial x_1}\right\} \\ = & \left\{(0, -s^{-1}\theta - d\theta \cdot x_1), \left(0, d\theta\frac{\partial}{\partial x_2}\right), \left(s\frac{\partial}{\partial x_1}, s \cdot d\theta\frac{\partial}{\partial x_1}\right)\right\} \end{aligned}$$

gives the orientation of D_p at $(sx_1, \theta(sx_1, x_2))$. Change the above base to the orientation equivalent base $\{(0, -\theta - s \cdot d\theta \cdot x_1), (0, d\theta\frac{\partial}{\partial x_2}), (\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1})\}$. When $s = 0$, it becomes

$$\left\{(0, -x_2), \left(0, \frac{\partial}{\partial x_2}\right), \left(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1}\right)\right\}. \tag{5.8}$$

Since $(\frac{\partial}{\partial x_1}, 0)$ is a positive base of $V_- \times \{0\} = T_r D_r$, $(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1})$ represents a positive base of $N_{(0,x_2)}(\mathcal{M}(p, r), S_p^-)$. At $(0, x_2)$, $(0, -x_2) = -\nabla f$, and $(0, \frac{\partial}{\partial x_2})$ is a positive base of $T_{(0,x_2)}\mathcal{M}(p, r)$. Thus (5.8) gives the orientation of D_r. $\square$

Remark 5.2. *It seems that, in the finite dimensional case, the paper [35] gets orientation relations by the same strategy as we have. The following key fact is pointed out without explanations in [35, p. 155]. "La variété* $W^s(c, \varepsilon) \times L_A(c, d)$ *est de codimension* 0 *dans le bord de* $\overline{W}^s(d, A + \varepsilon)$ *et la normale sortante* n_0 *à* $\overline{W}^s(d, A + \varepsilon)$ *en* $(c, l) \in W^s(c) \times L_A(c, d)$ *s'identifie au vecteur tangent à l orientée par* $-\xi$*." (Here* $\xi = \nabla f$*.) This is proved in the paper by moving* $-d\varphi\frac{\partial}{\partial s}$ *(see (5.7)) to be* $(0, -x_2) = -\nabla f$ *in (5.8). Thus our work may give the details omitted in [35].*

5.4 Proof of (3) of Theorem 5.1

The proof of (3) is a mixture of those of (1) and (2).

Proof. We shall prove that $\partial(\mathcal{W}(p,q) \sqcup \mathcal{M}(p,r) \times \mathcal{W}(r,q)) = \mathcal{M}(p,r) \times \mathcal{W}(r,q)$ and $\partial(\mathcal{W}(p,q) \sqcup \mathcal{W}(p,r) \times \mathcal{M}(r,q)) = (-1)^{\mathrm{ind}(p)-\mathrm{ind}(r)+1}\mathcal{W}(p,r) \times \mathcal{M}(r,q)$. Recall the evaluation map $e : \mathcal{W}(p,q) \sqcup \mathcal{M}(p,r) \times \mathcal{W}(r,q) \longrightarrow M$ (or $\mathcal{W}(p,q) \sqcup \mathcal{W}(p,r) \times \mathcal{M}(r,q) \longrightarrow M$) in (3) of Theorem 4.6. Define $M(r) = f^{-1}((f(r) - \epsilon, f(r) + \epsilon))$, $M(r)^+ = f^{-1}((f(r), f(r) + \epsilon))$ and $M(r)^- = f^{-1}((f(r) - \epsilon, f(r)))$. We have four cases. Just as the proofs of (1) and (2), we will define a manifold L which plays a important role all through this proof, where

$$L = \{(x, y) \mid x \text{ and } y \text{ are connected by a generalized flow line.}\},$$

and (x, y) is contained in some different manifolds in each case. Also, x and y will be connected by a unbroken flow line if and only if $(x, y) \in L^\circ$.

Case (a). The boundary is $\mathcal{M}(p,r) \times \mathcal{W}(r,q)$ and $r \neq q$.

We reduce this problem to considering the case of $e^{-1}(M(r)^-)$. Denote $e^{-1}(M(r)^-)$ by $\widehat{W_{p,q}}$, $\mathcal{W}(r,q) \cap M(r)^-$ by $W_{r,q}$, $\mathcal{D}(p) \cap M(r)^-$ by D_p and $\mathcal{D}(r) \cap M(r)^-$ by D_r. Clearly, as unoriented manifolds, $\partial\widehat{W_{p,q}} = \mathcal{M}(p,r) \times W_{r,q}$.

Define $L \subseteq S_p^- \times M(r)^-$. The natural projection $\pi_2 : L \longrightarrow M(r)^-$ identifies L° with D_p. $\partial L = \mathcal{M}(p,r) \times D_r$. The orientation of D_p gives L an orientation. In the proof of (2), it has been verified that the boundary orientation and the product orientation of ∂L are the same. We identify $\pi_2^{-1}(\mathcal{A}(q))$ with $\widehat{W_{p,q}}$ and identify $(\pi_2|_{\partial L})^{-1}(\mathcal{A}(q))$ with $\partial\widehat{W_{p,q}}$. An argument similar to that in (1) completes the proof.

Case (b). The boundary is $\mathcal{M}(p,q)\times\mathcal{W}(q,q)$.

Replace $M(r)^-$ by $M(q)$ in Case (a). The same argument gives a proof.

Case (c). The boundary is $\mathcal{W}(p,r)\times\mathcal{M}(r,q)$ and $p\neq r$.

Reduce to the case of $e^{-1}(M(r)^+)$. Denote $e^{-1}(M(r)^+)$ by $\widehat{W_{p,q}}$, $\mathcal{W}(p,r)\cap M(r)^+$ by $W_{p,r}$ and $\mathcal{D}(p)\cap M(r)^+$ by D_p.

Define $L\subseteq D_p\times M^-$, where $M^-=f^{-1}(f(r)-\epsilon)$. The projection $\pi_1: L\longrightarrow D_p$ identifies L° with $D_p-W_{p,r}$, and $\partial L=W_{p,r}\times S_r^-$. Then D_p gives L an orientation. Consider another projection $\pi_2: L\longrightarrow M^-$. Then $\pi_2^{-1}(S_q^+)$ can be identified with $\widehat{W_{p,q}}$ and $(\pi_2|_{\partial L})^{-1}(S_q^+)$ can be identified with $\partial\widehat{W_{p,q}}$. We reduce the proof to checking the difference of two orientations of ∂L.

Define $\widetilde{W}_{p,r}=\{v_2\mid(0,v_2)\in W_{p,r}\}$. Similar to Lemma 5.3 and 5.4, there is a neighborhood Ω_2 of x_2 in $\widetilde{W}_{p,r}$ and a parametrization $\tilde{\theta}: B_1(\delta)\times\Omega_2\longrightarrow D_p$ such that $\tilde{\theta}(v_1,v_2)=(v_1,\theta(v_1,v_2))$ and $\theta(0,v_2)=v_2$. We also have a local collar embedding $\varphi:[0,\delta)\times\Omega_2\times\widetilde{S}_r^-\longrightarrow V_-\times V_+\times V_-\times V_+$ such that

$$\begin{aligned}\varphi(s,v_2,v_1) \;&=\; \Big(sv_1,\theta(sv_1,v_2),(2\epsilon)^{-\frac{1}{2}}(\epsilon+(\epsilon^2+4s^2\epsilon\|\theta(sv_1,v_2)\|^2)^{\frac{1}{2}})^{\frac{1}{2}}v_1,\\ &\qquad s(2\epsilon)^{\frac{1}{2}}(\epsilon+(\epsilon^2+4s^2\epsilon\|\theta(sv_1,v_2)\|^2)^{\frac{1}{2}})^{-\frac{1}{2}}\theta(sv_1,v_2)\Big).\end{aligned}$$

Just as the proof of (1), we reduce the proof to checking the orientation of $\{-d\pi_1\cdot d\varphi\frac{\partial}{\partial s}, d\pi_1\cdot d\varphi\frac{\partial}{\partial x_2}, d\pi_1\cdot d\varphi\frac{\partial}{\partial x_1}\}$ and then that of $\{-(x_1,d\theta\cdot x_1),(0,\frac{\partial}{\partial x_2}),(\frac{\partial}{\partial x_1},d\theta\frac{\partial}{\partial x_1})\}$ in $T_{(0,x_2)}D_p$. Here, $\{(0,\frac{\partial}{\partial x_2})\}$ is a positive base of $T_{(0,x_2)}\,W_{p,r}$. It contains $\mathrm{ind}(p)-\mathrm{ind}(r)$

vectors. Thus the orientations are

$$\begin{aligned} & \mathrm{Or}\left\{-(x_1, d\theta\cdot x_1), \left(0, \frac{\partial}{\partial x_2}\right), \left(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1}\right)\right\} \\ = \quad & (-1)^{\mathrm{ind}(p)-\mathrm{ind}(r)+1}\mathrm{Or}\left\{\left(0, \frac{\partial}{\partial x_2}\right), (x_1, d\theta\cdot x_1), \left(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1}\right)\right\}. \end{aligned}$$

Since $\{(0, \frac{\partial}{\partial x_2}), (x_1, d\theta \cdot x_1), (\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1})\}$ is positive, the proof is complete.

Case (d). The boundary is $\mathcal{W}(p,p)\times\mathcal{M}(p,q)$.

Reduce to the case of $e^{-1}(M(p))$. Denote $e^{-1}(M(p))$ by $\widehat{W_{p,q}}$ and $\mathcal{D}(p)\cap M(p)$ by D_p. Then $\partial\widehat{W_{p,q}} = \mathcal{W}(p,p)\times\mathcal{M}(p,q) = \{p\}\times\mathcal{M}(p,q)$.

Define $L\subseteq D_p\times M^-$, where $M^- = f^{-1}(f(p)-\epsilon)$. Then $\pi_1 : L \longrightarrow D_p$ identifies L° with $D_p - \{p\}$, and $\partial L = \mathcal{W}(p,p)\times S_p^-$. Moreover, D_p gives L an orientation. Consider $\pi_2 : L \longrightarrow M^-$. Then $\pi_2^{-1}(S_q^+)$ can be identified with $\widehat{W_{p,q}}$ and $(\pi_2|_{\partial L})^{-1}(S_q^+)$ can be identified with $\partial\widehat{W_{p,q}}$. We reduce the proof to checking the two orientations of ∂L.

Consider the collar embedding $\varphi : [0,\sqrt{2})\times\widetilde{S}_p^- \longrightarrow V_-\times V_+\times V_-\times V_+$ such that $\varphi(s,v_1) = (sv_1, 0, v_1, 0)$. Since $\mathcal{W}(p,p)$ has orientation $+1$, we only need to check the orientation difference between $\{-d\varphi\frac{\partial}{\partial s}, d\varphi\frac{\partial}{\partial x_1}\}$ and L. When $s=1$, $\mathrm{Or}\{-d\pi_1\cdot d\varphi\frac{\partial}{\partial s}, d\pi_1\cdot d\varphi\frac{\partial}{\partial x_1}\} = -\mathrm{Or}\{-\nabla f, (\frac{\partial}{\partial x_1}, 0)\}$ is the negative orientation of $T_{(x_1,0)}D_p$. Thus $\partial\widehat{W_{p,q}} = -\mathcal{W}(p,p)\times\mathcal{M}(p,q)$. $\square$

Remark 5.3. *The papers [3] and [60] compute the cup product of $H^*(M;R)$ via Morse Theory. Both [3, (2.2)] and [60, lem. 2 and 3] neglect signs. Theorem 5.1, (3), can tell us the the signs if we do care about them. The following is an explanation of [60, lem. 3]. We shall use notation different from that in [60]. Our $\mathcal{W}(p,q)$ and $\#\mathcal{M}(p,q)$ are $\mathcal{M}(p,q)$ and $n(p,q)$ in [60] respectively. A real coefficients Thom-Smale cochain complex is defined in [60]* *as* $C^* = \bigoplus_n \bigoplus_{ind(p)=n} R[p]$ *with coboundary operator*

$$\delta q = \sum_{ind(p)=ind(q)+1} \#\mathcal{M}(p,q)p,$$

where $\#\mathcal{M}(p,q)$ *is defined in Theorem 6.3. Let* ω *be a differential form, in [60], a cup product action of* ω *on* C^* *is defined as*

$$\pi(\omega)q = \sum_p \left(\int_{\mathcal{W}(p,q)} \omega \right) p.$$

The paper [60, lem. 3] states that $\pi(d\omega) = \delta\pi(\omega) \pm \pi(\omega)\delta$. *Actually, (3) of Theorem 5.1 tells us*

$$\pi(d\omega) = \delta\pi(\omega) + (-1)^{|\omega|+1}\pi(\omega)\delta. \tag{5.9}$$

If α *and* β *are two singular cochains, then* $\delta\alpha\cup\beta = \delta(\alpha\cup\beta)+(-1)^{|\alpha|+1}\alpha\cup\delta\beta$. *By comparison with this, (5.9) is reasonable. The proof of (5.9) is as follows.*

$$\begin{aligned} \pi(d\omega)q &= \sum_p \left(\int_{\mathcal{W}(p,q)} d\omega \right) p \\ = \quad & \sum_p \left(\int_{\overline{\mathcal{W}(p,q)}} e^* d\omega \right) p = \sum_p \left(\int_{\partial^1 \overline{\mathcal{W}(p,q)}} e^* \omega \right) p \\ = \quad & \sum_p \left(\sum_r \int_{\mathcal{M}(p,r)\times\mathcal{W}(r,q)} e^*\omega + \sum_r (-1)^{ind(p)-ind(r)+1} \int_{\mathcal{W}(p,r)\times\mathcal{M}(r,q)} e^*\omega \right) p. \end{aligned}$$

Here e *is defined in (3) of Theorem 4.6. When* $dim(\mathcal{W}(r,q)) < |\omega|$ *(or* $dim(\mathcal{W}(p,r)) < |\omega|$*),*

$e^*\omega = 0$ on $\mathcal{M}(p,r) \times \mathcal{W}(r,q)$ (or $\mathcal{W}(p,r) \times \mathcal{M}(r,q)$). Thus

$$\begin{aligned}
\pi(d\omega)q &= \sum_p \left(\sum_{ind(r)=ind(p)-1} \#\mathcal{M}(p,r) \int_{\mathcal{W}(r,q)} \omega \right. \\
&\quad \left. + \sum_{ind(r)=ind(q)+1} (-1)^{ind(p)-ind(q)} \#\mathcal{M}(r,q) \int_{\mathcal{W}(p,r)} \omega \right) p \\
&= \delta\pi(\omega)q + (-1)^{ind(p)-ind(q)} \pi(\omega)\delta q.
\end{aligned}$$

This completes the proof since $ind(p) - ind(q) = |\omega| + 1$.

6 CW Structures (I)

In this chapter, under the assumption of the local triviality of the metric, we present results on CW structures arising from the negative gradient dynamical systems.

6.1 Main Theorems

We shall show that the compatification of $\mathcal{D}(p)$ results in a bona fide smooth CW decomposition of M.

Clearly, $\mathcal{D}(p)$ is diffeomorphic to an open disk of dimension $\mathrm{ind}(p)$, and $\mathcal{D}(p) \cap \mathcal{D}(q) = \emptyset$ when $p \neq q$. Recall the evaluation map $e : \overline{\mathcal{D}(p)} \longrightarrow M$ and that $\overline{\mathcal{D}(p)} = \bigsqcup_I \mathcal{M}_I \times \mathcal{D}(r_k)$ (see Theorem 4.5). The restriction of e to $\mathcal{M}_I \times \mathcal{D}(r_k)$ is just the coordinate projection onto $\mathcal{D}(r_k)$. Thus $e|_{\mathcal{D}(p)}$ is the identity map, and $e(\partial\overline{\mathcal{D}(p)})$ consists of finite number of $\mathcal{D}(q)$ such that $\mathrm{ind}(q) < \mathrm{ind}(p)$. Thus if $\overline{\mathcal{D}(p)}$ is homeomorphic to a closed disk for all p, then, $\forall a \in R$, $K^a = \bigsqcup_{f(p)\leq a} \mathcal{D}(p)$ is a finite CW complex with characteristic maps e. We shall prove the following theorems in this chapter.

Theorem 6.1 (Topology of $\overline{\mathcal{D}(p)}$)**.** *Under the assumption of Theorem 4.5, there is a homeomorphism $\Psi : (D^{ind(p)}, S^{ind(p)-1}) \longrightarrow (\overline{\mathcal{D}(p)}, \partial\overline{\mathcal{D}(p)})$, where $D^{ind(p)}$ is the ind(p) dimensional closed disk and $S^{ind(p)-1} = \partial D^{ind(p)}$.*

For the definition of simple homotopy equivalence and elementary expansion, see [14, p. 14-15]

Theorem 6.2 (CW Structure)**.** *Under the assumption of Theorem 4.5, let a be a regular value of f. Then $K^a = \bigsqcup_{f(p)\leq a} \mathcal{D}(p)$ is a finite CW complex with characteristic maps e :*

$\overline{\mathcal{D}(p)} \longrightarrow K^a$, *where* e *is defined in (3) of Theorem 4.5. In particular, if* f *is proper, then the inclusion* $K^a \hookrightarrow M^a$ *is a simple homotopy equivalence. In fact, in this special case, there is a CW decomposition of* M^a *such that* K^a *expands to* M^a *by elementary expansions.*

As mentioned before, $\dim(\mathcal{M}(p,q)) = \mathrm{ind}(p) - \mathrm{ind}(q) - 1$. If $\mathrm{ind}(q) = \mathrm{ind}(p) - 1$, then $\mathcal{M}(p,q)$ is a 0 dimensional manifold. Actually, $\mathcal{M}(p,q)$ consists of finitely many points because it is compact in this case.

The following theorem explicitly computes the boundary operator of the CW chain complex $C_*(K^a)$ associated with the CW structure.

Theorem 6.3 (Boundary Operator)**.** *Let* K^a *be the CW complex in Theorem 6.2 (we do NOT assume* f *is proper). Let* $C_*(K^a)$ *be the associated CW chain complex and* $[\overline{\mathcal{D}(p)}]$ *be the base element represented by* $\overline{\mathcal{D}(p)}$ *in* $C_*(K^a)$*. Then*

$$\partial[\overline{\mathcal{D}(p)}] = \sum_{ind(q)=ind(p)-1} \#\mathcal{M}(p,q)[\overline{\mathcal{D}(q)}],$$

where $\#\mathcal{M}(p,q)$ *is the sum of the orientations* ± 1 *of all points in* $\mathcal{M}(p,q)$ *defined in Theorem 5.1.*

Remark 6.1. *Theorem 6.3 shows that the boundary operator of* $C_*(K^a)$ *coincides with that of the Thom-Smale complex in Morse homology when* M *is compact. This shows Morse homology arises from a cellular chain complex. Morse homology was first formulated by Milnor ([38, cor. 7.3]), and was rediscovered by Witten (see [62]). For more details, see [9] and [55]. For some of its generalizations to Hilbert manifolds, see [52], [1] and [2]. However, unlike the assumption of Theorem 6.3, Morse homology does not require the local triviality of* *metrics. In Chapter 10, we shall extend the above theorems to the case of a general metric. Thus our results also fit Morse homology well in general (see Remark 10.2).*

6.2 Proof of Theorem 6.1

We present an elementary proof here. In Section 10.2, we shall give a non-elementary but quick proof.

Recall the evaluation map $e : \overline{\mathcal{D}(p)} \longrightarrow M$ in (3) of Theorem 4.5. We shall "pull back" the vector field $-\nabla f$ on M to $\overline{\mathcal{D}(p)}$ via e. First, we need to explain the definition of the pull back. We know $\overline{\mathcal{D}(p)} = \bigsqcup_I \mathcal{D}_I$, where I are critical sequences with head p. The restriction of e on $\mathcal{D}_I = \mathcal{M}_I \times \mathcal{D}(r_k)$ is the projection $\mathcal{M}_I \times \mathcal{D}(r_k) \longrightarrow \mathcal{D}(r_k)$. For all $(\alpha, x) \in \mathcal{M}_I \times \mathcal{D}(r_k)$, $\{0\} \times T_x\mathcal{D}(r_k) \subseteq T_\alpha\mathcal{M}_I \times T_x\mathcal{D}(r_k) = T_{(\alpha,x)}(\mathcal{M}_I \times \mathcal{D}(r_k))$ and the derivative of e gives an isomorphism $de : \{0\} \times T_x\mathcal{D}(r_k) \longrightarrow T_x\mathcal{D}(r_k)$. Thus there is a unique vector $(0, -\nabla f) \in \{0\} \times T_x\mathcal{D}(r_k)$ such that $de(0, -\nabla f) = -\nabla f$. Then $(0, -\nabla f(x)) \in T_{(\alpha,x)}(\mathcal{M}_I \times \mathcal{D}(r_k))$ is the pull back of $-\nabla f(x)$.

Lemma 6.4. *There is a smooth vector field X on $\overline{\mathcal{D}(p)}$ such that $\forall (\alpha, z) \in \mathcal{M}_I \times \mathcal{D}(r_k)$, $X(\alpha, z) \in \{0\} \times T_z\mathcal{D}(r_k)$ and $de(X) = -\nabla f$.*

Proof. Let X be the pull back of $-\nabla f$ as explained above. We only need to prove that X is smooth.

Suppose the critical values in $(-\infty, f(p)]$ are exactly $f(p) = c_0 > c_1 > \cdots > c_l$. Let $U(i) = e^{-1} \circ f^{-1}((c_{i+1}, c_{i-1}))$, where $c_{-1} = +\infty$ and $c_{l+1} = -\infty$. By Theorem 4.5, each $U(i)$ is open and $\bigcup_i U(i) = \overline{\mathcal{D}(p)}$, and we only need to prove that X is smooth in each $U(i)$. By (4) of Theorem 4.5, there is a smooth embedding $E(i) : U(i) \longrightarrow \prod_{j=0}^{i-1} f^{-1}(a_j) \times M(i)$, where $a_j \in (c_{j+1}, c_j)$ is a regular value and $M(i) = f^{-1}((c_{i+1}, c_{i-1}))$. Define a vector field $\widehat{X} = (0, \cdots, 0, -\nabla f) \in \prod_{j=0}^{i-1} Tf^{-1}(a_j) \times TM(i)$ on $\prod_{j=0}^{i-1} f^{-1}(a_j) \times M(i)$. Clearly, $\widehat{X}$ is smooth. For brevity, denote $E(i)$ by E. We shall prove that the restriction of $\widehat{X}$ on $E(U(i))$ is X.

Each $(\alpha, z) \in (\mathcal{M}_I \times \mathcal{D}(r_k)) \cap U(i)$ represents a pair (Γ, z), where Γ is a generalized flow line connecting p and z (see (4.7)). Suppose $\Gamma = (\gamma_0, \cdots, \gamma_n)$, where $\gamma_0 \equiv p$ and $\gamma_n(0) = z$. Suppose the intersection of Γ with $f^{-1}(a_j)$ is z_j. Then $\xi(t) = (z_0, \cdots, z_{i-1}, \gamma_n(t))$ is a curve in $E(U(i)) \subseteq \prod_{j=0}^{i-1} f^{-1}(a_j) \times M(i)$ such that

$$\xi'(0) = (0, \cdots, 0, -\nabla f) = \widehat{X}, \qquad de \cdot \xi'(0) = -\nabla f.$$

Moreover, since $\xi(t) \subseteq E(\{\alpha\} \times \mathcal{D}(r_k))$, we infer $\xi'(0) \in dE(\{0\} \times T_z\mathcal{D}(r_k))$. Identify $U(i)$ with $E(U(i))$, then $\widehat{X} = \xi'(0) = X$ at (α, z). This completes the proof. □

Definition 6.5. *Suppose L is a manifold with corners, $\partial^k L$ is the k-stratum ($k > 0$) of L, $x \in \partial^k L$ and $v \in T_x L$. v is in the corner if $v \in T_x \partial^k L$. v is outward if $v \notin A_x L$ (see Definition 2.21). v is strictly outward if $-v$ is in the interior of $A_x L$.*

Clearly, strictly outward implies outward. We know that $A_x L$ is linear isomorphic to $[0, +\infty)^k \times R^{n-k}$. Under this isomorphism, v is in the corner if and only if $v \in \{0\}^k \times R^{n-k}$; v is strictly outward if and only if $v \in (-\infty, 0)^k \times R^{n-k}$. This does not depend on the isomorphisms. It's easy to see the above vector field X is in the corner. We present the following easy lemma without proof.

Lemma 6.6. *If both v_1 and v_2 are strictly outward, so are $v_1 + v_2$ and lv_1 for $l > 0$. If v_1*

is strictly outward and v_2 is in the corner, then $v_1 + v_2$ is strictly outward.

Lemma 6.7. *Suppose L is a manifold with corners, and $g : L \longrightarrow H$ is a smooth map where H is a Hilbert space. If there exists a smooth map $\tilde{g} : L \longrightarrow S(H)$ such that $g(x) = \|g(x)\|\tilde{g}(x)$, then $\|g(x)\|$ is also smooth, where $S(H)$ is the unit sphere of H.*

Proof. Define $\varphi : [0, +\infty) \times S(H) \longrightarrow H \times S(H)$ by $\varphi(\lambda, v) = (\lambda v, v)$. Then

$$d\varphi \frac{\partial}{\partial \lambda} = (v, 0), \qquad d\varphi \frac{\partial}{\partial v} = \left(\lambda \frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right).$$

Thus $d\varphi$ is nonsingular everywhere.

Define $\theta : H \times S(H) \longrightarrow [0, +\infty) \times S(H)$ by $\theta(v_1, v_2) = (\|v_1\|, v_2)$. Then θ is continuous and $\theta\varphi = \text{Id}$. Thus φ is a smooth embedding. Then $\varphi^{-1} : \text{Im}\varphi \longrightarrow [0, +\infty) \times S(H)$ is also smooth.

Clearly, $\forall x \in L$, $(g(x), \tilde{g}(x)) \in \text{Im}\varphi$, and $\varphi^{-1}(g(x), \tilde{g}(x)) = (\|g(x)\|, \tilde{g}(x))$. Since φ^{-1}, $g(x)$ and $\tilde{g}(x)$ are smooth, then so is $\|g(x)\|$. $\square$

Let $\tilde{f} = f \circ e$ defined on $\overline{\mathcal{D}(p)}$ be the pull back of f, then $X \cdot \tilde{f} = -\|(\nabla f)e\|^2 \leq 0$.

Lemma 6.8. *Suppose $x \in \overline{\mathcal{D}(p)}$ be such that $e(x)$ is a critical point. Let U_x be a neighborhood of x. Then there is a smooth vector field Y_x on $\overline{\mathcal{D}(p)}$ such that its support* $\text{supp}(Y_x) \subseteq U_x$*, $Y_x(x) \neq 0$ and $Y_x \tilde{f} \leq 0$. In addition, for all $y \in \partial\overline{\mathcal{D}(p)}$, $Y_x(y)$ is strictly outward if $Y_x(y) \neq 0$.*

Proof. Suppose $e(x) = r_k$ for some critical point r_k and $x = (\alpha, r_k) \in \mathcal{M}_I \times \mathcal{D}(r_k)$, where $I = \{p, r_1, \cdots, r_k\}$. By Lemma 4.7, there exist a neighborhood W_1 of α in $\mathcal{M}_I$, a neighborhood W_2 of r_k in $\mathcal{D}(r_k)$, an $\epsilon > 0$ and a smooth embedding $\varphi : W_1 \times W_2 \times [0, \epsilon)^k \longrightarrow \overline{\mathcal{D}(p)}$ such that $\text{Im}\varphi \subseteq U_x$, and φ satisfies the stratum condition in Lemma 4.7.

By local triviality of the metric, choose a neighborhood U of r_k as (2.2) such that (2.3) and (2.4) hold. We identify U with B by h in (2.2). We may assume $e(\mathrm{Im}\varphi) \subseteq U$, and W_2 is a neighborhood of 0 in V_-. Identify $r_k \in \mathcal{D}(r_k)$ with $0 \in V_-$. The key part of the proof is to show φ can be modified so that

$$\tilde{f} \circ \varphi(\tilde{\alpha}, z, \rho_I, \sigma) = f(r_k) - \frac{1}{2}\langle z, z\rangle + \frac{1}{2}\sigma^2, \tag{6.1}$$

where $\tilde{\alpha} \in W_1$, $z \in W_2$, $\rho_I = (\rho_1, \cdots, \rho_{k-1}) \in [0, \epsilon)^{k-1}$ and $\sigma \in [0, \epsilon)$.

Denote $e \circ \varphi(\tilde{\alpha}, z, \rho_I, \sigma) = (e_1(\tilde{\alpha}, z, \rho_I, \sigma), e_2(\tilde{\alpha}, z, \rho_I, \sigma)) \in V_- \times V_+$. Consider the map $\theta : W_1 \times W_2 \times [0, \epsilon)^k \longrightarrow W_1 \times W_2 \times [0, \epsilon)^k$ defined by

$$\theta(\tilde{\alpha}, z, \rho_I, \sigma) = (\tilde{\alpha}, e_1(\tilde{\alpha}, z, \rho_I, \sigma), \rho_I, \|e_2(\tilde{\alpha}, z, \rho_I, \sigma)\|).$$

Firstly, we prove θ is smooth. It suffices to show $\|e_2\|$ is smooth. Since e_2 is smooth, by Lemma 6.7, we only need to find a smooth $\tilde{g}$ such that $e_2 = \|e_2\|\tilde{g}$. By (4.7), an element in $\overline{\mathcal{D}(p)}$ represents a pair (Γ, z), where Γ is a generalized flow line connecting p and $z \in M$. Let $c = f(r_k)$. Define $E : \overline{\mathcal{D}(p)} \cap e^{-1} \circ f^{-1}((c - \epsilon, c + \epsilon)) \longrightarrow f^{-1}(c + \frac{\epsilon}{2}) \times M$ to be the map $E(\Gamma, z) = (s(\Gamma), z)$, where $s(\Gamma)$ is the intersection of Γ with $f^{-1}(c + \frac{\epsilon}{2})$. By (4) of Theorem 4.5, E is smooth. Furthermore, $E\varphi(\tilde{\alpha}, z, \rho_I, \sigma) = ((\eta_1, \eta_2), (e_1, e_2)) \in V_- \times V_+ \times V_- \times V_+$. By the stratum condition in Lemma 4.7, $e\varphi(\tilde{\alpha}, z, \rho_I, \sigma) \in \mathcal{D}(r_k)$ or $e_2 = 0$ if and only if $\sigma = 0$. Thus, when $\sigma > 0$, $e_2 \neq 0$ and (e_1, e_2) is connected with (η_1, η_2) by a unbroken flow line. Thus $(e_1, e_2) = (\lambda^{-1}\eta_1, \lambda\eta_2)$ for some $\lambda > 0$ and $e_2/\|e_2\| = \eta_2/\|\eta_2\|$. However, $\eta_2 \neq 0$ even if $\sigma = 0$. Thus $\eta_2/\|\eta_2\|$ is smooth for all $\sigma \in [0, \epsilon)$. Let $\tilde{g}(\tilde{\alpha}, z, \rho_I, \sigma) = \eta_2/\|\eta_2\|$, then

$e_2 = \|e_2\|\tilde{g}$ for all $\sigma \in [0, \epsilon)$. Thus $\|e_2\|$ is smooth.

Secondly, we prove that $\frac{\partial}{\partial\sigma}\|e_2\| \neq 0$ at $(\alpha, 0, 0, 0)$. By the stratum condition, $d\varphi\frac{\partial}{\partial\sigma}$ represents an inward normal vector in $N_{(\alpha,r_k)}(\mathcal{M}_I \times \mathcal{D}(r_k), \mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})})$, where $J = \{p, r_1, \cdots, r_{k-1}\}$. Thus by Lemma 4.8, $0 \neq de_2\frac{\partial}{\partial\sigma} \in V_-$. Denote $de_2\frac{\partial}{\partial\sigma}$ by w. Since $e_2(\alpha, 0, 0, 0) = 0$, we see $e_2(\alpha, 0, 0, \sigma) = \sigma w + O(\sigma^2)$, and

$$\frac{\partial}{\partial\sigma}|_{\sigma=0}\|e_2\| = \lim_{\sigma\to 0+}\frac{\|\sigma w + O(\sigma^2)\|}{\sigma} = \|w\| \neq 0.$$

Thirdly, the Jacobian of θ at $(\alpha, 0, 0, 0)$ is

$$\begin{pmatrix} \frac{\partial}{\partial\tilde{\alpha}} & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial z} & de_1\frac{\partial}{\partial\rho_I} & de_2\frac{\partial}{\partial\sigma} \\ 0 & 0 & \frac{\partial}{\partial\rho_I} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial\sigma}\|e_2\| \end{pmatrix}.$$

Since $\frac{\partial}{\partial\sigma}\|e_2\| \neq 0$, $d\theta$ is nonsingular at $(\alpha, 0, 0, 0)$.

Since $\|e_2\|$ is smooth, $\frac{\partial}{\partial\sigma}|_{\sigma=0}\|e_2\| \neq 0$, and $\|e_2\|$ vanishes if and only if $\sigma = 0$, we can extend $\|e_2\|$ to be defined on $W_1 \times W_2 \times (-\epsilon, \epsilon)^k$ such that $\|e_2\| < 0$ when $\sigma < 0$. By the Inverse Function Theorem, shrinking W_1, W_2 and ϵ suitably, a smooth θ^{-1} can be defined in $W_1 \times W_2 \times [0, \epsilon)^k$. Modify φ to be $\varphi \circ \theta^{-1}$ to get a smooth embedding $\varphi : W_1 \times W_2 \times [0, \epsilon)^k \longrightarrow$

$\overline{\mathcal{D}(p)}$ such that $e \circ \varphi(\tilde{\alpha}, z, \rho_I, \sigma) = (z, e_2)$ and $\|e_2\| = \sigma$. This gives (6.1).

Consider the vector field $\widetilde{Y} = \sum_{i=1}^{k-1}(\rho_i - \epsilon)\frac{\partial}{\partial \rho_i} + (\sigma - \epsilon)\frac{\partial}{\partial \sigma}$ in $W_1 \times W_2 \times [0, \epsilon)^k$. It's strictly outward at corners, $\widetilde{Y}(\varphi^{-1}(x)) \neq 0$ and $\widetilde{Y}(\tilde{f} \circ \varphi) = (\sigma - \epsilon)\sigma \leq 0$.

By Lemma 6.6, using the partition of the unity, we can move $\widetilde{Y}$ to $\overline{\mathcal{D}(p)}$. This defines the desired smooth vector field Y_x. □

Lemma 6.9. *Suppose $x \in \overline{\mathcal{D}(p)}$ is such that $e(x)$ is a regular point. Let U_x be a neighborhood of x. Then there is a smooth vector field Y_x on $\overline{\mathcal{D}(p)}$ such that its support* $\mathrm{supp}(Y_x) \subseteq U_x$*, $Y_x(x) \neq 0$ and $Y_x\tilde{f} = 0$. In addition, $\forall y \in \partial\overline{\mathcal{D}(p)}$, $Y_x(y)$ is strictly outward if $Y_x(y) \neq 0$.*

Proof. Suppose $x \in \mathcal{M}_I \times \mathcal{D}(r_k)$. By Lemma 4.7, there is a smooth embedding $\varphi : W \times [0, \epsilon)^k \longrightarrow \overline{\mathcal{D}(p)}$ such that $\mathrm{Im}\varphi \subseteq U_x$ where W is a neighborhood of x in $\mathcal{M}_I \times \mathcal{D}(r_k)$. Since $e(x)$ is a regular point, $X\tilde{f}(x) = -\|\nabla f(e(x))\|^2 < 0$. Shrinking W and ϵ suitably, we may assume $X\tilde{f} < 0$ in $\mathrm{Im}\varphi$.

Denote the coordinates of $[0, \epsilon)^k$ by $(\rho_1, \cdots, \rho_k)$. Then $\sum_{i=1}^{k}(\rho_i - \epsilon)\frac{\partial}{\partial \rho_i}$ defines a vector field on $W \times [0, \epsilon)^k$ which is strictly outward at corners. Move this one to $\mathrm{Im}\varphi$ to get a strictly outward vector field Y_1 on $\mathrm{Im}\varphi$. Let $Y_2 = Y_1 - \frac{Y_1\tilde{f}}{X\tilde{f}}X$. Then $Y_2\tilde{f} = 0$. Since Y_1 is strictly outward, and X is in the corner, we get, by Lemma 6.6, Y_2 is strictly outward and $Y_2(x) \neq 0$. Using a partition of the unity, we get Y_x. □

As mentioned in Introduction, the following key lemma fulfills Milnor's suggestion of adding a vector field to X.

Lemma 6.10. *Suppose $K \subseteq \mathcal{D}(p) \subseteq \overline{\mathcal{D}(p)}$, K is closed and p is an interior point of K. Then there is a smooth vector field $\widetilde{X}$ on $\overline{\mathcal{D}(p)}$ such that $\widetilde{X}\tilde{f} \leq X\tilde{f} = (-\nabla f)f$, $\widetilde{X}$ equals X and $-\nabla f$ on K, and $\widetilde{X}$ is strictly outward on $\partial\overline{\mathcal{D}(p)}$.*

Proof. Since K is closed, $\overline{\mathcal{D}(p)} - K$ is open. Since $K \subseteq \mathcal{D}(p)$, then $\overline{\mathcal{D}(p)} - K \supseteq \partial\overline{\mathcal{D}(p)}$. Thus $\forall x \in \partial\overline{\mathcal{D}(p)}$, by Lemmas 6.8 and 6.9, there is a vector field Y_x such that $\mathrm{supp}(Y_x) \subseteq \overline{\mathcal{D}(p)} - K$ and satisfies the conclusions of those lemmas. Define $W_x = \{y | Y_x(y) \neq 0\}$, we have W_x is a neighborhood of x. Since $\partial\overline{\mathcal{D}(p)}$ is compact, it can be covered by finite many W_{x_i} $(i = 1, \cdots, n)$. Let $Y = \sum_{i=1}^{n} Y_{x_i}$. Since $Y_{x_i}\tilde{f} \leq 0$, we get $Y\tilde{f} \leq 0$. Since Y_{x_i} vanishes on K, so does Y. Also since $\{W_{x_i} \mid i = 1, \cdots, n\}$ covers $\partial\overline{\mathcal{D}(p)}$, and Y_{x_i} is strictly outward if it's nonzero, by Lemma 6.6, we have that Y is strictly outward. Recall that X is in the corner on $\partial\overline{\mathcal{D}(p)}$. We complete the proof by defining $\widetilde{X} = X + Y$. □

Lemma 6.11. *Let $\phi_t(x)$ be the flow line of $\widetilde{X}$ with initial value x and $x \neq p$. Then $\phi_t(x)$ reaches $\partial\overline{\mathcal{D}(p)}$ at a unique time $0 \leq \omega(x) < +\infty$. Furthermore, $\omega(x)$ is continuous with respect to x in $\overline{\mathcal{D}(p)} - \{p\}$.*

Proof. Above all, we prove the following claim: If $\phi_t(x)$ cannot reach $\partial\overline{\mathcal{D}(p)}$ when $t \geq 0$, then $\phi_t(x)$ exists for $t \in [0, +\infty)$.

If not, the maximal positive flow of $\phi_t(x)$ can only be defined in $[0, s]$ or $[0, s)$, where $s < +\infty$. If the domain is $[0, s]$, then $\phi_s(x) \in \partial\overline{\mathcal{D}(p)}$. This is a contradiction. If the domain is $[0, s)$, by the compactness of $\overline{\mathcal{D}(p)}$, $\phi_t(x)$ has a cluster point y_0 when $t \to s$. There are two cases. Case (1): $y_0 \in \mathcal{D}(p)$. In this case, there is a neighborhood U_{y_0} of y_0 such that there exists $\delta > 0$ such that, for all $y \in U_{y_0}$, and for all $t \in (-\delta, +\delta)$, $\phi_t(y)$ exists. Thus $\phi_t(x)$ can be defined in $[0, s + \delta)$. This is a contradiction. Case (2): $y_0 \in \partial\overline{\mathcal{D}(p)}$. In this case, a neighborhood U_{y_0} of y_0 is diffeomorphic to an open subset of $[0, +\infty)^k \times R^{n-k}$ for some k and n. The vector field in U_{y_0} can be smoothly extended to an open subset of R^n. Then we may consider y_0 as an interior point. This converts the argument to the first case. We can define $\phi_t(x)$ for $t \in [0, s]$ with $\phi_s(x) = y_0$. This is also a contradiction. This gives the claim.

Secondly, we prove that $\phi_t(x)$ reaches $\partial\overline{\mathcal{D}(p)}$ at some time $0 \leq \omega(x) < +\infty$ by contradiction.

Suppose $\phi_t(x)$ doesn't reach $\partial\overline{\mathcal{D}(p)}$. By the claim, $\phi_t(x)$ exists for $t \in [0, +\infty)$. By the assumption, $m = \inf_M f > -\infty$. For all $y \in \overline{\mathcal{D}(p)}$, $\tilde{f}(y) \leq \tilde{f}(p) = f(p)$. For all $T \geq 0$,

$$\int_0^T \widetilde{X}\tilde{f}(\phi_t(x))dt = \tilde{f}(\phi_T(x)) - \tilde{f}(\phi_0(x)) \geq m - f(p) > -\infty. \tag{6.2}$$

Since $\widetilde{X}\tilde{f} \leq X\tilde{f} \leq 0$, then there exists $\{t_n\} \subseteq [0, +\infty)$, $t_n \to +\infty$ and $\widetilde{X}\tilde{f}(\phi_{t_n}(x)) \to 0$. Since $\overline{\mathcal{D}(p)}$ is compact, we may assume $\phi_{t_n}(x) \to y_0$. Then $0 = \widetilde{X}\tilde{f}(y_0) \leq X\tilde{f}(y_0) \leq 0$. Since $X\tilde{f}(y_0) = -\|\nabla f(e(y_0))\|^2$, we see that $e(y_0)$ is a critical point. Thus $y_0 \in \partial\overline{\mathcal{D}(p)}$. Choose a neighborhood U_{y_0} of y_0 which is diffeomorphic to $[0, \epsilon)^k \times B(0, \epsilon)$, where $B(0, \epsilon) = \{v \in R^{n-k} \mid \|v\| < \epsilon\}$ and y_0 is identified with $0 \in [0, \epsilon)^k \times B(0, \epsilon)$. Identify U_{y_0} with $[0, \epsilon)^k \times B(0, \epsilon)$. We may assume $\widetilde{X}$ can be extended smoothly to $(-\epsilon, \epsilon)^k \times B(0, \epsilon)$. Denote the flow of the extended vector field by φ_t. Then $\varphi_t(y_0) = \varphi_t(0) = t\widetilde{X}(0) + O(t^2)$. Since $\widetilde{X}(0)$ is outward, there exists $\delta_1 > 0$, such that for all $\delta \in (0, \delta_1]$, $\varphi_\delta(0) \in (-\epsilon, \epsilon)^k \times B(0, \epsilon) - [0, \epsilon)^k \times B(0, \epsilon)$. Fixing δ, there exists $\epsilon_1 > 0$, for all $y \in [0, \epsilon_1)^k \times B(0, \epsilon_1)$, $\varphi_t(y)$ exists for $t \in [-\delta, \delta]$ and $\varphi_\delta(y) \in (-\epsilon, \epsilon)^k \times B(0, \epsilon) - [0, \epsilon)^k \times B(0, \epsilon)$. Since $(0, \epsilon)^k \times B(0, \epsilon)$ and $(-\epsilon, \epsilon)^k \times B(0, \epsilon) - [0, \epsilon)^k \times B(0, \epsilon)$ are disconnected, we have $\varphi_{t_0}(y) \in [0, \epsilon)^k \times B(0, \epsilon) - (0, \epsilon)^k \times B(0, \epsilon)$ at some time $t_0 \in [0, \delta)$. Since $\phi_{t_n}(x) \in [0, \epsilon_1)^k \times B(0, \epsilon_1)$ for some t_n, we have $\phi_{t_n + t_0}(x) \in \partial\overline{\mathcal{D}(p)}$ for some $t_0 \in [0, \delta)$. This gives a contradiction.

Finally, we prove that $\omega(x)$ is unique and continuous.

Since $\widetilde{X}$ is outward, $\phi_t(x)$ does not exist after it reaches $\partial\overline{\mathcal{D}(p)}$. Thus $\omega(x)$ is unique.

Denote $y_0 = \phi_{\omega(x_0)}(x_0) \in \partial\overline{\mathcal{D}(p)}$, by the argument at the end of the second step, we have, $\forall \delta > 0$, there is a neighborhood U_{y_0} of y_0 such that, for all $y \in U_{y_0}$, $\phi_{t_0}(y) \in \partial\overline{\mathcal{D}(p)}$ for some $t_0 \in [0, \delta)$. Then there exist a neighborhood U_{x_0} of x_0 and $\delta_2 > 0$ such that, for all $x \in U_{x_0}$, $\phi_{\omega(x_0)-\delta_2}(x)$ exists and is in U_{y_0}. Thus $\omega(x) \le \omega(x_0) + \delta$. Since $\omega(x) \ge 0$, and $\omega(x) = 0$ when $x \in \partial\overline{\mathcal{D}(p)}$, we get $\omega(x)$ is continuous at $x_0 \in \partial\overline{\mathcal{D}(p)}$. If $x_0 \in \mathcal{D}(p)$, then for all $\delta > 0$, there exists $\delta_2 \in (0, \delta)$, such that $\phi_{\omega(x_0)-\delta_2}(x_0)$ exists and is in $\mathcal{D}(p)$. Also, there exists U_{x_0} such that, for all $x \in U_{x_0}$, $\phi_{\omega(x_0)-\delta_2}(x)$ exists and is in $\mathcal{D}(p)$. Thus $\omega(x) \ge \omega(x_0) - \delta$. We have now proved $\omega(x)$ is continuous in general. $\square$

Actually, the above lemma only requires $\widetilde{X}$ to be outward. However, the following one requires $\widetilde{X}$ to be strictly outward.

Lemma 6.12. *Let $\phi_t(x)$ be the flow line of $\widetilde{X}$ with initial value x. Then $\phi_t(x)$ exists for $t \in (-\infty, 0]$ and $\lim_{t\to-\infty} \phi_t(x) = p$.*

Proof. Firstly, we prove that $\phi_t(x)$ exists for $t \in (-\infty, 0]$ by contradiction. If not, the maximal negative flow can only be defined for $[s, 0]$ or $(s, 0]$, where $s > -\infty$.

Suppose the domain is $[s, 0]$. If $\phi_s(x) \in \mathcal{D}(p)$, then $\phi_t(x)$ can be defined in $(s - \delta, 0]$ for some $\delta > 0$. This is a contradiction. Suppose $\phi_s(x) = x_0 \in \partial\overline{\mathcal{D}(p)}$. Like the proof of Lemma 6.11, a neighborhood of x_0 is identified with $[0, \epsilon)^k \times B(0, \epsilon)$ and x_0 is identified with 0. Extend the vector field in $[0, \epsilon)^k \times B(0, \epsilon)$ smoothly to be defined in $(-\epsilon, \epsilon)^k \times B(0, \epsilon)$. Denote the flow of the extended vector field by φ_t. Since $\widetilde{X}(0) = \widetilde{X}(x_0)$ is strictly outward, then $-\widetilde{X}(0) \in (0, +\infty)^k \times R^{n-k}$. Since $\varphi_t(x_0) = \varphi_t(0) = t\widetilde{X}(0) + O(t^2)$, there exists $\delta > 0$ such that, for all $t \in [-\delta, 0]$, we have $\varphi_t(0) \in [0, \epsilon)^k \times B(0, \epsilon)$. Thus $\phi_t(x)$ exists for $t \in [s - \delta, 0]$. This gives a contradiction.

Suppose the domain is $(s, 0]$. Using the same argument as in the proof of Lemma 6.11, we can extend the domain to be $[s, 0]$. This gives a contradiction.

As a result, we proved the first assertion.

Secondly, we prove by contradiction that $\phi_t(x)$ has no cluster point in $\partial\overline{\mathcal{D}(p)}$ when $t \to -\infty$.

Suppose $\phi_t(x)$ has a cluster point $x_0 \in \partial\overline{\mathcal{D}(p)}$. By the continuity of $\omega(x)$ in Lemma 6.11, there exists a neighborhood U_{x_0} of x_0 such that, for all $x \in U_{x_0}$, we have $\omega(x) \in [0, 1)$. Since x_0 is a cluster point, there exist $T < -1$, and $\phi_T(x) \in U_{x_0}$. Thus $\phi_{T+t_0}(x) \in \partial\overline{\mathcal{D}(p)}$ for some $t_0 \in [0, 1)$. Then $\phi_t(x)$ does not exist when $t > T + t_0$. In particular, $\phi_t(x)$ does not exist when $t = 0$. This gives a contradiction.

Thirdly, we prove by contradiction that $\phi_t(x)$ has no cluster point in $\mathcal{D}(p) - \{p\}$ when $t \to -\infty$.

Suppose $x_0 \in \mathcal{D}(p) - \{p\}$ is a cluster point. Clearly, $\widetilde{X}\tilde{f}(x_0) \le Xf(x_0) = -\|\nabla f(e(x_0))\|^2 = A < 0$. Thus there exists a neighborhood U_{x_0} of x_0, a $\delta > 0$, for all $x \in U_{x_0}$, such that $\phi_t(x)$ exists for $t \in [-\delta, \delta]$ and $\widetilde{X}\tilde{f}(\phi_t(x)) \le \frac{A}{2}$ in this interval. Since x_0 is a cluster point, there exists $\{t_n\} \subseteq (-\infty, 0]$ such that $t_{n+1} < t_n - \delta$ and $\phi_{t_n}(x) \in U_{x_0}$. Then

$$\int_{-\infty}^{0} \widetilde{X}\tilde{f}(\phi_t(x))dt \le \sum_{n=1}^{\infty} \int_{t_n-\delta}^{t_n} \widetilde{X}\tilde{f}(\phi_t(x))dt \le \sum_{n=1}^{\infty} \int_{t_n-\delta}^{t_n} \frac{A}{2} = -\infty.$$

On the other hand, similar to (6.2), we have for all $T < 0$, $\int_T^0 \widetilde{X}\tilde{f}(\phi_t(x))dt \ge \tilde{f}(x) - \tilde{f}(p) > -\infty$. This gives a contradiction.

Finally, since $\overline{\mathcal{D}(p)}$ is compact, $\forall\{t_n\} \subseteq (-\infty, 0]$, there must be a cluster point of $\phi_{t_n}(x)$. Thus $\phi_t(x) \to p$ when $t \to -\infty$. $\square$

Now we are ready to prove Theorem 6.1. The idea of this proof is as follows. Choose a closed neighborhood K of p in $\overline{\mathcal{D}(p)}$ which is diffeomorphic to $D^{\mathrm{ind}(p)}$. The flow line ϕ_t of the above $\widetilde{X}$ expands K homeomorphically onto $\overline{\mathcal{D}(p)}$. We also explain this idea by the previous example on T^2. The flow generated by X on $\overline{\mathcal{D}(p)}$ is as the right part of Figure 3. The flow generated by $\widetilde{X}$ is illustrated by Figure 4.

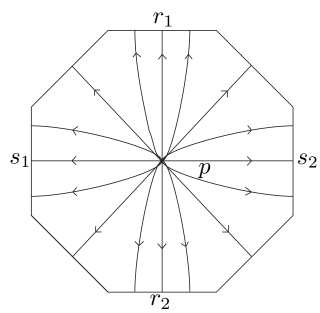

Figure 4: Flow Generated by $\widetilde{X}$

Proof of Theorem 6.1. Choose a closed neighborhood K of p in $\overline{\mathcal{D}(p)}$ satisfying the following two properties: (1). $K \subseteq \mathcal{D}(p)$. (2). There is a diffeomorphism $\theta : D(\epsilon) \longrightarrow K$ such that $\theta(0) = p$, $\tilde{f} \circ \theta(v) = f(p) - \frac{1}{2}\langle v, v\rangle$ and $((d\theta)^{-1}X)(v) = v$, where $D(\epsilon) = \{v \in R^{\mathrm{ind}(p)} \mid \|v\| \leq \epsilon\}$.

We only need to construct a homeomorphism $\Psi : (D(\epsilon), S(\epsilon)) \longrightarrow (\overline{\mathcal{D}(p)}, \partial\overline{\mathcal{D}(p)})$, where $S(\epsilon) = \partial D(\epsilon)$.

By Lemmas 6.10, 6.11 and 6.12, there is a vector field $\widetilde{X}$ on $\overline{\mathcal{D}(p)}$ satisfying the following four properties: (1). We have $\widetilde{X} = X$ in K. (2). We have $\widetilde{X}\tilde{f} < 0$ in $\mathcal{D}(p) - \{p\}$. (3). The flow $\phi_t(x)$ generated by $\widetilde{X}$ reaches the boundary at a unique time $\omega(x) \in [0, +\infty)$ when $x \neq p$, and $\omega(x)$ is continuous in $\overline{\mathcal{D}(p)} - \{p\}$. (4). For all x, $\phi_t(x) \to p$ when $t \to -\infty$.

Denote φ_t the flow generated by the vector field $Z(v) = v$ on $D(\epsilon)$. Then $\theta(\varphi_t(v)) =$

$\phi_t(\theta(v))$.

Define $\beta(s)$ in $[0,\epsilon]$ to be

$$\beta(s) = \begin{cases} 0 & t \in [0, \frac{\epsilon}{2}], \\ \frac{2t-\epsilon}{\epsilon} & t \in [\frac{\epsilon}{2}, \epsilon]. \end{cases}$$

Define $\Psi : D(\epsilon) \longrightarrow \overline{\mathcal{D}(p)}$ to be

$$\Psi(v) = \begin{cases} \theta(v) & \|v\| \in [0, \frac{\epsilon}{2}], \\ \phi[\omega[\theta(\frac{v}{\|v\|}\epsilon)]\beta(\|v\|), \theta(v)] & \|v\| \in [\frac{\epsilon}{2}, \epsilon]. \end{cases}$$

Here we use the notation $\phi(t,x) = \phi_t(x)$.

Firstly, Ψ is continuous, $\Psi(S(\epsilon)) \subseteq \partial\overline{\mathcal{D}(p)}$ and $\Psi^{-1}(\partial\overline{\mathcal{D}(p)}) \subseteq S(\epsilon)$.

Secondly, we prove that Ψ is injective. Consider the orbits of the flows. The orbits in $D(\epsilon)$ are $\{0\}$ and $\{sv \mid \|v\| = \epsilon, s \in (0,1]\}$. We have $\Psi(0) = p$ and $\Psi(sv) = \phi(l(s,v), \theta(v))$, where $l(s,v) = \omega(\theta(v))\beta(s\epsilon) + \log s$ and $\|v\| = \epsilon$. When $\|v\| \equiv \epsilon$, $\tilde{f}(\theta(v)) \equiv \frac{1}{2}f(p) - \frac{1}{2}\epsilon^2$, by the above property (2) of $\widetilde{X}$, we have Ψ maps distinct orbits to distinct orbits. Since $l(s,v)$ is a strictly increasing function with respect to s, by the above property (2) of $\widetilde{X}$ again, we have Ψ is injective.

Thirdly, we prove that Ψ is surjective. Clearly, $\Psi(0) = p$. For all $x \in \overline{\mathcal{D}(p)}$ and $x \neq p$, since $\phi_t(x) \to p$ when $t \to -\infty$, we have that there exist $t_0 \in R$ and $v_0 \in D(\epsilon)$ such that $\phi_{-t_0}(x) \in K$, $\|v_0\| = \epsilon$, and $v_0 = \theta^{-1}(\phi_{-t_0}(x))$. Then $t_0 \leq \omega(\theta(v_0))$. Since $l(s,v_0) \to -\infty$ when $s \to 0$ and $l(s,v)$ is continuous, the range of $l(s,v_0)$ is $(-\infty, \omega(\theta(v_0))]$. Then there exists s_0 such that $l(s_0,v_0) = t_0$. Thus $\Psi(s_0v_0) = x$. Therefore, Ψ is surjective.

Finally, Ψ is a map from a compact space to a Hausdorff space, so Ψ is a homeomorphism.

□

6.3 Proof of Theorem 6.2

As mentioned in Section 6.1, the CW complex structure of K^a immediately results from Theorems 4.5 and 6.1. We only need to prove that, when f is proper, M^a has the desired CW decomposition. By Theorem 6.1, we can always construct a CW decomposition from a good vector field. The key part of this proof is to find a good vector field for M^a (see Lemma 6.14). This is heavily based on Milnor's dealing with gradient-like dynamics in [38].

Up until now, we haven't assumed that M^a is compact and we have considered only negative gradient dynamics. In this section, we take M^a to be compact because we take f to be proper. The results proved before this section still hold for negative gradient-like dynamics when the underlying manifold is compact. There are two reasons. Both are sufficient. Firstly, Lemma 2.15 and the comment after Definition 2.7 show that and (M, f) is a CF pair automatically. Secondly, we can formally replace "gradient" by "gradient-like" in the above proofs when M is compact.

Since a is a regular value of f and f is proper, M^a is a compact manifold with boundary $f^{-1}(a)$. There is a smooth collar embedding $\varphi : [0, \epsilon_0) \times \partial M^a \longrightarrow M^a$ such that $f \circ \varphi(s, x) = a - s$. Clearly, all critical points of f are in $M^a - \mathrm{Im}\varphi$. Double M^a to be a compact manifold $2M^a$ without boundary such that the above φ can be extended in the obvious way to a smooth embedding $\varphi : (-\epsilon_0, \epsilon_0) \times \partial M^a \longrightarrow 2M^a$.

For convenience, we identify $(-\epsilon_0, \epsilon_0) \times \partial M^a$ with $\mathrm{Im}\varphi$ from now on.

There is an evident Z_2-symmetry group acting on $2M^a$. For all $x \in M^a \subseteq 2M^a$, denote

$\bar{x} \in 2M^a$ the copy of x. Define $\sigma : 2M^a \longrightarrow 2M^a$ by $\sigma(x) = \bar{x}$ and $\sigma(\bar{x}) = x$. Then

$$Z_2 = \{\mathrm{Id}, \sigma\} \tag{6.3}$$

is the group. By the smooth structure of $2M^a$, Z_2 acts smoothly. The set of fixed points of Z_2 is $\mathrm{Fix}(Z_2; 2M^a) = \partial M^a$.

We omit the proof of the following, which is straightforward.

Lemma 6.13. *There exists a Morse Function F on $2M^a$ satisfying the following properties. (1). It is invariant under the Z_2 action. (2). It equals f on $M^a - \mathrm{Im}\varphi$. (3). We have $F(s, x) = a - \frac{1}{2}s^2 + g(x)$ in $(-\delta, \delta) \times \partial M^a$ for some $\delta \in (0, \epsilon_0)$, and g (and then $F|_{\partial M^a}$) is a Morse function on ∂M^a. (4). The critical points of F are exactly the critical points of f (which are in $M^a - \mathrm{Im}\varphi$) together with their images under the Z_2 action, and the critical points of g. (5) The function values of F on ∂M^a are greater than the function values at critical points off ∂M^a.*

We can define a metric G on $2M^a$ satisfying the following properties. (1). It is invariant under the Z_2 action. (2). It equals the original metric on $M^a - \mathrm{Im}\varphi$. (3). It is a product metric on $(-\delta, \delta) \times \partial M^a$, where $(-\delta, \delta)$ is given the standard metric. (4). It is locally trivial.

Lemma 6.14. *There is a negative gradient-like vector field ξ of F on $2M^a$ satisfying the following properties. (1). The vector field ξ is invariant under the Z_2 action. (2). It equals $-\nabla f$ on $M^a - Im\varphi$. (3). It satisfies local triviality and transversality. (4). For all $x \in \partial M^a$, $\xi(x) \in T_x\partial M^a$, and $\xi|_{\partial M^a}$ is a negative gradient-like vector field of $F|_{\partial M^a}$ on ∂M^a satisfying local triviality and transversality.*

Proof. We shall modify $-\nabla F$ to be ξ. The proof follows closely those of [38, thm. 4.4, lem. 4.6 and thm. 5.2] plus arguing in the Z_2 invariant setting. The book [38] uses gradient-like vector fields, we use negative ones.

Clearly, if ξ is Z_2 invariant, then $\xi(x) \in T_x\partial M^a$ for all $x \in \partial M^a$. Since both F and the metric on $2M^a$ are Z_2 invariant, so is $-\nabla F$. By the constructions of F and the metric, $-\nabla F$ and $-\nabla F|_{\partial M^a}$ satisfy everything but transversality.

Suppose the critical points on ∂M^a have function values $c_1 < \cdots < c_l$. Suppose c_0 is the maximum of function values on critical points off ∂M^a. By (5) of Lemma 6.13, $c_0 < c_1$. By induction on k, we shall modify the vector field ξ on M^{a_k,b_k} for some a_k, $b_k \in (c_{k-1}, c_k)$ such that the vector field on M^{c_k} satisfies the conclusion (in M^{c_k}, we don't consider $\mathcal{D}(p) \cap M^{c_k}$ for $p \notin M^{c_k}$), and the vector field globally satisfies everything but transversality.

Firstly, by (2) and (4) of Lemma 6.13 and the construction of the metric, the vector field on M^{c_0} satisfies the conclusion automatically.

Secondly, supposing we have finished the construction for $M^{c_{k-1}}$, we shall modify ξ for M^{c_k} by the method in [38]. Suppose the critical points with function value c_k are exactly p_i $(i = 1, \cdots, n)$. Denote the descending and ascending manifolds of p in ∂M^a with respect to $\xi|_{\partial M^a}$ by $\widetilde{\mathcal{D}(p)}$ and $\widetilde{\mathcal{A}(p)}$ respectively.

By (3) of Lemma 6.13 and local triviality of ξ, there is a neighborhood U_i of p_i such that U_i has a coordinate chart (s, v_1, v_2), $s^2 < 4\epsilon$, $\|v_1\|^2 < 4\epsilon$, $\|v_2\|^2 < 4\epsilon$, $F(s, v_1, v_2) = c_k - \frac{1}{2}s^2 - \frac{1}{2}\|v_1\|^2 + \frac{1}{2}\|v_2\|^2$, the metric on U_i is standard, and the action of σ is $\sigma(s, v_1, v_2) = (-s, v_1, v_2)$. Here ϵ is uniform for all i. We may assume U_i are disjoint for different i. Then $\mathcal{D}(p_i) \cap U_i = \{(s, v_1, 0)\}$ and $\widetilde{\mathcal{D}(p_i)} \cap U_i = \{(0, v_1, 0)\}$. Denote $S_i^- = \mathcal{D}(p_i) \cap F^{-1}(c_k - \epsilon) = \{(s, v_1, 0) \mid s^2 + \|v_1\|^2 = 2\epsilon\}$ and $\widetilde{S_i^-} = \widetilde{\mathcal{D}(p_i)} \cap F^{-1}(c_k - \epsilon) = \{(0, v_1, 0) \mid \|v_1\|^2 = 2\epsilon\}$. Let

$B_2 = \{v_2 \mid \|v_2\|^2 < \epsilon\}$. Then we have a map $\alpha_i : S_i^- \times B_2 \longrightarrow F^{-1}(c_k - \epsilon) \cap U_i$ defined by

$$\alpha_i(s, v_1, 0, v_2) = ((\|v_2\|^2 + 2\epsilon)^{\frac{1}{2}}(2\epsilon)^{-\frac{1}{2}}s, (\|v_2\|^2 + 2\epsilon)^{\frac{1}{2}}(2\epsilon)^{-\frac{1}{2}}v_1, v_2).$$

Clearly, α_i is a diffeomorphism to its image, and $\alpha_i(\widetilde{S_i^-} \times B_2) \subseteq \partial M^a$. There is a $v_{2,i} \in B_2$ such that, for all critical points $q \in M^{c_{k-1}}$, $\alpha_i : S_i^- \times \{v_{2,i}\} \longrightarrow F^{-1}(c_k - \epsilon)$ is transverse to $\mathcal{A}(q) \cap F^{-1}(c_k - \epsilon)$, and $\alpha_i : \widetilde{S_i^-} \times \{v_{2,i}\} \longrightarrow F^{-1}(c_k - \epsilon) \cap \partial M^a$ is transverse to $\widetilde{\mathcal{A}(q)} \cap F^{-1}(c_k - \epsilon)$. Define $\alpha_{t,i} : S_i^- \longrightarrow F^{-1}(c_k - \epsilon)$ by $\alpha_{t,i}(s, v_1, 0) = \alpha_i(s, v_1, 0, tv_{2,i})$ for $t \in [0, 1]$. When t varies in $[0, 1]$, $\alpha_{t,i} : S_i^- \longrightarrow F^{-1}(c_k - \epsilon)$ is an isotopy of embeddings, and its restriction to $\widetilde{S_i^-}$ is also an isotopy of embeddings $\alpha_{t,i} : \widetilde{S_i^-} \longrightarrow F^{-1}(c_k - \epsilon) \cap \partial M^a$. Moreover, $\alpha_{t,i}$ is Z_2 equivariant. Following [38], we can extend $\alpha_{t,i}$ to be a Z_2 equivariant isotopy of $F^{-1}(c_k - \epsilon)$, which we still denote by $\alpha_{t,i}$, such that $\alpha_{0,i}$ is the identity, and $\alpha_{t,i}$ is the identity outside of U_i for all t.

Since U_i are disjoint for all i, composing these isotopies $\alpha_{t,i}$, we get a Z_2 equivariant isotopy α_t of $F^{-1}(c_k - \epsilon)$ such that $\alpha_t(U_i) = U_i$ and $\alpha_t|_{U_i} = \alpha_{t,i}|_{U_i}$. We have α_0 is the identity, and for all critical points $q \in M^{c_{k-1}}$, $\alpha_1 : S_i^- \longrightarrow F^{-1}(c_k - \epsilon)$ is transverse to $\mathcal{A}(q) \cap F^{-1}(c_k - \epsilon)$, and $\alpha_1 : \widetilde{S_i^-} \longrightarrow F^{-1}(c_k - \epsilon) \cap \partial M^a$ is transverse to $\widetilde{\mathcal{A}(q)} \cap F^{-1}(c_k - \epsilon)$.

By this isotopy α_t and its Z_2 equivariance, following [38], we can modify ξ in $M^{c_k - \epsilon, c_k - \frac{1}{2}\epsilon}$ such that the new ξ is still Z_2 invariant, and $\mathcal{D}(p_i)$ ($\widetilde{\mathcal{D}(p_i)}$) is transverse to $\mathcal{A}(q)$ ($\widetilde{\mathcal{A}(q)}$) for all p_i and all critical points $q \in M^{c_{k-1}}$. Since ξ only changed in $M^{c_k - \epsilon, c_k - \frac{1}{2}\epsilon}$, we have ξ and $\xi|_{\partial M^a}$ are still locally trivial, and on $M^{c_{k-1}}$ nothing has changed. Thus we get a desired ξ for M^{c_k}.

The above two steps complete the induction. □

By Lemma 6.14, ξ and $\xi|_{\partial M^a}$ give $2M^a$ and ∂M^a a CW decomposition respectively. We shall consider the relation between these two decompositions. Use the same notations as in the proof of Lemma 6.14, denote the descending and ascending manifolds of $\xi|_{\partial M^a}$ by $\widetilde{\mathcal{D}(p)}$ and $\widetilde{\mathcal{A}(p)}$. It's easy to see that $\mathcal{D}(p) \cap \mathcal{A}(q) = \widetilde{\mathcal{D}(p)} \cap \widetilde{\mathcal{A}(q)}$ when $p,\, q \in \partial M^a$. Thus the moduli spaces $\mathcal{M}(p,q)$ of ξ and $\xi|_{\partial M^a}$ are the same. Then $\overline{\mathcal{D}(p)} = \coprod_I \mathcal{M}_I \times \mathcal{D}(r_k)$ and $\overline{\widetilde{\mathcal{D}(p)}} = \coprod_{I \subseteq \partial M^a} \mathcal{M}_I \times \widetilde{\mathcal{D}(r_k)}$. Since $\widetilde{\mathcal{D}(r_k)} \subseteq \mathcal{D}(r_k)$, there is a natural embedding $\theta : \overline{\widetilde{\mathcal{D}(p)}} \hookrightarrow \overline{\mathcal{D}(p)}$. In addition, suppose Γ is a generalized flow line connecting p and x, then $\sigma\Gamma$ is a generalized flow line connecting $\sigma p = p$ and σx. Thus there is a Z_2 action on $\overline{\mathcal{D}(p)}$.

Lemma 6.15. *Suppose $p \in \partial M^a$. Then $\theta : \overline{\widetilde{\mathcal{D}(p)}} \hookrightarrow \overline{\mathcal{D}(p)}$ is a smooth embedding. The action of Z_2 on $\overline{\mathcal{D}(p)}$ is smooth and* $\mathrm{Im}\theta = \mathrm{Fix}(Z_2; \overline{\mathcal{D}(p)})$. *In addition, $\widetilde{e} = e\theta$, where $\widetilde{e}$ is the characteristic map $\widetilde{e} : \overline{\widetilde{\mathcal{D}(p)}} \longrightarrow \partial M^a$ and e is the characteristic map $e : \overline{\mathcal{D}(p)} \longrightarrow 2M^a$.*

Proof. Except for smoothness, this lemma is obviously true. We only need to prove smoothness. This is a local property.

Suppose the critical values in $(-\infty, f(p)]$ are $c_l < \cdots < c_0$. Denote $M(i) = F^{-1}((c_{i+1}, c_{i-1}))$, $U(i) = e^{-1}(M(i))$ and $\widetilde{U}(i) = \widetilde{e}^{-1}(M(i))$. Choose $a_i \in (c_{i-1}, c_{i+1})$, by (4) of Theorem 4.5, we have the following commutative diagram, and both $E(i)$ and $\widetilde{E}(i)$ are smooth embeddings. Thus θ is a smooth embedding.

$$\begin{tikzcd}
\widetilde{U}(i) \arrow[r, "\widetilde{E}(i)"] \arrow[d, "\theta"'] & \prod_{j=0}^{i-1} F^{-1}(a_j) \times M(i) \\
U(i) \arrow[ur, "E(i)"'] &
\end{tikzcd}$$

Since F is Z_2 invariant, there is a smooth Z_2 action on $\prod_{j=0}^{i-1} F^{-1}(a_j) \times M(i)$ and $E(i)$ is

Z_2 equivariant. Thus the action of Z_2 on $U(i)$ is smooth. $\square$

Proof of Theorem 6.2. For briefness, we shall not distinguish between a CW complex and its underlying space in this proof.

The function F in Lemma 6.13 and the vector fields ξ and $\xi|_{\partial M^a}$ in Lemma 6.14 give two CW decompositions. They are $2M^a = \bigsqcup_p \mathcal{D}(p)$ with characteristic maps $e : \overline{\mathcal{D}(p)} \longrightarrow 2M^a$ and $\partial M^a = \bigsqcup_{p\in\partial M^a} \widetilde{\mathcal{D}(p)}$ with characteristic maps $\widetilde{e} : \overline{\widetilde{\mathcal{D}(p)}} \longrightarrow \partial M^a$. The decomposition of $2M^a$ is Z_2 invariant, $K^a = \bigsqcup_{p\in M^a-\partial M^a} \mathcal{D}(p)$ is a subcomplex of $2M^a$, and $\bigsqcup_{p\in 2M^a-M^a} \mathcal{D}(p) = \sigma(K^a) \subseteq 2M^a - M^a$. However, there is still no CW structure on M^a. We shall expand K^a to M^a by a sequence of elementary expansions (compare [14, p. 14]), which gives M^a a CW structure.

For clarity, denote the characteristic map for $\overline{\mathcal{D}(p)}$ by e_p. Suppose $p \in \partial M^a$, and denote $e_p^{-1}(M^a)$ by $\frac{1}{2}\overline{\mathcal{D}(p)}$.

Construct a vector field $\widetilde{X}$ on $\overline{\mathcal{D}(p)}$ as Lemma 6.10, i.e., $\widetilde{X}(F\circ e_p) \leq \xi F$, $\widetilde{X}$ equals ξ near p in $\mathcal{D}(p)$, and $\widetilde{X}$ is strictly outward on $\partial\overline{\mathcal{D}(p)}$. By Lemma 6.15, $\sigma\widetilde{X}$ has the same property as $\widetilde{X}$ does. By Lemma 6.6, and replacing $\widetilde{X}$ by $\frac{1}{2}(\widetilde{X} + \sigma\widetilde{X})$ if necessary, we may assume $\widetilde{X}$ is Z_2 invariant. By the Z_2 invariance of F, Lemma 6.15 and the proof of Theorem 6.1, the Z_2 equivariant flow generated by $\widetilde{X}$ gives a homeomorphism

$$\Psi : \left(\frac{1}{2}D^{\mathrm{ind}(p)}, D^{\mathrm{ind}(p)-1}\right) \longrightarrow \left(\frac{1}{2}\overline{\mathcal{D}(p)}, \overline{\widetilde{\mathcal{D}(p)}}\right),$$

where $\frac{1}{2}D^{\mathrm{ind}(p)} = \{(s, v_1) \in [0, +\infty) \times V_- \mid s^2 + \|v_1\|^2 \leq \epsilon\}$, $D^{\mathrm{ind}(p)-1} = \{(0, v_1) \in \{0\} \times V_- \mid \|v_1\|^2 \leq \epsilon\}$, and $V_- \times \{0\}$ is the descending subspace of $T_p\partial M^a$.

Denote the k skeletons of $2M^a$ and ∂M^a by L_k and $\widetilde{L}_k$ respectively. If $\mathrm{ind}(p) = n$, then $e_p(\partial\widetilde{\overline{\mathcal{D}(p)}}) \subseteq \widetilde{L}_{n-2}$, $e_p(\partial(\frac{1}{2}\overline{\mathcal{D}(p)})) \subseteq (L_{n-1} \cap M^a) \cup \widetilde{L}_{n-1}$ and $e_p(\partial(\frac{1}{2}\overline{\mathcal{D}(p)}) - \widetilde{\mathcal{D}(p)}) \subseteq L_{n-1} \cap M^a$.

Expand K^a by attaching cell pairs $e_p : (\frac{1}{2}\overline{\mathcal{D}(p)}, \widetilde{\overline{\mathcal{D}(p)}}) \longrightarrow (M^a, \partial M^a)$ for critical points $p \in \partial M^a$ by induction on $\mathrm{ind}(p)$. Then K^a expands by elementary expansions to a CW complex N such that K^a and ∂M^a are its subcomplexes. Clearly, $N \subseteq M^a$. In addition, if $x \in M^a - K^a$, then $x \in \mathcal{D}(p)$ for some $p \in \partial M^a$ because $\mathcal{D}(q) \subseteq 2M^a - M^a$ when $q \notin M^a$. Since $\frac{1}{2}\overline{\mathcal{D}(p)} = e_p^{-1}(M^a)$, then $x \in e_p(\frac{1}{2}\overline{\mathcal{D}(p)}) \subseteq N$. Thus $N = M^a$ as sets. Finally, N and M^a share the same topology since N is a finite complex. $\square$

6.4 Proof of Theorem 6.3

Proof. In this proof, all critical points have function values less than a.

Suppose $\mathrm{ind}(p) = k$. $[\overline{\mathcal{D}(p)}]$ is a base of $H_k(\overline{\mathcal{D}(p)}, \partial\overline{\mathcal{D}(p)})$. It's well known that $\partial[\overline{\mathcal{D}(p)}]$ is the image of $[\overline{\mathcal{D}(p)}]$ under the following composition of homomorphisms

$$\begin{aligned} H_k(\overline{\mathcal{D}(p)}, \partial\overline{\mathcal{D}(p)}) \longrightarrow H_{k-1}(\partial\overline{\mathcal{D}(p)}) &\longrightarrow \widetilde{H}_{k-1}(\partial\overline{\mathcal{D}(p)}/e^{-1}(K^a_{k-2})) \\ &\longrightarrow \widetilde{H}_{k-1}(K^a_{k-1}/K^a_{k-2}) = H_{k-1}(K^a_{k-1}, K^a_{k-2}), \end{aligned}$$

where $K^a_n = \bigsqcup_{\mathrm{ind}(q)\le n} \mathcal{D}(q)$, and $\partial\overline{\mathcal{D}(p)} = \bigsqcup_i \partial^i\overline{\mathcal{D}(p)}$ is the full boundary of $\overline{\mathcal{D}(p)}$. The first homomorphism follows from the homology long exact sequence, the second one follows from the quotient map $\partial\overline{\mathcal{D}(p)} \longrightarrow \partial\overline{\mathcal{D}(p)}/e^{-1}(K^a_{k-2})$, and the third one follows from the map $\partial\overline{\mathcal{D}(p)}/e^{-1}(K^a_{k-2}) \longrightarrow K^a_{k-1}/K^a_{k-2}$ induced by e. Denote the first homomorphism by φ_1 and the composition of the first two by φ_2. The composition of all of them is the boundary operator ∂.

We have $e^{-1}(K^a_{k-2}) = \bigsqcup_{\mathrm{ind}(q)<k-1} \mathcal{M}(p,q) \times \mathcal{D}(q) \sqcup \bigsqcup_{|I|>0} \mathcal{D}_I$. Thus there is the following wadge of spheres with dimension $k-1$

$$\partial\overline{\mathcal{D}(p)}/e^{-1}(K^a_{k-2}) = \bigvee_{\mathrm{ind}(q)=k-1} \bigvee_{x\in\mathcal{M}(p,q)} \{x\} \times \overline{\mathcal{D}(q)}/\partial(\{x\} \times \overline{\mathcal{D}(q)}),$$

where the base points of spheres are $\partial(\{x\} \times \overline{\mathcal{D}(q)})/\partial(\{x\} \times \overline{\mathcal{D}(q)})$.

Clearly, $\overline{\mathcal{D}(p)}$ is a topological manifold with boundary $\partial\overline{\mathcal{D}(p)}$, and $[\overline{\mathcal{D}(p)}]$ represents an orientation of $\overline{\mathcal{D}(p)}$. So $\varphi_1([\overline{\mathcal{D}(p)}])$ represents the boundary orientation of $\partial\overline{\mathcal{D}(p)}$ induced from $[\overline{\mathcal{D}(p)}]$. Give $\{x\} \times \overline{\mathcal{D}(q)}$ the orientation $[\overline{\mathcal{D}(q)}]$ of $\overline{\mathcal{D}(q)}$ by the natural identification. Denote by $[\{x\} \times \overline{\mathcal{D}(q)}]$ the element in $\widetilde{H}_{k-1}(\{x\} \times \overline{\mathcal{D}(q)}/\partial(\{x\} \times \overline{\mathcal{D}(q)})) \subseteq \widetilde{H}_{k-1}(\partial\overline{\mathcal{D}(p)}/e^{-1}(K^a_{k-2}))$ which represents this orientation. Then by (2) of Theorem 5.1, we have

$$\varphi_2([\overline{\mathcal{D}(p)}]) = \sum_{\mathrm{ind}q=k-1} \sum_{x\in\mathcal{M}(p,q)} \varepsilon(x)[\{x\} \times \overline{\mathcal{D}(q)}],$$

where $\varepsilon(x)$ is the orientation ±1 at $x \in \mathcal{M}(p,q)$. Thus

$$\partial[\overline{\mathcal{D}(p)}] = \sum_{\mathrm{ind}(q)=k-1} \sum_{x\in\mathcal{M}(p,q)} \varepsilon(x)[\overline{\mathcal{D}(q)}] = \sum_{\mathrm{ind}(q)=\mathrm{ind}(p)-1} \#\mathcal{M}(p,q)[\overline{\mathcal{D}(q)}].$$

$\square$

7 Dynamical Aspects of Gradient Fields

In Chapters 8, 9 and 10, we shall extend the results in Chapters 4, 5 and 6 to the case of a general metric. The strategy is to reduce the case of a general metric to the one of a locally trivial metric.

This chapter yields the tools of the above reduction by working on dynamical systems. In particular, the important Theorem 7.7 on regular path is proved.

From this chapter to Chapter 10, we only consider the proper case, i.e., M is a finite ditmensional manifold and f is a proper Morse function. Actually, we deal with a negative gradient-like field X which is not assumed to be locally trivial.

7.1 Preliminaries

We shall deal with negative gradient-like vector fields in this chapter.

Definition 7.1. *Suppose p and q are critical points of a negative gradient-like vector field X, we say that p and q are transversal if the invariant manifolds of p are transverse to those of q. Suppose U is a subset of M, and these invariant manifolds meet transversally at each point in U (this includes the case that they don't meet at that point), we say that p and q are transversal in U.*

The following lemma is obvious.

Lemma 7.2. *If p and q are transversal in $f^{-1}((a,b))$ and $p \in f^{-1}((a,b))$, then p and q are transversal. If p and q are transversal in $f^{-1}(a)$ and $f(q) < a < f(p)$, then p and q are transversal.*

Now we introduce the definitions of topological conjugacy and topological equivalence in dynamical systems. The reader is to be forewarned that the definitions appearing the literature are not uniform. We follow the terminology of [50, p. 26]. In this dissertation, a topological conjugacy is a relation strictly stronger than a topological equivalence. This is *different* from the definition in [28]. The "topological conjugacy" in [28, p. 201] is actually the "topological equivalence" in this dissertation. Although a topological equivalence is good enough for us to prove results in the following chapters, we still introduce the notion of topological conjugacy in order to make the statement of Theorem 7.7 stronger.

Definition 7.3. *Suppose X_i ($i = 1, 2$) is a vector field on M_i and ϕ_t^i is the flow generated by X_i. Suppose $h : M_1 \to M_2$ is a homeomorphism. If $h\phi_t^1 = \phi_t^2 h$, then we call h a topological conjugacy between X_1 and X_2. If h maps the orbits of X_1 to the orbits of X_2 and h preserves the directions of orbits, then we call h is a topological equivalence between X_1 and X_2.*

Remark 7.1. *In dynamical systems, people usually consider the topological equivalence (or conjugacy) of vector fields on one manifold M, i.e. $M_1 = M_2$ in Definition 7.3. However, it seems beneficial for topology to allow that M_1 is not diffeomorphic to M_2. For example, choose a standard sphere S^n and an exotic sphere Σ^n. Let f_1 and f_2 be the height functions on S^n and Σ^n respectively. We can define a topological conjugacy between $-\nabla f_1$ and $-\nabla f_2$ as follows. Choose a homeomorphism (or even a diffeomorphism) $h_0 : S^{n-1} \to \Sigma^{n-1}$, where S^{n-1} and Σ^{n-1} are the equators of S^n and Σ^n respectively. Define h such that $h\phi_t^1(x) = \phi_t^2 h_0(x)$ for all $x \in S^{n-1}$, and h maps the maximum (minimum) point to the maximum (minimum) point. Clearly, this topological conjugacy h recovers the Alexander trick.*

The following definition of *filtration* is a special case of that in hyperbolic dynamical systems (see [43, p. 1029]).

Definition 7.4. *A compact submanifold M_1 with boundary inside M is a filtration for X if $dim(M_1) = dim(M)$, $\phi_t(M_1) \subseteq IntM_1$ for $t > 0$, and X is transverse to ∂M_1. Here $IntM_1$ is the interior of M_1, and ϕ_t is the flow generated by X.*

Lemma 7.5. *Suppose X satisfies transversality. If p and q are critical points such that $p \npreceq q$, then there exists a filtration M_1 such that $p \in M - M_1$ and $q \in IntM_1$.*

Lemma 7.5 can be proved as follows. The transversality implies " $\preceq$ " is a partial order. We have $p \npreceq q_1$ if $q_1 \preceq q$. Using [38, thm. 4.1] repeatedly, we can modify f to be a Morse function g such that X is a negative gradient-like field for g and $g(q) < g(p)$. The proof is finished.

7.2 A Strengthened Morse Lemma

In this section, we shall present a Strengthened Morse Lemma which is useful for the proof of Theorem 7.7 (See Remarks 7.2 and 7.3).

Suppose H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and U is an open subset of H. Define a smooth Riemannian metric (or smooth metric for brevity) on U in the usual sense. In other words, for each $x \in U$, assign a symmetric positive definite linear operator $A(x)$ such that $A(x)$ is a smooth function of x. For any v and w in $T_xU = H$, define $\langle v, w \rangle_{G(x)} = \langle A(x)v, w \rangle$.

Theorem 7.6 (Strengthened Morse Lemma)**.** *Suppose H is a Hilbert space, U is an open neighborhood of $0 \in H$. Suppose f is a smooth Morse function on U with a critical point 0, and G is a smooth metric on U. Let $-\nabla_G f$ be the negative gradient of f with respect to G, and*

ϕ_t be the flow generated by $-\nabla_G f$. Suppose $H = H_1 \oplus H_2$, where H_1 and H_2 are the negative and positive spectral spaces of $\nabla_G^2 f(0)$ respectively. Then there exist an open neighborhood V of 0 such that $V \subseteq U$, $B_1 = \{x_1 \in H_1 \mid \|x_1\| < \epsilon\}$, $B_2 = \{x_2 \in H_2 \mid \|x_2\| < \epsilon\}$, and a diffeomorphism $h : B_1 \times B_2 \to V$ such that the following holds. We have

$$h^* f(x_1, x_2) = f(0) - \frac{1}{2}\langle x_1, x_1\rangle + \frac{1}{2}\langle x_2, x_2\rangle, \tag{7.1}$$

$$\begin{aligned} h(B_1) &= \mathcal{D}_V(0; -\nabla_G f) = \{x \in V \mid \phi((-\infty, 0], x) \subseteq V\} \\ &= \left\{x \in V \mid \phi((-\infty, 0], x) \subseteq V, \lim_{t\to-\infty} \phi(t, x) = 0\right\}, \end{aligned}$$

and

$$\begin{aligned} h(B_2) &= \mathcal{A}_V(0; -\nabla_G f) = \{x \in V \mid \phi([0, +\infty), x) \subseteq V\} \\ &= \left\{x \in V \mid \phi([0, +\infty), x) \subseteq V, \lim_{t\to+\infty} \phi(t, x) = 0\right\}. \end{aligned}$$

Before proving it, we explain the statement of Theorem 7.6. In this theorem, $\mathcal{D}_V(0; -\nabla_G f)$ is the local unstable (descending) manifold of 0 in the neighborhood V, and $\mathcal{A}_V(0; -\nabla_G f)$ is the local stable (ascending) manifold. They certainly depend on the metric. The classical Morse Lemma shows that, by a coordinate transformation h, we get a new chart (we call it a Morse Chart) such that the function has the form (7.1) in it. Theorem 7.6 tells us more: No matter what the metric is, there exists a Morse chart such that the local invariant manifolds are standard in it. (Figure 5 illustrates this strengthened Morse chart, where the arrows indicate the directions of the flows.) This makes three objects, i.e. the function, the local invariant manifolds, and the coordinate chart fit well. In short, Theorem 7.6 strengthens the classical Morse Lemma by taking the dynamical system into account.

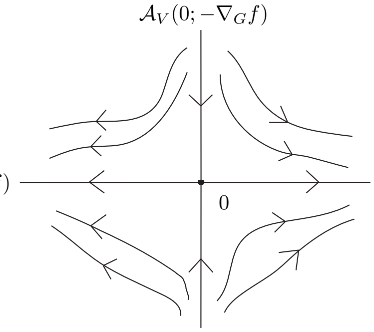

Figure 5: Strengthened Morse Chart

Proof of Theorem 7.6. We know that ϕ_1 is a smooth map defined on U_0 with a hyperbolic fixed point 0, where U_0 is a neighborhood of 0. By the Local Invariant Manifold Theorem (see [30] and [31, thm. 28]), shrinking U_0 suitably, there exists a diffeomorphism $h_1 : \widetilde{B}_1 \times \widetilde{B}_2 \to U_0$ such that

$$\begin{aligned} h_1(\widetilde{B}_1) &= \mathcal{D}_{U_0}(0;\phi_1) = \{x \in U_0 \mid \forall n \leq 0, (\phi_1)^n(x) \in U_0\} \\ &= \left\{ x \in U_0 \mid \forall n \leq 0, (\phi_1)^n(x) \in U_0, \lim_{n \to -\infty} (\phi_1)^n(x) = 0 \right\}, \end{aligned}$$

and $h_1(\widetilde{B}_2) = \mathcal{A}_{U_0}(0;\phi_1)$. Here the definition of $\mathcal{A}_{U_0}(0;\phi_1)$ is similar to that of $\mathcal{D}_{U_0}(0;\phi_1)$, and $(0,0) \in \widetilde{B}_1 \times \widetilde{B}_2 \subseteq H_1 \times H_2$.

Clearly, $h_1^* f|_{\widetilde{B}_1}$ and $h_1^* f|_{\widetilde{B}_2}$ are Morse functions on $\widetilde{B}_1$ and $\widetilde{B}_2$ respectively. By the Morse

Lemma, composing h_1 with a diffeomorphism if necessary, we may assume that

$$h_1^* f|_{\widetilde{B}_1} = f(0) - \frac{1}{2}\langle x_1, x_1 \rangle, \qquad h_1^* f|_{\widetilde{B}_2} = f(0) + \frac{1}{2}\langle x_2, x_2 \rangle.$$

Define

$$R(x) = h_1^* f(x) - \left(f(0) - \frac{1}{2}\langle x_1, x_1 \rangle + \frac{1}{2}\langle x_2, x_2 \rangle \right).$$

Here $x = (x_1, x_2)$. Denote the differential of R with order n by $D^n R$. Then $R(x_1, 0) \equiv 0$ and $R(0, x_2) \equiv 0$. In addition, for any $v_1 \in H_1$ and $v_2 \in H_2$, we have

$$\begin{aligned} D^2(h_1^* f)(0)(v_1, v_2) &= D^2 f(Dh_1 \cdot v_1, Dh_1 \cdot v_2) \\ &= \langle \nabla_G^2 f(0) Dh_1 \cdot v_1, Dh_1 \cdot v_2 \rangle_{G(0)}. \end{aligned}$$

We know that $Dh_1 \cdot v_1 \in H_1$, $Dh_1 \cdot v_2 \in H_2$, $\nabla_G^2 f(0)$ is symmetric with respect to $G(0)$, and H_1 and H_2 are negative and positive spectral spaces of $\nabla_G^2 f(0)$ respectively. Thus $D^2(h_1^* f)(0)(v_1, v_2) = 0$. We infer $D^2 R(0)(v_1, v_2) = 0$ and $D_{1,2}^2 R(0) = 0$.

Now we have

$$
\begin{aligned}
& R(x_1, x_2) \\
= \; & R(x_1, 0) + \int_0^1 \frac{d}{dt} R(x_1, tx_2) dt \\
= \; & \int_0^1 D_2 R(x_1, tx_2) dt \cdot x_2 \quad \text{(because } R(x_1, 0) = 0\text{)} \\
= \; & \int_0^1 \left[D_2 R(0, tx_2) + \int_0^1 \frac{d}{ds} D_2 R(sx_1, tx_2) ds \right] dt \cdot x_2 \\
= \; & \int_0^1 \int_0^1 D^2_{1,2} R(sx_1, tx_2) ds dt (x_1, x_2) \quad \text{(because } D_2 R(0, tx_2) = 0\text{)} \\
= \; & \int_0^1 \int_0^1 \left[D^2_{1,2} R(0, 0) + \int_0^1 \frac{d}{d\tau} D^2_{1,2} R(\tau s x_1, \tau t x_2) d\tau \right] ds dt (x_1, x_2) \\
= \; & \int_0^1 \int_0^1 \int_0^1 s D^3_{1,1,2} R(\tau s x_1, \tau t x_2) d\tau ds dt (x_1, x_1, x_2) \quad \text{(because } D^2_{1,2} R(0) = 0\text{)} \\
& + \int_0^1 \int_0^1 \int_0^1 t D^3_{1,2,2} R(\tau s x_1, \tau t x_2) d\tau ds dt (x_1, x_2, x_2).
\end{aligned}
$$

Since $D^3 R$ is a symmetric multilinear form, there exists symmetric operators $R_1(x)$ and $R_2(x)$ on H_1 and H_2 respectively such that, for any v_1 and w_1 in H_1,

$$
\int_0^1 \int_0^1 \int_0^1 s D^3_{1,1,2} R(\tau s x_1, \tau t x_2) d\tau ds dt (v_1, w_1, x_2) = \frac{1}{2} \langle R_1(x_1, x_2) v_1, w_1 \rangle;
$$

and, for any v_2 and w_2 in H_2,

$$
\int_0^1 \int_0^1 \int_0^1 t D^3_{1,2,2} R(\tau s x_1, \tau t x_2) d\tau ds dt (x_1, v_2, w_2) = \frac{1}{2} \langle R_2(x_1, x_2) v_2, w_2 \rangle.
$$

Here $R_1(x)$ and $R_2(x)$ are smooth with respect to x.

Clearly, $R_1(0) = 0$, $R_2(0) = 0$, and

$$h_1^* f(x) = f(0) - \frac{1}{2}\langle (I - R_1)(x)x_1, x_1\rangle + \frac{1}{2}\langle (I + R_2)(x)x_2, x_2\rangle.$$

Since $I - R_1$ and $I + R_2$ are symmetric, $I - R_1(0) = I$ and $I - R_2(0) = I$, shrinking $\widetilde{B}_1 \times \widetilde{B}_2$ if necessary, we have $I - R_1(x) = C_1(x)^2$ and $I + R_2(x) = C_2(x)^2$. Here $C_1(x)$ and $C_2(x)$ are symmetric and positive definite operators on H_1 and H_2 respectively, and they are smooth functions of x. Thus

$$h_1^* f(x) = f(0) - \frac{1}{2}\langle (C_1(x)x_1, C_1(x)x_1\rangle + \frac{1}{2}\langle C_2(x)x_2, C_2(x)x_2\rangle.$$

Define $h_2 : \widetilde{B}_1 \times \widetilde{B}_2 \to H_1 \times H_2$ by $h_2(x) = (C_1(x)x_1, C_2(x)x_2)$. Then $h_2(\widetilde{B}_1) \subseteq H_1$ and $h_2(\widetilde{B}_2) \subseteq H_2$. Since $Dh_2(0) = I$, there exists $\widehat{B}_1 \times \widehat{B}_2 \subseteq H_1 \times H_2$ such that h_2^{-1} exists and is smooth on $\widehat{B}_1 \times \widehat{B}_2$. Then we get

$$(h_2^{-1} \circ h_1)^* f(x) = f(0) - \frac{1}{2}\langle x_1, x_1\rangle + \frac{1}{2}\langle x_2, x_2\rangle.$$

Define $B_1 = \{x_1 \in H_1 \mid \|x_1\| < \epsilon\} \subseteq h_1^{-1}(\widehat{B}_1)$, $B_2 = \{x_2 \in H_2 \mid \|x_2\| < \epsilon\} \subseteq h_1^{-1}(\widehat{B}_2)$, $h = h_2^{-1} \circ h_1$ and $V = h(B_1 \times B_2)$.

We see that $h^{-1}(\mathcal{D}_{U_0}(0; \phi_1)) = B_1$ and $h^{-1}(\mathcal{A}_{U_0}(0; \phi_1)) = B_2$. By the fact that $-\nabla_G f \cdot f \leq 0$, it is straightforward to prove that $h(B_1) = \mathcal{D}_V(0; -\nabla_G f)$ and $h(B_2) = \mathcal{A}_V(0; -\nabla_G f)$. $\square$

7.3 A Regular Path

In this section, based on the idea outlined in [42, lem. 2], we shall prove Theorem 7.7. It shows that there exists a regular path connecting a generic gradient-like field to one with special singularities. This result is stated in [28, prop. 1.6] without proof.

Theorem 7.7 (Regular Path). *Suppose f is a Morse function on a compact manifold M. Suppose X is a negative gradient-like field for f, and X satisfies transversality. Then there is a continuous path $\mathcal{Y} : [0,1] \to \mathfrak{X}^\infty(M)$ such that, for all $s \in [0,1]$, $\mathcal{Y}_s$ is a negative gradient-like field for f, $\mathcal{Y}_s$ satisfies transversality, $\mathcal{Y}_0 = X$ and $\mathcal{Y}_1$ is locally trivial. In particular, there exists a topological conjugacy h between X and $\mathcal{Y}_1$ such that $h(p) = p$ for each critical point p. Here $\mathfrak{X}^\infty(M)$ is the set with the Whitney C^∞ topology consisting of C^∞ vector fields on M.*

We call a continuous path of negative gradient-like vector fields $\mathcal{Y} : [a,b] \to \mathfrak{X}^\infty(M)$ a regular path if $\mathcal{Y}_s$ satisfies transversality for all s.

We need the following classical Comparison Theorem for ODEs (see [61, p. 96]).

Theorem 7.8 (well-known). *Suppose $F(t,x)$ is a Lipschitz continuous function defined on $[t_0, t_1] \times [a,b]$. Let $x(t)$ be the solution of the equation $\dot{x} = F(t,x)$ with $x(t_0) = x_0$. Suppose $y(t)$ is a C^1 function defined on $[t_0, t_1]$ with $y(t_0) = x_0$. Then*

1. *if $\dot{y} \le F(t,y)$, then $y(t) \le x(t)$ on $[t_0, t_1]$;*

2. *if $\dot{y} \ge F(t,y)$, then $y(t) \ge x(t)$ on $[t_0, t_1]$.*

Suppose $H = H_1 \oplus H_2$ is a Hilbert space, $v = (v_1, v_2) \in H$, $v_1 \ne 0$, and $\lambda = \frac{\|v_2\|}{\|v_1\|}$. We call λ the inclination of v with respect to H_1. Suppose L is a closed subspace of H, and

$P : H \to H_1$ is the projection. If $P : L \to P(L)$ is a topological linear isomorphism, then there exists a bounded linear operator $A : P(L) \to H_2$ such that L is the graph of A, i.e., for any $v \in L$, we have $v = (v_1, Av_1)$, where $v_1 \in P(L)$. We call the supremum of the inclinations of all non-zero vectors in L the inclination of L with respect to H_1. Clearly, the inclination of L equals, $\|A\|$, the norm of A.

Suppose H, H_1 and H_2 are Hilbert spaces as above. Suppose A_0 and A_1 are linear operators on H_1, and B is a linear operator on H_2. There exist positive numbers $\alpha_0 > 0$, $\alpha_1 > 0$ and $\beta > 0$ such that

$$\alpha_0\langle w, w\rangle \leq \langle A_i w, w\rangle \leq \alpha_1\langle w, w\rangle \qquad (i = 0, 1), \tag{7.2}$$

and

$$\beta\langle w, w\rangle \leq \langle Bw, w\rangle. \tag{7.3}$$

Let ρ be a smooth bump function on $(-\infty, +\infty)$ such that $0 \leq \rho \leq 1$, $\rho(s) \equiv 1$ when $s \leq \frac{1}{2}$, and $\rho(s) \equiv 0$ when $s \geq 1$. Define $\rho_r(s) = \rho(\frac{s}{r})$ for $r > 0$. For convenience, we denote $\rho_r(\|x_i\|)$ by $\rho_r(x_i)$, where $x_i \in H_i$.

Define a smooth vector field X_r on H by

$$X_r(x_1, x_2) = (\rho_r(x_1)\rho_r(x_2)A_0x_1 + [1 - \rho_r(x_1)\rho_r(x_2)]A_1x_1, -Bx_2).$$

Denote the flow generated by X_r by $\phi_t(x_1, x_2)$. For a fixed t, ϕ_t is a diffeomorphism, thus $D\phi_t$ acts on the tangent vectors at each point (x_1, x_2), where $D\phi_t$ is the differential of ϕ_t with respect to $x = (x_1, x_2)$.

Lemma 7.9. *For any $\epsilon > 0$, there exists $\delta > 0$ such that the following holds. For any $r > 0$ and $v \in H$, if the inclination of v with respect to H_1 is less than δ, then we have the inclination of $D\phi_t \cdot v$ with respect to H_1 is less than ϵ for all $t \geq 0$. Here δ only depends on α_0, α_1, β and ϵ, and δ is independent of r.*

Proof. The flow $\phi_t = (\phi_t^1, \phi_t^2)$ satisfies the following ordinary differential equation

$$\begin{cases} \dot{\phi}^1 &= \rho_r(\phi^1)\rho_r(\phi^2)A_0\phi^1 + [1 - \rho_r(\phi^1)\rho_r(\phi^2)]A_1\phi^1, \\ \\ \dot{\phi}^2 &= -B\phi^2. \end{cases}$$

Denote $\rho_r(\phi^1)\rho_r(\phi^2)A_0 + [1 - \rho_r(\phi^1)\rho_r(\phi^2)]A_1$ by $A(\phi^1, \phi^2)$. We have

$$\frac{d}{dt}\langle \phi^1, \phi^1 \rangle = 2\langle \dot{\phi}^1, \phi^1 \rangle = 2\langle A(\phi^1, \phi^2)\phi^1, \phi^1 \rangle.$$

By (7.2), we have

$$0 \leq 2\alpha_0 \langle \phi^1, \phi^1 \rangle \leq \frac{d}{dt}\langle \phi^1, \phi^1 \rangle \leq 2\alpha_1 \langle \phi^1, \phi^1 \rangle.$$

Thus $\|\phi^1\|$ is increasing, and by Theorem 7.8, we have

$$e^{\alpha_0 t}\|\phi_0^1\| \leq \|\phi_t^1\| \leq e^{\alpha_1 t}\|\phi_0^1\|. \tag{7.4}$$

Similarly, $\|\phi^2\|$ is decreasing, $\phi_t^2 = e^{-Bt}\phi_0^2$, and $\|\phi_t^2\| \leq e^{-\beta t}\|\phi_0^2\|$.

Let $\mathbb{D}_1(r) = \{x_1 \in H_1 \mid \|x_1\| < r\}$, and $\mathbb{D}_2(r) = \{x_2 \in H_2 \mid \|x_2\| < r\}$. Clearly, $A(x_1, x_2)|_{H - (\mathbb{D}_1(r) \times \mathbb{D}_2(r))} = A_1$, and $A(x_1, x_2)|_{\overline{\mathbb{D}_1(\frac{r}{2}) \times \mathbb{D}_2(\frac{r}{2})}} = A_0$. Denote $\overline{\mathbb{D}_1(r) \times \mathbb{D}_2(r)} -$

$(\mathbb{D}_1(\frac{r}{2}) \times \mathbb{D}_2(\frac{r}{2}))$ by $E(r)$. When $\phi([0,t],x)$ is out of $E(r)$, we have $\phi(t,x) = (e^{A_i t}x_1, e^{-Bt}x_2)$, and

$$D\phi_t = \begin{pmatrix} e^{A_i t} & 0 \\ 0 & e^{-Bt} \end{pmatrix}.$$

Since $\|e^{A_i t}w\| \geq \|w\|$ and $\|e^{-Bt}w\| \leq \|w\|$ for $t \geq 0$, we have that the inclination of $D\phi_t \cdot v$ is decreasing when t is increasing. Thus it suffices to control the variation of the inclination when $\phi_t(x)$ passes through $E(r)$.

Suppose $t \geq 0$ and $\|\phi_t^1\| = 2\|\phi_0^1\|$, then by (7.4), we have $t \leq \frac{\ln 2}{\alpha_0}$. Similarly, if $\|\phi_t^2\| = \frac{1}{2}\|\phi_0^2\|$, then $t \leq \frac{\ln 2}{\beta}$. Since $\|\phi_t^1\|$ is increasing and $\|\phi_t^2\|$ is decreasing, we infer that ϕ_t enters $E(r)$ at most twice, and the time for it to stay in $E(r)$ is no more than

$$T = \frac{\ln 2}{\alpha_0} + \frac{\ln 2}{\beta}. \tag{7.5}$$

Suppose $\phi([0,t],x) \subset E(r)$, we have $0 \leq t \leq T$. Since $\phi_t^2(x) = e^{-Bt}x_2$, we have

$$D_1\phi_t^2 = 0, \qquad D_2\phi_t^2 = e^{-Bt}, \qquad \text{and} \quad \|D_2\phi_t^2 \cdot w\| \leq \|w\|. \tag{7.6}$$

Since

$$\dot{\phi}^1 = A(\phi^1, e^{-Bt}x_2)\phi^1,$$

we have

$$D_1\dot{\phi}^1 \cdot w = A(\phi^1, \phi^2)(D_1\phi^1 \cdot w) + D\rho_r(\phi^1)(D_1\phi^1 \cdot w)\rho_r(\phi^2)(A_0 - A_1)\phi^1.$$

Thus

$$\begin{aligned} \frac{d}{dt}\langle D_1\phi^1 \cdot w, D_1\phi^1 \cdot w\rangle &= 2\langle D_1\dot{\phi}^1 \cdot w, D_1\phi^1 \cdot w\rangle \\ &= 2\langle A(\phi^1, \phi^2)(D_1\phi^1 \cdot w), D_1\phi^1 \cdot w\rangle \\ &\quad +2\langle D\rho_r(\phi^1)(D_1\phi^1 \cdot w)\rho_r(\phi^2)(A_0 - A_1)\phi^1, D_1\phi^1 \cdot w\rangle. \end{aligned}$$

Clearly, $D\rho_r(\phi^1) = O(r^{-1})$, and $\|\phi^1\| \leq r$ when $D\rho_r(\phi^1) \neq 0$. So there exists a constant $C_1 > 0$ which is independent of r such that

$$|\langle D\rho_r(\phi^1)(D_1\phi^1 \cdot w)\rho_r(\phi^2)(A_0 - A_1)\phi^1, D_1\phi^1 \cdot w\rangle| \leq C_1\|D_1\phi^1 \cdot w\|^2.$$

Combining the above inequality with (7.2), we get

$$-2C_1\langle D_1\phi^1 \cdot w, D_1\phi^1 \cdot w\rangle \leq \frac{d}{dt}\langle D_1\phi^1 \cdot w, D_1\phi^1 \cdot w\rangle.$$

Since $D_1\phi^1_0 = I$ and $\|D_1\phi^1_0 \cdot w\| = \|w\|$, by Theorem 7.8, we have

$$\|D_1\phi^1_t \cdot w\| \geq e^{-C_1 t}\|w\| \geq e^{-C_1 T}\|w\|. \tag{7.7}$$

Similarly, we have

$$\begin{aligned}\frac{d}{dt}\langle D_2\phi^1\cdot w, D_2\phi^1\cdot w\rangle &= 2\langle A(\phi^1,\phi^2)(D_2\phi^1\cdot w), D_2\phi^1\cdot w\rangle\\ &\quad +2\langle D\rho_r(\phi^1)(D_2\phi^1\cdot w)\rho_r(\phi^2)(A_0-A_1)\phi^1, D_2\phi^1\cdot w\rangle\\ &\quad +2\langle \rho_r(\phi^1)D\rho_r(\phi^2)e^{-Bt}w(A_0-A_1)\phi^1, D_2\phi^1\cdot w\rangle,\end{aligned}$$

and

$$|\langle D\rho_r(\phi^1)(D_2\phi^1\cdot w)\rho_r(\phi^2)(A_0-A_1)\phi^1, D_2\phi^1\cdot w\rangle| \le C_1\|D_2\phi^1\cdot w\|^2.$$

In addition, $\rho_r(\phi^1)D\rho_r(\phi^2) = O(r^{-1})$, and $\|\phi^1\| \le r$ when $\rho_r(\phi^1)D\rho_r(\phi^1) \ne 0$. So there exists $C_2 > 0$ which is independent of r such that

$$\begin{aligned}&2|\langle \rho_r(\phi^1)D\rho_r(\phi^2)e^{-Bt}w(A_0-A_1)\phi^1, D_2\phi^1\cdot w\rangle|\\ \le\ & 2C_2\|D_2\phi^1\cdot w\|\|w\| \le C_2\|D_2\phi^1\cdot w\|^2 + C_2\|w\|^2.\end{aligned}$$

Thus by (7.2), we infer

$$\frac{d}{dt}\langle D_2\phi^1\cdot w, D_2\phi^1\cdot w\rangle \le (2\alpha_1 + 2C_1 + C_2)\langle D_2\phi^1\cdot w, D_2\phi^1\cdot w\rangle + C_2\|w\|^2.$$

Since $\|D_2\phi_0^1\cdot w\| = 0$, by Theorem 7.8 again, there exists a $C_3 > 0$ which is independent of r such that

$$\|D_2\phi_t^1\cdot w\| \le \left[\frac{C_2}{C_3}(e^{C_3T}-1)\right]^{\frac{1}{2}}\|w\|. \tag{7.8}$$

By (7.5), (7.7) and (7.8), there exist $K_1 > 0$ and $K_2 > 0$, which are independent of r, such that

$$\|D_1\phi_t^1 \cdot w\| \geq K_1\|w\|, \qquad \text{and} \quad \|D_2\phi_t^1 \cdot w\| \leq K_2\|w\|. \tag{7.9}$$

Suppose $v = (v_1, v_2) \in H_1 \oplus H_2$, and its inclination is $\lambda_0 = \frac{\|v_2\|}{\|v_1\|}$. By (7.6) and (7.9), we have the inclination of $D\phi_t^1 \cdot v$ is

$$\begin{aligned} \lambda_1 &= \frac{\|D_2\phi_t^2 \cdot v_2\|}{\|D_1\phi_t^1 \cdot v_1 + D_2\phi_t^1 \cdot v_2\|} \leq \frac{\|D_2\phi_t^2 \cdot v_2\|}{\|D_1\phi_t^1 \cdot v_1\| - \|D_2\phi_t^1 \cdot v_2\|} \\ &\leq \frac{\|v_2\|}{K_1\|v_1\| - K_2\|v_2\|} = \frac{\lambda_0}{K_1 - K_2\lambda_0}. \end{aligned}$$

Thus λ_1 tends to 0 when λ_0 tends to 0.

Since $\phi_t(x)$ enters $E(r)$ at most twice, and K_1 and K_2 are independent of r, the proof is completed. □

By Definition 2.14, we have the following obvious lemma.

Lemma 7.10. *Suppose X_1 and X_2 are negative gradient-like fields of f. Suppose $\sigma_1(x)$ and $\sigma_2(x)$ are nonnegative smooth functions on M such that $\sigma_1 + \sigma_2 > 0$. Then $\sigma_1 X_1 + \sigma_2 X_2$ is also a negative gradient-like field for f.*

Let p be a critical point. Suppose there exists a Morse chart near p (see (7.1)), and $X(x_1, x_2) = (Ax_1, -Bx_2)$, where A and B are symmetric positive definite linear operators. Similarly to Lemma 7.9, define

$$\mathcal{Y}_r(x_1, x_2) = (\rho_r(x_1)\rho_r(x_2)x_1 + [1 - \rho_r(x_1)\rho_r(x_2)]Ax_1, -Bx_2)$$

in this Morse chart and $\mathcal{Y}_r = X$ out of this Morse chart. For $s \in [0,1]$, define

$$\mathcal{Y}_{r,s} = (1-s)X + s\mathcal{Y}_r.$$

By Lemma 7.10, for all $s \in [0,1]$, $\mathcal{Y}_{r,s}$ is a negative gradient-like field for f.

Lemma 7.11. *Suppose X satisfies transversality. Then when r is small enough, we have the following conclusion.*

Suppose q_1 and q_2 are two critical points which are not of the following two cases: (1) $q_2 \prec p \prec q_1$; or (2) $q_1 \prec p \prec q_2$. Then we have that q_1 and q_2 are transversal with respect to $\mathcal{Y}_{r,s}$ for all $s \in [0,1]$. Here "$\prec$" is defined with respect to X.

Proof. Clearly, $\mathcal{Y}_{r,s}$ differs from X only in a neighborhood U_r of p. When r tends to 0, U_r shrinks to p.

We may assume that $f(q) \neq f(p)$ for any critical point q such that $q \neq p$. If this is not true, perturb f to be a Morse function $\tilde{f}$ such that X is a negative gradient-like field for $\tilde{f}$, and $\tilde{f}(x) = f(x) + C$ in a neighborhood U of p. Let r be small enough such that $U_r \subseteq U$. Then $\mathcal{Y}_{r,s}$ is also a negative gradient-like field for $\tilde{f}$. For the rest of the proof we make the above assumption.

Suppose $U_r \subseteq M^{a,b}$ and p is the unique singularity in $M^{a,b}$. As in Definition 2.4, we use notation $\mathcal{D}(*;*)$ and $\mathcal{A}(*;*)$ to indicate the vector fields.

It's easy to see that $\mathcal{D}(p;\mathcal{Y}_{r,s}) = \mathcal{D}(p;X)$. Suppose that $q \in M^a$. Since $\mathcal{Y}_{r,s}$ is identical to X in $M - M^{a,b}$, we have $\mathcal{A}(q;\mathcal{Y}_{r,s}) \cap M^a = \mathcal{A}(q;X) \cap M^a$. Since X satisfies transversality, we infer that p and q are transversal in M^a with respect to $\mathcal{Y}_{r,s}$. By Lemma 7.2, p and q are transversal globally. Similarly, if $q \in M - M^a$, p and q are also transversal. As a result, p

and q are transversal. It suffices to check the case that $q_1 \neq p$ and $q_2 \neq p$.

If $p \not\succ q$, by Lemma 7.5, there exists a filtration M_1 such that $q \in \mathrm{Int}M_1$ and $p \in M - M_1$. Let r be small enough such that $U_r \subseteq M - M_1$, then $\mathcal{Y}_{r,s}$ is identical to X on M_1. So $\mathcal{D}(q; \mathcal{Y}_{r,s}) = \mathcal{D}(q; X)$. Similarly, if $q \not\succ p$, we can get $\mathcal{A}(q; \mathcal{Y}_{r,s}) = \mathcal{A}(q; X)$ when r is small enough. Thus there exists $r_0 > 0$ such that the following holds. When $r < r_0$, we have, for all $s \in [0, 1]$, $\mathcal{D}(q; \mathcal{Y}_{r,s}) = \mathcal{D}(q; X)$ if $p \not\succ q$, and $\mathcal{A}(q; \mathcal{Y}_{r,s}) = \mathcal{A}(q; X)$ if $q \not\succ p$.

In order to complete this proof, we only need to check the following three cases.

(1). Case 1: q_1 and q_2 are in M^a.

Since $\mathcal{Y}_{r,s}$ is identical to X on M^a and X satisfies transversality, we have q_1 and q_2 are transversal in M^a. By Lemma 7.2, they are transversal globally.

(2). Case 2: q_1 and q_2 are in $M - M^a$.

Similarly to Case (1), this case is also true.

(3). Case 3: one of q_1 and q_2 is in $M - M^a$ and the other one is in M^a.

We may presume $q_1 \in M - M^a$ and $q_2 \in M^a$. By the assumption of this lemma, we have either $p \not\succ q_1$ or $q_2 \not\succ p$. Suppose $p \not\succ q_1$. We have $\mathcal{D}(q_1; \mathcal{Y}_{r,s}) = \mathcal{D}(q_1; X)$. Since X satisfies transversality, we have q_1 and q_2 are transversal in M^a with respect to $\mathcal{Y}_{r,s}$. By Lemma 7.2, they are transversal globally. Similarly, if $q_2 \not\succ p$, this is also true. Thus Case 3 is also verified. $\square$

We shall strengthen Lemma 7.11 to get the transversality of $\mathcal{Y}_{r,s}$. Recall a classical result on transversality at first.

Suppose U is a neighborhood of p such that U is identified with a neighborhood of 0 in $T_pM = H_1 \oplus H_2$, and p is identified with 0, where $H_1 = T_p\mathcal{D}(p; X)$ and $H_2 = T_p\mathcal{A}(p; X)$.

Furthermore, suppose $\mathcal{D}(p;X)\cap U \subseteq H_1$ and $\mathcal{A}(p;X)\cap U \subseteq H_2$. Then we have the following crucial fact: When U is small enough, there exists $\Lambda > 0$ such that for any $q_1 \succeq p$ and any $x \in \mathcal{D}(q_1;X)\cap U$, there exists a linear space $V_x^d \subseteq T_x\mathcal{D}(q_1;X)$ such that $\dim(V_x^d) = \dim(H_1)$ and the inclination of V_x^d with respect to H_1 is less than Λ. Similarly, for any $q_2 \preceq p$ and any $x \in \mathcal{A}(q_2;X)\cap U$, there exists $V_x^a \subseteq T_x\mathcal{A}(q_2;X)$ such that $\dim(V_x^a) = \dim(H_2)$ and the inclination of V_x^a with respect to H_2 is also less than Λ. In addition, Λ tends to 0 when U shrinks to p. This fact follows from the transversality of X and the estimate of the λ-Lemma. (Note: the λ-Lemma is also named the Inclination Lemma.) On the other hand, we assume this fact holds but do not assume the transversality of X. If $\Lambda < 1$, then, for any $x \in \mathcal{D}(q_1;X)\cap\mathcal{A}(q_2;X)\cap U$, we have

$$T_xM = H_1 \oplus H_2 = V_x^d \oplus V_x^a = T_x\mathcal{D}(q_1;X) + T_x\mathcal{A}(q_2;X).$$

So we infer that $\mathcal{D}(q_1;X)$ and $\mathcal{A}(q_2;X)$ are transversal in U. The above argument is the key part of the proof of that, for Morse-Smale dynamical systems, transversality is preserved under small C^1 perturbations. All of these are addressed in [49, lem. 1.11 and thm. 3.5]. In the proof of the following lemma, we shall apply a similar argument to large C^1 perturbations of X.

Lemma 7.12. *Suppose X satisfies transversality. When r is small enough, we have $\mathcal{Y}_{r,s}$ satisfies transversality for all $s \in [0,1]$.*

Proof. By Lemma 7.11, it suffices to prove that $\mathcal{D}(q_1;\mathcal{Y}_{r,s})$ is transverse to $\mathcal{A}(q_2;\mathcal{Y}_{r,s})$ if $q_2 \prec p \prec q_1$.

Similarly to the proof of Lemma 7.11, we assume that p is the unique critical point in

$M^{f(p)-\epsilon,f(p)+\epsilon}$. Let U be the neighborhood of p in the argument before this lemma. Let D be an open subset of $f^{-1}(f(p)+\epsilon)\cap U$ such that $D \supseteq f^{-1}(f(p)+\epsilon)\cap \mathcal{A}(p;X)$. Let $U_0 = [\phi([0,+\infty), D)\cup \mathcal{D}(p;X)]\cap M^{f(p)-\epsilon,f(p)+\epsilon}$. Then U_0 is a neighborhood of p and is relatively open in $M^{f(p)-\epsilon,f(p)+\epsilon}$. When ϵ tends to 0 and D shrinks, U_0 shrinks to p. (In Figure 6, the shadowed part is U_0, the arrows indicate the the directions of the flows.) Denote the flow generated by $\mathcal{Y}_{r,s}$ by $\phi_t^{r,s}$.

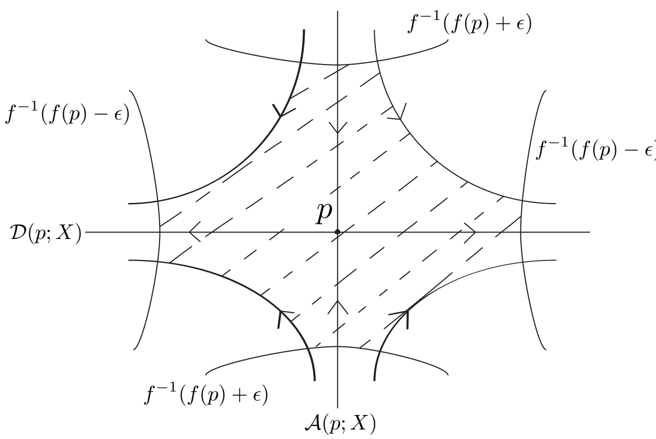

Figure 6: Neighborhood U_0

Both $M^{f(p)-\epsilon,f(p)+\epsilon} - U_0$ and U_0 are unions of some complete orbits generated by X in $M^{f(p)-\epsilon,f(p)+\epsilon}$. Let U_0 be small enough such that $U_0 \subseteq U$. Choose $U_1 \subseteq U_0$ such that U_1 is also a union of some complete orbits generated by X in $M^{f(p)-\epsilon,f(p)+\epsilon}$, and U_1 is a closed neighborhood of p. Let r be small enough such that $\mathcal{Y}_{r,s}$ is identical to X out of U_1. We have $M^{f(p)-\epsilon,f(p)+\epsilon} - U_0$ is still the union of some complete orbits generated by $\mathcal{Y}_{r,s}$ in $M^{f(p)-\epsilon,f(p)+\epsilon}$. Then so is U_0. Thus, for any $x \in [f^{-1}(f(p)+\epsilon)\cap U_0] - \mathcal{A}(p;X)$, we have $\phi^{r,s}(t,x) \in f^{-1}(f(p)-\epsilon)$ for some $t > 0$ and $\phi^{r,s}([0,t]) \subset U_0$.

We know that

$$\mathcal{Y}_{r,s}(x_1, x_2) = (\rho_r(x_1)\rho_r(x_2)(sI + (1-s)A)x_1 + [1 - \rho_r(x_1)\rho_r(x_2)]Ax_1, -Bx_2),$$

and there exist $\alpha_0 > 0$, $\alpha_1 > 0$ and $\beta > 0$ such that, for any $s \in [0,1]$, we have

$$\alpha_0 I \leq sI + (1-s)A \leq \alpha_1 I, \qquad \alpha_0 I \leq A \leq \alpha_1 I, \qquad \text{and} \quad \beta I \leq B.$$

By Lemma 7.9, there exists $\delta > 0$ such that the following holds. Suppose $x \in \mathcal{D}(q_1; X) \cap f^{-1}(f(p) + \epsilon) \cap U_0$, and $V_x^d \subseteq T_x\mathcal{D}(q_1; X)$ is the space described before this lemma. If the inclination of V_x^d with respect to H_1 is less than δ, then, in U_0, the inclination of $\phi_t^{r,s}(V_x^d)$ with respect to H_1 is less than 1. It's necessary to point out that δ is independent of r and s.

Clearly, $\mathcal{D}(q_1; X) \cap f^{-1}([f(p)+\epsilon, +\infty)) = \mathcal{D}(q_1; \mathcal{Y}_{r,s}) \cap f^{-1}([f(p)+\epsilon, +\infty))$ and $\mathcal{A}(q_2; X) \cap M^{f(p)-\epsilon} = \mathcal{A}(q_2; \mathcal{Y}_{r,s}) \cap M^{f(p)-\epsilon}$. Since X satisfies transversality, by the argument before this lemma, we can choose U_0 be small enough such that the following holds. For any $x \in \mathcal{D}(q_1; X) \cap f^{-1}(f(p) + \epsilon) \cap U_0$, the inclination of V_x^d with respect to H_1 is less than δ, and, for any $y \in \mathcal{A}(q_2; X) \cap f^{-1}(f(p) - \epsilon) \cap U_0$, the inclination of V_y^a with respect to H_2 is less than 1. Here $V_x^d \subseteq T_x\mathcal{D}(q_1; X) = T_x\mathcal{D}(q_1; \mathcal{Y}_{r,s})$ and $V_y^a \subseteq T_y\mathcal{A}(q_2; X) = T_y\mathcal{A}(q_2; \mathcal{Y}_{r,s})$. Thus, if $\phi_t^{r,s}(x) = y$, then the inclination of $V_y^d = D\phi_t^{r,s} \cdot V_x^d$ with respect to H_1 is less than 1. Here $V_y^d \subseteq T_y\mathcal{D}(q_1; \mathcal{Y}_{r,s})$. By the argument before this lemma again, we have $T_yM = V_y^d \oplus V_y^a$. So $\mathcal{D}(q_1; \mathcal{Y}_{r,s})$ and $\mathcal{A}(q_2; \mathcal{Y}_{r,s})$ are transversal in $f^{-1}(f(p) - \epsilon) \cap U_0$.

Furthermore, $\mathcal{D}(q_1; X) \cap (M^{f(p)-\epsilon, f(p)+\epsilon} - U_1) = \mathcal{D}(q_1; \mathcal{Y}_{r,s}) \cap (M^{f(p)-\epsilon, f(p)+\epsilon} - U_1)$ and

$\mathcal{A}(q_2; X) \cap (M^{f(p)-\epsilon, f(p)+\epsilon} - U_1) = \mathcal{A}(q_2; \mathcal{Y}_{r,s}) \cap (M^{f(p)-\epsilon, f(p)+\epsilon} - U_1)$. Thus $\mathcal{D}(q_1; \mathcal{Y}_{r,s})$ and $\mathcal{A}(q_2; \mathcal{Y}_{r,s})$ are transversal in $M^{f(p)-\epsilon, f(p)+\epsilon} - U_1$.

In summary, $\mathcal{D}(q_1; \mathcal{Y}_{r,s})$ and $\mathcal{A}(q_2; \mathcal{Y}_{r,s})$ are transversal in $f^{-1}(f(p) - \epsilon)$. By Lemma 7.2, they are transversal globally. $\square$

Proof of Theorem 7.7. First, we construct the regular path. It suffices to prove that, for any critical point p, we can construct a regular path $\mathcal{Y}$ such that $\mathcal{Y}_0 = X$ and $\mathcal{Y}_1$ is locally trivial at p.

By Theorem 7.6, there exists a coordinate chart U near p such that p has coordinate $(0,0)$,

$$f(x_1, x_2) = f(p) - \frac{1}{2}\langle x_1, x_1 \rangle + \frac{1}{2}\langle x_2, x_2 \rangle,$$

$\mathcal{D}(p; X) \cap U = \{(x_1, 0)\}$ and $\mathcal{A}(p; X) \cap U = \{(0, x_2)\}$. Clearly, $D^2X(p) = (A, -B)$, where A and B are symmetric and positive definite. Furthermore, $(Ax_1, -Bx_2)$ is also a gradient-like vector field for f near p.

Let ρ_r be the bump function defined before. For convenience, for all $x = (x_1, x_2)$, denote $\rho_r(\|x\|)$ by $\rho_r(x)$. Let $R(x) = X(x) - (Ax_1, -Bx_2)$. Then we have $\|\rho_r(x)R(x)\|$ and $\|D[\rho_r(x)R(x)]\|$ tend to 0 when r tends to 0. Since the transversality of X is preserved under small C^1 perturbations, we have $\mathcal{Z}_s = X - s\rho_r R$ is a regular path when r is small enough and $s \in [0,1]$. Clearly, $\mathcal{Z}_1(x) = (Ax_1, -Bx_2)$ near p. By Lemma 7.12, we can construct a regular path $\mathcal{Z}_{[1,2]}$ such that $\mathcal{Z}_2(x) = (x_1, -Bx_2)$ near p. Since $-\mathcal{Z}_2$ is a negative gradient-like field for $-f$, using Lemma 7.12 again, we can construct a regular path $\mathcal{Z}_{[2,3]}$ such that $\mathcal{Z}_3(x) = (x_1, -x_2)$ near p. We get the desired path by defining $\mathcal{Y}_s = \mathcal{Z}_{3s}$.

Second, we prove the existence of the conjugacy h.

By the proof in [51, thm. 5.2], we know that, for each $\mathcal{Y}_{s_0}$, there is a topological equivalence h_{s_0} between $\mathcal{Y}_{s_0}$ and $\mathcal{Y}_s$ such that $h_{s_0}(p) = p$ for all critical points p when s is close to s_0 enough. In addition, since the flow generated by $\mathcal{Y}_{s_0}$ has no closed orbits, by the comment in [51, p. 231], we know that h_{s_0} is actually a conjugacy. Thus it's easy to get the desired conjugacy h. □

Remark 7.2. *In the proof of Theorem 7.7, we need to choose a Morse chart* $U \subseteq H_1 \oplus H_2$ *such that* H_1 *and* H_2 *are respectively the tangent spaces of* $\mathcal{D}(p; X)$ *and* $\mathcal{A}(p; X)$ *at* p*. This is not granted because these tangent spaces depend on the metric. Theorem 7.6 provides this.*

Remark 7.3. *The regular path in [42] consists of the Morse-Smale vector fields without closed orbits. In this case,* $DX(p) = (A, -B)$ *for singularities* p*, where* A *and* B *are linear isomorphisms whose eigenvalues have positive real parts. The paper [42] claims that there exists a regular path connecting* X *with* Y *such that* $Y(x_1, x_2) = (2x_1, -2x_2)$ *near each singularity. Thus, in the setting of dynamical systems, this result is more general than Theorem 7.7. However, Theorem 7.7 has the advantage that its vector fields are negative gradient-like for* f*. This is the reason that we need Theorem 7.6. Furthermore, the argument in this paper can also be used to verify the result in [42]. This is because we can choose a metric near each critical point, for example, by the real Jordan canonical form, such that the above operators* A *and* B *satisfy (7.2) and (7.3).*

7.4 A Reduction Lemma

In the following chapters, we shall apply Theorem 7.7 to noncompact manifolds with proper Morse functions. However, the manifold in Theorem 7.7 is required to be compact. The following lemma reduces the proper case to the compact case.

Lemma 7.13. *Suppose M is a compact manifold with boundary $\partial M = M_1 \sqcup M_2$. Here M_i $(i = 1, 2)$ may be empty. Suppose f is a Morse function on M such that $f|_{M_1} \equiv a$, $f|_{M_2} \equiv b$, a and b are regular values of f, and $a < b$. Suppose X is a negative gradient-like vector field for f, and X satisfies transversality. Then there exist a compact manifold $\widetilde{M}$ without boundary and a smooth embedding $i : M \hookrightarrow \widetilde{M}$ such that the following holds. There exist a Morse function $\widetilde{f}$ and its negative gradient-like like vector field $\widetilde{X}$ on $\widetilde{M}$. They are extensions of f and X respectively, and $\widetilde{X}$ satisfies transversality. For any critical points p and q in M, we have $\mathcal{D}(p; \widetilde{X}) \cap \mathcal{A}(q; \widetilde{X}) = \mathcal{D}(p; X) \cap \mathcal{A}(q; X)$. Furthermore, $\mathcal{D}(p; \widetilde{X}) = \mathcal{D}(p; X)$ and $\widetilde{f}|_{\widetilde{M}-M} > b$ if $M_1 = \emptyset$; and $\mathcal{A}(p; \widetilde{X}) = \mathcal{A}(p; X)$ and $\widetilde{f}|_{\widetilde{M}-M} < a$ if $M_2 = \emptyset$.*

Proof. If $\partial M = \emptyset$, let $\widetilde{M} = M$, the proof is finished. Now we assume $\partial M \neq \emptyset$.

Let $\widetilde{M}$ be the double of M. Extend f to be $\widetilde{f}$ such that a and b are its regular values, and extend X to be $\widetilde{X}$ which is a negative gradient-like field for $\widetilde{f}$. (Figure 7 illustrates the manifold $\widetilde{M}$, where the Morse function is the height function and the shadowed part is M.) We shall modify $\widetilde{X}$ such that it satisfies transversality. The method of such a modification is Milnor's sliding invariant (descending or ascending) manifolds in [38, thm. 5.2]. Basically, there are two ways of sliding invariant manifolds in order to get transversality. Method 1 is sliding the descending manifolds one by one with the order from critical points with lower values to those with higher values. On the contrary, Method 2 is sliding the ascending manifolds one by one with the order from critical points with higher values to those with lower values. Our method is a combination of the above two methods.

In this proof, we say two critical points $\widetilde{p}$ and $\widetilde{q}$ of $\widetilde{f}$ are transversal if they are transversal

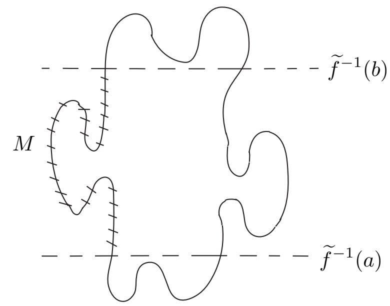

Figure 7: Manifold $\widetilde{M}$

with respect to $\widetilde{X}$.

Step 1: we show the transversality between $p \in M$ and $q \in M$. Since $\mathcal{D}(p;\widetilde{X}) \subseteq M \cup \mathrm{Int}\widetilde{M}^a$, we have $\mathcal{D}(p;\widetilde{X}) \cap \widetilde{M}^{a,b} = \mathcal{D}(p;\widetilde{X}) \cap M = \mathcal{D}(p;X)$. Similarly, $\mathcal{A}(p;\widetilde{X}) \cap \widetilde{M}^{a,b} = \mathcal{A}(p;X)$. Since X satisfies transversality, p and q are transversal in $\widetilde{M}^{a,b}$. By Lemma 7.2, they are transversal globally. This shows the transversality between p and q does not depend on the extension of X. So, no matter how $\widetilde{X}$ is changed outside of M, p and q are always transversal if they are in M.

Step 2: we modify $\widetilde{X}$ in $\widetilde{M}^a$. We made modifications near each critical point $\widetilde{p}$ in $\widetilde{M}^a$ with the order from critical points with higher values to those with lower values. Slide $\mathcal{A}(\widetilde{p};\widetilde{X})$ for each $\widetilde{p} \in \widetilde{M}^a$ such that $\widetilde{p}$ is transverse to each $\widetilde{q} \in M \cup \widetilde{M}^a$ with $\widetilde{f}(\widetilde{q}) > \widetilde{f}(\widetilde{p})$. (Here, for all $\widetilde{q} \in M$, we have $\widetilde{f}(\widetilde{q}) > \widetilde{f}(\widetilde{p})$.) Thus, for all $\widetilde{p}$ and $\widetilde{q}$ in $M \cup \widetilde{M}^a$, they are transversal globally after these modifications. By Lemma 7.2 and Step 1, no matter how $\widetilde{X}$ is changed outside of $M \cup \widetilde{M}^a$, $\widetilde{p}$ and $\widetilde{q}$ are still transversal globally because they are still transversal in $\widetilde{M}^a$.

Step 3: we modify $\widetilde{X}$ in $\widetilde{M}^b - [M \cup \widetilde{M}^a]$. To do this, we slide the descending manifolds with the order from critical points with lower values to those with higher values. More precisely, slide $\mathcal{D}(\widetilde{p}; \widetilde{X})$ for each $\widetilde{p} \in \widetilde{M}^b - [M \cup \widetilde{M}^a]$ such that $\widetilde{p}$ is transverse to all $\widetilde{q} \in \widetilde{M}^b - M$ with $\widetilde{f}(\widetilde{q}) < \widetilde{f}(\widetilde{p})$. (Here, for all $\widetilde{q} \in \widetilde{M}^a$, we have $\widetilde{f}(\widetilde{q}) < \widetilde{f}(\widetilde{p})$.) We claim that, for all $\widetilde{p}$ and $\widetilde{q}$ in $\widetilde{M}^b$, they are transversal. It suffices to prove that, for each $p \in M$ and $\widetilde{q} \in \widetilde{M}^{a,b} - M$, we have p and $\widetilde{q}$ are transversal. Clearly, $\mathcal{D}(\widetilde{q}; \widetilde{X}) \subseteq \widetilde{M}^b - M$, thus $\mathcal{D}(\widetilde{q}; \widetilde{X}) \cap \widetilde{M}^{a,b} \subseteq \widetilde{M}^{a,b} - M$. Since $\mathcal{A}(p; \widetilde{X}) \cap \widetilde{M}^{a,b} \subseteq M$, we get $\mathcal{D}(\widetilde{q}; \widetilde{X}) \cap \mathcal{A}(p; \widetilde{X}) \cap \widetilde{M}^{a,b} = \emptyset$. So $\mathcal{D}(\widetilde{q}; \widetilde{X}) \cap \mathcal{A}(p; \widetilde{X}) = \emptyset$. Similarly, $\mathcal{A}(\widetilde{q}; \widetilde{X}) \cap \mathcal{D}(p; \widetilde{X}) = \emptyset$. We infer that p and $\widetilde{q}$ are transversal. The above claim is proved. By Lemma 7.2 again, no matter how $\widetilde{X}$ is changed outside of $\widetilde{M}^b$, all critical points in $\widetilde{M}^b$ are still mutually transverse.

Step 4: we modify $\widetilde{X}$ on $\widetilde{M} - \widetilde{M}^b$. Slide the descending manifolds with the order from critical points with lower values to those with higher values. We eventually get that $\widetilde{X}$ satisfies transversality.

By the above argument, for all p and q in M, we have $\mathcal{D}(p; \widetilde{X}) \subseteq M \cup \widetilde{f}^{-1}((-\infty, a))$, $\mathcal{A}(q; \widetilde{X}) \subseteq M \cup \widetilde{f}^{-1}((b, +\infty))$, $\mathcal{D}(p; \widetilde{X}) \cap M = \mathcal{D}(p; X)$ and $\mathcal{A}(q; \widetilde{X}) \cap M = \mathcal{A}(q; X)$. Thus

$$\mathcal{D}(p; \widetilde{X}) \cap \mathcal{A}(q; \widetilde{X}) = (\mathcal{D}(p; \widetilde{X}) \cap M) \cap (\mathcal{A}(q; \widetilde{X}) \cap M) = \mathcal{D}(p; X) \cap \mathcal{A}(q; X).$$

Suppose $M_1 = \emptyset$. Clearly, we can construct $\widetilde{f}$ such that $\widetilde{f}|_{\widetilde{M} - M} > b$. Thus, for any $p \in M$, we have $\mathcal{D}(p; \widetilde{X}) \subseteq M$ and $\mathcal{D}(p; \widetilde{X}) = \mathcal{D}(p; X)$. Similarly, the conclusion is true in the case of $M_2 = \emptyset$. $\square$

8 Manifold Structures (II)

This chapter extends the results in Chapter 4 to the case of a general metric.

8.1 Moduli Spaces and Topological Equivalence

Since the vector field is not locally trivial, there are no natural smooth structures for $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$ (see Example 8.1). In other words, most parts of Theorems 4.4, 4.5 and 4.6 are not true for a general gradient-like vector field. However, they still have natural topologies. We shall equip them with topologies which are compatible with those defined by Theorems 4.4, 4.5 and 4.6.

Definition 8.1. *Define the set $\overline{\mathcal{M}(p,q)}$ as (4.3). Equip $\overline{\mathcal{M}(p,q)}$ with the unique topology such that the evaluation map $E : \overline{\mathcal{M}(p,q)} \to \prod_{i=0}^{l} f^{-1}(a_i)$ in (4.4) is a topological embedding. We call $\overline{\mathcal{M}(p,q)}$ the compactified moduli space of $\mathcal{M}(p,q)$.*

Definition 8.2. *Define the set $\overline{\mathcal{W}(p,q)}$ as (4.12). Define the topology of $\overline{\mathcal{W}(p,q)}$ as the unique topology such that $i : \overline{\mathcal{W}(p,q)} \to \overline{\mathcal{M}(p,q)} \times M$ is a topological embedding, where i is defined in (3) of Theorem 4.6. We call $\overline{\mathcal{W}(p,q)}$ the compactified space of $\mathcal{W}(p,q)$.*

Similarly, we define $\overline{\mathcal{D}(p)}$ as (4.6), define $U(i)$ as (4.8), and define $E(i)$ as (4) in Theorem 4.5.

Clearly, $E(i) : U(i) \to \prod_{j=0}^{i-1} f^{-1}(a_j) \times M$ is injective. Give $U(i)$ the unique topology such that $E(i)$ is a topological embedding. By an argument similar to the proof of Theorem 4.5, we get $U(i)$ and $U(j)$ share the same topology on $U(i) \cap U(j)$.

Definition 8.3. *Define the set* $\overline{\mathcal{D}(p)}$ *as (4.6). Define the topology of* $\overline{\mathcal{D}(p)} = \bigcup_i U(i)$ *as the coherent topology such that each* $U(i)$ *is an open subspace of* $\overline{\mathcal{D}(p)}$ *(see (4.8)). We call* $\overline{\mathcal{D}(p)}$ *the compactified space of* $\mathcal{D}(p)$*.*

Suppose f_1 and f_2 are Morse functions on M_1 and M_2. Suppose X_i is a negative gradient-like field for f_i, and X_i satisfies transversality. Suppose $h : M_1 \to M_2$ is a topological equivalence between X_1 and X_2. If p is a critical point of f_1, then $h(p)$ is a critical point of f_2. Furthermore, $h(\mathcal{D}(p)) = \mathcal{D}(h(p))$, $h(\mathcal{A}(p)) = \mathcal{A}(h(p))$, and $h(\mathcal{W}(p,q)) = \mathcal{W}(h(p), h(q))$. Thus h naturally induces maps $h_* : \overline{\mathcal{M}(p,q)} \to \overline{\mathcal{M}(h(p), h(q))}$, $h_* : \overline{\mathcal{W}(p,q)} \to \overline{\mathcal{W}(h(p), h(q))}$, and $h_* : \overline{\mathcal{D}(p)} \to \overline{\mathcal{D}(h(p))}$. Here, if $\Gamma \in \overline{\mathcal{M}(h(p), h(q))}$, then $h_*(\Gamma) = h(\Gamma)$; if $(\Gamma, x) \in \overline{\mathcal{W}(p,q)}$ (or $\overline{\mathcal{D}(p)}$), then $h_*(\Gamma, x) = (h(\Gamma), h(x))$. Clearly, h_* is a bijection and $(h_*)^{-1} = (h^{-1})_*$.

Theorem 8.4. *The maps* $h_* : \overline{\mathcal{M}(p,q)} \to \overline{\mathcal{M}(h(p), h(q))}$*,* $h_* : \overline{\mathcal{D}(p)} \to \overline{\mathcal{D}(h(p))}$*, and* $h_* : \overline{\mathcal{W}(p,q)} \to \overline{\mathcal{W}(h(p), h(q))}$ *are homeomorphisms.*

Proof. It suffices to prove that h_* is continuous because this implies h_*^{-1} is also continuous.

(1). We consider the case of $h_* : \overline{\mathcal{M}(p,q)} \to \overline{\mathcal{M}(h(p), h(q))}$.

By the definition, $\overline{\mathcal{M}(p,q)}$ is identified with a topological subspace of $\prod_{i=0}^{l} f_1^{-1}(a_i)$ and $\overline{\mathcal{M}(h(p), h(q))}$ is identified with a topological subspace of $\prod_{i=0}^{k} f_2^{-1}(b_i)$. By this identification, for any $\Gamma \in \overline{\mathcal{M}(p,q)}$, we have $\Gamma = (x_0(\Gamma), \cdots, x_l(\Gamma))$ and $h_*(\Gamma) = (y_0(h(\Gamma)), \cdots, y_k(h(\Gamma)))$. Suppose $x_0(\Gamma_0)$ is on $\gamma \in \mathcal{M}(p,r)$ and γ is a component of Γ_0, then $h(x_0(\Gamma_0))$ is on $h(\gamma) \in \mathcal{M}(h(p), h(r))$. Suppose the regular values in $[f_2(h(r)), f_2(h(p))]$ are $b_0, \cdots, b_s$. Then $h(\gamma)$ intersects with $f_2^{-1}(b_i)$ $(0 \le i \le s)$ at $y_i(h(\Gamma_0))$. When Γ converges to Γ_0, we have $h(x_0(\Gamma))$ converges to $h(x_0(\Gamma_0))$. Thus, when Γ is close to Γ_0 enough, the flow line passing through $h(x_0(\Gamma))$ intersects with $f_2^{-1}(b_i)$ $(0 \le i \le s)$ at $y_i(h(\Gamma))$ and $y_i(h(\Gamma))$ is continuous with respect to Γ.

By an induction, we can prove that, for all $0 \leq i \leq k$, $y_i(h(\Gamma))$ is continuous with respect to Γ. Thus h_* is continuous.

(2). Since $\overline{\mathcal{W}(p,q)}$ is a topological subspace of $\overline{\mathcal{M}(p,q)} \times M_1$, by (1), we infer that h_* is continuous on $\overline{\mathcal{W}(p,q)}$.

(3). We consider the case of $h_* : \overline{\mathcal{D}(p)} \to \overline{\mathcal{D}(h(p))}$.

It suffices to check the continuity of h_* on each $U(i)$. Suppose $(\Gamma_0, z_0) \in U(i)$ and $\widetilde{c}_{s+1} < f_2(h(z_0)) < \widetilde{c}_{s-1}$, where $\widetilde{c}_j$ are critical values of f_2. Then $h_*(\Gamma_0, z_0) \in \widetilde{U}(s)$, where $\widetilde{U}(s) \subseteq \overline{\mathcal{D}(h(p))}$ is defined similarly to $U(i)$. Thus, when (Γ, z) is close to (Γ_0, z_0) enough, we have $h_*(\Gamma, z) \in \widetilde{U}(s)$. Identify $\widetilde{U}(s)$ with a topological subspace of $\prod_{j=0}^{s-1} f_2^{-1}(b_j) \times M_2$, we have $h_*(\Gamma, z) = (y_0(h(\Gamma)), \cdots, y_{s-1}(h(\Gamma)), h(z))$. By an argument similar to that in (1), we can prove that $y_j(h(\Gamma))$ is continuous with respect to Γ. Since $h(z)$ is continuous with respect to z, we infer h_* is continuous. $\square$

8.2 Properties of Moduli Spaces

In this section, we establish the relevant properties of the compactified moduli spaces. Particularly, the manifold structures of these spaces will be emphasized.

Consider first the special case when M is compact. By Theorem 7.7, we can construct a negative gradient-like field Y for f such that Y is locally trivial and satisfies transversality. In addition, there exists a topological equivalence between X and Y such that $h(p) = p$ for each critical point p. Thus, by Theorem 8.4, X and Y have isomorphic compatified moduli spaces. Since the properties of these spaces for Y are proved in Chapter 4. We deduce certain properties of these spaces for X.

More generally, suppose that f is proper but M is not necessarily compact. For any pair of critical points (p, q), choose regular values a and b such that $M^{a,b}$ is compact and contains p and q. By Lemma 7.13, we can embed $M^{a,b}$ into $\widetilde{M}$, extend $f|_{M^{a,b}}$ to be $\widetilde{f}$ on $\widetilde{M}$, and extend $X|_{M^{a,b}}$ to be $\widetilde{X}$ on $\widetilde{M}$. Furthermore, $\mathcal{W}(p, q; X) = \mathcal{W}(p, q; \widetilde{X})$. Thus we get $\overline{\mathcal{M}(p, q; X)} = \overline{\mathcal{M}(p, q; \widetilde{X})}$ and $\overline{\mathcal{W}(p, q; X)} = \overline{\mathcal{W}(p, q; \widetilde{X})}$. If f is bounded below, we choose M^a such that $p \in M^a$. Do the above extension again to get $\overline{\mathcal{D}(p; X)} = \overline{\mathcal{D}(p; \widetilde{X})}$. Thus Lemma 7.13 reduces the proper case to the compact case.

Before formulating the property of $\overline{\mathcal{M}(p, q)}$, we introduce a map. Suppose $\Gamma_1 \in \overline{\mathcal{M}(p, r)}$ is a generalized flow line connecting p with r and $\Gamma_2 \in \overline{\mathcal{M}(r, q)}$ is a generalized flow line connecting r with q. Thus the combination of Γ_1 and Γ_2 gives a generalized flow line Γ connecting p with q. So we have the natural inclusion $i_{(p,r,q)} : \overline{\mathcal{M}(p, r)} \times \overline{\mathcal{M}(r, q)} \to \overline{\mathcal{M}(p, q)}$.

Theorem 8.5 (Property of $\overline{\mathcal{M}(p, q)}$)**.** *Suppose f is proper and X satisfies transversality. Then, for each pair of critical points (p, q), the space $\overline{\mathcal{M}(p, q)} = \bigsqcup_I \mathcal{M}_I$ has the flowing properties.*

(1). It is a compact topological manifold with boundary. Its interior is $\mathcal{M}(p, q)$.

(2). Its topology is compatible with those of $\mathcal{M}_I$, and the map $i_{(p,r,q)} : \overline{\mathcal{M}(p, r)} \times \overline{\mathcal{M}(r, q)} \to \overline{\mathcal{M}(p, q)}$ is a topological embedding.

(3). The evaluation map $E : \overline{\mathcal{M}(p, q)} \to \prod_{i=0}^{l} f^{-1}(a_i)$ is a topological embedding, where E is defined in (4.4).

(4). There exists a topological embedding $\iota : \overline{\mathcal{M}(p, q)} \to \prod_{i=0}^{l} f^{-1}(a_i)$ such that $\iota(\overline{\mathcal{M}(p, q)})$ is a smoothly embedded submanifold with faces inside $\prod_{i=0}^{l} f^{-1}(a_i)$ and the k-stratum of

$\iota(\overline{\mathcal{M}(p,q)})$ is $\bigsqcup_{|I|=k} \iota(\mathcal{M}_I)$.

In particular, if M is compact, then there exist homeomorphisms $h_i : f^{-1}(a_i) \to f^{-1}(a_i)$ such that $\iota = (\prod_{i=0}^{l} h_i) \circ E$ in (4).

Theorem 8.6 (Smooth Structure of $\overline{\mathcal{M}(p,q)}$)**.** *Under the assumption of Theorem 8.5, each $\overline{\mathcal{M}(p,q)}$ carries a smooth structure compatible with its topology such that $\overline{\mathcal{M}(p,q)}$ is a compact smooth manifold with faces and $\partial^k \overline{\mathcal{M}(p,q)} = \bigsqcup_{|I|=k} \mathcal{M}_I$. In particular, suppose M is compact, then $i_{(p,r,q)} : \overline{\mathcal{M}(p,r)} \times \overline{\mathcal{M}(r,q)} \to \overline{\mathcal{M}(p,q)}$ is a smooth embedding.*

Remark 8.1. *The (1) of Theorem 8.5 shows that we can add a boundary to $\mathcal{M}(p,q)$ such that it becomes a compact manifold with boundary. The following theorems show that this is also true for $\mathcal{W}(p,q)$ and $\mathcal{D}(p)$. Thus moduli spaces are special open manifolds (if they are open) because there exists an obstruction of adding a boundary to a general open manifold.*

Remark 8.2. *Example 8.1 shows that, if the metric is not locally trivial, then $E(\overline{\mathcal{M}(p,q)})$ usually is even not a C^1 embedded submanifold of $\prod_{i=0}^{l} f^{-1}(a_i)$. Here E is the evaluation map in the (3) of Theorem 8.5. However, the (4) of Theorem 8.5 shows that a suitable embedding ι makes the image good.*

Proof of Theorem 8.5. Choose regular values a and b such that $M^{a,b}$ is compact and contains p and q. As described in the above, construct $\widetilde{M}$, $\widetilde{f}$ and $\widetilde{X}$. We have $\overline{\mathcal{M}(p,q;X)} = \overline{\mathcal{M}(p,q;\widetilde{X})}$ and $\mathcal{M}_I(\widetilde{X}) = \mathcal{M}_I(X)$ for all critical sequences I with head p and tail q. There exists a topological equivalence $h : \widetilde{M} \to \widetilde{M}$ which maps the orbits of $\widetilde{X}$ to those of Y, where Y is locally trivial.

(1). By Theorem 4.4, we know that $\overline{\mathcal{M}(p,q;Y)}$ is a compact smooth manifold with faces whose k-stratum is $\bigsqcup_{|I|=k} \mathcal{M}_I(Y)$. Thus $\overline{\mathcal{M}(p,q;Y)}$ is a compact topological manifold with boundary, and its interior is $\mathcal{M}(p,q;Y)$. By Theorem 8.4, we know that h induces a homeomorphism $h_* : \overline{\mathcal{M}(p,q;\widetilde{X})} \to \overline{\mathcal{M}(p,q;Y)}$ such that $h_*(\mathcal{M}_I(\widetilde{X})) = \mathcal{M}_I(Y)$. This completes the proof of (1).

(2). The proof is easy and even does not need the comparison among $\overline{\mathcal{M}(p,q;X)}$, $\overline{\mathcal{M}(p,q;\widetilde{X})}$ and $\overline{\mathcal{M}(p,q;Y)}$. Similar details is also included in the proof of Theorem 4.4.

(3). This is the definition of the topology of $\overline{\mathcal{M}(p,q;X)}$.

(4). Let $E_Y : \overline{\mathcal{M}(p,q;Y)} \to \prod_{i=0}^{l} \widetilde{f}^{-1}(a_i)$ be the evaluation map. By Theorem 4.4, we know E_Y is a smooth embedding. We shall prove that $\mathrm{Im}(E_Y) \subseteq \prod_{i=0}^{l} f^{-1}(a_i) \subseteq \prod_{i=0}^{l} \widetilde{f}^{-1}(a_i)$. It suffices to prove that $\mathcal{W}(r_1,r_2;Y) \subseteq M^{a,b}$ for all r_1 and r_2 in $M^{a,b}$.

Suppose γ is a flow line in $\widetilde{M}$ such that $\gamma(t_0) \in M^{a,b}$ and $\gamma(t_1) \notin M^{a,b}$ for some t_0 and t_1. Then either $\widetilde{f}(\gamma(t_1)) > b > \widetilde{f}(r_1)$ or $\widetilde{f}(\gamma(t_1)) < a < \widetilde{f}(r_2)$. Thus $\mathcal{W}(r_1,r_2;Y) \subseteq M^{a,b}$.

Thus $\mathrm{Im}(E_Y) \subseteq \prod_{i=0}^{l} f^{-1}(a_i)$ and $\iota = E_Y \circ h_*$ is the desired map.

Finally, we consider the special case when M is compact.

We construct Y on M. The topological equivalence $h : M \to M$ induces the homeomorphism $h_* : \overline{\mathcal{M}(p,q;X)} \to \overline{\mathcal{M}(p,q;Y)}$. We consider the relation between $h(f^{-1}(a_i))$ and $f^{-1}(a_i)$. Denote by ϕ_t^1 the flow generated by X and by ϕ_t^2 the flow generated by Y. For any $x \in f^{-1}(a_i)$, we have $\phi^1(-\infty,x) = r_1$ for some $r_1 \in M - M^{a_i}$ and $\phi^1(+\infty,x) = r_2$ for some $r_2 \in M^{a_i}$. Since h is a topological equivalence fixing r_1 and r_2, we know that $\phi^2(-\infty,h(x)) = r_1$ and $\phi^2(+\infty,h(x)) = r_2$. Thus $\phi^2(t,h(x)) \in f^{-1}(a_i)$ for some $t \in (-\infty,+\infty)$. An isotopy along the flows generated by Y gives a homeomorphism $\psi_i : h(f^{-1}(a_i)) \to f^{-1}(a_i)$. We complete the proof by defining $h_i = \psi_i \circ h$. $\square$

The first half part of Theorem 8.6 is a corollary of Theorem 8.5. We can construct the topological equivalence on M when M is compact. Thus the second half part is also true because it is true in the special case.

By Theorems 4.6, 4.5 and 6.1, using an argument similar to the proof of Theorem 8.5, we can get the following results.

Theorem 8.7 (Property of $\overline{\mathcal{W}(p,q)}$)**.** *Suppose f is proper and X satisfies transversality. Then, for each pair of critical points (p,q), the space $\overline{\mathcal{W}(p,q)} = \bigsqcup_{(I,s)} \mathcal{W}_{I,s}$ has the flowing properties.*

(1). It is a compact topological manifold with boundary. Its interior is $\mathcal{W}(p,q)$.

(2). Its topology is compatible with that of $\mathcal{W}_{I,s}$. The maps $i^1_{(p,r,q)} : \overline{\mathcal{W}(p,r)} \times \overline{\mathcal{M}(r,q)} \to \overline{\mathcal{W}(p,q)}$ and $i^2_{(p,r,q)} : \overline{\mathcal{M}(p,r)} \times \overline{\mathcal{W}(r,q)} \to \overline{\mathcal{W}(p,q)}$ are topological embeddings.

(3). The inclusion $i : \overline{\mathcal{W}(p,q)} \to \overline{\mathcal{M}(p,q)} \times M$ and the map $\widetilde{E} : \overline{\mathcal{W}(p,q)} \to \prod_{i=0}^{l} f^{-1}(a_i) \times M$ are topological embeddings, where $\widetilde{E}$ is defined in (4.14).

(4). There exists a topological embedding $\iota : \overline{\mathcal{W}(p,q)} \to \prod_{i=0}^{l} f^{-1}(a_i) \times M$ such that $\iota(\overline{\mathcal{W}(p,q)})$ is a smoothly embedded submanifold with faces inside $\prod_{i=0}^{l} f^{-1}(a_i) \times M$ and the k-stratum of $\iota(\overline{\mathcal{W}(p,q)})$ is $\bigsqcup_{(I,s)} \iota(\mathcal{W}_{I,s})$, where (I,s) contains $k+2$ components.

In particular, if M is compact, then there exist homeomorphisms $h_i : f^{-1}(a_i) \to f^{-1}(a_i)$ such that $\iota = [(\prod_{i=0}^{l} h_i) \times h] \circ \widetilde{E}$ in (4).

Corollary 8.8 (Smooth Structures of $\overline{\mathcal{W}(p,q)}$)**.** *Under the assumption of Theorem 8.7, $\overline{\mathcal{W}(p,q)}$ carries a smooth structure compatible with its topology such that $\overline{\mathcal{W}(p,q)}$ is a compact smooth manifold with faces and $\partial^k \overline{\mathcal{W}(p,q)} = \bigsqcup_{(I,s)} \mathcal{W}_{I,s}$, where (I,s) contains $k+2$ components.*

Theorem 8.9 (Property of $\overline{\mathcal{D}(p)}$)**.** *Suppose f is proper and bounded below. Suppose X*

satisfies transversality. Then, for each critical point p, $\overline{\mathcal{D}(p)} = \bigsqcup_I \mathcal{D}_I$ has the following properties.

(1). It is homeomorphic to a closed disc. Its interior is $\mathcal{D}(p)$.

(2). Its topology is compatible with those of $\mathcal{D}_I$. The map $i_{(p,r)} : \overline{\mathcal{M}(p,r)} \times \overline{\mathcal{D}(r)} \to \overline{\mathcal{D}(p)}$ is a topological embedding.

(3). The evaluation map $e : \overline{\mathcal{D}(p)} \to M$ is continuous. The restriction of e on $\mathcal{D}_I = \mathcal{M}_I \times \mathcal{D}(r_k)$ is the coordinate projection onto $\mathcal{D}(r_k) \subseteq M$, where $I = \{p, r_1, \cdots, r_k\}$.

(4). It carries a smooth structure compatible with its topology such that it is a compact smooth manifold with faces and $\partial^k \overline{\mathcal{D}(p)} = \bigsqcup_{|I|=k-1} \mathcal{D}_I$.

8.3 An Example

In the previous section, we constructed smooth structures for $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$ in the case of a general Riemannian metric. However, these smooth structures are not natural, which is very different from the case of a locally trivial metric.

In this section, we shall show a remarkable difference between these two cases by an example.

Consider $E : \overline{\mathcal{M}(p,q)} \longrightarrow \prod_{i=0}^{l} f^{-1}(a_i)$ in (4.4). By (3) of Theorem 4.4, the image of E, $\mathrm{Im}(E) \subseteq \prod_{i=0}^{l} f^{-1}(a_i)$ is a smooth (C^∞) embedded submanifold of $\prod_{i=0}^{l} f^{-1}(a_i)$ when the metric is locally trivial. For a general metric, we have the following counterexample.

Example 8.1 (Not C^1). *Let CP^2 the complex projective plane. Then there exist a metric and a Morse function f on CP^2, where f has three critical points p, q and r such that $ind(p) = 4$, $ind(q) = 0$ and $ind(r) = 2$. $\overline{\mathcal{M}(p,q)} = \mathcal{M}(p,q) \sqcup (\mathcal{M}(p,r) \times \mathcal{M}(r,q))$. And*

$E(\overline{\mathcal{M}(p,q)})$ is NOT a C^1 embedded submanifold with boundary $E(\mathcal{M}(p,r) \times \mathcal{M}(r,q))$ of $\prod_{i=0}^{1} f^{-1}(a_i)$. In other words, it's impossible to give Im(E) a C^1 structure compatible with $\prod_{i=0}^{1} f^{-1}(a_i)$.

Proof. Clearly, there is a Morse function on CP^2 with such three critical points and $f(r) = 0$. By the Morse Lemma, in a neighborhood U of r, there is a local coordinate chart (v_1, v_2, v_3, v_4) such that r has the coordinate $(0,0,0,0)$, $\sum_{i=1}^{4} v_i^2 < 4\epsilon^2$ and, in the local chart, we have $f(v) = \frac{1}{2}(-v_1^2 - v_2^2 + v_3^2 + v_4^2)$. We can choose f such that $\epsilon = 1$. We equip CP^2 with a metric such that, in U, it has the form

$$(dx_1)^2 + \frac{1}{2}(dx_2)^2 + \frac{1}{4}(dx_3)^2 + \frac{1}{4}(dx_4)^2. \tag{8.1}$$

Then the flow with initial value (v_1, v_2, v_3, v_4) is $(e^t v_1, e^{2t} v_2, e^{-4t} v_3, e^{-4t} v_4)$.

Consider the map $E : \overline{\mathcal{M}(p,q)} \longrightarrow M_0 \times M_1$, where $M_0 = f^{-1}(\frac{1}{2})$ and $M_1 = f^{-1}(-\frac{1}{2})$. We shall prove Im$(E)$ is not a C^1 embedded submanifold with boundary $E(\mathcal{M}(p,r) \times \mathcal{M}(r,q))$ of $M_0 \times M_1$.

Clearly, $E(\mathcal{M}(p,r) \times \mathcal{M}(r,q)) = S^+ \times S^-$, where $S^+ = \{(0,0,v_3,v_4) \mid v_3^2 + v_4^2 = 1\}$ and $S^- = \{(v_1,v_2,0,0) \mid v_1^2 + v_2^2 = 1\}$. Let $\tilde{S}^+ = \{(v_3,v_4) \mid v_3^2 + v_4^2 = 1\}$ and $\tilde{S}^- = \{(v_1,v_2) \mid v_1^2 + v_2^2 = 1\}$.

The flow map gives a diffeomorphism from an open neighborhood W_0 of S^+ in M_0 onto $U \cap (R^2 \times \tilde{S}^+)$ and a diffeomorphism from an open neighborhood W_1 of S^- in M_1 onto $U \cap (\tilde{S}^- \times R^2)$. Thus there is a diffeomorphism $\psi : W_0 \times W_1 \longrightarrow (U \cap (R^2 \times \tilde{S}^+)) \times (U \cap$

$(\tilde{S}^- \times R^2))$. Denote $\psi(\mathrm{Im}(E) \cap (W_0 \times W_1))$ by P. Then

$$\begin{aligned} P &= \{((v_1, v_2, v_3, v_4), (v_5, v_6, v_7, v_8)) \mid (v_1, v_2, v_3, v_4) \in U \cap (R^2 \times \tilde{S}^+) \text{ and} \\ &\quad (v_5, v_6, v_7, v_8) \in U \cap (\tilde{S}^- \times R^2) \text{ are connected by a generalized flow line.}\}. \end{aligned}$$

In order to prove $\mathrm{Im}(E)$ is not a C^1 embedded submanifold of $M_1 \times M_2$, we only need to check P is not a C^1 embedded submanifold of $(U \cap (R^2 \times \tilde{S}^+)) \times (U \cap (\tilde{S}^- \times R^2))$.

Suppose $(v_1, v_2, v_3, v_4) \in U \cap (R^2 \times \tilde{S}^+)$ and $(v_1, v_2) \neq (0, 0)$, by a direct calculation, (v_1, v_2, v_3, v_4) is connected to

$$(d^{-\frac{1}{2}} v_1, d^{-1} v_2, d^2 v_3, d^2 v_4) \tag{8.2}$$

by an unbroken flow line, where

$$d = \frac{1}{2} v_1^2 + \frac{1}{2} (v_1^4 + 4 v_2^2)^{\frac{1}{2}}. \tag{8.3}$$

We prove our result by contradiction. If P were a C^1 embedded submanifold with boundary $\partial P = S^+ \times S^-$, then there is a C^1 collar embedding $\varphi : \tilde{S}^+ \times \tilde{S}^- \times [0, \epsilon) \longrightarrow (R^2 \times \tilde{S}^+) \times (\tilde{S}^- \times R^2)$ such that

$$\varphi(\cos\theta^+, \sin\theta^+, \cos\theta^-, \sin\theta^-, s) = ((v_1, v_2, v_3, v_4), (v_5, v_6, v_7, v_8)),$$

and

$$\varphi(\cos\theta^+, \sin\theta^+, \cos\theta^-, \sin\theta^-, 0) = ((0, 0, \cos\theta^+, \sin\theta^+), (\cos\theta^-, \sin\theta^-, 0, 0)).$$

When $s \neq 0$, $\mathrm{Im}(\varphi) \cap \partial P = \emptyset$, thus $(v_1, v_2) \neq (0,0)$ and (v_5, v_6, v_7, v_8) equals (8.2).

In the following four steps, we will use some estimates. The same notation C or C_i may stand for different constants in different steps.

Firstly, we prove that $\frac{\partial}{\partial s}|_{s=0} v_7 = \frac{\partial}{\partial s}|_{s=0} v_8 = 0$.

Fix θ^+ and θ^-, then v_1 and v_2 are C^1 functions of s, and $v_1 = v_2 = 0$ when $s = 0$. So there exist $C_1 > 0$ and $\delta > 0$ such that, for all $s \in [0, \delta)$, we have $|v_1| \leq C_1 s$ and $|v_2| \leq C_1 s$. Since $(v_3, v_4) \in \tilde{S}^+$, (v_3, v_4) is bounded, by (8.2) and (8.3), there exists $C_2 > 0$ such that $|v_7| \leq C_2 s^2$ and $|v_8| \leq C_2 s^2$. This proves our first claim.

Secondly, we claim that $\frac{\partial}{\partial s}|_{s=0}(v_1, v_2) \neq (0,0)$.

If not, then

$$(d\varphi)|_{s=0}\frac{\partial}{\partial s} = \left(0, 0, \frac{\partial}{\partial s}|_{s=0} v_3, \frac{\partial}{\partial s}|_{s=0} v_4, \frac{\partial}{\partial s}|_{s=0} v_5, \frac{\partial}{\partial s}|_{s=0} v_6, 0, 0\right) \in T(S^+ \times S^-),$$

So $(d\varphi)|_{s=0}\frac{\partial}{\partial s}$ is not a normal vector of $S^+ \times S^-$. This gives a contradiction.

Thirdly, we prove that $\frac{\partial}{\partial s}|_{s=0} v_2 = 0$.

By the continuity of $\frac{\partial}{\partial s}|_{s=0} v_2$, we only need to prove this is true when $\cos\theta^- \neq 0$. Fix θ^+ and θ^-, then $\lim\limits_{s \to 0} v_5 = \cos\theta^- \neq 0$ and $\lim\limits_{s \to 0} v_6 = \sin\theta^-$. Thus there exist $\delta > 0$, $C_2 > 0$ and $C_3 > 0$, such that, for all $s \in (0, \delta)$, we have $0 < C_2 \leq |v_5|$ and $|v_6| \leq C_3$. By (8.2), $C_2 \leq |d^{-\frac{1}{2}} v_1|$ and $|d^{-1} v_2| \leq C_3$. Then $C_2^2 d \leq |v_1|^2$ and $|v_2| \leq C_3 d$. So $|v_2| \leq C_3 C_2^{-2} |v_1|^2$. In the first step, we showed that there exists $C_1 > 0$, shrinking δ if necessary, we get $|v_1| \leq C_1 s$. Thus $|v_2| \leq C_3 C_2^{-2} C_1^2 s^2$. This gives our third claim.

Finally, we derive the contradiction.

Let $\cos\theta^- = 0$ and $\sin\theta^- = 1$. Fix θ^+. By the second and the third claims, $\frac{\partial}{\partial s}|_{s=0} v_1 \neq 0$.

Since $v_1 = 0$ when $s = 0$, then there exist $\delta_1 > 0$ and $C_1 > 0$ such that, for all $s \in [0, \delta_1)$, we have

$$|v_1| \geq C_1 s. \tag{8.4}$$

Since $v_5 = \cos\theta^- = 0$ when $s = 0$, and v_5 is a C^1 function, then there exist $\delta_2 > 0$ and $C_2 > 0$ such that, for all $s \in [0, \delta_2)$, we have

$$|v_5| \leq C_2 s. \tag{8.5}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Combining (8.2), (8.4) and (8.5), we can find $C > 0$ such that, for all $s \in (0, \delta)$, we have $|d^{-\frac{1}{2}} v_1| = |v_5| \leq C|v_1|$ and $v_1 \neq 0$. Thus by (8.3),

$$\frac{2}{v_1^2 + (v_1^4 + 4v_2^2)^{\frac{1}{2}}} = d^{-1} \leq C^2. \tag{8.6}$$

However, when $s \to 0$, we have $v_1 \to 0$, $v_2 \to 0$ and $v_1^2 + (v_1^4 + 4v_2^2)^{\frac{1}{2}} \to 0$, then $d^{-1} \to +\infty$. This gives a contradiction. $\square$

9 Orientations (II)

This chapter extends the results in Chapter 5 to the case of a general metric.

9.1 A Remark on Orientations

In this section, we shall compare two definitions of orientations. One is by the method of algebraic topology, the other is by the method of differential topology. This results in Lemma 9.6 which is important in the next section.

Suppose M is a n dimensional smooth manifold. The orientation of M at x can be defined by the method of either algebraic topology or differential topology. In algebraic topology, the orientation is a generator $\alpha \in H^n(M, M - \{x\})$. In differential topology, the orientation is an ordered base $\{e_1, \cdots, e_n\} \subseteq T_xM$. These two definitions are related as follows. Choose a smooth embedding $\varphi : V \to M$ such that $\varphi(0) = x$ and $D\varphi(0) = \mathrm{Id}$, where V is a neighborhood of 0 in T_xM. Then $\varphi^*\alpha \in H^n(V, V - \{0\}) = H^n(T_xM, T_xM - \{0\})$ is a generator. Here $\varphi^*\alpha$ does not depend on the choice of φ. Actually, if $\widetilde{\varphi}$ is another such embedding, then there exists an isotopy between φ and $\widetilde{\varphi}$ in a smaller neighborhood of 0. Denote by α_0 the preferred generator in $H^n(R^n, R^n - \{0\})$ (see [39, p. 266]). The ordered base $\{e_1, \cdots, e_n\}$ determines a linear isomorphism $A : T_xM \to R^n$, then $A^*\alpha_0 \in H^n(T_xM, T_xM - \{0\})$ is also a generator. We say that these two definitions give the same orientation if and only if $\varphi^*\alpha = A^*\alpha_0$. It's easy to prove that an orientation is continuous in the sense of algebraic topology if and only if it is continuous in the sense of differential topology.

Suppose L is a k dimensional embedded submanifold of M such that its normal bundle is orientable. We can also define the normal orientation of L with respect to M by either algebraic or differential method. First, choose a neighborhood U of L such that L is closed in U. Choose a Thom class $\beta \in H^{n-k}(U, U-L)$. The Thom class β defines the normal orientation in the algebraic sense. Second, for any $x \in L$. Choose an ordered base $\{\varepsilon_{k+1}, \cdots, \varepsilon_n\}$ of the normal space $N_x(L, M) = T_xM/T_xL$. This defines the normal orientation of L at x in the differential sense. These two definitions are related as follows. Let $\varphi : V \to M$ be a smooth embedding such that $\varphi(0) = x$ and $P \cdot D\varphi(0) = \mathrm{Id}$, where V is a neighborhood of 0 in $N_x(L, M)$ and $P : T_xM \to T_xM/T_xL = N_x(L, M)$ is the projection. Then $\varphi^*\beta \in H^{n-k}(V, V - \{0\}) = H^{n-k}(N_x, N_x - \{0\})$ is a generator. Here $\varphi^*\beta$ does not depend on the choice of φ. The ordered base determines an isomorphism $A : N_x \to R^{n-k}$. So $A^*\alpha_0$ is also a generator of $H^{n-k}(N_x, N_x - \{0\})$, where α_0 is the preferred generator of $H^{n-k}(R^{n-k}, R^{n-k} - \{0\})$. These two definitions coincide if and only if $\varphi^*\beta = A^*\alpha_0$. Clearly, a Thom class determines a continuous differential normal orientation on L and vice versa.

Lemma 9.1. *Suppose M is a n dimensional manifold and L is a k dimensional embedded submanifold of M. Let U be a tubular neighborhood of L with a smooth projection $\pi : U \to L$. Suppose $\alpha \in H^k(L, L - \{x\})$ is an orientation representing $\{e_1, \cdots, e_k\} \subseteq T_xL$ and $\beta \in H^{n-k}(U, U - L)$ is a normal orientation representing $\{\varepsilon_{k+1}, \cdots, \varepsilon_n\} \subseteq N_x(L, M)$. Then $\pi^*\alpha \cup \beta \in H^n(U, U - \{x\})$ is an orientation representing $\{e_1, \cdots, e_n\} \subseteq T_xM$, where $P(e_i) = \varepsilon_i$ for $k+1 \le i \le n$ and $P : T_xM \to N_x(L, M)$ is the projection.*

Proof. Choose a smooth embedding $\varphi : W_1 \times W_2 \to M$ such that $\varphi(0) = x$, $D \cdot \varphi(0) = \mathrm{Id}$, $\varphi(W_1) \subseteq L$ and $\varphi(W_2) \subseteq \pi^{-1}(x)$, where $W_1 \times W_2 \subseteq T_xL \times T_x\pi^{-1}(x) = T_xM$. Let $\pi_i : W_1 \times W_2 \to W_i$ be the projection and $j_i : W_i \to W_1 \times W_2$ be the inclusion, then we have $j_i\pi_i$

is homotopic to Id. Thus

$$\begin{aligned}\varphi^*(\pi^*\alpha \cup \beta) &= \varphi^*\pi^*\alpha \cup \varphi^*\beta = \pi_1^* j_1^* \varphi^* \pi^* \alpha \cup \pi_2^* j_2^* \varphi^* \beta \\ &= \pi_1^* \varphi^* \alpha \cup \pi_2^* j_2^* \varphi^* \beta = \varphi^* \alpha \times j_2^* \varphi^* \beta.\end{aligned}$$

Here $\varphi^*\alpha \in H^k(W_1, W_1 - \{0\})$ represents the orientation of $\{e_1, \cdots, e_k\}$ and $j_2^*\varphi^*\beta \in H^{n-k}(W_2, W_2 - \{0\})$ represents the orientation of $\{\varepsilon_{k+1}, \cdots, \varepsilon_n\}$. By the Künneth Formular, we know that $\varphi^*(\pi^*\alpha \cup \beta) \in H^n(T_xM, T_xM - \{0\})$ represents the orientation of $\{e_1, \cdots, e_n\}$.

□

The above lemma shows that $\pi^*\alpha \cup \beta$ does not depend on the smooth projection π.

Suppose $\{e_1, \cdots, e_k\}$ represents the orientation of L and $\{e_{k+1}, \cdots, e_n\}$ represents the normal orientation of L. We say the orientation $\{e_1, \cdots, e_n\}$ of M is defined by the orientation and the normal orientation of L.

Suppose L is closed in M for now on. Let U be a closed tubular neighborhood of L such that U is diffeomorphic to a closed disk bundle over L via the exponential map. Suppose $i : L \hookrightarrow U$ is the inclusion and $\pi : U \to L$ is the smooth projection. Clearly, i and π are proper. Furthermore, $\pi i = \mathrm{Id}$ and there exists a proper homotopy between $i\pi$ and Id. Thus $\pi^* : H_C^k(L) \to H_C^k(U)$ and $i^* : H_C^k(U) \to H_C^k(L)$ are isomorphisms and they are a pair of inverses, where H_C^* is the cohomology with compact support.

Lemma 9.2. *There exists a cup product homomorphism*

$$H_C^k(U) \otimes H^{n-k}(U, U - L) \xrightarrow{\ \cup\ } H_C^n(U, U - L),$$

where $H^n_C(U, U-L) = \varinjlim_{K \subseteq L} H^n(U, U-K)$ *and* K *are compact subsets of* L.

Proof. Let K be a compact subset of U. Then $K \cap L$ is a compact subset of L. We have the homomorphism

$$H^k(U, U-K) \otimes H^{n-k}(U, U-L) \xrightarrow{\cup} H^n(U, U-L \cap K) \longrightarrow H^n_C(U, U-L).$$

Passing K to the limit, we get the conclusion. □

Lemma 9.3. *Suppose* L *is connected. Then* $H^n_C(U, U-L) \cong Z$ *and the inclusion* $H^n(U, U-\{x\}) \to H^n_C(U, U-L)$ *is an isomorphism for any* $x \in L$.

Proof. Suppose K is a compact connected submanifold (with boundary) of L. By the Alexander Duality, for any coefficient field F, we have $H^n(U, U-K; F) \cong H_0(K; F) \cong F$. Thus $H^n(U, U-K; Q) \cong Q$ and $H^n(U, U-K; Z_p) \cong Z_p$. By the Universal Coefficient Theorem

$$0 \to H^n(U, U-K) \otimes F \longrightarrow H^n(U, U-K; F) \longrightarrow \mathrm{Tor}(H^{n+1}(U, U-K), F) \to 0\ ,$$

we get $H^n(U, U-K) \cong Z$.

We know that $H_n(U, U-K) \cong Z$ and its generator stands for the orientation of U on K. Again by the Universal Coefficient Theorem

$$0 \to \mathrm{Ext}(H_{n-1}(U, U-K), Z) \longrightarrow H^n(U, U-K) \longrightarrow \mathrm{Hom}(H_n(U, U-K), Z) \to 0\ ,$$

we get $H^n(U, U-K) = \mathrm{Hom}(H_n(U, U-K), Z)$.

We have the following commutative diagram.

$$\begin{array}{ccc} H^n(U, U-\{x\}) & \xrightarrow{\cong} & \mathrm{Hom}(H_n(U, U-\{x\}), Z) \\ \downarrow & & \downarrow \cong \\ H^n(U, U-K) & \xrightarrow{\cong} & \mathrm{Hom}(H_n(U, U-K), Z) \end{array}$$

Here $H^n(U, U - \{x\})$ is isomorphic to $\mathrm{Hom}(H_n(U, U - \{x\}), Z)$ obviously. We also know that $H_n(U, U - K) \to H_n(U, U - \{x\})$ is an isomorphism since it is an evaluation of the orientation. So $\mathrm{Hom}(H_n(U, U - \{x\}), Z) \to \mathrm{Hom}(H_n(U, U - K), Z)$ is an isomorphism.

Thus the inclusion $H^n(U, U - \{x\}) \to H^n(U, U - K)$ is an isomorphism. Passing K to the limit, we complete the proof. □

Lemma 9.4. *Suppose L is connected and $x \in L$. Then the inclusion $H^k(U, U - \pi^{-1}(x)) \to H^k_C(U)$ is an isomorphism.*

Proof. This immediately follows from the following commutative diagram.

$$\begin{array}{ccc} H^k(L, L-\{x\}) & \xrightarrow{\cong} & H^k_C(L) \\ {\scriptstyle \pi^*}\downarrow \cong & & {\scriptstyle \pi^*}\downarrow \cong \\ H^k(U, U-\pi^{-1}(x)) & \longrightarrow & H^k_C(U) \end{array}$$

□

Lemma 9.5. *Suppose L is connected. Suppose $\alpha \in H^k_C(L)$ represents the orientation of L and $\beta \in H^{n-k}(U, U - L)$ represents the normal orientation of L. Suppose the orientation and the normal orientation defines the orientation of M. Then we have the following cup* *product isomorphism*

$$H_C^k(U) \otimes H^{n-k}(U, U-L) \xrightarrow[\cong]{\cup} H_C^n(U, U-L).$$

Furthermore, via the isomorphism $H^n(U, U-\{x\}) \to H_C^n(U, U-L)$ in Lemma 9.3, we get $\pi^*\alpha \cup \beta \in H_C^n(U, U-L)$ represents the orientation of M in $H^n(U, U-\{x\})$.

Proof. Suppose $x \in L$. Consider the following commutative diagram.

$$\begin{tikzcd}
H^k(U, U-\pi^{-1}(x)) \otimes H^{n-k}(U, U-L) \arrow[r] \arrow[d, "\cong"'] & H^n(U, U-\{x\}) \arrow[d, "\cong"] \\
H_C^k(U) \otimes H^{n-k}(U, U-L) \arrow[r] & H_C^n(U, U-L)
\end{tikzcd}$$

By Lemmas 9.3 and 9.4, two vertical maps are isomorphisms. It suffices to prove the following claim: $H^k(U, U-\pi^{-1}(x)) \otimes H^{n-k}(U, U-L) \to H^n(U, U-\{x\})$ is an isomorphism such that, if $\alpha \in H^k(L, L-\{x\})$ represents the orientation of L, then $\pi^*\alpha \cup \beta \in H^n(U, U-\{x\})$ represents the orientation of U.

Near x, the normal bundle of N has the product structure $O_x = W_x \times \pi^{-1}(x) \subseteq U$, where W_x is an open neighborhood of x in L. We have the following commutative diagram,

$$\begin{tikzcd}
H^k(U, U-\pi^{-1}(x)) \otimes H^{n-k}(U, U-L) \arrow[r] \arrow[d, "\cong"'] & H^n(U, U-\{x\}) \arrow[d, "\cong"] \\
H^k(O_x, O_x-\pi^{-1}(x)) \otimes H^{n-k}(O_x, O_x-W_x) \arrow[r] & H^n(O_x, O_x-\{x\})
\end{tikzcd}$$

where the isomorphism $H^{n-k}(U, U-L) \to H^{n-k}(O_x, O_x-W_x)$ follows from the property of the Thom class, others follow from the excision. So it suffices to prove that $\pi^*\alpha|_{W_x} \cup \beta|_{O_x} \in$

$H^n(O_x, O_x - \{x\})$ represents the orientation of U.

Let $i_x : \pi^{-1}(x) \hookrightarrow W_x \times \pi^{-1}(x) = O_x$ be the inclusion and $p : O_x = W_x \times \pi^{-1}(x) \to \pi^{-1}(x)$ be the projection. Then we have $\beta|_{O_x} = p^* i_x^* \beta|_{O_x} = p^*\beta|_{\pi^{-1}(x)}$. Thus

$$\pi^*\alpha|_{W_x} \cup \beta|_{O_x} = \pi^*\alpha|_{W_x} \cup p^*\beta|_{\pi^{-1}(x)} = \alpha|_{W_x} \times \beta|_{\pi^{-1}(x)}.$$

By the Künneth Formula, $\pi^*\alpha|_{W_x} \cup \beta|_{O_x}$ represents the orientation of U. □

Lemma 9.6. *Suppose M_i ($i = 1, 2$) is a smooth orientable manifold, L_i is a closed orientable submanifold of M_i (which means L_i is a closed subset). Suppose the orientation and the normal orientation of L_i define the orientation of M_i. Let $\beta_i \in H^{n-k}(M_i, M_i - L_i)$ be the Thom class representing the normal orientation of L_i. Let $h : (M_1, L_1) \to (M_2, L_2)$ be a homeomorphism such that h preserves the orientation of M_i and $h^*\beta_2 = \beta_1$. Then h preserves the orientation of L_i.*

Proof. It suffices to prove the special case of that L_i is connected.

Let U_2 be a closed tubular neighborhood of L_2 with the smooth projection $\pi_2 : U_2 \to L_2$. Let $\alpha_2 \in H_C^k(L_2)$ be the orientation of L_2, by Lemma 9.5, we have the following cup product isomorphism

$$H_C^k(U_2) \otimes H^{n-k}(U_2, U_2 - L_2) \xrightarrow[\cong]{\cup} H_C^n(U_2, U_2 - L_2),$$

and $\pi_2^*\alpha_2 \cup \beta_2|_{U_2} = \gamma_2 \in H_C^n(U_2, U_2 - L_2)$ represents the orientation of M_2 on L_2.

Let $U_1' = h^{-1}(U_2)$. We have the following isomorphism

$$H_C^k(U_1') \otimes H^{n-k}(U_1', U_1' - L_1) \xrightarrow[\cong]{\cup} H_C^n(U_1', U_1' - L_1),$$

and $h^*\pi_2^*\alpha_2 \cup h^*\beta_2|_{U_2} = h^*\gamma_2$.

Choose a closed tubular neighborhood U_1 of L_1 such that $U_1 \subseteq \mathrm{Int}U_1'$ and $\pi_1 : U_1 \to L_1$ is the smooth projection. By Lemma 9.5 again, we have the following isomorphism

$$H_C^k(U_1) \otimes H^{n-k}(U_1, U_1 - L_1) \xrightarrow[\cong]{\cup} H_C^n(U_1, U_1 - L_1),$$

and

$$\pi_1^*\alpha_1 \cup \beta_1|_{U_1} = \gamma_1 \tag{9.1}$$

represents the orientation of M_1 on L_1.

Consider the following commutative diagram.

$$\begin{tikzcd}
H_C^k(U_2) \arrow[r, "h^*"] \arrow[d, "i_2^*"'] & H_C^k(U_1') \arrow[r, "\iota^*"] \arrow[d, "j^*"'] & H_C^k(U_1) \arrow[dl, "i_1^*"] \\
H_C^k(L_2) \arrow[r, "h^*"'] & H_C^k(L_1) &
\end{tikzcd}$$

Here, i_1, i_2, j and ι are inclusions. Since $h^*\pi_2^*\alpha_2 \cup h^*\beta_2|_{U_2} = h^*\gamma_2$, we have $\iota^*h^*\pi_2^*\alpha_2 \cup \iota^*h^*\beta_2|_{U_2} = \iota^*h^*\gamma_2$. Since h preserves the orientation of M_1 and the Thom class, we have $\iota^*h^*\gamma_2 = \gamma_1$ and $\iota^*h^*\beta_2|_{U_2} = \beta_1|_{U_1}$. Thus

$$\iota^*h^*\pi_2^*\alpha_2 \cup \beta_1|_{U_1} = \gamma_1. \tag{9.2}$$

Since the cup product is an isomorphism, by (9.1) and (9.2), we infer $\iota^*h^*\pi_2^*\alpha_2 = \pi_1^*\alpha_1$. So we have

$$\begin{aligned}\alpha_1 &= i_1^*\pi_1^*\alpha_1 = i_1^*\iota^*h^*\pi_2^*\alpha_2 \\ &= j^*h^*\pi_2^*\alpha_2 = h^*i_2^*\pi_2^*\alpha_2 = h^*\alpha_2.\end{aligned}$$

This completes the proof. $\square$

9.2 Orientation Formulas

In this section, we shall extend Theorem 5.1 to the case of a general negative gradient-like vector field.

Theorem 9.7 (Orientation Formulas)**.** *Suppose f is proper and X satisfies transversality. As oriented topological manifolds, we have*

(1). $\partial^1\overline{\mathcal{M}(p,q)} = \bigsqcup_{p\succ r\succ q}(-1)^{ind(p)-ind(r)}\mathcal{M}(p,r)\times\mathcal{M}(r,q)$*;*

(2). $\partial^1\overline{\mathcal{D}(p)} = \bigsqcup_{p\succ r}\mathcal{M}(p,r)\times\mathcal{D}(r)$*, where f is bounded below;*

(3). $\partial^1\overline{\mathcal{W}(p,q)} = \bigsqcup_{p\succeq r\succ q}(-1)^{ind(p)-ind(r)+1}\mathcal{W}(p,r)\times\mathcal{M}(r,q)\sqcup\bigsqcup_{p\succ r\succeq q}\mathcal{M}(p,r)\times\mathcal{W}(r,q)$*.*

In the above, $\partial^1\square$ are equipped with boundary orientations, $\square\times\square$ are equipped with product orientations.

Following Section 5.1, we define the orientations of $\mathcal{D}(p)$, $\mathcal{W}(p,q)$ and $\mathcal{M}(p,q)$ for all p and q.

By Theorems 8.5, we know $\overline{\mathcal{M}(p,q)}$ is a topological manifold with boundary, whose interior is $\mathcal{M}(p,q)$. Thus the orientation of $\mathcal{M}(p,q)$ gives $\partial\overline{\mathcal{M}(p,q)}$ the **boundary orientation** in the usual sense. In other words, the combination of the outward normal direction and the boundary orientation of the boundary gives the orientation of the manifold. Also by Theorem 8.5, we know that $\partial^1\overline{\mathcal{M}(p,q)} = \bigsqcup_{|I|=1} \mathcal{M}_I = \bigsqcup_{p\succ r\succ q} \mathcal{M}(p,r)\times\mathcal{M}(r,q)$ is an open subset of $\partial\overline{\mathcal{M}(p,q)}$. Thus $\partial^1\overline{\mathcal{M}(p,q)}$ has the boundary orientation. On the other hand, both $\mathcal{M}(p,r)$ and $\mathcal{M}(r,q)$ have orientations. Thus $\mathcal{M}(p,r)\times\mathcal{M}(r,q)$ has the **product orientation**. We shall consider the relation between these two orientations. Similarly, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{M}(p,q)}$ also have such two types of orientations. Theorem 9.7 indicates these relations.

Similarly to Chapter 8, by Lemma 7.13, we may assume that M is compact. By Theorem 7.7, we can construct the locally trivial field Y and the topological equivalence h mapping the orbits of X to those of Y.

However, since h is not assumed differentiable, we have to use the algebraic method to describe the orientation of $\mathcal{W}(p,q;X)$ again. Choose an open tubular neighborhood U_q of $\mathcal{A}(q;X)$ such that $\mathcal{A}(q;X)$ is closed in U_q. Suppose the index $\mathrm{ind}(q)=s$. We have the inclusion isomorphism

$$H^s(U_q, U_q-\mathcal{A}(q;X)) \xrightarrow{\cong} H^s(U_q\cap\mathcal{D}(q;X), U_q\cap\mathcal{D}(q;X)-\{q\}),$$

where $H^s(U_q\cap\mathcal{D}(q;X), U_q\cap\mathcal{D}(q;X)-\{q\}) = H^s(\mathcal{D}(q;X),\mathcal{D}(q;X)-\{q\})$. Thus the orientation of $\mathcal{D}(q;X)$, $\alpha_q\in H^s(\mathcal{D}(q;X),\mathcal{D}(q;X)-\{q\})$, determines a Thom class $\beta_q\in H^s(U_q,U_q-\mathcal{A}(q;X))$. Let $U_{p,q}=\mathcal{D}(p;X)\cap U_q$. Then $U_{p,q}$ is open in $\mathcal{D}(p;X)$ and $\mathcal{W}(p,q;X)$ is closed in $U_{p,q}$. By the inclusion monomorphism (it is an isomorphism if and only if $U_{p,q}$ is connected)

$$H^s(U_q,U_q-\mathcal{A}(q;X)) \longrightarrow H^s(U_{p,q},U_{p,q}-\mathcal{W}(p,q;X)),$$

we have that β_q determines a Thom class $\beta_{p,q} \in H^s(U_{p,q}, U_{p,q} - \mathcal{W}(p,q;X))$. Clearly, $U_{p,q}$ inherits the orientation from $\mathcal{D}(p;X)$. Thus $\beta_{p,q}$ and the orientation of $U_{p,q}$ give $\mathcal{W}(p,q;X)$ the orientation.

Since $h(\mathcal{D}(p;X)) = \mathcal{D}(p;Y)$, we can define the orientation of $\mathcal{D}(p;Y)$ as $\alpha'_p = (h^{-1})^*\alpha_p \in H^s(\mathcal{D}(p;Y), \mathcal{D}(p;Y) - \{p\})$ for each p. Then the orientations of $\mathcal{W}(p,q;Y)$ are defined. We also have $h(\mathcal{W}(p,q;X)) = \mathcal{W}(p,q;Y)$.

Lemma 9.8. *The topological equivalence h preserves the orientation of $\mathcal{W}(p,q;X)$.*

Proof. Choose the open tubular neighborhood U_q of $\mathcal{A}(q;X)$ and define $U_{p,q} = \mathcal{D}(p;X) \cap U_q$ as the above. Define $U'_q = h(U_q)$ and $U'_{p,q} = h(U_{p,q})$. We may assume $U_{p,q}$ is connected.

Suppose the orientation of $\mathcal{D}(q;Y)$ defines the Thom class $\beta'_q \in H^s(U'_q, U'_q - \mathcal{A}(q;Y))$ and the Thom class $\beta'_{p,q} \in H^s(U'_{p,q}, U'_{p,q} - \mathcal{W}(p,q;Y))$. We have the following commutative diagram.

$$\begin{tikzcd}
H^s(U'_{p,q}, U'_{p,q} - \mathcal{W}(p,q;Y)) \arrow[r, "h^*"] & H^s(U_{p,q}, U_{p,q} - \mathcal{W}(p,q;X)) \\
H^s(U'_q, U'_q - \mathcal{A}(q;Y)) \arrow[r, "h^*"] \arrow[u] \arrow[d] & H^s(U_q, U_q - \mathcal{A}(q;X)) \arrow[u] \arrow[d] \\
H^s(U'_q \cap \mathcal{D}(q;Y), U'_q \cap \mathcal{D}(q;Y) - \{q\}) \arrow[r, "h^*"] & H^s(U_q \cap \mathcal{D}(q;X), U_q \cap \mathcal{D}(q;X) - \{q\})
\end{tikzcd}$$

All of these maps are isomorphisms. The vertical maps are induced by inclusions. Since $h^*\alpha'_q = \alpha_q$, we have $h^*\beta'_q = \beta_q$. Thus we get $h^*\beta'_{p,q} = \beta_{p,q}$.

We also know that h preserves the orientation of $U_{p,q}$. By Lemma 9.6, the proof is completed. $\square$

As in Theorem 8.4, let $h_* : \overline{\mathcal{M}(p,q;X)} \to \overline{\mathcal{M}(p,q;Y)}$, $h_* : \overline{\mathcal{W}(p,q;X)} \to \overline{\mathcal{W}(p,q;Y)}$ and

$h_* : \overline{\mathcal{D}(p;X)} \to \overline{\mathcal{M}(p;Y)}$ be the maps induced by h. Since h preserves the direction of flow, by Lemma 9.8, we get the following immediately.

Lemma 9.9. *The map h_* preserves the orientation of $\mathcal{M}(p,q;X)$.*

Proof of Theorem 9.7. Consider the map h_* defined in the above. Clearly, h_* is identical to h on $\mathcal{D}(p;X)$ and $\mathcal{W}(p,q;X)$.

By the definition of the orientation of $\mathcal{D}(p;Y)$, we know h_* preserves the orientation of $\mathcal{D}(p;X)$. Combining this fact with Lemmas 9.8 and 9.9, we infer that h_* preserves both the boundary orientations and the product orientations. Thus h_* preserves the orientation relations. By Theorem 5.1, these formulas are true for Y, we infer that the orientation formulas are valid for X. $\square$

10 CW Structures (II)

This chapter extends the results in Chapter 6 to the case of a general metric.

10.1 Theorems

In this section, We shall extend the results in Chapter 6 to the case of a general negative gradient-like vector field.

Theorem 10.1. *Suppose f is proper and bounded below. Suppose X satisfies transversality. Suppose a is a regular value of f. Define $K^a = \bigsqcup_{f(p)\leq a} \mathcal{D}(p)$ with the topology induced from M. Then K^a is a finite CW complex with characteristic maps $e : \overline{\mathcal{D}(p)} \to K^a$, where e is defined in (3) of Theorem 8.9. The inclusion $K^a \hookrightarrow M^a$ is a simple homotopy equivalence. In fact, there is a CW decomposition of M^a such that K^a expands to M^a by elementary expansions.*

Theorem 10.2. *Under the assumption of Theorem 10.1, define $K = \bigsqcup_{p\in M} \mathcal{D}(p)$. Define the topology of K as the direct limit of that of K^a when a tends to $+\infty$. Then K is a countable CW complex with characteristic maps $e : \overline{\mathcal{D}(p)} \to K$, where e is defined in (3) of Theorem 8.9. Furthermore, the inclusion $i : K \hookrightarrow M$ is a homotopy equivalence.*

Theorem 10.3. *Let K^a (or K) be the CW complex in Theorem 10.1 (or 10.2). Let $C_*(K^a)$ (or $C_*(K)$) be the associated cellular chain complex and $[\overline{\mathcal{D}(p)}]$ be the base element represented by the oriented $\overline{\mathcal{D}(p)}$ in $C_*(K^a)$ (or $C_*(K)$). Then*

$$\partial[\overline{\mathcal{D}(p)}] = \sum_{ind(q)=ind(p)-1} \#\mathcal{M}(p,q)[\overline{\mathcal{D}(q)}],$$

where $\#\mathcal{M}(p,q)$ is the sum of the orientations ± 1 of all points in $\mathcal{M}(p,q)$ defined in Theorem 9.7.

Remark 10.1. *Consider the special case when M is compact. Theorem 10.1 shows that the compactified descending manifolds give a bona fide CW decomposition of M. Before the invention of the theory of Moduli spaces, this problem was addressed in [34, thm. 1] and [36, rem. 3], which show the existence of the characteristic maps under the assumption that the vector field is locally trivial. Besides the simple homotopy type, Theorem 10.1 strengthens their solution in two ways. Firstly, the characteristic maps here $e : \overline{\mathcal{D}(p)} \to M$ have the explicit formula defined in (3) of Theorem 8.9. Secondly, we drop the assumption of the local triviality of the vector field. In the case when f has only one critical point of index 0, the paper [5, lem. 2.15] also gives a answer similar to Theorem 10.1.*

Remark 10.2. *The above theorems show that $C_*(K)$ computes the homology of M. As mentioned in Remark 6.1, the boundary operator ∂ of $C_*(K)$ coincides with that of Morse homology. This shows Morse homology arises from a cellular chain complex.*

Proof of Theorem 10.1. By Theorem 8.9, $\overline{\mathcal{D}(p)}$ is a closed disc and e is continuous. Thus K^a is a finite CW complex with characteristic maps e.

We shall construct the desired CW decomposition of M^a.

Suppose M is not compact. By Lemma 7.13, we can embed M^a into $\widetilde{M}$ and extend $f|_{M^a}$ to be $\widetilde{f}$ on $\widetilde{M}$ such that $\widetilde{f}|_{\widetilde{M}-M^a} > a$. We get $\widetilde{M}^a = M^a$. As a result, we may assume M is compact.

By Theorem 7.7, we can construct a locally trivial field Y on M and a topological equivalence h which maps the orbits of Y to those of X. Clearly, $h(\mathcal{D}(p;Y)) = \mathcal{D}(p;X)$

and $h(K^a(Y)) = K^a$ where $K^a(Y) = \bigsqcup_{f(p)\le a} \mathcal{D}(p;Y)$. By Theorem 6.2, there exists a CW decomposition of M^a such that $K^a(Y)$ expands to M^a by elementary expansions. Thus it suffices to prove that there exists a homeomorphism $\widetilde{h} : M^a \to M^a$ such that $\widetilde{h}$ and h coincide on $K^a(Y)$.

Denote by ϕ_t^1 the flow generated by X and by ϕ_t^2 the flow generated by Y. For any $x \in f^{-1}(a)$, we have $\phi^2(-\infty, x) = r_1$ for some $r_1 \in M - M^a$ and $\phi^2(+\infty, x) = r_2$ for some $r_2 \in M^a$. Since h is a topological equivalence fixing r_1 and r_2, we have $\phi_t^1(h(x))$ is a flow line between r_1 and r_2. Thus, for any $x \in h(f^{-1}(a))$, $\phi^1(t(x), x) \in f^{-1}(a)$ for some $t(x)$ and $t(x)$ is continuous on $h(f^{-1}(a))$. Since $h(f^{-1}(a))$ is compact, there exists $T > 0$ such that $T > -t(x)$ for all $x \in h(f^{-1}(a))$. As a result, $\phi_T^1(M^a) \subseteq \mathrm{Int}[h(M^a)]$. (This is illustrated by Figure 8, $\phi_T^1(M^a)$ is the shadowed part, M^a is the part below $f^{-1}(a)$ and $h(M^a)$ is the part below $h(f^{-1}(a))$.) By an isotopy along the flows generated by X, we can construct a homeomorphism $\psi : h(M^a) \to M^a$ such that $\psi|_{\phi_T^1(M^a)} = \mathrm{Id}$ and $\psi(h(M^a) - \phi_T^1(M^a)) = M^a - \phi_T^1(M^a)$. Then $\widetilde{h} = \psi \circ h$ is the desired homeomorphism. □

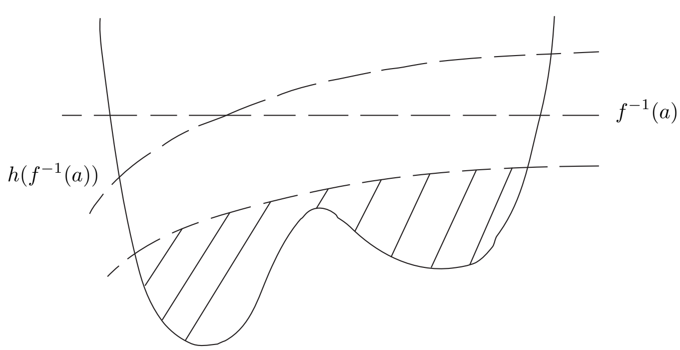

Figure 8: Construction of ψ

Proof of Theorem 10.2. The CW structure of K is obvious.

By Theorem 10.1, $i : K^a \hookrightarrow M^a$ is a homotopy equivalence for any regular value a. Thus, it's straightforward to check that $i : K \hookrightarrow M$ is a weak homotopy equivalence, i.e. i induces the isomorphisms between homotopy groups. Since M carries a triangulation, by Whitehead's Theorem, i is a homotopy equivalence. □

Proof of Theorem 10.3. There are two proofs.

First, duplicate the proof of Theorem 6.3. Certainly, the local triviality of the vector field X is assumed in Theorem 6.3. However, the only reason for making this assumption is that the (2) of Theorem 9.7 was proved under this assumption in Chapter 5. Now this orientation formula is true even if we drop this assumption. Thus, the first proof is valid.

Second, reduce it to the case of a locally trivial vector field Y. The map h_* in Theorem 8.4 induces an isomorphism between $C_*(K^a(X))$ and $C_*(K^a(Y))$. By Lemma 9.9, h_* preserves the orientation of $\mathcal{M}(p, q; X)$. Since this statement is true for $C_*(K^a(Y))$, the second proof is complete. □

10.2 An Alternative Proof

In order to get the CW structure, Theorem 6.1 is essential. The proof of it in Section 6.2 is elementary but complicated. In this section, we shall give is a quick but non-elementary proof of Theorem 6.1, which is based on the Poincaré Conjecture in all dimensions.

Second Proof of Theorem 6.1. By Theorem 4.5, $\overline{\mathcal{D}(p)}$ is a compact smooth manifold with corners whose interior is an open disk. Thus, it is a compact topological manifold with boundary whose interior is an open disk. (Actually, $\overline{\mathcal{D}(p)}$ carries a structure of smooth manifold with boundary since we can round the corner.)

Identify $\mathcal{D}(p)$ with R^n. Puncture an open disk K with center 0 from $\mathcal{D}(p)$. Let $W = \overline{\mathcal{D}(p)} - K$. Then W is a manifold with boundary $\partial\overline{\mathcal{D}(p)} \sqcup \partial K$, where $\partial K = S^{n-1}$.

Firstly, we prove that $(W; \partial\overline{\mathcal{D}(p)}, \partial K)$ is an h-cobordism.

Choose a collar neighborhood $\partial\overline{\mathcal{D}(p)} \times [0,1]$ of $\partial\overline{\mathcal{D}(p)}$ in W. Then $\overline{\mathcal{D}(p)} - \partial\overline{\mathcal{D}(p)} \times [0,1)$ is a compact subset in R^n, and it contains K. Thus, ∂K is a deformation retract of $W - \partial\overline{\mathcal{D}(p)} \times [0,1)$. Since $W - \partial\overline{\mathcal{D}(p)} \times [0,1)$ is a deformation retract of W, we infer that ∂K is a deformation retract of W. On the other hand, we can push $R^n - K$ to infinity. Thus there exists a compact set L such that $\partial\overline{\mathcal{D}(p)} \subseteq L \subseteq \partial\overline{\mathcal{D}(p)} \times [0,1)$ and L is a deformation retract of W. Since $\partial\overline{\mathcal{D}(p)}$ is a deformation retract of L, we infer that $\partial\overline{\mathcal{D}(p)}$ is a deformation retract of W.

Secondly, we know now $\partial\overline{\mathcal{D}(p)}$ is a homotopy sphere. By the Poincaré Conjecture in all dimensions, $\partial\overline{\mathcal{D}(p)}$ is a topological sphere.

Finally, by the Generalized Schoenflies Theorem (see [6, thm. 5]), we can prove that $\overline{\mathcal{D}(p)} - \partial\overline{\mathcal{D}(p)} \times [0, \frac{1}{2})$ is a closed disk. This completes the proof. □

The proof in Section 6.2 avoids the Poincaré Conjecture because it studies the speciality of $\overline{\mathcal{D}(p)}$. An interesting question is that whether or not the Poincaré Conjecture is necessary if we ignore the speciality of $\overline{\mathcal{D}(p)}$. Prof. Paul Kirk informed me that the 3 dimensional Poincaré Conjecture is necessary for proving the following claim.

Claim 10.1. *Suppose M^4 is a compact topological 4-manifold with boundary, and its interior is homeomorphic to R^4. Then M^4 is a closed topological disk.*

Before explaining this, we cite two theorems of Freedman ([29, thm. 1.4' & cor. 1.2]).

Theorem 10.4 (Freedman)**.** *Every 3 dimensional homology sphere bounds a contractible compact topological 4-manifold.*

Theorem 10.5 (Freedman)**.** *A contractible open topological 4-manifold which is simply connected at infinity is homeomorphic to R^4.*

Suppose Σ^3 is a 3 dimensional homotopy sphere. By Theorem 10.4, Σ^3 bounds a contractible compact topological 4-manifold M^4. By [7, thm. 2], there exists a collar neighborhood $\Sigma^3 \times [0,1]$ of Σ^3 in M^4. Since Σ^3 is simply connected, we known that the interior of M^4 is simply connected at infinity. Thus, by Theorem 10.5, the interior of M^4 is homeomorphic to R^4.

If Claim 10.1 is true, then M^4 is a closed topological disk. Therefore, Σ^3 is a topological sphere, which proves the 3 dimensional Poincaré Conjecture.

In summary, Claim 10.1 implies the 3 dimensional Poincaré Conjecture. Actually, by the argument of this section, they are equivalent to each other.

11 Associativity of Gluing

In this chapter, we shall show that the associativity of gluing exclusively follows from the compatible manifold structures of the compactified moduli spaces. This does *not* rely on any speciality of Morse theory.

Since this phenomenon does not depend on Morse theory directly, we are allowed to deal with $\mathcal{M}(p,q)$ formally. We shall assume that $\mathcal{M}(p,q)$ is defined without considering its definition.

11.1 Main Theorem

Recall Definition 2.9, the definition of the moduli spaces of flow lines, $\mathcal{M}(p,q)$, only requires the existence of $\mathcal{W}(p,q)$ and a flow action on it. Thus, $\mathcal{M}(p,q)$ can be defined in a very general setting, for example, in the case of Floer homology. In this chapter, we shall assume the moduli spaces are defined without considering the definition. In other words, $\mathcal{M}(p,q)$ is not necessarily defined by Definition 2.9.

Suppose M is a manifold such that a Morse function f can be defined on it. Suppose Ω is a countable set containing some critical points of f. Here we do *not* require that Ω contains all critical points. Suppose a relation " $\succeq$ " is defined on Ω. Suppose the moduli spaces $\mathcal{M}(p,q)$ are defined for all p, $q \in \Omega$ such that $p \succ q$.

Here we are not concerned with the property of M and the definitions of $\mathcal{M}(p,q)$ and " $\succeq$ ". Practically, $\mathcal{M}(p,q)$ and " $\succeq$ " are defined similarly to Definitions 2.9 and 2.12.

By the relation " $\succeq$ ", we can define a critical sequence I like Definition 2.13 and define the space $\mathcal{M}_I$ like (2.1) when $I \subseteq \Omega$.

Theorems 11.2 and 11.3 will be based on the following assumption. For the definitions of manifold with faces and the k-stratum, see Definitions 2.19 and 2.18.

Assumption 11.1. *Suppose Ω is a countable set consisting of some critical points of f. The relation "$\succeq$" defined on Ω is a partial order. Suppose $\mathcal{M}(p,q)$ are finite dimensional manifolds for all p, $q \in \Omega$ such that $p \succ q$ (see Remark 11.1). And $\mathcal{M}(p,q)$ can be compactified to be $\overline{\mathcal{M}(p,q)}$ which are compact smooth manifolds with faces. These $\overline{\mathcal{M}(p,q)}$ satisfy the following conditions.*

(1). We have $\overline{\mathcal{M}(p,q)} = \bigsqcup_I \mathcal{M}_I$, where the disjoint union is over all critical sequences with head p and tail q. The k-stratum of $\overline{\mathcal{M}(p,q)}$ is $\bigsqcup_{|I|=k} \mathcal{M}_I$, and each M_I is an open subset of the k-stratum. The smooth structure of $\overline{\mathcal{M}(p,q)}$ is compatible with those of $\mathcal{M}_I$. Here all I are contained in Ω.

(2). Suppose p, r and q are in Ω, and $p \succ r \succ q$. Then the natural inclusion $\overline{\mathcal{M}(p,r)} \times \overline{\mathcal{M}(r,q)} \hookrightarrow \overline{\mathcal{M}(p,q)}$ is a smooth embedding.

Remark 11.1. *If $\mathcal{M}(p,q)$ is defined as Definition 2.9, then, as we have seen before, $\mathcal{M}(p,q)$ induces a natural smooth structure from those of $\mathcal{D}(p)$ and $\mathcal{A}(q)$. However, in order to make Assumption 11.1 hold, we may give $\mathcal{M}(p,q)$ a smooth structure different from the above one (see Remark 11.3).*

In order to make the statement of gluing conceptual and strong, we shall have to introduce the following formal definitions.

Suppose $I_1 = \{r_0, \cdots, r_{k+1}\}$ and $I_2 = \{r'_0, \cdots, r'_{l+1}\}$ are two critical sequences. If $I_2 \subseteq I_1$, $r'_0 = r_0$ and $r'_{l+1} = r_{k+1}$, i.e. $I_2 = \{r_0, r_{i_1}, \cdots, r_{i_l}, r_{k+1}\}$, denote them by $I_2 \preceq I_1$.

We use the notation Λ_{I_1} to represent the gluing parameter for $\mathcal{M}_{I_1}$. Here $\Lambda_{I_1} = (\lambda_1, \cdots, \lambda_{|I_1|}) \in$

$\prod_{i=1}^{|I_1|}[0,+\infty) = [0,+\infty)^{|I_1|}$. By the relation between I_1 and I_2, we introduce the following definitions of the tuples induced from Λ_{I_1}. Define $\Lambda_{I_1,I_2} \in [0,+\infty)^{|I_2|}$ as

$$\Lambda_{I_1,I_2} = (\lambda_{i_1}, \cdots, \lambda_{i_l}). \tag{11.1}$$

Here we consider Λ_{I_1,I_2} as a gluing parameter for $\mathcal{M}_{I_2}$. Define $\Lambda_{I_1}(I_1 - I_2) \in [0,+\infty)^{|I_1|}$ as

$$\Lambda_{I_1}(I_1 - I_2)(i) = \begin{cases} 0 & r_i \in I_2, \\ \\ \lambda_i & r_i \notin I_2. \end{cases} \tag{11.2}$$

For example, suppose $I_1 = \{r_0, r_1, r_2, r_3, r_4\}$, $I_2 = \{r_0, r_2, r_4\}$ and $\Lambda_{I_1} = (5, 6, 7)$, then $\Lambda_{I_1,I_2} = (6)$ and $\Lambda_{I_1}(I_1 - I_2) = (5, 0, 7)$.

Suppose $I_1 = \{r_0, \cdots, r_{k+1}\}$, $I_2 = \{r'_0, \cdots, r'_{l+1}\}$ and $r_{k+1} = r'_0$. Define

$$I_1 \cdot I_2 = \{r_0, \cdots, r_{k+1}, r'_1, \cdots, r'_{l+1}\}. \tag{11.3}$$

If $x_1 = (a_1, \cdots, a_{k+1}) \in \mathcal{M}_{I_1}$ and $x_2 = (a'_1, \cdots, a'_{l+1}) \in \mathcal{M}_{I_2}$, then define

$$x_1 \cdot x_2 = (a_1, \cdots, a_{k+1}, a'_1, \cdots, a'_{l+1}) \in \mathcal{M}_{I_1} \times \mathcal{M}_{I_2} = \mathcal{M}_{I_1 \cdot I_2}. \tag{11.4}$$

Suppose $\Lambda_{I_1} = (\lambda_1, \cdots, \lambda_{|I_1|})$ and $\Lambda_{I_2} = (\lambda'_1, \cdots, \lambda'_{|I_2|})$, define

$$\Lambda_{I_1} \cdot \Lambda_{I_2} = (\lambda_1, \cdots, \lambda_{|I_1|}, 0, \lambda'_1, \cdots, \lambda'_{|I_2|}). \tag{11.5}$$

In particular, if $|I_1| = 0$, then $\Lambda_{I_1} \cdot \Lambda_{I_2} = (0, \lambda'_1, \cdots, \lambda'_{|I_2|})$. If $|I_2| = 0$, then $\Lambda_{I_1} \cdot \Lambda_{I_2} = (\lambda_1, \cdots, \lambda_{|I_1|}, 0)$. If $|I_1| = |I_2| = 0$, then $\Lambda_{I_1} \cdot \Lambda_{I_2} = (0)$.

Suppose $I = \{r_0, r_1, \cdots, r_{k+1}\}$ is a critical sequence. Recall that an element $x \in \mathcal{M}_I$ is a (un)broken flow line which is broken at the points r_i $(i = 1, \cdots, k)$. A gluing should be a map $G_I : \mathcal{M}_I \times [0, \epsilon_I)^{|I|} \longrightarrow \overline{\mathcal{M}(r_0, r_{|I|+1})}$ for some $\epsilon_I > 0$. For all $(x, \Lambda_I) \in \mathcal{M}_I \times [0, \epsilon_I)^{|I|}$, we have $\Lambda_I = (\lambda_1, \cdots, \lambda_{|I|})$ is a parameter of gluing, and $G_I(x, \Lambda_I)$ is the (un)broken flow line glued from x. We expected that $G_I(x, \Lambda_I)$ is not broken at r_i if and only if $\lambda_i > 0$. Thus we can interpret the gluing map as a collaring map, which leads to the following definition.

Definition 11.1. *A map $G_I : \mathcal{M}_I \times [0, \epsilon_I)^{|I|} \to \overline{\mathcal{M}(r_0, r_{|I|+1})}$ for some $\epsilon_I > 0$ is a gluing map if it satisfies the following properties. (1). It is a smooth embedding. In particular, if $|I| = 0$, $G_I : \mathcal{M}_I = \mathcal{M}(r_0, r_1) \to \overline{\mathcal{M}(r_0, r_1)}$ is the inclusion. (2). It satisfies the stratum condition, i.e., suppose $I = \{r_0, r_1, \cdots, r_{k+1}\}$, $\Lambda_I = (\lambda_1, \cdots, \lambda_{|I|}) \in [0, \epsilon_I)^{|I|}$, $I_1 \preceq I$, and $\lambda_i = 0$ if and only if $r_i \in I_1$, then for all $x \in \mathcal{M}_I$, we have $G_I(x, \Lambda_I) \in \mathcal{M}_{I_1}$.*

Now we give two examples to illustrate the compatibility issue of gluing.

Suppose the gluing maps are defined for all critical sequences. Suppose $I_1 = \{r_0, r_1, r_2, r_3, r_4\}$, $I_2 = \{r_0, r_2, r_4\}$, $\Lambda_{I_1} = (\lambda_1, \lambda_2, \lambda_3)$, $\lambda_1 > 0$, $\lambda_3 > 0$, and $x \in \mathcal{M}_{I_1}$. Gluing x at the points r_1 and r_3 at first, we get $y = G_{I_1}(x, \lambda_1, 0, \lambda_3) \in \mathcal{M}_{I_2}$. Do we have $G_{I_2}(y, \lambda_2) = G_{I_1}(x, \lambda_1, \lambda_2, \lambda_3)$? This is a question about the compatibility for a fixed critical pair (r_0, r_4).

Suppose $I_1 = \{r_0, r_1, r_2\}$, $I_2 = \{r_2, r_3, r_4\}$, $\Lambda_{I_1} = (\lambda_1)$, $\Lambda_{I_2} = (\lambda_2)$, $x_1 \in \mathcal{M}_{I_1}$ and $x_2 \in \mathcal{M}_{I_2}$. Gluing x_1 and x_2, we get $y_1 = G_{I_1}(x_1, \lambda_1) \in \mathcal{M}(r_0, r_2)$ and $y_2 = G_{I_2}(x_2, \lambda_2) \in \mathcal{M}(r_2, r_4)$. Do we have $G_{I_1 \cdot I_2}(x_1 \cdot x_2, \lambda_1, 0, \lambda_2) = (y_1, y_2)$? This is a question about the compatibility for different critical pairs.

The following theorem answers the above two questions.

Theorem 11.2. *Suppose M is a manifold such that a Morse function f is defined on it. Suppose a set Ω, a relation " $\succeq$ " and moduli spaces $\mathcal{M}(p,q)$ are defined which satisfy Assumption 11.1. Then the gluing maps (see Definition 11.1) can be defined for all critical sequences. These maps satisfy the following compatibility.*

(1). Compatibility for One critical Pair. Suppose $I_2 \preceq I_1$, let $\epsilon = \min\{\epsilon_{I_1}, \epsilon_{I_2}\}$. Then, for all $x \in \mathcal{M}_{I_1}$ and $\Lambda_{I_1} \in [0,\epsilon)^{|I_1|}$ such that $\lambda_i > 0$ when $r_i \notin I_2$, we have

$$G_{I_1}(x, \Lambda_{I_1}) = G_{I_2}(G_{I_1}(x, \Lambda_{I_1}(I_1 - I_2)), \Lambda_{I_1,I_2}). \tag{11.6}$$

(2). Compatibility for Critical Pairs. Suppose $I_1 = \{r_0, \cdots, r_{k+1}\}$ and $I_2 = \{r_{k+1}, \cdots, r_n\}$. Let $\epsilon = \min\{\epsilon_{I_1}, \epsilon_{I_2}, \epsilon_{I_1 \cdot I_2}\}$, then for all $x_1 \in \mathcal{M}_{I_1}$, $x_2 \in \mathcal{M}_{I_2}$, $\Lambda_{I_1} \in [0,\epsilon)^{|I_1|}$, and $\Lambda_{I_2} \in [0,\epsilon)^{|I_2|}$, we have

$$\begin{aligned} G_{I_1 \cdot I_2}(x_1 \cdot x_2, \Lambda_{I_1} \cdot \Lambda_{I_2}) &= (G_{I_1}(x_1, \Lambda_{I_1}), G_{I_2}(x_2, \Lambda_{I_2})) \qquad (11.7) \\ &\in \overline{\mathcal{M}(r_0, r_{k+1})} \times \overline{\mathcal{M}(r_{k+1}, r_n)}. \end{aligned}$$

Theorem 11.2 will follow from a more general Theorem 11.6.

We introduce a traditional notation of gluing as in the Introduction (see e.g. [25, p. 529]). Suppose $\gamma_1 \in \mathcal{M}(p,r)$ and $\gamma_2 \in \mathcal{M}(r,q)$ are two flow lines. We denote the gluing map $G_{\{p,r,q\}}(\gamma_1, \gamma_2, \lambda)$ by $\gamma_1 \#_\lambda \gamma_2$. Theorem 11.2 derives the following theorem immediately.

Theorem 11.3. *Under the assumption of Theorem 11.2, there exist $\epsilon_I > 0$ for all critical sequences I with $|I| = 1$ or $|I| = 2$. For all $\{r_0, r_1, r_2\}$, the gluing $\gamma_1 \#_\lambda \gamma_2$ can be defined* *for* $(\gamma_1, \gamma_2) \in \mathcal{M}(r_0, r_1) \times \mathcal{M}(r_1, r_2)$ *and* $\lambda \in [0, \epsilon_{\{r_0, r_1, r_2\}})$. *The gluing satisfies the following associativity.*

For all $\gamma_1 \in \mathcal{M}(p_1, p_2)$, $\gamma_2 \in \mathcal{M}(p_1, p_2)$, $\gamma_3 \in \mathcal{M}(p_2, p_3)$, *and* λ_1, $\lambda_2 \in (0, \epsilon)$, *where* $\epsilon = \min\{\epsilon_{\{p_0,p_1,p_2\}}, \epsilon_{\{p_1,p_2,p_3\}}, \epsilon_{\{p_0,p_1,p_2,p_3\}}\}$, *we have*

$$(\gamma_1 \#_{\lambda_1} \gamma_2) \#_{\lambda_2} \gamma_3 = \gamma_1 \#_{\lambda_1} (\gamma_2 \#_{\lambda_2} \gamma_3) \, . \tag{11.8}$$

Proof.

$$\begin{aligned} & (\gamma_1 \#_{\lambda_1} \gamma_2) \#_{\lambda_2} \gamma_3 \\ = \;& G_{\{p_0,p_2,p_3\}} \left(G_{\{p_0,p_1,p_2\}}(\gamma_1, \gamma_2, \lambda_1), \gamma_3, \lambda_2 \right) \\ = \;& G_{\{p_0,p_2,p_3\}} \left(G_{\{p_0,p_1,p_2,p_3\}}(\gamma_1, \gamma_2, \gamma_3, \lambda_1, 0), \lambda_2 \right) \\ = \;& G_{\{p_0,p_1,p_2,p_3\}}(\gamma_1, \gamma_2, \gamma_3, \lambda_1, \lambda_2). \end{aligned}$$

Here we have used the (2) of Theorem 11.2 in the second equality and the (1) of Theorem 11.2 in the third equality.

Similarly,

$$\gamma_1 \#_{\lambda_1} (\gamma_2 \#_{\lambda_2} \gamma_3) = G_{\{p_0,p_1,p_2,p_3\}}(\gamma_1, \gamma_2, \gamma_3, \lambda_1, \lambda_2).$$

This completes the proof. $\square$

Remark 11.2. *Suppose* $I = \{r_0, \cdots, r_{n+1}\}$ *is a critical sequence. Let* $\epsilon = \min\{\epsilon_J \mid J \subseteq I, \text{ and } |J| = 1 \text{ or } 2.\}$. *Then, for* $(\gamma_1, \gamma_2) \in \mathcal{M}(r_i, r_j) \times \mathcal{M}(r_j, r_k)$, *the gluing* $\gamma_1 \#_\lambda \gamma_2$ *in Theorem 11.3 can be defined for* $\lambda \in [0, \epsilon)$. *And the gluing satisfies the associativity. Thus* *we can define G_J on $\mathcal{M}_J \times (0,\epsilon)^{|J|}$ for any $J \subseteq I$ by inductive gluing of pairs of flow lines. The definition of G_J does not depend on the order of the pairwise gluing.*

By Theorems 4.4 and 8.6, Assumption 11.1 holds in certain practical cases. Thus Theorems 11.2 and 11.3 lead to the following two propositions, where $\mathcal{M}(p,q)$ and " $\succeq$ " are defined as Definitions 2.9 and 2.12.

Proposition 11.4. *Suppose M is a complete Hilbert manifold. The Morse function f satisfies Condition (C) and has finite indices. The metric on M is locally trivial and $-\nabla f$ satisfies transversality. Give $\mathcal{M}(p,q)$ the smooth structure induced from $\mathcal{D}(p)$ and $\mathcal{A}(q)$. Then there exist smooth structures on $\overline{\mathcal{M}(p,q)}$ and gluing maps which satisfy the compatibility and associativity in Theorems 11.2 and 11.3.*

Proposition 11.5. *Suppose M is compact and $-\nabla f$ satisfies tranversality. Then there exist smooth structures on $\mathcal{M}(p,q)$ and $\overline{\mathcal{M}(p,q)}$ and exist gluing maps which satisfy the compatibility and associativity in Theorems 11.2 and 11.3.*

Remark 11.3. *Proposition 11.5 is based on Theorem 8.6. In the case of a compact M, it has the advantage that the metric is allowed to be general. However, the smooth structure on $\mathcal{M}(p,q)$ may be different from the natural one when the metric is not locally trivial.*

11.2 Generalization

The statement of Theorem 11.2 actually does not directly depend on any speciality of Morse theory. Thus we can generalize it to be the following Theorem 11.6 on collaring maps of manifolds with faces.

Suppose Ω is a partially ordered set with a partial order " $\succeq$ ". Suppose $I = \{r_0, r_1, \cdots, r_{k+1}\}$ is a finite chain of Ω, i.e., $I \subseteq \Omega$ and $r_i \succ r_{i+1}$. We call r_0 the head of I and r_{k+1} the tail of I. Define the length of I as $|I| = k$. If $J \subseteq I$, $J = \{r'_0, \cdots, r'_{l+1}\}$, $r'_0 = r_0$ and $r'_{l+1} = r_{k+1}$, i.e. $J = \{r_0, r_{i_1}, \cdots, r_{i_l}, r_{k+1}\}$, denote them by $J \preceq I$. Suppose $I_1 = \{r_0, \cdots, r_{k+1}\}$ and $I_2 = \{r_{k+1}, \cdots, r_n\}$ are two chains. Define $I_1 \cdot I_2 = \{r_0, \cdots, r_n\}$, which is also a chain.

Suppose a finite dimensional manifold $\mathcal{M}(p, q)$ is defined for each pair $(p, q) \subseteq \Omega$ such that $p \succ q$. For the above chain I, define $\mathcal{M}_I = \prod_{i=0}^{|I|} \mathcal{M}(r_i, r_{i+1})$.

Assumption 11.2. *The partially ordered set Ω is countable. The finite dimensional manifolds $\mathcal{M}(p, q)$ can be compactified to be $\overline{\mathcal{M}(p, q)}$ which are compact smooth manifolds with faces. These $\overline{\mathcal{M}(p, q)}$ satisfy the following conditions.*

(1). We have $\overline{\mathcal{M}(p, q)} = \bigsqcup_I \mathcal{M}_I$, where the disjoint is over all finite chain I with head p and tail q. The k-stratum of $\overline{\mathcal{M}(p, q)}$ is $\bigsqcup_{|I|=k} \mathcal{M}_I$, and each $\mathcal{M}_I$ is an open subset of the k-stratum. The smooth structure of $\overline{\mathcal{M}(p, q)}$ is compatible with those of $\mathcal{M}_I$.

(3). Suppose $p \succ r \succ q$, then the natural inclusion $\overline{\mathcal{M}(p, r)} \times \overline{\mathcal{M}(r, q)} \hookrightarrow \overline{\mathcal{M}(p, q)}$ is a smooth embedding.

We introduce the following definitions similar to Section 11.1. Use $\Lambda_I = (\lambda_1, \cdots, \lambda_{|I|})$ to represent the collaring parameter for $\mathcal{M}_I$. Define $\Lambda_{I_1}(I_1 - I_2)$, Λ_{I_1, I_2} and $\Lambda_{I_1} \cdot \Lambda_{I_2}$. Also for $x_1 \in \mathcal{M}_{I_1}$ and $x_2 \in \mathcal{M}_{I_2}$, define $x_1 \cdot x_2 \in \mathcal{M}_{I_1 \cdot I_2}$.

Define the collaring map $G_I : \mathcal{M}_I \times [0, \epsilon_I)^{|I|} \to \overline{\mathcal{M}(r_0, r_{|I|+1})}$ as Definition 11.1.

The proof of the following theorem is given in Section 11.4.

Theorem 11.6. *Under Assumption 11.2, the collaring maps G_I can be defined for all finite chain I of Ω. These maps satisfy the following compatibility.*

(1). Suppose $I_2 \preceq I_1$*, let* $\epsilon = \min\{\epsilon_{I_1}, \epsilon_{I_2}\}$*. Then, for all* $x \in \mathcal{M}_{I_1}$ *and* $\Lambda_{I_1} \in [0,\epsilon)^{|I_1|}$ *such that* $\lambda_i > 0$ *when* $r_i \notin I_2$*, we have*

$$G_{I_1}(x, \Lambda_{I_1}) = G_{I_2}(G_{I_1}(x, \Lambda_{I_1}(I_1 - I_2)), \Lambda_{I_1,I_2}). \tag{11.9}$$

(2). Suppose $I_1 = \{r_0, \cdots, r_{k+1}\}$ *and* $I_2 = \{r_{k+1}, \cdots, r_n\}$*. Let* $\epsilon = \min\{\epsilon_{I_1}, \epsilon_{I_2}, \epsilon_{I_1 \cdot I_2}\}$*, then for all* $x_1 \in \mathcal{M}_{I_1}$*,* $x_2 \in \mathcal{M}_{I_2}$*,* $\Lambda_{I_1} \in [0,\epsilon)^{|I_1|}$*, and* $\Lambda_{I_2} \in [0,\epsilon)^{|I_2|}$*, we have*

$$G_{I_1 \cdot I_2}(x_1 \cdot x_2, \Lambda_{I_1} \cdot \Lambda_{I_2}) = (G_{I_1}(x_1, \Lambda_{I_1}), G_{I_2}(x_2, \Lambda_{I_2})). \tag{11.10}$$

11.3 Face Structures

In order to prove Theorem 11.6, we shall study the face structures at first.

By Definition 2.20, if F is a face of L, then $F = \bigsqcup_{\alpha \in \mathfrak{A}} C_\alpha$, where C_α is the closure of C_α° and C_α° is a component of $\partial^1 L$. As pointed in [32], F is still a manifold with corners. We have the following result which is trivial when $\mathfrak{A}$ is a finite set.

Lemma 11.7. *Using the notation as the above, we have that* F *is a smoothly embedded submanifold with corners inside* L*. The components of* F *are* C_α*. The interior of* F *(i.e.* $\partial^0 F$*) is* $\bigsqcup_{\alpha \in \mathfrak{A}} C_\alpha^\circ$ *and* F *is a closed subset of* L*.*

Proof. First, we show that C_α is a submanifold with corners and its 0-stratum is C_α°. It suffices to show that, for each $x \in C_\alpha$, there exists an open neighborhood U_x of x such that $U_x \cap C_\alpha$ has the desired corner structure.

We can choose U_x such that it has the chart $(-\epsilon, \epsilon)^{n-l} \times [0, \epsilon)^l$ and x has the coordinate

$(0, \cdots, 0)$. Clearly,

$$U_x \cap C_\alpha^\circ \subseteq \bigsqcup_{i=1}^{l} \left[(-\epsilon,\epsilon)^{n-l} \times (0,\epsilon)^{i-1} \times \{0\} \times (0,\epsilon)^{l-i}\right],$$

and $U_x \cap C_\alpha^\circ \neq \emptyset$. We may assume $[(-\epsilon,\epsilon)^{n-l} \times \{0\} \times (0,\epsilon)^{l-1}] \cap C_\alpha^\circ \neq \emptyset$. Since $(-\epsilon,\epsilon)^{n-l} \times \{0\} \times (0,\epsilon)^{l-1}$ is connected and contained in $\partial^1 L$, and C_α° is a component of $\partial^1 L$, we infer that $(-\epsilon,\epsilon)^{n-l} \times \{0\} \times (0,\epsilon)^{l-1} \subseteq C_\alpha^\circ$. By Definition 2.19, it's easy to see $U_x \cap C_\alpha^\circ = (-\epsilon,\epsilon)^{n-l} \times \{0\} \times (0,\epsilon)^{l-1}$. Since, U_x is open, we have $U_x \cap C_\alpha$ is the relative closure of $U_x \cap C_\alpha^\circ$ in U_x. In other words, $U_x \cap C_\alpha = (-\epsilon,\epsilon)^{n-l} \times \{0\} \times [0,\epsilon)^{l-1}$ and the 0-stratum of $U_x \cap C_\alpha$ is contained in C_α°. Thus we get the desired corner structure.

Second, we show that F is a manifold with corners.

Since C_α has no intersection with other C_β, by the above argument, we can see that the above open neighborhood U_x has no intersection with other C_β. Thus $\bigcup_{x \in C_\alpha} U_x$ is an open neighborhood of C_α which has no intersection with other C_β. So C_α is relatively open in F. This verifies the manifold structure of F.

Finally, we show that F is a closed subset of L. Suppose x is in the closure of F, then x can be approximated by points in F and thus by points in $\bigsqcup_{\alpha \in \mathfrak{A}} C_\alpha^\circ$. By the above argument, it's easy to see that x belongs to some C_α. □

Lemma 11.8. *Suppose L is an n dimensional manifold with faces. Suppose F_i $(i = 1, \cdots k)$ are faces of L such that their interiors are pair-wisely disjoint and $\bigcap_{i=1}^k F_i$ is nonempty. Then $\bigcap_{i=1}^k F_i$ is an $n-k$ dimensional smoothly embedded submanifold with corners inside L.*

Proof. Let x be an arbitrary point in $\bigcap_{i=1}^k F_i$. It suffices to prove that there exists an open neighborhood U of x such that $U \cap \bigcap_{i=1}^k F_i$ has a corner structure.

For each i, x belongs to an unique component of F_i. Since this component is relatively open in F_i, we can choose U small enough such that U has no intersection with other components. Thus we may assume F_i is connected.

By the proof of Lemma 11.7, we can choose U such that it has a chart $(-\epsilon, \epsilon)^{n-l} \times [0, \epsilon)^l$, x has the coordinate $(0, \cdots, 0)$ and $U \cap F_1 = (-\epsilon, \epsilon)^{n-l} \times \{0\} \times [0, \epsilon)^{l-1}$. Since the interior of F_i are pair-wisely disjoint, repeating this argument, we get $U \cap F_i = (-\epsilon, \epsilon)^{n-l} \times [0, \epsilon)^{i-1} \times \{0\} \times [0, \epsilon)^{l-i}$. Thus $U \cap \bigcap_{i=1}^k F_i = (-\epsilon, \epsilon)^{n-l} \times \{0\}^k \times [0, \epsilon)^{l-k}$. This verifies the corner structure. □

We introduce another concept following [24].

Definition 11.9. *Suppose L_1 is a submanifold without corners inside L and $x \in L_1$, we define the normal sector $A_x(L_1, L) = A_xL/T_xL_1$.*

In [24], $A_x(L_1, L)$ is called *secteur transverse.*

Define the tangent sector bundle AL as the subbundle of TL with fibers A_xL. Define the normal bundle $N(L_1, L)$ as the bundle whose fibers are the normal space $N_x(L_1, L) = T_xL/T_xL_1$. Define the normal sector bundle $A(L_1, L)$ as the subbundle of $N(L_1, L)$ with fiber $A_x(L_1, L)$ and $A_{L_1}L$ as the restriction of AL to L_1.

Lemma 11.10. *Under the assumption of Lemma 11.8, assume that L_1 is an open subset of $\partial^k L$ and $L_1 \subseteq \bigcap_{i=1}^k F_i$. Then there exist smooth sections e_i of $A_{L_1}L$ $(i = 1, \cdots, k)$ satisfying the following stratum condition: (1). $e_i \in A_{L_1}(\bigcap_{j\neq i} F_j)$; (2). $\{\pi e_1, \cdots, \pi e_k\}$ is linearly independent everywhere and all elements in $A_x(L_1, L)$ can be linearly represented by $\{\pi e_1(x), \cdots, \pi e_k(x)\}$ with nonnegative coefficients, where $\pi : A_{L_1}L \to A(L_1, L)$ is the natural projection.*

Proof. Suppose $x \in L_1$, by the proof of Lemma 11.8, there exists a neighborhood U of x such that U has a chart $(-\epsilon,\epsilon)^{n-k} \times [0,\epsilon)^k$, x has the coordinate $(0,\cdots,0)$, $U \cap L_1 = (-\epsilon,\epsilon)^{n-k} \times \{0\}^k$ and $U \cap F_i = (-\epsilon,\epsilon)^{n-k} \times [0,\epsilon)^{i-1} \times \{0\} \times [0,\epsilon)^{k-i}$. Thus $U \cap \bigcap_{j\neq i} F_j = (-\epsilon,\epsilon)^{n-k} \times \{0\}^{i-1} \times [0,\epsilon) \times \{0\}^{k-i}$. Obviously, for any vector $e_i(x) \in A_x(\bigcap_{j\neq i} F_j) - T_xL_1$, we have $\{\pi e_1(x),\cdots,\pi e_k(x)\}$ satisfies the desired property in $A_x(L_1,L)$.

Since L_1 is an open subset of the 1-stratum of $\bigcap_{j\neq i} F_j$, we can choose a smooth inward normal section e_i along L_1. □

In the case of Assumption 11.2, it's easy to see that $\overline{\mathcal{M}(p,q)}$ is a manifold with faces $\overline{\mathcal{M}(p,r)}\times\overline{\mathcal{M}(r,q)}$. The interiors of these faces are $\mathcal{M}(p,r)\times\mathcal{M}(r,q)$ which are pair-wisely disjoint. Suppose $I = \{p, r_1,\cdots,r_k,q\}$ is a chain of Ω. Let $I_i = \{p,r_1,\cdots,r_{i-1},r_{i+1},\cdots,r_k,q\}$. Then $\mathcal{M}_I$ is the interior of $\bigcap_{i=1}^k \overline{\mathcal{M}(p,r_i)} \times \overline{\mathcal{M}(r_i,q)}$, and $\bigcap_{j\neq i} \overline{\mathcal{M}(p,r_j)} \times \overline{\mathcal{M}(r_j,q)} = \overline{\mathcal{M}_{I_i}}$. By Lemma 11.10, we have the following corollary.

Corollary 11.11. *There exists a smooth frame $\{e_1,\cdots,e_k\}$ along $\mathcal{M}_I$ satisfying the following stratum condition: (1). $e_i \in A_{\mathcal{M}_I}\overline{\mathcal{M}_{I_i}}$; (2). $\{\pi e_1,\cdots,\pi e_k\}$ is linearly independent everywhere and all elements in $A_x(\mathcal{M}_I,\overline{\mathcal{M}(p,q)})$ can be linearly represented by $\{\pi e_1(x),\cdots,\pi e_k(x)\}$ with nonnegative coefficients, where $\pi : A_{\mathcal{M}_I}\overline{\mathcal{M}(p,q)} \to A(\mathcal{M}_I,\overline{\mathcal{M}(p,q)})$ is the natural projection.*

For a manifold L with corners, [24] shows that there exists a connection on L such that all strata are totally geodesic. (See [13, Chapter 4] for a detailed treatment of connections.) Suppose L_1 is a stratum of L. Then by the above connection and the exponential map, [24] shows that an open neighborhood of L_1 in $A(L_1,L)$ is diffeomorphic to an open neighborhood of L_1 in L. Thus by the frame in Corollary 11.11, we get the following lemma.

Lemma 11.12. *There is a smooth embedding* $\varphi_I : \mathcal{M}_I \times [0,1)^{|I|} \longrightarrow \overline{\mathcal{M}(p,q)}$ *satisfying the stratum condition (See (2) in Definition 11.1).*

In order to prove Theorem 11.6, we need to use some connections even better than the above one. This leads to the definition of the product connection. There are several ways to define a connection on a manifold L. One is as follows. A connection is to assign each smooth curve $\gamma : [0,1] \longrightarrow L$ a parallel transportation (displacement) $P_\gamma : T_{\gamma(0)}L \longrightarrow T_{\gamma(1)}L$ which is a linear isomorphism. Suppose L_1 and L_2 are two manifolds with corners. Clearly, $T(L_1 \times L_2) = TL_1 \times TL_2$. We define the product connection on $L_1 \times L_2$ as follows.

Definition 11.13. *Let* $\gamma = (\gamma_1, \gamma_2) : [0,1] \longrightarrow L_1 \times L_2$ *be a smooth curve. Define the parallel transportation* $P_\gamma : T_{\gamma(0)}(L_1 \times L_2) \longrightarrow T_{\gamma(1)}(L_1 \times L_2)$ *as* $P_\gamma(v_1, v_2) = (P_{\gamma_1}v_1, P_{\gamma_2}v_2)$, *where* P_{γ_i} *is the parallel transportation along* γ_i. *The connection assigning* P_γ *is the product connection.*

For a product connection, a curve γ in $L_1 \times L_2$ is a geodesic if and only if both γ_1 and γ_2 are geodesics. By Lemma 11.12, φ_I pulls back the connection on $\overline{\mathcal{M}(p,q)}$ to $\mathcal{M}_I \times [0,1)^{|I|}$. Let γ be a curve in $\mathcal{M}_I \times [0,1)^{|I|}$ such that $\gamma(t) = (x, \sigma(t))$, where $x \in \mathcal{M}_I$ and σ is a straight line in $[0,1)^{|I|}$. If σ passes through the origin, then γ is a geodesic because φ_I is defined by the exponential map. Since $\mathcal{M}_I$ is totally geodesic in $\overline{\mathcal{M}(p,q)}$, we infer that $\mathcal{M}_I$ has a connection. Moreover, $[0,1)^{|I|}$ also has its standard flat connection. We can define the product connection of $\mathcal{M}_I \times [0,1)^{|I|}$. The product connection coincides with the old one on $T(\mathcal{M}_I \times \{0\}^{|I|})$, and φ_I is still given by the exponential map under the new connection. This new connection has its advantage over the old one. In particular, for *every* straight line σ in $[0,1)^{|I|}$, *not* necessarily passing through the origin, $\gamma(t) = (x, \sigma(t))$ is a geodesic of the new connection. This is important in the proof of Theorem 11.6.

11.4 Proof of Theorem 11.6

Before proving Theorem 11.6, we shall introduce some definitions and notation.

Definition 11.14. *Suppose $(p, q) \subseteq \Omega$, where Ω is the set defined in Assumption 11.2. If $p \not\succ q$, then define the length of (p, q) as $|p, q| = -1$. Otherwise, define the length of (p, q) as $|p, q| = \sup\{|I| \mid I$ is a chain with head p and tail $q\}$.*

By (1) of Assumption 11.2, we know that $|p, q| \le \dim(\overline{\mathcal{M}(p, q)}) < +\infty$.

By the compactness of $\overline{\mathcal{M}(p, q)}$ and (1) of Assumption 11.2, there are only finitely many chains I with head p and tail q.

Suppose $I_1 = \{r_0, \cdots, r_{k+1}\}$ and $I_2 = \{r_0, r_{i_1}, \cdots, r_{i_l}, r_{k+1}\}$ are two chains of Ω such that $I_2 \preceq I_1$. Like Section 11.1, if $\Lambda_{I_2} = (\lambda_{i_1}, \cdots, \lambda_{i_l}) \in [0, +\infty)^{|I_2|}$ is a collaring parameter for for $\mathcal{M}_{I_2}$, then define $\Lambda_{I_2,I_1} \in [0, +\infty)^{|I_1|}$ as

$$\Lambda_{I_2,I_1}(i) = \begin{cases} \lambda_i & r_i \in I_2, \\ 0 & r_i \notin I_2. \end{cases} \tag{11.11}$$

Here we consider Λ_{I_2,I_1} as a collaring parameter for $\mathcal{M}_{I_1}$.

If $I_i \prec I$ $(i = 1, \cdots, n)$, then define

$$\Lambda_I + \Lambda_{I_1} + \cdots \Lambda_{I_n} = \Lambda_I + \Lambda_{I_1,I} + \cdots \Lambda_{I_n,I}. \tag{11.12}$$

Clearly,

$$\Lambda_{I_1} = \Lambda_{I_1}(I_1 - I_2) + \Lambda_{I_1,I_2}$$

For example, suppose $I_1 = \{r_0, r_1, r_2, r_3, r_4\}$, $I_2 = \{r_0, r_2, r_4\}$ and $\Lambda_{I_1} = (5, 6, 7)$, then $\Lambda_{I_1,I_2} = (6)$, and

$$\Lambda_{I_1}(I_1 - I_2) + \Lambda_{I_1,I_2} = (5, 0, 7) + (6) = (5, 0, 7) + (0, 6, 0) = (5, 6, 7) = \Lambda_{I_1}.$$

If $\Lambda_{I_2} = (8)$, then $\Lambda_{I_2,I_1} = (0, 8, 0)$ and

$$\Lambda_{I_1} + \Lambda_{I_2} = (5, 6, 7) + (8) = (5, 6, 7) + (0, 8, 0) = (5, 14, 7).$$

Proof of Theorem 11.6. We shall define G_I by exponential maps. This requires two things. First, a frame satisfying the stratum condition (See Corollary 11.11) in $A(\mathcal{M}_I, \overline{\mathcal{M}(p,q)})$. Second, a connection on $\overline{\mathcal{M}(p,q)}$. The proof is to construct the above two things by a double induction. The outer induction is on the length $|p,q|$. We construct the desired G_I in the case of $|p,q| = n$ based on the hypothesis that all G_I have been constructed and satisfy (11.9) and (11.10) for all $|p,q| < n$. The inner induction is the process to construct G_I for a fixed pair (p,q).

(1). The first step of the outer induction (the induction on $|p,q|$).

When $|p,q| = 0$, then $\mathcal{M}_I = \overline{\mathcal{M}(p,q)}$, define $G_I : \mathcal{M}_I \to \overline{\mathcal{M}(p,q)}$ as the identity.

(2). The second step of the outer induction (the induction on $|p,q|$).

Suppose we have constructed the desired G_I for all pair (p,q) such that $|p,q| < n$. We shall construct G_I in the case of $|p,q| = n$. The construction is the inner induction. Let

X_k be the union of all l-strata of $\overline{\mathcal{M}(p,q)}$ with $l \geq k$. Clearly, $X_{k+1} \subseteq X_k$, X_1 is the full boundary of $\overline{\mathcal{M}(p,q)}$. We shall construct a family of open sets U_k such that $U_{k+1} \subseteq U_k$ and $X_k \subseteq U_k$ by an inverse induction on k. In other words, construct U_k after having constructed U_{k+1}. For each k, we shall construct $G_I : (\mathcal{M}_I \cap U_k) \times [0,\epsilon)^{|I|} \to \overline{\mathcal{M}(p,q)}$ such that $ImG_I \subseteq U_k$, and all G_I satisfy (11.9) and (11.10). We call such a map G_I in U_k, denote it by $G_I|_{U_k}$. Extend G_I with the step of the inner induction. Clearly, U_1 contains all $\mathcal{M}_I$ such that $|I| > 0$. If the construction of $G_I|_{U_1}$ is finished, we shall complete the proof by defining $G_{\{p,q\}}$ as the inclusion.

Since $|p,q| = n$, the stratum with the lowest dimension is the n-strata.

(I). The first step of the inner induction (the induction on U_k).

We shall construct U_n, $G_I|_{U_n}$, frames for $\mathcal{M}_I \cap U_n$ and a connection providing all G_I via the exponential map. Moreover, $(\mathcal{M}_I \cap U_n) \times [0,\epsilon)^{|I|}$ will also have a product connection (see Definition 11.13 and the comment following it) if we pull back the connection on U_n via G_I.

We know that $X_n = \bigcup_{|J|=n} \mathcal{M}_J$. By Lemma 11.12, we can construct a smooth embedding $\varphi_J : \mathcal{M}_J \times [0,\epsilon_0)^{|J|} \to \overline{\mathcal{M}(p,q)}$ satisfying the stratum condition (See (2) in Definition 11.1). Furthermore, $\mathcal{M}_J$ is compact because it is closed (also open) in the lowest dimensional stratum. Choose ϵ_0 small enough, $Im\varphi_J$ are pair-wisely disjoint for all J such that $|J| = n$. Fix $J = \{p, r_1, \cdots, r_n, q\}$. Suppose $J_l = \{p, r_1, \cdots, r_l\}$ and $J_l' = \{r_l, \cdots, r_n, q\}$. Clearly, $|p, r_l| < n$ and $|r_l, q| < n$. By the outer induction on $|p,q|$, G_{J_l} and $G_{J_l'}$ have been defined.

Lemma 11.15. *There exists $\epsilon > 0$. And φ_J can be modified to be defined on $\mathcal{M}_J \times [0,\epsilon)^{|J|}$*

such that for all $l \in \{1, \cdots, n\}$*, we have*

$$\varphi_J(x_1 \cdot x_2, \Lambda_{J_l} \cdot \Lambda_{J'_l}) = (G_{J_l}(x_1, \Lambda_{J_l}), G_{J'_l}(x_2, \Lambda_{J'_l})).$$

Proof. For small ϵ, $G_{J_l} \times G_{J'_l}(\mathcal{M}_J \times \prod_{i=1, i\neq l}^{|J|}[0, \epsilon)) \subseteq Im\varphi_J$, where $G_{J_l} \times G_{J'_l}(x_1 \cdot x_2, \Lambda_{J_l}, \Lambda_{J'_l}) = (G_{J_l}(x_1, \Lambda_{J_l}), G_{J'_l}(x_2, \Lambda_{J'_l}))$.

Consider the following map $\phi_l = \varphi_J^{-1} \circ (G_{J_l} \times G_{J'_l})$,

$$\phi_l : \mathcal{M}_J \times \prod_{i=1, i\neq l}^{|J|}[0, \epsilon) \to Im\varphi_J \to \mathcal{M}_J \times [0, \epsilon_0)^{|J|}.$$

We only need to prove that φ_J can be modified such that for all l,

$$\phi_l(x, \lambda_1, \cdots, \lambda_{l-1}, \lambda_{l+1}, \cdots, \lambda_n) = (x, \lambda_1, \cdots, \lambda_{l-1}, 0, \lambda_{l+1}, \cdots, \lambda_n). \tag{11.13}$$

Denote $(\lambda_1, \cdots, \lambda_n)$ by Λ_J, $(\lambda_1, \cdots, \lambda_{l-1}, \lambda_{l+1}, \cdots, \lambda_n)$ by Λ_{J-l}, $(\lambda_1, \cdots, \lambda_{l-1})$ by Λ_{J_l}, and $(\lambda_{l+1}, \cdots, \lambda_n)$ by $\Lambda_{J'_l}$. Since $Im(G_{J_l} \times G_{J'_l}) \subseteq \overline{\mathcal{M}(p, r_l)} \times \overline{\mathcal{M}(r_l, q)}$ and φ_J satisfies the stratum condition, we have

$$\phi_l(x, \Lambda_{J-l}) = (a, c_1, \cdots, c_{l-1}, 0, c_{l+1}, \cdots, c_n)$$

where a and c_i are smooth functions of x and Λ_{J-l}.

Define $\theta_l : \mathcal{M}_J \times [0, \epsilon)^{|J|} \to \mathcal{M}_J \times [0, \epsilon_0)^{|J|}$ as

$$\theta_l(x, \Lambda_J) = (a, \cdots, c_{l-1}, \lambda_l, c_{l+1}, \cdots, c_n). \tag{11.14}$$

Since ϕ_l is a smooth embedding, so is θ_l. Since $\mathcal{M}_J$ is compact, shrink ϵ_0 if necessary, we may assume θ_l^{-1} can be defined on $\mathcal{M}_J \times [0, \epsilon_0)^{|J|}$. Thus

$$\begin{aligned}
& (\varphi_J \circ \theta_l)^{-1} \circ (G_{J_l} \times G_{J_l'})(x, \Lambda_{J-l}) \\
= \ & \theta_l^{-1} \circ \phi_l(x, \Lambda_{J-l}) \\
= \ & (x, \lambda_1, \cdots, \lambda_{l-1}, 0, \lambda_{l+1}, \cdots, \lambda_n) \\
= \ & (x, \Lambda_{J_l} \cdot \Lambda_{J_l'}).
\end{aligned}$$

Modify φ_J to be $\varphi_J \circ \theta_l$, we get (11.13) is true for a fixed l and some $\epsilon > 0$.

In general, suppose we have proved (11.13) is true for $l \in \{1, \cdots, j-1\}$, we shall modify φ_J such that (11.13) is true for all $l \in \{1, \cdots, j\}$. Let $x = x_1 \cdot x_2 \cdot x_3$, where $x_1 = (a_0, \cdots, a_{l-1})$, $x_2 = (a_l, \cdots, a_{j-1})$ and $x_3 = (a_j, \cdots, a_n)$. Denote $\{r_l, \cdots, r_j\}$ by $J_{(l,j)}$ and $(\lambda_{l+1}, \cdots, \lambda_{j-1})$ by $\Lambda_{J_{(l,j)}}$.

$$\phi_j(x, \Lambda_{J_l} \cdot \Lambda_{J_{(l,j)}}, \Lambda_{J_j'}) = \varphi_J^{-1}(G_{J_j}(x_1 \cdot x_2, \Lambda_{J_l} \cdot \Lambda_{J_{(l,j)}}), G_{J_j'}(x_3, \Lambda_{J_j'})).$$

Since $|p, r_l| < n$, by the outer inductive hypothesis, G_{J_j} satisfies (11.10). Shrink ϵ if necessary, we have

$$G_{J_j}(x_1 \cdot x_2, \Lambda_{J_l} \cdot \Lambda_{J_{(l,j)}}) = (G_{J_l}(x_1, \Lambda_{J_l}), G_{J_{(l,j)}}(x_2, \Lambda_{J_{(l,j)}})),$$

Similarly,

$$G_{J_l'}(x_2 \cdot x_3, \Lambda_{J_{(l,j)}} \cdot \Lambda_{J_j'}) = (G_{J_{(l,j)}}(x_2, \Lambda_{J_{(l,j)}}), G_{J_j'}(x_3, \Lambda_{J_j'})).$$

Thus

$$G_{J_j} \times G_{J'_j}(x, \Lambda_{J_l} \cdot \Lambda_{J_{(l,j)}}, \Lambda_{J'_j}) = G_{J_l} \times G_{J'_l}(x, \Lambda_{J_l}, \Lambda_{J_{(l,j)}} \cdot \Lambda_{J'_j}).$$

Then

$$\begin{aligned} & \phi_j(x, \Lambda_{J_l} \cdot \Lambda_{J_{(l,j)}}, \Lambda_{J'_j}) \\ = \; & \varphi_J^{-1} \circ (G_{J_j} \times G_{J'_j})(x, \Lambda_{J_l} \cdot \Lambda_{J_{(l,j)}}, \Lambda_{J'_j}) \\ = \; & \varphi_J^{-1} \circ (G_{J_l} \times G_{J'_l})(x, \Lambda_{J_l}, \Lambda_{J_{(l,j)}} \cdot \Lambda_{J'_j}) \\ = \; & \phi_l(x, \Lambda_{J_l}, \Lambda_{J_{(l,j)}} \cdot \Lambda_{J'_j}). \end{aligned}$$

Since ϕ_l satisfies (11.13), we have $\phi_l(x, \Lambda_{J_l}, \Lambda_{J_{(l,j)}} \cdot \Lambda_{J'_j}) = (x, \Lambda_{J_l} \cdot \Lambda_{J_{(l,j)}} \cdot \Lambda_{J'_j})$, or

$$\begin{aligned} & \phi_j(x, \lambda_1, \cdots, \lambda_{l-1}, 0, \lambda_{l+1}, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots, \lambda_n) \\ = \; & (x, \lambda_1, \cdots, \lambda_{l-1}, 0, \lambda_{l+1}, \cdots, \lambda_{j-1}, 0, \lambda_{j+1}, \cdots, \lambda_n). \end{aligned}$$

Define $\theta_j : \mathcal{M}_J \times [0, \epsilon)^{|J|} \to \mathcal{M}_J \times [0, \epsilon_0)^{|J|}$ as (11.14), we have

$$\theta_j(x, \lambda_1, \cdots, \lambda_{l-1}, 0, \lambda_{l+1}, \cdots, \lambda_n) = (x, \lambda_1, \cdots, \lambda_{l-1}, 0, \lambda_{l+1}, \cdots, \lambda_n).$$

The operation of θ_j on $\mathcal{M}_J \times \prod_{i=1,i\neq l}^{|J|}[0,\epsilon) \times \{0\}$ is the identity. Thus

$$\begin{aligned} & (\varphi_J \circ \theta_j)^{-1} \circ (G_{J_l} \times G_{J_l'}) \\ = \;& \theta_j^{-1} \circ (\varphi_J^{-1} \circ (G_{J_l} \times G_{J_l'})) \\ = \;& \varphi_J^{-1} \circ (G_{J_l} \times G_{J_l'}) \\ = \;& \phi_l. \end{aligned}$$

So if we modify φ_J to be $\varphi_J \circ \theta_j$, then ϕ_l $(l < j)$ will not change and still satisfy (11.13). However, ϕ_j may change and must satisfy (11.13) now. Thus we get a new φ_J such that (11.13) is true for $l \in \{1, \cdots, j\}$.

Repeat this process, we finish the proof of this lemma. □

Now we define G_I in $Im\varphi_J$. If $I \npreceq J$, then $Im\varphi_J \cap \mathcal{M}_I = \emptyset$, we don't need to consider it. We assume $I \preceq J$.

For all $y \in Im\varphi_J \cap \mathcal{M}_I$, there exist $x \in \mathcal{M}_J$ and $\Lambda_J \in [0,\epsilon)^n$ such that $y = \varphi_J(x, \Lambda_J)$ where x and Λ_J are unique and $\lambda_i = 0$ if and only if $r_i \in I$. Define $G_I(y, \Lambda_I) = \varphi_J(x, \Lambda_J + \Lambda_I)$. Since φ_J is a smooth embedding, so is G_I. (Actually, if we identify $Im\varphi_J$ with $\mathcal{M}_J \times [0,\epsilon)^{|J|}$ via φ_J, then G_I has the form $G_I((x, \Lambda_J), \Lambda_I) = (x, \Lambda_J + \Lambda_I)$.)

Lemma 11.16. *The maps G_I satisfy (11.9) in $Im\varphi_J$.*

Proof. Suppose $I_2 \preceq I_1 \preceq J$ and $y \in Im\varphi_J \cap \mathcal{M}_{I_1}$, we need to show that $G_{I_1}(y, \Lambda_{I_1}) = G_{I_2}(G_{I_1}(y, \Lambda_{I_1}(I_1 - I_2)), \Lambda_{I_1,I_2})$.

Suppose $y = \varphi_J(x, \Lambda_J)$, we have $G_{I_1}(y, \Lambda_{I_1}) = \varphi_J(x, \Lambda_J + \Lambda_{I_1})$, $G_{I_1}(y, \Lambda_{I_1}(I_1 - I_2)) =$

$\varphi_J(x, \Lambda_J + \Lambda_{I_1}(I_1 - I_2))$, and

$$
\begin{aligned}
& G_{I_2}(G_{I_1}(y, \Lambda_{I_1}(I_1 - I_2)), \Lambda_{I_1,I_2}) \\
= \;& G_{I_2}(\varphi_J(x, \Lambda_J + \Lambda_{I_1}(I_1 - I_2)), \Lambda_{I_1,I_2}) \\
= \;& \varphi_J(x, \Lambda_J + \Lambda_{I_1}(I_1 - I_2) + \Lambda_{I_1,I_2}) \\
= \;& \varphi_J(x, \Lambda_J + \Lambda_{I_1}) = G_{I_1}(y, \Lambda_{I_1}).
\end{aligned}
$$

This completes the proof of the lemma. □

Lemma 11.17. *The maps G_I satisfy (11.10) in $Im\varphi_J$.*

Proof. Suppose $I \preceq J$, $I = I_1 \cdot I_2$, $y_1 \in \mathcal{M}_{I_1}$, $y_2 \in \mathcal{M}_{I_2}$, and $y_1 \cdot y_2 \in Im\varphi_J$. We need to show that $G_I(y_1 \cdot y_2, \Lambda_{I_1} \cdot \Lambda_{I_2}) = (G_{I_1}(y_1, \Lambda_{I_1}), G_{I_2}(y_2, \Lambda_{I_2}))$.

Since $I \preceq J$, we have $J = J_l \cdot J_l'$, $I_1 \preceq J_l$ and $I_2 \preceq J_l'$ for some $J_l = \{p, r_1, \cdots, r_l\}$ and $J_l' = \{r_l, \cdots, r_n, q\}$. Since $y_1 \cdot y_2 \in \mathcal{M}_{I_1} \times \mathcal{M}_{I_2}$ and $y_1 \cdot y_2 = \varphi_J(x, \Lambda_J)$, we have $x = x_1 \cdot x_2$ for some $x_1 \in \mathcal{M}_{J_l}$ and $x_2 \in \mathcal{M}_{J_l'}$ and $\Lambda_J = \Lambda_{J_l} \cdot \Lambda_{J_l'}$ for some Λ_{J_l} and $\Lambda_{J_l'}$. Thus $y_1 \cdot y_2 = \varphi_J(x_1 \cdot x_2, \Lambda_{J_l} \cdot \Lambda_{J_l'})$. By Lemma 11.15, $y_1 = G_{J_l}(x_1, \Lambda_{J_l})$ and $y_2 = G_{J_l'}(x_2, \Lambda_{J_l'})$. Furthermore,

$$
\begin{aligned}
& G_I(y_1 \cdot y_2, \Lambda_{I_1} \cdot \Lambda_{I_2}) \\
= \;& \varphi_J(x_1 \cdot x_2, \Lambda_{J_l} \cdot \Lambda_{J_l'} + \Lambda_{I_1} \cdot \Lambda_{I_2}) \\
= \;& \varphi_J(x_1 \cdot x_2, (\Lambda_{J_l} + \Lambda_{I_1}) \cdot (\Lambda_{J_l'} + \Lambda_{I_2})).
\end{aligned}
$$

By Lemma 11.15,

$$\varphi_J(x_1 \cdot x_2, (\Lambda_{J_l} + \Lambda_{I_1}) \cdot (\Lambda_{J_l'} + \Lambda_{I_2})) = (G_{J_l}(x_1, \Lambda_{J_l} + \Lambda_{I_1}), G_{J_l'}(x_2, \Lambda_{J_l'} + \Lambda_{I_2})).$$

Since $|p, r_l| < n$ and $|r_l, q| < n$, by the outer inductive hypothesis, G_{J_l}, $G_{J_l'}$, G_{I_1} and G_{I_2} satisfy (11.9). Thus

$$\begin{aligned} & (G_{J_l}(x_1, \Lambda_{J_l} + \Lambda_{I_1}), G_{J_l'}(x_2, \Lambda_{J_l'} + \Lambda_{I_2})) \\ = \ & (G_{I_1}(G_{J_l}(x_1, \Lambda_{J_l}), \Lambda_{I_1}), G_{I_2}(G_{J_l'}(x_2, \Lambda_{J_l'}), \Lambda_{I_2})) \\ = \ & (G_{I_1}(y_1, \Lambda_{I_1}), G_{I_2}(y_2, \Lambda_{I_2})). \end{aligned}$$

This completes the proof of the lemma. $\square$

We have defined the desired G_I in $Im\varphi_J$ for all I such that $\mathcal{M}_I \cap Im\varphi_J \neq \emptyset$. Clearly, $(\mathcal{M}_I \cap Im\varphi_J) \times [0, \epsilon)^{|I|}$ has a frame $\{\frac{\partial}{\partial\lambda_1}, \cdots, \frac{\partial}{\partial\lambda_{|I|}}\}$. Then

$$\{\mathcal{N}_1(I), \cdots, \mathcal{N}_{|I|}(I)\} = dG_I|_{\Lambda_J = 0} \left\{ \frac{\partial}{\partial\lambda_1}, \cdots, \frac{\partial}{\partial\lambda_{|I|}} \right\}$$

serves a desired frame of $A((\mathcal{M}_I \cap Im\varphi_J), \overline{\mathcal{M}(p,q)})$. Identify $Im\varphi_J$ with $\mathcal{M}_J \times [0, \epsilon)^{|J|}$ via φ_J, give $Im\varphi_J$ the product connection (See Definition 11.13 and the comment following it.). Again, $G_I(y, \Lambda_I) = \varphi_J(x, \Lambda_J + \Lambda_I)$, and $\Lambda_J + t\Lambda_I$ for $t \in [0, 1]$ is a line segment in $[0, \epsilon)^{|J|}$. Then $G_I(y, t\Lambda_I)$ is a geodesic segment. Thus $G_I(y, \Lambda_I) = \exp(y, \sum_{i=1}^{|I|} \lambda_i \mathcal{N}_i(I))$ and this connection is the desired one.

Do the above construction for each J such that $|J| = n$. Clearly, $G_J = \varphi_J$ when $|J| = n$.

Let $U_n = \bigcup_{|J|=n} ImG_J$, then $U_n \supseteq X_n$. This completes the first step of the inner induction.

(II). The second step of the inner induction (the induction on U_k).

Suppose we have constructed $U_{k+1} = \bigcup_{|I_0|\geq k+1} ImG_{I_0}$. Suppose, for all I, we have constructed $G_I|_{U_{k+1}}$, the frames on $\mathcal{M}_I \cap U_{k+1}$ and the connection on U_{k+1} which provides G_I via exponential maps. Moreover, $(\mathcal{M}_I \cap U_{k+1}) \times [0,\epsilon)^{|I|}$ has a product connection if we pull back the connection on U_{k+1} via G_I. We shall extend the above things to those on U_k.

The construction shares many details with the first step. The essential point is that the definition of $G_I|_{U_k}$ should be an extension of $G_I|_{U_{k+1}}$.

Let $U_{k+1}(\delta) = \bigcup_{|I|\geq k+1} G_I|_{U_{k+1}}(\mathcal{M}_I \times [0,\delta)^{|I|})$ for $\delta \in (0,\epsilon)$. It's an open set such that $X_{k+1} \subset U_{k+1}(\delta) \subset U_{k+1}$. Let $\overline{U_{k+1}(\delta)} = \bigcup_{|I|\geq k+1} G_I|_{U_{k+1}}(\mathcal{M}_I \times [0,\delta]^{|I|})$.

Lemma 11.18. *The set $\overline{U_{k+1}(\delta)}$ is closed.*

Proof. For each I_0 such that $|I_0| \geq k+1$, we have $\overline{\mathcal{M}_{I_0}} = \bigsqcup_{I_0 \preceq I} \mathcal{M}_I$ is compact. Moreover, $G_{I_0}|_{U_{k+1}} : \mathcal{M}_{I_0} \times [0,\epsilon)^{|I_0|} \to \overline{\mathcal{M}(p,q)}$ has been defined.

Define $\overline{G_{I_0}} : \overline{\mathcal{M}_{I_0}} \times [0,\epsilon)^{|I_0|} \to \overline{\mathcal{M}(p,q)}$ as $\overline{G_{I_0}}(x,\Lambda_{I_0}) = G_I|_{U_{k+1}}(x,\Lambda_{I_0,I})$ for $(x,\Lambda_{I_0}) \in \mathcal{M}_I \times [0,\epsilon)^{|I_0|}$. Since the maps $G_I|_{U_{k+1}}$ satisfy (11.9), we infer that $\overline{G_{I_0}}$ is well defined and is a smooth embedding.

Thus $\overline{U_{k+1}(\delta)} = \bigcup_{|I_0|\geq k+1} \overline{G_{I_0}}(\overline{\mathcal{M}_{I_0}} \times [0,\delta]^{|I_0|})$ is compact. ☐

As the first step, by Lemma 11.12, for each J such that $|J| = k$, there is a smooth embedding $\varphi_J : \mathcal{M}_J \times [0,\epsilon_0)^{|J|} \to \overline{\mathcal{M}(p,q)}$ satisfying the stratum condition. Thus $d\varphi_J\{\frac{\partial}{\partial\lambda_1}, \cdots, \frac{\partial}{\partial\lambda_{|J|}}\}$ is frame satisfying the stratum condition (See Corollary 11.11). By the inner inductive hypothesis, $\mathcal{M}_J \cap U_{k+1}$ already has a frame $\{\mathcal{N}_1(J), \cdots, \mathcal{N}_{|J|}(J)\}$ satisfying the stratum condition. Both $N_i(J)$ and $d\varphi_J\frac{\partial}{\partial\lambda_i}$ represent nonzero elements in the same $A(\mathcal{M}_J, \mathcal{M}_I) \cong [0,+\infty)$

for some $I \prec J$ such that $|I| = |J| - 1$. Thus, for all $\alpha(x) \geq 0$, $\{\alpha(x)N_i(J) + (1-\alpha(x))d\varphi_J \frac{\partial}{\partial \lambda_i} \mid i = 1, \cdots, n\}$ is also a frame satisfying the stratum condition. By Lemma 11.18 and the partition of unity,, there is a frame satisfying the stratum condition and coinciding with the old one in $U_{k+1}(\delta)$ for some $\delta > 0$. Also by the same reason, there is a connection in $U_{k+1} \cup Im\varphi_J$ such that it coincides with the old one in $U_{k+1}(\delta)$. Then, by the above frame and connection, we can modify φ_J such that it coincides with $G_J|_{U_{k+1}}$ in $U_{k+1}(\delta)$. Since $\mathcal{M}_J - U_{k+1}(\delta) = \overline{\mathcal{M}_J} - U_{k+1}(\delta)$ is compact, and $G_J|_{U_{k+1}}$ is an embedding, by Lemma 11.18, we infer φ_J is an embedding defined on $\mathcal{M}_J \times [0, \epsilon_0)^{|J|}$ for some $\epsilon_0 \in (0, \delta]$. Just as the first step, we can modify φ_J furthermore such that it satisfies the conclusion of Lemma 11.15. Since originally φ_J and $G_J|_{U_{k+1}}$ coincide in $U_{k+1}(\delta)$ and $G_J|_{U_{k+1}}$ satisfies (11.10), the modification does not change $\varphi_J|_{U_{k+1}(\delta)}$. Thus the modified φ_J still coincides with $G_J|_{U_{k+1}}$ in $U_{k+1}(\delta)$.

The big difference between this step and the first step is as follows. In the first step, $Im\varphi_J$ are pair-wisely disjoint for $|J| = n$. Thus there is no contradiction of the definition when G_I is defined in each $Im\varphi_J$. Now it's impossible to make $Im\varphi_J$ pair-wisely disjoint. We shall control their pair-wise intersections. Suppose $J_1 \neq J_2$ and $|J_1| = |J_2| = k$. Then $(\mathcal{M}_{J_1} - U_{k+1}(\delta)) \cap (\mathcal{M}_{J_2} - U_{k+1}(\delta)) \subseteq \mathcal{M}_{J_1} \cap \mathcal{M}_{J_2} = \emptyset$. Since $\mathcal{M}_{J_i} - U_{k+1}(\delta)$ is compact, shrink ϵ_0 if necessary, we have

$$\varphi_{J_1}\left((\mathcal{M}_{J_1} - U_{k+1}(\delta)) \times [0, \epsilon_0)^{|J_1|}\right) \cap \varphi_{J_2}\left((\mathcal{M}_{J_2} - U_{k+1}(\delta)) \times [0, \epsilon_0)^{|J_2|}\right) = \emptyset.$$

Since

$$\varphi_{J_i}\left((\mathcal{M}_{J_i} \cap U_{k+1}(\delta)) \times [0, \epsilon_0)^{|J_i|}\right) \subseteq U_{k+1}(\delta),$$

we get $Im\varphi_{J_1} \cap Im\varphi_{J_2} \subseteq U_{k+1}(\delta)$.

Now we define G_I in each $Im\varphi_J$. We only need to consider I such that $I \preceq J$. For all $y \in \mathcal{M}_I \cap Im\varphi_J$, $y = \varphi_J(x, \Lambda_J)$, define $\widetilde{G}_I(J)(y, \Lambda_I) = \varphi_J(x, \Lambda_J + \Lambda_I)$. Given $\varphi_J = G_J|_{U_{k+1}}$ in $U_{k+1}(\delta)$, similarly to the argument in the first step, we get $\widetilde{G}_I(J) = G_I|_{U_{k+1}}$ in $U_{k+1}(\delta)$. Since $Im\varphi_{J_1} \cap Im\varphi_{J_2} \subseteq U_{k+1}(\delta)$, $\widetilde{G}_I(J_1)$ coincides with $\widetilde{G}_I(J_2)$ in their common domains. Define $G_I|_{Im\varphi_J} = \widetilde{G}_I(J)$. Then G_I is well defined on $U_{k+1}(\delta) \cup \bigcup_{|J|=k} Im\varphi_J$ and it coincides with $G_I|_{U_{k+1}}$ in $U_{k+1}(\delta)$.

Similarly to the first step, the maps $G_I|_{Im\varphi_J}$ satisfy (11.9) and (11.10).

Shrink U_{k+1} to be $U_{k+1}(\epsilon_0)$. Again, $G_J = \varphi_J$ when $|J| = k$. Let

$$U_k = U_{k+1} \cup \bigcup_{|J|=k} G_J(\mathcal{M}_J \times [0, \epsilon_0)^k).$$

The desired $G_I|_{U_k}$ is defined in the above. Shrink ϵ_I to be ϵ_0 for all I. Give frames to $\mathcal{M}_I \cap U_k$ as the first step. For $|J| = k$, give ImG_J the product connection via G_J. The old connection in U_{k+1} is the product connection. Thus the new connection in ImG_J coincides with the old one in U_{k+1}. This completes the second step of the inner induction.

(III). The completion of the second step of the outer induction (the induction on $|p, q|$).

For the fixed pair (p, q), the construction in U_k requires a shrink of ϵ_I for all I with head p and tail q. However, the inner induction stops in a finite number of steps. Eventually, we have $\epsilon_I > 0$ which are the same for all I with head p and tail q. And if $I_1 \cdot I_2 = I$, then $\epsilon_I \leq \epsilon_{I_i}$. Thus we have constructed the desired G_I for the pair (p, q) with length n. This completes the second step of the outer induction and also the proof of this theorem. $\square$

11.5 A Byproduct

The argument for Theorem 11.6 already gives the following Proposition 11.19 which gives a compatible collar structure for an arbitrary compact manifold with faces.

Suppose L is a smooth manifold with faces. Suppose F_i $(i = 1, \cdots, n)$ are its faces such that $\bigcup_{i=1}^n F_i = \bigcup_{k>0} \partial^k L$. In other words, $\bigcup_{i=1}^n F_i$ is the full boundary of L. Suppose the interiors of F_i are pair-wisely disjoint.

Let $I = \{i_1, \cdots, i_k\}$ be a subset of $\{1, \cdots, n\}$. Define $|I| = k$. Define $F_I = \bigcap_{i \in I} F_i$. In particular, when $I = \emptyset$, define $F_\emptyset = L$. Then, by Lemma 11.8, F_I is either empty or an $n-k$ dimensional smoothly embedded submanifold with corners insider L. Denote the interior of F_I by F_I°.

Let $V_I = \prod_{i \in I} [0, +\infty)$ be a factor space of $[0, +\infty)^n$. In other words, V_I is the product of the ith coordinate spaces of $[0, +\infty)^n$ such that $i \in I$. In particular, $V_\emptyset$ consists of one point. Let $V_I(\epsilon) = \prod_{i \in I} [0, \epsilon)$.

Let $\Lambda_I = \{\lambda_{i_1}, \cdots, \lambda_{i_k}\} \in V_I$ represent the collaring parameter for F_I°. Suppose $J \subseteq I$. Define $\Lambda_I(I - J) \in V_I$ as

$$\Lambda_I(I-J)(i) = \begin{cases} 0 & i \in J, \\ \lambda_i & i \in I - J. \end{cases}$$

Define $\Lambda_{I,J} \in V_J$ as $\Lambda_{I,J}(i) = \lambda_i$ for $i \in J$.

Proposition 11.19. *Suppose L is compact. Then collaring maps $G_I : F_I^\circ \times V_I(1) \to L$ can be defined for all I such that $F_I^\circ \neq \emptyset$. These maps satisfy the following conditions.*

(1). They are smooth embeddings which satisfy the following stratum condition. If $J \subseteq I = \{i_1, \cdots, i_k\}$, $\Lambda_I = \{\lambda_{i_1}, \cdots, \lambda_{i_k}\} \in V_I(1)$, and $\lambda_i = 0$ if and only if $i \in J$, then

$G_I(x, \Lambda_I) \in F_J^\circ$ for all $x \in F_I^\circ$. In particular, $G_\emptyset : F_\emptyset^\circ = \partial^0 L \to L$ is the inclusion.

(2). They satisfy the following compatibility. If $J \subseteq I$ and $\lambda_i > 0$ when $i \notin J$, then, for all $x \in F_I^\circ$, we have

$$G_I(x, \Lambda_I) = G_J(G_I(x, \Lambda_I(I - J)), \Lambda_{I,J}).$$

The assumption of Proposition 11.19 is more general than that of Theorem 11.6 in some sense. However, this proof is actually even easier than that one because we only deal with one manifold with faces. It only requires that (11.9) is true in a more general setting. We don't need any more the arguments related to (11.10) such as Lemmas 11.15 and 11.17. Instead of a double induction, it suffices to repeat the inner induction in the proof of Theorem 11.6. Since there are only finitely many set I, we can find $\epsilon > 0$ such that $\epsilon_I = \epsilon$ for all I. By a scaling of parameter, we get $\epsilon = 1$, which finishes the proof.

REFERENCES

[1] A. Abbondandolo and P. Majer, A Morse complex for infinite dimensional manifolds-part I, Adv. Math. **197** (2005), 321-410.

[2] A. Abbondandolo and P. Majer, Lectures on Morse complex for infinite-dimensional manifolds, Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology, 1-74, Montreal, Springer, 2006.

[3] D. Austin and P. Braam, Morse-Bott theory and equivariant cohomology, The Floer Memorial Volume, 123-183, Progress in Mathematics, **133**, Birkhäuser Verlag, 1995.

[4] J. Barraud and O. Cornea, Homotopic dynamics in symplectic topology, Morse theoretic methods in nonlinear analysis and in symplectic topology, 109-148, NATO Sci. Ser. II Math. Phys. Chem., **217**, Springer, Dordrecht, 2006.

[5] J. Barraud and O. Cornea, Lagrangian intersections and the Serre spectral sequence, Ann. of Math., **166** (2007), 657-722.

[6] M. Brown, A proof of the generalized Schoenflies theorem, Bull. Amer. Math. Soc., **66** (1960), 74-76.

[7] M. Brown, Locally flat imbeddings of topological manifolds, Ann. of Math., **75** (1962), 331-341.

[8] R. Bott, Morse theory indomitable, Publications Mathmatiques de l'I.H.É.S., **68** (1988), 99-114.

[9] A. Banyaga and D. Hurtubise, Lectures on Morse Homology, Kluwer Texts in Mathematical Sciences, **29**, Kluwer Academic Publishers, 2004.

[10] D. Burghelea, L. Friedlander and T. Kappeler, Witten-Helffer-Sjöstrand Theory, book in preparation.

[11] D. Burghelea and S. Haller, On the topology and analysis of a closed one form (Novikov's theory revisited), Essays on geometry and related topics, 133-175, Monogr. Enseign. Math., **38**, Enseignement Math., Geneva, 2001.

[12] J. Bismut and W. Zhang, An extension of a theorem by Cheeger and Müller, Astérisque, **205** (1992), 1-235.

[13] S. Chern, W. Chen and K. Lam, Lectures on Differential Geometry, Series on University Mathematics, **1**, World Scientific, 1998.

[14] M. Cohen, A Course in Simple-Homotopy Theory, Springer-Verlag, 1973.

[15] R. Cohen, Floer homotopy theory, realizing chain complexes by module spectra, and manifolds with corners, Algebraic topology, 39C59, Abel Symp., **4**, Springer, Berlin, 2009

[16] R. Cohen, The Floer homotopy type of the cotangent bundle, Pure Appl. Math. Q., **6** (2010), 391-438.

[17] R. Cohen, J. Jones, and G. Segal, Morse theory and classifying spaces, preprint.

[18] R. Cohen, J. Jones, and G. Segal, Floer's infinite dimensional Morse theory and homotopy theory, The Floer Memorial Volume, 297-325, Progress in Mathematics, **133**, Birkhäuser Verlag, 1995.

[19] O. Cornea, Homotopical dynamics: suspension and duality, Ergod. Th. & Dynam. Sys. **20** (2002), 379-391.

[20] O. Cornea, Homotopical dynamics, II: Hopf invariants, smoothings and the Morse complex, Ann. Sci. cole Norm. Sup. (4) **35** (2002), 549-573.

[21] O. Cornea, Homotopical dynamics, III: real sigularities and Hamiltonian flows, Duke Math. J., **109** (2001), 183C204

[22] O. Cornea, Homotopical dynamics, IV: Hopf invariants and Hamiltonian flows, Comm. Pure Appl. Math. **55** (2002), 1033-1088.

[23] O. Cornea, New obstructions to the thickening of CW-complexes, Proc. Amer. Math. Soc. **132** (2004), 2769-2781.

[24] A. Douady, Variétés à bord anguleux et voisinages tubulaires, Séminaire Henri Cartan, **14** (1961-1962), 1-11.

[25] A. Floer, Morse theory for Lagranigian intersections, J. Differential Geom. **28** (1988), 513-547.

[26] A. Floer, Symplectic fixed points and holomorphic spheres, Comm. Math. Phys. **120** (1989), 575-611.

[27] A. Floer and H. Hofer, Coherent orientations for periodic orbit problems in symplectic geometry, Math. Z. **212** (1993), 13-38.

[28] J. Franks, Morse-Smale flows and homotopy theory, Topology, **18** (1979), 199-215.

[29] M. Freedman, The topology of four-dimensional manifolds, J. Differential Geom., **17** (1982), 357-453.

[30] M. Irwin, On the stable manifold theorem, Bull. London Math. Soc. **2** (1970), 196-198.

[31] M. Irwin, On the smoothness of the composition map, Quart. J. Math. Oxford Ser. (2), **23** (1972), 113-133.

[32] K. Jänich, On the classification of $O(n)$-manifolds, Math. Annalen, **176** (1968), 53-76.

[33] G. Kalmbach, Deformation retracts and weak deformation retracts of noncompact manifolds, Proc. Amer. Math. Soc. **20** (1969), 539-544.

[34] G. Kalmbach, On some results in Morse theory. Canad. J. Math. **27** (1975), 88-105.

[35] F. Latour, Existence de 1-formes fermées non singulières dans une classe de cohomologie de de Rham, Publications mathématiques de l'I.H.É.S., **80** (1994), 135-194.

[36] F. Laudenbach, On the Thom-Smale complex, Appendix to [12].

[37] J. Milnor, Morse Theory, Princeton University Press, 1963.

[38] J. Milnor, Lectures on the h-cobordism theorem, Princeton University Press, 1965.

[39] J. Milnor and J. Stasheff, Characteristic Classes, Princeton University Press, 1974.

[40] M. Morse, Relations between the critical points of a real function of n independent variables, Trans. Amer. Math. Soc., **27** (1925), 345-396.

[41] M. Morse, The foundations of a theory in the calculus of variations in the large, Trans. Amer. Math. Soc., **30** (1928), 213-274.

[42] S. Newhouse and M. Peixoto, There is a simple arc joining any two Morse-Smale flows, Astérisque, **31** (1976), 16-41.

[43] Z. Nitecki and M. Shub, Filtrations, decompositions, and explosions, Amer. J. Math., **97** (1975), 1029-1047.

[44] R. Palais, Morse theory on Hilbert manifolds, Topology, **2** (1963), 299-340.

[45] R. Palais, Lectures on the Differential Topology of Infinite Dimensional Manifolds, Waltham, Mass., 1966.

[46] R. Palais and S. Smale, A generalized Morse theory, Bull. Amer. Math. Soc., **70** (1964), 165-172.

[47] R. Palais and C. Terng, Critical Point Theory and Submanifold Geometry, Lecture Notes in Mathematics, **1353**, Springer-Verlag, Berlin, 1988.

[48] J. Palis, On the local structure of hyperbolic points in Banach spaces, An. Acad. Brasil. Ciênc., **40** (1968), 263-266.

[49] J. Palis, On Morse-Smale dynamical systems, Topology, **8** (1968), 385-404.

[50] J. Palis and W. de Melo, Geometric Theorey of Dynamical Systems, Springer-Verlag, 1982.

[51] J. Palis and S. Smale, Structural stability theorems, 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), 223-231.

[52] X. Peng, On the Morse complex, Math. Z., **187** (1984), 89-96.

[53] C. Pugh, On a theorem of P. Hartman, Amer. J. Math., **91** (1969), 363-367.

[54] D. Salamon, Lectures on Floer homology. Symplectic geometry and topology (Park City, UT, 1997), 143-229, IAS/Park City Math. Ser., 7, Amer. Math. Soc., Providence, RI, 1999.

[55] M. Schwarz, Morse Homology, Progress in Mathematics, **111**, Birkhäuser Verlag, 1993.

[56] S. Smale, On gradient dynamical systems, Ann. of Math., **74** (1961), 199-206.

[57] S. Smale, On dynamical systems, Bol. Soc. Mat. Mexicana (2), **5** (1960), 195-198.

[58] S. Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc., **73** (1967), 747-817.

[59] R. Thom, Sur une partition en cellules associée à une fonction sur une variété, C.R. Acad. Sci. Paris, **228** (1949), 973-975.

[60] C. Viterbo, The cup-product on the Thom-Smale-Witten complex, and Floer cohomology, The Floer Memorial Volume, 609-625, Progress in Mathematics, **133**, Birkhäuser Verlag, 1995.

[61] W. Walter, Ordinary Differential Equations, Graduate Text in Mathematics 182, Springer-Verlag, 1998.

[62] E. Witten, Supersymmetry and Morse theory, J. Differential Geom., **17** (1982), 661-692.

[63] W. Zhang, Lectures on Chern-Weil Theory and Witten Deformations, Nankai Tracts in Mathematics, **4**, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

ABSTRACT

MODULI SPACES AND CW STRUCTURES ARISING FROM MORSE THEORY

by

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August 2011

Advisor: Professor John R. Klein

Major: Mathematics

Degree: Doctor of Philosophy

In this dissertation, we study the moduli spaces and CW Structures arising from Morse theory.

Suppose M is a smooth manifold and f is a Morse function on it. We consider the negative gradient flow of f. Suppose the flow satisfies transversality. This naturally defines the moduli spaces of flow lines and gives a stratification of M by its unstable manifolds. The gluing of broken flow lines can also be constructed.

We prove that, under certain assumptions, these moduli spaces can be compactified and the compactified spaces are smooth manifolds with corners. Moreover, these compactified manifolds satisfy certain orientation formulas. We also prove that the stratification of M is actually a CW decomposition of M with explicit characteristic maps, which has good properties. Finally, we show that the associativity of gluing of broken flow lines exclusively follows from the compatibility of the manifold structures of the compactified moduli spaces, which establishes the associativity of gluing in certain cases.

In order to obtain the above results, we also prove some results on the dynamical aspects of negative gradient flows, which may be of independent interest.

AUTOBIOGRAPHICAL STATEMENT

Lizhen Qin was born on September 17, 1981 in China. In 2002, he received a B.S. degree in mathematics at Nanjing Normal University in China. In 2007, he came to Wayne State University in Detroit, Michigan for his Ph.D. in mathematics. He graduated in August 2011.