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A Modified $\bar{X}$ Control Chart for Samples Drawn from Finite Populations

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The $\bar{X}$ chart works well under the assumption of random sampling from infinite populations. However, many process monitoring scenarios may consist of random sampling from finite populations. A modified $\bar{X}$ chart is proposed in this article to solve the problems encountered by the standard $\bar{X}$ chart when samples are drawn from finite populations.

Key words: $\bar{X}$ control chart, finite population, infinite population, average run length (ARL), in-control, out-of-control (o.o.c.), upper control limit (*UCL*), lower control limit (*LCL*).

Introduction

The Shewhart $\bar{X}$ control chart is widely used in manufacturing industries to monitor the stability of the mean of a process. Since its introduction in the late 1920's, numerous extensions and enhancements of the $\bar{X}$ chart have been suggested.

Nelson (1984) discussed eight types of runs rules which increase the sensitivity of the $\bar{X}$ chart for the detection of a shift in the mean of a normally distributed process. Wheeler (1983) provided tables of the power function of the $\bar{X}$ chart and the Type-I error probabilities of each of the four different sets of detection rules. False signal rates of the $\bar{X}$ chart incorporating each of the eight different runs rules are studied by Walker et al. (1991). Seven of these rules are discussed in Nelson (1984). Klein (2000) proposed two different runs rules for the chart, the 2-of-2 and 2-of-3 rules, based on a Markov chain approach in setting the limits of the chart. Using the same Markov chain approach. Khoo (2004a) extended the work of Klein (2000) by suggesting three additional rules, i.e., the 2-of-4, 3-of-3 and 3-of-4 rules.

Superior alternatives to the two rules of Klein (2000) are proposed by Khoo and Khotrun (2006) to enable a quicker detection of a big shift, while maintaining the same sensitivity towards a small shift. Shmueli and Cohen (2003) introduced a new method for computing the run length distribution of a Shewhart chart with runs and scans rules. Davis and Krehbiel (2002) compared the ARL performances of Shewhart charts with all possible combinations of supplementary runs rules and that of zone charts and found the latter to outperform the former.

The first optimum economic design of the $\bar{X}$ chart which considered statistical and cost considerations in the selection of design parameters, i.e., sample size, sampling intervals and location of control limits was proposed by Duncan (1956). Tagaras (1989) studied the statistical properties and the economic design of $\bar{X}$ charts with asymmetric control limits. Del Castillo et al. (1996) applied an interactive multicriteria nonlinear optimization algorithm to a model for the design of $\bar{X}$ charts where only the sampling cost needs to be specified while the cost of false alarms need not be specified. Jaraiedi and Zhuang (1991) presented a computer program to perform optimal cost-based design of $\bar{X}$ charts when multiple

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assignable causes can shift the process to an out-of-control state. McWilliams et al. (2001) gave a FORTRAN program that can be used to jointly determine the parameters of $\bar{X}$ charts used in combination with either the R or S charts. Waheba and Nickerson (2005) developed a comprehensive cost model to incorporate two cost functions, i.e., the reactive and proactive functions for obtaining economically optimum designs of $\bar{X}$ charts for controlling the process mean. Keats et al. (1995) presented and analyzed a methodology for using average production length (APL) and sampling constraints to aid in the design of $\bar{X}$ control schemes.

Costa (1994) studied the properties of the variable sample size (VSS) $\bar{X}$ chart when the size of each sample depends on what is observed in the preceding sample and compared its performance with the other methods. Sim et al. (2004) considered the occurrence of double assignable causes in a process, adopted the Markov chain approach to investigate the statistical properties of the VSS $\bar{X}$ chart and suggested a procedure to compute the optimal sample size. Lin and Chou (2005a) proposed the variable sample size and control limit (VSSCL) $\bar{X}$ chart which was shown to have a lower false alarm rate and to be quicker than the VSS $\bar{X}$ chart in detecting small and moderate shifts in a process involving non-normal populations. Reynolds and Stoumbos (2001) showed that the variable sampling interval (VSI) X chart which allows the sampling interval to be varied enables a substantial reduction in the expected times in detecting shifts in process parameters. Chen and Chiou (2005) developed an economic design of VSI $\bar{X}$ control charts. Lin and Chou (2005b) proposed two adaptive $\bar{X}$ charts, i.e., the variable sampling rate with sampling at fixed times (VSRFT) $\bar{X}$ chart and the variable parameters with sampling at fixed times (VPFT) chart.

Nedumaran and Pignatiello (2001) addressed the problem of estimating the $\bar{X}$ chart limits when the values of the process parameters are unknown. Nedumaran and Pignatiello (2005) also proposed the use of the analysis of means (ANOM) technique for constructing retrospective $\bar{X}$ control chart limits so as to control the overall probability of a false alarm at a desired level. Champ and Jones (2004) examined methods for obtaining probability limits of Phase-I $\bar{X}$ charts when the process mean and standard deviation are estimated.

Methods of making the $\bar{X}$ charts less influenced by extreme observations and hence more effective in the detection of outliers are the trimmed mean $\bar{X}$ and R charts proposed by Langenberg and Iglewicz (1986) and the robust $\bar{X}_Q$ and R_Q charts based on the sample interquartile range estimator suggested by Rocke (1989 and 1992). Among the procedures of using the charts for skewed populations are those based on the weighted variance concept proposed by Bai and Choi (1995) and Chang and Bai (2001), as well as that using the skewness correction method suggested by Chan and Cui (2003).

Other extensions of the $\bar{X}$ chart are as follows: The estimation of the time of a change in the mean following an out-of-control signal using the maximum likelihood estimation technique was proposed by Samuel et al. (1998). Park and Park (2004) suggested a maximum likelihood joint estimator of the change point to identify the time of a change in the process mean or variance when $\bar{X}$ and S control charts issue a signal. Daudin (1992) presented a double sampling $\bar{X}$ chart which offers better statistical efficiency than the standard $\bar{X}$ chart without increasing the sampling. Costa and Rahim (2004) suggested joint $\bar{X}$ and R charts with a two stage sampling procedure which speeds up the detection of process disturbances. Del Castillo (1996) presented a C program for the computation of the run length distribution and average run length of $\bar{X}$ charts with unknown process variance. Khoo (2004b) reviewed and studied some commonly used performance measures for the $\bar{X}$ charts. Maragah and Woodall (1992) showed the effect of autocorrelation on the retrospective X chart for individuals. Roes et al. (1993), Rigdon et al. (1994) and Trip and Wieringa (2006) showed that using the X chart alone is as efficient as the combined X-MR chart for detecting changes in the process variance. However, Rigdon et al. (1994) recommended that the limits on the individuals X chart be based on the moving range (MR) rather than the sample standard deviation. Rahardja (2005) found that adding the MR chart to an X chart is not helpful for detecting independently and identically distributed (i.i.d.) departures from standard conditions, but is beneficial in detecting some non-i.i.d. conditions. Combined $\bar{X}$ and S charts such as the semicircle and Max charts are proposed by Chao and Cheng (1996) and Chen and Cheng (1998) respectively.

A Modified $\bar{X}$ Control Chart

Suppose that a quality characteristic is normally distributed with mean μ and standard deviation σ, where both μ and σ are known. If $X_1, X_2, ..., X_n$ is a sample of size n, then the mean of this sample is

$$\bar{X} = \frac{X_1 + X_2 + ... + X_n}{n} \quad (1)$$

For sampling from infinite populations, which is usually assumed to be the case in process monitoring, $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$. Thus, the ±3 sigma limits of the standard $\bar{X}$ chart are $UCL_S / LCL_S = \mu_0 \pm 3\sigma_{\bar{X}} = \mu_0 \pm \frac{3\sigma}{\sqrt{n}}$, where μ_0 is the in-control mean of the process.

However, in some industrial settings, sampling is made from finite populations. Here, the use of the standard $\bar{X}$ chart's limits can lead to erroneous conclusions as it will cause an inflated Type-II error which will be discussed later. For sampling from finite populations (Bluman, 2004), the sample mean, $\bar{X}_i \sim N\left(\mu, \frac{\sigma^2 (N_i - n_i)}{n_i (N_i - 1)}\right)$, i =1, 2, …. Assuming that a manufacturing process is producing items at a steady rate such as in a conveyor belt system and that the number of items drawn for each sample are of equal size, then $\bar{X}_i \sim N\left(\mu, \frac{\sigma^2 (N - n)}{n(N - 1)}\right)$. The correction factor for the variance of $\bar{X}_i$, i.e., $\frac{N-n}{N-1}$ is necessary if relatively large samples are taken from a small population so that the sample mean will more accurately estimate the population mean and there will be less error in the estimation. The ±3 sigma limits of the modified $\bar{X}$ chart for samples drawn from finite populations are

$$UCL_M = \mu_0 + \frac{3\sigma}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}} \quad (2a)$$

and

$$LCL_M = \mu_0 - \frac{3\sigma}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}} \quad (2b)$$

If the process parameters μ_0 and σ are unknown, they are estimated from $\bar{\bar{X}}$ and $\bar{R}/d_2$ or $\bar{S}/c_4$ respectively, where $\bar{\bar{X}}$ is the grand average, $\bar{R}$ is the average range and $\bar{S}$ is the average standard deviation. Here $\bar{\bar{X}}$, $\bar{R}$ and, $\bar{S}$ are computed from the following formulae:

$$\bar{\bar{X}} = \frac{\bar{X}_1 + \bar{X}_2 + ... + \bar{X}_m}{m} \quad (3)$$

$$\bar{R} = \frac{R_1 + R_2 + ... + R_m}{m} \quad (4)$$

and

$$\bar{S} = \frac{S_1 + S_2 + ... + S_m}{m} \quad (5)$$

where $\bar{X}_1, \bar{X}_2, ..., \bar{X}_m$ denote the means of the m samples, $R_1, R_2, ..., R_m$ the ranges of the m samples while $S_1, S_2, ..., S_m$ the standard deviations of the m samples. The m samples from which $\bar{X}_i$, R_i, and S_i, i = 1, 2, …, m, are computed are assumed to be taken from an in-control process.

If μ_0 and σ are unknown, the limits in eqs. (2a) and (2b) when σ is estimated using $\bar{R}/d_2$ are

$$UCL_M = \bar{\bar{X}} + \frac{3\bar{R}}{d_2\sqrt{n}}\sqrt{\frac{N-n}{N-1}} = \bar{\bar{X}} + A_2'\bar{R} \quad (6a)$$

and

$$LCL_M = \bar{\bar{X}} - \frac{3\bar{R}}{d_2\sqrt{n}}\sqrt{\frac{N-n}{N-1}} = \bar{\bar{X}} - A_2'\bar{R}, \quad (6b)$$

respectively, where $A_2' = \frac{3}{d_2\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$.

If $\bar{S}/c_4$ is used to estimate σ, then the limits in eqs. (2a) and (2b) become

$$UCL_M = \bar{\bar{X}} + \frac{3\bar{S}}{c_4\sqrt{n}}\sqrt{\frac{N-n}{N-1}} = \bar{\bar{X}} + A_3'\bar{S} \quad (7a)$$

and

$$LCL_M = \bar{\bar{X}} - \frac{3\bar{S}}{c_4\sqrt{n}}\sqrt{\frac{N-n}{N-1}} = \bar{\bar{X}} - A_3'\bar{S}, \quad (7b)$$

respectively, where $A_3' = \frac{3}{c_4\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$.

Generally, the estimator $\hat{\sigma} = \bar{R}/d_2$ is used for small sample sizes, say $n < 10$ while the estimator $\hat{\sigma} = \bar{S}/c_4$ is used for big sample sizes, say $n \geq 10$. Factors A_2' and A_3' based on various values of n and N, for the construction of the limits of the modified $\bar{X}$ chart are given in Tables A1 and A2 respectively in the Appendix, where sample sizes of n = 2, 3, …, 25 and selected population sizes of $N \leq 1000$ are considered. Note that $N > 1000$ is not considered because it will be shown via Monte Carlo simulation in this article that the results of the standard and modified $\bar{X}$ charts are about the same for $N > 1000$.

Formulae for Computing the Type-I and Type-II Errors of the Modified and Standard $\bar{X}$ Charts

This section deals with the derivation of formulae for computing the probabilities of Type-I, α and Type-II, β errors of the modified and standard $\bar{X}$ charts. The exact in-control and out-of-control ARLs can be easily computed using formulae

$$\text{ARL}_0 = \frac{1}{\alpha} \quad (8)$$

and

$$\text{ARL}_1 = \frac{1}{1-\beta}, \quad (9)$$

respectively.

Assume that the out-of-control process mean is represented by $\mu = \mu_0 + \delta\sigma$, where μ_0 denotes the in-control mean. Note that δ = 0 shows that the process is in-control while δ > 0 or δ < 0 indicates that the process is out-of-control. For sampling from finite populations, it is known that $\bar{X} \sim N\left[\mu, \frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right)\right]$. The probability of a Type-I error of the modified $\bar{X}$ chart for sampling from finite populations is

$$\begin{aligned}\alpha_M &= P\left(\bar{X} > UCL_M \mid \mu = \mu_0\right) + P\left(\bar{X} < LCL_M \mid \mu = \mu_0\right)\\ &= P\left(\frac{\bar{X}-\mu_0}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}} > \frac{\mu_0 + \frac{3\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} - \mu_0}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}}\right) + \\ &\quad P\left(\frac{\bar{X}-\mu_0}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}} < \frac{\mu_0 - \frac{3\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} - \mu_0}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}}\right)\\ &= P(Z > 3) + P(Z < -3) \end{aligned} \quad (10)$$

while the corresponding probability of a Type-I error of the standard $\bar{X}$ chart is

$$\alpha_S = P\left(\bar{X} > UCL_S \mid \mu = \mu_0\right) + P\left(\bar{X} < LCL_S \mid \mu = \mu_0\right)$$
$$= P\left(\frac{\bar{X}-\mu_0}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}} > \frac{\mu_0 + \frac{3\sigma}{\sqrt{n}} - \mu_0}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}}\right) +$$
$$P\left(\frac{\bar{X}-\mu_0}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}} < \frac{\mu_0 - \frac{3\sigma}{\sqrt{n}} - \mu_0}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}}\right)$$
$$= P\left(Z > 3\sqrt{\frac{N-1}{N-n}}\right) + P\left(Z < -3\sqrt{\frac{N-1}{N-n}}\right) \tag{11}$$

The probability of a Type-II error of the modified $\bar{X}$ chart for sampling from finite populations is computed as follows:

$$\beta_M = P\left(LCL_M < \bar{X} < UCL_M \mid \mu = \mu_0 + \delta\sigma\right)$$
$$= P\left(\bar{X} < UCL_M \mid \mu = \mu_0 + \delta\sigma\right) - P\left(\bar{X} < LCL_M \mid \mu = \mu_0 + \delta\sigma\right)$$

$$= P\left(\frac{\bar{X}-\mu_0-\delta\sigma}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}} < \frac{\mu_0 + \frac{3\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} - \mu_0 - \delta\sigma}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}}\right)$$
$$- P\left(\frac{\bar{X}-\mu_0-\delta\sigma}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}} < \frac{\mu_0 - \frac{3\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} - \mu_0 - \delta\sigma}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}}\right)$$
$$= P\left(Z < 3 - \delta\sqrt{\frac{n(N-1)}{N-n}}\right) - P\left(Z < -3 - \delta\sqrt{\frac{n(N-1)}{N-n}}\right) \tag{12}$$

while that of the standard $\bar{X}$ chart is

$$\beta_S = P\left(LCL_S < \bar{X} < UCL_S \mid \mu = \mu_0 + \delta\sigma\right)$$
$$= P\left(\bar{X} < UCL_S \mid \mu = \mu_0 + \delta\sigma\right) - P\left(\bar{X} < LCL_S \mid \mu = \mu_0 + \delta\sigma\right)$$
$$= P\left(\frac{\bar{X}-\mu_0-\delta\sigma}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}} < \frac{\mu_0 + \frac{3\sigma}{\sqrt{n}} - \mu_0 - \delta\sigma}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}}\right)$$
$$- P\left(\frac{\bar{X}-\mu_0-\delta\sigma}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}} < \frac{\mu_0 - \frac{3\sigma}{\sqrt{n}} - \mu_0 - \delta\sigma}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}}\right)$$
$$= P\left[Z < \sqrt{\frac{N-1}{N-n}}\left(3 - \delta\sqrt{n}\right)\right] - P\left[Z < -\sqrt{\frac{N-1}{N-n}}\left(3 + \delta\sqrt{n}\right)\right] \tag{13}$$

A Comparison of the ARL Performances of the Modified and Standard $\bar{X}$ charts

The ARL profiles of the modified $\bar{X}$ chart can be easily computed using eqs. (8), (9), (10) and (12) while that of the standard $\bar{X}$ chart from eqs. (8), (9), (11) and (13). SAS version 9 is used in the computation of the ARL values. For ease of computation, the in-control process is assumed to follow a standard normal, N(0,1) distribution while the out-of-control process a normal, N(δ,1) distribution so that the out-of-control mean is $\mu = \mu_0 + \delta\sigma$ where $\mu_0 = 0$ and $\sigma = 1$. Values of $\delta \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.75, 1, 1.2, 1.4, 1.6, 1.8, 2\}$ are used so that a positive shift is considered. Due to the symmetrical limits of the modified and standard $\bar{X}$ charts, similar ARL profiles will be obtained for positive and negative values of δ. The sample sizes, $n \in \{1, 2, 5\}$ and population sizes, $N \in \{10, 25, 50, 100, 500, 1000, 2500, 5000, 7500, 10000\}$ are considered. Tables 1 and 2 give the ARL results of the modified and standard $\bar{X}$ charts respectively.

When n = 1, both the modified and standard $\bar{X}$ charts are reduced to the individuals X charts. From eqs. (10) and (11), it is observed that $\alpha_M = \alpha_S$ for n = 1 and similarly, from eqs. (12) and (13), $\beta_M = \beta_S$ for n = 1. Thus, the ARL profiles of the two charts in Tables 1 and 2 are exactly the same when n =1, where ARL_0 = 370.4 irrespective of the population size, N. Note that the results in Tables 1 and 2 for n = 1 are also similar to that of the standard $\bar{X}$ chart when samples are drawn from infinite populations because it can be shown easily that $\alpha_M = \alpha_S = \alpha$ and $\beta_M = \beta_S = \beta$, where α and β are the probabilities of the Type-I and Type-II errors of the standard $\bar{X}$ chart for sampling from infinite populations.

For bigger sample sizes of n = 2 or 5, it is observed that the modified $\bar{X}$ chart gives reliable results (see Table 1) compared to that of the standard $\bar{X}$ chart (see Table 2). The ARL_0 values of the modified $\bar{X}$ chart for n = 2 and 5 are all 370.4, irrespective of the value of N, i.e., similar to the case of the standard $\bar{X}$ chart when sampling is made from infinite populations. On the contrary, the ARL_0 values of the standard $\bar{X}$ chart for n = 2 and 5 are greatly larger than 370.4 for small values of N, which are more pronounced for n = 5. For example, when n = 5, ARL_0 = 17545.7, 985.2, 573.0, 455.6 and 385.4 for N = 10, 25, 50, 100 and 500 respectively. The ARL_0 values of the standard $\bar{X}$ chart in Table 2 for n = 2 and 5 decreases as N increases and approximates 370.4 when N > 1000. The ARL_1 values of the standard $\bar{X}$ chart in Table 2 for n = 2 and 5 involving small values of N and δ are greatly larger than the corresponding values in Table 1. For example, when n = 5, N = 10 and $\delta \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ the ARL_1 values of the standard $\bar{X}$ chart are 9495.2, 3232.5, 1124, 422.9, and 172.8 respectively, while that of the modified $\bar{X}$ chart are significantly smaller at 253.1, 119.7, 55.8, 27.8 and 15 respectively. Thus, using the standard $\bar{X}$ chart in the detection of process shifts when sampling is made from finite populations where N is small or of moderate size can lead to a significant delay in the detection of small shifts in the mean. The ARL_1 value of the standard $\bar{X}$ chart that corresponds to a fixed small value of δ for n = 2 and 5, say δ = 0.1 decreases as N increases and approximates that of the modified chart when N > 1000. From the above discussion, it is found that the use of the standard $\bar{X}$ chart can lead to erroneous conclusion and wrong understanding of the probabilities of Type-I and Type-II errors of the chart if sampling is made from finite populations of N < 1000. The use of the modified chart is justified in that it produces reliable in-control and out-of-control ARL values which are somewhat close to that of the standard $\bar{X}$ chart where sampling is made from infinite populations.

Conclusion

The standard $\bar{X}$ chart caters only for the case involving sampling from infinite populations. This article identifies the problems faced by the standard $\bar{X}$ chart when it is used in the monitoring of processes for samples drawn from finite populations or if the population which is supposedly assumed to be infinite consists of less than N = 1000 items of a certain part. As highlighted above, the problems arise include ARL_0s for $n \geq 2$ and N < 1000 are greatly larger than the target value of approximately 370 and the corresponding ARL_1s involving small shifts in the mean are also greatly larger than that of the modified $\bar{X}$ chart. In an industrial setting if the assumption of an infinite population size where sampling is made cannot be met, the modified $\bar{X}$ chart should be used in place of the standard $\bar{X}$ chart. Tables A1 and A2 in the Appendix provide factors A_2' and A_3' used in the computation of the control limits of the modified $\bar{X}$ chart if the process parameters need to be estimated from a preliminary set of data of in-control subgroups. These factors simplify the computation of the limits of the modified $\bar{X}$ chart.

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Table 1. ARL profiles of the modified $\overline{X}$ chart

n	δ	N = 10	25	50	100	500	1000	2500	5000	7500	10000
	0	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
	0.1	352.9	352.9	352.9	352.9	352.9	352.9	352.9	352.9	352.9	352.9
	0.2	308.4	308.4	308.4	308.4	308.4	308.4	308.4	308.4	308.4	308.4
	0.3	253.1	253.1	253.1	253.1	253.1	253.1	253.1	253.1	253.1	253.1
	0.4	200.1	200.1	200.1	200.1	200.1	200.1	200.1	200.1	200.1	200.1
	0.5	155.2	155.2	155.2	155.2	155.2	155.2	155.2	155.2	155.2	155.2
1	0.75	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2
	1	43.9	43.9	43.9	43.9	43.9	43.9	43.9	43.9	43.9	43.9
	1.2	27.8	27.8	27.8	27.8	27.8	27.8	27.8	27.8	27.8	27.8
	1.4	18.2	18.2	18.2	18.2	18.2	18.2	18.2	18.2	18.2	18.2
	1.6	12.4	12.4	12.4	12.4	12.4	12.4	12.4	12.4	12.4	12.4
	1.8	8.7	8.7	8.7	8.7	8.7	8.7	8.7	8.7	8.7	8.7
	2	6.3	6.3	6.3	6.3	6.3	6.3	6.3	6.3	6.3	6.3
	0	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
	0.1	333.1	335.6	336.2	336.6	336.8	336.9	336.9	336.9	336.9	336.9
	0.2	253.1	259.3	261.1	261.9	262.6	262.7	262.7	262.7	262.7	262.7
	0.3	176.5	184.1	186.3	187.3	188.1	188.2	188.3	188.3	188.3	188.3
	0.4	119.7	126.8	128.9	129.9	130.7	130.8	130.8	130.8	130.9	130.9
	0.5	81.2	87.2	89	89.8	90.5	90.6	90.6	90.6	90.6	90.6
2	0.75	32.9	36.1	37.1	37.6	38	38	38.1	38.1	38.1	38.1
	1	15	16.7	17.2	17.5	17.7	17.7	17.7	17.7	17.7	17.7
	1.2	8.7	9.7	10.1	10.2	10.4	10.4	10.4	10.4	10.4	10.4
	1.4	5.4	6.1	6.3	6.4	6.5	6.5	6.5	6.5	6.5	6.5
	1.6	3.6	4.1	4.2	4.3	4.3	4.3	4.3	4.3	4.3	4.3
	1.8	2.6	2.9	3	3	3.1	3.1	3.1	3.1	3.1	3.1
	2	2	2.2	2.3	2.3	2.3	2.3	2.3	2.3	2.3	2.3
	0	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
	0.1	253.1	284	290.4	293.2	295.3	295.5	295.7	295.7	295.7	295.7
	0.2	119.7	159.3	169.1	173.6	176.9	177.3	177.6	177.7	177.7	177.7
	0.3	55.8	84.4	92.3	96	98.9	99.2	99.4	99.5	99.5	99.5
	0.4	27.8	46	51.5	54.1	56.1	56.3	56.5	56.5	56.6	56.6
	0.5	15	26.4	29.9	31.7	33.1	33.2	33.3	33.4	33.4	33.4
5	0.75	4.4	8.2	9.5	10.1	10.6	10.7	10.7	10.7	10.8	10.8
	1	2	3.4	4	4.2	4.4	4.5	4.5	4.5	4.5	4.5
	1.2	1.4	2.1	2.4	2.5	2.6	2.6	2.7	2.7	2.7	2.7
	1.4	1.1	1.5	1.7	1.7	1.8	1.8	1.8	1.8	1.8	1.8
	1.6	1	1.2	1.3	1.3	1.4	1.4	1.4	1.4	1.4	1.4
	1.8	1	1.1	1.1	1.2	1.2	1.2	1.2	1.2	1.2	1.2
	2	1	1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1

Table 2. ARL profiles of the standard $\overline{X}$ chart

N	δ	N									
		10	25	50	100	500	1000	2500	5000	7500	10000
	0	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
	0.1	352.9	352.9	352.9	352.9	352.9	352.9	352.9	352.9	352.9	352.9
	0.2	308.4	308.4	308.4	308.4	308.4	308.4	308.4	308.4	308.4	308.4
	0.3	253.1	253.1	253.1	253.1	253.1	253.1	253.1	253.1	253.1	253.1
	0.4	200.1	200.1	200.1	200.1	200.1	200.1	200.1	200.1	200.1	200.1
	0.5	155.2	155.2	155.2	155.2	155.2	155.2	155.2	155.2	155.2	155.2
1	0.75	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2
	1	43.9	43.9	43.9	43.9	43.9	43.9	43.9	43.9	43.9	43.9
	1.2	27.8	27.8	27.8	27.8	27.8	27.8	27.8	27.8	27.8	27.8
	1.4	18.2	18.2	18.2	18.2	18.2	18.2	18.2	18.2	18.2	18.2
	1.6	12.4	12.4	12.4	12.4	12.4	12.4	12.4	12.4	12.4	12.4
	1.8	8.7	8.7	8.7	8.7	8.7	8.7	8.7	8.7	8.7	8.7
	2	6.3	6.3	6.3	6.3	6.3	6.3	6.3	6.3	6.3	6.3
	0	683.7	458.7	410.4	389.5	374.1	372.2	371.1	370.8	370.6	370.6
	0.1	607.6	413.9	371.9	353.6	340.1	338.5	337.5	337.2	337.1	337
	0.2	449.5	317.1	287.5	274.6	265	263.9	263.2	263	262.9	262.9
	0.3	304.1	222.7	204.1	195.9	189.8	189.1	188.6	188.5	188.4	188.4
	0.4	200.4	151.9	140.6	135.5	131.8	131.3	131	131	130.9	130.9
	0.5	132.4	103.5	96.6	93.5	91.2	90.9	90.8	90.7	90.7	90.7
2	0.75	50.4	42	39.9	39	38.3	38.2	38.1	38.1	38.1	38.1
	1	21.6	19	18.3	18	17.8	17.8	17.7	17.7	17.7	17.7
	1.2	12	10.9	10.6	10.5	10.4	10.4	10.4	10.4	10.4	10.4
	1.4	7.2	6.7	6.6	6.6	6.5	6.5	6.5	6.5	6.5	6.5
	1.6	4.6	4.4	4.4	4.4	4.3	4.3	4.3	4.3	4.3	4.3
	1.8	3.2	3.1	3.1	3.1	3.1	3.1	3.1	3.1	3.1	3.1
	2	2.3	2.3	2.3	2.3	2.3	2.3	2.3	2.3	2.3	2.3
	0	17545.7	985.2	573.0	455.6	385.4	377.8	373.3	371.9	371.4	371.1
	0.1	9495.2	722.4	441.1	357.6	306.8	301.2	297.9	296.8	296.5	296.3
	0.2	3232.5	375.5	248.5	208.5	183.3	180.5	178.8	178.3	178.1	178
	0.3	1124	185.5	131.5	113.6	102.1	100.8	100.1	99.8	99.7	99.7
	0.4	422.9	94.8	71.3	63.2	57.8	57.2	56.8	56.7	56.7	56.7
	0.5	172.8	51.0	40.3	36.5	34.0	33.7	33.5	33.5	33.4	33.4
5	0.75	26.3	13.6	11.9	11.3	10.9	10.8	10.8	10.8	10.8	10.8
	1	6.5	5	4.7	4.6	4.5	4.5	4.5	4.5	4.5	4.5
	1.2	3	2.7	2.7	2.7	2.7	2.7	2.7	2.7	2.7	2.7
	1.4	1.8	1.8	1.8	1.8	1.8	1.8	1.8	1.8	1.8	1.8
	1.6	1.3	1.4	1.4	1.4	1.4	1.4	1.4	1.4	1.4	1.4
	1.8	1.1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
	2	1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1

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Appendix

Table A1. Values of factor, A_2' for the Modified $\bar{X}$ chart

Sample Size, n	Population size, N												
	10	25	50	100	200	300	400	500	600	700	800	900	1000
2	1.773	1.841	1.861	1.871	1.876	1.877	1.878	1.879	1.879	1.879	1.879	1.880	1.880
3	0.902	0.980	1.002	1.013	1.018	1.020	1.021	1.021	1.021	1.022	1.022	1.022	1.022
4	0.595	0.681	0.706	0.717	0.723	0.725	0.726	0.726	0.727	0.727	0.727	0.727	0.727
5	0.430	0.527	0.553	0.565	0.571	0.573	0.574	0.574	0.575	0.575	0.575	0.576	0.576
6	0.322	0.430	0.458	0.471	0.477	0.479	0.480	0.481	0.481	0.482	0.482	0.482	0.482
7	0.242	0.363	0.393	0.406	0.413	0.415	0.416	0.417	0.417	0.418	0.418	0.418	0.418
8	0.176	0.316	0.345	0.359	0.366	0.368	0.369	0.370	0.370	0.371	0.371	0.371	0.371
9	0.112	0.275	0.308	0.323	0.330	0.332	0.333	0.334	0.334	0.335	0.335	0.335	0.335
10		0.244	0.278	0.294	0.301	0.304	0.305	0.305	0.306	0.306	0.306	0.307	0.307
11		0.218	0.254	0.270	0.278	0.280	0.281	0.282	0.283	0.283	0.283	0.283	0.284
12		0.196	0.234	0.251	0.258	0.261	0.262	0.263	0263	0.264	0.264	0.264	0.264
13		0.176	0.217	0.234	0.242	0.244	0.246	0.246	0.247	0.247	0.248	0.248	0.248
14		0.159	0.202	0.219	0.228	0.230	0.231	0.232	0.233	0.233	0.233	0.234	0.234
15		0.144	0.189	0.207	0.215	0.218	0.219	0.220	0.220	0.221	0.221	0.221	0.222
16		0.130	0.177	0.196	0.204	0.207	0.208	0.209	0.210	0.210	0.210	0.211	0.211
17		0.117	0.166	0.186	0.194	0.197	0.199	0.200	0.200	0.200	0.201	0.201	0.201
18		0.105	0.157	0.177	0.186	0.189	0.190	0.191	0.191	0.192	0.192	0.192	0.193
19		0.093	0.148	0.169	0.178	0.181	0.182	0.183	0.184	0.184	0.184	0.185	0.185
20		0.082	0.141	0.161	0.171	0.174	0.175	0.176	0.177	0.177	0.177	0.178	0.178
21		0.071	0.133	0.155	0.164	0.167	0.169	0.170	0.170	0.171	0.171	0.171	0.172
22		0.059	0.127	0.149	0.158	0.161	0.163	0.164	0.165	0.165	0.165	0.166	0.166
23		0.047	0.120	0.143	0.153	0.156	0.158	0.159	0.159	0.160	0.160	0.160	0.160
24		0.032	0.115	0.138	0.148	0.151	0.153	0.154	0.154	0.155	0.155	0.155	0.155
25			0.109	0.133	0.143	0.146	0.148	0.149	0.150	0.150	0.150	0.151	0.151

Table A2. Values of factor, A_3' for the Modified $\bar{X}$ chart

Sample Size, n	Population size, N												
	10	25	50	100	200	300	400	500	600	700	800	900	1000
2	2.507	2.603	2.631	2.645	2.652	2.654	2.655	2.656	2.656	2.657	2.657	2.657	2.657
3	1.724	1.871	1.914	1.935	1.945	1.948	1.950	1.951	1.951	1.952	1.952	1.952	1.953
4	1.329	1.523	1.578	1.603	1.616	1.620	1.622	1.623	1.624	1.625	1.625	1.625	1.626
5	1.064	1.303	1.368	1.398	1.413	1.418	1.420	1.422	1.423	1.423	1.424	1.424	1.424
6	0.858	1.145	1.220	1.254	1.271	1.276	1.279	1.281	1.282	1.283	1.283	1.284	1.284
7	0.682	1.024	1.107	1.146	1.164	1.170	1.173	1.175	1.176	1.177	1.177	1.178	1.178
8	0.518	0.925	1.018	1.060	1.080	1.086	1.089	1.091	1.093	1.094	1.094	1.095	1.095
9	0.344	0.842	0.944	0.989	1.011	1.018	1.021	1.023	1.025	1.026	1.026	1.027	1.028
10		0.771	0.881	0.930	0.953	0.961	0.964	0.966	0.968	0.969	0.970	0.970	0.971
11		0.708	0.827	0.879	0.904	0.912	0.916	0.918	0.920	0.921	0.922	0.922	0.923
12		0.652	0.780	0.835	0.861	0.869	0.874	0.876	0.878	0.879	0.880	0.880	0.881
13		0.601	0.738	0.796	0.824	0.832	0.837	0.839	0.841	0.842	0.843	0.844	0.844
14		0.553	0.701	0.762	0.790	0.799	0.804	0.807	0.808	0.810	0.811	0.811	0.812
15		0.509	0.666	0.731	0.760	0.770	0.775	0.777	0.779	0.781	0.782	0.782	0.783
16		0.467	0.635	0.702	0.733	0.743	0.748	0.751	0.753	0.754	0.755	0.756	0.757
17		0.427	0.607	0.677	0.709	0.719	0.724	0.727	0.729	0.731	0.732	0.732	0.733
18		0.388	0.580	0.653	0.686	0.697	0.702	0.705	0.707	0.709	0.710	0.711	0.711
19		0.349	0.555	0.631	0.666	0.677	0.682	0.685	0.687	0.689	0.690	0.691	0.692
20		0.310	0.532	0.611	0.646	0.658	0.663	0.667	0.669	0.670	0.672	0.673	0.673
21		0.271	0.510	0.592	0.629	0.640	0.646	0.649	0.652	0.653	0.655	0.655	0.656
22		0.229	0.489	0.575	0.612	0.624	0.630	0.633	0.636	0.637	0.639	0.640	0.640
23		0.183	0.470	0.558	0.597	0.609	0.615	0.619	0.621	0.623	0.624	0.625	0.626
24		0.126	0.451	0.542	0.582	0.595	0.601	0.605	0.607	0.609	0.610	0.611	0.612
25			0.433	0.528	0.569	0.581	0.588	0.592	0.594	0.596	0.597	0.598	0.599