Wayne State University
DigitalCommons@WayneState

Wayne State University Dissertations

1-1-2011

Asymptotic expansions and stability of hybrid systems with two-time scales

Dung Tien Nguyen
Wayne State University,

Follow this and additional works at: http://digitalcommons.wayne.edu/oa_dissertations

Recommended Citation
Nguyen, Dung Tien, "Asymptotic expansions and stability of hybrid systems with two-time scales" (2011). *Wayne State University Dissertations.* Paper 324.

This Open Access Dissertation is brought to you for free and open access by DigitalCommons@WayneState. It has been accepted for inclusion in Wayne State University Dissertations by an authorized administrator of DigitalCommons@WayneState.

ASYMPTOTIC EXPANSIONS AND STABILITY OF HYBRID SYSTEMS WITH TWO-TIME SCALES

by

DUNG TIEN NGUYEN

DISSERTATION

Submitted to the Graduate School,

of Wayne State University,

Detroit, Michigan

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

2011

MAJOR: MATHEMATICS

Approved by:

Advisor Date

DEDICATION

This dissertation is dedicated to my parents and my wife, and in memory of my beloved grandmothers.

ACKNOWLEDGEMENTS

It is a great pleasure to acknowledge to those whose help and support have made this dissertation possible. First and foremost, I would like to express deep gratitude and appreciation to my advisor, Professor George Yin, who introduced hybrid systems and stochastic processes to me, for kindly offering me the opportunity and the privilege to do research with him, and for his wonderful supervision, inspiration and thoughtfulness he showed me over the years. I very much appreciate his ways of supporting, building and developing my self-confidence and critical thinking skills in science. This dissertation, which was partially supported by a Research Assistantship directed by Professor George Yin, would not have been completed without his excellent guidance and endless support. I am truly indebted to him for his great influence on the beginning of my career as a scientist.

For this dissertation, I am happy to thank my committee members: Professor Guozhen Lu, Professor Pei-Yong Wang, and Professor Feng Lin for their time and insightful comments.

It is an honor for me to show my respect and gratefulness to Professor Rafail Khasminskii, Professor Alex Korostelev, and Professor Pao Liu Chow whose interesting lectures and discussions explore and solidify my knowledge in Theory of Probability and Stochastic Processes. Also, I have benefited from cooperation with Professor Xuerong Mao and Professor Chenggui Yuan, whose useful reviews and discussions are much appreciated, on a project which is now chapter 4.

I would like to give my thanks to Professor Le Yi Wang, Ms. Mary Klamo, Ms. Patricia Bonesteel, Mr. Donald Sherry, Professor Daniel Drucker, Professor John Klein, Professor John Breckenridge, Mr. James Veneri, and especially, Ms. Tiana Fluker for their very kind and quick support in many ways during the period of my graduate study. My thanks also go to Dr. Son Nguyen who have been sharing many interesting scientific discussions and a great friendship with me. Besides, I want to express my acknowledgement to the entire Department of Mathematics for their hospitality and services. I really enjoy active academic activities and friendly atmosphere at the department.

Besides, I am much obliged Wayne State University for awarding me the assistantship which allowed me to follow its Ph.D. program. Regarding to this scholarship, I am grateful to Professor Duong Minh Duc, Professor Dang Duc Trong, Profesor Nguyen Thanh Long, and Professor Dinh Ngoc Thanh for their critical comments and recommendations. Especially, I owe many thanks and much appreciation to Professor Dang Dinh Ang for his profound advices, enthusiasm and total support.

I am truly thankful to my colleagues at Department of Applied Science, University of Technology in Ho Chi Minh City and my friends, who always trust, encourage and support me whenever I need them.

Lastly, I would like to express my thankfulness to my beloved family for their everlasting and unconditional love and support.

TABLE OF CONTENTS

Chapter 1

Introduction

Considering hybrid stochastic systems of switching diffusion type, this dissertation develops asymptotic properties of solutions for systems of Kolmogorov's backward equations and stability analysis of singular systems with switching. By hybrid systems we mean such dynamical systems in which continuous dynamics and discrete events coexist. Basically, the discrete events can change value only through discrete "jump" while continuous processes dynamically evolve according to a differential equation. Hybrid systems provide a convenient framework for modeling systems in many emerging applications arising in physical sciences, biological sciences, engineering, and finance. In the past 15 years, there has been much effort in understanding such systems. In fact, the coexistence of continuous and discrete events, in turn, introduces new challenges for modeling, analysis, and computation. A new trend is to use a continuous-time Markov chain to represent the discrete events.

Our analysis is partly based on Markov processes. The theory of Markov chains can be traced back to the work of Markov, who introduced the concept in 1907. Kolmogorov systematized the theory in the early 1930s. It has experienced significant advances in the last few decades. To mention some of the important contributions, we mention the work of Doeblin, Doob, Levy, Chung, Dynkyn, and so on. Especially, fundamental work on continuous-time Markov chains was done by Doob in the 1940s and Levy in the 1950s. A new modern treatment of the theory is in Ethier and Kurtz [14].

The premise of the model can be seen from the following example. Many important movements in economy arise from discrete events. For example, a nation's economy sometimes appear quite calm and at other instances are rather volatile. To describe how this volatility changes over time is by far important. It is easily seen that monetary, fiscal, or income policies, often change in a way referred to as shocks in economics. These shifts cannot be observed directly. These discrete events are often governed by hidden random processes. Since late 80's, increasing interests on using Markov-based models in economics have been shown. Although most of these efforts are devoted to time series analysis (see Brunner [6], Cai [7], Hamilton and Susmel [18], Hansen [19, 20] and the references therein), it is conceivable that the use of Markov-based models will play a more and more prominent role in the future. Similar to the consideration of stock market, a continuous-time Markov chain can be used to formulate the trend of the economy. For example, suppose that the economy has two possible "states," fast growth phase (denoted by 2) and slow growth phase (denoted by 1). At any given time t, the economy will be in either the fast growth state or the slow growth state governed by the outcome of a Markov chain. Similarly, consider another example with the use of unemployment data, the economy may be said to be in state

1 if the unemployment rate is rising and in state 2 if the unemployment rate is falling. Corresponding to the two states, either the economic growth or the unemployment rates, the regimes or configurations of system differ resulting in different coefficients of the model. This then leads to a hybrid or switching model modulated by a Markov chain with finite state space.

In addition, another challenge arises from state space of discrete events. Frequently, dynamic systems in the real world are very large and complex. For example, in a multi-sector economy, it is likely that the state space of $\alpha(\cdot)$ is large. The large number of states of the underlying chain gives a detailed representation of the position of the economy. Mathematically, to model complex world scenarios, the state space of the Markov chain is often large. This causes the underlying computational tasks infeasible. For instance, treating controlled linear quadratic systems that are modulated by a continuous-time Markov chain, one needs to solve a system of Riccati equations in lieu of a single Riccati equation. The large-scale nature of the Markov chain often makes the amount of computation insurmountable. Reducing computational complexity becomes an important task. In this dissertation, we use two-time scale approach to overcome this challenge.

It has long been recognized that in large-scale systems, not all components or subsystems evolve at the same speed. Some of them vary rapidly, whereas others change slowly. Using the contrast of different rates of variations, one may introduce two-time scales. The main premise is: One may split a large state space of the underlying Markov chain into smaller subspaces so that within each subspace, the states switch back and forth at essentially the same rate. It would be ideal, if we could completely separate these subspaces. However, this cannot be done in general. The state space cannot be separated as isolated subspaces. They can only be split to weakly coupled subspaces, termed as nearly completely decomposable spaces. We may use the different rates of change to aggregate the states into clusters or subspaces so that within each subspace, the process changes relatively frequently, whereas from one subspace to another, the transitions take place relatively infrequently. Using a representative in each subspace to represent all of its states (or aggregating all the states in the subspace into one super state), we obtain an aggregated process. Using aggregation, the effective state space becomes a substantial reduction of complexity can be achieved. To formulate such problems, we introduce a small parameter into the system resulting in two distinct time scales, the normal time and the fast time.

In the literature, there have been much effort devoted to singular perturbation methods and their applications. Aggregation in dynamic systems were considered in Simon and Ando [47] (see also Courtois [12]), in which the term nearly completely decomposability was coined. In Phillips and Kokotovic [44], singularly perturbed Markov chains were examined thoroughly. General discussion on singular perturbation techniques can be found in Bogoliubov and Mitropolski [3] and many other references. Singular perturbation theory has a wide range of applications. In Sethi and Zhang [45], Sethi et al. [46], the authors made effective use of hierarchical structures of Markovian systems in production planning and manufacturing. For applications in chemical physics, we refer the reader to Kampen [25] among others. Related applications can be found in Kokotovic et al. [32], Phillips and Kokotovic [44], and references therein. Recent works on two-time-scale modeling using diffusions could be found in Fouque et al. [17], where stochastic volatility was modeled by use of the fast-slow diffusions. An up-to-date treatment of switching diffusions can be found in Mao and Yuan [37], Yin and Zhu [53], and references therein. Two-time-scale expansion has emerging applications in communication theory (see Tse et al. [49]), physics, and so on.

Owing to their applications across different disciplines, asymptotic properties and stability of hybrid systems have drawn much attention. We often need to study Kolmogorov forward equations (KFEs) and Kolmogorov backward equations (KBEs), which usually describe the density of processes and the expected cost functions, respectively. In literature, weak convergence methods were used in Anisimov [1], Khasminskii [26], Pardoux and Veretennikov [41, 42, 43], Skorokhod [48] among others. Asymptotic expansions of solutions of forward equations were presented in Khasminskii et al. [30, 31], Yin and Zhang [50], in which matched asymptotic expansions were constructed to approximate the solutions. The problems of systems of forward equations for switching diffusions have been worked out in Il'in et al. [22, 23]. In many applications, instead of treating the forward equations, we need to deal with the backward systems. The study of certain asymptotic properties of the backward equations was carried out in Matkowsky and Schuss [38], Papanicolaou [39, 40] for small perturbation with the main focus on the leading term of the asymptotic expansions. Most recent results dealing with singularly perturbed backward equations for diffusions can be found in Khasminskii and Yin [29] where the fast diffusion was considered.

The second part of the dissertation is devoted to stability. It is concerned with regular systems and singular systems. Some of the recent effort in stability of jump systems can be found in Feng et al. [16], Ji and Chizeck [24], Mao [36] (see also Khasminskii [27], Kushner [33], Mao [35], and the references therein for general discussion on stochastic stability). Linear systems were treated in Feng et al. [16], Ji and Chizeck [24], whereas nonlinear systems were dealt with in Mao [36]. Stability of hybrid dynamic systems containing singularly perturbed random processes was studied in Badowski and Yin [2]. On the other hand, singular systems, which have many synonyms such as descriptor systems, generalized systems, and implicit systems, are featured in differential-algebraic equations (DAEs). They arise in various applications in physical sciences, engineering, and economic systems. Owing to their importance, such systems have been studied extensively and used widely in control and optimization tasks. For some recent literature, we refer the reader to Campbell [8, 9], Cheng et al. [10], Dai [13], Lewis [34] among others. While the references mentioned above are all concerned with deterministic systems, recent works also include formulation, analysis, and computation involving stochastic systems; see Boukas [4], Boukas et al.

[5], Huang and Mao [21], Yin and Zhang [51], among others.

The rest of this dissertation begins with a review of hybrid systems and the regime-switching models. Chapter 3 is concerned with the construction of asymptotic expansions of solutions of systems of Kolmogorov backward equations. We treat Kolmogorov backward equations with terminal value conditions; both fast switching and rapid diffusion are considered. In addition, the corresponding errors bound is obtained. In the second part of the dissertation, we study the stability of singular jump-linear systems with a large state space. In chapter 4, we established sufficient conditions for stability of solutions of singular linear hybrid systems. Using the limit of the system as a guide, we employ perturbed Liapunov function methods to show that if the limit system is stable so is the original system in a suitable sense for ε small enough. A few further remarks are provided in Chapter 5.

Chapter 2

Preliminaries

This chapter is devoted to certain background materials used in the rest of this dissertation. In what follows, we focus on Markov chains in Section 2.1, irreducibility and quasi-stationary distribution in Section 2.2, and switching-diffusion processes in Section 2.3.

2.1 Markov Chains

Throughout the dissertation, we denote $\mathcal{M} = \{1, 2, \cdots, m\}$. For any matrix A, we use A' to represent its transpose. A jump process is a stochastic process with right continuous and piecewise constant sample paths.

Definition 2.1.1. *Let $\alpha(\cdot) = \{\alpha(t) : t \geq 0\}$ be a jump process defined on $(\Omega, \mathcal{F}, P)$ taking values in $\mathcal{M}$.Then $\alpha(\cdot)$ is a Markov chain with state space if*

$$P(\alpha(t) = i | \alpha(r) : r \leq s) = P(\alpha(t) = i | \alpha(s))$$

for all $0 \leq s < t$ and $i \in \mathcal{M}$.

For any $i, j \in \mathcal{M}$ and $t \geq t \geq s \geq 0$, denote $P(t, s)$ the transition matrix $(p_{ij}(t, s))$ of the Markov chain $\alpha(\cdot)$, where $p_{ij}(t, s) = P(\alpha(t) = j | \alpha(s) = i)$. If this transition probability depends only on $(t-s)$, then $\alpha(\cdot)$ is called stationary. Otherwise, it is non-stationary.

Definition 2.1.2. *The matrix $Q(t) = (q_{ij}(t))$ for $t \geq 0$ is said to satisfy q-property if for all $i, j \in \mathcal{M}$ and $t \geq 0$, $q_{ii}(t) = -\sum_{k \neq i} q_{ik}(t)$ and $q_{ij}(t)$ is Borel measurable, uniformly bounded, and positive.*

Any matrix that satisfies q-property may be called a generator. Let $Q(x,t)$ be an x-dependant generator; that is, $Q(x,\cdot)$ is a generator for each x. From now on, for $i \in \mathcal{M}$ and an appropriate function u on $\mathcal{M}$, we denote

$$Q(x,t)u(x,\cdot)(i) = \sum_{j \in \mathcal{M}} q_{ij}(x)u(x,j) = \sum_{j \in \mathcal{M}, j \neq i} q_{ij}(x)\left(u(x,j) - u(x,i)\right)$$

2.2 Irreducibility and quasi-stationary distribution

Definition 2.2.1. *(irreducibility)*

1. *A generator $Q(t)$ is said to be weakly irreducible if the system of equations*

$$\nu(t)Q(t) = 0, \qquad \sum_{i}^{m} \nu_i(t) = 1 \tag{2.2.1}$$

   has a unique solution and all coordinates of this solution is non-negative.

2. *A generator $Q(t)$ is said to be strongly irreducible, or simply irreducible, if all coordinate of the unique solution of (2.2.1) is positive.*

For example, $Q_1 = \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$ is weakly irreducible, but it is not irreducible, whereas $Q_2 = \begin{pmatrix} -2 & 2 \\ 4 & -4 \end{pmatrix}$ is irreducible.

Definition 2.2.2. *For $t \geq 0$, $\nu(t)$ is termed a quasi-stationary distribution if it is the unique solution of (2.2.1) with non-negative coordinates.*

2.3 Switching-diffusion processes

Let $B(\cdot)$ be an $\mathbb{R}^d$-valued standard Brownian motion defined in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. For suitable functions $b(\cdot,\cdot,\cdot)$ and $\sigma(\cdot,\cdot,\cdot)$, the two-component $(X(\cdot), \alpha(\cdot))$ is called a switching diffusion or a regime-switching diffusion if the continuous dynamics satisfies stochastic differential equations

$$dX(t) = b(X(t), \alpha(t), t)dt + \sigma(X(t), \alpha(t), t)dB(t), \qquad X(0) = x, \alpha(0) = \iota, \quad (2.3.1)$$

and the pure jump process $\alpha(\cdot)$ satisfies the transition law

$$P(\alpha(t+\Delta) = j | \alpha(t) = i, X(s), \alpha(s), s \leq t) = q_{ij}(X(t))\Delta + o(\Delta). \qquad (2.3.2)$$

Associated with (2.3.1) and (2.3.2), there is an operator $\mathcal{L}$ defined by

$$
\begin{aligned}
\mathfrak{L}f(x,\iota,t) &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}E\Big([f(X(t+\Delta t),\alpha(t+\Delta t),t+\Delta t)-f(x,\iota,t)]\Big| \\
&\qquad X(t)=x,\alpha(t)=\iota\} \\
&= \frac{\partial f(x,\iota,t)}{\partial t}+\nabla f(x,\iota,t)b(x,\iota,t)+\mathrm{Tr}(\nabla^2 f(x,\iota,t)A(x,\iota,t)) \\
&\qquad +Q(x,t)f(x,\cdot,t)(\iota) \qquad\qquad (2.3.3) \\
&= \frac{\partial f(x,\iota,t)}{\partial t}+\sum_{i=1}^{m}b_i(x,\iota,t)\frac{\partial f(x,\iota,t)}{\partial x_i} \\
&\qquad +\frac{1}{2}\sum_{i,j=1}^{m}a_{ij}(x,\iota,t)\frac{\partial^2 f(x,\iota)}{\partial x_i\partial x_j}+Q(x,t)f(x,\cdot,t)(\iota),
\end{aligned}
$$

where ∇ and ∇^2 denote the gradient and Hessian operator respectively and $A = (a_{ij}) = \sigma\sigma'$.

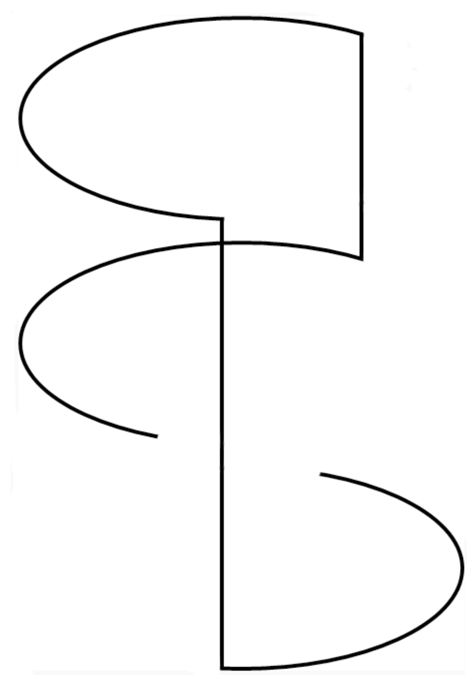


Figure 2.1: Switching-diffusion process

Figure (2.1) illustrates the motion of switching-diffusion processes. Suppose that there are m identical circles. On each circle, the underlying process evolves as a diffusion, and it jumps from one circle at position x to the same point on another circle instantaneously. The law of transitions satisfies equation (2.3.2). If it may jump to any point on another circle instantaneously, then integro-differential equations involve (see Il'in et al. [22]). More details about switching-diffusion can be found in Yin and Zhu [53, chapter 2].

Chapter 3

Asymptotic Expansions of Solutions of Systems of Kolmogorov Backward Equations for Two-Time-Scale Switching Diffusions

3.1 Introduction

This chapter is concerned with systems of Kolmogorov backward equations, which arise in switching diffusions and describe properties of the associated functionals. Is it possible to construct asymptotic expansions for hybrid switching systems? In this chapter, we answer this question by analytic methods. The structure of the chapter is as follows.

Section 3.2 presents the problem formulation. One distinct feature considered here is: We do not assume that the jump process to be Markov, but rather, the switching component has a generator $Q(x,t)$ that depends both on the continuous state x and the time t. That is, the switching process is not homogeneous in time and is coupled with the continuous dynamics. For recent results on switching diffusions, we refer the reader to Zhu and Yin [54].

In light of the different rates of changes, we treat two distinct cases, namely, the fast-varying switching in Section 3.3 and the rapidly-changing diffusion in Section 3.4. In the first case, although the discrete component lives in a finite set, the set is rather large owing to various modeling considerations of complex systems and random environments. As a result, one often has to treat a large dimensional system of partial differential equations. Aiming at reducing the computational complexity, we introduce a two-time-scale formulation in the models. We may divide the large state space of the discrete component to subspaces such that the interactions within each subspace are frequent, but the changes from one subspace to another are relatively rare. Lumping the states in each subspace into a single super-state leading to a reduced system. Corresponding to the reduced system, the total number of Kolmogorov PDEs is substantially less than that of the original one. Thus, we achieve the goal of reduction of complexity by aggregating states and by taking appropriate averaging.

In the second case, the diffusion part has two diffusion processes. One of them is fast varying, whereas the other is slowly changing. Suppose that we are interested in finding the optimal controls of a suitable cost function for this switching diffusion.

It is difficult to solve the problem directly due to the different time scales and the interactions of the continuous dynamics with that of the discrete events. Nevertheless, under suitable conditions, the fast-varying diffusion does not blowup, but it has an invariant probability measure. As a result, it may be viewed as a noise and can be averaged out with respect to the invariant measure leading to a limit system. We can proceed to use the optimal control of the limit system (assuming that it has an optimal control) to construct controls of the original system. This leads to near-optimal controls of the original systems with reduced computational effort.

For both cases, our approach is constructive. It provides a step-by-step and inductive procedure. Using averaging techniques, we derive asymptotic expansions leading to a reduction of complexity. Upon obtaining the formal asymptotic expansions, we derive the error bounds. This enables us to show that the asymptotic series so constructed are uniformly valid with desired uniform error bounds.

The novelty of our approaches includes the following aspects. (1) In both the aforementioned papers, the switching takes place in an irreducible finite set, whereas the switching is allowed to evolve in several irreducible classes in our work. (2) The solutions of the forward equations are probability measures, whereas those of the backward equations are functionals. To facilitate the analysis, new techniques are developed in our work. (3) For the forward equations, the probabilistic nature enables us to use the orthogonality (with respect to the invariant measure) directly, whereas in our work, we need to bring out certain orthogonality from tangled information. We note that the asymptotic expansions constructed will be of utility for many control and optimization problems of large-scale and complex systems.

Section 3.5 gives some illustrations and remarks.

3.2 Problem formulation

Consider a switching diffusion, a Markov process $Y(t)$ having two components, a continuous component $X(t)$ and a switching component $\alpha(t)$. The state space of the process is

$$\mathcal{X} = \mathcal{S} \times \mathcal{M}$$

where $\mathcal{S}$ is the unit circle and $\mathcal{M} = \{1, \ldots, m\}$. By identifying the endpoints 0 and 1, let $x \in [0,1]$ be the coordinates in $\mathcal{S}$. Suppose

$$b(\cdot,\cdot,\cdot) : [0,1] \times \mathcal{M} \times [0,T] \to \mathbb{R}$$

$$\sigma(\cdot,\cdot,\cdot) : [0,1] \times \mathcal{M} \times [0,T] \to \mathbb{R}.$$

The dynamics of the process can be represented by the following stochastic differential equation

$$dX(t) = b(X(t), \alpha(t), t)dt + \sigma(X(t), \alpha(t), t)dB(t), \tag{3.2.1}$$

together with transition law for the second component $\alpha(t)$

$$P(\alpha(t+\Delta) = \ell | \alpha(t) = k, X(t) = x) = q_{k\ell}(x,t)\Delta + o(\Delta), \tag{3.2.2}$$

where $o(\Delta)/\Delta \to 0$ as $\Delta \to 0$. In the above, $B(\cdot)$ is a standard real-valued Brownian motion, and $Q(x,t) = (q_{k\ell}(x,t))$ is an x and t dependent generator for the jump process satisfying for each $k, \ell \in \mathcal{M}$,

$$q_{k\ell}(x,t) \geq 0, \text{ when } k \neq \ell, \quad \text{and} \sum_{\ell \in \mathcal{M}} q_{k\ell}(x,t) = 0.$$

Associated with (3.2.1) and (3.2.2), there is an operator $\mathcal{L}$ defined by

$$\begin{aligned} &\mathcal{L}(x,t)u(x,t) = (\mathcal{L}_1(x,t)u(x,1,t), \ldots, \mathcal{L}_l(x,t)u(x,m,t))' \quad \text{where} \\ &\mathcal{L}_k(x,t)u(x,k,t) = \frac{1}{2}a(x,k,t)\frac{\partial^2}{\partial x^2}u(x,k,t) + b(x,k,t)\frac{\partial}{\partial x}u(x,k,t), \quad k \in \mathcal{M}, \end{aligned} \tag{3.2.3}$$

$u(x,k,t)$ is a real-valued function for each $k \in \mathcal{M}$,

$$u(x,t) = (u(x,1,t), \ldots, u(x,m,t))' \in \mathbb{R}^m,$$

and

$$a(x,k,t) = \sigma^2(x,k,t), \ k \in \mathcal{M}.$$

Consider the following system of equations

$$\begin{aligned} &-\frac{\partial}{\partial t}u(x,k,t) = \mathcal{L}_k(x,t)u(x,k,t) + Q(x,t)u(x,\cdot,t)(k), \ k \in \mathcal{M}, \\ &u(x,k,T) = g(x,k), \ k \in \mathcal{M}, \end{aligned} \tag{3.2.4}$$

where for each $k = 1, \ldots, m$, $u(\cdot,k,\cdot) \in C^{2,1}([0,1] \times [0,T])$ (twice continuously differentiable with respect to x and continuously differentiable with respect to t), and

$$Q(x,t)u(x,\cdot,t)(k) = \sum_{\ell \in \mathcal{M}} q_{k\ell}(x,t)u(x,\ell,t).$$

System (3.2.4) is the well-known system of Kolmogorov backward equations.

3.3 Switching Diffusions with Rapid Switching

Let $\varepsilon > 0$ be a small parameter, $\alpha^{\varepsilon}(\cdot)$ be a jump process with state space $\mathcal{M}$ and $Q(x,t)$ is of the form

$$Q^{\varepsilon}(x,t) = \frac{\widetilde{Q}(x,t)}{\varepsilon} + \widehat{Q}(x,t). \tag{3.3.1}$$

Henceforth, we relabel the states of $\mathcal{M}$ so that

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cdots \cup \mathcal{M}_l \cup \mathcal{M}_* \tag{3.3.2}$$

where

$$\mathcal{M}_\imath = \{s_{\imath 1}, \ldots, s_{\imath m_\imath}\},$$

for $\iota = 1, \ldots, l$ and

$$\mathcal{M}_* = \{s_{*1}, \ldots, s_{*m_*}\}.$$

In what follows, we will use $s_{\imath\jmath}$ with $\imath = 1, \ldots, l, *$ and $\jmath = 1, \ldots, m_\imath$ to denote a state in $\mathcal{M}$. Sometimes, when we use $k \in \mathcal{M}$, we mean k is one of the $s_{\imath\jmath}$'s. This convention will be used throughout the section.

Assume that $\widetilde{Q}(x,t)$ is of the form:

$$\widetilde{Q}(x,t) = \begin{pmatrix} \widetilde{Q}^1(x,t) & & & \\ & \ddots & & \\ & & \widetilde{Q}^l(x,t) & \\ \widetilde{Q}^1_*(x,t) & \ldots & \widetilde{Q}^l_*(x,t) & \widetilde{Q}_*(x,t) \end{pmatrix}. \tag{3.3.3}$$

Denote

$$\begin{aligned} u^{\varepsilon}(x,t) &= (u^{\varepsilon}(x, s_{\imath\jmath}, t) : \ s_{\imath\jmath} \in \mathcal{M}, \ \imath = 1, \ldots, l, *, \ \jmath = 1, \ldots, m_\imath)', \\ g(x) &= (g(x, s_{\imath\jmath}) : \ s_{\imath\jmath} \in \mathcal{M}, \ \imath = 1, \ldots, l, *, \ \jmath = 1, \ldots, m_\imath)'. \end{aligned}$$

Then the system of different equations (3.2.4) has the form

$$-\frac{\partial}{\partial t}u^{\varepsilon}(x,t) = \mathcal{L}(x,t)u^{\varepsilon}(x,t) + Q^{\varepsilon}(x,t)u^{\varepsilon}(x,t), \tag{3.3.4}$$

$$u^{\varepsilon}(x,T) = g(x). \tag{3.3.5}$$

We make the following assumptions.

(A1) For each $\imath = 1,\ldots,l$, $t \in [0,T]$, and each $x \in [0,1]$, $\widetilde{Q}^{\imath}(x,t)$ is weakly irreducible in that for any $\imath = 1,\ldots,l$ and $x \in [0,1]$,

$$\nu^{\imath}(x,t)\widetilde{Q}^{\imath}(x,t) = 0, \qquad \nu^{\imath}(x,t)\mathbb{1}_{m_\imath} = \sum_{\jmath=1}^{m_\imath} \nu^{\imath\jmath}(x,t) = 1,$$

where $\mathbb{1}_{m_\imath} = (1,\ldots,1)' \in \mathbb{R}^{m_\imath}$, has a unique solution, and the solution is termed a quasi-stationary distribution.

(A2) For some positive integer n, $\widetilde{Q}(\cdot,\cdot)$ and $\widehat{Q}(\cdot,\cdot) \in C^{2(n+2),n+2}([0,1]\times[0,T])$. That is, $\widetilde{Q}(\cdot,\cdot)$ and $\widehat{Q}(\cdot,\cdot)$ are $2(n+2)$-times continuously differentiable with respect to x and $(n+2)$-times continuously differentiable with respect to t.

(A3) For each $t \in [0,T]$ and $x \in [0,1]$, $\widetilde{Q}_*(x,t)$ is Hurwitz (i.e., all of its eigenvalues have negative real parts).

(A4) For each $k \in \mathcal{M}$, $g(\cdot,k)$ are periodic in x with period 1 and $g(\cdot,k) \in C^{2(n+2)}([0,1])$.

(A5) For each $k \in \mathcal{M}$, $a(\cdot,k,t)$, $b(\cdot,k,t)$ are periodic in x with period 1 for all $t \in [0,T]$ and $a(\cdot,k,\cdot), b(\cdot,k,\cdot) \in C^{2(n+2),n+2}([0,1]\times[0,T])$.

3.3.1 Construction of Asymptotic Expansions

For convenience, we use a stretched variable

$$\tau = \frac{T-t}{\varepsilon}, \tag{3.3.6}$$

which magnifies the details of the solution near the terminal time T. Denote

$$\begin{aligned}
\Phi_n^\varepsilon(x,t) &= (\Phi_n^\varepsilon(x,k,t):\ k = s_{\imath\jmath},\ \imath = 1,\ldots,l,*,\ \jmath = 1,\ldots,m_\imath)', \\
\Psi_n^\varepsilon(x,\tau) &= (\Psi_n^\varepsilon(x,k,t):\ k = s_{\imath\jmath},\ \imath = 1,\ldots,l,*,\ \jmath = 1,\ldots,m_\imath)'.
\end{aligned}$$

We aim to approximate the solution $u^\varepsilon(x,t)$ of (3.3.4) by

$$\begin{aligned}
&\Phi_n^\varepsilon(x,t) + \Psi_n^\varepsilon(x,\tau), \quad \text{where} \\
&\Phi_n^\varepsilon(x,t) = \sum_{j=0}^{n} \varepsilon^j \phi_j(x,t), \quad \Psi_n^\varepsilon(x,\tau) = \sum_{j=0}^{n} \varepsilon^j \psi_j(x,\tau).
\end{aligned} \tag{3.3.7}$$

In constructing the asymptotic expansions, to obtain the desired estimates, we need to compute a couple of more terms. Thus, we need to compute $\Phi_i^\varepsilon(x,t)$ for $i \le n+2$. Substituting $\Phi_i^\varepsilon(x,t)$ for $i = 0,\ldots,n+2$ into (3.2.4) and equating coefficients powers of ε^i, we obtain:

$$\begin{aligned}
&\widetilde{Q}(x,t)\phi_0(x,t) = 0, \\
&\widetilde{Q}(x,t)\phi_1(x,t) = -\frac{\partial}{\partial t}\phi_0(x,t) - (\mathcal{L} + \widehat{Q})(x,t)\phi_0(x,t), \\
&\cdots\ \cdots \\
&\widetilde{Q}(x,t)\phi_{i+1}(x,t) = -\frac{\partial}{\partial t}\phi_i(x,t) - (\mathcal{L} + \widehat{Q})(x,t)\phi_i(x,t),
\end{aligned} \tag{3.3.8}$$

for $i = 1,\ldots,n+2$, where

$$\mathcal{L}(x,t)\phi_i(x,t) = (\mathcal{L}_k(x,t)\phi_i(x,k,t) : k = s_{\imath\jmath},\ \imath = 1,\ldots,l,*,\ \jmath = 1,\ldots,m_\imath)'. \tag{3.3.9}$$

Likewise, substituting $\Psi_i(x,\tau)$ for $i \le n+2$ into (3.3.4), we obtain

$$\frac{\partial}{\partial \tau}\left(\sum_{j=0}^{i} \varepsilon^j \psi_j(x,\tau)\right) = \sum_{j=0}^{i} \varepsilon^j \Big(\widetilde{Q}(x,T-\varepsilon\tau) + \varepsilon(\mathcal{L}+\widehat{Q})(x,T-\varepsilon\tau)\Big)\psi_j(x,\tau). \tag{3.3.10}$$

For simplicity, we denote the jth-order partial derivative w.r.t. t by

$$f^{(j)}(x,t) = \frac{\partial^j f(x,t)}{\partial t^j}$$

in what follows. By means of the Taylor expansion, we have

$$\begin{aligned}
\widetilde{Q}(x,T-\varepsilon\tau) &= \sum_{j=0}^{i} \frac{\widetilde{Q}^{(j)}(x,T)}{j!}(-\varepsilon\tau)^j + \widetilde{R}_i(x,\varepsilon\tau),\\
\varepsilon(\mathcal{L}+\widehat{Q})(x,T-\varepsilon\tau) &= \sum_{j=0}^{i-1} \frac{(\mathcal{L}+\widehat{Q})^{(j)}(x,T)}{j!}\varepsilon(-\varepsilon\tau)^j + \widehat{R}_{i-1}(x,\varepsilon\tau),
\end{aligned}$$

where $\widetilde{R}_i(x,\varepsilon\tau) = O(\varepsilon^{i+1})$ and $\widehat{R}_{i-1}(x,\varepsilon\tau) = O(\varepsilon^{i+1})$ uniformly in $x \in [0,1]$ for any $\tau > 0$. Equating coefficients of powers of ε^i, for $i = 0,1,\ldots,n+2$ and using the Taylor expansions above, we obtain

$$\begin{aligned}
\frac{\partial \psi_0(x,\tau)}{\partial \tau} &= \widetilde{Q}(x,T)\psi_0(x,\tau),\\
\frac{\partial \psi_1(x,\tau)}{\partial \tau} &= \widetilde{Q}(x,T)\psi_1(x,\tau) + \Big(-\tau\widetilde{Q}^{(1)}(x,T) + (\mathcal{L}+\widehat{Q})(x,T)\Big)\psi_0(x,\tau),\\
\cdots\ \cdots&\\
\frac{\partial \psi_i(x,\tau)}{\partial \tau} &= \widetilde{Q}(x,T)\psi_i(x,\tau) + r_i(x,\tau)\\
r_i(x,\tau) &= \sum_{j=0}^{i-1}\left((-\tau)^{i-j}\frac{\widetilde{Q}^{(i-j)}(x,T)}{(i-j)!} + (-\tau)^{i-j-1}\frac{(\mathcal{L}+\widehat{Q})^{(i-j-1)}(x,T)}{(i-j-1)!}\right)\psi_j(x,\tau).
\end{aligned} \tag{3.3.11}$$

From the initial condition, we derive

$$\phi_0(x,T) + \psi_0(x,0) = g(x) \text{ and } \phi_i(x,T) + \psi_i(x,0) = 0, \text{ for } i > 0.$$

Therefore, we obtain

$$\begin{aligned}\psi_0(x,\tau) &= \exp(\widetilde{Q}(x,T)\tau)(g(x)-\phi_0(x,T)),\\ \psi_i(x,\tau) &= -\exp(\widetilde{Q}(x,T)\tau)\phi_i(x,T)+\int_0^\tau \exp(\widetilde{Q}(x,T)(\tau-s))r_i(x,s)ds, \text{ for } i>0.\end{aligned} \tag{3.3.12}$$

Denote

$$\widetilde{\mathbb{1}}(x,t)=\begin{pmatrix} \mathbb{1}_{m_1} & & \\ & \ddots & \\ & & \mathbb{1}_{m_l} \\ d^1(x,t) & \dots & d^l(x,t)\end{pmatrix}, \tag{3.3.13}$$

where $d^\imath(x,t)=-\widetilde{Q}_*^{-1}(x,t)\widetilde{Q}_*^\imath(x,t)\mathbb{1}_{m_\imath}$, for $\imath=1,\dots,l$.

In what follows, we will prove the smoothness of φ_i for $0\le i\le n+2$ and the exponential decay of ψ_i for $0\le i\le n+1$ which implies the desired error bound by Lemma 3.3.9.

Lemma 3.3.1. *The solutions of the equation $\widetilde{Q}(x,t)\phi(x,t)=0$ are given by*

$$\phi(x,t)=\widetilde{\mathbb{1}}(x,t)\beta(x,t)$$

with $\beta(x,t)=(\beta^1(x,t),\dots,\beta^l(x,t))'\in\mathbb{R}^l$. More precisely, $\phi(x,t)$ is of the partitioned form

$$\phi(x,t)=([\phi^1(x,t)]',\dots,[\phi^l(x,t)]',[\phi^*(x,t)]')',$$

such that $\phi^\imath(x,t)\in\mathbb{R}^{m_\imath\times 1}$ and $\phi^*(x,t)\in\mathbb{R}^{m_*\times 1}$ satisfy

$$\begin{aligned}\phi^\imath(x,t)&=\beta^\imath(x,t)\mathbb{1}_{m_\imath},\\ \phi^*(x,t)&=\sum_{\imath=1}^l\beta^\imath(x,t)d^\imath(x,t).\end{aligned}$$

Proof. Let $\phi(x,t) = ([\phi^1(x,t)]', \ldots, [\phi^l(x,t)]', [\phi^*(x,t)]')'$ be a solution of the above equation. Then for any $\imath = 1, \ldots, l$,

$$\begin{aligned} &\widetilde{Q}^\imath(x,t)\phi^\imath(x,t) = 0, \\ &\sum_{\imath=1}^{l} \widetilde{Q}_*^\imath(x,t)\phi^\imath(x,t) + \widetilde{Q}_*(x,t)\phi^*(x,t) = 0. \end{aligned}$$

Thus

$$\begin{aligned} \phi^\imath(x,t) &= \beta^\imath(x,t)\mathbb{1}_{m_\imath}, \\ \phi^*(x,t) &= \sum_{\imath=1}^{l} -\beta^\imath(x,t)\widetilde{Q}_*^{-1}(x,t)\widetilde{Q}_*^\imath(x,t)\mathbb{1}_{m_\imath}. \end{aligned}$$

The lemma is proved. $\square$

Denote

$$\overline{P}(x) = \widetilde{\mathbb{1}}(T)\nu(x,T) = \begin{pmatrix} \mathbb{1}_{m_1}\nu^1(x,T) & & & \\ & \ddots & & \\ & & \mathbb{1}_{m_l}\nu^l(x,T) & \\ d^1(x,T)\nu^1(x,T) & \ldots & d^l(x,T)\nu^l(x,T) & 0_{m_*\times m_*} \end{pmatrix} \tag{3.3.14}$$

with

$$\nu(x,t) = (\mathrm{diag}(\nu^1(x,t), \ldots, \nu^l(x,t), 0_{l\times m_*})) = \begin{pmatrix} \nu^1(x,t) & & & 0_{1\times m_*} \\ & \ddots & & \vdots \\ & & \nu^l(x,t) & 0_{1\times m_*} \end{pmatrix}. \tag{3.3.15}$$

Lemma 3.3.2. *For each $\imath = 1, \ldots, l$, suppose that $\widetilde{Q}^\imath(x,T)$ is weakly irreducible. Then there exist constants C and γ such that*

$$|\exp(\widetilde{Q}^\imath(x,T)\tau) - \mathbb{1}_{m_\imath}\nu^\imath(x,T)| \leq Ce^{-\gamma\tau}.$$

Proof. See Yin and Zhang [52, Lemma A.2].

Lemma 3.3.3. *There exists positive constants γ and C such that*

$$|\exp(\widetilde{Q}(x,T)\tau) - \overline{P}(x)| \leq Ce^{-\gamma\tau}, \text{ for all } \tau,$$

where $|A|$ is the matrix norm, e.g., $|A| = \|A\|_\infty$.

Proof. To prove this lemma, it suffices to show for all m-column vector z

$$\left|\left[\exp(\widetilde{Q}(x,T)\tau) - \overline{P}(x)\right] z\right| \leq Ce^{-\gamma\tau}|z|.$$

Given $z = (z^1, \ldots, z^l, z^*)' \in \mathbb{R}^{m\times 1}$, set

$$y(x,\tau) = (y^1(x,\tau), \ldots, y^l(x,\tau), y^*(x,\tau))' = \exp(\widetilde{Q}(x,T)\tau)z.$$

Then

$$\underline{\mathrm{P}}(x)z = \begin{pmatrix} \mathbb{1}_{m_1}\nu^1(x,T)z^1 \\ \vdots \\ \mathbb{1}_{m_l}\nu^1(x,T)z^l \\ \displaystyle\sum_{\imath=1}^{l} d^\imath(x,T)\nu^\imath(x,T)z^\imath \end{pmatrix}$$

and $y(x,\tau)$ is a solution to

$$\frac{dy(x,\tau)}{d\tau} = \widetilde{Q}(x,T)y(x,\tau), \qquad y(x,0) = z.$$

It follows that

$$\frac{dy^*(x,\tau)}{d\tau} = \widetilde{Q}_*(s,T)y^*(x,\tau) + \sum_{\imath=1}^{l} \widetilde{Q}_*^\imath(x,T)y^\imath(x,\tau), \qquad y^*(x,0) = z^*,$$

and for $\imath = 1, \dots, l$,

$$\frac{dy^\imath(x,\tau)}{d\tau} = \widetilde{Q}^\imath(x,T)y^\imath(x,\tau), \qquad y^\imath(x,0) = z^\imath.$$

Then

$$y^*(x,\tau) = \exp(\widetilde{Q}_*(x,T)\tau)z^* + \sum_{\imath=1}^{l}\int_0^\tau \exp(\widetilde{Q}_*(x,T)(\tau-s))\widetilde{Q}_*^\imath(x,T)y^\imath(x,s)ds,$$

and for each $\imath = 1, \dots, l$,

$$y^\imath(x,\tau) = \exp(\widetilde{Q}^\imath(x,T)\tau)z^\imath.$$

By Lemma 3.3.2,

$$|y^\imath(x,\tau) - \mathbb{1}_{m_\imath}\nu^\imath(x,T)z^\imath| = |\exp(\widetilde{Q}^\imath(x,T)\tau) - \mathbb{1}_{m_\imath}\nu^\imath(x,T)||z^\imath| \le C\,\mathrm{e}^{-\gamma\tau}\,|z|.$$

Since $\widetilde{Q}_*(x,\tau)$ is a Hurwitz matrix, we have

$$\begin{aligned}\widetilde{Q}_*^{-1}(x,T) &= -\int_0^\infty \exp(\widetilde{Q}_*(x,T)s)ds \\ &= -\int_0^\tau \exp(\widetilde{Q}_*(x,T)(\tau-s))ds - \int_\tau^\infty \exp(\widetilde{Q}_*(x,T)s)ds.\end{aligned}$$

Therefore,

$$\begin{aligned}y^*(x,\tau)-&\sum_{\imath=1}^{l} d^\imath(x,T)\nu^\imath(x,T)z^\imath \\ &= \exp(\widetilde{Q}_*(x,T)\tau)z^* + \sum_{\imath=1}^{l}\int_0^\tau \exp(\widetilde{Q}_*(x,T)(\tau-s))\widetilde{Q}_*^\imath(x,T)y^\imath(x,s)ds \\ &\quad + \sum_{\imath=1}^{l}\widetilde{Q}_*^{-1}(x,T)\widetilde{Q}_*^\imath(x,T)\mathbb{1}_{m_\imath}\nu^\imath(x,T)z^\imath \\ &= A_0 + A_\infty + \sum_{\imath=1}^{l} A_\imath,\end{aligned}$$

where

$$
\begin{aligned}
A_0 &= \exp(\widetilde{Q}_*(x,T)\tau)z^*,\\
A_\infty &= -\sum_{\imath=1}^{l}\int_\tau^\infty \exp(\widetilde{Q}_*(x,T)s)\widetilde{Q}_*^\imath(x,T)\mathbb{1}_{m_\imath}\nu^\imath(x,T)z^\imath ds\\
A_\imath &= \int_0^\tau \exp(\widetilde{Q}_*(x,T)(\tau-s))\widetilde{Q}_*^\imath(x,T)[\exp(\widetilde{Q}^\imath(x,T)\tau)-\mathbb{1}_{m_\imath}\nu^\imath(x,T)]z^\imath ds.
\end{aligned}
$$

for $\iota = 1,\ldots,l$. For the first two terms, we have

$$
\begin{aligned}
|A_0| &\le C\,\mathrm{e}^{-\gamma\tau}\,|z|\\
|A_\infty| &\le C\int_\tau^\infty \mathrm{e}^{-\gamma s}\,|z|ds = C\,\mathrm{e}^{-\gamma\tau}\,|z|.
\end{aligned}
$$

Moreover, owing to Lemma 3.3.2, for each $\imath = 1,\ldots,l$,

$$
|A_\imath| \le C|z|\int_0^\tau \mathrm{e}^{-\gamma(\tau-s)}\,\mathrm{e}^{-\gamma s}\,ds = C\tau\,\mathrm{e}^{-\gamma\tau}\,|z| \le C\,\mathrm{e}^{-\widetilde{\gamma}\tau}\,|z|
$$

for some $0 < \widetilde{\gamma} < \gamma$. These inequalities lead to the desired result. □

3.3.2 Leading Term $\phi_0(x,t)$ and Zero-order Terminal Layer Term $\psi_0(x,\tau)$

Since, in view of (3.3.8),

$$
\widetilde{Q}(x,t)\phi_0(x,t) = 0
$$

We derive from Lemma 3.3.1 that

$$
\phi_0(x,t) = \widetilde{\mathbb{1}}(x,t)\beta_0(x,t).
$$

Recall that for a suitable function $f(x,t)$, $\dot{f}(x,t) = \dfrac{\partial f(x,t)}{\partial t}$. Then

$$
\dot{\phi}_0(x,t) = \widetilde{\mathbb{1}}(x,t)\dot{\beta}_0(x,t),
$$

so

$$\widetilde{Q}(x,t)\phi_1(x,t) = -\widetilde{\mathbb{1}}(x,t)\dot{\beta}_0(x,t) - (\mathcal{L}+\widehat{Q})(x,t)(\widetilde{\mathbb{1}}(x,t)\beta_0(x,t)) \stackrel{\text{def}}{=} \widetilde{b}_0(x,t). \quad (3.3.16)$$

By definition, we have $\nu(x,t)\widetilde{Q}(x,t) = 0$ and $\nu(x,t)\widetilde{\mathbb{1}} = I_l \in \mathbb{R}^{l\times l}$, the $l \times l$ identity matrix with $\nu(x,t)$ given in (3.3.15). Multiplying both sides of equation (3.3.16) from the left by $\nu(x,t)$, we obtain

$$\dot{\beta}_0(x,t) = -\nu(x,t)(\mathcal{L}+\widehat{Q})(x,t)(\widetilde{\mathbb{1}}(x,t)\beta_0(x,t)). \quad (3.3.17)$$

In view of (3.3.12),

$$\psi_0(x,\tau) = \exp(\widetilde{Q}(x,T)\tau)(g(x) - \phi_0(x,T)). \quad (3.3.18)$$

We demand that $\psi_0(x,\tau) \to 0$ as $\tau \to \infty$. Letting $\tau \to \infty$ in (3.3.18) and noting $\exp(\widetilde{Q}(x,T)\tau) \to \overline{P}(x)$ with $\overline{P}(x)$ given in (3.3.14), we obtain

$$\overline{P}(x)\psi_0(x,0) = 0. \quad (3.3.19)$$

Multiplying both sides from the left by $\nu(x,T)$, (3.3.19) is equivalent to

$$\nu(x,T)\psi_0(x,0) = 0. \quad (3.3.20)$$

On the other hand,

$$\begin{aligned} \nu(x,T)\psi_0(x,0) &= \nu(x,T)(g(x) - \phi_0(x,T)) \\ &= \nu(x,T)g(x) - \beta_0(x,T). \end{aligned} \quad (3.3.21)$$

Thus

$$\beta_0(x,T) = \nu(x,T)g(x). \quad (3.3.22)$$

Conversely, condition (3.3.19) holds provided $\beta_0(x,T)$ satisfies (3.3.22). As a result, $\beta_0(x,t)$ can be determined from differential equation (3.3.17) and terminal condition (3.3.22) uniquely. Moreover, with this $\beta_0(x,t)$, we also have $\nu(x,t)\widetilde{b}_0(x,t) = 0$.

3.3.3 Higher-order Terms

Define $Q_v(x,t) = \begin{pmatrix} \widetilde{Q}(x,t) \\ \nu(x,t) \end{pmatrix}$. Then we have the following Lemma.

Lemma 3.3.4. *Under condition* (A1)*,* $\mathrm{rank}(Q_v(x,t)'Q_v(x,t)) = m$*.*

Proof. Let $w(x,t) \in \mathbb{R}^{m\times 1}$ be a solution of

$$Q_v(x,t)w(x,t) = 0. \tag{3.3.23}$$

Then (3.3.23) yields

$$\widetilde{Q}(x,t)w(x,t) = 0 \quad \text{and} \quad \nu(x,t)w(x,t) = 0.$$

For the first equation above, in view of Lemma 3.3.1,

$$w(x,t) = \widetilde{\mathbb{1}}(x,t)\eta(x,t).$$

Substituting this into the second equation, we obtain

$$\nu(x,t)w(x,t) = \nu(x,t)\widetilde{\mathbb{1}}(x,t)\eta(x,t) = \eta(x,t).$$

So $\eta(x,t) = 0$ and hence $w(x,t) \equiv 0$. Thus the only $w(x,t) = (w_1(x,t),\ldots,w_m(x,t))' \in \mathbb{R}^{m\times 1}$ satisfying

$$w_1(x,t)Q_v^1(x,t) + \cdots + w_m(x,t)Q_v^m(x,t) = 0$$

is 0, where $Q_v^k(x,t)$ is the kth-column of $Q_v(x,t)$ for each $k=1,\dots,m$. Thus the m columns of $Q_v(x,t)$ are linearly independent. Hence

$$\operatorname{rank}(Q_v(x,t)) = m.$$

As a result,

$$\operatorname{rank}(Q_v'(x,t)Q_v(x,t)) = \operatorname{rank}(Q_v(x,t)) = m. \qquad \square$$

To proceed to the error bound, for $i>0$, we construct $\phi_i(x,t)$ and $\psi_i(x,\tau)$ by induction. Suppose that the terms $\phi_j(x,t)$ and $\psi_j(x,\tau)$ for $j<i$ have been constructed such that $\psi_j(x,\tau)$ decays exponentially fast and $\phi_j(x,t)$ are smooth. Moreover, assume $\nu(x,t)\widetilde{b}_j(x,t)=0$ for all $j<i$. Using (3.3.8), we have

$$\widetilde{Q}(x,t)\phi_i(x,t) = -\dot{\phi}_{i-1}(x,t) - (\mathcal{L}+\widehat{Q})(x,t)\phi_{i-1}(x,t) \stackrel{\text{def}}{=} \widetilde{b}_{i-1}(x,t). \tag{3.3.24}$$

Thus, by Lemma 3.3.1, $\phi_i(x,t)$ is the sum of solutions to the homogeneous equation and a particular solution $\widehat{\phi}_i(x,t)$ of the nonhomogeneous equation. It is of the form

$$\phi_i(x,t) = \widetilde{\mathbb{1}}(x,t)\beta_i(x,t) + \widehat{\phi}_i(x,t) \tag{3.3.25}$$

such that

$$\widetilde{Q}(x,t)\widehat{\phi}_i(x,t) = \widetilde{b}_{i-1}(x,t).$$

The Fredholm alternative leads to $\nu(x,t)\widetilde{b}_{i-1}(x,t)=0$. Denote $Q_v(x,t)$ as defined in Lemma 3.3.4 and $\widehat{b}_{i-1}(x,t) = \begin{pmatrix} \widetilde{b}_{i-1}(x,t) \\ 0 \end{pmatrix}$. We can find a unique solution $\widehat{\phi}_i(x,t)$ of (3.3.24) such that $\widetilde{Q}(x,t)\widehat{\phi}_i(x,t) = \widetilde{b}_{i-1}(x,t)$ and $\nu(x,t)$ is orthogonal to $\widehat{\phi}_i(x,t)$.

That is,

$$Q_v(x,t)\widehat{\phi}_i(x,t) = \widehat{b}_{i-1}(x,t).$$

Lemma 3.3.4 implies that the particular solution is uniquely determined by

$$\widehat{\phi}_i(x,t) = (Q_v(x,t)'Q_v(x,t))^{-1}Q_v(x,t)'\widehat{b}_{i-1}(x,t). \tag{3.3.26}$$

On the other hand

$$\widetilde{Q}(x,t)\phi_{i+1}(x,t) = -\dot{\phi}_i(x,t) - (\mathcal{L}+\widehat{Q})(x,t)\phi_i(x,t) \stackrel{\text{def}}{=} \widetilde{b}_i(x,t).$$

Multiplying both sides by $\nu(x,t)$ from the left and noting (3.3.25), we deduce

$$\dot{\beta}_i(x,t) = -\nu(x,t)\dot{\widehat{\phi}}_i(x,t) - \nu(x,t)(\mathcal{L}+\widehat{Q})(x,t)\widehat{\phi}_i(x,t) - \nu(x,t)(\mathcal{L}+\widehat{Q})(x,t)\Big(\widetilde{\mathbb{1}}(x,t)\beta_i(x,t)\Big). \tag{3.3.27}$$

Equation (3.3.27) is uniquely solvable if the terminal condition is specified. We need to use the terminal layer term to determine the terminal condition. In view of (3.3.12),

$$\psi_i(x,\tau) = -\exp(\widetilde{Q}(x,T)\tau)\phi_i(x,T) + \int_0^\tau \exp(\widetilde{Q}(x,T)(\tau-s))r_i(x,s)ds. \tag{3.3.28}$$

We demand that $\psi_i(x,\tau) \to 0$ as $\tau \to \infty$. Letting $\tau \to \infty$ in (3.3.28) and noting that

$$\exp(\widetilde{Q}(x,T)\tau) \to \overline{P}(x)$$

with $\overline{P}(x)$ given in (3.3.14) and that $r_i(x,t)$ decays exponentially fast, we obtain

$$\overline{P}(x)\psi_i(x,0) + \int_0^\infty \overline{P}(x)r_i(x,s)ds = 0. \tag{3.3.29}$$

By multiplying both sides from the left by $\nu(x,T)$, the above equation is equivalent to

$$\nu(x,T)\psi_i(x,0) + \int_0^\infty \nu(x,T)r_i(x,s)ds = 0. \tag{3.3.30}$$

We have

$$
\begin{aligned}
&\nu(x,T)\psi_i(x,0) + \int_0^\infty \nu(x,T)r_i(x,s)ds \\
&\quad = -\nu(x,T)\phi_i(x,T) + \int_0^\infty \nu(x,T)r_i(x,s)ds \\
&\quad = -\beta_i(x,T) - \nu(x,T)\widehat{\phi}_i(x,T) + \int_0^\infty \nu(x,T)r_i(s)ds \\
&\quad = -\beta_i(x,T) + \int_0^\infty \nu(x,T)r_i(x,s)ds.
\end{aligned}
\tag{3.3.31}
$$

Note that the integral involving $r_i(x,s)$ is well defined since $|r_i(x,s)| \le C\,\mathrm{e}^{-\gamma s}$ by induction hypothesis. By virtue of (3.3.29) and (3.3.31), we obtain

$$
\beta_i(x,T) = \int_0^\infty \nu(x,T)r_i(x,s)ds. \tag{3.3.32}
$$

Conversely, when $\beta_i(x,T)$ satisfies (3.3.27), condition (3.3.29) holds as desired. Then

$$
\phi_i(x,t) = \widetilde{\mathbb{1}}(x,t)\beta_i(x,t) + \widehat{\phi}_i(x,t) = \widetilde{\mathbb{1}}(x,t)\beta_i(x,t) + (Q_v(x,t)'Q_v(x,t))^{-1}Q_v(x,t)'\widehat{b}_{i-1} \tag{3.3.33}
$$

with $\beta_i(x,t)$ uniquely determined by the differential equations (3.3.27) and the terminal condition (3.3.32). In addition, $\nu(x,t)\widetilde{b}_i(x,t) = 0$. Moreover, by the construction, it is readily seen that $\psi_i(x,\tau)$ decays exponentially fast.

Proposition 3.3.5. *$\phi_i \in C^{2(n+2-i),n+2-i}([0,1]\times[0,T])$ for any $i = 0,\dots,n+2$.*

Proof. We prove this by induction. First, denote $Q_a(x,t) = (\widetilde{Q}(x,t)\ \widetilde{\mathbb{1}}(x,t))$. Then

$$
\nu(x,t)Q_a = (0_{l\times m}\ I_l).
$$

Moreover, using irreducibility of $\widetilde{Q}^\imath(x,t)$ for $\imath = 1,\dots,l$, we can follow the proof of Lemma 3.3.4 to prove that $\mathrm{rank}(Q_a'(x,t)Q_a(x,t)) = m$. So

$$
\nu(x,t) = (0_{l\times m}\ I_l)Q_a'(x,t)(Q_a(x,t)Q_a'(x,t))^{-1}.
$$

Thus $\nu(\cdot,\cdot) \in C^{2(n+2),n+2}([0,1]\times[0,T])$ and $\widetilde{\mathbb{1}}(\cdot,\cdot) \in C^{2(n+2),n+2}([0,1]\times[0,T])$. Therefore, (3.3.17) implies that $\beta_0(\cdot,\cdot) \in C^{2(n+2),n+2}([0,1]\times[0,T])$. So $\phi_0 \in C^{2(n+2),n+2}([0,1]\times[0,T])$. Assume that $\phi_j \in C^{2(n+2-j),n+2-j}([0,1]\times[0,T])$ for any $j<i$. In view of (3.3.24), we deduce $\widehat{b}_{i-1} \in C^{2(n+2-i),n+2-i}([0,1]\times[0,T])$. Then we derive from (3.3.26) and (3.3.27) that $\widehat{\phi}_i \in C^{2(n+2-i),n+2-i}([0,1]\times[0,T])$ and $\beta_i \in C^{2(n+2-i),n+2-i}([0,1]\times[0,T])$. Thus (3.3.33) implies $\phi_i \in C^{2(n+2-i),n+2-i}([0,1]\times[0,T])$. $\square$

Lemma 3.3.6. *For a fixed integer i and an integer h satisfying $0 \le h \le 2(n+2-i)$, put*

$$w_i^h(x,\tau) = \frac{\partial^h \psi_i(x,\tau)}{\partial x^h}.$$

Assume for any τ, x,

$$|\psi_i(x,\tau)| \le C\,\mathrm{e}^{-\gamma\tau}$$

$$\max_{h=0,\dots,2(n+2-i)} \left|\frac{\partial^h r_i(x,\tau)}{\partial x^h}\right| \le C\,\mathrm{e}^{-\gamma\tau}\,.$$

Then for any τ, x,

$$\max_{h=0,\dots,2(n+2-i)} \left|w_i^h(x,\tau)\right| \le C\,\mathrm{e}^{-\gamma\tau}\,.$$

Proof. First,

$$|w_i^0(x,\tau)| = |\psi_i(x,\tau)| \le C\,\mathrm{e}^{-\gamma\tau}\,.$$

Suppose for any $h_1 < h$,

$$\left|w_i^{h_1}(x,\tau)\right| \le C\,\mathrm{e}^{-\gamma\tau}, \tag{3.3.34}$$

Then (3.3.11) implies

$$\begin{aligned} \frac{\partial w_i^h(x,\tau)}{\partial \tau} &= \widetilde{Q}(x,T)w_i^h(x,\tau) + \sum_{h_1=0}^{h-1} \frac{h!}{h_1!(h-h_1)!} \frac{\partial^{h-h_1}\widetilde{Q}(x,T)}{\partial x^{h-h_1}} w_i^{h_1}(x,\tau) + \frac{\partial^h r_i(x,\tau)}{\partial x^h} \\ w_i^h(x,0) &= \frac{\partial^h \psi_i(x,0)}{\partial x^h}. \end{aligned} \tag{3.3.35}$$

It follows that

$$\begin{aligned} w_i^h(x,\tau) &= \exp(\widetilde{Q}(x,T)\tau)w_i^h(x,0) \\ &\quad + \sum_{h_1=0}^{h-1} \int_0^\tau \frac{h!}{h_1!(h-h_1)!} \exp(\widetilde{Q}(x,T)(\tau-s)) \frac{\partial^{h-h_1}\widetilde{Q}(x,T)}{\partial x^{h-h_1}} w_i^{h_1}(x,s)ds \\ &\quad + \int_0^\tau \exp(\widetilde{Q}(x,T)(\tau-s)) \frac{\partial^h r_i(x,s)}{\partial x^h} ds. \end{aligned} \tag{3.3.36}$$

Define

$$\begin{aligned} \widetilde{w}_i^h(x) \stackrel{\text{def}}{=} \nu(x,T)w_i^h(x,0) &+ \sum_{h_1=0}^{h-1} \int_0^\infty \frac{h!}{h_1!(h-h_1)!} \nu(x,T) \frac{\partial^{h-h_1}\widetilde{Q}(x,T)}{\partial x^{h-h_1}} w_i^{h_1}(x,s)ds \\ &+ \int_0^\infty \nu(x,T) \frac{\partial^h r_i(x,s)}{\partial x^h} ds. \end{aligned}$$

We claim that

$$\widetilde{w}_i^h(x) = 0. \tag{3.3.37}$$

Note that (3.3.36) implies

$$
\begin{aligned}
|w_i^h(x,\tau)| \le & \,|\exp(\widetilde{Q}(x,T)\tau) - \overline{P}(x)||w_i^h(x,0)| \\
& + \sum_{h_1=0}^{h-1} \int_0^\tau \frac{h!}{h_1!(h-h_1)!} \left|\exp(\widetilde{Q}(x,T)(\tau-s)) - \overline{P}(x)\right| \left| \frac{\partial^{h-h_1}\widetilde{Q}(x,T)}{\partial x^{h-h_1}} \right| \\
& \times \left|w_i^{h_1}(x,s)\right| ds \\
& + \sum_{h_1=0}^{h-1} \int_\tau^\infty \frac{h!}{h_1!(h-h_1)!} \left|\overline{P}(x)\right| \left| \frac{\partial^{h-h_1}\widetilde{Q}(x,T)}{\partial x^{h-h_1}} \right| \left|w_i^{h_1}(x,s)\right| ds \\
& + \int_0^\tau \left|\exp(\widetilde{Q}(x,T)(\tau-s)) - \overline{P}(x)\right| \left| \frac{\partial^h r_i(x,s)}{\partial x^h} \right| ds \\
& + \int_\tau^\infty \left|\overline{P}(x)\right| \left| \frac{\partial^h r_i(x,s)}{\partial x^h} \right| ds \\
\le & \, C\,\mathrm{e}^{-\gamma\tau} + (h+1)\int_0^\tau C\,\mathrm{e}^{-\gamma(\tau-s)}\,\mathrm{e}^{-\gamma s}\,ds + (h+1)\int_\tau^\infty C\,\mathrm{e}^{-\gamma s}\,ds \\
= & \, C\,\mathrm{e}^{-\gamma\tau}.
\end{aligned}
$$

Note that we use γ to represent a generic positive constant, whose value may be different for different appearances. Now we prove (3.3.37). In fact, (3.3.30) implies for any h_1,

$$
\begin{aligned}
& \sum_{\iota=0}^{h} \frac{h!}{\iota!(h-\iota)!} \frac{\partial^{h-\iota}\nu(x,T)}{\partial x^{h-\iota}} w_i^\iota(x,0) + \sum_{\iota=0}^{h} \int_0^\infty \frac{h!}{\iota!(h-\iota)!} \frac{\partial^{h-\iota}\nu(x,T)}{\partial x^{h-\iota}} \frac{\partial^\iota r_i(x,s)}{\partial x^\iota} ds = 0, \\
& \sum_{\iota=0}^{h_1} \frac{h_1!}{\iota!(h_1-\iota)!} \frac{\partial^{h_1-\iota}\nu(x,T)}{\partial x^{h_1-\iota}} \frac{\partial^\iota \widetilde{Q}(x,T)}{\partial x^\iota} = 0.
\end{aligned}
\tag{3.3.38}
$$

We derive from (3.3.11), (3.3.38), and (3.3.35) that

$$
\begin{aligned}
\widetilde{w}_i^h(x) = & -\sum_{\iota=0}^{h-1} \frac{h!}{\iota!(h-\iota)!} \frac{\partial^{h-\iota}\nu}{\partial x^{h-\iota}}(x,T) w_i^\iota(x,0) - \sum_{\iota=0}^{h-1} \int_0^\infty \frac{h!}{\iota!(h-\iota)!} \frac{\partial^{h-\iota}\nu(x,T)}{\partial x^{h-\iota}} \frac{\partial^\iota r_i(x,s)}{\partial x^\iota} ds \\
& - \sum_{h_1=0}^{h-1} \int_0^\infty \frac{h!}{h_1!(h-h_1)!} \sum_{\iota=0}^{h-h_1-1} \frac{h-h_1!}{\iota!(h-h_1-\iota)!} \frac{\partial^{h-h_1-\iota}\nu(x,T)}{\partial x^{h-h_1-\iota}} \frac{\partial^\iota \widetilde{Q}(x,T)}{\partial x^\iota} w_i^{h_1}(x,s) ds.
\end{aligned}
$$

On the other hand, the last term of the above equation equals

$$
\begin{aligned}
&-\sum_{h_1=0}^{h-1}\int_0^\infty\sum_{\iota=1}^{h-h_1}\frac{h!}{h_1!\iota!(h-h_1-\iota)!}\frac{\partial^\iota\nu(x,T)}{\partial x^\iota}\frac{\partial^{h-h_1-\iota}\widetilde{Q}(x,T)}{\partial x^{h-h_1-\iota}}w_i^{h_1}(x,s)ds\\
&\quad=-\sum_{\iota=1}^{h}\int_0^\infty\sum_{h_1=0}^{h-\iota}\frac{h!}{(h-\iota)!\iota!}\frac{(h-\iota)!}{h_1!(h-\iota-h_1)!}\frac{\partial^\iota\nu(x,T)}{\partial x^\iota}\frac{\partial^{h-h_1-\iota}\widetilde{Q}(x,T)}{\partial x^{h-h_1-\iota}}w_i^{h_1}(x,s)ds\\
&\quad=-\sum_{\iota=0}^{h-1}\frac{h!}{\iota!(h-\iota)!}\int_0^\infty\sum_{h_1=0}^{\iota}\frac{\iota!}{h_1!(\iota-h_1)!}\frac{\partial^{h-\iota}\nu(x,T)}{\partial x^{h-\iota}}\frac{\partial^{\iota-h_1}\widetilde{Q}(x,T)}{\partial x^{\iota-h_1}}w_i^{h_1}(x,s)ds.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\widetilde{w}_i^h(x)&=-\sum_{\iota=0}^{h-1}\frac{h!}{\iota!(h-\iota)!}\frac{\partial^{h-\iota}\nu(x,T)}{\partial x^{h-\iota}}w_i^\iota(x,0)\\
&\quad-\sum_{\iota=0}^{h-1}\frac{h!}{\iota!(h-\iota)!}\frac{\partial^{h-\iota}\nu(x,T)}{\partial x^{h-\iota}}\int_0^\infty\Big(\sum_{h_1=0}^{\iota}\frac{\iota!}{h_1!(\iota-h_1)!}\frac{\partial^{\iota-h_1}\widetilde{Q}(x,T)}{\partial x^{\iota-h_1}}w_i^{h_1}(x,s)\\
&\quad+\frac{\partial^\iota r_i(x,s)}{\partial x^\iota}\Big)ds\\
&=-\sum_{\iota=0}^{h-1}\frac{h!}{\iota!(h-\iota)!}\frac{\partial^{h-\iota}\nu}{\partial x^{h-\iota}}(x,T)w_i^\iota(x,0)-\sum_{\iota=0}^{h-1}\frac{h!}{\iota!(h-\iota)!}\frac{\partial^{h-\iota}\nu(x,T)}{\partial x^{h-\iota}}\int_0^\infty\frac{\partial w_i^\iota(x,s)}{\partial s}ds\\
&=-\sum_{\iota=0}^{h-1}\frac{h!}{\iota!(h-\iota)!}\frac{\partial^{h-\iota}\nu(x,T)}{\partial x^{h-\iota}}w_i^\iota(x,0)-\sum_{\iota=0}^{h-1}\frac{h!}{\iota!(h-\iota)!}\frac{\partial^{h-\iota}\nu(x,T)}{\partial x^{h-\iota}}\int_0^\infty\frac{\partial w_i^\iota(x,s)}{\partial s}ds\\
&=-\sum_{\iota=0}^{h-1}\frac{h!}{\iota!(h-\iota)!}\frac{\partial^{h-\iota}\nu(x,T)}{\partial x^{h-\iota}}\lim_{s\to\infty}w_i^\iota(x,s)=0.
\end{aligned}
$$

The last equation holds owning to (3.3.34). Thus (3.3.37) is valid. Therefore,

$$
\left|w_i^h(x,\tau)\right|\le C\,\mathrm{e}^{-\gamma\tau}
$$

for any h. $\square$

Proposition 3.3.7. *There exist constants C and $0<\gamma_i<\gamma$ such that for any $0\le i\le n+1$,*

$$
\max_{h=0,\dots,2(n+2-i)}\left|\frac{\partial^h\psi_i(x,\tau)}{\partial x^h}\right|\le C\,\mathrm{e}^{-\gamma_i\tau},\qquad\forall\tau\ge0,\ 0\le x\le1. \tag{3.3.39}
$$

Proof. First, under condition (3.3.19), we have

$$\begin{aligned}
|\psi_0(x,\tau)| &= |\underline{\mathrm{P}}(x)\psi_0(x,0) + (\exp(\widetilde{Q}(x,T)\tau) - \underline{\mathrm{P}}(x))\psi_0(x,0)| \\
&= |\exp(\widetilde{Q}(x,T)\tau) - \underline{\mathrm{P}}(x)||\psi_0(x,0)| \\
&\le C\varepsilon^{-\gamma\tau}.
\end{aligned}$$

Applying Lemma 3.3.6 with $r_0 = 0$, we deduce that (3.3.39) is valid for $i = 0$. Assume that for any $j < i$,

$$\max_{h=0,\ldots,2n+2} \left| \frac{\partial^h \psi_j(x,\tau)}{\partial x^h} \right| \le C\varepsilon^{-\gamma_j \tau}.$$

Then

$$\max_{h=0,\ldots,2n+2} \left| \frac{\partial^h r_i(x,\tau)}{\partial x^h} \right| \le C\varepsilon^{-\gamma\tau}.$$

with $\gamma = \min(\gamma_1, \ldots, \gamma_{i-1})$. Under condition (3.3.29), we deduce

$$\begin{aligned}
|\psi_i(x,\tau)| \le& \left|(\exp(\widetilde{Q}(x,T)\tau) - \underline{\mathrm{P}})\psi_i(x,0)\right| \\
&+ \left|\int_0^\tau (\exp(\widetilde{Q}(x,T)(\tau-s)) - \underline{\mathrm{P}})r_i(x,s)ds\right| + \left|\int_\tau^\infty -\underline{\mathrm{P}}r_i(x,s)ds\right| \\
\le& \, C\varepsilon^{-\gamma\tau} + C\int_0^\tau \varepsilon^{-\gamma(\tau-s)}\varepsilon^{-\gamma s}ds + C\int_\tau^\infty \varepsilon^{-\gamma s}ds.
\end{aligned}$$

Thus

$$|\psi_i(x,\tau)| \le C\,\mathrm{e}^{-\gamma\tau}.$$

Lemma 3.3.6 again shows that (3.3.39) holds for i. This completes the proof by induction. □

3.3.4 Error Estimates

For a suitable function f, define

$$\mathbf{L}^\varepsilon f = \frac{\partial f}{\partial t} + Q^\varepsilon f + \mathcal{L}f. \tag{3.3.40}$$

Lemma 3.3.8. *Let $\omega(x,s)$ be the solution of the following equation*

$$\begin{aligned} &\mathbf{L}^{\varepsilon}(x,t)\omega(x,t)=\zeta(x,t), \qquad \text{for } t<T,\\ &\omega(x,T)=0. \end{aligned} \tag{3.3.41}$$

Then

$$\omega_i(x,t)=-E\int_t^T \zeta(Y^{\varepsilon,x,i}(s))ds.$$

where $Y^{\varepsilon,x,i}(t)=(X^{\varepsilon}(t),\alpha^{\varepsilon}(t))$ satisfies

$$X^{\varepsilon}(T)=x, \qquad \alpha^{\varepsilon}(T)=i\in\mathcal{M}.$$

Proof. Since $\mathbf{L}^{\varepsilon}$ is the generator of Y^{ε}, by virtue of Itô's formula,

$$\begin{aligned} \omega(x,t) &= \omega(Y^{\varepsilon}(T))-\omega(Y^{\varepsilon}(t))\\ &= -\int_t^T \mathbf{L}^{\varepsilon}\omega(Y^{\varepsilon}(s))ds+M(t)\\ &= -\int_t^T \zeta(Y^{\varepsilon}(s))ds+M(t), \end{aligned} \tag{3.3.42}$$

where $M(t)$ is a martingale. Taking expectation in (3.3.42) leads to the desired assertion. □

Lemma 3.3.9. *Suppose that $\zeta\in C([0,1]\times[0,T])$ is periodic in $x\in[0,1]$, satisfying*

$$\sup_{(x,t)\in[0,1]\times[0,T]}|\zeta(x,t)|\le C\varepsilon^{\kappa}.$$

Let $\xi^{\varepsilon}(x,t)$ be a solution to

$$\mathbf{L}^{\varepsilon}\xi^{\varepsilon}(x,t)=\zeta(x,t), \qquad \xi^{\varepsilon}(x,T)=0, \qquad \forall x\in[0,1]. \tag{3.3.43}$$

Then

$$\sup_{(x,t)\in[0,1]\times[0,T]}|\xi^{\varepsilon}(x,t)|\le C\varepsilon^{\kappa}.$$

Proof. The desired result follows from the previous Lemma. □

Theorem 3.3.10. *There exists a $C > 0$ such that*

$$\sup_{(x,t)\in[0,1]\times[0,T]} |p^\varepsilon(x,t) - \Phi_n^\varepsilon(x,t) - \Psi_n^\varepsilon(x,\tau)| \le C\varepsilon^{n+1}.$$

Proof. Recall the definition of τ in (3.3.6). Put

$$e^{\varepsilon,\kappa}(x,t) = p^\varepsilon(x,t) - \Phi_\kappa^\varepsilon(x,t) - \Psi_\kappa^\varepsilon(x,\tau).$$

Then $\mathbf{L}^\varepsilon u^\varepsilon(x,t) = 0$ and therefore,

$$\mathbf{L}^\varepsilon e^{\varepsilon,\kappa}(x,t) = -\mathbf{L}^\varepsilon\Phi_\kappa^\varepsilon(x,t) - \mathbf{L}^\varepsilon\Psi_\kappa^\varepsilon(x,\tau).$$

Moreover

$$\begin{aligned}
\mathbf{L}^\varepsilon\Phi_\kappa^\varepsilon(x,t) &= \sum_{i=0}^{\kappa}\varepsilon^i\dot{\phi}_i(x,t) + \sum_{i=0}^{\kappa}\varepsilon^{i-1}\widetilde{Q}(x,t)\phi_i(x,t) + \sum_{i=0}^{\kappa}\varepsilon^i(\mathcal{L}+\widehat{Q})(x,t)\phi_i(x,t)\\
&= \sum_{i=0}^{\kappa}\varepsilon^i(-\widetilde{Q}(x,t)\phi_{i+1}(x,t) - (\mathcal{L}+\widehat{Q})(x,t)\phi_i(x,t)) + \sum_{i=0}^{\kappa}\varepsilon^{i-1}\widetilde{Q}(x,t)\phi_i(x,t)\\
&\quad + \sum_{i=0}^{\kappa}\varepsilon^i(\mathcal{L}+\widehat{Q})(x,t)\phi_i(x,t)\\
&= -\varepsilon^\kappa\widetilde{Q}(x,t)\phi_{\kappa+1}(x,t) + \underbrace{\varepsilon^{-1}\widetilde{Q}(x,t)\phi_0(x,t)}_{0}.
\end{aligned}$$

So

$$|\mathbf{L}^\varepsilon\Phi_\kappa^\varepsilon(x,t)| \le C\varepsilon^\kappa.$$

Note that

$$\varepsilon\frac{d}{dt}\psi_i(\frac{T-t}{\varepsilon}) = -\frac{d}{d\tau}\psi_i(x,\tau),$$

which yields

$$
\begin{aligned}
\mathbf{L}^\varepsilon \Psi_\kappa^\varepsilon\Big(x,\tau\Big) &= \sum_{i=0}^{\kappa} -\varepsilon^{i-1}\frac{\partial \psi_i(x,\tau)}{\partial \tau} + \sum_{i=0}^{\kappa} \varepsilon^{i-1}\widetilde{Q}(x,t)\psi_i(x,\tau) + \sum_{i=0}^{\kappa}\varepsilon^i(\mathcal{L}+\widehat{Q})(x,t)\psi_i(x,\tau) \\
&= \sum_{i=0}^{\kappa}\varepsilon^{i-1}(-\widetilde{Q}(x,T)\psi_i(x,\tau) - r_i(x,\tau)) + \sum_{i=0}^{\kappa}\varepsilon^{i-1}\widetilde{Q}(x,t)\psi_i(x,\tau) \\
&\qquad\qquad + \sum_{i=0}^{\kappa}\varepsilon^i(\mathcal{L}+\widehat{Q})(x,t)\psi_i(x,\tau) \\
&= \sum_{i=0}^{\kappa}\varepsilon^{i-1}(-\widetilde{Q}(x,T)+\widetilde{Q}(x,t))\psi_i(x,\tau) - \sum_{i=0}^{\kappa}\varepsilon^{i-1}r_i(x,\tau) \\
&\qquad\qquad + \sum_{i=0}^{\kappa}\varepsilon^i(\mathcal{L}+\widehat{Q})(x,t)\psi_i(x,\tau).
\end{aligned}
$$

For the second term, we have

$$
\begin{aligned}
&\sum_{i=0}^{\kappa}\varepsilon^{i-1}r_i(x,\tau) \\
&= \sum_{i=1}^{\kappa}\varepsilon^{i-1}\sum_{j=0}^{i-1}\left((-\tau)^{i-j}\frac{\widetilde{Q}^{(i-j)}(x,T)}{(i-j)!} + (-\tau)^{i-j-1}\frac{(\mathcal{L}+\widehat{Q})^{(i-j-1)}(x,T)}{(i-j-1)!}\right)\psi_j(x,\tau) \\
&= \sum_{j=0}^{\kappa-1}\sum_{i=j+1}^{\kappa}\left(\varepsilon^{i-1}(-\tau)^{i-j}\frac{\widetilde{Q}^{(i-j)}(x,T)}{(i-j)!} + \varepsilon^{i-1}(-\tau)^{i-j-1}\frac{(\mathcal{L}+\widehat{Q})^{(i-j-1)}(x,T)}{(i-j-1)!}\right)\psi_j(x,\tau) \\
&= \sum_{j=0}^{\kappa-1}\sum_{i=j+1}^{\kappa}\left(\varepsilon^{j-1}(t-T)^{i-j}\frac{\widetilde{Q}^{(i-j)}(x,T)}{(i-j)!} + \varepsilon^{j}(t-T)^{i-j-1}\frac{(\mathcal{L}+\widehat{Q})^{(i-j-1)}(x,T)}{(i-j-1)!}\right)\psi_j(x,\tau) \\
&= \sum_{j=0}^{\kappa-1}\varepsilon^{j-1}\left(\sum_{i=1}^{\kappa-j}(t-T)^i\frac{\widetilde{Q}^{(i)}(x,T)}{i!}\right)\psi_j(x,\tau) \\
&\qquad\qquad + \sum_{j=0}^{\kappa-1}\varepsilon^j\left(\sum_{i=0}^{\kappa-j-1}(t-T)^i\frac{(\mathcal{L}+\widehat{Q})^{(i)}(x,T)}{i!}\right)\psi_j(x,\tau).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbf{L}^\varepsilon \Psi_\kappa^\varepsilon\Big(x,\tau\Big) \;=\;& \varepsilon^{\kappa-1}(-\widetilde{Q}(x,T)+\widetilde{Q}(x,t))\psi_\kappa(x,\tau) + \varepsilon^\kappa(\mathcal{L}+\widehat{Q})(x,t)\psi_\kappa(x,\tau) \\
&+ \sum_{j=0}^{\kappa-1}\varepsilon^{j-1}\left(\widetilde{Q}(x,t) - \sum_{i=0}^{\kappa-j}(t-T)^i\frac{\widetilde{Q}^{(i)}(x,T)}{i!}\right)\psi_j(x,\tau) \\
&+ \sum_{j=0}^{\kappa-1}\varepsilon^j\left((\mathcal{L}+\widehat{Q})(x,t) - \sum_{i=0}^{\kappa-j-1}(t-T)^i\frac{(\mathcal{L}+\widehat{Q})^{(i)}(x,T)}{i!}\right)\psi_j(x,\tau).
\end{aligned}
\tag{3.3.44}
$$

Using Taylor expressions and Proposition 3.3.7, we obtain

$$\begin{aligned}\Big|\mathbf{L}^{\varepsilon}\Psi_{\kappa}^{\varepsilon}\Big(x,\tau\Big)\Big| &\le C\varepsilon^{\kappa-1}|t-T|\,\mathrm{e}^{-\gamma\tau} + C\varepsilon^{\kappa} + C\sum_{j=0}^{\kappa-1}\varepsilon^{j-1}\,|t-T|^{\kappa-j+1}\,\mathrm{e}^{-\gamma\tau}\\ &\quad + C\sum_{j=0}^{\kappa-1}\varepsilon^{j}\,|t-T|^{\kappa-j}\,\mathrm{e}^{-\gamma\tau}\\ &= C\varepsilon^{\kappa}\tau\,\mathrm{e}^{-\gamma\tau} + C\varepsilon^{\kappa} + C\sum_{j=0}^{\kappa-1}\varepsilon^{\kappa}\tau^{\kappa-j+1}\,\mathrm{e}^{-\gamma\tau} + C\sum_{j=0}^{\kappa-1}\varepsilon^{\kappa}\tau^{\kappa-j}\,\mathrm{e}^{-\gamma\tau}\\ &\le C\varepsilon^{\kappa}.\end{aligned}$$

Piecing this together with the estimates on $\mathbf{L}^{\varepsilon}\Phi_{\kappa}(x,t)$, we have shown that

$$\sup_{(x,t)\in[0,1]\times[0,T]}|\mathbf{L}^{\varepsilon}e^{\varepsilon,\kappa}(x,t)| \le C\varepsilon^{\kappa}.$$

Note the terminal condition $e^{\varepsilon,\kappa}(x,T)=0$. Thus Lemma 3.3.9 implies

$$\sup_{(x,t)\in[0,1]\times[0,T]}|e^{\varepsilon,\kappa}(x,t)| \le C\varepsilon^{\kappa}.$$

Taking $\kappa = n+1$, we obtain

$$\sup_{(x,t)\in[0,1]\times[0,T]}|e^{\varepsilon,n+1}(x,t)| = O(\varepsilon^{n+1}).$$

Finally, note that

$$e^{\varepsilon,n+1}(x,t) = e^{\varepsilon,n}(x,t) + \varepsilon^{n+1}\phi_{n+1}(x,t) + \varepsilon^{n+1}\psi_{n+1}(x,\tau). \tag{3.3.45}$$

The continuity of $\phi_{n+1}(x,t)$ and the exponential decay properties of $\psi_{n+1}(x,\tau)$ yield that

$$\sup_{(x,t)\in[0,1]\times[0,T]}|\varepsilon^{n+1}\phi_{n+1}(x,t) + \varepsilon^{n+1}\psi_{n+1}(x,\tau)| \le C\varepsilon^{n+1}.$$

Substituting this into (3.3.45), we obtain

$$\sup_{(x,t)\in[0,1]\times[0,T]}|e^{\varepsilon,n}(x,t)| \le C\varepsilon^{n+1}. \qquad \square$$

3.4 Fast Diffusion

Suppose that $\alpha(t)$ is a jump process with generator $Q(x,t)$. Let two operators $\widetilde{\mathcal{L}}$ and $\widehat{\mathcal{L}}$ be defined as in (3.2.3) where a and b are correspondingly replaced by $\widetilde{a}$ and $\widetilde{b}$ ($\widehat{a}$ and $\widehat{b}$ respectively). That is,

$$\begin{aligned}\widetilde{\mathcal{L}}(x,t)u(x,k,t) &= \frac{1}{2}\widetilde{a}(x,k,t)\frac{\partial^2}{\partial x^2}u(x,k,t)+\widetilde{b}(x,k,t)\frac{\partial}{\partial x}u(x,k,t)\\ \widehat{\mathcal{L}}(x,t)u(x,k,t) &= \frac{1}{2}\widehat{a}(x,k,t)\frac{\partial^2}{\partial x^2}u(x,k,t)+\widehat{b}(x,k,t)\frac{\partial}{\partial x}u(x,k,t).\end{aligned}$$

Let $\mathcal{L}(x,t)$ in (3.3.4) be of the form

$$\mathcal{L}^\varepsilon(x,t)=\frac{\widetilde{\mathcal{L}}(x,t)}{\varepsilon}+\widehat{\mathcal{L}}(x,t).$$

Throughout this section, in addition to assumptions (A4), we also assume that

(A6) $\widetilde{a}(x,k,t)>0$ for all x,t and k. That is, the fast changing part of the diffusion is uniformly elliptic.

(A7) For each $k\in\mathcal{M}$,

- $\widetilde{a}(\cdot,k,t)$, $\widehat{a}(\cdot,k,t)$, $\widetilde{b}(\cdot,k,t)$, $\widehat{b}(\cdot,k,t)$ are periodic in x with period 1 for each $t\in[0,T]$ and $\widetilde{a}(\cdot,k,\cdot),\widehat{a}(\cdot,k,\cdot),\widetilde{b}(\cdot,k,\cdot),\widehat{b}(\cdot,k,\cdot)\in C^{2(n+2),n+2}([0,1]\times[0,T])$.
- $\widetilde{a}(\cdot,k,T)\in C^{2n+6}([0,1])$ and $\widetilde{b}(\cdot,k,T)\in C^{2n+5}([0,1])$.

(A8) $Q(\cdot,\cdot)\in C^{2(n+2),n+2}([0,1]\times[0,T])$.

We consider

$$-\frac{\partial\widetilde{u}^\varepsilon}{\partial t}=Q(x,t)\widetilde{u}^\varepsilon+\mathcal{L}^\varepsilon(x,t)\widetilde{u}^\varepsilon,\qquad \widetilde{u}^\varepsilon(x,T)=g(x). \tag{3.4.1}$$

Similarly to Section 3.3, we seek asymptotic expansions of the form (3.3.7). Substituting the expansions in (3.4.1), we obtain

$$\begin{aligned}
&\widetilde{\mathcal{L}}(x,t)\phi_0(x,t) = 0,\\
&\widetilde{\mathcal{L}}(x,t)\phi_1(x,t) = -\frac{\partial}{\partial t}\phi_0(x,t) - (\widehat{\mathcal{L}}+Q)(x,t)\phi_0(x,t) \stackrel{\text{def}}{=} \varsigma_0(x,t),\\
&\cdots\ \cdots\\
&\widetilde{\mathcal{L}}(x,t)\phi_{i+1}(x,t) = -\frac{\partial}{\partial t}\phi_i(x,t) - (\widehat{\mathcal{L}}+Q)(x,t)\phi_i(x,t) \stackrel{\text{def}}{=} \varsigma_i(x,t),
\end{aligned} \tag{3.4.2}$$

where $i = 2, \ldots, n+2$. Likewise, substituting $\Psi_\kappa(x,\tau)$ for $\kappa \le n+2$ into (3.4.1) and applying Taylor expansion for $\widetilde{\mathcal{L}}(x, T-\varepsilon\tau)$, $\widehat{\mathcal{L}}(x, T-\varepsilon\tau)$ and $Q(x, T-\varepsilon\tau)$, we arrive at

$$\begin{aligned}
\frac{\partial\psi_0(x,\tau)}{\partial\tau} &= \widetilde{\mathcal{L}}(x,T)\psi_0(x,\tau),\\
\frac{\partial\psi_1(x,\tau)}{\partial\tau} &= \widetilde{\mathcal{L}}(x,T)\psi_1(x,\tau) + \Big(-\tau\widetilde{\mathcal{L}}^{(1)}(x,T) + (\widehat{\mathcal{L}}+Q)(x,T)\Big)\psi_0(x,\tau),\\
\cdots\ \cdots\\
\frac{\partial\psi_i(x,\tau)}{\partial\tau} &= \widetilde{\mathcal{L}}(x,T)\psi_i(x,\tau) + r_i(x,\tau)\\
r_i(x,\tau) &= \sum_{j=0}^{i-1}\left((-\tau)^{i-j}\frac{\widetilde{\mathcal{L}}^{(i-j)}(x,T)}{(i-j)!} + (-\tau)^{i-j-1}\frac{(\widehat{\mathcal{L}}+Q)^{(i-j-1)}(x,T)}{(i-j-1)!}\right)\psi_j(x,\tau),
\end{aligned} \tag{3.4.3}$$

where

$$\widetilde{\mathcal{L}}^{(i)}(x,T) = \frac{\partial^i\widetilde{\mathcal{L}}(x,T)}{\partial t^i},\quad \widehat{\mathcal{L}}^{(i)}(x,T) = \frac{\partial^i\widehat{\mathcal{L}}(x,T)}{\partial t^i},\quad Q^{(i)}(x,T) = \frac{\partial^i Q(x,T)}{\partial t^i}.$$

From the initial condition, we derive

$$\phi_0(x,T) + \psi_0(x,0) = g(x) \text{ and } \phi_i(x,T) + \psi_i(x,0) = 0, \text{ for } i > 0.$$

We recall that the adjoint operator of $\widetilde{\mathcal{L}}_k$ has the form

$$\widetilde{\mathcal{L}}_k^*(x,t)u(x,k,t) = \frac{\partial^2}{\partial x^2}\left[\frac{1}{2}\widetilde{a}(x,k,t)u(x,k,t)\right] - \frac{\partial}{\partial x}\left[\widetilde{b}(x,k,t)u(x,k,t)\right],\quad k\in\mathcal{M}.$$

In what follows, we will prove the smoothness of φ_i for $0 \le i \le n+2$ and the exponential decay of ψ_i for $0 \le i \le n+1$ which implies the desired error bound by Lemma 3.4.13. Let us consider the layer terms by starting with some lemmas.

Lemma 3.4.1. *For each $k \in \mathcal{M}$, there exists a unique solution μ_k to the following equations*

$$\begin{aligned} &\widetilde{\mathcal{L}}_k^*(x,t)\mu_k(x,t) = 0 \\ &\int_0^1 \mu_k(x,t)dx = 1 \\ &\mu_k(0,t) = \mu_k(1,t). \end{aligned} \tag{3.4.4}$$

Remark 3.4.2. By the smoothness and the periodicity of the boundary conditions in (3.4.4), the function μ_k defined above also satisfies

$$\frac{\partial}{\partial x}(\widetilde{a}(0,k,t)\mu_k(0,t)) = \frac{\partial}{\partial x}(\widetilde{a}(1,k,t)\mu_k(1,t)).$$

Definition 3.4.3. *For any functions $\xi(x), \zeta(x)$ on $[0,1]$, define*

$$[\xi,\zeta] = \left(\int_0^1 \xi_1(x)\zeta_1(x)dx, \ldots, \int_0^1 \xi_m(x)\zeta_m(x)dx\right)'$$

and

$$\langle \xi,\zeta\rangle = \sum_{k=1}^m \int_0^1 \xi_k(x)\zeta_k(x)dx.$$

Lemma 3.4.4. *Let $X_x^s(k,\tau)$ be a Markov process corresponding to the generator $\widetilde{\mathcal{L}}_k(x,s)$ and $\omega_k(x)$ be a bounded measurable real-valued function. Then*

$$\left|E\omega_k(X_x^s(k,\tau)) - \int_0^1 \omega_k(x)\mu_k(x,s)dx\right| \le C\,\mathrm{e}^{-\gamma\tau}.$$

Proof.

The quasi-stationary density function μ_k of the diffusion process verifies the so-called Doeblin condition which implies the desired result.

Lemma 3.4.5. *Consider the following Poisson equation with periodic boundary conditions*

$$\begin{aligned}
&\widetilde{\mathcal{L}}_k(x,t)\phi(x,k,t) = \zeta_k(x,t)\\
&\phi(0,k,t) = \phi(1,k,t)\\
&\frac{\partial}{\partial x}\phi(0,k,t) = \frac{\partial}{\partial x}\phi(1,k,t),\\
&\textit{for each} \quad k \in \mathcal{M}.
\end{aligned} \tag{3.4.5}$$

Then

1. *The only solution to* $\widetilde{\mathcal{L}}_k\phi(x,k,t) = 0$ *is* $\phi(x,k,t) = \varphi_k(t)$.

2. (3.4.5) *has a solution if and only if*

$$[\zeta(\cdot,t), \mu(\cdot,t)] = 0.$$

Proof.

1. Assume $\phi(0,k,t) = \phi(1,k,t) = \varphi_k(t)$. Put $\xi_k(x,t) = \phi(x,k,t) - \varphi_k(t)$. Then

$$\widetilde{\mathcal{L}}(x,k,t)\xi_k(x,t) = 0, \qquad \xi_k(0,t) = \xi_k(1,t) = 0.$$

It could be verified by the maximum principle for the elliptic operator $\widetilde{\mathcal{L}}$ that $\xi_k = 0$; see Evans [15, Chapter 6]. So $\phi(x,k,t) = \varphi_k(t)$. That is, $\phi(x,k,t)$ is independent of x.

2. The system of equations (3.4.5) is solvable then

$$\langle \zeta_k(\cdot,t), \mu_k(\cdot,t)\rangle = \left\langle \widetilde{\mathcal{L}}_k(\cdot,t)\phi_k(\cdot,t), \mu_k(\cdot,t)\right\rangle = \left\langle \phi_k(\cdot,t), \widetilde{\mathcal{L}}_k^*(\cdot,t)\mu_k(\cdot,t)\right\rangle = \langle \phi_k(\cdot,t), 0\rangle = 0.$$

Thus

$$[\zeta(\cdot,t),\mu(\cdot,t)]=0.$$

Conversely, assume that $[\zeta(\cdot,t),\mu(\cdot,t)]=0$. Under the uniform-ellipticity condition, it can be shown that the following equation

$$\begin{aligned}&\widetilde{\mathcal{L}}_k(x,t)\phi(x,k,t)=\zeta_k(x,t)\\&\phi(0,k,t)=\phi(1,k,t)\end{aligned}$$

has a solution. In view of (3.4.2) and using integration by part,

$$\begin{aligned}\langle\zeta_k(\cdot,t),\mu_k(\cdot,t)\rangle &= \int_0^1\left(\widetilde{\mathcal{L}}_k(x,t)\phi_k(x,t)\right)\mu_k(x,t)dx\\
&=\widetilde{a}(x,k,t)\mu_k(x,t)\frac{\partial}{\partial x}\phi_k(x,t)\Big|_{x=1}-\widetilde{a}(x,k,t)\mu_k(x,t)\frac{\partial}{\partial x}\phi_k(x,t)\Big|_{x=0}\\
&\quad+\widetilde{b}(x,k,t)\mu_k(x,t)\phi_k(x,t)\Big|_{x=1}-\widetilde{b}(x,k,t)\mu_k(x,t)\phi_k(x,t)\Big|_{x=0}\\
&\quad-\phi_k(x,t)\frac{\partial}{\partial x}(\widetilde{a}(x,k,t)\mu_k(x,t))\Big|_{x=1}+\phi_k(x,t)\frac{\partial}{\partial x}(\widetilde{a}(x,k,t)\mu_k(x,t))\Big|_{x=0}\\
&\quad+\int_0^1\phi_k(x,t)\left(\widetilde{\mathcal{L}}_k^*(x,t)\mu_k(x,t)\right)dx\\
&=\widetilde{a}(0,k,t)\mu_k(0,t)\left(\frac{\partial}{\partial x}\phi_k(1,t)-\frac{\partial}{\partial x}\phi_k(0,t)\right).\end{aligned}$$

Since $\langle\zeta_k(\cdot,t),\mu_k(\cdot,t)\rangle=0$ and $\widetilde{a}(0,k,t)\mu_k(0,t)>0$, we obtain

$$\frac{\partial}{\partial x}\phi_k(1,t)=\frac{\partial}{\partial x}\phi_k(0,t).$$

The proof is concluded. $\square$

3.4.1 Leading Term $\phi_0(x,t)$ and Zero-order Terminal Layer Term $\psi_0(x,\tau)$

Note that (3.4.2) gives

$$\widetilde{\mathcal{L}}(x,t)\phi_0(x,t)=0,$$

which, by Lemma 3.4.5, implies

$$\phi_0(x,t) = \varphi_0(t). \tag{3.4.6}$$

Moreover, we derived from (3.4.2) that

$$\begin{aligned}\widetilde{\mathcal{L}}(x,t)\phi_1(x,t) &= -\frac{\partial \phi_0(x,t)}{\partial t} - (\widehat{\mathcal{L}} + Q)(x,t)\phi_0(x,t) \\ &= -\dot{\varphi}_0(t) - Q(x,t)\varphi_0(t).\end{aligned}$$

Again, Lemma 3.4.5 implies that

$$\dot{\varphi}_0(t) + [Q(x,t)\varphi_0(t), \mu(x,t)] = [\dot{\varphi}_0(t), \mu(x,t)] + [Q(x,t)\varphi_0(t), \mu(x,t)] = 0. \tag{3.4.7}$$

The $\varphi_0(T)$ is to be determined. Also, the zero-order terminal layer term is uniquely determined by

$$\begin{aligned}&\frac{\partial \psi_0(x,\tau)}{\partial \tau} = \widetilde{\mathcal{L}}(x,T)\psi_0(x,\tau) \\ &\psi_0(x,0) = g(x) - \phi_0(x,T).\end{aligned} \tag{3.4.8}$$

Then $\psi_0(x,\tau) = E\psi_0(X_x^T(\tau),0)$ where $X_x^T(\tau)$ is a Markov process corresponding to the generator $\widetilde{\mathcal{L}}(x,T)$. We demand $\lim_{\tau\to\infty}\psi_0(x,\tau) = 0$. By Lemma 3.4.4, we deduce

$$[\psi_0(\cdot,0), \mu(\cdot,T)] = 0, \tag{3.4.9}$$

which is equivalent to

$$\varphi_0(T) = [g(\cdot), \mu(\cdot,T)]\,. \tag{3.4.10}$$

Hence $\phi_0(x,t)$ is uniquely determined by (3.4.6), (3.4.7), and (3.4.10).

3.4.2 Higher-order Terms

Before proceeding further, we need to verify the following lemmas.

Lemma 3.4.6. *Let $\psi(x,\tau)$ be a solution of*

$$\begin{aligned}&\frac{\partial \psi(x,\tau)}{\partial \tau} = \widetilde{\mathcal{L}}(x,t)\psi(x,\tau) + r(x,\tau)\\ &\psi(x,0) = \omega(x),\end{aligned}$$

where $\omega(x)$ is a l-dimensional vector function and $r(x,\tau)$ decays exponentially fast, i.e., there exist $C, \gamma > 0$ such that

$$\sup_{x\in[0,1]} |r(x,\tau)| \le C\,\mathrm{e}^{-\gamma\tau}\,.$$

Then

$$\left|\psi(x,\tau) - [\omega(\cdot),\mu(\cdot,t)] - \int_0^\infty [r(\cdot,s),\mu(\cdot,t)]\,ds\right| \le C\,\mathrm{e}^{-\gamma\tau}\,.$$

Proof. In fact, for each $k \in \mathcal{M}$, $\psi(x,k,\tau)$ satisfies

$$\begin{aligned}&\frac{\partial \psi(x,k,\tau)}{\partial \tau} = \widetilde{\mathcal{L}}_k(x,t)\psi(x,k,\tau) + r_k(x,\tau)\\ &\psi(x,0) = \omega_k(x),\end{aligned}$$

Thus

$$\psi(x,k,\tau) = E\omega_k(X_x(k,\tau)) + \int_0^\tau Er_k(X_x(k,\tau-s),s)ds,$$

where $X_x(k,\tau)$ is the diffusion process associated with the generator $\widetilde{\mathcal{L}}_k(x,t)$ satisfying $X_x(k,0) = x$. It follows from Lemma 3.4.4 that

$$|E\omega_k(X_x(k,\tau)) - \langle \omega_k(\cdot),\mu_k(\cdot,t)\rangle\,| \le C\,\mathrm{e}^{-\gamma\tau}\,.$$

Moreover,

$$
\begin{aligned}
&\left|\int_0^{\tau} Er_k(X_x(k,\tau-s),s)ds-\int_0^{\infty}\langle r_k(\cdot,s),\mu_k(\cdot,t)\rangle\,ds\right| \\
&\qquad\le\left|\int_0^{\tau/2} Er_k(X_x(k,\tau-s),s)ds-\int_0^{\tau/2}\langle r_k(\cdot,s),\mu_k(\cdot,t)\rangle\,ds\right| \\
&\qquad\qquad+\int_{\tau/2}^{\tau}|Er_k(X_x(k,\tau-s),s)|\,ds+\int_{\tau/2}^{\infty}|\langle r_k(\cdot,s),\mu_k(\cdot,t)\rangle|\,ds \\
&\qquad\le C\,\mathrm{e}^{-\gamma\tau}+\int_{\tau/2}^{\tau}C\,\mathrm{e}^{-\gamma(\tau-s)}\,ds+\int_{\tau/2}^{\infty}C\,\mathrm{e}^{-\gamma s}\,ds \\
&\qquad\le C\,\mathrm{e}^{-\gamma\tau}\,.
\end{aligned}
$$

So the proof of the lemma is completed. $\square$

By using (3.4.2), we have

$$\widetilde{\mathcal{L}}(x,t)\phi_i(x,t)=\varsigma_{i-1}(x,t). \tag{3.4.11}$$

By Lemma 3.4.5, we arrive at

$$\phi_i(x,t)=\varphi_i(t)+\widehat{\varphi}_i(x,t), \tag{3.4.12}$$

where $\widehat{\varphi}_i(x,t)$ is a particular solution of (3.4.11) satisfying

$$[\widehat{\varphi}_i(\cdot,t),\mu(\cdot,t)]=0. \tag{3.4.13}$$

Denote $\overline{a}(x,k,t) = 2\widetilde{a}^{-1}(x,k,t)\widetilde{b}(x,k,t)$ and $\overline{\varsigma}_{i-1}(x,k,t) = \widetilde{a}^{-1}(x,k,t)\varsigma_{i-1}(x,k,t)$. Then

$$\frac{\partial^2\widehat{\varphi}_i(x,k,t)}{\partial x^2}+\overline{a}(x,k,t)\frac{\partial\widehat{\varphi}_i(x,k,t)}{\partial x}=\overline{\varsigma}_{i-1}(x,k,t),\qquad \widehat{\varphi}_i(0,k,t)=\widehat{\varphi}_i(1,k,t).$$

Then we obtain

$$\widehat{\varphi}_i(x,k,t)=\widetilde{\varphi}_i(k,t)+\overline{\varphi}_i(k,t)\int_0^x\mathrm{e}^{-\rho_k(y,t)}\,dy+\int_0^x\int_0^y\mathrm{e}^{\rho_k(z,t)-\rho_k(y,t)}\,\overline{\varsigma}_{i-1}(z,k,t)dzdy \tag{3.4.14}$$

where $\rho_k(x,t) = \displaystyle\int_0^x \overline{a}(y,k,t)dy$. By the periodicity of $\widehat{\varphi}_i(\cdot,t)$ (i.e. $\widehat{\varphi}_i(0,t) = \widehat{\varphi}_i(1,t)$), we deduce

$$\overline{\varphi}_i(t) = -\frac{\displaystyle\int_0^1 \int_0^y \mathrm{e}^{\rho_k(z,t)-\rho_k(y,t)}\,\varsigma_{i-1}(z,t)dzdy}{\displaystyle\int_0^1 \mathrm{e}^{-\rho_k(y,t)}\,dy}. \tag{3.4.15}$$

Moreover, it follows from (3.4.13) that

$$\begin{aligned}\widetilde{\varphi}_i(k,t) = &-\int_0^1 \int_0^x \overline{\varphi}_i(k,t)\,\mathrm{e}^{-\rho_k(y,t)}\,\mu(x,k,t)dydx \\ &-\int_0^1 \int_0^x \int_0^y \mathrm{e}^{\rho_k(z,t)-\rho_k(y,t)}\,\overline{\varsigma}_{i-1}(z,k,t)\mu(x,k,t)dzdydx.\end{aligned} \tag{3.4.16}$$

On the other hand,

$$\begin{aligned}\widetilde{\mathcal{L}}(x,t)\phi_{i+1}(x,t) &= -\frac{\partial}{\partial t}\phi_i(x,t) - (\widehat{\mathcal{L}}+Q)(x,t)\phi_i(x,t) \\ &= -\dot{\varphi}_i(t) - \frac{\partial \widehat{\varphi}_i(x,t)}{\partial t} - Q(x,t)\varphi_i(t) - (\widehat{\mathcal{L}}+Q)(x,t)\widehat{\varphi}_i(x,t).\end{aligned}$$

Using Lemma 3.4.5, we obtain

$$\dot{\varphi}_i(t)+[Q(x,t)\varphi_i(t),\mu(x,t)] = \left[-\frac{\partial \widehat{\varphi}_i(x,t)}{\partial t} - (\widehat{\mathcal{L}}+Q)(x,t)\widehat{\varphi}_i(x,t),\mu(x,t)\right]. \tag{3.4.17}$$

The terminal layer term is uniquely determined by

$$\begin{aligned}&\frac{\partial \psi_i(x,\tau)}{\partial \tau} = \widetilde{\mathcal{L}}(x,T)\psi_i(x,\tau) + r_i(x,\tau) \\ &\psi_i(x,0) = -\phi_0(x,T).\end{aligned} \tag{3.4.18}$$

Denote by $X_x^T(k,\tau)$ the Markov process corresponding to the generator $\widetilde{\mathcal{L}}_k(x,T)$. Then

$$\psi_i(x,k,\tau) = E\psi_i(X_x^T(k,\tau),k,0) + \int_0^\tau Er_i(X_x^T(k,\tau-s),k,s)ds.$$

Furthermore, by demanding $\lim_{\tau\to\infty}\psi_i(x,\tau) = 0$ and using Lemma 3.4.4, we obtain

$$[\psi_i(\cdot,0),\mu(\cdot,T)] + \int_0^\infty [r_i(\cdot,s),\mu(\cdot,T)]\,ds = 0. \tag{3.4.19}$$

In virtue of the initial condition in (3.4.18), we arrive at

$$\varphi_i(T) = \int_0^\infty [r_i(\cdot, s), \mu(\cdot, T)]\, ds. \tag{3.4.20}$$

Thus $\phi_i(x,t)$ is uniquely determined by (3.4.12), (3.4.14), (3.4.17) and (3.4.20).

Lemma 3.4.7. *There exists a Green function for the following problem*

$$\begin{aligned} &\frac{\partial \psi}{\partial t} = \widetilde{\mathcal{L}}(x,t)\psi \\ &\psi(0,t) = \psi(1,t) \\ &\psi(x,0) = \omega(x). \end{aligned}$$

Proof. Let $\widehat{G}$ be the Green function for the corresponding parabolic equation in the unbounded domain. Then there exist positive constants C_1, C_2, K_1, and K_2, such that for all $x, y \in R$ and $t > s$,

$$C_1 F_1(t-s, y-x) \le \widehat{G}(s,x,t,y) \le C_2 F_2(t-s, y-x). \tag{3.4.21}$$

Here for $h_1 = 1, 2$, $F_{h_1}(y,t)$ is the fundamental solution of the equation

$$K_{h_1} \Delta w = \frac{\partial w}{\partial t},$$

where Δw denotes the Laplacian of w. Define

$$G(s,x,t,y) = \sum_{\iota=-\infty}^{\infty} \widehat{G}(s,x,t,y+\iota).$$

In view of (3.4.21), G is well-defined, since when $t > s$,

$$G(s,x,t,y) \le C_2 \sum_{\iota=-\infty}^{\infty} \frac{1}{\sqrt{4\pi K_2(t-s)}} \exp\left(-\frac{|y+\iota-x|^2}{4K_2(t-s)}\right) \le \infty.$$

Moreover, by virtue of estimates on the derivatives of $\widehat{G}$, there exist $C > 0$ and $K > 0$ such that

$$\left|\frac{\partial^h}{\partial y^h}\widehat{G}(s,x,t,y)\right| \le C\frac{1}{(t-s)^{(1+h)/2}}\exp\left(-\frac{K|y-x|^2}{(t-s)}\right),$$
$$\left|\frac{\partial}{\partial t}\widehat{G}(s,x,t,y)\right| \le C\frac{1}{(t-s)^{3/2}}\exp\left(-\frac{K|y-x|^2}{(t-s)}\right).$$

Thus G is differentiable with respect to y and t, and the series

$$\frac{\partial G}{\partial t} = \sum_{\iota=-\infty}^{\infty}\frac{\partial \widehat{G}(s,x,t,y+\iota)}{\partial t}$$

and

$$\frac{\partial^h G}{\partial t^h} = \sum_{\iota=-\infty}^{\infty}\frac{\partial^h \widehat{G}(s,x,t,y+\iota)}{\partial t^h}$$

converge uniformly. Furthermore, G is periodic with period 1 and satisfies the differential equation. $\square$

Remark 3.4.8. *If $\widetilde{\mathcal{L}}(x,t)$ does not depend on t then we can write the Green function as $G(x,t-s,y)$. In this case, the solution has the form*

$$\psi(x,t) = \int_0^1 G(x,t,y)\omega(y)dy.$$

Lemma 3.4.9. *Consider*

$$\frac{\partial\psi}{\partial t} = \widetilde{\mathcal{L}}(x,T)\psi$$
$$\psi(0,t) = \psi(1,t)$$
$$\psi(x,0) = \omega(x).$$

Then there exists an invariant density $\mu(x)$ such that for some $\gamma > 0$ and for $h = 0,1,2$,

$$\sup_{x\in[0,1]}\left|\frac{\partial^h G(y,\tau,x)}{\partial x^h} - \frac{\partial^h \mu(x)}{\partial x^h}\right| \le C\,\mathrm{e}^{-\gamma\tau}.$$

Proof. We have (see Khasminskii and Yin [28, Lemma 5.1])

$$\sup_{x\in[0,1]} |G(x,\tau,y)-\mu(y)| \le C\,\mathrm{e}^{-\gamma\tau}\,.$$

Then

$$\begin{aligned}
\left|\frac{\partial^h}{\partial y^h}\left(G(x,\tau,y)-\mu(y)\right)\right| &= \left|\frac{\partial^h}{\partial y^h}\left(\int_0^1 G(x,\tau-1,z)G(z,1,y)-\mu(y)\right)\right| \\
&= \left|\frac{\partial^h}{\partial y^h}\int_0^1 \left(G(x,\tau-1,z)-\mu(z)\right)G(z,1,y)\right| \\
&= \left|\int_0^1 \left(G(x,\tau-1,z)-\mu(z)\right)\frac{\partial^h G(z,1,y)}{\partial y^h}\right| \\
&\le C\,\mathrm{e}^{-\gamma\tau}\,.
\end{aligned}$$

Since C does not depend on x, we obtain the desired result. □

Lemma 3.4.10. *For any $i=0,\dots,n+2$,*

$$\phi_i \in C^{2(n+2),n+2-i}([0,1]\times[0,T]).$$

Proof. First of all, from (3.4.4), we obtain $\mu \in C^{2(n+2),n+2}([0,1]\times[0,T])$. We will prove this lemma by induction. For $i=0$, (3.4.2) implies $\phi_0 \in C^{2(n+2),n+2}([0,1]\times[0,T])$ Now assume $\phi_j \in C^{2(n+2),n+2-j}([0,1]\times[0,T])$ for $j\le i$. In view of (3.4.2), $\varsigma_i \in C^{2(n+1),n+1-i}([0,1]\times[0,T])$. Then, by (3.4.14), $\widehat{\varphi}_{i+1} \in C^{2(n+2),n+1-i}([0,1]\times[0,T])$. On the other hand, we can conclude from (3.4.17) that $\varphi_{i+1}\in C^{n+1-i}([0,T])$. Therefore, $\phi_{i+1}\in C^{2(n+2),n+1-i}([0,1]\times[0,T])$. This completes the proof of this lemma. □

Lemma 3.4.11. *Let $0\le i\le n+1$ be a fixed integer. For a nonnegative integer h with $0\le h\le 2(n+2-i)$, put $f_i^h(x,\tau)=\dfrac{\partial^h\psi_i(x,\tau)}{\partial x^h}$. Assume for any τ,x,*

$$|\psi_i(x,\tau)| \le C\,\mathrm{e}^{-\gamma\tau}$$

and

$$\max_{h=0,\ldots,2(n+2-i)} \left| \frac{\partial^h r_i(x,\tau)}{\partial x^h} \right| \leq C\,\mathrm{e}^{-\gamma\tau}\,.$$

Then for any τ, x,

$$\max_{h=0,\ldots,2(n+2-i)} \left| f_i^h(x,\tau) \right| \leq C\,\mathrm{e}^{-\gamma\tau}\,.$$

Proof. First,

$$|f_i^0(x,\tau)| = |\psi_i(x,\tau)| \leq C\,\mathrm{e}^{-\gamma\tau}\,.$$

Suppose for any $h_1 < h$,

$$\left| f_i^{h_1}(x,\tau) \right| \leq C\,\mathrm{e}^{-\gamma\tau}\,. \tag{3.4.22}$$

Then (3.4.3) implies

$$\begin{aligned}
\frac{\partial f_i^h(x,\tau)}{\partial \tau} &= \frac{\partial^h}{\partial x^h}\Big(\widetilde{a}(x,T)\frac{\partial^2 \psi}{\partial x^2}_i(x,\tau) + \widetilde{b}(x,T)\frac{\partial \psi_i(x,\tau)}{\partial x}\Big) + \frac{\partial^h r_i(x,\tau)}{\partial x^h} \\
&= \widetilde{\mathcal{L}}(x,T) f_i^h(x,\tau) + \frac{\partial^h r_i(x,\tau)}{\partial x^h} \\
&\quad + \sum_{h_1=0}^{h-1} \frac{h!}{h_1!(h-h_1)!}\Big(\frac{\partial^{h-h_1}\widetilde{a}(x,T)}{\partial x^{h-h_1}} f_i^{h_1+2}(x,\tau) + \frac{\partial^{h-h_1}\widetilde{b}(x,T)}{\partial x^{h-h_1}} f_i^{h_1+1}(x,\tau)\Big) \\
&\stackrel{\text{def}}{=} \widetilde{\mathcal{L}}(x,T) f_i^h(x,\tau) + \frac{\partial^h r_i(x,\tau)}{\partial x^h} + \widetilde{f}_i^h(x,\tau) \\
f_i^h(x,0) &= \frac{\partial^h \psi(x,0)}{\partial x^h}.
\end{aligned} \tag{3.4.23}$$

We claim that

$$\widehat{f}_i^h \stackrel{\text{def}}{=} \int_0^1 f_i^h(x,0)\mu(x,T)dx + \int_0^\infty \int_0^1 \widetilde{f}_i^h(x,s)\mu(x,T)dxds = 0.$$

Let G_i^h and μ_i^h be the Green function for equation (3.4.23) and its associated invariant density respectively. Put $w_i^h(x,s) = G_i^h(x,s) - \mu_i^h(x,T)$. Then

$$
\begin{aligned}
f_i^h(x,\tau) &= \int_0^1 G_i^h(x,\tau,y) f_i^h(y,0)dy + \int_0^\tau \int_0^1 G_i^h(x,\tau-s,y)\frac{\partial^h r_i(y,s)}{\partial x^h}dxds \\
&\quad + \int_0^\tau \int_0^1 \sum_{h_1=0}^{h-1} \frac{h!}{h_1!(h-h_1)!} G_i^h(x,\tau-s,y)\Big(\frac{\partial^{h-h_1}\widetilde{a}(y,T)}{\partial y^{h-h_1}} f_i^{h_1+2}(y,s) \\
&\quad + \frac{\partial^{h-h_1}\widetilde{b}(y,T)}{\partial y^{h-h_1}} f_i^{h_1+1}(y,s)\Big)dyds \\
&= \int_0^1 w_i^h(x,\tau,y) f_i^h(y,0)dy + \int_0^\tau \int_0^1 w_i^h(x,\tau-s,y)\frac{\partial^h r_i(y,s)}{\partial x^h}dyds \\
&\quad + \int_0^\tau \int_0^1 \sum_{h_1=0}^{h-1} \frac{h!}{h_1!(h-h_1)!} w_i^h(x,\tau-s,y)\frac{\partial^{h-h_1}\widetilde{a}(y,T)}{\partial y^{h-h_1}} f_i^{h_1+2}(y,s)dyds \\
&\quad + \int_0^\tau \int_0^1 \sum_{h_1=0}^{h-1} \frac{h!}{h_1!(h-h_1)!} w_i^h(x,\tau-s,y)\frac{\partial^{h-h_1}\widetilde{b}(y,T)}{\partial y^{h-h_1}} f_i^{h_1+1}(y,s)dyds \\
&\quad + \int_\tau^\infty \int_0^1 \mu_i^h(y,T)\frac{\partial^h r_i(y,s)}{\partial x^h}dyds \\
&\quad + \int_\tau^\infty \int_0^1 \sum_{h_1=0}^{h-1} \frac{h!}{h_1!(h-h_1)!} \mu_i^h(y,T)\frac{\partial^{h-h_1}\widetilde{a}(y,T)}{\partial y^{h-h_1}} f_i^{h_1+2}(y,s)dyds \\
&\quad + \int_\tau^\infty \int_0^1 \sum_{h_1=0}^{h-1} \frac{h!}{h_1!(h-h_1)!} \mu_i^h(y,T)\frac{\partial^{h-h_1}\widetilde{b}(y,T)}{\partial y^{h-h_1}} f_i^{h_1+1}(y,s)dyds.
\end{aligned}
$$

On the other hand, by Lemma 3.4.9, for all $s > 0$,

$$
\max_{j=0,1,2} \sup_{x\in[0,1]} \left| \frac{\partial^j w_i^h(x,s)}{\partial y^j} \right| \leq C\,\mathrm{e}^{-\gamma s}.
$$

Therefore, we derive from (3.4.22) that

$$
\begin{aligned}
&\left| \int_0^\tau \int_0^1 \sum_{h_1=0}^{h-1} \frac{h!}{h_1!(h-h_1)!} w_i^h(x,\tau-s,y)\frac{\partial^{h-h_1}\widetilde{a}(y,T)}{\partial y^{h-h_1}} f_i^{h_1+2}(y,s)dyds \right| \\
&\quad = \left| \int_0^\tau \int_0^1 \sum_{h_1=0}^{h-1} \frac{h!}{h_1!(h-h_1)!} \frac{\partial^2}{\partial y^2}\left(w_i^h(x,\tau-s,y)\frac{\partial^{h-h_1}\widetilde{a}(y,T)}{\partial y^{h-h_1}} \right) f_i^{h_1}(y,s)dyds \right| \\
&\quad \leq \int_0^\tau C\,\mathrm{e}^{-\gamma(\tau-s)}\,\mathrm{e}^{-\gamma s}\,ds \leq C\,\mathrm{e}^{-\widetilde{\gamma}\tau},
\end{aligned}
$$

and also the following inequality

$$\begin{aligned}
&\left|\int_\tau^\infty \int_0^1 \sum_{h_1=0}^{h-1} \frac{h!}{h_1!(h-h_1)!}\mu_i^h(y,T)\frac{\partial^{h-h_1}\widetilde{a}(y,T)}{\partial y^{h-h_1}} f_i^{h_1+2}(y,s)dyds\right| \\
&\qquad = \left|\int_\tau^\infty \int_0^1 \sum_{h_1=0}^{h-1} \frac{h!}{h_1!(h-h_1)!}\frac{\partial^2}{\partial y^2}\left(\mu_i^h(y,T)\frac{\partial^{h-h_1}\widetilde{a}(y,T)}{\partial y^{h-h_1}}\right) f_i^{h_1}(y,s)dyds\right| \\
&\qquad \le \int_\tau^\infty C\,\mathrm{e}^{-\gamma s}\, ds = C\,\mathrm{e}^{-\gamma\tau}
\end{aligned}$$

for some $0<\widetilde{\gamma}<\gamma$. Similarly, there exists $0<\widehat{\gamma}<\gamma$ such that

$$\begin{aligned}
&\left|\int_0^\tau \int_0^1 \sum_{h_1=0}^{h-1} \frac{h!}{h_1!(h-h_1)!}w_i^h(x,\tau-s,y)\frac{\partial^{h-h_1}\widetilde{b}(y,T)}{\partial y^{h-h_1}} f_i^{h_1+1}(y,s)dyds\right| \le C\,\mathrm{e}^{-\widehat{\gamma}\tau}, \\
&\left|\int_\tau^\infty \int_0^1 \sum_{h_1=0}^{h-1} \frac{h!}{h_1!(h-h_1)!}\mu_i^h(y,T)\frac{\partial^{h-h_1}\widetilde{b}(y,T)}{\partial y^{h-h_1}} f_i^{h_1+1}(y,s)dyds\right| \le C\,\mathrm{e}^{-\widehat{\gamma}\tau}, \\
&\left|\int_0^\tau \int_0^1 w_i^h(x,\tau-s,y)\frac{\partial^h r_i(y,s)}{\partial x^h}dyds\right| \le C\,\mathrm{e}^{-\widehat{\gamma}\tau}, \\
&\left|\int_\tau^\infty \int_0^1 \mu_i^h(y,T)\frac{\partial^h r_i(y,s)}{\partial x^h}dyds\right| \le C\,\mathrm{e}^{-\widehat{\gamma}\tau}, \\
&\left|\int_0^1 w_i^h(x,\tau,y)f_i^h(y,0)dy\right| \le C\,\mathrm{e}^{-\widehat{\gamma}\tau}\,.
\end{aligned}$$

Put $\overline{\gamma}=\min(\widetilde{\gamma},\widehat{\gamma})$. Then

$$\sup_{x\in[0,1]}\left|f_i^h(x,\tau)\right| \le C\,\mathrm{e}^{-\overline{\gamma}\tau}\,.$$

Thus the desired result follows immediately by induction. Now we will verify the above claim. In fact, for any $h_1 \le h$,

$$\begin{aligned}
\frac{\partial f_i^{h_1-1}(x,s)}{\partial s} &= \frac{\partial^{h_1-1}}{\partial x^{h_1-1}}(\widetilde{\mathcal{L}}(x,T)\psi_i(x,s)) + \frac{\partial^{h_1-1}r_i(x,s)}{\partial x^{h_1-1}} \\
&= \sum_{\iota=0}^{h_1-1}\frac{(h_1-1)!}{\iota!(h_1-1-\iota)!}\Big(\frac{\partial^{h_1-1-\iota}}{\partial x^{h_1-1-\iota}}\widetilde{\mathcal{L}}(x,T)\Big)f_i^\iota(x,s) + \frac{\partial^{h_1-1}r_i(x,s)}{\partial x^{h_1-1}}.
\end{aligned}$$

Integrating the above equation, we have

$$-f_i^{h-1}(x,0) = \int_0^\infty \Big(\sum_{h_1=0}^{h-1}\frac{(h-1)!}{h_1!(h-1-h_1)!}\Big(\frac{\partial^{h-1-h_1}}{\partial x^{h-1-h_1}}\widetilde{\mathcal{L}}(x,T)\Big)f_i^{h_1}(x,s) + \frac{\partial^{h-1}r_i(x,s)}{\partial x^{h-1}}\Big)ds.$$

Differentiating with respect to x and noting that

$$\frac{(h-1)!}{h_1!(h-1-h_1)!}+\frac{(h-1)!}{(h_1-1)!(h-h_1)!}=\frac{h!}{h_1!(h-h_1)!},$$

we obtain

$$\begin{aligned}
-f_i^h(x,0) &= \int_0^\infty \Big(\sum_{h_1=0}^{h-1}\frac{(h-1)!}{h_1!(h-1-h_1)!}\Big(\Big(\frac{\partial^{h-h_1}}{\partial x^{h-h_1}}\widetilde{\mathcal{L}}(x,T)\Big)f_i^{h_1}(x,s)\\
&\quad+\Big(\frac{\partial^{h-1-h_1}}{\partial x^{h-1-h_1}}\widetilde{\mathcal{L}}(x,T)\Big)f_i^{h_1+1}(x,s)\Big)+\frac{\partial^h r_i(x,s)}{\partial x^h}\Big)ds\\
&= \int_0^\infty \Big(\sum_{h_1=0}^{h-1}\frac{(h-1)!}{h_1!(h-1-h_1)!}\Big(\frac{\partial^{h-h_1}}{\partial x^{h-h_1}}\widetilde{\mathcal{L}}(x,T)\Big)f_i^{h_1}(x,s)\\
&\quad+\sum_{h_1=1}^{h}\frac{(h-1)!}{(h_1-1)!(h-h_1)!}\Big(\frac{\partial^{h-h_1}}{\partial x^{h-h_1}}\widetilde{\mathcal{L}}(x,T)\Big)f_i^{h_1}(x,s)+\frac{\partial^h r_i(x,s)}{\partial x^h}\Big)ds\\
&= \int_0^\infty \Big(\sum_{h_1=0}^{h}\frac{h!}{h_1!(h-h_1)!}\Big(\frac{\partial^{h-h_1}}{\partial x^{h-h_1}}\widetilde{\mathcal{L}}(x,T)\Big)f_i^{h_1}(x,s)+\frac{\partial^h r_i(x,s)}{\partial x^h}\Big)ds\\
&= \int_0^\infty \Big(\widetilde{\mathcal{L}}(x,T)f_i^h(x,s)+\widetilde{f}_i^h(x,s)\Big)ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
-\big\langle f_i^h(x,0),\mu(x,T)\big\rangle &= \int_0^\infty <\widetilde{\mathcal{L}}(x,T)f_i^h(x,s),\mu(x,T)> ds\\
&\quad+\int_0^\infty <\widetilde{f}_i^h(x,s),\mu(x,T)> ds\\
&= \int_0^\infty <\widetilde{f}_i^h(x,s),\mu(x,T)> ds.
\end{aligned}$$

Hence, the claim is proved. $\square$

Lemma 3.4.12. *There exist constants C and $0<\widetilde{\gamma}<\gamma$ such that for any $0\le i\le n+1$ and $\tau>0$,*

$$\max_{h=0,\ldots,2(n+2-i)}\max_{x\in[0,1]}\left|\frac{\partial^h\psi_i(x,\tau)}{\partial x^h}\right|\le C\,\mathrm{e}^{-\widetilde{\gamma}\tau}\,. \tag{3.4.24}$$

As a result, for any $0\le i\le n+1$ and $\tau>0$,

$$\max_{0\le h_1\le n+2}|\widetilde{\mathcal{L}}^{(h_1)}(x,T)\psi_i(x,\tau)|\le C\,\mathrm{e}^{-\widetilde{\gamma}\tau}$$

and

$$\max_{0\le h_1\le n+2} |(\widehat{\mathcal{L}}+Q)^{(h_1)}(x,T)\psi_i(x,\tau)| \le C\,\mathrm{e}^{-\widetilde{\gamma}\tau}\,.$$

Proof. First, under condition (3.4.9) and by Lemma 3.4.6, we have

$$|\psi_0(x,\tau)| = |\psi_0(x,\tau) - [\psi_0(\cdot,0),\mu(\cdot,T)]\,| \le C\,\mathrm{e}^{-\gamma\tau}\,.$$

Applying Lemma 3.4.11 with $r_0 = 0$, we verify that (3.4.24) holds for $i = 0$. Assume that for any $j < i$,

$$\max_{h=0,\ldots,2(n+2-j)} \sup_{x\in[0,1]} \left|\frac{\partial^h \psi_j(x,\tau)}{\partial x^h}\right| \le C\,\mathrm{e}^{-\gamma\tau}\,.$$

Then for some $0 < \widetilde{\gamma} < \gamma$,

$$\max_{h=0,\ldots,2(n+2-i)} \sup_{x\in[0,1]} \left|\frac{\partial^h r_i(x,\tau)}{\partial x^h}\right| \le C\,\mathrm{e}^{-\gamma\tau}\,.$$

Now, under condition (3.4.19), we again derive from Lemma 3.4.6 that

$$|\psi_i(x,\tau)| = \left|\psi_i(x,\tau) - [\psi_i(\cdot,0),\mu(\cdot,T)] - \int_0^\infty [r_i(\cdot,t),\mu(\cdot,T)]\,dt\right| \le C\,\mathrm{e}^{-\widetilde{\gamma}\tau}\,.$$

Thus Lemma 3.4.11 yields (3.4.24). This completes the proof by induction. □

3.4.3 Error Estimates

For any function f, define

$$\mathcal{D}^\varepsilon f = \frac{\partial f}{\partial t} + Qf + \mathcal{L}^\varepsilon f. \tag{3.4.25}$$

Lemma 3.4.13. *Suppose that $\zeta \in C([0,1]\times[0,T])$ is periodic in $x \in [0,1]$, satisfying*

$$\sup_{(x,t)\in[0,1]\times[0,T]} |\zeta(x,t)| \le C\varepsilon^{\kappa}.$$

Let $\xi^{\varepsilon}(x,t)$ be a solution to

$$\mathcal{D}^{\varepsilon}\xi^{\varepsilon}(x,t) = \zeta(x,t), \qquad \xi^{\varepsilon}(x,T) = 0, \qquad \forall x \in [0,1]. \tag{3.4.26}$$

Then

$$\sup_{(x,t)\in[0,1]\times[0,T]} |\xi^{\varepsilon}(x,t)| \le C\varepsilon^{\kappa}.$$

Proof. Using $\tau = (T-t)/\varepsilon$ and putting $F^{\varepsilon}(x,\tau) = \xi^{\varepsilon}(x,\varepsilon\tau)$, (3.4.26) becomes

$$\frac{\partial F^{\varepsilon}(x,\tau)}{\partial \tau} = (\widetilde{\mathcal{L}}(x,\varepsilon\tau) + \varepsilon\widehat{\mathcal{L}}(x,\varepsilon\tau))F^{\varepsilon}(x,\tau) + \varepsilon Q(x,\varepsilon\tau)F^{\varepsilon}(x,\tau) + \varepsilon\zeta(x,\varepsilon\tau).$$

Since $\widetilde{\mathcal{L}}(x,\varepsilon\tau) + \varepsilon\widehat{\mathcal{L}}(x,\varepsilon\tau)$ is elliptic, there exists a Green's function G_{ε} such that

$$F^{\varepsilon}(x,\tau) = \int_0^{\tau}\int_0^1 G_{\varepsilon}(s,y,\tau,x)\varepsilon\zeta(y,\varepsilon s)dyds + \int_0^{\tau}\int_0^1 G_{\varepsilon}(s,y,\tau,x)\varepsilon Q(y,\varepsilon s)F^{\varepsilon}(y,s)dyds.$$

We have

$$\begin{aligned}
\int_0^1 G_{\varepsilon}(s,y,\tau,x)dy &\le \int_0^1 \sum_{\iota=-\infty}^{\infty} \frac{1}{\sqrt{4\pi c(\tau-s)}} \exp\left(-\frac{(x+\iota-y)^2}{4\pi c(\tau-s)}\right) dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi c(\tau-s)}} \exp\left(-\frac{(x+y)^2}{4\pi c(\tau-s)}\right) dy \\
&= \int_{-\infty}^{\infty} e^{-z^2} dz, \qquad \text{with } z = \frac{x+y}{\sqrt{4\pi c(\tau-s)}} \\
&= \frac{\pi}{2\sqrt{2}}.
\end{aligned}$$

Put $f(\tau) = \sup_{x\in[0,1]} |F^{\varepsilon}(x,\tau)|$. Then

$$f(\tau) \le C\varepsilon^{\kappa+1}\tau + C\varepsilon\int_0^{\tau} f(s)ds \le C\varepsilon^{\kappa} + C\varepsilon\int_0^{\tau} f(s)ds.$$

Using Gronwall's inequality, we obtain

$$f(\tau) \leq C\varepsilon^{\kappa} \exp\left(C\varepsilon \int_0^{\tau} ds\right) \leq C\varepsilon^{\kappa}.$$

Lemma 3.4.13 is thus proved. $\square$

Theorem 3.4.14. *There exists $C > 0$ such that*

$$\sup_{(x,t)\in[0,1]\times[0,T]} \left| u^{\varepsilon}(x,t) - \Phi_n^{\varepsilon}(x,t) - \Psi_n^{\varepsilon}(x, \frac{T-t}{\varepsilon}) \right| \leq C\varepsilon^{n+1}.$$

Proof. Put

$$e^{\varepsilon,\kappa}(x,t) = u^{\varepsilon}(x,t) - \Phi_{\kappa}^{\varepsilon}(x,t) - \Psi_{\kappa}^{\varepsilon}(x,\tau).$$

Then $\mathcal{D}^{\varepsilon} u^{\varepsilon}(x,t) = 0$ and therefore,

$$\mathcal{D}^{\varepsilon} e^{\varepsilon,\kappa}(x,t) = -\mathcal{D}^{\varepsilon}\Phi_{\kappa}^{\varepsilon}(x,t) - \mathcal{D}^{\varepsilon}\Psi_{\kappa}^{\varepsilon}(x, \frac{T-t}{\varepsilon}).$$

Moreover

$$\begin{aligned}
\mathcal{D}^{\varepsilon}\Phi_{\kappa}^{\varepsilon}(x,t) &= \sum_{i=0}^{\kappa} \varepsilon^i \dot{\phi}_i(x,t) + \sum_{i=0}^{\kappa} \varepsilon^{i-1}\widetilde{\mathcal{L}}(x,t)\phi_i(x,t) + \sum_{i=0}^{\kappa} \varepsilon^i (\widehat{\mathcal{L}} + Q)(x,t)\phi_i(x,t) \\
&= \sum_{i=0}^{\kappa} \varepsilon^i (-\widetilde{\mathcal{L}}(x,t)\phi_{i+1}(x,t) - (\widehat{\mathcal{L}} + Q)(x,t)\phi_i(x,t)) + \sum_{i=0}^{\kappa} \varepsilon^{i-1}\widetilde{\mathcal{L}}(x,t)\phi_i(x,t) \\
&\qquad + \sum_{i=0}^{\kappa} \varepsilon^i (\widehat{\mathcal{L}} + Q)(x,t)\phi_i(x,t) \\
&= -\varepsilon^{\kappa}\widetilde{\mathcal{L}}(x,t)\phi_{\kappa+1}(x,t) + \underbrace{\varepsilon^{-1}\widetilde{\mathcal{L}}(x,t)\phi_0(x,t)}_{0}.
\end{aligned}$$

So, by Lemma 3.4.10,

$$|\mathcal{D}^{\varepsilon}\Phi_{\kappa}^{\varepsilon}(x,t)| \leq C\varepsilon^{\kappa}.$$

Using the strethed variable τ, $\varepsilon\frac{\partial}{\partial t}\psi_i(x,\tau) = -\frac{d}{d\tau}\psi_i(x,\tau)$ which yields

$$\begin{aligned}
\mathcal{D}^\varepsilon\Psi^\varepsilon_\kappa\Big(x,\tau\Big) &= \sum_{i=0}^{\kappa} -\varepsilon^{i-1}\frac{d}{d\tau}\psi_i(x,\tau) + \sum_{i=0}^{\kappa}\varepsilon^{i-1}\widetilde{\mathcal{L}}(x,t)\psi_i(x,\tau) + \sum_{i=0}^{\kappa}\varepsilon^{i}(\widehat{\mathcal{L}}+Q)(x,t)\psi_i(x,\tau) \\
&= \sum_{i=0}^{\kappa}\varepsilon^{i-1}(-\widetilde{\mathcal{L}}(x,T)\psi_i(x,\tau) - r_i(x,\tau)) + \sum_{i=0}^{\kappa}\varepsilon^{i-1}\widetilde{\mathcal{L}}(x,t)\psi_i(x,\tau) \\
&\qquad + \sum_{i=0}^{\kappa}\varepsilon^{i}(\widehat{\mathcal{L}}+Q)(x,t)\psi_i(x,\tau) \\
&= \sum_{i=0}^{\kappa}\varepsilon^{i-1}(-\widetilde{\mathcal{L}}(x,T) + \widetilde{\mathcal{L}}(x,t))\psi_i(x,\tau) - \sum_{i=0}^{\kappa}\varepsilon^{i-1}r_i(x,\tau) \\
&\qquad + \sum_{i=0}^{\kappa}\varepsilon^{i}(\widehat{\mathcal{L}}+Q)(x,t)\psi_i(x,\tau).
\end{aligned}$$

The second term is equal to

$$\begin{aligned}
&\sum_{i=0}^{\kappa}\varepsilon^{i-1}r_i(x,\tau) \\
&= \sum_{i=1}^{\kappa}\varepsilon^{i-1}\sum_{j=0}^{i-1}\left((-\tau)^{i-j}\frac{\widetilde{\mathcal{L}}^{(i-j)}(x,T)}{(i-j)!} + (-\tau)^{i-j-1}\frac{(\widehat{\mathcal{L}}+Q)^{(i-j-1)}(x,T)}{(i-j-1)!}\right)\psi_j(x,\tau) \\
&= \sum_{j=0}^{\kappa-1}\sum_{i=j+1}^{\kappa}\left(\varepsilon^{i-1}(-\tau)^{i-j}\frac{\widetilde{\mathcal{L}}^{(i-j)}(x,T)}{(i-j)!} + \varepsilon^{i-1}(-\tau)^{i-j-1}\frac{(\widehat{\mathcal{L}}+Q)^{(i-j-1)}(x,T)}{(i-j-1)!}\right)\psi_j(x,\tau) \\
&= \sum_{j=0}^{\kappa-1}\sum_{i=j+1}^{\kappa}\left(\varepsilon^{j-1}(t-T)^{i-j}\frac{\widetilde{\mathcal{L}}^{(i-j)}(x,T)}{(i-j)!} + \varepsilon^{j}(t-T)^{i-j-1}\frac{(\widehat{\mathcal{L}}+Q)^{(i-j-1)}(x,T)}{(i-j-1)!}\right)\psi_j(x,\tau) \\
&= \sum_{j=0}^{\kappa-1}\varepsilon^{j-1}\left(\sum_{i=1}^{\kappa-j}(t-T)^{i}\frac{\widetilde{\mathcal{L}}^{(i)}(x,T)}{i!}\right)\psi_j(x,\tau) \\
&\qquad + \sum_{j=0}^{\kappa-1}\varepsilon^{j}\left(\sum_{i=0}^{\kappa-j-1}(t-T)^{i}\frac{(\widehat{\mathcal{L}}+Q)^{(i)}(x,T)}{i!}\right)\psi_j(x,\tau).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{D}^\varepsilon\Psi^\varepsilon_\kappa(x,\tau) \;=\;& \varepsilon^{\kappa-1}(-\widetilde{\mathcal{L}}(x,T) + \widetilde{\mathcal{L}}(x,t))\psi_\kappa(x,\tau) + \varepsilon^{\kappa}(\widehat{\mathcal{L}}+Q)(x,t)\psi_\kappa(x,\tau) \\
&+ \sum_{j=0}^{\kappa-1}\varepsilon^{j-1}\left(\widetilde{\mathcal{L}}(x,t) - \sum_{i=0}^{\kappa-j}(t-T)^{i}\frac{\widetilde{\mathcal{L}}^{(i)}(x,T)}{i!}\right)\psi_j(x,\tau) \\
&+ \sum_{j=0}^{\kappa-1}\varepsilon^{j}\left((\widehat{\mathcal{L}}+Q)(x,t) - \sum_{i=0}^{\kappa-j-1}(t-T)^{i}\frac{(\widehat{\mathcal{L}}+Q)^{(i)}(x,T)}{i!}\right)\psi_j(x,\tau).
\end{aligned} \tag{3.4.27}$$

Using Taylor expression and Proposition 3.4.12, we obtain

$$\begin{aligned}|\mathcal{D}^\varepsilon \Psi_\kappa^\varepsilon(x,\tau)| &\le C\varepsilon^{\kappa-1}|t-T|\,\mathrm{e}^{-\gamma\tau} + C\varepsilon^\kappa + C\sum_{j=0}^{\kappa-1}\varepsilon^{j-1}\,|t-T|^{\kappa-j+1}\,\mathrm{e}^{-\gamma\tau}\\ &\quad + C\sum_{j=0}^{\kappa-1}\varepsilon^{j}\,|t-T|^{\kappa-j}\,\mathrm{e}^{-\gamma\tau}\\ &= C\varepsilon^\kappa\tau\,\mathrm{e}^{-\gamma\tau} + C\varepsilon^\kappa + C\sum_{j=0}^{\kappa-1}\varepsilon^\kappa\tau^{\kappa-j+1}\,\mathrm{e}^{-\gamma\tau} + C\sum_{j=0}^{\kappa-1}\varepsilon^\kappa\tau^{\kappa-j}\,\mathrm{e}^{-\gamma\tau}\\ &\le C\varepsilon^\kappa.\end{aligned}$$

Piecing this together with the estimates on $\mathcal{D}^\varepsilon\Phi_\kappa(x,t)$, we have shown that

$$\sup_{(x,t)\in[0,1]\times[0,T]} |\mathcal{D}^\varepsilon e^{\varepsilon,\kappa}(x,t)| \le C\varepsilon^\kappa$$

for $k \le n+1$. Note the terminal condition $e^{\varepsilon,\kappa}(x,T)=0$. Thus Lemma 3.4.13 implies

$$\sup_{(x,t)\in[0,1]\times[0,T]} |e^{\varepsilon,\kappa}(x,t)| \le C\varepsilon^\kappa.$$

Taking $\kappa = n+1$, we obtain

$$\sup_{(x,t)\in[0,1]\times[0,T]} |e^{\mathrm{e},n+1}(x,t)| = O(\varepsilon^{n+1}).$$

Finally, note that

$$e^{\varepsilon,n+1}(x,t) = e^{\varepsilon,n}(x,t) + \varepsilon^{n+1}\phi_{n+1}(x,t) + \varepsilon^{n+1}\psi_{n+1}(x,\tau). \tag{3.4.28}$$

The continuity of $\phi_{n+1}(x,t)$ and the exponential decay properties of $\psi_{n+1}(x,\tau)$ yield that

$$\sup_{(x,t)\in[0,1]\times[0,T]} |\varepsilon^{n+1}\phi_{n+1}(x,t) + \varepsilon^{n+1}\psi_{n+1}(x,\tau)| \le C\varepsilon^{n+1}.$$

Substituting this into (3.4.28), we obtain

$$\sup_{(x,t)\in[0,1]\times[0,T]} |e^{\varepsilon,n}(x,t)| \le C\varepsilon^{n+1}$$

as desired. □

3.5 Illustrations and Remarks

3.5.1 Illustrations

Asymptotic expansions have been obtained in this paper. In this section, we provide some interpretations of our results.

To illustrate, let us begin with a simple case. Consider (3.2.1) together with (3.2.2), in which $Q^\varepsilon(x) = Q(x)/\varepsilon$ and $Q(x,t)$ is weakly irreducible. Now (3.2.1) and (3.2.2) can be written as

$$\begin{aligned} &dX^\varepsilon(t) = b(X^\varepsilon(t), \alpha^\varepsilon(t))dt + \sigma(X^\varepsilon(t), \alpha^\varepsilon(t))dB(t), \\ &P(\alpha^\varepsilon(t+\Delta) = \ell | \alpha^\varepsilon(t) = k, X(t) = x) = q^\varepsilon_{k\ell}(x)\Delta + o(\Delta). \end{aligned} \tag{3.5.1}$$

Using weak convergence methods (see e.g., Yin and Zhang [52]), one can show that $X^\varepsilon(\cdot)$ converges weakly to $X(\cdot)$ such that $X(\cdot)$ is the solution of

$$dX(t) = \overline{b}(X(t))dt + \overline{\sigma}(X(t))dB(t), \tag{3.5.2}$$

where

$$\begin{aligned} &\overline{b}(x) = \sum_{i=1}^m b(x,i)\nu_i(x), \quad \overline{\sigma}(x) = \sqrt{\sum_{i=1}^m \sigma^2(x,i)\nu_i(x)}, \quad \text{and} \\ &\nu(x) = (\nu_1(x), \dots, \nu_m(x)) \end{aligned} \tag{3.5.3}$$

is the quasi-stationary distribution. The asymptotic results obtained in this paper gives us more than those only obtained by the weak convergence. It provides new insight even for the leading term in the asymptotic expansion.

Suppose that $U(x, \alpha)$ is a smooth functional. Our asymptotic expansions (e.g., Theorem 3.3.10) and the probabilistic interpretation of the solution of the backward equation enable us to conclude that for any $t > K\varepsilon \ln(1/\varepsilon)$ and some $K > 0$,

$$EU(X^\varepsilon(t), \alpha^\varepsilon(t)) \to E\overline{U}(X(t)) = E\sum_{i=1}^{m} U(X(t), i)\nu_i(X(t)).$$

Next, consider $\widetilde{U}(x, \alpha, t) = 1_A 1_B$ (the indicators of A and B, resp.), which can be thought of as an approximation to the smooth function U. Then we have

$$P(X^\varepsilon(t) \in A, \alpha^\varepsilon(t) \in B) \to \sum_{i\in B} \int_A P(X(t) \in dx)\nu_i(x) \quad \text{as} \quad \varepsilon \to 0.$$

In particular, when $A = [0, 1]$,

$$P(\alpha^\varepsilon(t) \in B) \to \sum_{i\in B} E\nu_i(X(t)) \quad \text{as} \quad \varepsilon \to 0.$$

As a convention, $X^{x,i}(t)$ denotes the process $X(t)$ starting at $X(0) = x$ and $\alpha(t) = i$. By virtue of the Markov property of $(X^\varepsilon(t), \alpha^\varepsilon(t))$, for $0 < t_1 < t_2$ not depending on ε,

$$\begin{aligned}
&P(\alpha^{\varepsilon,x,i}(t_1) \in A_1, \alpha^{\varepsilon,x,i}(t_2) \in A_2) \\
&= \sum_{i_1\in A_1} \int_0^1 P(X^{\varepsilon,x,i}(t) \in dx_1, \alpha^{\varepsilon,x,i}(t) = i_1)P(\alpha^{\varepsilon,x_1,i_1}(t_2 - t_1) \in A_2) \\
&\to \sum_{i_1\in A_1} \int_0^1 P(X^x(t) \in dx_1)\nu_{i_1}(x_1) \sum_{i_2\in A_2} \nu_{i_2}(X^{x_1}(t_2 - t_1)) \quad \text{as} \quad \varepsilon \to 0 \\
&= \sum_{i_1\in A_1} \sum_{i_2\in A_2} E\nu_{i_1}(X^x(t_1))E\nu_{i_2}(X^{X^x(t_1)}(t_2 - t_1)) \\
&= \sum_{i_1\in A_1} \sum_{i_2\in A_2} E[\nu_{i_1}(X^x(t_1))E(\nu_{i_2}(X^x(t_2))|X^x(t_1))] \\
&= E\sum_{i_1\in A_1} \sum_{i_2\in A_2} \nu_{i_1}(X^x(t_1))\nu_{i_2}(X^x(t_2)).
\end{aligned}$$

In fact, by induction, we obtain the finite dimensional distributions

$$\begin{aligned}
&P(\alpha^{\varepsilon,x,i}(t_1) \in A_1, \alpha^{\varepsilon,x,i}(t_2) \in A_2, \dots, \alpha^{\varepsilon,x,i}(t_n) \in A_n) \\
&\to E\sum_{i_1\in A_1} \sum_{i_2\in A_2} \cdots \sum_{i_n\in A_n} \nu_{i_1}(X^x(t_1))\nu_{i_2}(X^x(t_2)) \cdots \nu_{i_n}(X^x(t_n)) \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

As another illustration, consider a control problem with the cost function given by

$$J^{\varepsilon}(x,\alpha,u(\cdot)) = E_{x,\alpha}\int_0^T C(X^{\varepsilon}(t),\alpha^{\varepsilon}(t),u(t))dt, \tag{3.5.4}$$

where $(X^{\varepsilon}(\cdot),\alpha^{\varepsilon}(\cdot))$ is given by (3.5.1). Generally, the problem is difficult to solve due to the complexity of the problem setup. Using our asymptotic expansions, we can show that there is an associated cost function for the limit problem

$$J(x,u(\cdot)) = E_x\int_0^T \overline{C}(X(t),u(t))dt, \tag{3.5.5}$$

where $X(t)$ is given in (3.5.2) and $\overline{C}(x,u) = \sum_{i=1}^{m} C(x,i)\nu_i(x)$ as defined in (3.5.3). We can then find optimal control of the limit problem. Using this optimal control in the original system, we can obtain asymptotic optimal control under suitable conditions.

3.5.2 Remarks

Let us remark on the case that the switching process has a more complex structure.

$$\widetilde{Q}(x,t) = \begin{pmatrix} \widetilde{Q}^1(x,t) & & & & \\ & \ddots & & & \\ & & \widetilde{Q}^l(x,t) & & \\ \widetilde{Q}^1_*(x,t) & \dots & \widetilde{Q}^l_*(x,t) & \widetilde{Q}_*(x,t) & \\ & & & & 0_{m_a\times m_a} \end{pmatrix}, \tag{3.5.6}$$

That is, the process includes recurrent states, absorbing states, and transient states. We denote $m = m_1 + \cdots + m_l + m_* + m_a$, and

$$\widetilde{\mathbb{1}}(x,t) = \begin{pmatrix} \mathbb{1}_{m_1} & & & & \\ & \ddots & & & \\ & & \mathbb{1}_{m_l} & & \\ d^1(x,t) & \dots & d^l(x,t) & & \\ & & & & I_{m_a} \end{pmatrix},$$

where

$$d_k(x,t) = -\widetilde{Q}_*^{-1}(x,t)\widetilde{Q}_*^k(x,t)\mathbb{1}_{m_k}(x,t) \in \mathbb{R}^{m_* \times m_k}, \quad \text{for} \quad k = 1, \dots, l.$$

It is readily seen that $\widetilde{Q}(x,t)\widetilde{\mathbb{1}}(x,t) = 0$ for each $x \in [0,1]$ and $t \in [0,T]$. Denote also

$$\nu(x,t) = \begin{pmatrix} \nu^1(x,t) & & & 0_{1\times m_*} & \\ & \ddots & & \vdots & \\ & & \nu^l(x,t) & 0_{1\times m_*} & \\ & & & & I_{m_a} \end{pmatrix} \in \mathbb{R}^{l\times m},$$

$$\overline{P}(x) = \widetilde{\mathbb{1}}(x,T)\nu(x,T) = \begin{pmatrix} \mathbb{1}_{m_1}\nu^1(x,T) & & & 0_{m_1\times m_*} & \\ & \ddots & & \vdots & \\ & & \mathbb{1}_{m_l}\nu^l(x,T) & 0_{m_l\times m_*} & \\ d^1(x,T)\nu^1(x,T) & \dots & d^l(x,T)\nu^l(x,T) & 0_{m_*\times m_*} & \\ & & & & I_{m_a} \end{pmatrix}.$$

Analogue calculations can be carried out and a similar asymptotic expansion can be constructed.

For non-diffusion case, we focus on the system of equations (see Chiang [11, p.

402]),

$$\begin{aligned} &\frac{d}{dt}u^{\varepsilon}(t) = -Q(t)u^{\varepsilon}(t), \\ &u^{\varepsilon}(T) = u_0, \end{aligned} \tag{3.5.7}$$

for some $0 < T < \infty$, where $u^{\varepsilon}(t) \in \mathbb{R}^{m\times 1}$, $Q(t) = \dfrac{\widetilde{Q}(t)}{\varepsilon} + \widehat{Q}(t)$ for some generators $\widetilde{Q}(t)$ and $\widehat{Q}(t)$, and

$$\widetilde{Q}(t) = \begin{pmatrix} \widetilde{Q}^1(t) & & & \\ & \ddots & & \\ & & \widetilde{Q}^l(t) & \\ \widetilde{Q}^1_*(t) & \dots & \widetilde{Q}^l_*(t) & \widetilde{Q}_*(t) \end{pmatrix}.$$

Under the following conditions

- For each $i = 1, \dots, l$, and each $t \in [0, T]$, $\widetilde{Q}^i(t)$ is weakly irreducible with the associated quasi-stationary distribution denoted by $\nu^i(t)$,
- For some positive integer n, $\widetilde{Q}(\cdot)$ and $\widehat{Q}(\cdot)$ are $(n+2)$-times continuously differentiable,
- For each $t \in [0, T]$, $\widetilde{Q}_*(t)$ is Hurwitz (i.e., all of its eigenvalues have negative real parts),

A similar asymptotic expansion can also be constructed and the corresponding error bound can also be obtained.

Chapter 4

Stability of Singular Jump-Linear Systems with A Large State Space: A Two-time-scale Approach

4.1 Introduction

Singular systems, which have many synonyms such as descriptor systems, generalized systems, and implicit systems, are featured in differential-algebraic equations (DAEs). They arise in various applications in physical sciences, engineering, and economic systems. Owing to their importance, such systems have been studied extensively and used widely in control and optimization applications. For some recent literature, we refer the reader to Campbell [8, 9], Cheng et al. [10], Dai [13], Huang and Mao [21], Lewis [34] among others. While the references mentioned above are all concerned with deterministic systems, recent works also include formulation, analysis, and computation involving stochastic systems; see for instance, Boukas [4], Boukas et al. [5], Huang and Mao [21], Yin and Zhang [51], among others.

The main motivations of this chapter are from the following two aspects. First it is motivated by the recent stability analysis in Huang and Mao [21] for analyzing stability of stochastic systems with Markov regime switching. Second, it is motivated by the two-time-scale formulation of Markov chains; see for example, Yin and Zhang [51, 52]. In this chapter, we treat a system similar to Huang and Mao [21], but the discrete state space is very large. We focus on stability analysis. By sending $\varepsilon \to 0$, we obtain a limit system with reduced state space for an aggregated switching process. Knowing the stability of the limit system, we aim to obtain stability of the original system under suitable conditions.

The rest of this chapter is organized as follows. The precise problem formulation is given next. Section 4.3 presents a number of preliminary results. Section 4.4 focuses on stability of the underlying singular systems. Our approach is along the line of two-time-scale approach. Under broad conditions, we show that by use of the limit system, we can obtain stability of the original system. Section 4.5 presents a couple of examples for demonstration. Finally, Section 4.6 gives some further remarks and concludes the paper.

4.2 Problem formulation

Suppose that $\alpha(t)$ is a continuous-time Markov chain taking values in a finite state space $\mathcal{M}$. In this paper, we consider the switching process $\alpha(t)$ having fast and slowly varying transitions in that the generator of the Markov chain is given by

$$Q^{\varepsilon} = \frac{\widetilde{Q}}{\varepsilon} + \widehat{Q}, \tag{4.2.1}$$

where

$$\widetilde{Q} = \mathrm{diag}(\widetilde{Q}^1, \ldots, \widetilde{Q}^l), \tag{4.2.2}$$

where $\mathrm{diag}(D^1, \ldots, D^l)$ denotes a diagonal block matrix with entries $D^1, \ldots, D^l$, and $\widehat{Q}$ is another generator without specific structure. Because of the structure of the matrix $\widetilde{Q}$ in (4.2.2), we write the state space $\mathcal{M}$ as

$$\begin{aligned} &\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_l, \quad \text{where} \\ &\mathcal{M}_i = \{s_{i1}, \ldots, s_{im_i}\} \ \text{ for } \ i = 1, \ldots, l. \end{aligned}$$

To indicate the ε-dependence of the Markov chain, we write it as $\alpha^{\varepsilon}(t)$ henceforth. Let $B(\cdot)$ be a standard real-valued Brownian motion.

Throughout the paper, we use the following condition for the fast changing part of the generator $\widetilde{Q}$.

(B1) For $i = 1, \ldots, l$, the generator $\widetilde{Q}^i$ is irreducible.

Here, by irreducibility, we meant that the systems of equations

$$\begin{aligned} \nu^i \widetilde{Q}^i &= 0 \\ \nu^i \mathbb{1}_i &= 1 \end{aligned}$$

has a unique nonnegative solution. In the above $\nu^i \in \mathbb{R}^{1\times m_i}$ and $\mathbb{1}_i = (1, \ldots, 1)' \in \mathbb{R}^{m_i \times 1}$. The ν^i is nothing but the stationary distribution associated with the generator $\widetilde{Q}^i$. In what follows, we also use the notation $q_{s_{ij}s_{\kappa\iota}}$ to denote the $(m_1 + \cdots + m_i + j, m_1 + \cdots + m_\kappa + \iota)^{th}$ entry of a given matrix Q.

Suppose that for each $s_{ij} \in \mathcal{M}$, $A(s_{ij})$, $G(s_{ij})$, and $H(s_{ij})$ are $n \times n$ matrices such that $G(s_{ij})$ is singular. Our interest lies in the following system of switching linear systems:

$$
\begin{aligned}
&G(\alpha^\varepsilon(t))dx^\varepsilon(t) = A(\alpha^\varepsilon(t))x^\varepsilon(t)dt + H(\alpha^\varepsilon(t))x^\varepsilon(t)dB(t),\\
&x^\varepsilon(0) = \xi, \quad \alpha^\varepsilon(0) = \iota = s_{ij},
\end{aligned}
\tag{4.2.3}
$$

for some $i \in \{1, \ldots, l\}$ and $j = 1, \ldots, m_i$, where $B(\cdot)$ is an n-dimensional standard Brownian motion. We aim at studying the stability of the system above. The difficulty lies in that the system is singular, so that the standard stability analysis techniques do not carry over.

4.3 Preliminary Results

We will use the following assumptions.

(B2) For any $i = 1, \ldots, l$ and any $j \in \mathcal{M}_i$, the triplet (G, A, H) satisfies one of the following conditions

a. $\det(sG(s_{ij}) - A(s_{ij})) \neq 0$ for some s, $\deg(\det(sG(s_{ij}) - A(s_{ij}))) = r_{ij}$ and $\operatorname{rank}([G(s_{ij})\ H(s_{ij})]) = rij$.

b. $\det(sG(s_{ij}) - H(s_{ij})) \neq 0$ for some s, $\deg(\det(sG(s_{ij}) - H(s_{ij}))) = r_{ij}$ and $\operatorname{rank}([G(s_{ij})\ A(s_{ij})]) = r_{ij}$.

Denote by $\{\tau_k\}$ a sequence of jump times of the Markov chain $\alpha^\varepsilon(t)$, namely, $\tau_0^\varepsilon = 0$ and $\tau_{k+1}^\varepsilon = \inf\{t > \tau_k^\varepsilon : \alpha^\varepsilon(t) \neq \alpha^\varepsilon(\tau_k)\}$. Then $\alpha^\varepsilon(t) = \alpha^\varepsilon(\tau_k)$ on $[\tau_k^\varepsilon, \tau_{k+1}^\varepsilon)$. Moreover, $\tau_k \to \infty$ as $k \to \infty$.

Lemma 4.3.1. *If* (B2) *holds, then* (4.2.3) *has a unique solution.*

Proof. Assume that (B2)-b is valid. The following argument is similar if (B2)-a holds. For convenience, denote $s_{ij} = \alpha^{\varepsilon}_{\tau_0}$. There exist nonsingular $n \times n$ matrices $L(s_{ij})$, $R(s_{ij})$ such that

$$L(s_{ij})G(s_{ij})R(s_{ij}) = \begin{pmatrix} I_{r_{ij}} & 0 \\ 0 & 0 \end{pmatrix},$$

$$L(s_{ij})A(s_{ij})R(s_{ij}) = \begin{pmatrix} A_1(s_{ij}) & A_2(s_{ij}) \\ 0 & 0 \end{pmatrix},$$

$$L(s_{ij})H(s_{ij})R(s_{ij}) = \begin{pmatrix} H_1(s_{ij}) & 0 \\ 0 & I_{n-r_{ij}} \end{pmatrix},$$

where $A_1(s_{ij})$ and $H_1(s_{ij})$ are $r_{ij} \times r_{ij}$ matrices and $A_2(s_{ij})$ is an $r_{ij} \times (n - r_{ij})$ matrix. Let

$$w^{\varepsilon}(t) = R^{-1}x^{\varepsilon}(t) = [[w_1^{\varepsilon}(t)]' \; [w_2^{\varepsilon}(t)]']'.$$

Then (4.2.3) is equivalent to

$$\begin{cases} dw_1^{\varepsilon}(t) = [A_1(\alpha^{\varepsilon}(t))w_1^{\varepsilon}(t) + A_2(\alpha^{\varepsilon}(t))z_2^{\varepsilon}(t)]\,dt + H_1(\alpha^{\varepsilon}(t))w_1^{\varepsilon}(t)dB_t, \\ 0 = w_2^{\varepsilon}(t)dB_t, \\ w^{\varepsilon}(0) = R^{-1}(\iota)\xi, \alpha^{\varepsilon}(0) = \iota \in \mathcal{M}, \end{cases} \tag{4.3.1}$$

or

$$\begin{cases} dw_1^{\varepsilon}(t) = A_1(\alpha^{\varepsilon}(t))w_1^{\varepsilon}(t)dt + H_1(\alpha^{\varepsilon}(t))w_1^{\varepsilon}(t)dB_t, \\ w_2^{\varepsilon}(t) = 0, \\ w^{\varepsilon}(0) = R^{-1}(\iota)\xi, \alpha^{\varepsilon}(0) = \iota \in \mathcal{M}, \end{cases} \tag{4.3.2}$$

which has a unique solution on interval $[\tau_0, \tau_1]$. Continuing this process, we can prove that (4.2.3) has a unique solution for all $t \geq 0$ by induction. $\square$

4.4 Stability

In order to find stability conditions on the limit processes instead of the original processes, we lump the states in each $\mathcal{M}_i$ into a single state and define

$$\overline{\alpha}^{\varepsilon}(t) = i \text{ if } \alpha^{\varepsilon}(t) \in \mathcal{M}_i.$$

Denote the state space of $\overline{\alpha}^{\varepsilon}(\cdot)$ by

$$\overline{\mathcal{M}} = \{1, \ldots, l\}, \quad \text{and denote} \quad \widetilde{\nu} = \operatorname{diag}\left(\nu^1, \ldots, \nu^l\right),$$

where ν^k is the stationary distribution corresponding to $\widetilde{Q}^k$. Define

$$\begin{aligned}
&\overline{Q} = \widetilde{\nu}\widehat{Q}\mathbb{1}, \quad \text{where} \\
&\mathbb{1} = \operatorname{diag}\left(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l}\right), \\
&\mathbb{1}_k = (1, \ldots, 1)' \in \mathbb{R}^{k\times 1}.
\end{aligned}$$

For $i \in \overline{\mathcal{M}}_i$, denote

$$\begin{aligned}
\widehat{G}(i) &= \sum_{j=1}^{m_i} \nu_j^i G(s_{ij}), \\
\widehat{A}(i) &= \sum_{j=1}^{m_i} \nu_j^i A(s_{ij}), \\
\widehat{H}(i) &= \sum_{j=1}^{m_i} \nu_j^i H(s_{ij}).
\end{aligned}$$

We need the following assumption for our further analysis

(B3) For $i \in \overline{\mathcal{M}}$, $\operatorname{rank}(G(s_{i1})) = \cdots = \operatorname{rank}(G(s_{im_i})) = \operatorname{rank}\left([G'(s_{i1})|\cdots|G'(s_{im_i})]\right)$,

Remark 4.4.1. (B3) is equivalent to

(B3') For $i = 1, \ldots, l$, there exists a corresponding sequence of elementary row operations transforming $\{G'(s_{ij})\}_{j\in\mathcal{M}_i}$ into row echelon matrices.

We derive from (B3) that, for any $s_{ij} \in \mathcal{M}$, there exist non-singular $n \times n$ matrices $L(s_{ij})$ and $R(s_{ij})$ such that

$$L(s_{ij})G(s_{ij})R(s_{ij}) = \begin{pmatrix} I_{r_i} & 0 \\ 0 & 0 \end{pmatrix},$$

where $r_{ij} = r_i$ for all $j = 1, \ldots, m_i$ and $R(s_{ij})$ could be chosen to be the same matrix, denoted by $\widehat{R}(i)$, for all $s_{ij} \in \mathcal{M}_i$.

For any $i \in \overline{\mathcal{M}}$, put

$$\widehat{L}(i) = \left(\sum_{j=1}^{m_i} \nu_j^i L^{-1}(s_{ij}) \right)^{-1}.$$

Then

$$\widehat{L}(i)\widehat{G}(i)\widehat{R}(i) = \begin{pmatrix} I_{r_i} & 0 \\ 0 & 0 \end{pmatrix}.$$

For any $s_{ij} \in \mathcal{M}$, denote

$$\breve{G}(s_{ij}) = \widehat{G}(i), \breve{L}(s_{ij}) = \widehat{L}(i), \ \breve{R}(s_{ij}) = \widehat{R}(i).$$

Then

$$\breve{L}(s_{ij})\breve{G}(s_{ij}) = L(s_{ij})G(s_{ij}).$$

Given any $U \in \mathbb{R}^{n\times n}$, $\breve{U} = \frac{1}{2}(U + U')$ is a symmetric matrix. Let

$$V(x, s_{ij}) = x'\breve{G}'(s_{ij})U\breve{G}(s_{ij})x.$$

Then

$$V(x, s_{ij}) = x'\breve{G}'(s_{ij})\breve{U}\breve{G}(s_{ij})x = x'G'(s_{ij})\widehat{U}(s_{ij})G(s_{ij})x,$$

where

$$\widehat{U}(s_{ij}) = L'(s_{ij})\breve{L}'^{-1}(s_{ij})\breve{U}\breve{L}^{-1}(s_{ij})L(s_{ij}).$$

For a suitable function V, define

$$\mathcal{L}^{\varepsilon}V(x,\kappa) = \lim_{s\to 0^+}\left[E\left(V(x^{\varepsilon}(t+s),\alpha^{\varepsilon}(t+s))|x^{\varepsilon}(t)=x,\alpha^{\varepsilon}(t)=\kappa\right) - V(x,\kappa)\right]$$

Thus

$$\begin{aligned}\mathcal{L}^{\varepsilon}V(x,\kappa) = x'\Big\{&A'(\kappa)\widehat{U}(\kappa)G(\kappa) + G'(\kappa)\widehat{U}(\kappa)A(\kappa) + H'(\kappa)\widehat{U}(\kappa)H(\kappa)\\ &+\sum_{\iota\in\mathcal{M}}\widehat{q}_{\kappa\iota}G'(\iota)\widehat{U}(\iota)G(\iota)\Big\}x\end{aligned} \tag{4.4.1}$$

Denote by $\otimes$ the Kronecker product of matrices. We shall use operators $\langle\cdot\rangle$ and $\langle\cdot,\cdot\rangle$ defined by

$$\langle A\rangle = A\otimes A \ \text{ and } \ \langle A,B\rangle = A\otimes B + B\otimes A.$$

Let $z^{\varepsilon}(t) = \mathbb{E}\langle x^{\varepsilon}(t)\rangle$. Using a similar argument as that in Huang and Mao [21], we obtain

$$\begin{aligned}&\widetilde{G}(\alpha^{\varepsilon}(t))\dot{z}^{\varepsilon}(t) = \widetilde{A}(\alpha^{\varepsilon}(t))z^{\varepsilon}(t),\\ &z^{\varepsilon}(0) = \widetilde{\xi} = E\langle\xi\rangle,\ \alpha^{\varepsilon}(0) = \iota.\end{aligned} \tag{4.4.2}$$

where

$$\begin{aligned}&z^{\varepsilon}(0) = \widetilde{\xi} = \langle\xi\rangle,\\ &\widetilde{G}(s_{ij}) = \left\langle\breve{G}(s_{ij})\right\rangle = \left\langle\widehat{G}(i)\right\rangle \equiv \overline{G}(i),\\ &\breve{A}(s_{ij}) = \breve{L}^{-1}(s_{ij})L(s_{ij})A(s_{ij}),\\ &\breve{H}(s_{ij}) = \breve{L}^{-1}(s_{ij})L(s_{ij})H(s_{ij}),\\ &\widetilde{A}(s_{ij}) = \left\langle\breve{G}(s_{ij}),\breve{A}(s_{ij})\right\rangle + \left\langle\breve{H}(s_{ij})\right\rangle + \sum_{\kappa=1}^{l}\sum_{\iota=1}^{m_\kappa}\widehat{q}^{\varepsilon}_{s_{ij}\ s_{\kappa\iota}}\widetilde{G}(s_{\kappa\iota}).\end{aligned}$$

For $i\in\overline{\mathcal{M}}$, denote

$$\overline{A}(i) = \left\langle\widehat{G}(i),\widehat{A}(i)\right\rangle + \left\langle\widehat{H}(i)\right\rangle + \sum_{\kappa=1}^{l}\overline{q}_{i\kappa}\widehat{G}(\kappa).$$

The following result can be proved similar to the development in Yin and Zhang [51]; the details are omitted.

Proposition 4.4.2. *Let $z^\varepsilon(t)$ be the solution of (4.4.2). Then as $\varepsilon \to 0$, $z^\varepsilon(\cdot)$ weakly converges to $z(\cdot)$, a solution of the singular system of differential equations*

$$\begin{aligned} &\overline{G}(\overline{\alpha}(t))\dot{z}(t) = \overline{A}(\overline{\alpha}(t))z(t), \\ &z(0) = \widetilde{\xi}, \end{aligned} \tag{4.4.3}$$

To carry out the stability analysis, we need more assumptions.

(B4) There exists a nonsingular matrix $\overline{P}(i)$ for each $i = 1, \ldots, l$ such that

a. $\overline{G}'(i)\overline{P}(i) = \overline{P}'(i)\overline{G}'(i) \geq 0$.

b. $\overline{A}'(i)\overline{P}(i) + \overline{P}'(i)\overline{A}(i) + \sum_{i_1 \in \overline{\mathcal{M}}} \overline{q}_{ii_1}\overline{G}'(i_1)\overline{P}(i_1) < 0$.

In the above, the notation ≥ 0 and < 0 are in the sense of ordering for positive definite matrices.

(B5) For each $s_{ij} \in \mathcal{M}$, $\left(\widetilde{G}(s_{ij}), \widetilde{A}(s_{ij})\right)$ is impulse-free, i.e.,

$$\deg\left[\det\left(s\widetilde{G}(s_{ij}) - \widetilde{A}(s_{ij})\right)\right] = \operatorname{rank}\widetilde{G}(s_{ij}).$$

Remark 4.4.3. *Note that $\widetilde{G}(s_{ij}) = \overline{G}(i)$. Besides, we can compute $L(s_{ij})$ involved in (B5) for any $s_{ij} \in \mathcal{M}_i$ in the following way: Use elementary row operations to transform*

$$\left[G'(s_{i1})\middle|\cdots\middle|G'(s_{im_i})\right]$$

into a row echelon matrix

$$\left[\c{G}'(s_{i1})\middle|\cdots\middle|\c{G}'(s_{im_i})\right].$$

Again, use elementary row operations to transform $\left[\c{G}(s_{ij})\middle|I_n\right]$ into the reduced row echelon matrix $\left[\begin{pmatrix} I_{r_i} & 0 \\ 0 & 0 \end{pmatrix}\middle|L(s_{ij})\right]$, where I_n is an $n\times n$ identity matrix. In addition, *if $R(s_{ij})$'s are the same for any $s_{ij} \in \mathcal{M}_i$ then $L(i)$ could be relaxed and*

$$\widetilde{A}(s_{ij}) = \left\langle \widehat{G}(i), A(s_{ij}) \right\rangle + \langle H(s_{ij}) \rangle + \sum_{\kappa=1}^{l} \sum_{\iota=1}^{m_\kappa} \widehat{q}^{\varepsilon}_{s_{ij} \ s_{\kappa\iota}} \widetilde{G}(s_{\kappa\iota})$$

Lemma 4.4.4. *Assume* (B1), (B3)–(B5) *hold. Then there exists constants $c_1 > 0$ and $c_2 > 0$ such that*

(i) $|\dot{z}^\varepsilon(t)| \le c_1 |z^\varepsilon(t)|$.

(ii) $[z^\varepsilon(t)]' \overline{G}'(\overline{\alpha}^\varepsilon(t)) \overline{P}(\overline{\alpha}^\varepsilon(t)) z^\varepsilon(t) \ge c_2 |z^\varepsilon(t)|^2$.

Remark 4.4.5. The inequalities involves matrices. Thus, the second inequality is in the sense of matrix bound. The order of the matrices is determined by the ordering of positive definite matrices.

Proof. The proof is divided into a couple of steps.

To prove (i) above, we derive from condition (B5) that

$$\underline{G}(s_{ij}) = \widetilde{L}(s_{ij}) \widetilde{G}(s_{ij}) \widetilde{R}(s_{ij}) = \begin{pmatrix} I_{r_{ij}^2} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\underline{A}(s_{ij}) = \widetilde{L}(s_{ij}) \widetilde{A}(s_{ij}) \widetilde{R}(s_{ij}) = \begin{pmatrix} \breve{A}(s_{ij}) & 0 \\ 0 & I_{n^2 - r_{ij}^2} \end{pmatrix},$$

where $\breve{A}(s_{ij})$ is a nonsingular $r_{ij}^2 \times r_{ij}^2$ matrix. Let

$$y^\varepsilon(t) = \widetilde{R}^{-1}(\alpha^\varepsilon(t)) z^\varepsilon(t) = [y_1'(t) \ y_2'(t)]'.$$

Then

$$
\begin{aligned}
&\utilde{G}(\alpha^{\varepsilon}(t))\dot{y}^{\varepsilon}(t) = \utilde{A}(\alpha^{\varepsilon}(t))y^{\varepsilon}(t),\\
&y^{\varepsilon}(0) = \widetilde{R}^{-1}(\iota)\widetilde{\xi},\\
&\alpha^{\varepsilon}(0) = \iota.
\end{aligned}
\tag{4.4.4}
$$

So

$$
\begin{aligned}
&\dot{y}_1^{\varepsilon}(t) = \breve{A}(\alpha^{\varepsilon}(t))y_1^{\varepsilon}(t),\\
&y_2^{\varepsilon}(t) = 0,\\
&y^{\varepsilon}(0) = \widetilde{R}^{-1}(\iota)\widetilde{\xi}, \alpha^{\varepsilon}(0) = \iota.
\end{aligned}
\tag{4.4.5}
$$

Thus $\dot{y}_2(t) = 0$. Denote

$$
\begin{aligned}
&\mathring{A}(i) = \begin{pmatrix} \breve{A}(i) & 0 \\ 0 & 0_{n^2-r_i^2} \end{pmatrix} \quad \text{and}\\
&\text{\c{A}}(i) = R(i)\mathring{A}(i)R(i)^{-1}.
\end{aligned}
$$

Then

$$
\dot{z}^{\varepsilon}(t) = \text{\c{A}}(\alpha^{\varepsilon}(t))z^{\varepsilon}(t).
$$

Furthermore, $\text{\c{A}}(i)$ is bounded. Hence, the proof is completed.

As for (ii), put

$$
\utilde{P}(s_{ij}) = (\widetilde{L}^{-1})'(s_{ij})\overline{P}(i)\widetilde{R}(s_{ij}) = \begin{pmatrix} \utilde{P}_{11}(s_{ij}) & \utilde{P}_{12}(s_{ij}) \\ \utilde{P}_{21}(s_{ij}) & \utilde{P}_{22}(s_{ij}) \end{pmatrix}, \qquad \forall s_{ij} \in \mathcal{M}.
$$

Then

$$
\utilde{G}'(s_{ij})\utilde{P}(s_{ij}) = \utilde{P}'(s_{ij})\utilde{G}(s_{ij}) \geq 0,
$$

which implies

$$
\utilde{P}_{12}(s_{ij}) = \utilde{P}_{21}(s_{ij}) = 0 \;\; \text{and} \;\; \utilde{P}_{11}(s_{ij}) > 0.
$$

Thus

$$\begin{aligned}
[z^\varepsilon(t)]' \,\overline{G}'(\overline{\alpha}^\varepsilon(t))\overline{P}(\overline{\alpha}^\varepsilon(t))z^\varepsilon(t) &= [z^\varepsilon(t)]' \,\widetilde{G}'(\alpha^\varepsilon(t))\overline{P}(\overline{\alpha}^\varepsilon(t))z^\varepsilon(t) \\
&= [y^\varepsilon(t)]' \,\underset{\sim}{G}'(\alpha^\varepsilon(t))\underset{\sim}{P}(\alpha^\varepsilon(t))y^\varepsilon(t) \\
&= [y_1^\varepsilon(t)]' \,\underset{\sim}{P}_{11}(\alpha^\varepsilon(t))y_1^\varepsilon(t) \\
&\geq c_3 \left|y_1^\varepsilon(t)\right|^2 = c_3 \left|y^\varepsilon(t)\right|^2 \geq c_2 \left|z^\varepsilon(t)\right|^2.
\end{aligned}$$

Thus the assertion is proved. $\square$

Theorem 4.4.6. *Assume* (B3)–(B5) *hold. There exists constants* $\gamma > 0$ *and* $c > 0$ *such that*

$$E\left|x^\varepsilon(t)\right|^2 \leq ce^{-\gamma t}\sqrt{\varepsilon}.$$

Proof. For any $\alpha \in \mathcal{M}$, define

$$V_0(z,\alpha) = z'\overline{G}'(\overline{\alpha})\overline{P}(\overline{\alpha})z = z'\widetilde{G}'(\alpha)\overline{P}(\overline{\alpha})z$$

Moreover, by the irreducibility of $\widetilde{Q}^i$,

$$\widetilde{Q}V_0(z,\cdot)(\alpha) = 0.$$

Therefore

$$\mathcal{L}^\varepsilon V_0(z^\varepsilon(t),\alpha^\varepsilon(t)) = [z^\varepsilon(t)]' \,[\widetilde{A}'(\alpha^\varepsilon(t))\overline{P}(\alpha^\varepsilon(t)) + \overline{P}'(\alpha^\varepsilon(t))\widetilde{A}(\alpha^\varepsilon(t)) + g(\alpha^\varepsilon(t))]z^\varepsilon(t), \tag{4.4.6}$$

where

$$g(s_{ij}) = \sum_{\kappa=1}^{l}\sum_{\iota=1}^{m_\kappa} \widehat{q}_{s_{ij}s_{\kappa\iota}}\overline{G}'(s_{\kappa\iota})\overline{P}(s_{\kappa\iota}).$$

Denote

$$L = [z^\varepsilon(t)]' \left[\overline{A}'(\overline{\alpha}^\varepsilon(t))\overline{P}(\overline{\alpha}^\varepsilon(t)) + \overline{P}'(\overline{\alpha}^\varepsilon(t))\overline{A}(\overline{\alpha}^\varepsilon(t)) + \overline{g}(\overline{\alpha}^\varepsilon(t))\right] z^\varepsilon(t). \tag{4.4.7}$$

where

$$\overline{g}(i) = \sum_{\kappa=1}^{l} \overline{q}_{i\kappa}\overline{G}'(\kappa)\overline{P}(\kappa).$$

In order to obtain the desired stability result, we use the methods of perturbed Liapunov functions. The main idea lies in introducing perturbations to an appropriate Liapunov function. The perturbations are small in magnitude compared with the original Liapunov function, and that it results in desired cancellation of the unwanted terms.

To proceed, define the perturbations

$$\begin{aligned}
V_1^\varepsilon(z,t) &= E_t \int_t^\infty e^{t-u} z' \left[\widetilde{A}'(\alpha^\varepsilon(u)) - \overline{A}'(\overline{\alpha}^\varepsilon(u))\right] \overline{P}(\overline{\alpha}^\varepsilon(u)) z du, \\
V_2^\varepsilon(z,t) &= E_t \int_t^\infty e^{t-u} z'\overline{P}'(\overline{\alpha}^\varepsilon(u)) \left[\widetilde{A}(\alpha^\varepsilon(u)) - \overline{A}(\overline{\alpha}^\varepsilon(u))\right] z du, \\
V_3^\varepsilon(z,t) &= E_t \int_t^\infty e^{t-u} z' \left[g(\alpha^\varepsilon(u)) - \overline{g}(\overline{\alpha}^\varepsilon(u))\right] y du.
\end{aligned}$$

In order to estimate $V_2^\varepsilon(z,t)$ and $\mathcal{L}^\varepsilon V_2^\varepsilon(z^\varepsilon(t),t)$, we consider

$$\begin{aligned}
V_{2A}^\varepsilon(z,t) &= E_t \int_t^\infty e^{t-u} z'\overline{P}'(\overline{\alpha}^\varepsilon(u)) \left[\left\langle \breve{G}(\alpha^\varepsilon(u)), \breve{A}(\alpha^\varepsilon(u)) \right\rangle \right. \\
&\quad \left. - \left\langle \widehat{G}(\overline{\alpha}^\varepsilon(u)), \widehat{A}(\overline{\alpha}^\varepsilon(u)) \right\rangle \right] z du, \\
V_{2H}^\varepsilon(z,t) &= E_t \int_t^\infty e^{t-u} z'\overline{P}'(\overline{\alpha}^\varepsilon(u)) \left[\left\langle \breve{H}(\alpha^\varepsilon(u)) \right\rangle - \left\langle \widehat{H}(\overline{\alpha}^\varepsilon(u)) \right\rangle \right] z du, \\
V_{2Q}^\varepsilon(z,t) &= E_t \int_t^\infty e^{t-u} z'\overline{P}'(\overline{\alpha}^\varepsilon(u)) \left[h(\alpha^\varepsilon(u)) - \overline{h}(\overline{\alpha}^\varepsilon(u))\right] z du,
\end{aligned}$$

where $h(s_{ij}) = \sum_{\kappa=1}^{l} \sum_{\iota=1}^{m_i} \widehat{q}^{\varepsilon}_{s_{ij}\ \ s_{\kappa\iota}} \widetilde{G}(s_{\kappa\iota})$ and $\overline{h}(i) = \sum_{\kappa=1}^{l} \overline{q}_{i\kappa} \overline{G}(\kappa)$. Then

$$V_2^{\varepsilon}(z,t) = V_{2A}^{\varepsilon}(z,t) + V_{2H}^{\varepsilon}(z,t) + V_{2Q}^{\varepsilon}(z,t).$$

On one hand,

$$\begin{aligned}
\Big\langle \breve{G}(\alpha^{\varepsilon}(u)), \breve{A}(\alpha^{\varepsilon}(u)) \Big\rangle &- \Big\langle \widehat{G}(\overline{\alpha}^{\varepsilon}(u)), \widehat{A}(\overline{\alpha}^{\varepsilon}(u)) \Big\rangle \\
&= \Big\langle \widehat{G}(\overline{\alpha}^{\varepsilon}(u)), \breve{A}(\alpha^{\varepsilon}(u)) - \widehat{A}(\overline{\alpha}^{\varepsilon}(u)) \Big\rangle \\
&= \Big\langle \widehat{G}(\overline{\alpha}^{\varepsilon}(u)), \breve{L}^{-1}(\alpha^{\varepsilon}(u)) L(\alpha^{\varepsilon}(u)) \Big[A(\alpha^{\varepsilon}(u)) - \widehat{A}(\overline{\alpha}^{\varepsilon}(u)) \Big] \Big\rangle \\
&\quad + \Big\langle \widehat{G}(\overline{\alpha}^{\varepsilon}(u)), \Big[\breve{L}^{-1}(\alpha^{\varepsilon}(u)) - L^{-1}(\alpha^{\varepsilon}(u)) \Big] L(\alpha^{\varepsilon}(u)) \widehat{A}(\overline{\alpha}^{\varepsilon}(u)) \Big\rangle
\end{aligned}$$

On the other hand,

$$A(\alpha^{\varepsilon}(u)) - \widehat{A}(\overline{\alpha}^{\varepsilon}(u)) = \sum_{i=1}^{l} \sum_{j=1}^{m_i} A(s_{ij}) \left[\chi(\alpha^{\varepsilon}(u) = s_{ij}) - \nu_j^i \chi(\alpha^{\varepsilon}(u) \in \mathcal{M}_i) \right]$$

$$\breve{L}^{-1}(\alpha^{\varepsilon}(u)) - L^{-1}(\alpha^{\varepsilon}(u)) = -\sum_{i=1}^{l} \sum_{j=1}^{m_i} L^{-1}(s_{ij}) \left[\chi(\alpha^{\varepsilon}(u) = s_{ij}) - \nu_j^i \chi(\alpha^{\varepsilon}(u) \in \mathcal{M}_i) \right]$$

and for $u \geq t$,

$$E_t^{\varepsilon} \left[\chi(\alpha^{\varepsilon}(u) = s_{ij}) - \nu_j^i \chi(\overline{\alpha}^{\varepsilon}(u) = i) \right] = O\left(\varepsilon + e^{-k_0(u-t)/\varepsilon} \right),$$

Therefore

$$\begin{aligned}
|V_{2A}^{\varepsilon}(z,t)| &\leq \sum_{i=1}^{l} \sum_{j=1}^{m_i} |z|^2 \int_t^{\infty} e^{t-u} O\left(\varepsilon + e^{-k_0(u-t)/\varepsilon} \right) du \\
&= O(\varepsilon) \, |z|^2 .
\end{aligned} \tag{4.4.8}$$

Similarly, from the fact that

$$\begin{aligned}
\Big\langle \breve{H}(\alpha^{\varepsilon}(u)) \Big\rangle - \Big\langle \widehat{H}(\overline{\alpha}^{\varepsilon}(u)) \Big\rangle &= \Big\langle \breve{H}(\alpha^{\varepsilon}(u)), \breve{H}(\alpha^{\varepsilon}(u)) - \widehat{H}(\overline{\alpha}^{\varepsilon}(u)) \Big\rangle \\
&\quad + \Big\langle \breve{H}(\alpha^{\varepsilon}(u)) - \widehat{H}(\overline{\alpha}^{\varepsilon}(u)), \widehat{H}(\overline{\alpha}^{\varepsilon}(u)) \Big\rangle \\
h(\alpha^{\varepsilon}(u)) - \overline{h}(\overline{\alpha}^{\varepsilon}(t)) &= \sum_{i=1}^{l} \sum_{j=1}^{m_i} \widehat{Q}\widetilde{G}(\cdot)(s_{ij}) E_t^{\varepsilon} \left[\chi(\alpha^{\varepsilon}(u) = s_{ij}) - \nu_j^i \chi(\overline{\alpha}^{\varepsilon}(u) = i) \right]
\end{aligned}$$

we derive

$$\left|V_{2H}^{\varepsilon}(z,t)\right| \le O(\varepsilon)\left|z\right|^{2}, \qquad \left|V_{2Q}^{\varepsilon}(z,t)\right| \le O(\varepsilon)\left|z\right|^{2}.$$

Thus

$$\left|V_{2}^{\varepsilon}(z,t)\right| \le O(\varepsilon)\left|z\right|^{2}.$$

Furthermore,

$$\begin{aligned}
\mathcal{L}^{\varepsilon}V_{2}^{\varepsilon}(z^{\varepsilon}(t),t) &= \lim_{\delta\downarrow 0} E_t^{\varepsilon}\left[V_2^{\varepsilon}(z^{\varepsilon}(t+\delta),t+\delta) - V_2^{\varepsilon}(z^{\varepsilon}(t),t)\right] \\
&= \lim_{\delta\downarrow 0} E_t^{\varepsilon}\left[V_2^{\varepsilon}(z^{\varepsilon}(t+\delta),t+\delta) - V_2^{\varepsilon}(z^{\varepsilon}(t),t+\delta)\right] \\
&\qquad + \lim_{\delta\downarrow 0} E_t^{\varepsilon}\left[V_2^{\varepsilon}(z^{\varepsilon}(t),t+\delta) - V_2^{\varepsilon}(z^{\varepsilon}(t),t)\right] \\
&= -\left[z^{\varepsilon}(t)\right]'\overline{P}'(\overline{\alpha}^{\varepsilon}(t))\left[\widetilde{A}(\alpha^{\varepsilon}(t)) - \overline{A}(\overline{\alpha}^{\varepsilon}(t))\right]z^{\varepsilon}(t) + V_2^{\varepsilon}(z^{\varepsilon}(t),t) \\
&\qquad + E_t\int_t^{\infty} e^{t-u}\left[\dot{z}^{\varepsilon}(t)\right]'\overline{P}'(\overline{\alpha}^{\varepsilon}(t))\left[\widetilde{A}(\alpha^{\varepsilon}(u)) - \overline{A}(\overline{\alpha}^{\varepsilon}(u))\right]z^{\varepsilon}(t)du \\
&\qquad + E_t\int_t^{\infty} e^{t-u}\left[z^{\varepsilon}(t)\right]'\overline{P}'(\overline{\alpha}^{\varepsilon}(t))\left[\widetilde{A}(\alpha^{\varepsilon}(u)) - \overline{A}(\overline{\alpha}^{\varepsilon}(u))\right]\dot{z}^{\varepsilon}(t)du \\
&\le -\left[z^{\varepsilon}(t)\right]'\overline{P}'(\overline{\alpha}^{\varepsilon}(t))\left[\widetilde{A}(\alpha^{\varepsilon}(t)) - \overline{A}(\alpha^{\varepsilon}(t))\right]z^{\varepsilon}(t) + O(1)V_2^{\varepsilon}(z^{\varepsilon}(t),t) \\
&\le -\left[z^{\varepsilon}(t)\right]'\overline{P}'(\overline{\alpha}^{\varepsilon}(t))\left[\widetilde{A}(\alpha^{\varepsilon}(t)) - \overline{A}(\alpha^{\varepsilon}(t))\right] + O(\varepsilon)\left|z^{\varepsilon}(t)\right|^{2}
\end{aligned} \tag{4.4.9}$$

Using a similar argument, we obtain

$$\begin{aligned}
&\left|V_1^{\varepsilon}(z,t)\right| = O(\varepsilon)\left|z\right|^{2}, \\
&\left|V_3^{\varepsilon}(z,t)\right| = O(\varepsilon)\left|z\right|^{2} \\
&\mathcal{L}^{\varepsilon}V_1^{\varepsilon}(z^{\varepsilon}(t),t) \le -\left[z^{\varepsilon}(t)\right]'\left[\widetilde{A}'(\alpha^{\varepsilon}(t)) - \overline{A}'(\overline{\alpha}^{\varepsilon}(t))\right]\overline{P}(\overline{\alpha}^{\varepsilon}(t))z^{\varepsilon}(t) + O(\varepsilon)\left|z^{\varepsilon}(t)\right|^{2} \\
&\mathcal{L}^{\varepsilon}V_3^{\varepsilon}(z^{\varepsilon}(t),t) \le -[z^{\varepsilon}(t)]'\left[g(\alpha^{\varepsilon}(u)) - \overline{g}(\overline{\alpha}^{\varepsilon}(u))\right]z^{\varepsilon}(t) + O(\varepsilon)\left|z^{\varepsilon}(t)\right|^{2}.
\end{aligned} \tag{4.4.10}$$

Define

$$V^{\varepsilon}(t) = V_0(z^{\varepsilon}(t),\alpha^{\varepsilon}(t)) + \sum_{i=1}^{3} V_i^{\varepsilon}(z^{\varepsilon}(t),t). \tag{4.4.11}$$

Then

$$V^\varepsilon(t) = V_0(z^\varepsilon(t), \alpha^\varepsilon(t)) + O(\varepsilon)\,|z^\varepsilon(t)|^2\,. \tag{4.4.12}$$

In addition,

$$\mathcal{L}^\varepsilon V^\varepsilon(t) \le L + O(\varepsilon)\,|z^\varepsilon(t)|^2\,. \tag{4.4.13}$$

By assumption, there exists a constant $\gamma > 0$ such that

$$L + \gamma V_0(z^\varepsilon(t), \alpha^\varepsilon(t)) \le L + \gamma c\,|z^\varepsilon(t)|^2 \le 0.$$

Using

$$\begin{aligned} \mathcal{L}^\varepsilon\left(e^{\gamma t} V^\varepsilon(t)\right) &= e^{\gamma t}(\gamma V^\varepsilon(t) + \mathcal{L}^\varepsilon V^\varepsilon(t)) \\ &\le e^{\gamma t} O(\varepsilon)\,|z^\varepsilon(t)|^2\,, \end{aligned}$$

we obtain

$$\begin{aligned} E\left[e^{\gamma t} V^\varepsilon(t)\right] &\le EV^\varepsilon(0) + E\int_0^t e^{\gamma u} O(\varepsilon)\,|z^\varepsilon(u)|^2\,du \\ &\le O(\varepsilon)\left|\widetilde{\xi}\right|^2 + E\int_0^t e^{\gamma u} O(\varepsilon)\,|z^\varepsilon(u)|^2\,du. \end{aligned}$$

On the other hand,

$$V_0(z^\varepsilon(t), \alpha^\varepsilon(t)) > |z^\varepsilon(t)|^2/c.$$

Therefore,

$$E\left[e^{\gamma t}\,|z^\varepsilon(t)|^2\right] \le O(\varepsilon)\left|\widetilde{\xi}\right|^2 + E\int_0^t e^{\gamma u} O(\varepsilon)\,|z^\varepsilon(u)|^2\,du.$$

Gronwall's inequality yields that

$$E\left[e^{\gamma t}\,|z^\varepsilon(t)|^2\right] \le O(\varepsilon)\left|\widetilde{\xi}\right|^2.$$

Hence

$$E\,|z^\varepsilon(t)|^2 \le e^{-\gamma t} O(\varepsilon)\left|\widetilde{\xi}\right|^2.$$

Therefore,

$$E\left|x^{\varepsilon}(t)\right|^2 \le CE\left|\widehat{x}^{\varepsilon}(t)\right|^2 \le C\sqrt{E\left|z^{\varepsilon}(t)\right|^2} \le e^{-\gamma t/2}O(\sqrt{\varepsilon})\left|\xi\right|^2.$$

The proof is completed. $\square$

4.5 Examples

In this section, we provided a couple of examples to illustrate the two-time-scale singular systems. In the examples, the matrix manipulations were done by use of symbolic computation techniques through the use of Maple.

Example 4.5.1. Let $\alpha^{\varepsilon}(t)$ be a switching process taking values in $\mathcal{M} = \{1, 2\}$ with generator

$$Q^{\varepsilon} = \frac{\widetilde{Q}}{\varepsilon}, \quad \text{where} \quad \widetilde{Q} = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \quad \text{and} \quad \varepsilon = 0.1.$$

Consider the singular system

$$G(\alpha^{\varepsilon}(t))\dot{x}(t) = A(\alpha^{\varepsilon}(t))x(t)dt + H(\alpha^{\varepsilon}(t))x(t)dB(t), \tag{4.5.1}$$

where

$$G(1) = G(2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ A(1) = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix}, \ A(2) = \begin{bmatrix} -3.5 & 0 \\ 0 & 0 \end{bmatrix},$$
$$H(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \ H(2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\widetilde{G}(1) = \widetilde{G}(2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{A}(1) = \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}, \ \widetilde{A}(2) = \begin{bmatrix} -3.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now let us consider the corresponding limit singular system

$$\overline{G}\ \dot{\overline{z}}(t) = \overline{A}\ \overline{z}(t), \tag{4.5.2}$$

where

$$\overline{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \overline{A} = \begin{bmatrix} -2.3333 & 0.1667 & 0 & 0 \\ 0 & 0.1667 & 0 & 0 \\ 0 & 0 & 0.1667 & 0 \\ 0 & 0 & 0 & 0.75 \end{bmatrix}.$$

Then

$$\overline{G}'P = P'\overline{G} = \begin{bmatrix} P_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and}$$

$$P_{1,2} = P_{1,3} = P_{1,4} = 0.$$

Thus

$$
\begin{aligned}
&\overline{A}'P + P'\overline{A} \\
&= \begin{bmatrix}
-4.0\,P_{1,1} & 0.1\bar{6}\,P_{1,1} + 0.1\bar{6}\,P_{2,1} & 0.1\bar{6}\,P_{3,1} & 0.75\,P_{4,1} \\
0.1\bar{6}\,P_{1,1} + 0.1\bar{6}\,P_{2,1} & 0.\bar{3}\,P_{2,2} & 0.1\bar{6}\,P_{2,3} + 0.1\bar{6}\,P_{3,2} & 0.1\bar{6}\,P_{2,4} + 0.75\,P_{4,2} \\
0.1\bar{6}\,P_{3,1} & 0.1\bar{6}\,P_{2,3} + 0.1\bar{6}\,P_{3,2} & 0.\bar{3}\,P_{3,3} & 0.1\bar{6}\,P_{3,4} + 0.75\,P_{4,3} \\
0.75\,P_{4,1} & 0.1\bar{6}\,P_{2,4} + 0.75\,P_{4,2} & 0.1\bar{6}\,P_{3,4} + 0.75\,P_{4,3} & 1.50\,P_{4,4}
\end{bmatrix}.
\end{aligned}
$$

Hence $\overline{G}$ and $\overline{A}$ satisfy (B3)–(B5) with

$$
\overline{P} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}.
$$

Therefore, (4.5.1) is asymptotically mean-square stable.

For further demonstration, we next plot the sample paths of the systems. This is done by use of Matlab. We set the steps $h = 0.0001$. We then obtain the following figures for the first coordinate.

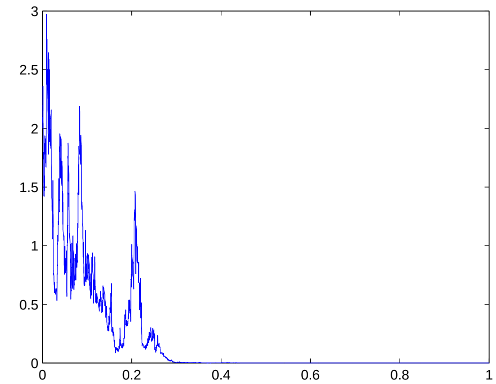
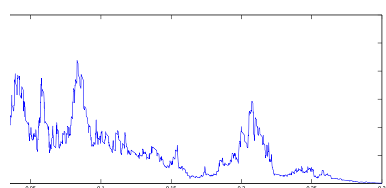
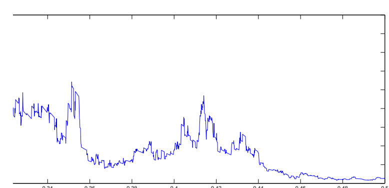


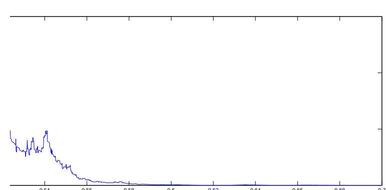
Figure 4.1: A trajectory of singular dynamic system (4.5.1): Time between 0 and 1; $\varepsilon = 0.001$



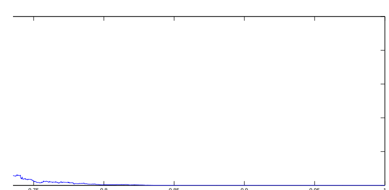
(a) Time between 0 and 0.3



(b) Time between 0.3 and 0.5



(c) Time between 0.5 and 0.7



(d) Time between 0.7 and1

Figure 4.2: A trajectory of singular dynamic system (4.5.1) in $[0, 1]$, $\varepsilon = 0.001$

The system quickly comes to its limit position. Fig. 4.1 shows the results for $\varepsilon = 0.001$, whereas Fig. 4.3 displays the sample path and trajectory corresponding to $\varepsilon = 0.1$. Moreover, it is easily seen that the smaller the ε, the faster the system

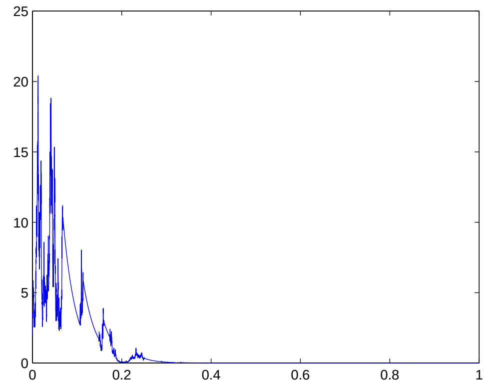


Figure 4.3: A trajectory of singular dynamic system (4.5.1): Time between 0 and 1; $\varepsilon = 0.1$

decays.

Example 4.5.2. Let $\alpha^\varepsilon(t)$ be a switching process taking values in $\mathcal{M} = \{1, 2, 3, 4\}$ with generator

$$Q^\varepsilon = \frac{\widetilde{Q}}{\varepsilon} + \widehat{Q}$$

where

$$\widetilde{Q} = \begin{bmatrix} -4 & 4 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 3 & -3 \end{bmatrix}, \widehat{Q} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

and $\varepsilon = 0.1$. Consider the singular system

$$G(\alpha^\varepsilon(t))\dot{x}(t) = A(\alpha^\varepsilon(t))x(t)dt + H(\alpha^\varepsilon(t))x(t)dB(t), \tag{4.5.3}$$

where

$$G(1)=G(2)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, G(3)=G(4)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

$$A(1)=\begin{bmatrix}0 & 0.5\\ 0 & 0\end{bmatrix},\ A(2)=\begin{bmatrix}-3.5 & 0\\ 0 & 0\end{bmatrix},\ A(3)=\begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix},\ A(4)=\begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix},$$

$$H(1)=\begin{bmatrix}0 & 0\\ 0 & 0.5\end{bmatrix},\ H(2)=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\ H(3)=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\ H(4)=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

As $\varepsilon \to 0$, $E\left\langle x^{\varepsilon}(t)\right\rangle \to y(t)$ such that

$$\overline{G}(\overline{\alpha}(t))dz(t)=\overline{A}(\overline{\alpha}(t))y(t)dt,$$

where $\overline{\alpha}(t)$ is the Markov chain generated by

$$\overline{Q}=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix} \quad \text{and}$$

$$\overline{G}(1)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\ \overline{G}(2)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$

$$\overline{A}(1)=\begin{bmatrix}-2.8 & 0.1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1.85\end{bmatrix},\ \overline{A}(2)=\begin{bmatrix}-1.4 & 0 & 0 & 0.6\\ 0 & 0.6 & 0 & \\ 0 & 0 & -2.4 & 0\\ 0 & 0 & 0 & 0.6\end{bmatrix}.$$

Since $\overline{G}$ and $\overline{A}$ satisfy $(B3)-(B5)$, (4.5.3) is asymptotically mean-square stable as demonstrated in Fig.4.4.

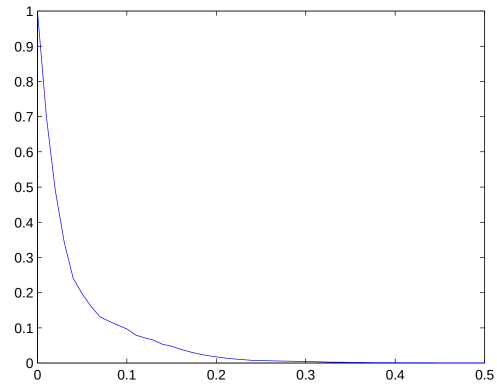


Figure 4.4: A trajectory of the singular dynamic system (4.5.3) in $[0, 0.5]$, $\varepsilon = 0.1$

4.6 Remarks

This chapter has been devoted to singular jump-linear systems whose switching process has a large state space. Alternatively, it could be called singular system with singularly perturbed Markov chain. The multi-scale structure and two-time-scale formulation are used to reflect that the discrete event process in the system has a large state space. We have established reduction of complexity results from the angle of stability analysis. We have used perturbed Liapunov function methods to carry out the desired task. The conclusion shows that as the small parameter goes to 0, we can use the stability of the limit system to infer that of the original system. The original system is normally difficult to analyze because of its large dimensionality, whereas the limit system is relatively simpler. Thus the result provides a practical guideline for treating many such systems.

Chapter 5

Discussion

This dissertation is adherent to two-time-scale switching diffusions where the switching processes live in a large but finite state space. In this dissertation, we have studied asymptotic expansions of Kolmogorov backward equations for switching diffusions and stability of singular linear systems. There are still a number of interesting open problems.

In the fast diffusion case mentioned in Chapter 3, only positive recurrent case has been Considered. We may deal with null recurrent processes or transient processes. These are challenging problems.

Also, we may study stability of singular systems with time delays, which was not considered in Chapter 4. Furthermore, random delay may also be incorporated into the formulation.

In the future, stability associated with numerical solutions of switching diffusions may be studied. Developing and analyzing numerical schemes for singularly perturbed hybrid systems are also very difficult problems, which require much attention. These problems deserve in-depth study and devoted attention.

BIBLIOGRAPHY

[1] V. V. Anisimov. Switching processes: Averaging principle, diffusion approximation and applications. *Acta Appl. Math.*, 40(2):95–141, 1995.

[2] G. Badowski and G. G. Yin. Stability of hybrid dynamic systems containing singularly perturbed random processes. *Ieee Transactions on Automatic Control*, 47(12):2021–2032, 2002.

[3] N. N. Bogoliubov and Y. A. Mitropolski. *Asymptotic Methods in the Theory of Non-Linear Oscillations*. Gordon and Breach, New York, 1961.

[4] E. K. Boukas. Stabilization of stochastic singular nonlinear hybrid systems. *Nonlinear Analysis-Theory Methods & Applications*, 64(2):217–228, 2006.

[5] E. K. Boukas, S. Xu, and J. Lam. On stability and stabilizability of singular stochastic systems with delays. *Journal of Optimization Theory and Applications*, 127(2):249–262, 2005.

[6] A. Brunner. Testing for structural breaks in us post-war inflation data. *Mimeograph, Board of Governors of the Federal Reserve System, Washington, DC*, 1991.

[7] J. Cai. A markov model of unconditional variance in arch. *Journal of Economic and Business Statistics*, 12:309–316, 1994.

[8] S. L. Campbell. *Singular Systems of Differential Equations*, volume 40. Pitman (Advanced Publishing Program), Boston, Mass., 1980.

[9] S. L. Campbell. *Singular Systems of Differential Equations. Ii*, volume 61. Pitman (Advanced Publishing Program), Boston, Mass., 1982.

[10] Z. L. Cheng, H. M. Hong, and J. F. Zhang. The optimal regulation of generalized state-space systems with quadratic cost. *Automatica*, 24(5):707–710, 1988.

[11] C. L. Chiang. *An Introduction to Stochastic Processes and Their Applications*. R. E. Krieger Pub. Co., Huntington, N.Y., 1980.

[12] P. J. Courtois. *Decomposability : Queueing and Computer System Applications*. Acm Monograph Series. Academic Press, New York, 1977.

[13] L. Dai. *Singular Control Systems*. Lecture Notes in Control and Information Sciences. Springer-Verlag, Berlin ; New York, 1989.

[14] S. N. Ethier and T. G. Kurtz. *Markov Processes : Characterization and Convergence*. Wiley Series in Probability and Mathematical Statistics. Wiley, New York, 1986.

[15] L. C. Evans. *Partial Differential Equations*. Graduate Studies in Mathematics. American Mathematical Society, Providence, R.I., 1998.

[16] X. B. Feng, K. A. Loparo, Y. D. Ji, and H. J. Chizeck. Stochastic stability

properties of jump linear-systems. *Ieee Transactions on Automatic Control*, 37 (1):38–53, 1992.

[17] J.-P. Fouque, G. Papanicolaou, and K. R. Sircar. *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge University Press, Cambridge [England]; New York, 2000.

[18] J. D. Hamilton and R. Susmel. Autoregressive conditional heteroskedasticity and changes in regime. *Journal of Econometrics*, 64(1-2):307–333, 1994.

[19] B. E. Hansen. The likelihood ratio test under nonstandard conditions - testing the markov switching model of gnp. *Journal of Applied Econometrics*, 7:S61–S82, 1992.

[20] B. E. Hansen. The likelihood ratio test under nonstandard conditions: Testing the markov switching model of gnp (vol 7, pg 61, 1992). *Journal of Applied Econometrics*, 11(2):195–198, 1996.

[21] L. R. Huang and X. R. Mao. Stability of singular stochastic systems with markovian switching. *Ieee Transactions on Automatic Control*, 56(2):424–429, 2011.

[22] A. M. Il'in, R. Z. Khasminskii, and G. Yin. Asymptotic expansions of solutions of integro-differential equations for transition densities of singularly perturbed switching diffusions: Rapid switchings. *Journal of Mathematical Analysis and Applications*, 238(2):516–539, 1999.

[23] A. M. Il'in, R. Z. Khasminskii, and G. Yin. Singularly perturbed switching diffusions: Rapid switchings and fast diffusions. *Journal of Optimization Theory and Applications*, 102(3):555–591, 1999.

[24] Y. D. Ji and H. J. Chizeck. Controllability, stabilizability, and continuous-time markovian jump linear quadratic control. *Ieee Transactions on Automatic Control*, 35(7):777–788, 1990.

[25] N. G. v. Kampen. *Stochastic Processes in Physics and Chemistry.* North-Holland Personal Library. North-Holland, Amsterdam ; New York, rev. and enl. edition, 1992.

[26] R. Z. Khasminskii. On an averaging principle for ito stochastic differential equations. *Kybernetika*, 4:260–279, 1968.

[27] R. Z. Khasminskii. *Stochastic Stability of Differential Equations.* Monographs and Textbooks on Mechanics of Solids and Fluids. Sijthoff & Noordhoff, Alphen aan den Rijn, Netherlands ; Rockville, Md., 1980.

[28] R. Z. Khasminskii and G. Yin. Asymptotic series for singularly perturbed kolmogorov-fokker-planck equations. *Siam Journal on Applied Mathematics*, 56 (6):1766–1793, 1996.

[29] R. Z. Khasminskii and G. Yin. On averaging principles: An asymptotic expansion approach. *Siam Journal on Mathematical Analysis*, 35(6):1534–1560, 2004.

[30] R. Z. Khasminskii, G. Yin, and Q. Zhang. Asymptotic expansions of singularly perturbed systems involving rapidly fluctuating markov chains. *Siam Journal on Applied Mathematics*, 56(1):277–293, 1996.

[31] R. Z. Khasminskii, G. Yin, and Q. Zhang. Constructing asymptotic series for probability distributions of markov chains with weak and strong interactions. *Quart. Appl. Math.*, 55(1):177–200, 1997.

[32] P. V. Kokotovic, A. Bensoussan, and G. Blankenship. *Singular Perturbations and Asymptotic Analysis in Control Systems*. Lecture Notes in Control and Information Sciences. Springer-Verlag, Berlin ; New York, 1986.

[33] H. J. Kushner. *Stochastic Stability and Control*. Mathematics in Science and Engineering. Academic Press, New York, 1967.

[34] F. L. Lewis. A survey of linear singular systems. *Circuits Systems and Signal Processing*, 5(1):3–36, 1986.

[35] X. Mao. *Exponential Stability of Stochastic Differential Equations*. Monographs and Textbooks in Pure and Applied Mathematics. M. Dekker, New York, 1994.

[36] X. Mao. Stability of stochastic differential equations with markovian switching. *Stochastic Processes and Their Applications*, 79(1):45–67, 1999.

[37] X. Mao and C. Yuan. *Stochastic Differential Equations with Markovian Switching*. Imperial College Press, London, 2006.

[38] B. J. Matkowsky and Z. Schuss. Exit problem for randomly perturbed dynamical-systems. *Siam Journal on Applied Mathematics*, 33(2):365–382, 1977.

[39] G. C. Papanicolaou. Some probabilistic problems and methods in singular perturbations. *Rocky Mountain J. Math.*, 6(4):653–674, 1976.

[40] G. C. Papanicolaou. Introduction to the asymptotic analysis of stochastic equations. *Modern modeling of continuum phenomena (Ninth Summer Sem. Appl. Math., Rensselaer Polytech. Inst., Troy, N.Y., 1975)*, pages 109–147. Lectures in Appl. Math., Vol. 16, 1977.

[41] E. Pardoux and A. Y. Veretennikov. On the poisson equation and diffusion approximation 1. *Annals of Probability*, 29(3):1061–1085, 2001.

[42] E. Pardoux and A. Y. Veretennikov. On the poisson equation and diffusion approximation 3. *Annals of Probability*, 33(3):1111–1133, 2005.

[43] E. Pardoux and Y. Veretennikov. On poisson equation and diffusion approximation 2. *Annals of Probability*, 31(3):1166–1192, 2003.

[44] R. G. Phillips and P. V. Kokotovic. A singular perturbation approach to modeling and control of markov chains. *Ieee Transactions on Automatic Control*, 26(5): 1087–1094, 1981.

[45] S. P. Sethi and Q. Zhang. Hierarchical production and setup scheduling in

stochastic manufacturing systems. *Proceedings of the 33rd Ieee Conference on Decision and Control, Vols 1-4*, pages 1571–1576, 1994.

[46] S. P. Sethi, Q. Zhang, and H. Zhang. *Average-Cost Control of Stochastic Manufacturing Systems*. Stochastic Modelling and Applied Probability. Springer, New York, 2005.

[47] H. A. Simon and A. Ando. Aggregation of variables in dynamic systems. *Econometrica*, 29(2):111–138, 1961.

[48] A. V. Skorokhod. *Asymptotic Methods in the Theory of Stochastic Differential Equations*. Translations of Mathematical Monographs. American Mathematical Society, Providence, R.I., 1989.

[49] D. N. C. Tse, R. G. Gallager, and J. N. Tsitsiklis. Statistical multiplexing of multiple time-scale markov streams. *Ieee Journal on Selected Areas in Communications*, 13(6):1028–1038, 1995.

[50] G. Yin and H. Q. Zhang. Two-time-scale markov chains and applications to quasi-birth-death queues. *Siam Journal on Applied Mathematics*, 65(2):567–586, 2005.

[51] G. Yin and J. F. Zhang. Hybrid singular systems of differential equations. *Science in China Series F*, 45(4):241–258, 2002.

[52] G. Yin and Q. Zhang. *Continuous-Time Markov Chains and Applications : A*

Singular Perturbation Approach. Applications of Mathematics 37. Springer, New York, 1998.

[53] G. Yin and C. Zhu. *Hybrid Switching Diffusions : Properties and Applications*. Stochastic Modelling and Applied Probability. Springer, New York, 2010.

[54] C. Zhu and G. Yin. Asymptotic properties of hybrid diffusion systems. *Siam Journal on Control and Optimization*, 46(4):1155–1179, 2007.

ABSTRACT

ASYMPTOTIC EXPANSIONS AND STABILITY OF HYBRID SYSTEMS WITH TWO-TIME SCALES

by

DUNG TIEN NGUYEN

August 2011

Advisor: Professor G. George Yin

Major: Mathematics (Applied)

Degree: Doctor of Philosophy

In this dissertation, we consider solutions of hybrid systems in which both continuous dynamics and discrete events coexists. One of the main ingredients of our models is the two-time-scale formulation. Under broad conditions, asymptotic expansions are developed for the solutions of the systems of backward equations for switching diffusion in two classes of models, namely, fast switching systems and fast diffusion systems. To prove the validity of the asymptotic expansions, uniform error bounds are obtained.

In the second part of the dissertation, a singular linear system is considered. Again a two-time-scale formulation is used. Under suitable conditions, the system has a limit. Using the limit system as a guide, our effort is devoted to deriving a sufficient condition for the stability of the original system. These results present a perspective on reduction of complexity from stability and asymptotic analysis points of view.

AUTOBIOGRAPHICAL STATEMENT

DUNG TIEN NGUYEN

Education

- Ph.D. in Applied Mathematics, August 2010 (expected)
  Wayne State University, Detroit, Michigan
- M.A. in Mathematical Statistics, December, 2010
  Wayne State University, Detroit, Michigan
- B.S. in Mathematics, April 2001
  University of Natural Science - Vietnam National University at Ho Chi Minh city, Vietnam

Honors and Awards

1. Alfred L. Nelson Award (in recognition of outstanding achievement in the Ph.D. Program), awarded by Department of Mathematics, Wayne State University, 2011

2. Maurice J. Zenlonka Endowed Scholarship (in recognition of outstanding academic record), awarded by Department of Mathematics, Wayne State University, 2009 and 2010

3. SIAM Student Travel Award, to attend SIAM Conference on Control and Its Applications, Colorado, Denver, 2009

4. Graduate Research Assistantship, directed by Prof. George Yin, Wayne State University, 2008-2009

5. Graduate Teaching Assistantship, awarded by Department of Mathematics, Wayne State University, 2006-2008 and 2009-2011

List of Publications

1. D.T. Nguyen, X. Mao, G. Yin, and C. Yuan, Stability of Singular Jump-Linear Systems with A Large State Space: A Two-time-scale Approach, submitted.

2. D.T. Nguyen and G. Yin, Asymptotic Expansions of Solutions of Systems of Kolmogorov Backward Equations for Two-Time-Scale Switching Diffusions, submitted.

3. D.T. Nguyen and G. Yin, Asymptotic Expansions for Solutions of Systems of Kolmogorov Backward Equations of Two-Time-Scale Switching Jump Diffusions, to appear in Communications in Statistics.

4. G. Yin and D.T. Nguyen, Asymptotic expansions of backward equations for two-time-scale Markov chains in continuous time. Acta Math. Appl. Sinica, 25 (2009), no. 3, 457-476.