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Comparison Of Some Simple Estimators Of The Lognormal Parameters Based On Censored Samples

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Point estimation of the parameters of the lognormal distribution with censored data is considered. The often employed maximum likelihood estimator does not exist in closed form and iterative methods that require very good starting points are needed. In this article, some techniques of finding closed form estimators to this situation are presented and extended. An extensive simulation study is carried out to investigate and compare the performance of these techniques. The results show that some of them are highly efficient as compared with the maximum likelihood estimator.

Keywords: Modified maximum likelihood estimator, least squares estimators, lognormal distribution, mean squared error, Persson Rootzen estimators

Introduction

Let the random variable Y be normally distributed with mean μ and variance σ^2. Let $T = e^Y$, then T is said to have a lognormal distribution. The probability density function of T is given by (Lawless, 1982);

$$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln t - \mu)^2}{2\sigma^2}\right), \quad 0 < t < \infty . \tag{1}$$

The many special features of the lognormal distribution together with its relation with the normal distribution have allowed it to be used as a model in various real life applications. It is used in analyzing biological data (Koch, 1966), and for analyzing data in workplace exposure to contaminants (Lyles & Kupper, 1996). It is also of importance in modeling lifetimes of products and individuals (Lawless, 1982). Various other motivations and applications of the lognormal distribution can be found in Johnson et al. (1994) and Schneider (1986).

In most life testing experiments, one is faced with censored data (Lawless, 1982) arising from either terminating the experiment at a certain prespecified time (Type 1 censoring) or when a predetermined number of failures occur (Type 2 censoring). Censoring is often employed because of time and cost considerations. However, complications do often arise in inference from censored data and usually likelihood based inference procedures are used. Assume that the data is Type 2 censored, whereby the following is observed: $t_{(1)}, \ldots, t_{(r)}$, $r \le n$. The likelihood function is given by

$$L(\mu, \sigma) = \left(\prod_{i=1}^{r} \frac{1}{\sigma t_{(i)}} \phi\left(\frac{\ln t_{(i)} - \mu}{\sigma}\right)\right)\left(Q\left(\frac{\ln t_{(r)} - \mu}{\sigma}\right)\right)^{n-r} \tag{2}$$

where $\phi(\ )$ and $Q(\ )$ are the probability density and the survival functions of the standard normal

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distribution. The likelihood function corresponding to Type 1 censoring is obtained by replacing $\ln t_{(r)}$ by $\ln t_0$, the censoring time under Type 1 censoring. The maximum likelihood estimator is obtained by finding $\hat{\mu}$ and $\hat{\sigma}$ that maximize the likelihood function. This is often done by equating the first partial derivatives of the log-likelihood function to zero and solving for μ and σ simultaneously by applying an iterative numerical procedure for root finding like the Newton-Raphson method. However, this is problematic unless very good starting values are available (Lawless, 1982); the problem becomes serious when the proportion of censored observations is large, especially when the total sample size is relatively small to moderate. In such cases, alternatives to the maximum likelihood estimator are needed, either on their own or as initial approximations to the maximum likelihood estimators. The books of Lawless (1982), Schneider (1986) and Balakrishnan and Cohen (1991) survey much of the work in this area.

In this article, the performances of three techniques for point estimation of parameters in the case of censored data from a lognormal distribution will be extended, investigated, and compared. The first technique is based on finding the least squares estimator by regressing certain estimators of the linearized distribution function on a function of the observations themselves. This approach is used in Hossain and Howlader (1996) and Hossain and Zimmer (2003) for the parameters of the Weibull distribution. Their results showed that the estimators are a reasonable substitute for the maximum likelihood estimator in most situations.

The second technique is due to Perrson and Rootzen (1977) where they presented some modified likelihood function with Type 1 censored data whose maximizing point does not require iterative techniques. The last technique is based on expanding certain terms in the first derivatives of the log-likelihood function in an appropriate Taylor series to get a new system of likelihood equations whose solution exists in closed form. This last approach was studied for Type 2 censored data. An account of this work can be found in Balakrishnan and Cohen (1991). Recently Al-Haj Ebarahem and Baklizi (2005) used the first and the last techniques to estimate the parameters of the Log-Logistic distribution based on complete and censored samples

Least Squares Estimators

The distribution function of the lognormal random variable is given by

$$F(t) = \Phi\left(\frac{\ln t - \mu}{\sigma}\right).$$

Linearization of this distribution function gives $\Phi^{-1}(F(t)) = -\frac{\mu}{\sigma} + \frac{1}{\sigma}\ln t$ .which is a linear regression model between $\Phi^{-1}(F(t))$ and $\ln t$. Let $T_{(1)},\ldots,T_{(r)}$ be the observed censored sample and let S_i be an estimate of $\Phi^{-1}(F(T_{(i)}))$, then the least squares estimators of $b = \frac{1}{\sigma}$ and $a = -\frac{\mu}{\sigma}$ are given respectively by

$$\hat{b} = \frac{\sum_{i=1}^{r} S_i \ln T_i - r\overline{\ln T}\,\overline{S}}{\sum_{i=1}^{r} (\ln T_i)^2 - r\left(\overline{\ln T}\right)^2}$$

and

$$\hat{a} = \overline{S} - \hat{b}\overline{\ln T},$$

where

$$\overline{\ln T} = \sum_{i=1}^{r} \ln T_i \Big/ r$$

and

$$\overline{S} = \sum_{i=1}^{r} S_i \Big/ r.$$

An estimate of S_i, $i=1,\dots,r$ is now required. Two methods of estimation of $F\left(T_{(i)}\right)$ and hence S_i will be considered:

a) Let $\hat{F}\left(T_{(i)}\right)=1-R_{(i)}, i=1,\dots,r$

where

$$R_{(i)}=\frac{r_i}{r_i+1}R_{(i-1)},\ R_{(0)}=1$$

and

$$r_i=n-r_i'+1$$

where r_i' is the rank of the i-th failure in the original sample. Hence, $S_i=\Phi^{-1}\left(1-R_{(i)}\right)$. Substituting these values in $\hat{b}$ and $\hat{a}$, one obtains the estimators $\hat{\mu}_1$ and $\hat{\sigma}_1$.

b) Use $R_{(i)}=\dfrac{r_i-0.5}{r_{i-1}-0.5}R_{(i-1)}$. In this case the new least squares based estimators are based on $\hat{\mu}_2$ and $\hat{\sigma}_2$.

Approximate Maximum Likelihood Estimators

Let $T_{(1)}\le T_{(2)}\le\dots\le T_{(r)}$ be a Type 2 censored sample consisted of the smallest r ordered observations obtained from the lognormal population with probability distribution function given by (1), the remaining $(n-r)$ observations being censored at $T_{(r)}$. Let $Y_i=\ln T_{(i)}, i=1,\dots,r$ be the corresponding order statistics from the normal distribution. The likelihood function of (μ,σ) is given by equation (2). The maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ are given as the solution to the following simultaneous system of nonlinear equations (Lawless, 1982);

$$\frac{\partial\log L}{\partial\mu}=\frac{1}{\sigma^2}\sum_{i=1}^{r}(y_i-\mu)+\frac{(n-r)}{\sigma}\left(\frac{\phi\left(\frac{y_r-\mu}{\sigma}\right)}{Q\left(\frac{y_r-\mu}{\sigma}\right)}\right)$$

$$\frac{\partial\log L}{\partial\sigma}=-\frac{r}{\sigma}+\frac{1}{\sigma^3}\sum_{i=1}^{r}(y_i-\mu)^2+\frac{(n-r)}{\sigma}\left(\frac{\frac{y_r-\mu}{\sigma}\phi\left(\frac{y_r-\mu}{\sigma}\right)}{Q\left(\frac{y_r-\mu}{\sigma}\right)}\right) \tag{3}$$

The likelihood equations corresponding to Type 1 censoring are obtained by replacing $y_r=\ln t_{(r)}$ by $y_0=\ln t_0$, the censoring time under Type 1 censoring. As stated in the introduction, the system of equations (3) does not admit a closed form solution and a numerical method is needed to find the solution (the MLE). In the following two subsections, some modifications of these likelihood equations will be presented to obtain a closed form solution.

The Persson-Rootzen Approach

Consider the likelihood function (2) given by

$$L(\mu,\sigma)=\left(\prod_{i=1}^{r}\frac{1}{\sigma}\phi\left(\frac{y_{(i)}-\mu}{\sigma}\right)\right)\left(Q\left(\frac{y_{(r)}-\mu}{\sigma}\right)\right)^{n-r}$$

Putting

$$x_i=y_i-y_L$$

and

$$\theta=\frac{y_L-\mu}{\sigma} \tag{4}$$

where

$$y_L=\begin{cases}\ln t_0, & \text{for type 1 censoring}\\ y_{(r)}, & \text{for type 2 censoring}\end{cases}$$

where t_0 is the censoring time, write:

$$L(\mu,\sigma)=\frac{1}{\sigma^r}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{r}(x_i+\theta\sigma)^2\right)Q(\theta)^{n-r} \tag{5}$$

Persson and Rootzen (1977) suggested replacing the survival function $Q(\theta)$ in (4) by its nonparameteric estimator $\frac{n-r}{n}$ and therefore replacing θ by $\theta^* = Q^{-1}\left(\frac{n-r}{n}\right)$, the $(r/n)th$ quantile of the standard normal distribution. Substituting these quantities in (4), one obtains a function of σ alone which is maximized by

$$\hat{\sigma}_3 = \frac{1}{2}\left(\frac{\theta^*}{r}\sum_{i=1}^{r}x_i + \left(\left(\frac{\theta^*}{r}\sum_{i=1}^{r}x_i\right)^2 + \frac{4}{r}\sum_{i=1}^{r}x_i^2\right)^{1/2}\right) \tag{6}$$

Substituting $\hat{\sigma}_3$ in (4) yields

$$\hat{\mu}_3 = y_L - \theta^*\hat{\sigma}_3 \tag{7}$$

Approximate MLE Based on Taylor Series Expansion

Consider the likelihood equations given by (3)

$$\frac{\partial \log L}{\partial\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{r}(y_i-\mu) + \frac{(n-r)}{\sigma}\left(\frac{\phi\left(\frac{y_r-\mu}{\sigma}\right)}{Q\left(\frac{y_r-\mu}{\sigma}\right)}\right)$$

$$\frac{\partial \log L}{\partial\sigma} = -\frac{r}{\sigma}$$

$$+\frac{1}{\sigma^3}\sum_{i=1}^{r}(y_i-\mu)^2 + \frac{(n-r)}{\sigma}\left(\frac{\frac{y_r-\mu}{\sigma}\phi\left(\frac{y_r-\mu}{\sigma}\right)}{Q\left(\frac{y_r-\mu}{\sigma}\right)}\right)$$

Let $Z_i = \frac{y_i-\mu}{\sigma}$, $i = 1,\ldots,r$ and noting that $\phi'(z) = -z\phi(z)$ obtains

$$\frac{\partial \log L}{\partial\mu} = \frac{1}{\sigma}\left(\sum_{i=1}^{r} z_i + (n-r)\frac{\phi(z_r)}{Q(z_r)}\right) = 0$$

$$\frac{\partial \log L}{\partial\sigma} = -\frac{1}{\sigma}\left(r-(n-r)z_r\frac{\phi(z_r)}{Q(z_r)} - \sum_{i=1}^{r} z_i^2\right) = 0 \tag{8}$$

Expanding the function $\frac{\phi(z_r)}{Q(z_r)}$ in a Taylor series about the point $\xi_r = \Phi^{-1}(p_r)$, where $\Phi^{-1}(.)$ is the inverse of the distribution function of the standard normal distribution and $p_r = \frac{r}{n+1}$. Setting $q_r = 1-p_r$ obtains $\frac{\phi(z_r)}{Q(z_r)} \cong \gamma + \delta z_r$, where

$$\gamma = \phi(\zeta_r)\left(1+\xi_r^2 - \xi_r\phi(\zeta_r)/q_r\right)/q_r$$

and

$$\delta = \phi(\zeta_r)\left(\phi(\zeta_r) - q_r\xi_r\right)/q_r^2 .$$

Substituting these quantities in the likelihood equations obtains

$$\frac{\partial \log L}{\partial\mu} \cong \frac{1}{\sigma}\left(\sum_{i=1}^{r} z_i + (n-r)\gamma + (n-r)\delta z_r\right) = 0$$

$$\frac{\partial \log L}{\partial\sigma} \cong -\frac{1}{\sigma}\left(r-(n-r)\gamma z_r - (n-r)\delta z_r^2 - \sum_{i=1}^{r} z_i^2\right) = 0 \tag{9}$$

Solving these equations yields the following:

$$\hat{\mu}_4 = B - \hat{\sigma}_4 C$$

$$\hat{\sigma}_4 = \left(-D + \left(D^2 + 4rE\right)^{1/2}\right)/2r, \tag{10}$$

where

$$B = \left(\sum_{i=1}^{r} y_i + (n-r)\delta y_r\right)/m,$$

$$C = -(n-r)\gamma/m\,,$$

$$D = -(n-r)\gamma(y_r - B),$$

$$E = \sum_{i=1}^{r} y_i^2 + (n-r)\delta y_r^2 - mB^2$$

and

$$m = r + (n-r)\delta\,.$$

Performance of the Estimators

A simulation study is conducted to investigate the performance of the estimators. The simulation indices are the sample size $n = 10,15,20,30,40,50,60,80,100,150$. The censoring proportion cp: 0.1, 0.3, 0.5, $a = 1 - cp$. For each combination of the simulation indices, 2,000 pairs of samples are generated and the maximum likelihood estimator $(\hat{\mu}, \hat{\sigma})$ and the closed form estimators $(\hat{\mu}_i, \hat{\sigma}_i), i = 1\ldots,4$ are calculated. Their biases $B\hat{\mu}, B\hat{\sigma}$ and $B\hat{\mu}_i, B\hat{\sigma}_i, i = 1,\ldots,4$ and their mean squared errors and the relative efficiencies $ef\hat{\mu}_i = \dfrac{MSE(\hat{\mu})}{MSE(\hat{\mu}_i)}$ and $ef\hat{\sigma}_i = \dfrac{MSE(\hat{\sigma})}{MSE(\hat{\sigma}_i)}$ $, i = 1\ldots,4$ are obtained.

Results

The results are given in Tables 1 – 4. The biases of the estimators are given in Tables 1 – 2 and the efficiencies of the estimators are given in tables 3 – 4. Inspection of the simulation numerical results lead to the following observations and conclusions. It appears that, under Type 1 censoring, $\hat{\mu}_1$ and $\hat{\mu}_2$ are positively biased when the censoring proportion is moderate to heavy. This is true for all sample sizes. In all other cases, all estimators tend to be negatively biased, regardless of the sample size. It appears that $\hat{\mu}_3$ has the highest bias, and the least bias is achieved by $\hat{\mu}_3$ for light censoring and $\hat{\mu}_2$ and $\hat{\mu}_5$ for moderate to heavy censoring.

For estimators of the scale parameter σ under Type 1 censoring, it appears that $\hat{\sigma}$ has the least bias followed by $\hat{\sigma}_3$ and $\hat{\sigma}_4$. The performances of $\hat{\sigma}_3$ and $\hat{\sigma}_4$ in terms of bias is about similar. However, $\hat{\sigma}_1$ tends to have the largest bias among the estimators considered.

The relative performance of estimators under Type 2 censoring is similar to that of Type 1 censoring. In all cases, the bias decreases as the sample size increases. It is also smaller for lighter censoring.

Concerning the relative efficiencies of the estimators under Type 1 censoring, it appears that the following schemes hold, $\hat{\mu}_4 > \hat{\mu}_3 > \hat{\mu}_2 > \hat{\mu}_1$ under heavy censoring regardless of the sample size and $\hat{\mu}_4 > \hat{\mu}_1 > \hat{\mu}_2 > \hat{\mu}_3$ for moderate to light censoring, where (>) means more efficient. It also appears that the relative efficiencies of $\hat{\mu}_1, \hat{\mu}_2$ and $\hat{\mu}_3$ do not depend on the sample size. However, the relative efficiency of $\hat{\mu}_4$ increases as sample size increases. The relative efficiencies of $\hat{\mu}_2$ and $\hat{\mu}_3$ increase as the censoring proportion becomes smaller, while it decreases for $\hat{\mu}_4$.

The results show that, under Type 1 censoring $\hat{\mu}_4$ are more efficient than the MLE. With regard to scale estimators under Type 1 censoring, it appears that $\hat{\sigma}_4 > \hat{\sigma}_3 > \hat{\sigma}_2 > \hat{\sigma}_1$, whereas before (>) indicated more efficient. It appears that the relative efficiencies of the scale estimators do not depend on n; however, they depend on the censoring proportion. As the censoring proportion becomes smaller, the relative efficiencies of $\hat{\sigma}_1, \hat{\sigma}_2$ and $\hat{\sigma}_4$ increases and it decreases for $\hat{\sigma}_3$. Surprisingly, in all cases considered, the approximate estimators $\hat{\sigma}_4$ are more efficient than the corresponding MLE.

Table 1. Bias of the Estimators Under Type 1 Censoring

n	a	$B\hat{\mu}_1$	$B\hat{\mu}_2$	$B\hat{\mu}_3$	$B\hat{\mu}_4$	$B\hat{\mu}$	$B\hat{\sigma}_1$	$B\hat{\sigma}_2$	$B\hat{\sigma}_3$	$B\hat{\sigma}_4$	$B\hat{\sigma}$
10	0.5	0.106	0.039	-0.121	-0.114	-0.099	0.268	0.239	-0.194	-0.195	-0.188
10	0.7	0.040	-0.033	-0.083	-0.055	-0.041	0.231	0.193	-0.141	-0.138	-0.125
10	0.9	-0.000	-0.086	-0.103	-0.016	-0.015	0.191	0.131	-0.149	-0.108	-0.099
15	0.5	0.088	0.043	-0.075	-0.066	-0.056	0.221	0.203	-0.118	-0.118	-0.112
15	0.7	0.030	-0.019	-0.047	-0.032	-0.018	0.175	0.149	-0.085	-0.086	-0.073
15	0.9	-0.007	-0.069	-0.103	-0.010	-0.014	0.153	0.107	-0.115	-0.062	-0.057
20	0.5	0.079	0.047	-0.062	-0.059	-0.051	0.184	0.171	-0.094	-0.096	-0.091
20	0.7	0.027	-0.008	-0.041	-0.028	-0.021	0.139	0.122	-0.074	-0.074	-0.066
20	0.9	0.005	-0.038	-0.050	-0.009	-0.001	0.126	0.096	-0.069	-0.055	-0.041
30	0.5	0.078	0.057	-0.036	-0.033	-0.026	0.147	0.139	-0.063	-0.064	-0.061
30	0.7	0.025	0.001	-0.024	-0.018	-0.010	0.108	0.096	-0.046	-0.049	-0.041
30	0.9	0.007	-0.021	-0.031	-0.007	0.003	0.098	0.079	-0.041	-0.035	-0.021
40	0.5	0.051	0.036	-0.039	-0.033	-0.033	0.117	0.111	-0.050	-0.050	-0.049
40	0.7	0.013	-0.004	-0.026	-0.019	-0.016	0.089	0.081	-0.033	-0.033	-0.029
40	0.9	-0.000	-0.022	-0.030	-0.008	-0.003	0.071	0.057	-0.038	-0.030	-0.022
50	0.5	0.050	0.038	-0.030	-0.025	-0.024	0.102	0.097	-0.041	-0.041	-0.040
50	0.7	0.015	0.001	-0.020	-0.013	-0.010	0.079	0.072	-0.025	-0.025	-0.022
50	0.9	0.002	-0.014	-0.022	-0.006	0.000	0.066	0.054	-0.026	-0.021	-0.012
60	0.5	0.051	0.041	-0.022	-0.019	-0.016	0.103	0.099	-0.024	-0.024	-0.022
60	0.7	0.013	0.002	-0.014	-0.011	-0.007	0.065	0.060	-0.023	-0.024	-0.020
60	0.9	0.001	-0.012	-0.019	-0.005	-0.001	0.053	0.044	-0.025	-0.020	-0.014
80	0.5	0.035	0.027	-0.019	-0.016	-0.016	0.076	0.074	-0.020	-0.020	-0.019
80	0.7	0.014	0.006	-0.008	-0.005	-0.003	0.050	0.047	-0.019	-0.020	-0.017
80	0.9	-0.002	-0.012	-0.016	-0.004	-0.003	0.036	0.029	-0.022	-0.019	-0.015
100	0.5	0.034	0.028	-0.014	-0.012	-0.011	0.069	0.067	-0.014	-0.015	-0.014
100	0.7	0.009	0.003	-0.010	-0.007	-0.005	0.048	0.045	-0.011	-0.011	-0.009
100	0.9	-0.001	-0.010	-0.014	-0.006	-0.002	0.036	0.030	-0.013	-0.012	-0.007
150	0.5	0.026	0.022	-0.008	-0.007	-0.006	0.048	0.046	-0.011	-0.012	-0.011
150	0.7	0.005	0.001	-0.009	-0.005	-0.005	0.035	0.033	-0.008	-0.007	-0.006
150	0.9	-0.001	-0.006	-0.008	-0.004	-0.001	0.025	0.022	-0.008	-0.008	-0.004

Table 2. Bias of the Estimators Under Type 2 Censoring

n	a	$B\hat{\mu}_1$	$B\hat{\mu}_2$	$B\hat{\mu}_3$	$B\hat{\mu}_4$	$B\hat{\mu}$	$B\hat{\sigma}_1$	$B\hat{\sigma}_2$	$B\hat{\sigma}_3$	$B\hat{\sigma}_4$	$B\hat{\sigma}$
10	0.5	0.117	0.049	-0.114	-0.092	-0.091	0.285	0.256	-0.185	-0.178	-0.178
10	0.7	0.050	-0.023	-0.064	-0.028	-0.027	0.221	0.184	-0.141	-0.128	-0.127
10	0.9	-0.000	-0.086	-0.108	-0.016	-0.015	0.201	0.140	-0.145	-0.094	-0.093
15	0.5	0.100	0.057	-0.109	-0.093	-0.092	0.231	0.214	-0.143	-0.138	-0.138
15	0.7	0.059	0.011	-0.040	-0.014	-0.014	0.193	0.170	-0.089	-0.081	-0.080
15	0.9	0.007	-0.048	-0.064	-0.009	-0.009	0.160	0.124	-0.087	-0.059	-0.059
20	0.5	0.088	0.056	-0.060	-0.048	-0.047	0.192	0.180	-0.092	-0.089	-0.089
20	0.7	0.034	-0.001	-0.032	-0.013	-0.013	0.140	0.122	-0.072	-0.065	-0.065
20	0.9	0.007	-0.036	-0.048	0.000	0.001	0.123	0.094	-0.071	-0.043	-0.043
30	0.5	0.078	0.057	-0.039	-0.029	-0.029	0.149	0.141	-0.065	-0.062	-0.062
30	0.7	0.027	0.003	-0.025	-0.010	-0.010	0.115	0.104	-0.043	-0.037	-0.037
30	0.9	0.002	-0.026	-0.034	-0.001	-0.001	0.084	0.065	-0.052	-0.033	-0.033
40	0.5	0.063	0.047	-0.034	-0.026	-0.026	0.123	0.117	-0.049	-0.047	-0.047
40	0.7	0.022	0.005	-0.018	-0.007	-0.007	0.089	0.081	-0.035	-0.030	-0.030
40	0.9	0.004	-0.017	-0.025	0.001	0.001	0.069	0.055	-0.039	-0.024	-0.024
50	0.5	0.047	0.035	-0.035	-0.028	-0.028	0.101	0.097	-0.043	-0.042	-0.041
50	0.7	0.020	0.007	-0.013	-0.004	-0.004	0.076	0.069	-0.027	-0.024	-0.024
50	0.9	-0.000	-0.017	-0.024	-0.002	-0.002	0.061	0.050	-0.029	-0.016	-0.016
60	0.5	0.041	0.031	-0.027	-0.023	-0.023	0.090	0.086	-0.033	-0.032	-0.032
60	0.7	0.019	0.007	-0.014	-0.004	-0.004	0.067	0.061	-0.025	-0.022	-0.022
60	0.9	-0.001	-0.015	-0.020	-0.003	-0.003	0.053	0.043	-0.023	-0.013	-0.013
80	0.5	0.040	0.033	-0.014	-0.011	-0.011	0.076	0.073	-0.022	-0.021	-0.021
80	0.7	0.011	0.002	-0.012	-0.006	-0.006	0.054	0.050	-0.016	-0.014	-0.014
80	0.9	0.001	-0.009	-0.015	-0.001	-0.001	0.039	0.032	-0.022	-0.013	-0.013
100	0.5	0.034	0.028	-0.012	-0.009	-0.009	0.060	0.058	-0.022	-0.021	-0.021
100	0.7	0.016	0.009	-0.005	0.000	0.000	0.048	0.045	-0.012	-0.010	-0.010
100	0.9	0.002	-0.005	-0.009	0.001	0.001	0.035	0.030	-0.014	-0.008	-0.008
150	0.5	0.028	0.024	-0.007	-0.005	-0.005	0.050	0.049	-0.010	-0.009	-0.009
150	0.7	0.010	0.005	-0.004	-0.001	-0.001	0.031	0.029	-0.010	-0.009	-0.009
150	0.9	0.001	-0.004	-0.006	0.001	0.001	0.026	0.022	-0.008	-0.004	-0.004

Table 3. Efficiencies of the Estimators Under Type 1 Censoring

n	a	$ef\hat{\mu}_1$	$ef\hat{\mu}_2$	$ef\hat{\mu}_3$	$ef\hat{\mu}_4$	$ef\hat{\sigma}_1$	$ef\hat{\sigma}_2$	$ef\hat{\sigma}_3$	$ef\hat{\sigma}_4$
10	0.5	0.741	0.835	0.977	1.718	0.388	0.419	0.994	1.054
10	0.7	0.917	0.957	0.932	2.095	0.421	0.477	0.975	1.143
10	0.9	1.000	0.937	0.822	1.563	0.472	0.600	0.863	1.200
15	0.5	0.745	0.811	0.981	2.109	0.399	0.425	0.991	1.097
15	0.7	0.930	0.952	0.927	2.208	0.459	0.510	0.964	1.180
15	0.9	0.999	0.935	0.755	1.438	0.495	0.623	0.792	1.278
20	0.5	0.732	0.787	0.966	2.320	0.446	0.467	0.989	1.098
20	0.7	0.891	0.915	0.936	2.373	0.496	0.537	0.959	1.179
20	0.9	0.997	0.971	0.810	1.485	0.535	0.626	0.856	1.293
30	0.5	0.674	0.714	0.989	2.521	0.439	0.454	1.000	1.100
30	0.7	0.878	0.902	0.939	2.520	0.534	0.565	0.971	1.243
30	0.9	0.983	0.973	0.832	1.438	0.551	0.625	0.855	1.335
40	0.5	0.736	0.767	0.966	2.727	0.489	0.503	0.993	1.126
40	0.7	0.897	0.910	0.925	2.753	0.548	0.575	0.968	1.291
40	0.9	0.989	0.973	0.814	1.494	0.635	0.701	0.837	1.377
50	0.5	0.725	0.752	0.973	2.847	0.512	0.524	0.994	1.132
50	0.7	0.890	0.905	0.930	2.827	0.571	0.594	0.972	1.291
50	0.9	0.986	0.978	0.813	1.505	0.613	0.670	0.852	1.358
60	0.5	0.707	0.734	0.970	3.018	0.490	0.501	0.992	1.145
60	0.7	0.884	0.898	0.935	2.867	0.601	0.624	0.963	1.306
60	0.9	0.991	0.982	0.804	1.528	0.663	0.715	0.859	1.354
80	0.5	0.712	0.730	0.977	3.119	0.518	0.528	0.993	1.145
80	0.7	0.910	0.924	0.911	3.171	0.625	0.643	0.969	1.277
80	0.9	0.991	0.980	0.801	1.571	0.754	0.798	0.836	1.447
100	0.5	0.702	0.718	0.975	3.224	0.532	0.541	0.993	1.145
100	0.7	0.901	0.911	0.919	3.152	0.616	0.632	0.975	1.307
100	0.9	0.988	0.978	0.801	1.482	0.725	0.764	0.821	1.437
150	0.5	0.719	0.733	0.972	3.309	0.588	0.595	0.998	1.158
150	0.7	0.913	0.918	0.923	3.307	0.677	0.691	0.956	1.351
150	0.9	0.988	0.983	0.806	1.528	0.758	0.789	0.833	1.436

Table 4. Efficiencies of the Estimators Under Type 2 Censoring

n	a	$ef\hat{\mu}_1$	$ef\hat{\mu}_2$	$ef\hat{\mu}_3$	$ef\hat{\mu}_4$	$ef\hat{\sigma}_1$	$ef\hat{\sigma}_2$	$ef\hat{\sigma}_3$	$ef\hat{\sigma}_4$
10	0.5	0.723	0.821	0.978	0.999	0.370	0.400	0.992	0.999
10	0.7	0.921	0.972	0.929	1.000	0.445	0.505	0.961	0.999
10	0.9	0.999	0.934	0.807	0.999	0.452	0.577	0.869	0.999
15	0.5	0.688	0.753	0.980	0.999	0.395	0.416	0.996	0.999
15	0.7	0.853	0.910	0.954	0.999	0.425	0.463	0.980	0.999
15	0.9	0.978	0.950	0.866	0.999	0.487	0.575	0.900	0.999
20	0.5	0.709	0.764	0.975	0.999	0.429	0.449	0.992	0.999
20	0.7	0.908	0.939	0.917	1.000	0.507	0.547	0.963	0.999
20	0.9	0.982	0.961	0.856	1.000	0.531	0.622	0.842	1.000
30	0.5	0.693	0.733	0.974	0.999	0.439	0.454	0.996	0.999
30	0.7	0.880	0.907	0.919	1.000	0.499	0.529	0.975	0.999
30	0.9	0.990	0.970	0.814	1.000	0.621	0.698	0.851	0.999
40	0.5	0.687	0.720	0.982	0.999	0.455	0.468	1.001	0.999
40	0.7	0.896	0.919	0.919	1.000	0.549	0.576	0.971	0.999
40	0.9	0.986	0.976	0.825	1.000	0.639	0.703	0.864	0.999
50	0.5	0.700	0.725	0.978	0.999	0.503	0.515	0.991	1.000
50	0.7	0.890	0.909	0.936	1.000	0.572	0.595	0.974	1.000
50	0.9	0.992	0.977	0.796	1.000	0.652	0.710	0.846	0.999
60	0.5	0.716	0.738	0.977	0.999	0.492	0.502	1.001	0.999
60	0.7	0.882	0.900	0.926	0.999	0.590	0.611	0.972	0.999
60	0.9	0.994	0.981	0.795	1.000	0.670	0.722	0.847	1.000
80	0.5	0.709	0.729	0.970	1.000	0.525	0.534	0.995	1.000
80	0.7	0.903	0.915	0.912	1.000	0.610	0.629	0.968	1.000
80	0.9	0.986	0.981	0.833	1.000	0.722	0.765	0.844	0.999
100	0.5	0.728	0.745	0.974	1.000	0.572	0.581	0.994	1.000
100	0.7	0.902	0.917	0.917	1.000	0.616	0.633	0.973	1.000
100	0.9	0.987	0.985	0.815	1.000	0.725	0.765	0.830	1.000
150	0.5	0.713	0.726	0.981	1.000	0.573	0.580	0.998	1.000
150	0.7	0.912	0.921	0.918	1.000	0.689	0.701	0.967	1.000
150	0.9	0.988	0.986	0.823	1.000	0.751	0.780	0.865	1.000

The performance of the estimators under Type 2 censoring is similar to their performance under Type 1 censoring. However it appears that $\hat{\sigma}_3$ and $\hat{\sigma}_4$ are about as efficient as the MLE for all sample sizes and censoring proportions, except for $\hat{\sigma}_3$ when the censoring proportion is small, in which case $\hat{\sigma}_3$ is less efficient.

Conclusion

It appears that good substitutes to the MLE in closed form do exist. The performance of some of them is highly competent with that of the MLE and sometimes they are better, as is the case with the approximation based on the Taylor series expansion $\hat{\mu}_4$ and $\hat{\sigma}_4$.

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