1-1-2011

Numerical methods for problems arising in risk management and insurance

Zhuo Jin
Wayne State University,

NUMERICAL METHODS FOR PROBLEMS ARISING IN RISK MANAGEMENT AND INSURANCE

by

ZHUO JIN

DISSERTATION

Submitted to the Graduate School

of Wayne State University,

Detroit, Michigan

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

2011

MAJOR: MATHEMATICS

Approved by:

Advisor Date

DEDICATION

To my parents

ACKNOWLEDGEMENTS

It is my great pleasure to thank those who provided me numerous supports for the completion of the dissertation. In particular, I would like to express my sincerest appreciation and deepest gratitude to my advisor, professor George Yin, for standing beside me during my graduate study. His endless care, consistent support, inspiring guidance, valuable suggestions enabled me to develop a comprehensive understanding of the stochastic control and actuarial science. Furthermore, I benefited a lot from his distinct and elegant personality during the five-year pursuit of my PhD degree.

I am taking this opportunity to thank Professor Paul Chow, Professor Tze Chien Sun, and Professor Stephen Spurr for serving on my committee.

I owe a great debt of gratitude to Professor Hailiang Yang, who have provided timely suggestions and support in both research and career. I am also grateful to Professor Shigeng Hu, who helped me a lot when I studied in China.

I owe my thanks to Ms. Mary Klamo, Ms. Patricia Bonesteel, Professor Choon-Jai Rhee, Professor John Breckenridge, Mr. James Veneri, Ms. Tiana Fluker, Professor Rafail Khasminskii, Professor Alex Korostelev, Dr. Qingshuo Song, Dr. Chao Zhu, Dr. Shaobai Kan, Dr. Son Luu Nguyen, and Mr. Lizhen Qin for their help and encouragement in one way or another.

I would like to thank my parents for their unconditional and unlimited love and support. My special gratitude goes to my wife for her love, encouragement, and sharing with me everything.

Finally, but not least, I would like to express my appreciation to the entire Department of Mathematics for the hospitality and services. I enjoyed the warm and friendly atmosphere in the department, and I appreciated the support I received during my study at Wayne State University.

TABLE OF CONTENTS

1 Introduction

This dissertation focuses on numerical methods for stochastic optimization problems arising in risk management and insurance. Both analytic properties and numerical results are presented for several insurance models.

Due to the recent economic crisis, more and more people are concerned with their future after retirement. According to the Transamerica Center for Retirement Studies, the number of U.S. workers who are confident in their ability to retire comfortably has declined significantly in the past year. Thus, to better manage retirement has become an urgent issue. One of the problems faced by numerous retirees is to find an optimal annuity-purchasing strategy to minimize the probability that the individual outlives his or her wealth termed the probability of lifetime ruin. Assuming the retiree maintains a pre-specified (exogenous) consumption level, one aims to determine the optimal investment strategy, as well as the optimal time to annuitize to minimize the probability that wealth will reach zero while the individual is still alive.

Taking up the retirement management issue, in the first part of the dissertation, we focus on the problem of investing in a risky financial market and purchasing annuities to minimize the probability of lifetime ruin. We consider a regime-switching diffusion model, which includes both continuous dynamics and discrete events. The modulating stochastic process is assumed to be a continuous-time Markov chain representing the random environment and other random factors not included in the usual diffusion formulation. For example, the rate of return and the volatility, and the insurance charge are modulated by a finite-state Markov chain $\alpha(\cdot)$, which represents the market modes and other economic conditions. For example, when $\alpha(t) \in \{1,2\}$, we use 1 to represent the bullish (up-trend) market and 2 the bearish (down-trend) market. In general, $\mathcal{M} = \{1,2,\ldots,m\}$ for some positive integer m. As is widely recognized, this regime-switching model appears to be more versatile and more realistic than the previous literature. The retirement management problem can be formulated as a stochastic control problem. The solution of the resulting stochastic control problem rests upon the solution of the associated systems of Hamilton-Jacobi-Bellman (HJB) equations and/or variational inequalities. Because of regime switching and the nonlinearity, it is virtually impossible to obtain closed-form or analytic solutions for our problems. Thus we are seeking viable alternative. In reference to the powerful methods of Markov chain approximation initiated by Kushner and developed more extensively in Kushner and Dupuis [13], we aim to find a good approximation to the underlying problems. By good approximation, we mean that the numerical methods should be consistent with the systems of interests and should converge to the right value.

In the second part of the dissertation, we consider dividend policy for regime-switching compound Poisson models. In the literature, De Finetti suggested that a company would seek to maximize the expectation of the present value of all dividends before possible ruin and showed the optimal dividend-payment strategy is a barrier strategy in 1957, see [4]. Since then a host of researchers tried to address this optimality question under more general and more realistic model assumptions. Nowadays, dividend optimization becomes a rich and challenging field of research, which needs the combination of tools from analysis, probability, and stochastic control. Similar to many papers in the literature, our objective is to maximize the expected discounted total dividends until ruin. We model the surplus process using a jump diffusion with regime-switching process. The process describing the regime switching is assumed to be a continuous-time Markov chain representing the random environment. As mentioned above, this model appears to be more versatile and more realistic than the classical compound Poisson and diffusion models. However, to solve the problem under this model, we need to solve a system of Hamilton-Jaccobi-Bellman (HJB) partial differential equations instead of a single HJB equation. Solving the associated system of HJB equations is a difficult task. Analytic solutions can not be obtained. A viable alternative is to construct feasible numerical approximation schemes for finding a good approximation to the underlying problems. Using the Markov chain approximation methods in [13] and in reference to the numerical methods developed for general regime-switching jump diffusions in [18], we develop an approximation procedure. The main ingredient is that we approximate the optimal dividend payout strategy by a controlled Markov chain. To prove the convergence, we use the methods of weak convergence. In addition to proving the convergence, we also provide numerical results for demonstration. Note that in the actual computation, we can simply use the well-known value or policy iteration techniques.

The rest of the dissertation is arranged as follows. Chapter 2 discusses Markov chain approximation for annuity-purchasing decision making to minimize the probability of financial ruin for regime-switching wealth models. Chapter 3 focuses on optimal dividend policy for regime-switching jump-diffusion model. A few further remarks are made in Chapter 4.

2 Optimal Annuity-Purchasing Strategies

To secure the post-retirement life, purchasing annuities from insurance companies is of crucial importance. The recipient of the annuity could receive a continuous fixed payment throughout the life. This life steam income could guarantee the retiree a given level of consumption. On the other hand, since the Swedish actuary Filip Lundberg [14] introduced the classical compound-Poisson risk model in 1903, probability of ruin is among the prime quantities to measure the insurance risk. Therefore, to measure the financial risk of purchasing annuity and managing portfolio becomes a big issue, that is what we are interested in.

A fruitful of results related to annuity were achieved in economics literature. Yaari [21] proved that in the absence of bequest motives and in a deterministic financial economy consumers will annuitize all of their liquid wealth. This result is generalized to a stochastic environment by Richard in [16], and recently T. Davidoff and Diamond demonstrated the robustness of Yaaris result in [6]. Similarly, Kapur and Orszag [12] and Brown [3] provide theoretical and empirical guidance on the optimal time to annuitize under various market structures. Optimal investment strategy to minimize the probability of lifetime ruin is considered by Young in [25], and Milevsky, Moore, and Young provide the annuity-purchasing strategies to minimize the probability of lifetime ruin in [15]. The so-called regime-switching models can be found in [7, 26, 27]; see also the related work [10, 22]. A comprehensive treatment of switching diffusions can be found in [23].

Unlike the previous work, the wealth is modeled as a regime-switching diffusion modulated by a continuous-time Markov chain. Based on Markov chain approximation techniques, an approximation procedure to find optimal annuity-purchasing strategies for minimizing the probability of lifetime ruin is constructed. Several interesting results are obtained that are consistent with the economics intuition.

2.1 Formulation

We use a controlled hybrid switching diffusion to represent the wealth. For simplicity, assume the system to be one dimensional. Let $(\Omega, \mathcal{F}, P)$ be the probability space and $\{\mathcal{F}_t\}$ be a filtration defined on it. Suppose that the discrete events take values in a finite set $\mathcal{M} = \{1, \ldots, m\}$ and that $\alpha(\cdot)$ is a continuous-time Markov chain having state space $\mathcal{M}$ and generator $Q = (q_{ij})$. Let $\omega(\cdot)$ be a standard $\mathcal{F}_t$-Wiener process, and $u(\cdot)$ be an $\mathcal{F}_t$-adapted control, taking value in a compact set U. Such controls are said to be *admissible*.

Now we set up the optimal annuity-purchasing and investment problem for an individual who seeks to minimize the probability that she or he outlives her or his wealth. The wealth of the individual consists of investment incomes from the riskless asset, the risky assets, and the income from the annuity after the purchase. To maintain a constant consumption rate, the individual manages the portfolio to avoid the financial ruin before she or he dies.

Initial income could include social security benefits and defined benefit pension benefits. We assume that the income variation only comes from buying life annuities by using money from current wealth. We assume that with $\alpha(s) = i$, the interest rate at time s is given by $r(i)$, and the individual can invest in a riskless asset with the yield rate $r(i)$ and a risky asset with the price $H(s,i)$ at time s such that

$$\begin{cases} dH(s,\alpha(s)) = \mu(s,\alpha(s))H(s,\alpha(s))ds + \sigma(s,\alpha(s))H(s,\alpha(s))d\omega, \\ H(t,\alpha(t)) = H_0 > 0. \end{cases} \tag{2.1}$$

where $\mu(s,i) > r(i)$ and $\sigma(s,i) > 0$ for all $i \in \mathcal{M}$. We use $\lambda(t)$ to denote the hazard rate at age t and $\widetilde{\lambda}(t)$ to denote the particular hazard rate function used to price the annuity. The actuarial present value of perpetuity with the life stream payment of 1 dollar per year by the interest rate $r(\alpha(t))$ and the hazard rate $\widetilde{\lambda}$ with the discount is

$$\begin{aligned} a(t) &= \int_0^\infty \exp(-r(\alpha(t))s)\exp(-\int_t^{t+s} \widetilde{\lambda}(v)dv)ds \\ &= \sum_{i\in\mathcal{M}} I_{\{\alpha(t)=i\}} \int_0^\infty \exp(-r(i)s)\exp(-\int_t^{t+s} \widetilde{\lambda}(v)dv)ds, \end{aligned} \tag{2.2}$$

where I_A is the indicator function of the set A.

For each $i \in \mathcal{M}$, $c(i)$ denotes a constant rate that the individual consumes, $W(s,i)$ denotes the wealth of the individual at time s, and $A(s,i)$ denotes a nonnegative income rate at time s of after any annuity purchases at that time. Let $u(s)$ be the amount that the decision maker invests in the risky asset at time s, and $0 \le u \le W$. The dynamic system is

$$\begin{cases} dW(s,\alpha(s)) = (r(\alpha(s))W(s,\alpha(s)) + (\mu(s,\alpha(s)) - r(\alpha(s))u(s) \\ \quad -c(\alpha(s)) + A(s,\alpha(s)))ds + \sigma(s,\alpha(s))u(s)d\omega - a(s)dA(s,\alpha(s)), \\ W(t,\alpha(t)) = w \ge 0, \\ A(t,\alpha(t)) = A \ge 0. \end{cases} \tag{2.3}$$

To simplify the dynamic system, we define the excess consumption $Z(s,i) = c(i) - A(s,i)$ with $Z(s,i)$ being the net income the decision maker inquires with $i \in \mathcal{M}$. Then the dynamic system (2.3) can be rewritten as

$$\begin{cases} dW(s,\alpha(s)) = (r(\alpha(s))W(s,\alpha(s)) + (\mu(s,\alpha(s)) - r(\alpha(s))u(s) - Z(s,\alpha(s)))ds \\ \qquad\qquad + \sigma(s,\alpha(s))u(s)d\omega + a(s)dZ(s), \\ W(t,\alpha(t)) = w \ge 0, \\ Z(t,\alpha(t)) = z \ge 0, \end{cases} \tag{2.4}$$

Since if $w \ge za(t)$, the individual can purchase the annuity immediately to guarantee a net income of z to avoid the lifetime ruin. Let τ_0 be the time when the wealth reaches zero and τ_d be the random time of death of the individual. With $\alpha(t) = i$, denote the cost function by

$$EI_{\{\tau_0<\tau_d|W(t,i)=w,\ Z(t,i)=z,\ \tau_0>t,\tau_d>t\}} = P[\tau_0 < \tau_d|W(t,i) = w,\ Z(t,i) = z,\ \tau_0 > t, \tau_d > t]$$

$$= \varphi(w, z, t, i, u), \quad i \in \mathcal{M}. \tag{2.5}$$

Then, the probability of lifetime ruin ψ at time t (with $\alpha(t) = i$)) can be represented on the domain $D = \{(w, z, t, i) : 0 \leq w \leq za(t), z \geq 0, t \geq 0, i \in \mathcal{M}\}$ as

$$\psi(w, z, t, i) = \inf_{\{u,Z\}} \varphi(w, z, t, i, u),\ i \in \mathcal{M}. \tag{2.6}$$

Then $\psi(w, z, t, \alpha(t)) = 0$ when $w \geq za(t)$. Note that $\tau_0 = \tau_0(w, z, u)$. That is, it depends on (w, z) as well on the control u. However, for notational simplicity, in what follows, we suppress the (w, z, u) dependence.

Note that this problem is a combination of the continuous control (the investment strategy u) and the singular control (the excess consumption Z). Combining stochastic control techniques used in Milevsky, Moore, and Young (2006), and methods for treating regime-switching diffusions in Yin and Zhu (2010), we can derive the Hamilton-Jacobi-Bellman variational inequality, which is given in Proposition 2.1. Furthermore, the Hamilton-Jacobi-Bellman equation with boundary conditions are presented in Proposition 2.2 and Proposition 2.3. The detailed derivations are omitted for brevity.

Proposition 2.1. *The probability of lifetime ruin is a constrained viscosity solution of the* *system of Hamilton-Jacobi-Bellman variational inequalities*

$$\max\left[\lambda(t)\psi-\psi_t-(rw-z)\psi_w-\min_u[(\mu-r)u\psi_w+\tfrac{1}{2}\sigma^2u^2\psi_{ww}]+Q\psi(w,z,t,\cdot)(i),\right.$$
$$\left. a(t)\psi_w+\psi_z\right]=0,\ \ i\in\mathcal{M}. \tag{2.7}$$

We can simplify the variational inequality by transferring the ruin probability $\psi(w,z,t,i)$ to a function of three variables with the barrier $W(t,i)=Z(t,i)a(t)$. Denote $x=w/z$. It is observed that the probability of lifetime ruin depends only on the ratio of current wealth to desired consumption; see Milevsky and Robinson (2000). That is, $\psi(x,1,t,i)=\psi(w,z,t,i)$. Define $V(x,t,i)=\psi(x,1,t,i)$. Then $V(x,t,i)=\psi(w,z,t,i)$.

$$V(x,t,i)=\inf_{u\in U}\varphi(w,z,t,i,u),\ \ i\in\mathcal{M}. \tag{2.8}$$

For an arbitrary $u\in U$, $i=\alpha(t)\in\mathcal{M}$, and $V(\cdot,\cdot,i)\in C^{2,1}(\mathbb{R}\times[0,\infty))$, define an operator $\mathcal{L}_t^u$ by

$$\mathcal{L}_t^uV(x,t,i)=V_t+V_x(x,i)(r(i)x-1+(\mu(t,i)-r(i))u)+\tfrac{1}{2}V_{xx}(x,i)(\sigma(t,i)u)^2$$
$$+QV(x,\cdot)(i), \tag{2.9}$$

where V_x and V_{xx} denote the first and second derivatives with respect to x, V_t is the derivative with respect to t, and

$$QV(x,t,\cdot)(i)=\sum_{j\neq i}q_{ij}(V(x,t,j)-V(x,t,i)),\ \ i\in\mathcal{M}.$$

The individual will not buy the annuity until the annuity can cover the excess consumption to avoid the lifetime ruin, then this bang-bang strategy lead to the formation of probability of lifetime ruin. Let $\mathcal{U}$ be the collection of admissible controls, the value functions have the following properties.

Proposition 2.2. *The probability of lifetime ruin can be written as*

$$\lambda(t)V(x,t,i) - \inf_{u\in U} \mathcal{L}_t^u V(x,t,i) = 0, \;\; i \in \mathcal{M}, \tag{2.10}$$

for $x < a(t)$ *with boundary conditions* $V(0,t,i) = 1$ *and* $V(a(t),t,i) = 0$ *with the transversality condition* $\lim_{s\to\infty} \exp(-\int_t^s \lambda(v)dv) E[V(X_s^*, s, i)|X_t = x] = 0$, *in which* X_s^* *is the optimally controlled* X_s.

Proposition 2.3. *Defining*

$$f(x,t,i) = V(x,t,i)\exp(-\int_0^t \lambda(v)dv),$$

equation (2.10) *becomes*

$$\inf_{u\in U} \mathcal{L}_t^u f(x,t,i) = 0, \;\; i \in \mathcal{M} \tag{2.11}$$

with the boundary condition $f(0,t,i) = \exp(-\int_0^t \lambda(v)dv)$ *and* $f(a(t),t,i) = 0$ *and with the transversality condition* $\lim_{s\to\infty} E[V(X_s^*, s)|X_t = x] = 0$. *This transversality condition can be rewritten as* $\lim_{t\to\infty} f(x,t,i) = 0$ *with probability* 1.

2.2 Constant Hazard Rate

In this section, we assume the forces of mortality to be a constant. That is, $\lambda(t) = \lambda$ and $\widetilde{\lambda}(t) = \widetilde{\lambda}$ for all $t \geq 0$. Define an operator $\mathcal{L}^u$

$$\begin{aligned} \mathcal{L}^u V(x,i) = {} & V_x(x,i)(r(i)x - 1 + (\mu(t,i) - r(i))u) + \tfrac{1}{2}V_{xx}(x,i)(\sigma(t,i)u)^2 \\ & + QV(x,\cdot)(i), i \in \mathcal{M}. \end{aligned} \tag{2.12}$$

Using (2.12), (2.10) becomes

$$\lambda V(x,i) - \inf_{u\in U} \mathcal{L}^u V(x,i) = 0, \tag{2.13}$$

and the boundary conditions are $V(0,i) = 1$ and $V(1/(\min_i r(i) + \widetilde{\lambda}), i) = 0$.

2.2.1 Approximating Markov Chain

We construct a discrete-time, finite-state, controlled Markov chain to approximate the controlled diffusion progress with regime switching. The discrete-time and finite-state controlled Markov chain is so defined that it is locally consistent with (2.4). We will show that the weak limit of the Markov chain satisfies (2.4).

For each $h > 0$, define $S_h = \{x : x = kh, k = 0, \pm 1, \pm 2, \ldots\}$. Let $\{(\tilde{\xi}_n^h, \alpha_n^h), n < \infty\}$ be a controlled discrete-time Markov chain on a discrete state space $S_h \times \mathcal{M}$ with transition probabilities from a state $(x,i) \in \mathcal{M}$ to another state $(y,j) \in \mathcal{M}$ denoted by $p^h((x,i),(y,j)|u)$. The u is a control parameter and takes values in the compact set U. We use u_n^h to denote the random variable that is the actual control action for the chain at discrete time n. To approximate the continuous-time Markov chain, we need another approximation sequence. Suppose that there is an $\Delta t^h(x,\alpha,u) > 0$ and define the "interpolation interval" as $\Delta t_n^h = \Delta t^h(\tilde{\xi}_n^h, \alpha_n^h, u_n^h)$ on $S_h \times \mathcal{M} \times \mathcal{U}$. Define the interpolation time $t_n^h = \sum_{k=0}^{n-1} \Delta t_k^h(\tilde{\xi}_k^h, \alpha_k^h, u_k^h)$. The piecewise constant interpolations $\tilde{\xi}^h(\cdot)$, $\alpha^h(\cdot)$, and $u^h(\cdot)$, are defined as

$$\tilde{\xi}^h(t) = \tilde{\xi}_n^h,\ \ \alpha^h(t) = \alpha_n^h,\ \ u^h(t) = u_n^h,\ \ \beta^h(t) = n \ \text{ for } \ t \in [t_n^h, t_{n+1}^h). \tag{2.14}$$

We need the approximating Markov chain constructed to satisfy local consistency. First let us recall the notion of local consistency.

Definition 2.4. Let $\{p^h((x,i),(y,j))|u\}$ for $(x,i),(y,j) \in S^h \times \mathcal{M}$ and $u \in U$ be a collection of well defined transition probabilities for the Markov chain $(\tilde{\xi}_n^h, \alpha_n^h)$, an approximation to $(x(\cdot), \alpha(\cdot))$. Define the difference $\Delta\tilde{\xi}_n^h = \tilde{\xi}_{n+1}^h - \tilde{\xi}_n^h$. Assume $\inf_{x,i,u} \Delta t^h(x,i,u) > 0$ for each $h > 0$ and $\lim_{h\to\infty} \Delta t^h(x,i,u) \to 0$. Let $E_{x,i,n}^{u,h}$, $\text{var}_{x,i,n}^{u,h}$, and $p_{x,i,n}^{u,h}$ denote the conditional expectation, variance, and marginal probability given $\{\tilde{\xi}_k^h, \alpha_k^h, u_k^h, k \le n, \tilde{\xi}_n^h = x, \alpha_n^h = i, u_n^h = u\}$, respectively. The sequence $\{(\tilde{\xi}_n^h, \alpha_n^h)\}$ is said to be locally consistent with (2.4), if

$$
\begin{aligned}
&E_{x,i,n}^{u,h}\Delta\tilde{\xi}_n^h = (r(i)x - 1 + (\mu(t,i) - r(i))u)\Delta t^h(x,i,u) + o(\Delta t^h(x,i,u)),\\
&\text{var}_{x,i,n}^{u,h}\Delta\tilde{\xi}_n^h = (\sigma(t,i)u)^2\Delta t^h(x,i,u) + o(\Delta t^h(x,i,u)),\\
&p_{x,i,n}^{u,h}\{\alpha_{n+1}^u = j\} = \Delta t^h(x,i,u)q_{ij} + o(\Delta t^h(x,i,u)), \quad \text{for } j \ne i,\\
&p_{x,i,n}^{u,h}\{\alpha_{n+1}^u = i\} = \Delta t^h(x,i,u)(1 + q_{ii}) + o(\Delta t^h(x,i,u)),\\
&\sup_{n,w\in\Omega} |\Delta\tilde{\xi}_n^h| \to 0 \text{ as } h \to 0.
\end{aligned}
\tag{2.15}
$$

Once we have a locally consistent approximating Markov chain, we can approximate the value function. Let $\mathcal{U}^h$ denote the collection of controls, which are determined by a sequence of measurable functions $F_n^h(\cdot)$ such that

$$
u_n^h = F_n^h(\tilde{\xi}_k^h, \alpha_k^h, k \le n; u_k^h, k \le n). \tag{2.16}
$$

Let $G_h^o = S_h \cap G^o$. Then $G_h \times \mathcal{M}$ is a finite state space. Practically, we compute $V^h(x,i)$ by solving the corresponding dynamic programming equation using the iteration method. In fact, for $i \in \mathcal{M}$, we can use

$$V^h(x,i) = \begin{cases} \min_{u\in U}[\frac{1}{1+\lambda\Delta t^h(x,i,u)} \sum_{y,j} (p^h((x,i),(y,j))|u) V^h(y,j)], \text{ for } x \in G_h^o, \\ \\ \widetilde{g}(x,i), \text{ for } x = 0, B, \end{cases} \quad (2.17)$$

where $\frac{1}{1+\lambda\Delta t^h(x,i,u)}$ is a discount factor. When the control space has only one element $u^h \in \mathcal{U}^h$, the min in (3.24) can be dropped. That is,

$$V^h(x,i) = \begin{cases} \frac{1}{1+\lambda\Delta t^h(x,i,u)} \sum_{y,j} (p^h((x,i),(y,j))|u) V^h(y,j), \text{ for } x \in G_h^o, \\ \\ \widetilde{g}(x,i), \text{ for } x = 0, B. \end{cases} \quad (2.18)$$

Similarly, the inf in (2.13) can also be dropped with $u = u(0)$ in $\mathcal{L}^u$. That is,

$$V_x(x,i)(r(i)x - 1 + (\mu(t,i) - r(i))u) + \frac{1}{2}V_{xx}(x,i)(\sigma(t,i)u)^2 + \sum_j V(x,\cdot)q_{ij} - \lambda V(x,i) = 0. \quad (2.19)$$

Define the approximation to the first and the second derivatives of $V(\cdot,i)$ by finite difference method using stepsize $h > 0$ as:

$$\begin{aligned} &V(x,i) \to V^h(x,i) \\ &V_x(x,i) \to \frac{V^h(x+h,i) - V^h(x,i)}{h} \quad \text{for} \quad r(i)x - 1 + (\mu(t,i) - r(i))u > 0, \\ &V_x(x,i) \to \frac{V^h(x,i) - V^h(x-h,i)}{h} \quad \text{for} \quad r(i)x - 1 + (\mu(t,i) - r(i))u < 0, \\ &V_{xx}(x,i) \to \frac{V^h(x+h,i) - 2V^h(x,i) + V^h(x-h,i)}{h^2}. \end{aligned} \quad (2.20)$$

Together with the boundary conditions, it leads to

$$
\begin{aligned}
&V^h(x,i) = \tilde{g}(x,i), \text{ for } x = 0, B,\\
&\frac{V^h(x+h,i) - V^h(x,i)}{h}(r(i)x - 1 + (\mu(t,i) - r(i))u)^+ - \frac{V^h(x,i) - V^h(x-h,i)}{h}(r(i)x - 1\\
&\quad +(\mu(t,i) - r(i))u)^- + \frac{V^h(x+h,i) - 2V^h(x,i) + V^h(x-h,i)}{h^2} \cdot \frac{(\sigma(t,i)u)^2}{2}\\
&\quad + \sum_j^m V^h(x,\cdot)q_{ij} - \lambda V^h(x,i) = 0, \quad \forall x \in G_h^o, i \in \mathcal{M},
\end{aligned}
\tag{2.21}
$$

where G_h^o denotes the interior of G_h, and $(r(i)x - 1 + (\mu(t,i) - r(i))u)^+$ and $(r(i)x - 1 + (\mu(t,i) - r(i))u)^-$ are the positive and negative parts of $r(i)x - 1 + (\mu(t,i) - r(i))u$, respectively. Simplifying (3.28) and comparing the result with (3.25), we have

$$
\begin{aligned}
&p^h((x,i),(x+h,i)|u) = \frac{((\sigma(t,i)u)^2/2) + h(r(i)x - 1 + (\mu(t,i) - r(i))u)^+}{\widetilde{D} - \lambda h^2},\\
&p^h((x,i),(x-h,i)|u) = \frac{((\sigma(t,i)u)^2/2) + h(r(i)x - 1 + (\mu(t,i) - r(i))u)^-}{\widetilde{D} - \lambda h^2},\\
&p^h((x,i),(x,j)|u) = \frac{h^2}{\widetilde{D} - \lambda h^2} q_{ij}, \quad \text{for} \quad j \neq i,\\
&p^h(\cdot) = 0, \quad \text{otherwise},\\
&\Delta t^h(x,i,u) = \frac{h^2}{\widetilde{D} - \lambda h^2},
\end{aligned}
\tag{2.22}
$$

with

$$\widetilde{D} = (\sigma(t,i)u)^2 + h|(r(i)x - 1 + (\mu(t,i) - r(i))u)| + h^2(\lambda - q_{ii})$$

being well defined.

Here, we present the local consistency for our approximating Markov chain.

Lemma 2.5. *The Markov chain* $\{\tilde{\xi}_n^h, \alpha_n^h\}$ *with transition probabilities* $(p^h(\cdot))$ *defined in* (3.29) *is locally consistent with the stochastic differential equation in* (2.4).

Proof. Using (3.29), it is readily seen that

$$E_{x,i,n}^{u,h}\Delta\tilde{\xi}_n^h = hp^h((x,i),(x+h,i)|u) - hp^h((x,i),(x-h,i)|u)$$

$$= (r(i)x - 1 + (\mu(t,i) - r(i))u)\Delta t^h(x,i,u) + o(\Delta t^h(x,i,u)).$$

Likewise, we obtain

$$E_{x,i,n}^{u,h}(\Delta\tilde{\xi}_n^h)^2 = h^2p^h((x,i),(x+h,i)|u) - h^2p^h((x,i),(x-h,i)|u)$$

$$= (\sigma(t,i)u)^2\Delta t^h(x,i,u) + \Delta t^h(x,i,u)O(h).$$

As a result,

$$\text{var}_{x,i,n}^{u,h}\Delta\tilde{\xi}_n^h = (\sigma(t,i)u)^2\Delta t^h(x,i,u) + o(\Delta t^h(x,i,u))$$

Thus both equations in (3.20) are verified. The desired local consistency follows. □

Based on the Markov chain approximation constructed in the last section, piecewise constant interpolation is obtained here with appropriately chosen interpolation intervals. Using $(\tilde{\xi}_n^h, \alpha_n^h)$ to approximate the continuous-time process $(x(\cdot), \alpha(\cdot))$, we defined the continuous-time interpolation $(\tilde{\xi}^h(\cdot), \alpha^h(\cdot))$, $u^h(\cdot)$ and $\beta^h(t)$ in (3.19). Define $\mathcal{D}_t^h$ as the smallest σ-algebra generated by $\{\tilde{\xi}^h(s), \alpha^h(s), u^h(s), \beta^h(s),\ s \le t\}$. In addition, $\mathcal{U}^h$ defined by (3.21) is equivalent to the collection of all piecewise constant admissible controls with respect to $\mathcal{D}_t^h$. To proceed, we need the following assumptions.

(A1) For each $i \in \mathcal{M}$, $\sigma(\cdot, i) > 0$.

(A2) For each $i \in \mathcal{M}$ and each $u \in U$, the function $\tilde{g}(\cdot, i)$ is continuous in G.

Use E_n^h to denote the conditional expectation given $\{\tilde{\xi}_k^h, \alpha_n^k, u_n^h, k \le n\}$. Define

$$M^h(t) = M_n^h, t \in [t_n^h, t_{n+1}^h), \text{ where } M_n^h = \sum_{k=0}^{n-1}(\Delta\tilde{\xi}_k^h - E_k^h\Delta_k^h). \tag{2.23}$$

The local consistency leads to

$$\begin{aligned}\tilde{\xi}^h(t) &= x + \sum_{k=0}^{\beta^h(t)-1} [E_k^h\Delta\tilde{\xi}_k^h + (\Delta\tilde{\xi}_k^h - E_k^h\Delta\tilde{\xi}_k^h)] \\ &= x + \sum_{k=0}^{\beta^h(t)-1} (r(\alpha_k^h)\tilde{\xi}_k^h - 1 + (\mu(t, \alpha_k^h) - r(\alpha_k^h))u_k^h)\Delta t^h(\tilde{\xi}_k^h, \alpha_k^h, u_k^h)) \\ &\qquad + \sum_{k=0}^{\beta^h(t)-1} (\Delta\tilde{\xi}_k^h - E_k^h\Delta\tilde{\xi}_k^h) + \varepsilon^h(t) \\ &= x + \int_0^t (r(\alpha^h(s))\tilde{\xi}^h(s) - 1 + (\mu(t, \alpha^h(s)) - r(\alpha^h(s)))u^h(s))ds + M^h(t) + \varepsilon^h(t),\end{aligned} \tag{2.24}$$

where $\varepsilon^h(t)$ is a negligible error satisfying

$$\lim_{h\to\infty} \sup_{0\le t\le T} E|\varepsilon^h(t)|^2 \to 0 \ \text{ for any } \ 0 < T < \infty. \tag{2.25}$$

Note that $M^h(\cdot)$ is a martingale with respect to $\mathcal{D}_t^h$, and its discontinuity goes to zero as

$h \to 0$. We attempt to represent $M^h(t)$ similar to the diffusion term in (2.4). Define $\omega^h(\cdot)$ as

$$\begin{aligned} \omega^h(t) &= \sum_{k=0}^{\beta^h(t)-1} (\Delta\tilde{\xi}_k^h - E_k^h \Delta\tilde{\xi}_k^h)/\sigma(\tilde{\xi}_k^h, \alpha_k^h), \\ &= \int_0^t \sigma^{-1}(\tilde{\xi}^h(s), \alpha^h(s)) dM^h(s). \end{aligned} \tag{2.26}$$

We can now rewrite (2.24) as

$$\begin{aligned} \tilde{\xi}^h(t) = x + \textstyle\int_0^t (r(\alpha^h(s))\tilde{\xi}^h(s) - 1 + (\mu(s, \alpha^h(s)) - r(\alpha^h(s)))u^h(s))ds \\ + \textstyle\int_0^t \sigma(\tilde{\xi}^h(s), \alpha^h(s)) d\omega^h(s) + \varepsilon^h(t). \end{aligned} \tag{2.27}$$

Since $\sigma(\cdot) > 0$ in the compact set G, $\sigma^{-1}(\cdot)$ is uniformly bounded, which ensures the weak limit has continuous path with probability one.

Consider the cost function

$$\begin{aligned} \varphi(x, i, u) &= P[\tau_0 < \tau_d | X(t, i) = x, \tau_0 > t, \tau_d > t] \\ &= P[\tau_0 < \tau_d | W(t, i) = w, Z(t, i) = z, \tau_0 > t, \tau_d > t] \\ &= E_{x,i}^u [I_{\{\tau_0 < \tau_d | W(t,i) = w, Z(t,i) = z, \tau_0 > t, \tau_d > t\}}] \\ &= E_{x,i}^u [I_{\{\tau_0 < \tau_d | x(t,i) = x, \tau_0 > t, \tau_d > t\}}]. \end{aligned} \tag{2.28}$$

Note that using the interpolation, the cost function can be rewritten as

$$\varphi^h(x,i,u^h) = P[\tau_0 < \tau_d | \tilde{\xi}_0^h = x, \tau_0 > t, \tau_d > t] \tag{2.29}$$
$$= E_{x,i}^{u^h}[I_{\{\tau_0<\tau_d|\tilde{\xi}_0^h = x, \tau_0>t, \tau_d>t\}}].$$

To proceed, we use the relaxed control representation; see Kushner and Dupuis (2001).

Let $\mathcal{B}(U \times [0,\infty))$ be the σ-algebra of Borel subsets of $U \times [0,\infty)$. An *admissible relaxed control* (or deterministic relaxed control) $m(\cdot)$ is a measure on $\mathcal{B}(U \times [0,\infty))$ such that $m(U \times [0,t]) = t$ for each $t \geq 0$. Given a relaxed control $m(\cdot)$, there is an $m_t(\cdot)$ such that $m(d\chi dt) = m_t(d\chi)dt$. We can define $m_t(B) = \lim_{\delta\to 0} \frac{m(B\times[t-\delta,t])}{\delta}$ for $B \in \mathcal{B}(U)$. With the given probability space, we say that $m(\cdot)$ is an admissible relaxed (stochastic) control for $(\omega(\cdot), \alpha(\cdot))$ or $(m(\cdot), \omega(\cdot), \alpha(\cdot))$ is admissible, if $m(\cdot, \omega)$ is a deterministic relaxed control with probability one and if $m(A \times [0,t])$ is $\mathcal{F}_t$-adapted for all $A \in \mathcal{B}(U)$. There is a derivative $m_t(\cdot)$ such that $m_t(\cdot)$ is $\mathcal{F}_t$-adapted for all $A \in \mathcal{B}(U)$.

Given a relaxed control $m(\cdot)$ of $u^h(\cdot)$, we define the derivative $m_t(\cdot)$ such that

$$m^h(B) = \int_{U\times[0,\infty)} I_{\{(u^h)\in B\}} m_t(d\chi)dt \tag{2.30}$$

for all $B \in \mathcal{B}(U \times [0,\infty))$, and that for each t, $m_t(\cdot)$ is a measure on $\mathcal{B}(U)$ satisfying $m_t(U) = 1$. For example, we can define $m_t(\cdot)$ in any convenient way for $t = 0$ and as the left-hand derivative for $t > 0$,

$$m_t(A) = \lim_{\delta\to 0} \frac{m(A \times [t-\delta, t])}{\delta}, \ \forall A \in \mathcal{B}(U). \tag{2.31}$$

Note that $m(d\chi dt) = m_t(d\chi)dt$. It is natural to define the relaxed control representation

$m^h(\cdot)$ of $u^h(\cdot)$ by

$$m_t^h(A) = I_{\{u^h(t)\in A\}},\ \forall A \in \mathcal{B}(U). \tag{2.32}$$

Let $\mathcal{F}_t^h$ be a filtration, which denotes the minimal $\sigma-$algebra that measures

$$\{\tilde{\xi}^h(s), \alpha^h(\cdot), m_s^h(\cdot), \omega^h(s), \beta^h(s), s \le t\}. \tag{2.33}$$

Use Γ^h to denote the set of admissible relaxed controls $m^h(\cdot)$ with respect to $(\alpha^h(\cdot), \omega^h(\cdot))$ such that $m_t^h(\cdot)$ is a fixed probability measure in the interval $[t_n^h, t_{n+1}^h)$ given $\mathcal{F}_t^h$. Then Γ^h is a larger control space containing $\mathcal{U}^h$. With the notation of relaxed control given above, we can write (2.27), (2.4) and the value function (2.8) as

$$\begin{aligned}\tilde{\xi}^h(t) = x + \int_0^t \int_U (r(\alpha^h(s))\tilde{\xi}^h(s) - 1 + (\mu(s,\alpha^h(s)) - r(\alpha^h(s)))\chi) m_s^h(d\chi) ds \\ + \int_0^t \sigma(\tilde{\xi}^h(s), \alpha^h(s)) d\omega^h(s) + \varepsilon^h(t),\end{aligned} \tag{2.34}$$

$$\begin{aligned}x(t) = x + \textstyle\int_0^t \int_U (r(\alpha(s))x(s) - 1 + (\mu(s,\alpha(s)) - r(\alpha(s))\chi) m_s(d\chi) ds \\ + \textstyle\int_0^t \sigma(x(s), \alpha(s)) d\omega(s),\end{aligned} \tag{2.35}$$

and

$$V^h(x,i) = \inf_{m^h \in \Gamma^h} \varphi^h(x,i,m^h). \tag{2.36}$$

Now we give the definition of existence and uniqueness of weak solution.

Definition 2.6. By a weak solution of (2.35), we mean that there exists a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathcal{F}_t$-Wiener process, and process $(x(\cdot), \alpha(\cdot), m(\cdot), \omega(\cdot))$ such that $\omega(\cdot)$ is a standard $\mathcal{F}_t$-Wiener process, $\alpha(\cdot)$ is a Markov chain with generator Q and state space $\mathcal{M}$, $m(\cdot)$ is admissible with respect to $x(\cdot)$ is $\mathcal{F}_t$-adapted, and (2.35) is satisfied. For an initial condition (x, i), by the weak sense uniqueness, we mean that the probability law of the admissible process $(\alpha(\cdot), m(\cdot), \omega(\cdot))$ determines the probability law of solution $(x(\cdot), \alpha(\cdot), m(\cdot), \omega(\cdot))$ to (2.35), irrespective of probability space.

To proceed, we need more assumptions.

(A3) Let $u(\cdot)$ be an admissible ordinary control with respect to $\omega(\cdot)$ and $\alpha(\cdot)$, and suppose that $u(\cdot)$ is piecewise constant and takes only a finite number of values. For each initial condition, there exists a solution to (2.35) where $m(\cdot)$ is the relaxed control representation of $u(\cdot)$. This solution is unique in the weak sense.

2.2.2 Main Results

This part deals with convergence of a sequence of wealth processes.

Lemma 2.7. *Using the transition probabilities $\{p^h(\cdot)\}$ defined in* (3.29)*, the interpolated process of the constructed Markov chain $\{\alpha^h(\cdot)\}$ converges weakly to $\alpha(\cdot)$, the Markov chain with generator $Q = (q_{ij})$.*

Proof. The proof can be obtained similar to Theorem 3.1 in Yin et. al (2003). □

Theorem 2.8. *Assume* (A1)*. Let $\{\tilde{\xi}^h_n, \alpha^h_n, n < \infty\}$ be constructed with transition probabilities defined in* (3.29)*, $\{u^h_n, n < \infty\}$ be a sequence of admissible controls, $(\tilde{\xi}^h(\cdot), \alpha^h(\cdot))$ be the continuous-time interpolation defined in* (3.19)*, $m^h(\cdot)$ be the relaxed control representation of $\{u^h_n, n < \infty\}$. Then $(\tilde{\xi}^h(\cdot), \alpha^h(\cdot), m^h(\cdot), \omega^h(\cdot))$ is tight. Denote the limit of weakly convergent subsequence by $(\tilde{\xi}(\cdot), \alpha(\cdot), m(\cdot), \omega(\cdot))$ and by $\mathcal{F}_t$ the σ-algebra generated by*

$\{x(s), \alpha(s), m(s), \omega(s), s \le t\}$. *Then* $\omega(\cdot)$ *is a standard* $\mathcal{F}_t$*-Wiener process, and* $m(\cdot)$ *is an admissible control. Moreover,* (2.35) *is satisfied.*

Proof. In view of Lemma 3.6, $\{\alpha^h(\cdot)\}$ is tight. Thus, it suffices to prove that the tightness of $\{\omega^h(\cdot)\}$ and $\{\tilde{\xi}^h(\cdot)\}$. By local consistency, and the definition of $\omega^h(\cdot)$ in (3.37), we obtain

$$E(\omega^h(t+\delta) - \omega^h(t))^2 = E[\sum_{j=\beta^h(t)}^{\beta^h(t+\delta)-1} (\Delta\tilde{\xi}^h_j - E^h_j \Delta\tilde{\xi}^h_j)/\sigma(\tilde{\xi}^h_j, \alpha^h_j)]^2 \qquad (2.37)$$

$$= O(\delta) + \varepsilon^h(\delta),$$

where $\varepsilon^h(\cdot)$ is a continuous function defined in (3.39). Taking $\limsup_{h\to 0}$ followed by $\lim_{\delta\to 0}$ yield the tightness of $\{\omega^h(\cdot)\}$.

Next, we prove the tightness of $\{\tilde{\xi}^h(\cdot)\}$. Let $E^h_{x,i}$ be the expectation for the interpolated process with interpolation stepsize h and initial data (x, i). By (2.34), we obtain

$$\begin{aligned}
&E^h_{x,i}|\tilde{\xi}^h(t) - x|^2 \\
&= E^h_{x,i}\Big|\int_0^t \int_U (r(\alpha^h(s))\tilde{\xi}^h(s) - 1 + (\mu(s, \alpha^h(s)) - r(\alpha^h(s)))\chi) m^h_s(d\chi)ds \\
&\qquad + \int_0^t \sigma(\tilde{\xi}^h(s), \alpha^h(s)) d\omega^h(s) + \varepsilon^h(t)\Big|^2 \qquad (2.38)\\
&\le 3E^h_{x,i}\Big|\int_0^t \int_U (r(\alpha^h(s))\tilde{\xi}^h(s) - 1 + (\mu(s, \alpha^h(s)) - r(\alpha^h(s)))\chi) m^h_s(d\chi)ds\Big|^2 \\
&\qquad + 3E^h_{x,i}\Big|\int_0^t \sigma(\tilde{\xi}^h(s), \alpha^h(s)) d\omega^h(s)\Big|^2 + 3|\varepsilon^h(t)|^2 \\
&\le \tilde{K}t^2 + \tilde{K}t + 3E^h_{x,i}|\varepsilon^h(t)|^2,
\end{aligned}$$

where $\tilde{K}$ is a generic positive constant. Similar to the argument of (2.37), we also obtain

$$E^{m^h}|\tilde{\xi}^h(t+\delta) - \tilde{\xi}^h(t)|^2 = O(\delta) + O(E^{m^h}|\varepsilon^h(t)|^2), \text{ as } \delta \to 0. \tag{2.39}$$

This establishes the tightness of $\tilde{\xi}^h(\cdot)$. Hence, we have proved that $\{\tilde{\xi}^h(\cdot), \alpha^h(\cdot), m^h(\cdot), \omega^h(\cdot)\}$ is tight.

Since $(\tilde{\xi}^h(\cdot), \alpha^h(\cdot), m^h(\cdot), \omega^h(\cdot))$ is tight, we can extract a weakly convergent subsequence by Prohorov's theorem. Still index the subsequence by h for notational simplicity. Denote the limit by $(x(\cdot), \alpha(\cdot), m(\cdot), \omega(\cdot))$. The process $\omega^h(\cdot)$ has continuous sample paths w.p.1. Thus the process $\omega(\cdot)$ also has continuous sample paths w.p.1. The weak convergence implies that $m(U,t) = t$ for all t. We shall prove that $x(\cdot)$ is a solution of a stochastic differential equation with driving processes $\alpha(\cdot)$, $m(\cdot)$, and $\omega(\cdot)$. By means of the Skorohod representation, without changing notation, we may assume that $(\tilde{\xi}^h(\cdot), \alpha^h(\cdot), m^h(\cdot), \omega^h(\cdot))$ converges to $(x(\cdot), \alpha(\cdot), m(\cdot), \omega(\cdot))$ w.p.1 and the convergence is uniform on compact set.

To characterize $\omega(\cdot)$, let $t > 0$, $\delta > 0$, p, q, $\{t_k : k \le p\}$ be given such that $t_k \le t \le t + t'$ for all $k \le p$, $g_j(\cdot)$ for $j \le q$ is real-valued and continuous functions on $U \times [0, \infty)$ and having compact support for all $j \le q$. Define

$$(g_j, m)_t = \int_0^t \int_U g_j(r, s) m(d\chi ds). \tag{2.40}$$

Let $K(\cdot)$ be a real-valued and continuous function of its arguments with compact support. By (3.37), $\omega^h(\cdot)$ is an $\mathcal{F}_t$-martingale. Thus we have

$$EK(\tilde{\xi}^h(t_k), \alpha^h(t_k), \omega^h(t_k), (g_j, m^h)_{t_k}), j \le q, k \le p)[\omega^h(t+t') - \omega^h(t)] = 0. \tag{2.41}$$

By using the Skorohod representation and the dominant convergence theorem, letting $h \to 0$, we obtain

$$EK(x(t_k),\alpha(t_k),\omega(t_k),(g_j,m)_{t_k}),j\le q,k\le p)[\omega(t+t')-\omega(t)]=0. \tag{2.42}$$

Since $\omega(\cdot)$ has continuous sample paths, (3.43) implies that $\omega(\cdot)$ is a continuous $\mathcal{F}_t$-martingale. On the other hand, since $E[((\omega^h(t+\delta))^2-(\omega^h(t))^2]=E[(\omega^h(t+\delta)-\omega^h(t))^2]$, by using the Skorohod representation and the dominant convergence theorem together with (2.37), we have

$$EK(x(t_k),\alpha(t_k),\omega(t_k),(g_j,m)_{t_k}),j\le q,k\le p)[\omega^2(t+\delta)-\omega^2(t)-\delta]=0. \tag{2.43}$$

The quadratic variation of the martingale $\omega(t)$ is t, then $\omega(\cdot)$ is an $\mathcal{F}_t$-Wiener process.

For $\delta>0$, define the process $q(\cdot)$ by $q^{h,\delta}(t)=q^h(n\delta),t\in[n\delta,(n+1)\delta)$. Then, by the tightness of $\{\tilde{\xi}^h(\cdot),\alpha^h(\cdot)\}$, (2.34) can be rewritten as

$$\begin{aligned}\tilde{\xi}^h(t)=x+\int_0^t\int_U(r(\alpha^h(s))\tilde{\xi}^h(s)-1+(\mu(s,\alpha^h(s))-r(\alpha^h(s)))\chi)m_s^h(d\chi)ds\\ +\int_0^t\sigma(\tilde{\xi}^{h,\delta}(s),\alpha^{h,\delta}(s))d\omega^h(s)+\varepsilon^{h,\delta}(t),\end{aligned} \tag{2.44}$$

where

$$\lim_{\delta\to0}\limsup_{h\to0}E|\varepsilon^{h,\delta}(t)|=0. \tag{2.45}$$

Let $h\to0$, by using the Skorohod representation, we obtain

$$\begin{aligned}E|\int_0^t\int_U(r(\alpha^h(s))\tilde{\xi}^h(s)-1+(\mu(s,\alpha^h(s))-r(\alpha^h(s)))\chi)m_s^h(d\chi)ds\\ -\int_0^t\int_U(r(\alpha(s))x(s)-1+(\mu(s,\alpha(s))-r(\alpha(s)))\chi)m_s^h(d\chi)ds|=0\end{aligned} \tag{2.46}$$

uniformly in t with probability one. On the other hand, $\{m^h(\cdot)\}$ converges in the compact weak topology, that is, for any bounded and continuous function $g(\cdot)$ with compact support,

$$\int_0^\infty \int_U g(r,s)m^h(d\chi ds) \to \int_0^\infty \int_U g(r,s)m(d\chi ds). \tag{2.47}$$

Again, the Skorohod representation implies that as $h \to 0$,

$$\begin{aligned}&\int_0^t \int_U (r(\alpha(s))x(s) - 1 + (\mu(s,\alpha(s)) - r(\alpha(s)))\chi)m_s^h(d\chi ds) \to \\ &\qquad \int_0^t \int_U (r(\alpha(s))x(s) - 1 + (\mu(s,\alpha(s)) - r(\alpha(s)))\chi)m_s(d\chi ds)\end{aligned} \tag{2.48}$$

uniformly in t with probability one on any bounded interval.

Since $\tilde{\xi}^{h,\delta}(\cdot)$ and $\alpha^{h,\delta}(\cdot)$ are piecewise constant functions, we obtain

$$\begin{aligned}\int_0^t \sigma(\tilde{\xi}^{h,\delta}(s),\alpha^{h,\delta}(s))d\omega^h(s) &= \sum_{i=0}^{t/\delta} \sigma(\tilde{\xi}^{h,\delta}(i\delta),\alpha^{h,\delta}(i\delta))(\omega^h((i+1)\delta) - \omega^h(i\delta)) \\ &\to \int_0^t \sigma(\tilde{\xi}^{\delta}(s),\alpha^{\delta}(s))d\omega(s) \quad \text{as} \quad h \to 0\end{aligned} \tag{2.49}$$

with probability one. Combining (2.40)-(3.50), we have

$$\begin{aligned}x(t) = x + &\textstyle\int_0^t \int_U (r(\alpha(s))x(s) - 1 + (\mu(s,\alpha(s)) - r(\alpha(s)))\chi)m_s^h(d\chi)ds + \\ &\textstyle\int_0^t \sigma(\tilde{\xi}^{\delta}(s),\alpha^{\delta}(s))d\omega(s) + \varepsilon^{\delta}(t),\end{aligned} \tag{2.50}$$

where $\lim_{\delta\to 0} E|\varepsilon^{\delta}(t)| = 0$. Finally, taking limits in the above equation as $\delta \to 0$, (2.35) is obtained. □

This part deals with the approximation of relaxed controls by ordinary controls. As is well-known that the relaxed controls are a device that is mainly used for mathematical analysis purpose. They can always be approximated by ordinary controls. This fact, is referred to as a chattering lemma. Here we present a result of chattering lemma for our problem.

Theorem 2.9. *Let* $(m(\cdot), \omega(\cdot))$ *be admissible for the problem given in* (2.35)*. Then given* $\eta > 0$*, there is a finite set* $\{\gamma_1^\eta, \ldots, \gamma_{l_\eta}^\eta\} = U^\eta \subset U$*, and an* $\varepsilon > 0$ *such that there is a probability space on which are defined* $(x^\eta(\cdot), \alpha^\eta(\cdot), u^\eta(\cdot), \omega^\eta(\cdot))$*, where* $\omega^\eta(\cdot)$ *are standard Brownian motions, and* $u^\eta(\cdot)$ *is an admissible* U^η*-valued ordinary control on the interval* $[k\varepsilon, k\varepsilon + \varepsilon)$*. Moreover,*

$$\begin{aligned} &P_x^m(\sup_{s \le T} |x^\eta(s) - x(s)| > \eta) \le \eta, \text{ and} \\ &|\varphi_x^m(\cdot) - \varphi_x^{u^\eta}(\cdot)| \le \eta. \end{aligned} \tag{2.51}$$

Coming back to the approximation to the optimal control, to show the discrete approximation of the value function $V^h(x, i)$ converges to the value function $V(x, i)$, we shall use the comparison control techniques. In doing so, we need to verify certain continuity properties. The details of the proof is presented in the appendix.

Proposition 2.10. *For* (2.35)*, let* $\widetilde{\eta} > 0$ *be given and* $(x(\cdot), \alpha(\cdot), m(\cdot), \omega(\cdot))$ *be an* $\widetilde{\eta}$*-optimal control. For each* $\eta > 0$*, there is an* $\varepsilon > 0$ *and a probability space on which are defined* $\omega^\eta(\cdot)$*, a control* $u^\eta(\cdot)$ *as in Theorem* 2.9*, and a solution* $x^\eta(\cdot)$ *such that the following assertions hold:*

(i)

$$|\varphi_x^m(\cdot) - \varphi_x^{u^\eta}(\cdot)| \le \eta. \tag{2.52}$$

(ii) *Moreover, there is a $\theta > 0$ such that the approximating $u^\eta(\cdot)$ can be chosen so that its probability law at $n\varepsilon$, conditioned on $\{\omega^\eta(\tau), \alpha^\eta(\tau), \tau \le n\varepsilon; u^\eta(k\varepsilon), k < n\}$ depends only on the samples $\{\omega^\eta(p\theta), \alpha^\eta(p\theta), p\theta \le n\varepsilon; u^\eta(k\varepsilon), k < n\}$, and is continuous in the $\omega^\varepsilon(p\theta)$ arguments.*

This part deals with the convergence of the cost and value functions. Note that the cost $\varphi^h(x,i,m^h)$ is given by (2.29), where $m^h(\cdot)$ is a sequence of admissible relaxed controls for $\{\tilde{\xi}^h(\cdot), \alpha^h(\cdot)\}$. Each sequence $\{\tilde{\xi}^h(\cdot), \alpha^h(\cdot), m^h(\cdot), \omega^h(\cdot)\}$ has a weakly convergent subsequence with the limit satisfying (2.35). By using the Skorohod representation, as $h \to 0$,

$$E^{u^h}_{x,i}[I_{\{\tau_0<\tau_d|\tilde{\xi}^h_0=x,\tau_0>t,\tau_d>t\}}] \to E^u_{x,i}[I_{\{\tau_0<\tau_d|x(t,i)=x,\tau_0>t,\tau_d>t\}}]. \tag{2.53}$$

This leads to

$$\varphi^h(x,i,m^h) \to \varphi(x,i,m). \tag{2.54}$$

Theorem 2.11. *Assume* (A1)-(A3). *$V^h(x,i)$ and $V(x,i)$ are value functions defined in* (2.36) *and* (2.8)*, respectively. Then $V^h(x,i) \to V(x,i)$ as $h \to 0$.*

Proof. Since $V(x,i)$ is the minimizing cost function, for any admissible control $m(\cdot)$,

$$\varphi(x,i,m) \ge V(x,i).$$

Let $\widetilde{m}^h(\cdot)$ be an optimal relaxed control for $\{\tilde{\xi}^h(\cdot)\}$. That is,

$$V^h(x,i) = \varphi^h(x,i,\widetilde{m}^h) = \inf_{m^h} \varphi^h(x,i,m^h).$$

Choose a subsequence $\{\widetilde{h}\}$ of $\{h\}$ such that

$$\lim_{\widetilde{h}\to 0} V^{\widetilde{h}}(x,i) = \liminf_{\widetilde{h}\to 0} V^{\widetilde{h}}(x,i) = \lim_{\widetilde{h}\to 0} \varphi^{\widetilde{h}}(x,i,\widetilde{m}^{\widetilde{h}}).$$

Without loss of generality (passing to an additional subsequence if needed), we may assume that $(\xi^{\widetilde{h}}(\cdot), \alpha^{\widetilde{h}}(\cdot), m^{\widetilde{h}}(\cdot), \omega^{\widetilde{h}}(\cdot))$ converges weakly to $(x(\cdot), \alpha(\cdot), m(\cdot), \omega(\cdot))$, where $m(\cdot)$ is an admissible related control. Then the weak convergence and the Skorohod representation yield that

$$\liminf_{h} V^h(x,i) = \varphi(x,i,m) \geq V(x,i). \tag{2.55}$$

We proceed to prove the reverse inequality.

We claim that

$$\limsup_{h} V^h(x,i) \leq V(x,i). \tag{2.56}$$

Suppose that $\overline{m}$ is an optimal control with Brownian motion $\omega(\cdot)$ such that $\overline{x}(\cdot)$ is the associated trajectory. By the chattering lemma, given any $\eta > 0$, there are an $\varepsilon > 0$ and an ordinary control $u^\eta(\cdot)$ that takes only finite many values, that $u^\eta(\cdot)$ is a constant on $[k\varepsilon, k\varepsilon + \varepsilon)$, that $\overline{m}^\eta(\cdot)$ is its relaxed control representation, that $(\overline{x}^\eta(\cdot), \overline{m}^\eta(\cdot))$ converges weakly to $(x(\cdot), \overline{m}(\cdot))$, and that $\varphi(x,i,\overline{m}^\eta) \leq V(x,i) + \eta$.

For each $\eta > 0$, and the corresponding $\varepsilon > 0$ as in the chattering lemma, consider an optimal control problem as in (2.4) with piecewise constant on $[k\varepsilon, k\varepsilon+\varepsilon)$. For this controlled diffusion process, we consider its η-skeleton. By that we mean we consider the process $(x^\eta(k\varepsilon), m^\eta(k\varepsilon))$. Let $\widehat{u}^\eta(\cdot)$ be the optimal control, $\widehat{m}^\eta(\cdot)$ the relaxed control representation, and $\widehat{x}^\eta(\cdot)$ the associated trajectory. Since $\widehat{m}^\eta(\cdot)$ is optimal control, $\varphi(x,i,\widehat{m}^\eta) \leq \varphi(x,i,\overline{m}^\eta) \leq V(x,i) + \eta$. We next approximate $\widehat{u}^\eta(\cdot)$ by a suitable function of $(\omega(\cdot), \alpha(\cdot))$. Moreover, $V^h(x,i) \leq \varphi^h(x,i,\overline{m}^h) \to \varphi(x,i,\overline{m}^{\eta,\theta})$ Thus,

$$\limsup_{h} V^h(x,i) \leq \varphi^h(x,i,\overline{m}^h) \to \varphi(x,i,\overline{m}^{\eta,\theta}).$$

Using the result obtained in Proposition 2.10,

$$\limsup V^h(x,i) \le V(x,i) + 2\eta.$$

The arbitrariness of η then implies that $\limsup_h V^h(x,i) \le V(x,i)$.

Using (3.54) and (3.55) together with the weak convergence and the Skorohod representation, we obtain the desired result. The proof of the theorem is concluded. □

2.3 General Hazard Rate

In this section, we assume the forces of mortality are not constant, but a continuous function with respect to t for all $t \ge 0$. Define another function $\hat{g}(x,T)$ to approximate the transversality condition of (2.2), and $\hat{g}(x,T) \to 0$ as $T \to \infty$.

Under this condition, (2.10) becomes

$$\lambda(t)V(x,t,i) - \inf_{u\in U} \mathcal{L}_t^u V(x,t,i) = 0 \tag{2.57}$$

with the boundary condition $V(0,t,i) = 1$ and $V(a(t),t,i) = 0$. and terminal condition as $V(x,T,i) = \hat{g}(x,T)$

2.3.1 Approximating Markov Chain

Similar to the constant hazard rate case, we construct a discrete-time, finite-state, controlled Markov chain to approximate the controlled diffusion progress with regime switching. We use $h > 0$ as the stepsize for the state and $\delta > 0$ as the stepsize for time. In fact, for the given $T > 0$, we use $N = N(\delta) = \lfloor T/\delta \rfloor$, where $\lfloor z \rfloor$ denotes the integer part of $z \in \mathbb{R}$. As a convention, in what follows, we often suppress the $\lfloor \cdot \rfloor$ notation and write for example, $\lfloor T/\delta \rfloor$ simply as T/δ. However, it is understood that the integer part is used.

For each $h > 0$, recall $S_h = \{x : x = kh, k = 0, \pm 1, \pm 2, \ldots\}$. Let $\{(\tilde{\xi}_n^{h,\delta}, \alpha_n^{h,\delta}), n < \infty\}$ be a controlled discrete-time Markov chain on a discrete state space $S_h \times \mathcal{M}$ with transition probabilities from a state $(x, i) \in \mathcal{M}$ to another state $(y, j) \in \mathcal{M}$ denoted by $p^{h,\delta}((x, i), (y, j)|u)$. The u is a control parameter and takes values in the compact set U. We use $u_n^{h,\delta}$ to denote the random variable which is the actual control action for the chain at discrete time n. We need the approximating Markov chain constructed satisfying local consistency.

Definition 2.12. Let $\{p^{h,\delta}((x, i), (y, j))|u\}$ for $(x, i), (y, j) \in S^h \times \mathcal{M}$ and $u \in U$ be a collection of well defined transition probabilities for the Markov chain $(\tilde{\xi}_n^{h,\delta}, \alpha_n^{h,\delta})$, an approximation to $(x(\cdot), \alpha(\cdot))$. Define the difference $\Delta\tilde{\xi}_n^{h,\delta} = \tilde{\xi}_{n+1}^{h,\delta} - \tilde{\xi}_n^{h,\delta}$. Let $E_{x,i,n}^{u,h,\delta}$, $\text{var}_{x,i,n}^{u,h,\delta}$, and $p_{x,i,n}^{u,h,\delta}$ denote the conditional expectation, variance, and marginal probability given $\{\tilde{\xi}_k^{h,\delta}, \alpha_k^{h,\delta}, u_k^{h,\delta}, k \le n, \tilde{\xi}_n^{h,\delta} = x, \alpha_n^{h,\delta} = i, u_n^{h,\delta} = u\}$, respectively. The sequence $\{(\tilde{\xi}_n^{h,\delta}, \alpha_n^{h,\delta})\}$ is said to be locally consistent with (2.4), if

$$
\begin{aligned}
&E_{x,i,n}^{u,h,\delta} \Delta\tilde{\xi}_n^{h,\delta} = (r(i)x - 1 + (\mu(t, i) - r(i))u)\delta + o(\delta), \\
&\text{var}_{x,i,n}^{u,h,\delta} \Delta\tilde{\xi}_n^{h,\delta} = (\sigma(t, i)u)^2\delta + o(\delta), \\
&p_{x,i,n}^{u,h,\delta}\{\alpha_{n+1}^u = j\} = \delta q_{ij} + o(\delta), \text{for } j \ne i, \\
&p_{x,i,n}^{u,h,\delta}\{\alpha_{n+1}^u = i\} = \delta(1 + q_{ii}) + o(\delta), \\
&\sup_{n,\omega\in\Omega} |\Delta\tilde{\xi}_n^h| \to 0 \text{ as } h \to 0.
\end{aligned}
\tag{2.58}
$$

To approximate the wealth $x(\cdot)$, we need to use an appropriate continuous-time interpolation. The piecewise constant interpolations, denoted by $\tilde{\xi}^{h,\delta}(\cdot)$, $\alpha^{h,\delta}(\cdot)$, $\lambda^{h,\delta}(\cdot)$ and $u^{h,\delta}(\cdot)$, are defined as

$$\tilde{\xi}^{h,\delta}(t) = \tilde{\xi}_n^{h,\delta} \alpha^{h,\delta}(t) = \alpha_n^{h,\delta}, \lambda^{h,\delta}(t) = \lambda_n^{h,\delta}, u^{h,\delta}(t) = u_n^{h,\delta}, \quad \text{for} \quad t \in [n\delta, n\delta + \delta). \tag{2.59}$$

First suppose the control space has a single element. In this case, inf in (2.10) can also be dropped with $u = u(0)$ in $\mathcal{L}_t^u$. That is,

$$\begin{aligned} V_t(x,t,i) + V_x(x,t,i)(r(i)x - 1 + (\mu(t,i) - r(i))u) + \frac{1}{2}V_{xx}(x,t,i)(\sigma(t,i)u)^2 \\ + QV(x,t,\cdot)(i) - \lambda(t)V(x,t,i) = 0. \end{aligned} \tag{2.60}$$

Define the approximation to the first and the second derivatives of $V(\cdot, i)$ by finite difference method using stepsize $h > 0$ and $\delta > 0$ such that $\delta = O(h^2)$ as:

$$\begin{aligned} &V(x,t,i) \to V^{h,\delta}(x,t,i) \\ &V_t(x,t,i) \to \frac{V^{h,\delta}(x,t,i) - V^{h,\delta}(x,t-\delta,i)}{\delta} \\ &V_x(x,t,i) \to \frac{V^{h,\delta}(x+h,t,i) - V^{h,\delta}(x,t,i)}{h} \quad \text{for} \quad r(i)x - 1 + (\mu(t,i) - r(i))u > 0, \\ &V_x(x,t,i) \to \frac{V^{h,\delta}(x,t,i) - V^{h,\delta}(x-h,t,i)}{h} \quad \text{for} \quad r(i)x - 1 + (\mu(t,i) - r(i))u < 0, \\ &V_{xx}(x,t,i) \to \frac{V^{h,\delta}(x+h,t,i) - 2V^{h,\delta}(x,t,i) + V^{h,\delta}(x-h,t,i)}{h^2}. \end{aligned} \tag{2.61}$$

After some detailed calculations, we obtain

$$
\begin{aligned}
&V^{h,\delta}(x,n\delta,i,u)\\
&= V^{h,\delta}(x+h,n\delta+\delta,i,u)\Big[\frac{(\sigma(n\delta,i)u)^2}{2}\frac{\delta}{h^2}+(r(i)x-1+(\mu(n\delta,i)-r(i))u)^{+}\frac{\delta}{h}\Big]\\
&\quad +V^{h,\delta}(x-h,n\delta+\delta,i,u)\Big[\frac{(\sigma(n\delta,i)u)^2}{2}\frac{\delta}{h^2}+(r(i)x-1+(\mu(n\delta,i)-r(i))u)^{-}\frac{\delta}{h}\Big]\\
&\quad +V^{h,\delta}(x,n\delta+\delta,i,u)\Big[1-(\sigma(n\delta,i)u)^2\frac{\delta}{h^2}-\Big|r(i)x-1+(\mu(n\delta,i)-r(i))u\Big|\frac{\delta}{h}\\
&\quad -\lambda(n\delta)\delta+q_{ii}\delta\Big]+\sum_{j\neq i}q_{ij}V^{h,\delta}(x,n\delta+\delta,\cdot,u)\delta.
\end{aligned}
\tag{2.62}
$$

To proceed, define

$$
\begin{aligned}
p^{h,\delta}((x,i),(x,i),n\delta|u) &= \frac{\Big[1-(\sigma(n\delta,i)u)^2\frac{\delta}{h^2}-\Big|r(i)x-1+(\mu(n\delta,i)-r(i))u\Big|\frac{\delta}{h}\Big]}{\widetilde{G}}\\
p^{h,\delta}((x,i),(x+h,i),n\delta|u) &= \frac{\Big[\frac{(\sigma(n\delta,i)u)^2}{2}\frac{\delta}{h^2}+(r(i)x-1+(\mu(n\delta,i)-r(i))u)^{+}\frac{\delta}{h}\Big]}{\widetilde{G}}\\
p^{h,\delta}((x,i),(x-h,i),n\delta|u) &= \frac{\Big[\frac{(\sigma(n\delta,i)u)^2}{2}\frac{\delta}{h^2}+(r(i)x-1+(\mu(n\delta,i)-r(i))u)^{-}\frac{\delta}{h}\Big]}{\widetilde{G}},
\end{aligned}
\tag{2.63}
$$

with $\widetilde{G}=1-\lambda(n\delta)\delta+q_{ii}\delta$. By choosing δ and h appropriately, we can make $p^{h,\delta}(\cdot|u)$ be nonnegative and well defined transition probability.

2.3.2 Main Results

Here, we present the local consistency for our approximating Markov chain.

Lemma 2.13. *The Markov Chain* $\{\tilde{\xi}_n^{h,\delta}, \alpha_n^{h,\delta}\}$ *with transition probabilities* $(p^{h,\delta}(\cdot))$ *defined in* (2.63) *is locally consistent with the stochastic differential equation in* (2.4).

Proof. Using (3.29), it is readily seen that

$$E_{x,i,n}^{u,h,\delta}(\Delta\tilde{\xi}_n^{h,\delta}) = hp^{h,\delta}((x,i),(x+h,i),n\delta|u) - hp^{h,\delta}((x,i),(x-h,i),n\delta|u)$$

$$= (r(i)x - 1 + (\mu(n\delta,i) - r(i))u)\delta + o(\delta).$$

Likewise, we obtain

$$E_{x,i,n}^{u,h,\delta}(\Delta\tilde{\xi}_n^{h,\delta})^2 = h^2p^{h,\delta}((x,i),(x+h,i),n\delta|u) + h^2p^{h,\delta}((x,i),(x-h,i),n\delta|u)$$

$$= (\sigma(n\delta,i)u)^2\delta + O(h\delta).$$

and as a result

$$\text{var}_{x,i,n}^{u,h,\delta}\Delta\tilde{\xi}_n^{h,\delta} = (\sigma(n\delta,i)u)^2\delta + O(h\delta).$$

Thus both equations in (3.20) are verified. The desired local consistency follows. □

In this part, we deal with the weak convergence of the approximating Markov chain. We have an approximating controlled Markov chain $\{\tilde{\xi}_n^{h,\delta}\}$ that is locally consistent. Define the relaxed control representation $m^{h,\delta}(\cdot)$ of $u^{h,\delta}(\cdot)$ by using its derivative $m_s^{h,\delta}(A) = I_{\{u^{h,\delta}(s)\in A\}}$. That is $m_s^{h,\delta}(\{\chi\}) = 1$ if $u^{h,\delta}(s) = \chi$. We proceed to show that $\tilde{\xi}^{h,\delta}(\cdot)$ converges weakly to the controlled wealth process given in the stochastic differential equation (2.4).

First note that

$$
\begin{aligned}
&\tilde{\xi}^{h,\delta}(s)\\
&= x + \sum_{k=\lfloor t/\delta\rfloor}^{\lfloor s/\delta\rfloor -1} [E^{\chi,h,\delta}_{x,k,\iota} \Delta\tilde{\xi}^{h,\delta}_k + (\Delta\tilde{\xi}^{h,\delta}_k - E^{\chi,h,\delta}_{x,k,\iota}\Delta\tilde{\xi}_k)]\\
&= x + \sum_{k=\lfloor t/\delta\rfloor}^{\lfloor s/\delta\rfloor -1} [r(\alpha^{h,\delta}_k)\tilde{\xi}^{h,\delta}_k - 1 + (\mu(k\delta, \alpha^{h,\delta}_k) - r(\alpha^{h,\delta}_k))u]\delta + M^{h,\delta}(s) - M^{h,\delta}(t) + o(\delta)\\
&= x + \delta \sum_{k=\lfloor t/\delta\rfloor}^{\lfloor s/\delta\rfloor -1} \int_U [r(\alpha^{h,\delta}_k)\tilde{\xi}^{h,\delta}_k - 1 + (\mu(k\delta, \alpha^{h,\delta}_k) - r(\alpha^{h,\delta}_k))\chi] m^{h,\delta}_\tau(d\chi) + M^{h,\delta}(s)\\
&\qquad - M^{h,\delta}(t) + e^{h,\delta}(s),
\end{aligned}
\tag{2.64}
$$

where

$$
M^{h,\delta}(s) = \sqrt{\delta} \sum_{k=0}^{\lfloor s/\delta\rfloor -1} \frac{\Delta\tilde{\xi}^{h,\delta}_k - E^{\chi,h,\delta}_{x,k,\iota}\Delta\tilde{\xi}_k}{\sqrt{\delta}}, \tag{2.65}
$$

and $e^{h,\delta}(s)$ satisfies

$$
\limsup_{t\le s\le T} E|e^{h,\delta}(s)|^2 = 0.
$$

Theorem 2.14. *Assume* (A1). *Let the approximating chain* $\{\tilde{\xi}^{h,\delta}_n, \alpha^{h,\delta}_n, n < \infty\}$ *be constructed with transition probabilities defined in* (2.63)*,* $\{u^{h,\delta}_n, n < \infty\}$ *be a sequence of admissible controls,* $(\tilde{\xi}^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot), \lambda^{h,\delta}(\cdot))$*be the continuous-time interpolation defined in* (2.59)*,* $m^{h,\delta}(\cdot)$ *be the relaxed control representation of* $\{u^{h,\delta}_n, n < \infty\}$*. Then* $(\tilde{\xi}^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot), m^{h,\delta}(\cdot),$ $\omega^{h,\delta}(\cdot), \lambda^{h,\delta}(\cdot))$ *is tight. Denote by* $\mathcal{F}_t$ *the limit of weakly convergent subsequence by* $(\tilde{\xi}(\cdot), \alpha(\cdot),$ $m(\cdot), \omega(\cdot), \lambda(\cdot))$ *and denote the* σ*-algebra generated by* $\{x(s), \alpha(s), m(s), \omega(s), \lambda(s), s \le t\}$*. Then* $\omega(\cdot)$ *is a standard* $\mathcal{F}_t$*-Wiener process, and* $m(\cdot)$ *is an admissible control. Moreover,* (2.35) *is satisfied.*

Proof. The proof is divided to several steps. First we prove the tightness of the sequence $(\tilde{\xi}^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot), m^{h,\delta}(\cdot), \omega^{h,\delta}(\cdot), \lambda^{h,\delta}(\cdot))$. By using the topology of the relaxed control space, $\{m^{h,\delta}(\cdot)\}$ is tight, and $\lambda(t)$ is continuous, then $\{\lambda^{h,\delta}(\cdot)\}$ is tight and converge to $\lambda(\cdot)$. Similar to the proof in (2.3), $\{\alpha^{h,\delta}(\cdot)\}$ is tight. Thus, we can concentrate on the tightness of $\tilde{\xi}^{h,\delta}(\cdot)$. For each $\Delta > 0$, each s, $s_1 > 0$ with $s_1 < \Delta$ and $s + s_1 \le T$, we have from (2.64),

$$
\begin{aligned}
&E|\tilde{\xi}^{h,\delta}(s+s_1) - \tilde{\xi}^{h,\delta}(s)|^2 \\
&\le \left[E\left|\delta \sum_{k=s/\delta}^{((s+s_1)/\delta)-1} \int_U [r(\alpha_k^{h,\delta})\tilde{\xi}_k^{h,\delta} - 1 + (\mu(k\delta, \alpha_k^{h,\delta}) - r(\alpha_k^{h,\delta}))\chi] m_\tau^{h,\delta}(d\chi)\right|^2 \right. \qquad (2.66) \\
&\quad \left. + \delta E\left| \sum_{k=s/\delta}^{((s+s_1)/\delta)-1} \frac{\Delta\tilde{\xi}_k^{h,\delta} - E_{x,k,\iota}^{\chi,h,\delta}\Delta\tilde{\xi}_k}{\sqrt{\delta}} \right|^2 + E|e^{h,\delta}(s+s_1) - e^{h,\delta}(s)|^2 \right].
\end{aligned}
$$

For the term on the second line of (2.66), it is readily seen that for sufficiently small Δ,

$$
\begin{aligned}
&E\left|\delta \sum_{k=s/\delta}^{((s+s_1)/\delta)-1} \int_U [r(\alpha_k^{h,\delta})\tilde{\xi}_k^{h,\delta} - 1 + (\mu(k\delta, \alpha_k^{h,\delta}) - r(\alpha_k^{h,\delta}))\chi] m_\tau^{h,\delta}(d\chi)\right|^2 \\
&\le K s_1^2 \le O(\Delta^2) \le O(\Delta).
\end{aligned}
$$

It is also easily seen that for the last term of (2.66), we have

$$
\begin{aligned}
&\limsup_{h,\delta} E|e^{h,\delta}(s+s_1) - e^{h,\delta}(s)|^2 \\
&\le K \limsup_{h,\delta} [E|e^{h,\delta}(s+s_1)|^2 + E|e^{h,\delta}(s)|^2] = 0.
\end{aligned}
$$

As for the next to the last term in (2.66), note that $\{\Delta\tilde{\xi}_k^{h,\delta} - E_{x,k,\iota}^{\chi,h,\delta}\Delta\tilde{\xi}_k\}$ is a martingale difference sequence and hence it is orthogonal. Thus the orthogonality together with the local consistency implies that

$$\begin{aligned}
&\delta E\Big|\sum_{k=s/\delta}^{((s+s_1)/\delta)-1}\frac{\Delta\tilde{\xi}_k^{h,\delta} - E_{x,k,\iota}^{\chi,h,\delta}\Delta\tilde{\xi}_k}{\sqrt{\delta}}\Big|^2 \\
&\quad = \delta E\sum_{k=s/\delta}^{((s+s_1)/\delta)-1}\frac{[\Delta\tilde{\xi}_k^{h,\delta} - E_{x,k,\iota}^{\chi,h,\delta}\Delta\tilde{\xi}_k][\Delta\tilde{\xi}_k^{h,\delta} - E_{x,k,\iota}^{\chi,h,\delta}\Delta\tilde{\xi}_k]}{\delta} \\
&\quad = \delta E\sum_{k=s/\delta}^{((s+s_1)/\delta)-1}\{[\sigma(k\delta,\alpha_k^{h,\delta})\chi]^2 + o(\delta)\} \le Ks_1 \le K\Delta.
\end{aligned}$$

Note that the above bound holds uniformly in h, δ and s. Putting the above estimates together, we arrive at

$$\begin{aligned}
&\lim_{\Delta\to 0}\limsup_{h,\delta\to 0} E|\tilde{\xi}^{h,\delta}(s+s_1) - \tilde{\xi}^{h,\delta}(s)|^2 \le \lim_{\Delta\to 0} K\Delta = 0 \quad \text{and} \\
&\lim_{\Delta\to 0}\limsup_{h,\delta\to 0} E|M^{h,\delta}(s+s_1) - M^{h,\delta}(s)|^2 = 0.
\end{aligned} \tag{2.67}$$

The tightness of the processes $(\tilde{\xi}^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot), m^{h,\delta}(\cdot), M^{h,\delta}(\cdot)), \lambda^{h,\delta}(\cdot))$ then follows from [p. 47, Theorem 3] of Kushner (1984).

Next note that $e^{h,\delta}(\cdot)$ is asymptotically unimportant owing to Lemma 5 in [p. 50] of Kushner (1984). Thus in the following consideration, we shall discard this term for notational simplicity.

Note also that we can show (using the Kolmogorov continuity criterion) that $M(\cdot)$ is a process with continuous sample paths w.p.1. In addition, using the definition in (2.65), it is easily seen that $M(\cdot)$ is martingale, whose quadratic variation (with relaxed control representation) is given by

$$M(s) = \int_0^s \int_U [\sigma(\tau, \alpha(\tau))\chi]^2 m_\tau(d\chi)d\tau. \tag{2.68}$$

Thus, the limit is a square integrable continuous martingale. Then the standard results (see Ikeda and Watanabe (1981), and [p. 16] of Kushner (1984)) imply that there is a standard Brownian motion $w(\cdot)$ such that

$$M(s) = M(t) + \int_t^s \int_U (\sigma(\tau, \alpha(\tau))\chi)^2 m_\tau(d\chi)d\omega(\tau). \tag{2.69}$$

For any $s \geq t$, $s_1 \geq 0$ with $s+s_1 \leq T$, any $C_0^{1,2}$ function $f(\cdot)$ (functions that have compact support whose first partial derivative w.r.t. the time variable and the partial derivatives with respect to the state variable x up to the second order are continuous), bounded and continuous function $h(\cdot)$, any positive integer κ, any t_i satisfying $0 \leq t_i \leq s$ and $i \leq \kappa$, the weak convergence and the Skorohod representation imply that

$$\begin{aligned} &Eh(\tilde{\xi}^{h,\delta}(t_i), \alpha^{h,\delta}(t_i), i \leq \kappa)[f(s+s_1, \tilde{\xi}^{h,\delta}(s+s_1), \alpha(s+s_1)) - f(s, \tilde{\xi}^{h,\delta}(s), \alpha(s))] \\ &\to Eh((\tilde{\xi}(t_i), \alpha(t_i)), i \leq \kappa)[f(s+s_1, \tilde{\xi}(s+s_1), \alpha(s+s_1)) - f(s, \tilde{\xi}(s), \alpha(s))] \\ &\qquad \text{as } h, \delta \to 0. \end{aligned} \tag{2.70}$$

Choose a sequence $\{n^\delta\}$ such that $n^\delta \to \infty$ but $\Delta^\delta = \delta n^\delta \to 0$. Direct calculations show that

$$Eh(\tilde{\xi}^{h,\delta}(t_i), \alpha^{h,\delta}(t_i), i \le \kappa)\Big[f(s+s_1, \tilde{\xi}^{h,\delta}(s+s_1), \alpha(s+s_1)) - f(s, \tilde{\xi}^{h,\delta}(s), \alpha(s))\Big]$$

$$= Eh(\tilde{\xi}^{h,\delta}(t_i), \alpha^{h,\delta}(t_i), i \le \kappa)\Big[\sum_{ln^\delta = s/\delta}^{(s+s_1)/\delta} f(\delta(\ln^\delta + n^\delta), \tilde{\xi}^{h,\delta}(\delta(ln^\delta + n^\delta), \alpha(\delta(\ln^\delta + n^\delta)))$$

$$-f(\delta ln^\delta, \tilde{\xi}^{h,\delta}(\delta(ln^\delta + n^\delta)), \alpha(\delta ln^\delta)) + f(\delta ln^\delta, \tilde{\xi}^{h,\delta}(\delta(ln^\delta + n^\delta)), \alpha(\delta ln^\delta))$$

$$-f(\delta ln^\delta, \tilde{\xi}^{h,\delta}(\delta(ln^\delta)), \alpha(\delta ln^\delta))\Big]. \tag{2.71}$$

Note that

$$\sum_{ln^\delta = s/\delta}^{(s+s_1)/\delta} [f(\delta(\ln^\delta + n^\delta), \tilde{\xi}^{h,\delta}(\delta(ln^\delta + n^\delta), \alpha(\delta(\ln^\delta + n^\delta)))$$

$$-f(\delta ln^\delta, \tilde{\xi}^{h,\delta}(\delta(ln^\delta + n^\delta)), \alpha(\delta ln^\delta))]$$

$$= \sum_{ln^\delta = s/\delta}^{(s+s_1)/\delta} \sum_{k=ln^\delta}^{ln^\delta + n^\delta - 1} [f(\delta(k+1), \tilde{\xi}^{h,\delta}(\delta(ln^\delta + n^\delta)), \alpha(\delta(k+1)))$$

$$-f(\delta k, \tilde{\xi}^{h,\delta}(\delta(ln^\delta + n^\delta)), \alpha(\delta k))]$$

$$= \sum_{ln^\delta = s/\delta}^{(s+s_1)/\delta} \frac{\partial f(\delta ln^\delta, \tilde{\xi}^{h,\delta}(\delta(ln^\delta + n^\delta)), \alpha(\delta ln^\delta))}{\partial \tau} \Delta^\delta + o(1),$$

where $o(1) \to 0$ in mean uniformly in t as $h, \delta \to 0$. Letting $\delta ln^\delta \to \tau$ as $\delta \to 0$, then $\delta(ln^\delta + n^\delta) \to \tau$ since $\Delta^\delta = \delta n^\delta \to 0$ as $\delta \to 0$. Consequently, by the weak convergence and the Skorohod representation, the continuity of $h(\cdot)$ and the smoothness of $f(\cdot)$ imply that

$$Eh(\tilde{\xi}^{h,\delta}(t_i),\alpha^{h,\delta}(t_i),i\le\kappa)\Big[\sum_{ln^\delta=s/\delta}^{(s+s_1)/\delta} f(\delta(\ln^\delta+n^\delta),\tilde{\xi}^{h,\delta}(\delta(ln^\delta+n^\delta),\alpha(\delta(\ln^\delta+n^\delta)))$$
$$-f(\delta ln^\delta,\tilde{\xi}^{h,\delta}(\delta(ln^\delta+n^\delta)),\alpha(\delta ln^\delta))\Big] \tag{2.72}$$
$$\to Eh(\tilde{\xi}^{h,\delta}(t_i),\alpha^{h,\delta}(t_i),i\le\kappa)\Big[\int_s^{s+s_1}\frac{\partial f(\tau,\tilde{\xi}(\tau),\alpha(\tau))}{\partial\tau}\Big]d\tau \ \text{ as } h,\delta\to 0.$$

As for the last term in (2.71), it can be seen that

$$\sum_{ln^\delta=s/\delta}^{(s+s_1)/\delta} f(\delta ln^\delta,\tilde{\xi}^{h,\delta}(\delta(ln^\delta+n^\delta)),\alpha(\delta ln^\delta))-f(\delta ln^\delta,\tilde{\xi}^{h,\delta}(\delta(ln^\delta)),\alpha(\delta ln^\delta))$$
$$=\sum_{ln^\delta=s/\delta}^{(s+s_1)/\delta}\Big\{f_{\tilde{\xi}}(\delta ln^\delta,\tilde{\xi}(\delta(ln^\delta))\alpha(\delta ln^\delta))\sum_{k=ln^\delta}^{ln^\delta+n^\delta-1}\int_U[r(\alpha_k^{h,\delta})\tilde{\xi}_k^{h,\delta}-1+(\mu(k\delta,\alpha_k^{h,\delta})$$
$$-r(\alpha_k^{h,\delta}))\chi]\cdot m_{ln^\delta}(d\chi)\delta+f_{\tilde{\xi}}(\delta ln^\delta,\tilde{\xi}(\delta(ln^\delta)),\alpha(\delta ln^\delta))$$
$$\sum_{k=ln^\delta}^{ln^\delta+n^\delta-1}[M^{h,\delta}(\delta(ln^\delta+n^\delta))-M^{h,\delta}(\delta ln^\delta)]+\frac{1}{2}f_{\tilde{\xi}\tilde{\xi}}(\delta ln^\delta,\tilde{\xi}(\delta(ln^\delta)),\alpha(\delta ln^\delta))$$
$$\sum_{k=ln^\delta}^{ln^\delta+n^\delta-1}[\tilde{\xi}^{h,\delta}(\delta(ln^\delta+n^\delta))-\tilde{\xi}^{h,\delta}(\delta ln^\delta)]^2\Big\}+\tilde{e}^{h,\delta}(s+s_1)-\tilde{e}^{h,\delta}(s),$$

where $f_{\tilde{\xi}}$ and $f_{\tilde{\xi}\tilde{\xi}}$ denote the first and second partial derivatives with respect to $\tilde{\xi}$, and

$$\sup_{t\le s\le T}E|\tilde{e}^{h,\delta}(s)|^2\to 0 \ \text{ as } h,\delta\to 0.$$

It can be seen that the martingale limit and the limit quadratic variation lead to

$$\sum_{ln^\delta=s/\delta}^{(s+s_1)/\delta} f_{\tilde{\xi}}(\delta ln^\delta, \tilde{\xi}(\delta(ln^\delta)), \alpha(\delta ln^\delta)) \sum_{k=ln^\delta}^{ln^\delta+n^\delta-1} [M^{h,\delta}(\delta(ln^\delta+n^\delta)) - M^{h,\delta}(\delta ln^\delta)] \quad (2.73)$$

$$\to 0 \quad \text{as } h, \delta \to 0,$$

$$\sum_{ln^\delta=s/\delta}^{(s+s_1)/\delta} f_{\tilde{\xi}\tilde{\xi}}(\delta ln^\delta, \tilde{\xi}(\delta(ln^\delta)), \alpha(\delta ln^\delta)) \sum_{k=ln^\delta}^{ln^\delta+n^\delta-1} [\tilde{\xi}^{h,\delta}(\delta(ln^\delta+n^\delta)) - \tilde{\xi}^{h,\delta}(\delta ln^\delta)]^2 \quad (2.74)$$

$$\to \int_s^{s+s_1} \int_U f_{\tilde{\xi}\tilde{\xi}}(\tau, \tilde{\xi}(\tau), \alpha(\tau))[\sigma(\tau,\alpha(\tau))u]^2 m_\tau(d\chi)d\tau.$$

Moreover, the limit of

$$f_{\tilde{\xi}}(\delta ln^\delta, \tilde{\xi}(\delta(ln^\delta)), \alpha(\delta ln^\delta)) \sum_{k=ln^\delta}^{ln^\delta+n^\delta-1} [\tilde{\xi}^{h,\delta}(\delta(ln^\delta+n^\delta)) - \tilde{\xi}^{h,\delta}(\delta ln^\delta)]$$

is the same as that of

$$f_{\tilde{\xi}}(\delta ln^\delta, \tilde{\xi}(\delta(ln^\delta)), \alpha(\delta ln^\delta)) \sum_{k=ln^\delta}^{ln^\delta+n^\delta-1} [r(\alpha_k^{h,\delta})\tilde{\xi}_k^{h,\delta} - 1 + (\mu(k\delta, \alpha_k^{h,\delta}) - r(\alpha_k^{h,\delta}))\chi]\delta$$

$$= f_{\tilde{\xi}}(\delta ln^\delta, \tilde{\xi}(\delta(ln^\delta)), \alpha(\delta ln^\delta)) \sum_{k=ln^\delta}^{ln^\delta+n^\delta-1} [r(\alpha_k^{h,\delta})\tilde{\xi}_k^{h,\delta} - 1 + (\mu(k\delta, \alpha_k^{h,\delta}) - r(\alpha_k^{h,\delta}))\chi]\delta$$

$$+o(1),$$

where $o(1) \to 0$ in probability as $h, \delta \to 0$. Thus, using the approximation techniques used in stochastic approximation as in [p. 169] of Kushner and Yin (2003), we can show that as

$h, \delta \to 0$,

$$Eh(\tilde{\xi}^{h,\delta}(t_i), \alpha^{h,\delta}(t_i), i \le \kappa)$$

$$\Big[\sum_{ln^\delta = s/\delta}^{(s+s_1)/\delta} f_{\tilde{\xi}}(\delta ln^\delta, \tilde{\xi}(\delta(ln^\delta)), \alpha(\delta ln^\delta)) \sum_{k=ln^\delta}^{ln^\delta + n^\delta - 1} [\tilde{\xi}^{h,\delta}(\delta(ln^\delta + n^\delta)) - \tilde{\xi}^{h,\delta}(\delta ln^\delta)]\Big] \tag{2.75}$$

$$\to Eh(\tilde{\xi}^{h,\delta}(t_i), \alpha^{h,\delta}(t_i), i \le \kappa)\Big[\int_s^{s+s_1}\int_U f_{\tilde{\xi}}(\tau, \tilde{\xi}(\tau), \alpha(\tau))[r(\alpha(\tau))\tilde{\xi}(\tau) - 1$$

$$+(\mu(\tau, \alpha(\tau)) - r(\alpha(\tau)))\chi]m_\tau(d\chi)d\tau\Big].$$

Finally, since the solution of (2.35) is unique in the sense in distribution, $\tilde{\xi}(s) = X(s)$ w.p.1. This completes the proof of the theorem. □

We have shown the convergence of wealth processes, with the similar method, we can establish the result about desired convergence to the cost and value function. Note that with the interpolation process, the cost function and value function can be written as

$$\varphi^{h,\delta}(x, t, i, u^h) = P[\tau_0 < \tau_d | \tilde{\xi}_0^h = x, \tau_0 > t, \tau_d > t]$$
$$= E_{x,t,i}^{u^h}[I_{\{\tau_0 < \tau_d | \tilde{\xi}_0^h = x, \tau_0 > t, \tau_d > t\}}]. \tag{2.76}$$

and

$$V^{h,\delta}(x, t, i) = \inf_{m^h \in \Gamma^h} \varphi^{h,\delta}(x, t, i, m^h). \tag{2.77}$$

The proofs of the convergence of and value functions are similar to Theorem 3.9, and is thus omitted.

Theorem 2.15. *Assume* (A1)-(A3). *$V^{h,\delta}(x, t, i)$ and $V(x, t, i)$ are value functions defined in* (2.77) *and* (2.8), *respectively. Then $V^{h,\delta}(x, t, i) \to V(x, t, i)$ as $h, \delta \to 0$.*

2.4 Examples

In this section, we consider a couple examples with constant and more general hazard rates with two regimes, respectively. For simplicity, we deal with systems that are linear in the wealth. The (2.4) becomes

$$\begin{cases} dW(s,\alpha(s)) = rA(\alpha(s))W(s) + B(\alpha(s))(\mu(s)-r))u(s) - Z(s))ds \\ \qquad\qquad +C(\alpha(s))\sigma(s)u(s)d\omega + a(s)dZ(s), \\ W(t,\alpha(t)) = w \geq 0, \\ Z(t,\alpha(t)) = z \geq 0. \end{cases} \tag{2.78}$$

Suppose $r = 0.02$ (the yield rate of riskless asset), $\mu = 0.06$ (the yield rate of risky asset), $\sigma = 0.2$ (the volatility of the risky asset), $z = 1$ (the individual consumes one unit wealth per year).

2.4.1 Constant Hazard Rate

Example 2.16. Take $\lambda = \widetilde{\lambda} = 0.04$, the hazard rate is 0.04 such that the expected future lifetime of individual is 25 years. The Markov Chain $\alpha(\cdot) \in \mathcal{M}$ with $\mathcal{M} = \{1, 2\}$ and generator Q, and

$$Q = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}, \begin{cases} A(1) = 1 \\ A(2) = 10, \end{cases} \begin{cases} B(1) = 4 \\ B(2) = 1, \end{cases} \begin{cases} C(1) = 2 \\ C(2) = 1. \end{cases} \tag{2.79}$$

We use the value iteration to numerically solve the optimal control problem, then we obtain the relationship between wealth and the probability of lifetime ruin as in Figure 1. In addition, we compute the probability of lifetime ruin under the assumption of exponential future lifetime for an individual with wealth \$1 who invests in the riskless asset only with constant interest rate and self-annuitizes. Then we obtain

$$dW(s) = (rW(s) - Z(s))ds.$$

The individual with initial wealth \$1 who self-annuitize will consumes

$$z = \int_0^\infty \exp(-rt)\exp(-\lambda t)dt = r + \lambda.$$

Then the time of financial ruin when wealth reaches 0 is $\tau_d = ln(1+r/\lambda)/r$, so the probability of lifetime ruin will be

$$P[\tau_0 < \tau_d] = \exp(-r\tau_d) = (1 + r/\lambda)^{-\frac{\lambda}{r}} = 0.444.$$

Moreover, Figure 2 shows that the probability of ruin with life annuity purchase is less than 0.444 when the initial wealth $w \in (0.5, 1)$.

Comparing to the probability of lifetime ruin without life annuity purchasing and the consumption $z = r+\lambda = 0.06$, if the individual buys the life annuity as in (2.4), the individual will have less probability of financial ruin even with lower wealth than the individual with self annuitization to maintain the same consumption.

2.4.2 General Hazard Rate

Example 2.17. In this example, we consider Gompertz hazard rate $\lambda(t) = \widetilde{\lambda}(t) = \exp(\frac{t-\bar{m}}{b})/b$, where $\bar{m}$ is a model value and b is a scale parameter, we choose $\bar{m} = 90$ and $b = 9$. We

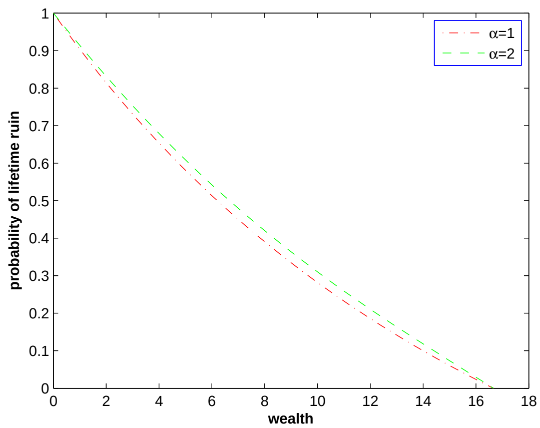

Figure 1: Probability of lifetime ruin versus wealth

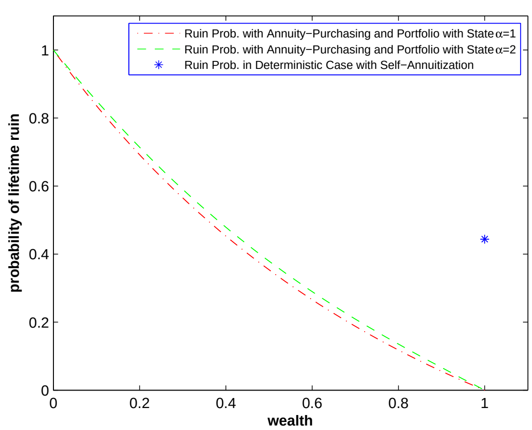

Figure 2: Comparison between Ruin Prob. with Annuity-Purchasing and Portfolio and Ruin Prob. in Deterministic Case with Self-Annuitization

also consider the terminal condition to be exponentially decay as $\hat{g}(x,T) = \exp(-xT)$. The Markov Chain $\alpha(\cdot) \in \mathcal{M}$ with $\mathcal{M} = \{1,2\}$ and generator Q, and

$$Q = \begin{pmatrix} -0.4 & 0.4 \\ \\ 0.8 & -0.8 \end{pmatrix}, \quad \begin{cases} A(1) = 1 \\ \\ A(2) = 10, \end{cases} \begin{cases} B(1) = -1 \\ \\ B(2) = -10, \end{cases} \begin{cases} C(1) = 10 \\ \\ C(2) = 1. \end{cases} \tag{2.80}$$

To illustrate the impact of ages of the investors on the probability of lifetime ruin, three age levels are presented as $t = 30, t = 50, t = 70$. From Figure 3 to 5, we can see that the individual with the same wealth but younger age will more likely to outlives his or her wealth.

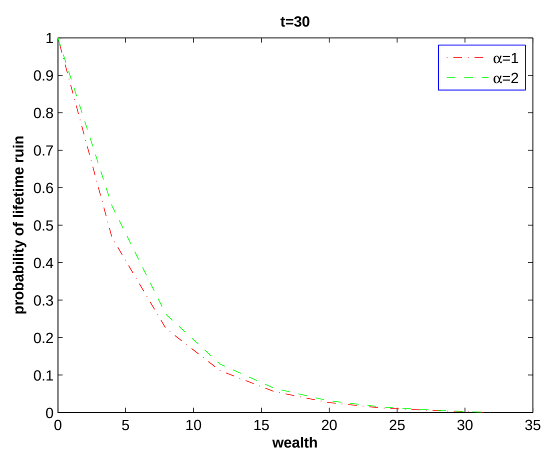

Figure 3: Probability of lifetime ruin versus wealth with age 30

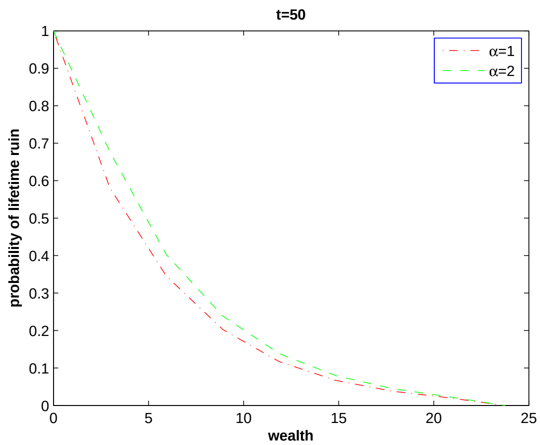

Figure 4: Probability of lifetime ruin versus wealth with age 50

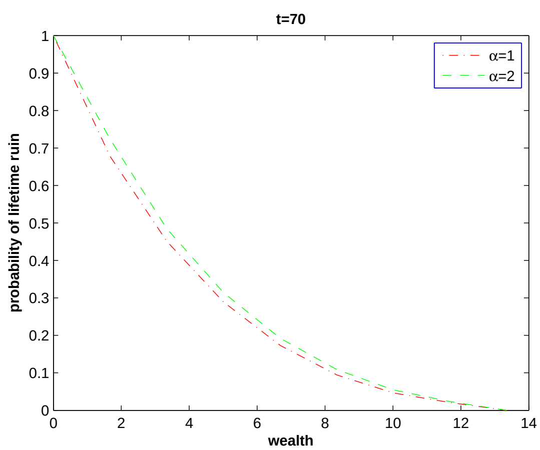

Figure 5: Probability of lifetime ruin versus wealth with age 70

3 Dividend Optimization

Probability of ruin is an efficient method to analyze the safety aspect of investment. However, it is not effective enough to consider the dividend payout strategies. Dividend policies affect an insurer's capital structure and are important to policyholders. The insurance companies prefer to pay out the dividend when the surplus is high, whereas leave the funds to companies for growth when. Instead of focussing on financial safety, performance of an insurance investment is measured by the present value of the dividend payout throughout the lifetime of the investment.

In recent years, there has been a growing effort on applying advanced methods of stochastic control to study the optimal dividend policy. Although the classical compound Poisson model were used in Gerber and Shiu [9], and Schmidli [17] among others, many papers use diffusion to model the surplus process; see, for example, Asmussen and Taksar [2], Asmussen, Høgaard and Taksar [1], Cadenillas, Choulli, Taksar and Zhang [5], Gerber and Shiu [8].

In this work, we have developed a numerical approximation scheme to maximize the present value of dividend with optimal dividend rate selection. Although one could derive the associate system of HJB equations by using the usual dynamic programming approach together with the use of properties of switching jump diffusions, solving them analytically is very difficult. As an alternative, one may try to discretize the system of HJB equations directly, but this relies on the properties of the HJB equations. We present a viable alternative. Our Markov chain approximation method uses mainly probabilistic methods that do not need any analytic properties of the solutions of the system of HJB equations. In the actual computation, the optimal control can be obtained by using the value or policy iteration methods.

3.1 Formulation

To delineate the random environment and other random factors, we use a continuous-time Markov chain $\alpha(t)$ whose generator is $Q = (q_{ij}) \in \mathbb{R}^{m\times m}$ and state space is $\mathcal{M} = \{1, \ldots, m\}$. Let ν_n be the arrival time of the n-th claim. Corresponding to each $i \in \mathcal{M}$, $N_i(t) = \max\{n \in \mathbb{N} : \nu_n \le t\}$ is the number of claims up to time t, which is a Poisson counting process.

The surplus process under consideration is a regime-switching jump diffusion. For each $i \in \mathcal{M}$, the premium rate is $c(i) > 0$ and the volatility is $\sigma(i) > 0$. Let $R_i(t)$ for each $i \in \mathcal{M}$ be a jump process representing claims with arrival rate λ_i, claim size distribution F_i, and zero initial surplus. The function $q(x, i, \rho)$ is assumed to be the magnitude of claim size, where ρ have the distribution $\Pi(\cdot)$. Then the Poisson measure $N_i(\cdot)$ has intensity $\lambda_i dt \times \Pi_i(d\rho)$ where $\Pi_i(d\rho) = f_i(\rho)d\rho$. The surplus process before dividend payment is given by

$$\begin{aligned} d\widetilde{x}(t) &= \sum_{i\in\mathcal{M}} I_{\{\alpha(s)=i\}}(c(i)dt + \sigma(i)dw(t) - dR_i(t)) \\ &= \Big[c(\alpha(t))dt + \sigma(\alpha(t))dw(t)\Big] - \int_{\mathbb{R}_+} q(x(t^-), \alpha(t), \rho) N_{\alpha(t)}(dt, d\rho), \end{aligned} \tag{3.1}$$

where I_A is the indicator function of the set A, $c(i) > 0$ and $\sigma(i) > 0$ for each $i \in \mathcal{M}$, and $w(t)$ is a standard Brownian motion. Assume that $q(\cdot, i, \rho)$ is continuous for each ρ and each $i \in \mathcal{M}$. We are working on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, where $\mathcal{F}_t$ is the σ-algebra generated by $\{\alpha(s), w(s), N_i(s) : 0 \le s \le t,\ i \in \mathcal{M}\}$.

Note that the drift c describes the premium magnitude collected by the insurance company, and is modulated by a finite Markov Chain $\alpha(t)$, which represents the market mode and other economic conditions. It is used to determine the amount charged by the insurer and mainly depends on the insurance coverage, not surplus. The volatility σ refers to measures of risk in the market here. Like the drift c, it is mainly affected by the market mode. From a numerical approximation point of view, making c and σ x-dependent will not introduce any essential difficulty.

A dividend strategy $D(\cdot)$ is an $\mathcal{F}_t$-adapted process $\{D(t) : t \geq 0\}$ corresponding to the accumulated amount of dividends paid up to time t such that $D(t)$ is a nonnegative and nondecreasing stochastic process that is right continuous and have left limits with $D(0^-) = 0$. In general, a dividend process is not necessarily an absolutely continuous process. In this dissertation, we consider the optimal dividend strategy, which is either a barrier strategy or a band strategy. In both cases, the dividend rate is the same as the premium rate. As a result, $D(t)$ is absolutely continuous. Denote $\Gamma = [0, C]$. Since the optimal dividends policy is either a barrier or a band strategy, $D(t)$ is an absolutely continuous process. We write $D(t)$ as

$$dD(t) = u(t)dt, \quad 0 \leq u(t) \leq C, \tag{3.2}$$

where $u(t)$ is an $\mathcal{F}_t$-adapted process and $0 < C < \infty$. Note that if $C < c(i)$ for some $i \in \mathcal{M}$, this formulation will lead to a threshold strategy. If $C \geq c(i)$ for all $i \in \mathcal{M}$, the optimal strategy is either a barrier or band strategy. Then the surplus process in the presence of dividend payments is given by

$$dx(t) = d\widetilde{x}(t) - dD(t), \ x(0) = x \geq 0 \tag{3.3}$$

for all $t < \tau$ and we impose $x(t) = 0$ for all $t > \tau$, where $\tau = \inf\{t \geq 0 : x(t) \leq 0\}$ represents the time of ruin. Denote $\Gamma = [0, C]$, $0 < C < \infty$. Suppose the dividend is paid at a rate $u(t)$, where $u(t)$ is an $\mathcal{F}_t$-adapted process, and the optimal payout strategy is applied subsequently. Then the expected discounted dividend until ruin is given by

$$J(x, i, u(\cdot)) = E_{x,i}\Big[\int_0^\tau e^{-rt}u(t)dt\Big],\ i \in \mathcal{M}, \tag{3.4}$$

where $E_{x,i}$ denotes the expectation conditioned on $x(0) = x$ and $\alpha(0) = i$.

Combing (3.1) and (3.3), we can rewrite the surplus process with the dividend payment as

$$\begin{aligned} &dx(t) = \Big[c(\alpha(t)) - u(t)\Big]dt + \sigma(\alpha(t))dw(t) - dR(t), \\ &R(t) = \sum_{i\in\mathcal{M}} I_{\{\alpha(s)=i\}}R_i(t) = \int_0^t \int_{\mathbb{R}_+} q(x(t^-), \alpha(t), \rho)N_{\alpha(t)}(dt, d\rho), \\ &x(0) = x. \end{aligned} \tag{3.5}$$

Admissible Strategies. A strategy $u(\cdot) = \{u(t) : t \geq 0\}$ satisfying $u(t) \in \Gamma$ being progressively measurable with respect to $\sigma\{\alpha(s), w(s), N_i(s) : 0 \leq s \leq t,\ i \in \mathcal{M}\}$ is called an admissible strategy. Denote the collection of all admissible strategies or admissible controls by $\mathcal{A}$. A Borel measurable function $u(x, \alpha)$ is an admissible feedback strategy or feedback control if (3.5) has a unique solution.

We are interested in finding the optimal dividend rate $u(t)$ that is bounded and is a function of x and α to maximize the expected utility function $J(x, i, u(\cdot))$. Define $V(x, i)$ as the optimal value of the corresponding problem. That is,

$$V(x, i) = \sup_{u(\cdot)\in\mathcal{A}} J(x, i, u(\cdot)). \tag{3.6}$$

Setting $u(t)$ to any quantity such that it does not change the value of $V(x(\tau), \alpha(\tau))$ for $t \geq \tau$, that is, $u(t) = 0$ for $t \geq \tau$, Therefore, (3.4) can be rewritten as

$$J(x, i, u(\cdot)) = E_{x,i}\Big[\int_0^\infty e^{-rt}u(t)dt\Big]. \tag{3.7}$$

The optimal dividend problem is formulated as

$$\begin{cases} \text{maximize} : J(x, i, u(\cdot)) = E_{x,i}\displaystyle\int_0^\infty e^{-rt}u(t)dt, \\ \text{subject to} : dx(t) = [c(\alpha(t)) - u(t)]dt + \sigma(\alpha(t))dw(t) \\ \qquad\qquad\qquad - \displaystyle\int_{\mathbb{R}_+} q(x(t^-), \alpha(t), \rho)N_{\alpha(t)}(dt, d\rho), \\ \qquad x(0) = x, \quad \alpha(0) = i, \quad u(\cdot) \in \mathcal{A}, \\ \text{value function} : V(x, i) = \displaystyle\sup_{u(\cdot)\in\mathcal{A}} J(x, i, u(\cdot)), \quad \text{for each } \ i \in \mathcal{M}. \end{cases} \tag{3.8}$$

For an arbitrary $u \in \mathcal{A}$, $i = \alpha(t) \in \mathcal{M}$, and $V(\cdot, i) \in C^2(\mathbb{R})$, define an operator $\mathcal{L}^u$ by

$$\begin{aligned} \mathcal{L}^u V(x, i) = V_x(x, i)(c(i) - u) + \frac{1}{2}\sigma(i)^2 V_{xx}(x, i) + QV(x, \cdot)(i) \\ +\lambda_i \int_0^x [V(x - q(x, i, \rho), i) - V(x, i)]f_i(\rho)d\rho, \end{aligned} \tag{3.9}$$

where V_x and V_{xx} denote the first and second derivatives with respect to x, and

$$QV(x, \cdot)(i) = \sum_{j\neq i} q_{ij}(V(x, j) - V(x, i)).$$

Note that

$$J(x, i, u) = E_{x,i}\Big[\int_0^\infty e^{-rt}u(t)dt\Big] \leq E_{x,i}\Big[\int_0^\infty e^{-rt}Cdt\Big] \leq \frac{C}{r}.$$

Taking $\sup_u$ in the above inequality leads to that $V(x,i)$ is bounded. Furthermore, by the concavity of $V(x,i)$ and monotonicity (nondecreasing) of $V_x(x,i)$ (see [19]), we have

$$\lim_{x\to\infty} V_x(x,i) = 0.$$

Formally, the value function (3.6) satisfies the HJB equations

$$\begin{cases} \max_{u\in[0,C]} \{\mathcal{L}^u V(x,i) - rV(x,i) + u\} = 0, \quad \forall i \in \mathcal{M}, \\ V(0,i) = 0, \quad \forall i \in \mathcal{M}. \end{cases} \tag{3.10}$$

In view of (3.10), the system of HJB equations can be rewritten as

$$\max_{u\in[0,C]} \Big\{(c(i)-u)V_x(x,i) + \frac{1}{2}\sigma(i)^2 V_{xx}(x,i) + \lambda_i \int_0^x V(x-q(x,i,\rho),i) f_i(\rho)d\rho \\ -(\lambda_i + r)V(x,i) + QV(x,\cdot)(i) + u\Big\} = 0 \tag{3.11}$$

$$V(0,i) = 0 \ \text{ for each } \ i \in \mathcal{M}.$$

Remark 3.1. Suppose there is an admissible feedback control $u^*(\cdot)$ that is the maximizer of (3.11). Then it can be shown that $V(x,i)$ is indeed the optimal cost and $u^*(t)$ is the optimal control. In fact, let $\hat{u}(t)$ be an arbitrary admissible control whose trajectory is $\hat{x}(t)$. In view of (3.11),

$$0 = \mathcal{L}^{u^*} V(x,i) - rV(x,i) + u^*,$$

$$0 \geq \mathcal{L}^{\hat{u}} V(\hat{x},i) - rV(\hat{x},i) + \hat{u}$$

for all values of $x \in (0, \infty)$, $t > 0$, w, and $i \in \mathcal{M}$. Applying Itô's formula to $\widetilde{V}(t, x, i) = e^{-rt}V(x, i)$, we have

$$\begin{aligned}
&-e^{-rt}E_{x,i}V(x(t), \alpha(t)) + V(x, i) \\
&= E_{x,i}\int_0^t e^{-rs}(-\mathcal{L}^{u^*}V(x(s), \alpha(s)) + rV(x(s), \alpha(s)))ds \\
&= E_{x,i}\int_0^t e^{-rs}u^*(s)ds.
\end{aligned} \tag{3.12}$$

In view of (3.12), we obtain

$$V(x, i) = e^{-rt}E_{x,i}V(x(t), \alpha(t)) + E_{x,i}\int_0^t e^{-rs}u^*(s)ds. \tag{3.13}$$

Similarly to (3.12), we also have

$$\begin{aligned}
&-e^{-rt}E_{x,i}V(\hat{x}(t), \alpha(t)) + V(x, i) \\
&= E_{x,i}\int_0^t e^{-rs}(-\mathcal{L}^{\hat{u}}V(\hat{x}(s), \alpha(s)) + rV(\hat{x}(s), \alpha(s)))ds \\
&\geq E_{x,i}\int_0^t e^{-rs}\hat{u}(s)ds.
\end{aligned} \tag{3.14}$$

Hence, we obtain

$$V(x, i) \geq e^{-rt}E_{x,i}V(\hat{x}(t), \alpha(t)) + E_{x,i}\int_0^t e^{-rs}\hat{u}(s)ds. \tag{3.15}$$

By virtue of the boundedness of $V(\cdot, i)$ for each $i \in \mathcal{M}$, $e^{-rt}E_{x,i}V(x(t), \alpha(t)) \to 0$ and $e^{-rt}E_{x,i}V(\hat{x}(t), \alpha(t)) \to 0$ as $t \to \infty$. Thus

$$J(x, i, \hat{u}) = E_{x,i}\int_0^\infty e^{-rs}\hat{u}(s)ds \leq V(x, i) = E_{x,i}\int_0^\infty e^{-rs}u^*(s)ds = J(x, i, u^*).$$

Hence the maximizing control $u^*(\cdot)$ is optimal.

3.2 Numerical Algorithm

In this section we construct a locally consistent Markov chain approximation for the jump diffusion model with regime-switching. The discrete-time and finite-state controlled Markov chain is so defined that it is locally consistent with (3.5). First let us recall some facts of Poisson random measure which is useful for constructing the approximating Markov chain and for the convergence theorem.

There is an equivalent way to define the process (3.5) by working with the claim times and values. To do this, set $\nu_0 = 0$, and let ν_n, $n \geq 1$, denote the time of the nth claim, and $q(\cdot, \cdot, \rho_n)$ is the corresponding claim intensity with a suitable function of $q(\cdot)$. Let $\{\nu_{n+1} - \nu_n, \rho_n, n < \infty\}$ be mutually independent random variables with $\nu_{n+1} - \nu_n$ being exponentially distributed with mean $1/\lambda$, and let ρ_n have a distribution $\Pi(\cdot)$. Furthermore, let $\{\nu_{k+1} - \nu_k, \rho_k, k \geq n\}$ be independent of $\{x(s), \alpha(s), s < \nu_n, \nu_{k+1} - \nu_k, \rho_k, k < n\}$, then the nth claim term is $q(x(\nu_n^-), \alpha(\nu_n), \rho_n)$, and the claim amount $R(t)$ can be written as

$$R(t) = \sum_{\nu_n \leq t} q(x(\nu_n^-), \alpha(\nu_n), \rho_n).$$

We note the local properties of claims for (3.5). Because $\nu_{n+1} - \nu_n$ is exponentially distributed, we can write

$$P\{\text{claim occurs on } \ [t, t+\Delta) | x(s), \alpha(s), w(s), N(s, \cdot), s \leq t\} = \lambda\Delta + o(\Delta). \tag{3.16}$$

By the independence and the definition of ρ_n, for any $H \in \mathcal{B}(\mathbb{R}_+)$, we have

$$\begin{aligned} &P\{x(t) - x(t^-) \in H | t = \nu_n \ \text{ for some n}; w(s), x(s), \alpha(s), N(s, \cdot), s < t; x(t^-) = x, \\ &\alpha(t) = \alpha\} = \Pi(\rho : q(x(t^-), \alpha(t), \rho) \in H). \end{aligned} \tag{3.17}$$

It is implied by the above discussion that $x(\cdot)$ satisfying (3.5) can be viewed as a process that involves regime-switching diffusion with claims according to the claim rate defined by (3.16). Given that the nth claim occurs at time ν_n, we construct the values according to the conditional probability law (3.17) or, equivalently, write it as $q(x(\nu_n^-), \alpha(\nu_n), \rho_n)$. Then the process given in (3.5) is a switching diffusion process until the time of the next claim. To begin, we construct a discrete-time, finite-state, controlled Markov chain to approximate the controlled diffusion process with regime-switching, with the dynamic system

$$dx(t) = \Big[c(\alpha(t)) - u(t)\Big]dt + \sigma(\alpha(t))dw(t), \quad x(0) = x. \tag{3.18}$$

For each $h > 0$, define S_h to be the approximation of the state space for the surplus. It is a finite set since in computation only finitely many values can be dealt with. We let S_h contain x of the form $x = kh$, i.e., constant multiple of h for $k \geq 0$. Let $\{(\tilde{\xi}_n^h, \alpha_n^h), n < \infty\}$ be a controlled discrete-time Markov chain on a discrete state space $S_h \times \mathcal{M}$ with transition probabilities from a state $(x, i) \in \mathcal{M}$ to another state $(y, j) \in \mathcal{M}$ denoted by $p^h((x, i), (y, j)|u)$. The u is a control parameter and takes values in the compact set U. We use u_n^h to denote the random variable that is the actual control action for the chain at discrete time n. To approximate the continuous-time Markov chain, we need another approximation sequence. Suppose that there is an $\Delta t^h(x, \alpha, u) > 0$ and define the "interpolation interval" as $\Delta t_n^h = \Delta t^h(\tilde{\xi}_n^h, \alpha_n^h, u_n^h)$ on $S_h \times \mathcal{M} \times \mathcal{U}$. Define the interpolation time $t_n^h = \sum_{k=0}^{n-1} \Delta t_k^h(\tilde{\xi}_k^h, \alpha_k^h, u_k^h)$. The piecewise constant interpolations $(\tilde{\xi}^h(\cdot), \alpha^h(\cdot))$, $u^h(\cdot)$ and $\beta^h(t)$ are defined as

$$\tilde{\xi}^h(t) = \tilde{\xi}_n^h, \;\; \alpha^h(t) = \alpha_n^h, \;\; u^h(t) = u_n^h, \;\; \beta^h(t) = n \;\; \text{for} \;\; t \in [t_n^h, t_{n+1}^h). \tag{3.19}$$

Definition 3.2. Let $\{p_D^h((x, i), (y, j)|u)\}$ for $(x, i), (y, j) \in S^h \times \mathcal{M}$, and $u \in U$ be a collection of well defined transition probabilities for the Markov chain $(\tilde{\xi}_n^h, \alpha_n^h)$, an approximation to

$(x(\cdot), \alpha(\cdot))$. Define the difference $\Delta\tilde{\xi}^h_n = \tilde{\xi}^h_{n+1} - \tilde{\xi}^h_n$. Assume $\inf_{x,i,u} \Delta t^h(x,i,u) > 0$ for each $h > 0$ and $\lim_{h\to\infty} \Delta t^h(x,i,u) \to 0$. Let $E^{u,h}_{x,i,n}$, $\text{var}^{u,h}_{x,i,n}$, and $p^{u,h}_{x,i,n}$ denote the conditional expectation, variance, and marginal probability given $\{\tilde{\xi}^h_k, \alpha^h_k, u^h_k, k \le n, \tilde{\xi}^h_n = x, \alpha^h_n = i, u^h_n = u\}$, respectively. The sequence $\{(\tilde{\xi}^h_n, \alpha^h_n)\}$ is said to be locally consistent with diffusion and regime switching, if

$$
\begin{aligned}
&E^{u,h}_{x,i,n}\Delta\tilde{\xi}^h_n = (c(i) - u)\Delta t^h(x,i,u) + o(\Delta t^h(x,i,u)),\\
&\text{var}^{u,h}_{x,i,n}\Delta\tilde{\xi}^h_n = \sigma(i)^2\Delta t^h(x,i,u) + o(\Delta t^h(x,i,u)),\\
&p^{u,h}_{x,i,n}\{\alpha^h_{n+1} = j\} = \Delta t^h(x,i,u)q_{ij} + o(\Delta t^h(x,i,u)), \text{for } j \ne i,\\
&p^{u,h}_{x,i,n}\{\alpha^h_{n+1} = i\} = \Delta t^h(x,i,u)(1 + q_{ii}) + o(\Delta t^h(x,i,u)),\\
&\sup_{n,\omega} |\Delta\tilde{\xi}^h_n| \to 0 \text{ as } h \to 0.
\end{aligned}
\tag{3.20}
$$

Once we have a locally consistent approximating Markov chain, we can approximate the value function. Let $\mathcal{U}^h$ denote the collection of controls, which are determined by a sequence of measurable functions $F^h_n(\cdot)$ such that

$$
u^h_n = F^h_n(\tilde{\xi}^h_k, \alpha^h_k, k \le n; u^h_k, k \le n). \tag{3.21}
$$

Let $G^o_h = S_h \cap (0,\infty)$. Then $G^o_h \times \mathcal{M}$ is a finite state space. Let N_h denote the first time that $\{\tilde{\xi}^h_n\}$ leaves G^o_h. Then the first exit time of $\tilde{\xi}^h(\cdot)$ from G^o_h is $\tau_h = t^h_{N_h}$. Natural reward functions for the chain that approximate (3.4) is

$$
J^h(x,i,u^h) = E_{x,i} \sum_{n=0}^{N_h - 1} e^{-r\Delta t^h_n} u^h_n \Delta t^h_n. \tag{3.22}
$$

Denote

$$V^h(x,i) = \sup_{u^h \in \mathcal{U}^h} J^h(x,i,u^h). \tag{3.23}$$

Practically, we compute $V^h(x,i)$ by solving the corresponding dynamic programming equation using either value iteration or policy iteration. In fact, for $i \in \mathcal{M}$, we can use

$$V^h(x,i) = \begin{cases} \max_{u \in U} \left[e^{-r\Delta t^h(x,i,u)} \sum_{y,j} (p^h((x,i),(y,j))|u) V^h(y,j) + u\Delta t^h(x,i,u) \right], \text{ for } x \in G_h^o, \\ \\ 0, \text{ for } x = 0, \end{cases} \tag{3.24}$$

where $e^{-r\Delta t^h(x,i,u)}$ represents the discount. When the control space has only one element $u^h \in \mathcal{U}^h$, the max in (3.24) can be dropped. That is,

$$V^h(x,i) = \begin{cases} \sum_{y,j} e^{-r\Delta t^h(x,i,u)} (p^h((x,i),(y,j))|u) V^h(y,j) + u\Delta t^h(x,i,u), \text{ for } x \in G_h^o, \\ \\ 0, \text{ for } x = 0. \end{cases} \tag{3.25}$$

On the other hand, the HJB equation with only diffusion and regime switching can be written as

$$V_x(x,i)(c(i) - u) + \frac{1}{2} V_{xx}(x,u,i)\sigma^2(i) + \sum_j V(x,\cdot)q_{ij} - rV(x,i) + u = 0. \tag{3.26}$$

Define the approximation to the first and the second derivatives of $V(\cdot,i)$ by finite difference method using stepsize $h > 0$ as:

$$
\begin{aligned}
&V(x,i) \to V^h(x,i) \\
&V_x(x,i) \to \frac{V^h(x+h,i) - V^h(x,i)}{h} \quad \text{for} \quad c(i) - u > 0, \\
&V_x(x,i) \to \frac{V^h(x,i) - V^h(x-h,i)}{h} \quad \text{for} \quad c(i) - u < 0, \\
&V_{xx}(x,i) \to \frac{V^h(x+h,i) - 2V^h(x,i) + V^h(x-h,i)}{h^2}.
\end{aligned}
\tag{3.27}
$$

Together with the boundary conditions, it leads to

$$
\begin{aligned}
&V^h(x,i) = 0, \ \text{for } x = 0, \\
&\frac{V^h(x+h,i) - V^h(x,i)}{h}(c(i)-u)^+ - \frac{V^h(x,i) - V^h(x-h,i)}{h}(c(i)-u)^- \\
&\quad + \frac{V^h(x+h,i) - 2V^h(x,i) + V^h(x-h,i)}{h^2} \cdot \frac{\sigma^2(i)}{2} \\
&\quad + \sum_{j}^{m} V^h(x,\cdot)q_{ij} - rV^h(x,i) + u = 0, \quad \forall x \in G_h^o, i \in \mathcal{M},
\end{aligned}
\tag{3.28}
$$

where $(c(i)-u)^+$ and $(c(i)-u)^-$ are the positive and negative parts of $c(i)-u$, respectively.

Simplifying (3.28) and comparing the result with (3.25), we have

$$
\begin{aligned}
& p_D^h((x,i),(x+h,i)|u) = \frac{(\sigma^2(i)/2) + h(c(i)-u)^+}{\widehat{D} - rh^2}, \\
& p_D^h((x,i),(x-h,i)|u) = \frac{(\sigma^2(i)/2) + h(c(i)-u)^-}{\widehat{D} - rh^2}, \\
& p_D^h((x,i),(x,j)|u) = \frac{h^2}{\widehat{D} - rh^2} q_{ij}, \quad \text{for} \quad j \neq i, \\
& p_D^h(\cdot) = 0, \quad \text{otherwise}, \\
& \Delta t^h(x,i,u) = \frac{h^2}{\widehat{D}},
\end{aligned}
\tag{3.29}
$$

with

$$\widetilde{D} = \sigma^2(i) + h|c(i) - u| + h^2(r - q_{ii})$$

being well defined.

Suppose that the current state is $\tilde{\xi}_n^h = x$, $\alpha_n^h = i$, and control is $u_n^h = u$. The next interpolation interval $\Delta t^h(x,i,u)$ is determined by (3.29). To present the claim terms, we determine the next state $(\tilde{\xi}_{n+1}^h, \alpha_{n+1}^h)$ by noting:

1. No claims occur in $[t_n^h, t_{n+1}^h)$ with probability $(1 - \lambda \Delta t^h(x,i,u) + o(\Delta t^h(x,i,u)))$; we determine $(\tilde{\xi}_{n+1}^h, \alpha_{n+1}^h)$ by transition probability $p_D^h(\cdot)$ as in (3.29).

2. There is a claim in $[t_n^h, t_{n+1}^h)$ with probability $\lambda \Delta t^h(x,i,u) + o(\Delta t^h(x,i,u)))$, we determine $(\tilde{\xi}_{n+1}^h, \alpha_{n+1}^h)$ by

$$\tilde{\xi}_{n+1}^h = \tilde{\xi}_n^h - q_h(x,i,\rho), \alpha_{n+1}^h = \alpha_n^h,$$

where $\rho \sim \Pi(\cdot)$, and $q_h(x,i,\rho) \in S_h \subseteq \mathbb{R}_+$ such that $q_h(x,i,\rho)$ is the nearest value of $q(x,i,\rho)$ so that $\tilde{\xi}^h_{n+1} \in S_h$. Then $|q_h(x,i,\rho) - q(x,i,\rho)| \to 0$ as $h \to 0$, uniformly in x.

Let H^h_n denote the event that $(\tilde{\xi}^h_{n+1}, \alpha^h_{n+1})$ is determined by the first alternative above and use T^h_n to denote the event of the second case. Let $I_{H^h_n}$ and $I_{T^h_n}$ be corresponding indicator functions, respectively. Then $I_{H^h_n} + I_{T^h_n} = 1$. Then we need a new definition of the local consistency for Markov chain approximation of compound Poisson process with diffusion and regime-switching.

Definition 3.3. A controlled Markov chain $\{(\tilde{\xi}^h_n, \alpha^h_n), n < \infty\}$ is said to be locally consistent with (3.5), if there is an interpolation interval $\Delta t^h(x,i,u) \to 0$ as $h \to 0$ uniformly in x,i, and u such that

1. there is a transition probability $p^h_D(\cdot)$ that is locally consistent with (3.18) in the sense that (3.20) holds.

2. there is a $\delta^h(x,i,u) = o(\Delta t^h(x,i,u))$ such that the one-step transition probability $\{p^h((x,i),(y,j))|u\}$ is given by

$$\begin{aligned} p^h(((x,i),(y,j))|u) &= (1 - \lambda\Delta t^h(x,i,u) + \delta^h(x,i,u))p^h_D((x,i),(y,j)) \\ &\quad +(\lambda\Delta t^h(x,i,u) + \delta^h(x,i,u))\Pi\{\rho : q_h(x,i,\rho) = x - y\}. \end{aligned} \tag{3.30}$$

Furthermore, the system of dynamic programming equations is a modification of (3.24).

That is

$$V^h(x,i) = \begin{cases} \max_{u\in U} \Big[(1-\lambda\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)} \\ \sum_{y,j}(p^h_D((x,i),(y,j))|u)V^h(y,j) + (\lambda\Delta t^h(x,i,u)+\delta^h(x,i,u)) \\ e^{-r\Delta t^h(x,i,u)} \int_0^x V^h(x-q_h(x,i,\rho),i)\Pi(d\rho) + u\Delta t^h(x,i,u)\Big], \text{ for } x\in G_h^o, \\ 0, \text{ for } x=0. \end{cases} \tag{3.31}$$

3.3 Convergence of Numerical Approximation

This section focuses on the asymptotic properties of the approximating Markov chain proposed in the last section. The main techniques are methods of weak convergence. This section is divided into several subsections. In Section 4.1, we show that the Markov chain constructed is locally consistent. Section 4.2 is concerned with the interpolation of the approximation sequences, weak convergence is also introduced. Section 4.3 deals with weak convergence of a sequence of $(x^h(\cdot),\alpha^h(\cdot),u^h(\cdot),w^h(\cdot),N^h(\cdot),\widetilde{\tau})$, which yields that a sequence of controlled surplus processes converges to a limit surplus process. Section 4.3 takes up the issue of the weak convergence of the surplus process. Section 4.4 deals with the convergence of the reward and value functions.

3.3.1 Local Consistency

To proceed, we first present the local consistency for our approximating Markov chain. Basically, it says that the approximation we constructed is consistent with the given dynamic system.

Lemma 3.4. *The Markov chain* $\{\tilde{\xi}_n^h, \alpha_n^h\}$ *with transition probabilities* $(p_D^h(\cdot))$ *defined in* (3.29) *is locally consistent with the stochastic differential equation in* (3.5).

Proof. Using (3.29), it is readily seen that

$$E_{x,i,n}^{u,h}\Delta\tilde{\xi}_n^h = hp_D^h((x,i),(x+h,i)|u) - hp_D^h((x,i),(x-h,i)|u)$$

$$= h\frac{(\sigma^2(i)/2) + h(c(i)-u)^+}{\widehat{D} - rh^2} - h\frac{(\sigma^2(i)/2) + h(c(i)-u)^-}{\widehat{D} - rh^2}$$

$$= (c(i)-u)\Delta t^h(x,i,u) + (c(i)-u)\Delta t^h(x,i,u)\frac{rh^2}{\widehat{D} - rh^2}$$

$$= (c(i)-u)\Delta t^h(x,i,u) + o(\Delta t^h(x,i,u)),$$

Likewise, we obtain

$$E_{x,i,n}^{u,h}(\Delta\tilde{\xi}_n^h)^2 = h^2p_D^h((x,i),(x+h,i)|u) - h^2p_D^h((x,i),(x-h,i)|u)$$

$$= \frac{h^2}{\widehat{D} - rh^2}(\sigma^2(i) + h|(c(i)-u)|)$$

$$= \sigma^2(i)\Delta t^h(x,i,u) + \Delta t^h(x,i,u)O(h).$$

As a result,

$$\mathrm{var}_{x,i,n}^{u,h}\Delta\tilde{\xi}_n^h = \sigma^2(i)\Delta t^h(x,i,u) + \Delta t^h(x,i,u)O(h) - (c(i)-u)\Delta t^h(x,i,u)$$

$$+o(\Delta t^h(x,i,u)))^2$$

$$= \sigma^2(i)\Delta t^h(x,i,u) + o(\Delta t^h(x,i,u))$$

Thus both equations in (3.20) are verified. The desired local consistency follows with the use of local properties of claims specified. □

3.3.2 Interpolations of Approximation Sequences

Based on the Markov chain approximation constructed in the last section, piecewise constant interpolation is obtained here with appropriately chosen interpolation intervals. Using $(\tilde{\xi}_n^h, \alpha_n^h)$ to approximate the continuous-time process $(x(\cdot), \alpha(\cdot))$, we defined the continuous-time interpolation $(\tilde{\xi}^h(\cdot), \alpha^h(\cdot))$, $u^h(\cdot)$ and $\beta^h(t)$ as in (3.19). Recall N_h is defined in the paragraph above (3.22), we define the first exit time of $\tilde{\xi}^h(\cdot)$ from G_h^o by

$$\tau_h = t_{N_h}^h. \tag{3.32}$$

Let the discrete times at which claims occur be denoted by ν_j^h, $j = 1, 2, \dots$ Then we have

$$\tilde{\xi}_{\nu_j^h - 1}^h - \tilde{\xi}_{\nu_j^h}^h = q_h(\tilde{\xi}_{\nu_j^h - 1}^h, \alpha_{\nu_j^h - 1}^h, \rho).$$

Define $\mathcal{D}_n^h$ as the smallest σ-algebra of $\{\tilde{\xi}_k^h, \alpha_k^h, u_k^h, H_k^h, k \le n; \nu_k^h, \rho_k^h : \nu_k^h \le t_n\}$. Then τ_h is a $\mathcal{D}_n^h$-stopping time. Using the interpolation process, we can rewrite (3.22) as

$$J^h(x,i,u^h) = E_{x,i}\int_0^{\tau_h} e^{-rs}u^h(s)ds. \tag{3.33}$$

Let $\tilde{\xi}_0^h = x$, $\alpha_0^h = \alpha$, E_n^h denote the expectation conditioned on the information up to time n, that is, conditioned on $\mathcal{D}_n^h$. In addition, $\mathcal{U}^h$ defined by (3.21) is equivalent to the collection of all piecewise constant admissible controls with respect to $\mathcal{D}_n^h$.

Then we can write

$$\begin{aligned} \tilde{\xi}_n &= x + \sum_{k=0}^{n-1}[\Delta\tilde{\xi}_k^h I_{H_k^h} + (\Delta\tilde{\xi}_k^h(1 - I_{H_k^h}))] \\ &= x + \sum_{k=0}^{n-1} E_k^h \Delta\tilde{\xi}_k^h I_{H_k^h} + \sum_{k=0}^{n-1}(\Delta\tilde{\xi}_k^h - E_k^h\Delta\tilde{\xi}_k^h) I_{H_k^h} + \sum_{k=0}^{n-1}(\Delta\tilde{\xi}_k^h(1 - I_{H_k^h})). \end{aligned} \tag{3.34}$$

The local consistency leads to

$$\begin{aligned} \sum_{k=0}^{n-1} E_k^h \Delta\tilde{\xi}_k^h I_{H_k^h} &= \sum_{k=0}^{n-1}((c(\alpha_k^h) - u_k^h)\Delta t_k^h + o(\Delta t_k^h)) I_{H_k^h} \\ &= \sum_{k=0}^{n-1}(c(\alpha_k^h) - u_k^h)\Delta t_k^h + o(\Delta t_k^h)) - (\max_{k' \le n} \Delta t_{k'}^h) O(\sum_{k=0}^{n-1} I_{T_k^h}) \end{aligned} \tag{3.35}$$

Denote

$$\begin{aligned} M_n^h &= \sum_{k=0}^{n-1}(\Delta\tilde{\xi}_k^h - E_k^h \Delta\tilde{\xi}_k^h) I_{H_k^h}, \\ R_n^h &= -\sum_{k=0}^{n-1}(\Delta\tilde{\xi}_k^h(1 - I_{H_k^h})) = \sum_{k:\nu_k < n} q_h(\tilde{\xi}_{\nu_k}^h, \alpha_{\nu_k}^h, \rho_k), \end{aligned} \tag{3.36}$$

where M_n^h is a martingale with respect to $\mathcal{D}_n^h$. Note that

$$E\sum_{k=0}^{n-1} I_{T_k^h} = E[\text{number of } \; n : \nu_n^h \le t] \to \lambda t \quad \text{as} \quad h \to 0.$$

This implies

$$(\max_{k' \le n} \Delta t_{k'}^h) O(\sum_{k=0}^{n-1} I_{T_k^h}) \to 0 \;\text{ in probability as }\; h \to 0.$$

Hence we can drop the term involving $I_{H_k^h}$ without affecting the limit in (3.35). We attempt to represent $M^h(t)$ similar to the diffusion term in (3.5). Define $w^h(\cdot)$ as

$$\begin{aligned} w^h(t) &= \sum_{k=0}^{n-1} (\Delta\tilde{\xi}_k^h - E_k^h \Delta\tilde{\xi}_k^h)/\sigma(\alpha_k^h), \\ &= \int_0^t \sigma^{-1}(\alpha^h(s)) dM^h(s). \end{aligned} \tag{3.37}$$

Combining (3.35)-(3.37), we rewrite (3.34) by

$$\begin{aligned} \tilde{\xi}^h(t) &= x + \int_0^t (c(\alpha^h(s)) - u^h(s))dt + \int_0^t \sigma(\alpha^h(s))dw^h(s) - R^h(t) + \varepsilon^h(t) \\ R^h(t) &= \sum_{\nu_n^h \le t} q_h(\tilde{\xi}^h_{\nu_n -}, \alpha^h_{\nu_n}, \rho_n), \end{aligned} \tag{3.38}$$

where $\varepsilon^h(t)$ is a negligible error satisfying

$$\lim_{h\to\infty} \sup_{0\le t\le T} E|\varepsilon^h(t)| \to 0 \quad \text{for any} \quad 0 < T < \infty. \tag{3.39}$$

We can also rewrite (3.5) as

$$x(t) = x + \int_0^t (c(\alpha(s)) - u)dt + \int_0^t \sigma(\alpha(s))dw(s) - R(t), \tag{3.40}$$

where

$$R(t) = \sum_{\nu_n \le t} q(x(\nu_n^-), \alpha(\nu_n), \rho_n) = \int_0^t \int_{\mathbb{R}_+} q(x(s^-), \alpha(s), \rho) N(ds d\rho).$$

Now we give the definition of existence and uniqueness of weak solution.

Definition 3.5. By a weak solution of (3.40), we mean that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, a filtration $\mathcal{F}_t$, and process $(x(\cdot), \alpha(\cdot), u(\cdot), w(\cdot), N(\cdot))$ such that $w(\cdot)$ is a standard $\mathcal{F}_t$-Wiener process, $N(\cdot)$ is a $\mathcal{F}_t$-Poisson measure with claim rate λ and claim size distribution $\Pi(\cdot)$, $\alpha(\cdot)$ is a Markov chain with generator Q and state space $\mathcal{M}$, $u(\cdot)$ is admissible with respect to $(\alpha(\cdot), w(\cdot), N(\cdot))$, $x(\cdot)$ is $\mathcal{F}_t$-adapted, and (3.40) is satisfied. For an initial condition (x, i), by the weak sense uniqueness, we mean that the probability law of the admissible process $(\alpha(\cdot), u(\cdot), w(\cdot), N(\cdot))$ determines the probability law of solution $(x(\cdot), \alpha(\cdot), u(\cdot), w(\cdot), N(\cdot))$ to (3.40), irrespective of probability space.

We need one more assumption.

(A1) Let $\hat{\tau}(\phi) = \infty$, if $\phi(t) \in G^o$, for all $t < \infty$, otherwise, define $\hat{\tau}(\phi) = \inf\{t : \phi \notin G^o\}$. The function $\hat{\tau}(\cdot)$ is continuous (as a map from $D[0, \infty)$, the space of functions that are right continuous and have left limits endowed with the Skorohod topology to the interval $[0, \infty]$ (the extended and compactified positive real numbers)) with probability one relative to the measure induced by any solution to (3.40) with initial condition (x, α).

3.3.3 Convergence of Surplus Processes

This section deals with convergence of surplus processes.

Lemma 3.6. *Using the transition probabilities* $\{p^h(\cdot)\}$ *defined in* (3.20) and (3.30), *the interpolated process of the constructed Markov chain* $\{\alpha^h(\cdot)\}$ *converges weakly to* $\alpha(\cdot)$, *the Markov chain with generator* $Q = (q_{ij})$.

Proof. The proof can be obtained similar to [24, Theorem 3.1]. □

Theorem 3.7. *Let the approximating chain* $\{\tilde{\xi}^h_n, \alpha^h_n, n < \infty\}$ *constructed with transition probabilities defined in* (3.29) *be locally consistent with* (3.5), $\{u^h_n, n < \infty\}$ *be a sequence of* *admissible controls, and* $(\tilde{\xi}^h(\cdot), \alpha^h(\cdot))$ *be the continuous-time interpolation defined in* (3.19). *Let* $\{\widetilde{\tau}_h\}$ *be a sequence of* $\mathcal{F}^h_t$*-stopping times. Then* $\{\tilde{\xi}^h(\cdot), \alpha^h(\cdot), u^h(\cdot), w^h(\cdot), N^h(\cdot), \widetilde{\tau}_h\}$ *is tight.*

Proof. Using one point compactification, $\widetilde{\tau} \in [0, \infty]$. In view of Lemma 3.6, $\{\alpha^h(\cdot)\}$ is tight. The sequences $\{u^h(\cdot), \widetilde{\tau}_h\}$ are always tight since their range spaces are compact. Let $T < \infty$, and let $\widetilde{\nu_h}$ be an $\mathcal{F}_t$-stopping time which is no bigger than T. Then for $\delta > 0$,

$$E^{u^h}_{\widetilde{\nu}_h}(w^h(\widetilde{\nu}_h + \delta) - w^h(\widetilde{\nu}_h))^2 = \delta + \widetilde{\varepsilon}_h, \tag{3.41}$$

where $\widetilde{\varepsilon}_h \to 0$ uniformly in $\widetilde{\nu}_h$. Taking $\limsup_{h\to 0}$ followed by $\lim_{\delta\to 0}$ yield the tightness of $\{w^h(\cdot)\}$. A similar argument yields the tightness of $M^h(\cdot)$. In view of [13, Theorem 9.2.1], the sequence $\{N^h(\cdot)\}$ is tight because the mean number of claims on any bounded interval $[t, t+s]$ is bounded by $\lambda s + \delta^h_1(s)$, where $\delta^h_1(s)$ goes to zero as $h \to 0$, and

$$\lim_{\delta\to 0} \inf_{h,n} P\{\nu^h_{n+1} - \nu^h_n > \delta | \text{data up to } \nu^h_n\} = 1.$$

This also implies the tightness of $\{R^h(\cdot)\}$. These results and the boundedness of $c(\cdot)$ and $u(\cdot)$ implies the tightness of $\{\tilde{\xi}^h(\cdot)\}$. Thus, $\{\tilde{\xi}^h(\cdot), \alpha^h(\cdot), u^h(\cdot), w^h(\cdot), N^h(\cdot), \widetilde{\tau}_h\}$ is tight. □

Theorem 3.8. *Let* $(\tilde{\xi}(\cdot), \alpha(\cdot), u(\cdot), w(\cdot), N(\cdot), \widetilde{\tau})$ *be the limit of weakly convergent subsequence and* $\mathcal{F}_t$ *the* σ*-algebra generated by* $\{x(s), \alpha(s), u(s), w(s), N(s), s \le t, \widetilde{\tau} I_{\{\widetilde{\tau}<t\}}\}$. *Then* $w(\cdot)$ *and* $N(\cdot)$ *are a standard* $\mathcal{F}_t$*-Wiener process and Poisson measure, respectively, and* $\widetilde{\tau}$ *is an* $\mathcal{F}_t$*-stopping time and* $u(\cdot)$ *is an admissible control. Let the claim times and claim sizes of* $N(\cdot)$ *be denoted by* ν_n, ρ_n*. Then,* (3.40) *is satisfied.*

Proof. Since $\{\tilde{\xi}^h(\cdot), \alpha^h(\cdot), u^h(\cdot), w^h(\cdot), N^h(\cdot), \widetilde{\tau}_h\}$ is tight, we can extract a weakly convergent subsequence by Prohorov's theorem. Denote the limit by $(\tilde{\xi}(\cdot), \alpha(\cdot), u(\cdot),$

$w(\cdot), N(\cdot), \widetilde{\tau})$. To characterize $w(\cdot)$, let $t > 0$, $\delta > 0$, p, κ, $\{t_k : k \leq p\}$ be given such that $t_k \leq t \leq t + \widetilde{t}$ for all $k \leq p$, $\mathrm{P}(\widetilde{\tau}_h = t_k)$ is zero. Let $\{\Gamma_j^\kappa, j \leq \kappa\}$ be a sequence of nondecreasing partition of Γ such that $\Pi(\partial\Gamma_j^\kappa) = 0$ for all j and all κ, where $\partial\Gamma_j^\kappa$ is the boundary of the set Γ_j^κ. As $\kappa \to \infty$, let the diameter of the sets Γ_j^κ go to zero. By (3.37), $w^h(\cdot)$ is an $\mathcal{F}_t$-martingale. Thus we have

$$EK(\tilde{\xi}^h(t_k), \alpha^h(t_k), w^h(t_k), u^h(t_k), N^h(t_k, \Gamma_j^\kappa), j \leq \kappa, k \leq p, \widetilde{\tau}_h I_{\{\widetilde{\tau}_h \leq t\}}) \times[w^h(t+\widetilde{t}) - w^h(t)] = 0. \tag{3.42}$$

By using the Skorohod representation and the dominant convergence theorem, letting $h \to 0$, we obtain

$$EK(x(t_k), \alpha(t_k), w(t_k), u(t_k), N(t_k, \Gamma_j^\kappa), j \leq \kappa, k \leq p, \widetilde{\tau} I_{\{\widetilde{\tau} \leq t\}})[w(t+\widetilde{t}) - w(t)] = 0. \tag{3.43}$$

Since $w(\cdot)$ has continuous sample paths, (3.43) implies that $w(\cdot)$ is a continuous $\mathcal{F}_t$-martingale. On the other hand, since $E[(w^h(t+\delta))^2 - (w^h(t))^2] = E[(w^h(t+\delta) - w^h(t))^2]$, by using the Skorohod representation and the dominant convergence theorem together with (3.41), we have

$$EK(x(t_k), \alpha(t_k), w(t_k), u(t_k), N(t_k, \Gamma_j^\kappa), j \leq \kappa, k \leq p, \widetilde{\tau} I_{\{\widetilde{\tau} \leq t\}})[w^2(t+\delta) - w^2(t) - \delta] = 0. \tag{3.44}$$

The quadratic variation of the martingale $w(t)$ is t. Then $w(\cdot)$ is an $\mathcal{F}_t$-Wiener process.

Now we need to show that $N(\cdot)$ is an $\mathcal{F}_t$-Poisson measure. Let $\theta(\cdot)$ be a continuous function on $\mathbb{R}_+$, and define the process

$$\Theta_N(t) = \int_0^t \int_{\mathbb{R}_+} \theta(\rho) N(ds d\rho).$$

By an argument which is similar to the Wiener process above, if $f(\cdot)$ is a continuous function with compact support, then

$$EK(x(t_k),\alpha(t_k),w(t_k),u(t_k),N(t_k,\Gamma_j^{\kappa}),j\le\kappa,k\le p,\widetilde{\tau}I_{\{\widetilde{\tau}\le t\}})\times\Big[f(\Theta_N(t+\widetilde{t}))$$
$$-f(\Theta_N(t))-\lambda\int_t^{t+\widetilde{t}}\int_{\mathbb{R}_+}[f(\Theta_N(s)+\theta(\rho))-f(\Theta_N(s))]\Pi(dsd\rho)\Big]=0. \tag{3.45}$$

Equation (3.45) and the arbitrariness of $K(\cdot),p,\kappa,t_k,\Gamma_j^{\kappa},f(\cdot)$ and $\theta(\cdot)$ imply that $N(\cdot)$ is an $\mathcal{F}_t$-Poisson measure.

For $\delta>0$, define the process $\phi(\cdot)$ by $\phi^{h,\delta}(t)=\phi^h(n\delta),t\in[n\delta,(n+1)\delta)$. Then, by the tightness of $\{\tilde{\xi}^h(\cdot),\alpha^h(\cdot)\}$, (3.38) can be rewritten as

$$\tilde{\xi}^h(t)=x+\int_0^t(c(\alpha^h(s))-u^h(s))dt+\int_0^t\sigma(\alpha^{h,\delta}(s))dw^h(s)-R^h(t)+\varepsilon^{h,\delta}(t), \tag{3.46}$$

where

$$\lim_{\delta\to0}\limsup_{h\to0}E|\varepsilon^{h,\delta}(t)|=0. \tag{3.47}$$

Letting $h\to0$, by using the Skorohod representation, we obtain

$$E|\int_0^t(c(\alpha^h(s))-u^h(s))ds-\int_0^t(c(\alpha(s))-u(s))ds|=0 \tag{3.48}$$

uniformly in t with probability one. Furthermore, the Skorohod representation implies that as $h\to0$,

$$\int_0^t(c(\alpha^h(s))-u^h(s))ds\to\int_0^t(c(\alpha(s))-u(s))ds \tag{3.49}$$

uniformly in t with probability one on any bounded interval.

Since $\tilde{\xi}^{h,\delta}(\cdot)$ and $\alpha^{h,\delta}(\cdot)$ are piecewise constant functions, we obtain

$$\int_0^t\sigma(\alpha^{h,\delta}(s))dw^h(s)\to\int_0^t\sigma(\alpha^{\delta}(s))dw(s)\ \text{ as }\ h\to0 \tag{3.50}$$

with probability one. Combining (3.42)-(3.50), we have

$$x(t) = x + \int_0^t (c(\alpha(s)) - u(s))dt + \int_0^t \sigma(\alpha^\delta(s))dw(s) - R(t) + \varepsilon^\delta(t), \tag{3.51}$$

where $\lim_{\delta\to 0} E|\varepsilon^\delta(t)| = 0$. Finally, taking limits in the above equation as $\delta \to 0$, (3.40) is obtained. □

3.3.4 Convergence of Reward and Value Functions

This section deals with the convergence of the reward and value functions. Note that the reward $J^h(x, i, u^h)$ is given by (3.33), By virtue of Theorem 3.7, with the use of τ_h in (3.32), each sequence $\{\tilde{\xi}^h(\cdot), \alpha^h(\cdot), u^h(\cdot), w^h(\cdot), N^h(\cdot), \tau_h\}$ has a weakly convergent subsequence with the limit satisfying (3.40). Abusing notation, still index the convergent subsequence by h with the limit denoted by $(x(\cdot), \alpha(\cdot), u(\cdot), w(\cdot), N(\cdot), \widetilde{\tau})$. By assumption (A1), $\{\tau_h\}$ is uniformly integrable. Using the Skorohod representation and the weak convergence, as $h \to 0$,

$$E_{x,i} \int_0^{\tau_h} e^{-rs} u^h(s)ds \to E_{x,i} \int_0^{\widetilde{\tau}} e^{-rs} u(s)ds. \tag{3.52}$$

Assumption (A1) guarantees that the exit time of $x(\cdot)$ from G^o is $\widetilde{\tau} = \tau$. This leads to

$$J^h(x, i, u^h) \to J(x, i, u) \ \text{ as } \ h \to 0. \tag{3.53}$$

Theorem 3.9. *Assume* (A1). $V^h(x, i)$ *and* $V(x, i)$ *are value functions defined in* (3.23) *and* (3.6)*, respectively. Then* $V^h(x, i) \to V(x, i)$ *as* $h \to 0$.

Proof. Since $V(x, i)$ is the maximizing reward function, for any admissible control $u(\cdot)$,

$$J(x, i, m) \le V(x, i).$$

Let $\widetilde{u}^h(\cdot)$ be an optimal control for $\{\tilde{\xi}^h(\cdot)\}$. That is,

$$V^h(x,i) = J^h(x,i,\widetilde{u}^h) = \sup_{u^h} J^h(x,i,u^h).$$

Choose a subsequence $\{\widetilde{h}\}$ of $\{h\}$ such that

$$\limsup_{h\to 0} V^h(x,i) = \lim_{\widetilde{h}\to 0} V^{\widetilde{h}}(x,i) = \lim_{\widetilde{h}\to 0} J^{\widetilde{h}}(x,i,\widetilde{u}^{\widetilde{h}}).$$

Without loss of generality (passing to an additional subsequence if needed), we may assume that $(\tilde{\xi}^{\widetilde{h}}(\cdot), \alpha^{\widetilde{h}}(\cdot), u^{\widetilde{h}}(\cdot), w^{\widetilde{h}}(\cdot), N^{\widetilde{h}}(\cdot), \tau^{\widetilde{h}})$ converges weakly to $(x(\cdot), \alpha(\cdot), u(\cdot),$ $w(\cdot), N(\cdot), \tau)$, where $u(\cdot)$ is an admissible related control. Then the weak convergence and the Skorohod representation yield that

$$\limsup_{h} V^h(x,i) = J(x,i,u) \le V(x,i). \tag{3.54}$$

We proceed to prove the reverse inequality.

We claim that

$$\liminf_{h} V^h(x,i) \ge V(x,i). \tag{3.55}$$

Suppose that $\overline{u}$ is an optimal control with respect to $(\alpha(\cdot), w(\cdot), N(\cdot))$ such that $\overline{x}(\cdot)$ and $\overline{\tau}$ are the associated trajectory and the stopping time, and $J(x,i,\overline{u}) = V(x,i)$. Given any $h > 0$, there are an $\varepsilon > 0$ and an ordinary control $\overline{u}^h(\cdot)$ that takes only finite many values, that $\overline{u}^h(\cdot)$ is a constant on $[k\varepsilon, k\varepsilon+\varepsilon)$, that $\overline{u}^h(\cdot)$ is its corresponding optimal control representation, and let $\overline{x}^h(\cdot)$ and $\overline{\tau}^h$ be the associated solution and stopping time. Then if $(\overline{u}^h(\cdot), \alpha(\cdot), w(\cdot), N(\cdot))$ converges weakly to $(\overline{u}(\cdot), \alpha(\cdot), w(\cdot), N(\cdot))$, we also have $(\overline{x}^h(\cdot), \overline{u}^h(\cdot), \alpha(\cdot), w(\cdot), N(\cdot), \overline{\tau}^h)$ converges weakly to $(x(\cdot), \overline{u}(\cdot), \alpha(\cdot), w(\cdot), N(\cdot), \overline{\tau})$, where (3.40) holds for the limit and $\overline{\tau}$ is the associate stopping time by Theorem 3.7. With assumption (A1), $J^h(x,i,\overline{u}^h) \to J(x,i,\overline{u})$, and that $J^h(x,i,\overline{u}^h) \geq V(x,i) - h$. Thus,

$$\liminf_h V^h(x,i) \geq J^h(x,i,\overline{u}^h) \geq V(x,i) - h.$$

The arbitrariness of h then implies that $\liminf_h V^h(x,i) \geq V(x,i)$.

Using (3.54) and (3.55) together with the weak convergence and the Skorohod representation, we obtain the desired result. The proof of the theorem is concluded. □

3.4 Numerical example

This section is devoted to a couple of examples. For simplicity, we consider the case the discrete event has two states. That is, the continuous-time Markov chain has two states.

Example 3.10. The Markov chain $\alpha(t)$ representing the discrete event state has generator Q

$$Q = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix},$$

and takes values in $\mathcal{M} = \{1,2\}$. The premium depends on the discrete state with $c(1) = 2$ and $c(2) = 3$. The dividend rate $u(t)$ taking its value in $[0,2]$ is a control parameter, $\sigma(\alpha(t))dw(t)$ is interpreted as small claim fluctuation and/or fluctuations due to premium incomes with $\sigma(1) = 0.2$ and $\sigma(2) = 2$, and $R(t)$ is a Poisson process interpreted as claims with $R(t) = \sum_{\nu_n \leq t} \rho_n$, where $\rho_n \in \{0.01, 0.02\}$, with distribution $\Pi(0.01) = 0.6, \Pi(0.02) = 0.4$. Let $\lambda_i = 4$, for $i = 1,2$. Then $\{\nu_{n+1} - \nu_n\}$ is a sequence of exponentially distributed random variables with mean $1/4$. Furthermore, the initial surplus x is supposed to have the maximum 100 and the minimum 0. We use policy iteration methods to numerically solve the optimal control problems. This provides us with the advantage that we trace out the optimal policy for the portfolio selection. we obtain the computation results depicted in Figure 6 and Figure 7 as follows.

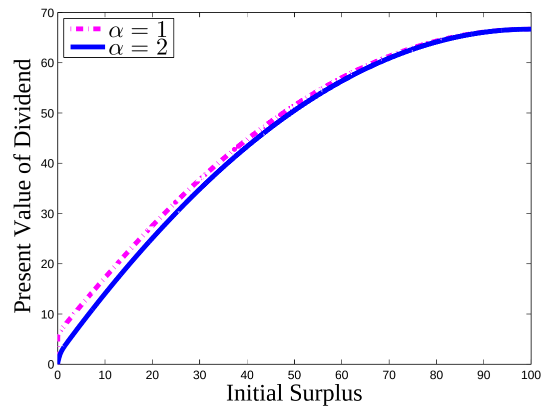

Figure 6: Maximal expected present value of dividend versus initial surplus

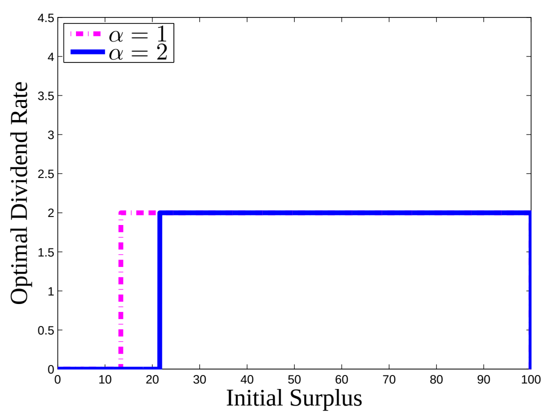

Figure 7: Optimal dividend rate versus initial surplus

Example 3.11. Comparing to Example 3.10, we consider the case that the dividend rate is more then the premium rate. Use data exactly the same as above, but change the range of dividend rate to $[0, 4]$. Then we obtain the computation results depicted in Figure 8 and Figure 9 as follows.

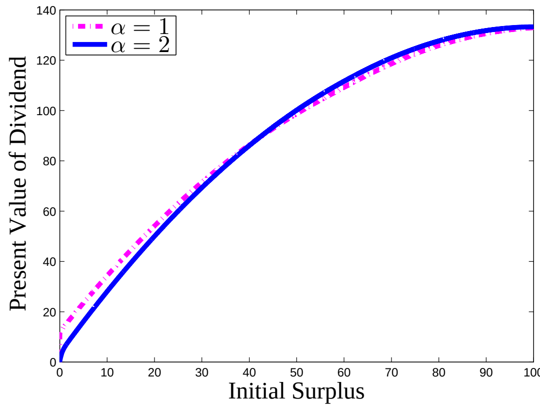

Figure 8: Maximal expected present value of dividend versus initial surplus

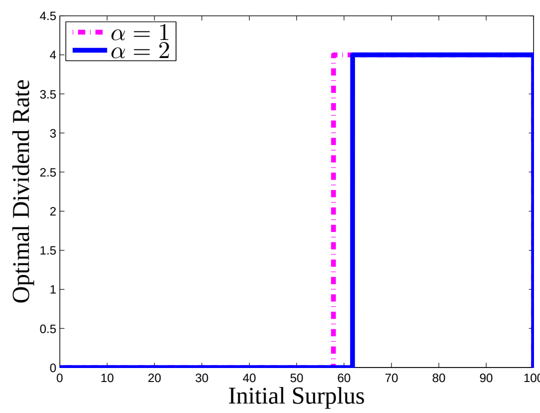

Figure 9: Optimal dividend rate versus initial surplus

Example 3.12. In this example, we assume the difference of the volatilities in the two regimes is larger comparing to Example 3.10. That is, taking $\sigma(1) = 0.1$ and $\sigma(2) = 4$. Then we obtain the computation results given in Figure 10 and Figure 11 as follows.

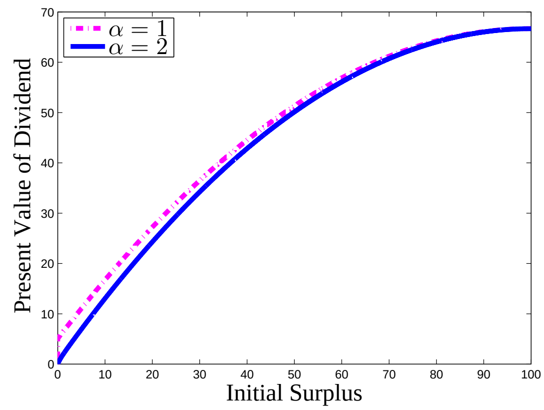

Figure 10: Maximal expected present value of dividend versus initial surplus

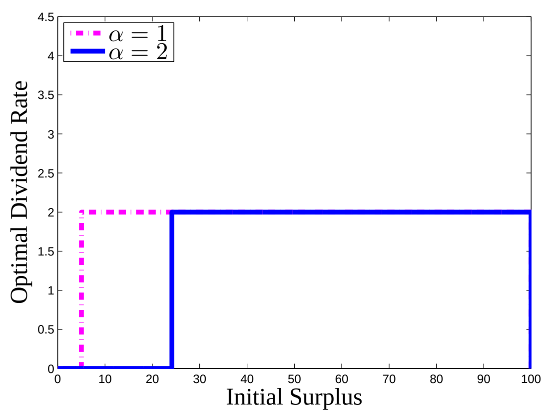

Figure 11: Optimal dividend rate versus initial surplus

Figure 6 to Figure 11 show the expected present value of dividends versus initial surplus and dividend rate versus initial surplus. From these figures, the expected present value of dividends is an increasing function of the initial surplus, this is intuitively obvious. We can also see that if the initial surplus is larger than certain level, the company should pay as much dividend as allowable (this will result a threshold strategy due to the upper bound of the dividend rate). From this we can deduce that, for $t > 0$, the optimal strategy for Example 3.10 is a threshold strategy due to the restriction of maximum dividend rate being less than the premium rate if the Markov chain is at state 2, and the optimal strategy for Example 3.11 is a band strategy. The dividend is paid when $V_x(x, i) < 1$, in which case the company is "inefficient" and cash surplus is high, otherwise, the company is considered "efficient" when $V_x(x, i) > 1$. It is best to pay no dividend when the company is efficient and the cash surplus is low, then funds should be left to company for growth.

By examining the graphs, the following observations are in order. Figure 6 and Figure 7, and Figure 8 and Figure 9 show that the dividend payment rates reach the thresholds depending on the sign of $V_x(x, i) - 1$ no matter whether the ceiling of the dividend payment rate is greater than premium rate or not. However, since the cap of dividend rate is larger in Example 3.11, the dividend will be paid at the rate of premium rate. This will be a kind band strategy. In Example 3.10, the cap of dividend rate is less than the premium rate if the state is 2, this will lead to a threshold strategy.

In addition, the difference of volatilities in Example 3.10 is 1.8 and the difference of volatilities in Example 3.12 is 3.9. From Figure 10 and Figure 11, we can see that the difference of the dividend payment strategies is bigger comparing to Figure 6 and Figure 7, in which case the difference of the volatilities is smaller. So the optimal dividend strategies are sensitive to the market regimes. This indicates that the regime-switching models are appropriate for the intended modeling and optimization.

4 Further Remarks

This dissertation has been devoted to numerical methods for problems arising in risk management and insurance. By choosing Markov regime-switching technique, the models are more realistic but more complicate. More often than not, closed-form solutions are not obtainable. Thus developing numerical solutions is necessary.

In Chapter 2, a numerical approximation scheme to minimize the probability of lifetime ruin for annuity purchase, has been developed. Although one could derive the associate systems of variational inequalities together with the use of properties of switching diffusions, solving them analytically is very difficult. Thus a numerical approach for solving such problems is a necessary step. One may directly discretize the system of variational inequalities, but this relies on the properties of the variational inequalities. We provide a viable alternative. Our Markov chain approximation method uses mainly probabilistic methods and does not need any analytic properties of the solutions of the system of variational inequalities. In the examples, for the constant hazard rate, we show that it is more advantage to purchase the life annuity than self annuitization so that the individual will have less probability of financial ruin even though he or she is less wealthy and maintains the same consumption. For the more general hazard rate such as Gompertz, we show that the individual with the same wealth but younger age will more likely to outlives his or her wealth.

In Chapter 3, we have developed a numerical approximation scheme to maximize the present value of dividend with optimal dividend rate selection. Although one could derive the associate system of HJB equations by using the usual dynamic programming approach together with the use of properties of switching jump diffusions, solving them analytically is very difficult. As an alternative, one may try to discretize the system of HJB equations directly, but this relies on the properties of the HJB equations. As is mentioned above, the powerful Markov chain approximation method could guarantee the consistent of the interpolation sequence with the original dynamic system. In the actual computation, the optimal control can be obtained by using the value or policy iteration methods.

For future study, singular control in dividend payout problem can be considered. For such cases, the dividend payout rate could not be obtained. With the regime-switching technique, we will need to consider the corresponding quasi-variational inequalities. Furthermore, one may consider the Markov chain approximation method to study the numerical solution of the optimal control policy.

REFERENCES

[1] S. Asmussen, B. Høgaard, and M. Taksar, Optimal risk control and dividend distribution policies. Example of excess-of loss reinsurance for an insurance corporation. *Finance and Stochastics*, **4** (2000), 299–324.

[2] S. Asmussen and M. Taksar, Controlled diffusion models for optimal dividend pay-Out. *Insurance: Math. and Economics*, **20** (1997), 1–15.

[3] J.R. Brown, Private pensions, mortality risk, and the decision to annuitize, *J. Public Econ.*, **82** (2001), 29–62.

[4] B. De Finetti, Su unimpostazione alternativa della teoria collettiva del rischio. *Transactions of the XVth International Congress of Actuaries* **2** (1957), 433–443.

[5] A. Cadenillas, T. Choulli, M. Taksar, and L. Zhang, Classical and impulse stochastic control for the optimization of the dividend and risk policies of an insurance firm, *Mathematical Finance*, **16** (2006), 181–202.

[6] T. Davidoff, J. Brown, and P. Diamond, Annuities and individual welfare, M.I.T. Department of Economics Working Paper Series, Working Paper 03–15.

[7] G.B. Di Masi, Y.M. Kabanov, and W.J. Runggaldier, Mean variance hedging of options on stocks with Markov volatility, *Theory Probab. Appl.*, **39** (1994), 173–181.

[8] H. Gerber and E. Shiu, Optimal dividends: analysis with Brownian motion. *North American Actuarial Journal*, **8** (2004), 1–20.

[9] H. Gerber and E. Shiu, On optimal dividend strategies in the compound Poisson model. *North American Actuarial Journal*, **10** (2006), 76–93.

[10] M.K. Ghosh, A. Arapostathis and S.I. Marcus, Optimal control of switching diffusions with application to flexible manufacturing systems, *SIAM J. Control Optim.*, **31** (1993), 1183–1204.

[11] Z. Jin, G. Yin, Numerical Methods for Dividend Policy of Regime-Switching Jump-Diffusion Models, to appear in International Journal of Computer Mathematics.

[12] S. Kapur and M. Orszag, A Portfolio Approach to Investment and Annuitization during Retirement, Working Paper, Birkbeck College, University of London.

[13] H.J. Kushner and P. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time*, 2nd Ed., Springer, New York, 2001.

[14] F. Lundberg. Approximerad Framställning av Sannolikehetsfunktionen, Aterförsäkering av Kollektivrisker, Almqvist & Wiksell, Uppsala. Akad. Afhandling. Almqvist o. Wiksell, Uppsala, 1903.

[15] M. Milevsky, K. Moore, and V. R .Young, Asset allocation and annuity-purchase strategies to minimize the probability of financial ruin, *Mathematical Finance*, **16** 2006, 647–671.

[16] S. Richard, Optimal Consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model, *J. Financ. Econ.*, **2** (1975), 187–203.

[17] H. Schmidli, *Stochastic Control in Insurance*. Springer Verlag, 2008.

[18] Q. S. Song, G. Yin, and Z. Zhang, Numerical method for controlled regime-switching diffusions and regime-switching jump diffusions, *Automatica*, **42** (2006), 1147–1157.

[19] L. Sotomayor and A. Cadenillas, Classical, Singular, and Impulse Stochastic Control for the Optimal Dividend Policy when There is Regime Switching. Preprint, (2008), Download at http://papers.ssrn.com/sol3/papers.cfm?abstract-id=1139444.

[20] M. Yaari, Uncertain lifetime, life insurance and the theory of the consumer, Review of Economic Studies 32,

[21] M. Yaari (1965). Uncertain lifetime, life insurance and the theory of the consumer, *Review of Economic Studies*, **32**, (1965) 137C150.

[22] G. Yin, V. Krishnamurthy, and C. Ion, Regime switching stochastic approximation algorithms with application to adaptive discrete stochastic optimization, *SIAM J. Optim.*, **14** (2004), 1187–1215.

[23] G. Yin and C. Zhu, *Hybrid Switching Diffusions: Properties and Applications*, Springer, New York, 2010.

[24] G. Yin, Q. Zhang, and G. Badowski, Discrete-time singularly perturbed Markov chains: Aggregation, occupation measures, and switching diffusion limit, *Adv. Appl. Probab.* **35** (2003), 449–476.

[25] V.R. Young, Optimal investment strategy to minimize the probability of lifetime ruin, *N. Amer. Actuarial J.*, **8** (2004), 106–126.

[26] Q. Zhang, Stock trading: An optimal selling rule, *SIAM J. Control Optim*, **40** (2001), 64–87.

[27] X.Y. Zhou and G. Yin, Markowitz mean-variance portfolio selectionwith regime switching: A continuous-time model, *SIAM J. Control Optim*, **42** (2003), 1466–1482.

ABSTRACT

NUMERICAL METHODS FOR PROBLEMS ARISING IN RISK MANAGEMENT AND INSURANCE

by

ZHUO JIN

August 2011

Advisor: Dr. G. George Yin

Major: Mathematics (Applied)

Degree: Doctor of Philosophy

In this dissertation we investigate numerical methods for problems annuity purchasing and dividend optimization arising in risk management and insurance. We consider the models with Markov regime-switching process. The regime-switching model contains both continuous and discrete components in their evolution and is referred to as a hybrid system. The discrete events are used to model the random factors that cannot formulated by differential equations. The switching process between regimes is modulated as a finite state Markov chain.

As is widely recognized, this regime-switching model appears to be more versatile and more realistic. However, because of the regime switching and the nonlinearity, it is virtually impossible to obtain closed-form or analytic solutions for our problems. Thus we are seeking numerical solutions by using Markov chain approximation methods.

Focusing on numerical solutions of the regime-switching models in the area of actuarial science, and based on the theory of weak convergence of probability measures, the convergence of the approximating sequences is obtained. In fact, under very broad conditions, we prove that the sequences of approximating Markov chain, the cost functions, and the value functions all converge to that of the underlying original processes. The proofs are

purely probabilistic. It need not appeal to regularity properties of or even explicitly use the Bellman equation. Moreover, the feasibility of regime-switching model and Markov chain approximation method are illustrated by the examples.

AUTOBIOGRAPHICAL STATEMENT

ZHUO JIN

Education

- Ph.D. in Applied Mathematics, December 2011 (expected)
  Wayne State University, Detroit, Michigan
- M.A. in Mathematical Statistics, May, 2010
  Wayne State University, Detroit, Michigan
- B.S. in Applied Mathematics, June 2005
  Huazhong University of Science and Technology, Wuhan, China

Awards

1. The Alfred L. Nelson Award in Recognition of Outstanding Achievement, Department of Mathematics, Wayne State University, April 2011.
2. The Maurice J. Zelonka Endowed Mathematics Scholarship, Department of Mathematics, Wayne State University, April 2010.
3. Graduate Student Professional Travel Award, Department of Mathematics, Wayne State University, August 2010.

List of Publications

1. Zhuo Jin, G. Yin, and H.L. Yang, Numerical methods for dividend optimization using regime-switching jump-diffusion models, Mathematical Control and Related Fields, 1 (2011), 21-40.
2. Zhuo Jin, Yumin Wang, and G. Yin, Numerical solutions of quantile hedging for guaranteed minimum death benefits under a regime-switching-jump-diffusion formulation, Journal of Computational and Applied Mathematics, 235 (2011) 2842-2860.
3. Zhuo Jin, G. Yin, A numerical method for annuity-purchasing decision making to minimize the probability of financial ruin for regime-switching wealth Models, International Journal of Computer Mathematics, 88 (2011), 1256-1282.
4. G. Yin, Zhuo Jin, and H.L. Yang, Asymptotically optimal dividend policy for regime-switching compound Poisson models, *Acta Mathematicae Applicatae Sinica*, 26 (2010), 529-542.
5. G. Yin, H. Jin, and Zhuo Jin, Numerical methods for portfolio selection with bounded constraints, *Journal of Computational and Applied Mathematics*, 233 (2009), 564-581.