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CONSTRAINT QUALIFICATIONS AND OPTIMALITY CONDITIONS FOR NONCONVEX SEMI-INFINITE AND INFINITE PROGRAMS¹

BORIS S. MORDUKHOVICH² and T. T. A. NGHIA³

Dedicated to Jon Borwein in honor of his 60th birthday

Abstract. The paper concerns the study of new classes of nonlinear and nonconvex optimization problems of the so-called infinite programming that are generally defined on infinite-dimensional spaces of decision variables and contain infinitely many of equality and inequality constraints with arbitrary (may not be compact) index sets. These problems reduce to semi-infinite programs in the case of finite-dimensional spaces of decision variables. We extend the classical Mangasarian-Fromovitz and Farkas-Minkowski constraint qualifications to such infinite and semi-infinite programs. The new qualification conditions are used for efficient computing the appropriate normal cones to sets of feasible solutions for these programs by employing advanced tools of variational analysis and generalized differentiation. In the further development we derive first-order necessary optimality conditions for infinite and semi-infinite programs, which are new in both finite-dimensional and infinite-dimensional settings.

1 Introduction

The paper mainly deals with constrained optimization problems formulated as follows:

$$\begin{cases} \text{minimize } f(x) \text{ subject to} \\ g_t(x) \leq 0 \text{ with } t \in T \text{ and } h(x) = 0, \end{cases} \quad (1.1)$$

where $f : X \rightarrow \overline{\mathbb{R}} := (-\infty, \infty]$ and $g_t : X \rightarrow \overline{\mathbb{R}}$ as $t \in T$ are extended-real-valued functions defined on Banach space X , and where $h : X \rightarrow Y$ is a mapping between Banach spaces. An important feature of problem (1.1) is that the index set T is *arbitrary*, i.e., may be infinite and also noncompact. When the spaces X and Y are finite-dimensional, the constraint system in (1.1) can be formed by finitely many equalities and infinite inequalities. These optimization problems belong to the well-recognized area of *semi-infinite programming* (SIP); see, e.g., the books [13, 14] and the references therein. When the dimension of the decision space X as well as the cardinality of T are infinite, problem (1.1) belongs to the so-called *infinite programming*; cf. the terminology in [1, 9] for linear and convex problems of this type. We also refer the reader to more recent developments [5, 6, 10, 11, 12, 20] concerning linear and convex problems of infinite programming with inequality constraints.

To the best of our knowledge, this paper is the first one in the literature to address *nonlinear* and *nonconvex* problems of infinite programming. Our primary goal in what follows is to find verifiable *constraint qualifications* that allow us to establish efficient *necessary optimality conditions* for local optimal solutions to nonconvex infinite programs of type

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(1.1) under certain differentiability assumptions on the constraint (while not on the cost) functions. In this way we obtain a number of results, which are new not only for infinite programs, but also for SIP problems with noncompact (e.g., countable) index sets.

It has been well recognized in semi-infinite programming that the *Extended Mangasarian-Fromovitz Constraint Qualification* (EMFCQ), first introduced in [18], is particularly useful when the index set T is a compact subset of a finite-dimensional space and when $g(x, t) := g_t(x) \in \mathcal{C}(T)$ for each $x \in X$; see, e.g., [2, 7, 17, 15, 19, 21, 26, 28, 29] for various applications of the EMFCQ in semi-infinite programming. Without the compactness of the index set T and the continuity of the inequality constraint function $g(x, t)$ with respect to the index variable t , problem (1.1) changes dramatically and—as shown below—does not allow us to employ the EMFCQ condition anymore. That motivates us to seek for new qualification conditions, which are more appropriate in applications to infinite programs as well as to SIP problems with noncompact index sets and infinite collections of inequality constraints defined by discontinuous functions.

In this paper we introduce two new qualification conditions, which allow us to deal with infinite and semi-infinite programs of type (1.1) without the convexity/linearity and compactness assumptions discussed above. The first condition, called the *Perturbed Mangasarian-Fromovitz Constraint Qualification* (PMFCQ), turns out to be an appropriate counterpart of the EMFCQ condition for infinite and semi-infinite programs (1.1) with noncompact index sets T and discontinuous functions $g(x, \cdot)$. The second condition, called the *Nonlinear Farkas-Minkowski Constraint Qualification* (NFMFCQ), is a new qualification condition of the *closedness* type, which is generally independent of both EMFCQ and PMFCQ conditions even for countable inequality constraints in finite dimensions.

Our approach is based on advanced tools of variational analysis and generalized differentiation that can be found in [22, 23]. Considerably new ingredients of this approach relate to computing appropriate *normal cones* to the set of *feasible solutions* for the infinite/semi-infinite program (1.1) given by

$$\Omega := \{x \in X \mid h(x) = 0, g_t(x) \leq 0 \text{ as } t \in T\}. \quad (1.2)$$

Since the feasible solution set Ω is generally nonconvex, we need to use some normal cone constructions for nonconvex sets. In this paper we focus on the so-called *Fréchet/regular normal cone* and the *basic/limiting normal cone* introduced by Mordukhovich; see [22] with the references and commentaries therein. Developing general principles of variational analysis, we employ this approach to derive several necessary optimality conditions for the class of nonlinear infinite programs under consideration.

The rest of the paper is organized as follows. In Section 2 we present basic definitions as well as some preliminaries from variational analysis and generalized differentiation widely used in this paper. Section 3 is mainly devoted to the study of the new PMFCQ and NFMFCQ conditions for infinite programs in Banach spaces. Relationships between the new qualification conditions and other well-recognized constraint qualifications for SIP and infinite programs are discussed here.

In Section 4, we provide exact computations for the Fréchet and limiting normal cones to the feasible set of (1.1) under the PMFCQ and NFMFCQ conditions. This part plays a crucial role for the subsequent results of the paper. Following this way, Section 5 concerns the derivation of necessary optimality conditions for local minimizers of the infinite and semi-infinite programs under consideration.

Our notation and terminology are basically standard and conventional in the area of

variational analysis and generalized differentials.; see, e.g., [22, 24]. As usual, $\|\cdot\|$ stands for the norm of Banach space X and $\langle \cdot, \cdot \rangle$ signifies for the canonical pairing between X and its topological dual X^* with the symbol $\xrightarrow{w^*}$ indicating the convergence in the weak* topology of X^* and the symbol cl^* standing for the weak* topological closure of a set. For any $x \in X$ and $r > 0$, denote by $\mathcal{B}_r(x)$ the closed ball centered at x with radius r while \mathcal{B}_X stands for the closed unit ball in X .

Given a set $\Omega \subset X$, the notation $\text{co}\Omega$ signifies the convex hull of Ω while that of $\text{cone}\Omega$ stands for the *convex conic hull* of Ω , i.e., for the convex cone generated by $\Omega \cup \{0\}$. Depending on the context, the symbols $x \xrightarrow{\Omega} \bar{x}$ and $x \xrightarrow{\varphi} \bar{x}$ mean that $x \rightarrow \bar{x}$ with $x \in \Omega$ and $x \rightarrow \bar{x}$ with $\varphi(x) \rightarrow \varphi(\bar{x})$ respectively. Given finally a set-valued mapping $F: X \rightrightarrows X^*$ between X and X^* , recall that the symbol

$$\text{Lim sup}_{x \rightarrow \bar{x}} F(x) := \left\{ x^* \in X^* \mid \exists x_n \rightarrow \bar{x}, \exists x_n^* \xrightarrow{w^*} x^* \text{ with } x_n^* \in F(x_n), n \in \mathbb{N} \right\} \quad (1.3)$$

stands for the *sequential Painlevé-Kuratowski outer/upper limit* of F as $x \rightarrow \bar{x}$ with respect to the norm topology of X and the weak* topology of X^* , where $\mathbb{N} := \{1, 2, \dots\}$.

2 Preliminaries from Generalized Differentiation

In this preliminary section we briefly review some constructions of generalized differentiation used in what follows; see [3, 22, 24, 25] for more details and related material. Throughout this paper, unless otherwise stated, all the spaces under consideration are Banach.

Given an extended-real-valued function $\varphi: X \rightarrow \overline{\mathbb{R}} := (-\infty, \infty]$, we always assume that it is proper, i.e., $\varphi \not\equiv \infty$. The notation

$$\text{dom } \varphi := \{x \in X \mid \varphi(x) < \infty\} \text{ and } \text{epi } \varphi := \{(x, r) \in X \times \mathbb{R} \mid r \geq \varphi(x)\}$$

are used for the domain and the epigraph of φ , respectively,

Define the *analytic ε -subdifferential* of φ at $\bar{x} \in \text{dom } \varphi$ by

$$\widehat{\partial}_\varepsilon \varphi(\bar{x}) := \left\{ x^* \in X^* \mid \liminf_{x \rightarrow \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq -\varepsilon \right\}, \quad \varepsilon \geq 0 \quad (2.1)$$

and let $\widehat{\partial}_\varepsilon \varphi(\bar{x}) := \emptyset$ for $\bar{x} \notin \text{dom } \varphi$. If $\varepsilon = 0$, the construction $\widehat{\partial} \varphi(\bar{x}) := \widehat{\partial}_0 \varphi(\bar{x})$ in (2.1) is known as the *Fréchet* or *regular subdifferential* of φ at \bar{x} ; it reduces in the convex case to the classical subdifferential of convex analysis. The sequential regularization of (2.1) defined via the outer limit (1.3) by

$$\partial \varphi(\bar{x}) := \text{Lim sup}_{\substack{x \xrightarrow{\varphi} \bar{x} \\ \varepsilon \downarrow 0}} \widehat{\partial}_\varepsilon \varphi(x), \quad (2.2)$$

is known as the *limiting*, or *basic*, or *Mordukhovich subdifferential* of φ at $\bar{x} \in \text{dom } \varphi$. It can be equivalently described with $\varepsilon = 0$ in (2.2) if φ is lower semicontinuous (l.s.c.) around \bar{x} and if X is an Asplund space, i.e., each of its separable subspace has a separable dual (in particular, any reflexive space is Asplund; see, e.g., [3, 22] for more details and references). We have $\partial \varphi(\bar{x}) \neq \emptyset$ for every locally Lipschitzian function on an Asplund space.

A complementary construction to (2.2), known as the *singular* or *horizontal subdifferential* of φ at \bar{x} , is defined by

$$\partial^\infty \varphi(\bar{x}) := \text{Lim sup}_{\substack{x \xrightarrow{\varphi} \bar{x} \\ \lambda, \varepsilon \downarrow 0}} \lambda \widehat{\partial}_\varepsilon \varphi(x), \quad (2.3)$$

where we can equivalently put $\varepsilon = 0$ if φ is l.s.c. around \bar{x} and X is Asplund. Note that $\partial^\infty\varphi(\bar{x}) = \{0\}$ if φ is locally Lipschitzian around \bar{x} . The converse implication also holds provided that φ is l.s.c. around \bar{x} , that X is Asplund, and that φ satisfies the so-called “sequential normal epi-compactness” property at \bar{x} (see below), which is always the case when X is finite-dimensional.

Given a set $\Omega \subset X$ with its indicator function $\delta(\cdot; \Omega)$ defined by $\delta(x; \Omega) := 0$ for $x \in \Omega$ and by $\delta(x; \Omega) := \infty$ otherwise, we construct the *Fréchet/regular normal cone* and *limiting/basic/Mordukhovich normal cone* to Ω at $\bar{x} \in \Omega$ by, respectively,

$$\widehat{N}(\bar{x}; \Omega) := \widehat{\partial}\delta(\bar{x}; \Omega) \quad \text{and} \quad N(\bar{x}; \Omega) := \partial\delta(\bar{x}; \Omega) \quad (2.4)$$

via the corresponding subdifferential of the indicator function. It follows from (2.4) that $\widehat{N}(\bar{x}; \Omega) \subset N(\bar{x}; \Omega)$. A set Ω is *normally regular* at \bar{x} if $\widehat{N}(\bar{x}; \Omega) = N(\bar{x}; \Omega)$; the latter is the case of convex and some other “nice” sets.

Recall further that Ω is *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for any sequences $\varepsilon_n \downarrow 0$, $x_n \xrightarrow{\Omega} \bar{x}$, and $x_n^* \in \widehat{N}_{\varepsilon_n}(x_n; \Omega) := \widehat{\partial}_{\varepsilon_n}\delta(\bar{x}; \Omega)$ we have

$$[x_n^* \xrightarrow{w^*} 0] \implies [\|x_n^*\| \rightarrow 0] \quad \text{as } n \rightarrow \infty,$$

where ε_n can be omitted if Ω is locally closed around \bar{x} and the space X is Asplund. A function $\varphi : X \rightarrow \overline{\mathbb{R}}$ is *sequentially normally epi-compact* (SNEC) at a point $\bar{x} \in \text{dom } \varphi$ if its epigraph is SNC at $(\bar{x}, \varphi(\bar{x}))$. Besides the finite dimensionality, the latter properties hold under certain Lipschitzian behavior; see, e.g., [22, Subsections 1.1.4 and 1.2.5].

Having an arbitrary (possibly infinite and noncompact) index set T as in (1.1), we consider the product space of multipliers $\mathbb{R}^T := \{\lambda = (\lambda_t) \mid t \in T\}$ with $\lambda_t \in \mathbb{R}$ for $t \in T$ and denote by $\widetilde{\mathbb{R}}^T$ the collection of $\lambda \in \mathbb{R}^T$ such that $\lambda_t \neq 0$ for finitely many $t \in T$. The *positive cone* in $\widetilde{\mathbb{R}}^T$ is defined by

$$\widetilde{\mathbb{R}}_+^T := \{\lambda \in \widetilde{\mathbb{R}}^T \mid \lambda_t \geq 0 \text{ for all } t \in T\}. \quad (2.5)$$

3 Qualification Conditions for Infinite Constraint Systems

This section is devoted to studying the set of *feasible solutions* to the original optimization problem (1.1) defined by the infinite constraint systems of inequalities and equalities

$$\begin{cases} g_t(x) \leq 0, & t \in T, \\ h(x) = 0, \end{cases} \quad (3.1)$$

where T is an arbitrary index set, and where the functions $g_t : X \rightarrow \overline{\mathbb{R}}$, $t \in T$, and the mapping $h : X \rightarrow Y$ are differentiable but may not be linear and/or convex. As in (1.2), the set of feasible solutions to (1.1), i.e., those $x \in X$ satisfying (3.1), is denoted by Ω .

Our *standing assumptions* throughout the paper (unless otherwise stated) are as follows:

(SA) For any $\bar{x} \in \Omega$ the functions g_t , $t \in T$, are Fréchet differentiable at \bar{x} and the mapping h is strictly differentiable at \bar{x} . The set $\{\nabla g_t(\bar{x}) \mid t \in T\}$ is bounded in X^* .

Recall that a mapping $h : X \rightarrow Y$ is *strictly differentiable* at \bar{x} with the (strict) derivative $\nabla h(\bar{x}) : X \rightarrow Y$ if

$$\lim_{x, x' \rightarrow \bar{x}} \frac{h(x) - h(x') - \nabla h(\bar{x})(x - x')}{\|x - x'\|} = 0.$$

The latter holds automatically when h is continuously differentiable around \bar{x} .

In addition to the standing assumptions (SA), we often impose some stronger requirements on the inequality constraint functions g_t that postulate a certain uniformity of their behavior with respect to the index parameter $t \in T$. We say that the functions $\{g_t\}_{t \in T}$ are *uniformly Fréchet differentiable* at \bar{x} if

$$s(\eta) := \sup_{t \in T} \sup_{\substack{x \in B_\eta(\bar{x}) \\ x \neq \bar{x}}} \frac{|g_t(x) - g_t(\bar{x}) - \langle \nabla g_t(\bar{x}), x - \bar{x} \rangle|}{\|x - \bar{x}\|} \rightarrow 0 \text{ as } \eta \downarrow 0. \quad (3.2)$$

Similarly, the functions $\{g_t\}_{t \in T}$ are *uniformly strictly differentiable* at \bar{x} if condition (3.2) above is replaced by a stronger one:

$$r(\eta) := \sup_{t \in T} \sup_{\substack{x, x' \in B_\eta(\bar{x}) \\ x \neq x'}} \frac{|g_t(x) - g_t(x') - \langle \nabla g_t(\bar{x}), x - x' \rangle|}{\|x - x'\|} \rightarrow 0 \text{ as } \eta \downarrow 0, \quad (3.3)$$

which clearly implies the strict differentiability of each function g_t , $t \in T$, at \bar{x} .

Let us present some sufficient conditions ensuring the fulfillment of all the assumptions formulated above for infinite families of inequality constraint functions.

Proposition 3.1 (compact index sets). *Let T be a compact metric space, let the functions g_t in (3.1) be Fréchet differentiable around \bar{x} for each $t \in T$, and let the mapping $(x, t) \in X \times T \mapsto \nabla g_t(x) \in X^*$ be continuous on $B_\eta(\bar{x}) \times T$ for some $\eta > 0$. Then the standing assumptions (SA) as well as (3.2) and (3.3) are satisfied.*

Proof. It is easy to see that our standing assumptions (SA) hold, since $\|\nabla g_t(\bar{x})\|$ is assumed to be continuous on the compact space T being hence bounded. It suffices to prove that (3.3) holds, which surely implies (3.2).

Arguing by contradiction, suppose that (3.3) fails. Then there are $\varepsilon > 0$, sequences $\{t_n\} \subset T$, $\{\eta_n\} \downarrow 0$, and $\{x_n\}, \{x'_n\} \subset B_{\eta_n}(\bar{x})$ such that

$$\frac{|g_{t_n}(x_n) - g_{t_n}(x'_n) - \langle \nabla g_{t_n}(\bar{x}), x_n - x'_n \rangle|}{\|x_n - x'_n\|} \geq \varepsilon - \frac{1}{n} \text{ for all large } n \in \mathbb{N}. \quad (3.4)$$

Since T is a compact metric space, there is a subsequence of $\{t_n\}$ converging (without relabeling) to some $\bar{t} \in T$. Applying the classical Mean Value Theorem to (3.4), we find $\theta_n \in [x_n, x'_n] := \text{co}\{x_n, x'_n\}$ such that

$$\begin{aligned} \frac{\varepsilon}{2} &< \frac{|\langle \nabla g_{t_n}(\theta_n), x_n - x'_n \rangle - \langle \nabla g_{t_n}(\bar{x}), x_n - x'_n \rangle|}{\|x_n - x'_n\|} \leq \|\nabla g_{t_n}(\theta_n) - \nabla g_{t_n}(\bar{x})\| \\ &< \|\nabla g_{t_n}(\theta_n) - \nabla g_{\bar{t}}(\bar{x})\| + \|\nabla g_{\bar{t}}(\bar{x}) - \nabla g_{t_n}(\bar{x})\| \end{aligned}$$

for all large $n \in \mathbb{N}$. This contradicts the continuity of the mapping $(x, t) \in X \times T \mapsto \nabla g_t(x)$ on $B_\eta(\bar{x}) \times T$ and thus completes the proof of the proposition. \triangle

Next we recall a well-recognized constraint qualification condition, which is often used in problems of nonlinear and nonconvex semi-infinite programming.

Definition 3.2 (Extended Mangasarian-Fromovitz Constraint Qualification). *The infinite system (3.1) satisfies the EXTENDED MANGASARIAN-FROMOVITZ CONSTRAINT*

QUALIFICATION (EMFCQ) at $\bar{x} \in \Omega$ if the derivative operator $\nabla h(\bar{x}): X \rightarrow Y$ is surjective and if there is $\tilde{x} \in X$ such that $\nabla h(\bar{x})\tilde{x} = 0$ and that

$$\langle \nabla g_t(\bar{x}), \tilde{x} \rangle < 0 \text{ for all } t \in T(\bar{x}) := \{t \in T \mid g_t(\bar{x}) = 0\}. \quad (3.5)$$

It is clear that in the case of a finite index set T and a finite-dimensional space Y in (3.1) the EMFCQ condition reduced to the classical Mangasarian-Fromovitz Constraint Qualification (MFCQ) in nonlinear programming. In the case of SIP problems the EMFCQ was first introduced in [18] and then extensively studied and applied in semi-infinite frameworks with $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$; see, e.g., [15, 19, 21, 27], where the reader can find its relationships with other constraint qualifications for SIP problems.

To the best of our knowledge, the vast majority of nonconvex semi-infinite programs are usually considered with the general assumptions that the index set T is compact, the functions g_t are continuously differentiable, and the mapping $(x, t) \mapsto \nabla g_t(x)$ is continuous on $X \times T$. Under these assumptions and the EMFCQ formulated above, several authors derive the Karush-Kuhn-Tucker (KKT) necessary optimality conditions of the following type: If \bar{x} is an optimal solution to (1.1) with $f \in C^1$ and $h = (h_1, h_2, \dots, h_n)$, then there are $\lambda \in \widetilde{\mathbb{R}}_+^T$ from (2.5) and $\mu \in \mathbb{R}^n$ such that

$$0 = \nabla f(\bar{x}) + \sum_{t \in T(\bar{x})} \lambda_t \nabla g_t(\bar{x}) + \sum_{j=1}^n \mu_j \nabla h_j(\bar{x}). \quad (3.6)$$

We are not familiar with any results in the literature on nonconvex infinite programming that apply to problems with noncompact index sets T . The following example shows that the KKT optimality conditions in form (3.6) may fail for nonconvex SIP with countable constraints even under the fulfillment of the EMFCQ.

Example 3.3 (violation of KKT for nonconvex SIP with countable sets under EMFCQ). Consider problem (1.1) with countable inequality constraints given by

$$\begin{cases} \text{minimize } (x_1 + 1)^2 + x_2 & \text{subject to} \\ x_1 + 1 \leq 0 \text{ and } \frac{1}{3n}x_1^3 - x_2 \leq 0 & \text{for all } n \in \mathbb{N} \setminus \{1\} \text{ with } (x_1, x_2) \in \mathbb{R}^2. \end{cases} \quad (3.7)$$

Let $X := \mathbb{R}^2$, $Y := \{0\}$, $f(x_1, x_2) := (x_1 + 1)^2 + x_2$, $T := \mathbb{N}$, $g_1(x_1, x_2) := x_1 + 1$, and $g_n(x_1, x_2) := \frac{1}{3n}x_1^3 - x_2$ for all $n \in \mathbb{N} \setminus \{1\}$. Observe that $\bar{x} := (-1, 0)$ is a global minimizer for problem (3.7) and that $T(\bar{x}) = \{1\}$ for the active index set in (3.5). It is easy to check that the EMFCQ holds at \bar{x} while there is no Lagrange multiplier $\lambda \in \mathbb{R}_+$ satisfying the KKT optimality condition (3.6) at \bar{x} . Indeed, we have $\langle \nabla g_1(\bar{x}), (-1, 0) \rangle = -1 < 0$, and the following equation does not admit any solution for $\lambda \geq 0$:

$$(0, 0) = \nabla f(\bar{x}) + \lambda \nabla g_1(\bar{x}) = (0, 1) + (\lambda, 0).$$

Now we introduce a new extension of the MFCQ condition to the infinite programs under consideration, which plays a crucial role throughout the paper.

Definition 3.4 (Perturbed Mangasarian-Fromovitz Constraint Qualification). We say that the infinite system (3.1) satisfies the PERTURBED MANGASARIAN-FROMOVITZ

CONSTRAINT QUALIFICATION (PMFCQ) at $\bar{x} \in \Omega$ if the derivative operator $\nabla h(\bar{x}): X \rightarrow Y$ is surjective and if there is $\tilde{x} \in X$ such that $\nabla h(\bar{x})\tilde{x} = 0$ and that

$$\inf_{\varepsilon > 0} \sup_{t \in T_\varepsilon(\bar{x})} \langle \nabla g_t(\bar{x}), \tilde{x} \rangle < 0 \quad \text{with } T_\varepsilon(\bar{x}) := \{t \in T \mid g_t(\bar{x}) \geq -\varepsilon\}. \quad (3.8)$$

In contrast to the EMFCQ, the active index set in (3.8) is *perturbed* by a small $\varepsilon > 0$. Since $T(\bar{x}) \subset T_\varepsilon(\bar{x})$ for all $\varepsilon > 0$, the PMFCQ is stronger than the EMFCQ. However, as shown in Section 4 and Section 5, the new condition is much more appropriate for applications to semi-infinite and infinite programs with general (including compact) index sets than the EMFCQ.

The following proposition reveals some assumptions on the initial data of (3.1) ensuring the equivalence between the PMFCQ and EMFCQ.

Proposition 3.5 (PMFCQ from EMFCQ). *Let T be a compact metric space, and let $\bar{x} \in \Omega$ in (3.1). Assume that the function $t \in T \mapsto g_t(\bar{x})$ is upper semicontinuous (u.s.c.) on T , that the derivative mapping $\nabla h(\bar{x}): X \rightarrow Y$ is surjective, and that there is $\tilde{x} \in X$ with the following properties: $\nabla h(\bar{x})\tilde{x} = 0$, the function $t \in T \mapsto \langle \nabla g_t(\bar{x}), \tilde{x} \rangle$ is u.s.c., and $\langle \nabla g_t(\bar{x}), \tilde{x} \rangle < 0$ for all $t \in T(\bar{x})$. Then the PMFCQ condition holds at \bar{x} , being thus equivalent to the EMFCQ condition at this point.*

Proof. Arguing by contradiction, suppose that the PMFCQ fails at \bar{x} . Then it follows from (3.8) that there exist sequences $\{\varepsilon_n\} \downarrow 0$ and $\{t_n\} \subset T$ such that $t_n \in T_{\varepsilon_n}(\bar{x})$ and

$$\langle \nabla g_{t_n}(\bar{x}), \tilde{x} \rangle \geq -\frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

Since T is a compact metric space, we find a subsequence of $\{t_n\}$ (no relabeling), which converges to some $\bar{t} \in T$. Observe from the continuity assumptions made imply that

$$\begin{aligned} g_{\bar{t}}(\bar{x}) &\geq \limsup_{n \rightarrow \infty} g_{t_n}(\bar{x}) \geq \limsup_{n \rightarrow \infty} -\varepsilon_n = 0 \quad \text{and} \\ \langle \nabla g_{\bar{t}}(\bar{x}), \tilde{x} \rangle &\geq \limsup_{n \rightarrow \infty} \langle \nabla g_{t_n}(\bar{x}), \tilde{x} \rangle \geq \limsup_{n \rightarrow \infty} -\frac{1}{n} = 0. \end{aligned}$$

Thus we have that $\bar{t} \in T(\bar{x})$ and $\langle \nabla g_{\bar{t}}(\bar{x}), \tilde{x} \rangle \geq 0$, which is a contradiction that completes the proof of the proposition. \triangle

The following example shows that the EMFCQ does not imply the PMFCQ (while not ensuring in this case the validity of the required necessary optimality conditions as will be seen in Sections 4 and 5) even for simple frameworks of nonconvex semi-infinite programs with *compact* index sets.

Example 3.6 (EMFCQ does not imply PMFCQ for semi-infinite programs with compact index sets). Let $X = \mathbb{R}^2$ and $T = [0, 1]$ in (3.1) with $h = 0$ and

$$g_0(x) := x_1 + 1 \leq 0, \quad g_t(x) := tx_1 - x_2^3 \leq 0 \quad \text{for } t \in T \setminus \{0\}.$$

It is easy to check that the functions g_t , $t \in T$, satisfy our standing assumptions and that they are strictly uniformly differentiable at the feasible point $\bar{x} = (-1, 0)$. Observe

furthermore that $T(\bar{x}) = \{0\}$, that $T_\varepsilon(\bar{x}) = [0, \varepsilon]$ for all $\varepsilon \in (0, 1)$, and that the EMFCQ holds at \bar{x} . However, for any $d = (d_1, d_2) \in \mathbb{R}^2$ we have

$$\begin{aligned} \inf_{\varepsilon > 0} \sup_{t \in T_\varepsilon(\bar{x})} \langle \nabla g_t(\bar{x}), d \rangle &= \inf_{\varepsilon > 0} \sup \left\{ \langle \nabla g_0(\bar{x}), d \rangle, \sup \{ \langle \nabla g_t(\bar{x}), d \rangle \mid t \in (0, \varepsilon] \} \right\} \\ &= \inf_{\varepsilon > 0} \sup \left\{ d_1, \sup \{ t d_1 \mid t \in (0, \varepsilon] \} \right\} \geq 0, \end{aligned}$$

which shows that the PMFCQ does not satisfy at \bar{x} . Note that the u.s.c. assumption with respect of t in Propositions 3.5 does not hold in this example.

It is well known in the classical nonlinear programming (when the index set T in (3.1) is finite), that the MFCQ condition is equivalent to the Slater condition provided that all the functions g_t are convex and differentiable and that h is a linear operator. The next proposition shows that a similar equivalence holds in the semi-infinite and infinite programming frameworks with replacing the MFCQ by our new PMFCQ condition and replacing the Slater by its strong counterpart well recognized in the SIP community; see, e.g., [13] and [5] for more references and discussions.

Proposition 3.7 (equivalence between PMFCQ and SSC for differentiable convex systems). *Assume that in (3.1) all the functions g_t , $t \in T$, are convex and uniformly Fréchet differentiable at \bar{x} and that $h = A$ is a surjective continuous linear operator. Then the PMFCQ condition is equivalent to the following strong Slater condition (SSC): there is $\hat{x} \in X$ such that $A\hat{x} = 0$ and*

$$\sup_{t \in T} g_t(\hat{x}) < 0. \quad (3.9)$$

Proof. Suppose first that the SSC holds at \bar{x} , i.e., there are $\hat{x} \in X$ and $\delta > 0$ such that $A\hat{x} = 0$ and $g_t(\hat{x}) < -2\delta$ for all $t \in T$. By the assumptions made this implies that for each $\varepsilon \in (0, \delta)$ and $t \in T_\varepsilon(\bar{x})$ we have

$$\langle \nabla g_t(\bar{x}), \hat{x} - \bar{x} \rangle \leq g_t(\hat{x}) - g_t(\bar{x}) \leq -2\delta + \varepsilon \leq -\delta.$$

Define further $\tilde{x} := \hat{x} - \bar{x}$ and get $A\tilde{x} = A\hat{x} - A\bar{x} = 0$ with $\langle \nabla g_t(\bar{x}), \tilde{x} \rangle \leq -\delta$ for all $t \in T_\varepsilon(\bar{x})$ and $\varepsilon \in (0, \delta)$. This clearly implies the PMFCQ condition at \bar{x} .

Conversely, assume that the PMFCQ condition holds at \bar{x} . Then there are $\varepsilon, \eta > 0$ and $\tilde{x} \in X$ such that $\langle \nabla g_t(\bar{x}), \tilde{x} \rangle \leq -\eta$ for all $t \in T_\varepsilon(\bar{x})$ and that $A\tilde{x} = 0$. It follows from the assumed uniform Fréchet differentiability (3.2) of g_t at \bar{x} that for each $\lambda > 0$ we have

$$g_t(\bar{x} + \lambda\tilde{x}) \leq g_t(\bar{x}) + \lambda \langle \nabla g_t(\bar{x}), \tilde{x} \rangle + \lambda \|\tilde{x}\| s(\lambda \|\tilde{x}\|), \quad (3.10)$$

which readily implies that $g_t(\bar{x} + \lambda\tilde{x}) \leq \lambda(-\eta + \|\tilde{x}\| s(\lambda \|\tilde{x}\|))$ for all $t \in T_\varepsilon(\bar{x})$. For $t \notin T_\varepsilon(\bar{x})$ we observe from (3.10) that

$$g_t(\bar{x} + \lambda\tilde{x}) \leq -\varepsilon + \lambda \sup_{\tau \in T} \|\nabla g_\tau(\bar{x})\| \cdot \|\tilde{x}\| + \lambda \|\tilde{x}\| s(\lambda \|\tilde{x}\|),$$

which gives, combining with the above, that

$$\sup_{t \in T} g_t(\bar{x} + \lambda\tilde{x}) \leq \max \left\{ \lambda(-\eta + \|\tilde{x}\| s(\lambda \|\tilde{x}\|)), -\varepsilon + \lambda \|\tilde{x}\| \left(\sup_{\tau \in T} \|\nabla g_\tau(\bar{x})\| + s(\lambda \|\tilde{x}\|) \right) \right\}.$$

The latter implies the existence of $\lambda_0 > 0$ sufficiently small such that $\sup_{t \in T} g_t(\hat{x}) < 0$ with $\hat{x} := \bar{x} + \lambda_0 \tilde{x}$. Furthermore, it is easy to see that $A\hat{x} = A\bar{x} + \lambda_0 A\tilde{x} = 0$. This concludes that the SSC holds at \hat{x} and thus completes the proof of the proposition. \triangle

Next we introduce another qualification condition of the *closedness/Farkas-Minkowski type* for infinite inequality constraints in (1.1).

Definition 3.8 (Nonlinear Farkas-Minkowski Constraint Qualification). *We say that system (3.1) with $h(x) = 0$ satisfies the NONLINEAR FARKAS-MINKOWSKI CONSTRAINT QUALIFICATION (NFMCCQ) at \bar{x} if the set*

$$\text{cone} \{ \langle \nabla g_t(\bar{x}), \langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x}) \rangle \mid t \in T \} \quad (3.11)$$

is weak* closed in the product space $X^* \times \mathbb{R}$.

In the linear case of $g_t(x) = \langle a_t^*, x \rangle - b_t$ for some $(a_t^*, b_t) \in X^* \times \mathbb{R}$, $t \in T$, the NFMCCQ condition above reduces to the classical Farkas-Minkowski qualification condition meaning that the set $\text{cone} \{ (a_t^*, b_t) \mid t \in T \}$ is weak* closed in $X^* \times \mathbb{R}$. It is well recognized that the latter condition plays an important role in linear semi-infinite and infinite optimization; see, e.g., [4, 6, 8, 10, 11, 13] for more details and references. Observe that the NFMCCQ condition can be represented in the following equivalent form: the set

$$\text{cone} \{ (\nabla g_t(\bar{x}), g_t(\bar{x})) \mid t \in T \} \text{ is weak* closed in } X^* \times \mathbb{R}.$$

Let us compare the new NFMCCQ condition with the other qualification conditions discussed in this section in the case of infinite inequality constraints.

Proposition 3.9 (sufficient conditions for NFMCCQ). *Consider the constraint inequality system (3.1) with $h = 0$ therein. Then the NFMCCQ condition is satisfied at $\bar{x} \in \Omega$ in each of the following settings:*

(i) *The index T is finite and the MFCQ condition holds at \bar{x} .*

(ii) *$\dim X < \infty$, the set $\{ (\nabla g_t(\bar{x}), \langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})) \mid t \in T \}$ is compact, and the PMFCQ condition holds at \bar{x} .*

(iii) *The index T is a compact metric space, $\dim X < \infty$, the mappings $t \in T \mapsto g_t(\bar{x})$ and $t \in T \mapsto \nabla g_t(\bar{x})$ are continuous, and the EMFCQ condition holds at \bar{x} .*

Proof. Define $\tilde{g}_t(x) := \langle \nabla g_t(\bar{x}), x - \bar{x} \rangle + g_t(\bar{x})$ for all $x \in X$. To justify (i), suppose that T is finite and that the MFCQ condition holds at \bar{x} for the inequality system in (3.1). It is clear that \tilde{g}_t also satisfy the MFCQ at \bar{x} . Since the functions \tilde{g}_t are linear, we observe from Proposition 3.7 that there is $\hat{x} \in X$ such that $\tilde{g}_t(\hat{x}) = \langle \nabla g_t(\bar{x}), \hat{x} - \bar{x} \rangle + g_t(\bar{x}) < 0$ for all $t \in T$. Thus it follows from [10, Proposition 6.1] that the NFMCCQ condition holds.

Next we consider case (ii) with $X = \mathbb{R}^d$ therein. Suppose that the PMFCQ condition holds at \bar{x} and that the set $\{ (\nabla g_t(\bar{x}), \langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})) \mid t \in T \}$ is compact in $\mathbb{R}^d \times \mathbb{R}$. Noting that the functions \tilde{g}_t also satisfy the PMFCQ at \bar{x} , we apply Proposition 3.7 to these functions and find $\hat{x} \in X$ such that $\nabla h(\bar{x})\hat{x} = 0$ and that

$$\sup_{t \in T} \tilde{g}_t(\hat{x}) = \sup_{t \in T} \langle \nabla g_t(\bar{x}), \hat{x} - \bar{x} \rangle + g_t(\bar{x}) < 0. \quad (3.12)$$

Let us check that $(0, 0) \notin \text{co} \{ (\nabla g_t(\bar{x}), \langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})) \mid t \in T \}$. Indeed, otherwise ensures the existence of $\lambda \in \tilde{\mathbb{R}}_+^T$ with $\sum_{t \in T} \lambda_t = 1$ such that

$$(0, 0) = \sum_{t \in T} \lambda_t (\nabla g_t(\bar{x}), \langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})).$$

Combining the latter with (3.12) gives us that

$$0 = \sum_{t \in T} \lambda_t \langle \nabla g_t(\bar{x}), \hat{x} \rangle - \sum_{t \in T} \lambda_t (\langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})) = \sum_{t \in T} \lambda_t \tilde{g}_t(\hat{x}) \leq \sup_{t \in T} \tilde{g}_t(\hat{x}) < 0,$$

which is a contradiction. Hence employing [16, Theorem 1.4.7] in this setting, we have that the conic hull cone $\{(\nabla g_t(\bar{x}), \langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})) \mid t \in T\}$ is closed in \mathbb{R}^{d+1} . This fully justifies (ii). Observing finally that (iii) follows from (ii) and Proposition 3.5, we complete the proof of the proposition. \triangle

To conclude this section, let us show that the NFMFCQ and PMFCQ conditions are independent for infinite inequality systems in finite dimensions.

Example 3.10 (independence of NFMFCQ and PMFCQ). It is easy to check that for the constraint inequality system from Example 3.6 the NFMFCQ is satisfied at $\bar{x} = (-1, 0)$, since the corresponding conic hull

$$\begin{aligned} \text{cone} \{(\nabla g_t(\bar{x}), \langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})) \mid t \in T\} &= \text{cone} \left((1, 0, -1) \cup \{(t, 0, 0) \mid t \in (0, 1]\} \right) \\ &= \{x \in \mathbb{R}^3 \mid x_1 + x_3 \geq 0, x_1 \geq 0 \geq x_3, x_2 = 0\} \end{aligned}$$

is closed in \mathbb{R}^3 . On the other hand, Example 3.6 demonstrates that the PMFCQ does not hold for this system at \bar{x} .

To show that the NFMFCQ does not generally follow from the PMFCQ (and even from the EMFCQ), consider the countable system of inequality constraints (3.7) in \mathbb{R}^2 discussed in Example 3.3. When $\bar{x} = (-1, 0)$, we get $T_\varepsilon(\bar{x}) = \{n \in \mathbb{N} \setminus \{1\} \mid n \leq \frac{1}{\varepsilon}\} \cup \{1\}$ for the the perturbed active index set in (3.8). It shows that the PMFCQ (and hence the EMFCQ) hold at \bar{x} . On the other hand, the conic hull

$$\text{cone} \{(\nabla g_t(\bar{x}), \langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})) \mid t \in T\} = \text{cone} \left[(1, 0, -1) \cup \left\{ \left(\frac{1}{n}, -1, \frac{-2}{3n} \right) \mid n \in \mathbb{N} \setminus \{1\} \right\} \right]$$

is not closed in \mathbb{R}^3 , i.e., the NFMFCQ condition is not satisfied at \bar{x} .

4 Normal Cones to Feasible Sets of Infinite Constraints

This section is devoted to computing both normal cones (2.4) to the feasible solution sets (1.2) for the class of nonconvex semi-infinite/infinite programs (1.1) under consideration in the paper. These calculus results are certainly of independent interest while they play a crucial role in deriving necessary optimality conditions for (1.1) in Section 5.

The first main theorem gives precise calculations of both Fréchet and limiting normal cones to the set Ω of feasible solutions in (1.2) under the new Perturbed Mangasarian-Fromovitz Constraint Qualification of Definition 3.4. Preliminary we present a known result from functional analysis whose simple proof is given for the reader's convenience.

Lemma 4.1 (weak* closed images of adjoint operators). *Let $A : X \rightarrow Y$ be a surjective continuous linear operator. Then the image of its adjoint operator $A^*(Y^*)$ is a weak* closed subspace of X^* .*

Proof. Define $C := A^*(Y^*) \subset X^*$ and pick any $n \in \mathbb{N}$. We claim that the set $A_n := C \cap nB_{X^*}$ is weak* closed in X^* . Considering a net $\{x_\nu^*\}_{\nu \in \mathcal{N}} \subset A_n$ weak* converging to $x^* \in X^*$ and taking into account that the ball B_{X^*} is weak* compact in X^* , we get $x^* \in nB_{X^*}$. By construction there is a net $\{y_\nu^*\}_{\nu \in \mathcal{N}} \subset Y^*$ satisfying $x_\nu^* = A^*y_\nu^*$ whenever $\nu \in \mathcal{N}$. It follows from the surjectivity of A that

$$\|x_\nu^*\| = \|A^*y_\nu^*\| \geq \kappa \|y_\nu^*\| \text{ for all } \nu \in \mathcal{N},$$

where $\kappa := \inf\{\|A^*y^*\| \text{ over } \|y^*\| = 1\} \in (0, \infty)$; see, e.g., [22, Lemma 1.18]. Hence $\|y_\nu^*\| \leq n\kappa^{-1}$ for all $\nu \in \mathcal{N}$. By passing to a subnet, suppose that y_ν^* weak* converges to some $y^* \in Y^*$ for which $x^* = A^*y^* \in A_n$. Thus we have that the set $A_n = C \cap nB_{X^*}$ is weak* closed for all $n \in \mathbb{N}$. The classical Banach-Dieudonné-Krein-Šmulian theorem yields therefore that the set C is weak* closed in X^* . \triangle

Now we are ready to establish the main result of this section.

Theorem 4.2 (Fréchet and limiting normals to infinite constraint systems). *Let $\bar{x} \in \Omega$ for the set of feasible solutions (1.2) to the infinite system (3.1) satisfying the PMFCQ at \bar{x} . Assume in addition that the inequality constraint functions g_t , $t \in T$, are uniformly Fréchet differentiable at \bar{x} . Then the Fréchet normal cone to Ω at \bar{x} is computed by*

$$\hat{N}(\bar{x}; \Omega) = \bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \} + \nabla h(\bar{x})^*(Y^*). \quad (4.1)$$

If furthermore the functions g_t , $t \in T$, are uniformly strictly differentiable at \bar{x} , then the limiting normal cone to Ω at \bar{x} is also computed by

$$N(\bar{x}; \Omega) = \bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \} + \nabla h(\bar{x})^*(Y^*), \quad (4.2)$$

and thus the set Ω of feasible solutions is normally regular at \bar{x} .

Proof. First we justify (4.1) under the assumptions made. It follows from the PMFCQ and the uniform Fréchet differentiability of g_t at \bar{x} that there are $\tilde{\varepsilon} > 0$, $\delta > 0$, and $\tilde{x} \in X$ such that $\nabla h(\bar{x})\tilde{x} = 0$ and

$$\sup_{t \in T_\varepsilon(\bar{x})} \langle \nabla g_t(\bar{x}), \tilde{x} \rangle < -\delta \text{ for all } \varepsilon \leq \tilde{\varepsilon}. \quad (4.3)$$

Let us prove the inclusion “ \supset ” in (4.1). To proceed, fix any $\varepsilon \in (0, \tilde{\varepsilon})$ and pick an arbitrary element x^* belonging to the right-hand side of (4.1). Then there exist a net $(\lambda_\nu)_{\nu \in \mathcal{N}} \subset \tilde{\mathbb{R}}_+^T$ and a dual element $y^* \in Y^*$ satisfying

$$x^* = w^* - \lim_\nu \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{t\nu} \nabla g_t(\bar{x}) + \nabla h(\bar{x})^* y^*. \quad (4.4)$$

Combining the latter with (4.3) gives us

$$\begin{aligned} \langle x^*, \tilde{x} \rangle &= \lim_\nu \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{t\nu} \langle \nabla g_t(\bar{x}), \tilde{x} \rangle + \langle \nabla h(\bar{x})^* y^*, \tilde{x} \rangle \\ &\leq \lim_\nu \inf \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{t\nu} (-\delta) + \langle y^*, \nabla h(\bar{x})\tilde{x} \rangle = -\delta \lim_\nu \sup \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{t\nu}. \end{aligned} \quad (4.5)$$

It follows further that for each $\eta > 0$ and $x \in \Omega \cap B_\eta(\bar{x})$ we have

$$\begin{aligned}
\langle x^*, x - \bar{x} \rangle &= \lim_{\nu} \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{t\nu} \langle \nabla g_t(\bar{x}), x - \bar{x} \rangle + \langle \nabla h(\bar{x})^* y^*, x - \bar{x} \rangle \\
&\leq \limsup_{\nu} \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{t\nu} \left(g_t(x) - g_t(\bar{x}) + \|x - \bar{x}\| s(\eta) \right) + \langle y^*, \nabla h(\bar{x})(x - \bar{x}) \rangle \\
&\leq \limsup_{\nu} \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{t\nu} \left(\varepsilon + \|x - \bar{x}\| s(\eta) \right) + \|y^*\| \left(\|h(x) - h(\bar{x})\| + o(\|x - \bar{x}\|) \right) \\
&\leq \left(\varepsilon + \|x - \bar{x}\| s(\eta) \right) \limsup_{\nu} \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{t\nu} + \|y^*\| o(\|x - \bar{x}\|).
\end{aligned}$$

Taking now the estimate (4.5) into account implies that

$$\langle x^*, x - \bar{x} \rangle \leq -\frac{\langle x^*, \tilde{x} \rangle}{\delta} \left(\varepsilon + \|x - \bar{x}\| s(\eta) \right) + o(\|x - \bar{x}\|) \|y^*\|,$$

which yields in turn by $\varepsilon \downarrow 0$ that

$$\langle x^*, x - \bar{x} \rangle \leq -\frac{\langle x^*, \tilde{x} \rangle}{\delta} \|x - \bar{x}\| s(\eta) + o(\|x - \bar{x}\|) \|y^*\|.$$

Since $s(\eta) \downarrow 0$ as $\eta \downarrow 0$, it follows from the latter inequality that

$$\limsup_{x \rightarrow \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0,$$

which means that $x^* \in \widehat{N}(\bar{x}; \Omega)$ and thus justifies the inclusion “ \supset ” in (4.1).

Next we prove the inclusion “ \subset ” in (4.2) under the assumption that g_t are uniformly strictly differentiable at \bar{x} . This immediately implies the inclusion “ \subset ” in (4.1) under the latter assumption, while we note that similar arguments justify the inclusion “ \subset ” in (4.1) under merely the uniform Fréchet differentiability of g_t at \bar{x} .

To proceed with proving the inclusion “ \subset ” in (4.2), define the set

$$A_\varepsilon := \text{cl}^* \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \} + \nabla h(\bar{x})^*(Y^*) \text{ for } \varepsilon > 0. \quad (4.6)$$

Arguing by contradiction, pick an arbitrary element $x^* \in N(\bar{x}; \Omega) \setminus \{0\}$ and suppose that $x^* \notin A_\varepsilon$ for some $\varepsilon \in (0, \tilde{\varepsilon})$. It follows by the structure of A_ε in (4.6) that

$$\left(x^* - \nabla h(\bar{x})^*(Y^*) \right) \cap \text{cl}^* \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \} = \emptyset.$$

Since the subspace $\nabla h(\bar{x})^*(Y^*)$ is weak* closed in X^* by Lemma 4.1, we conclude from the classical separation theorem that there are $x_0 \in X$ and $c > 0$ satisfying

$$\langle x^*, x_0 \rangle - \langle y^*, \nabla h(\bar{x}) x_0 \rangle = \langle x^*, x_0 \rangle - \langle \nabla h(\bar{x})^* y^*, x_0 \rangle \geq 2c > 0 \geq \langle \nabla g_t(\bar{x}), x_0 \rangle \quad (4.7)$$

for all $t \in T_\varepsilon(\bar{x})$ and $y^* \in Y^*$; hence $\nabla h(\bar{x}) x_0 = 0$. Define further

$$\widehat{x} := x_0 + \frac{c}{\|x^*\| \cdot \|\tilde{x}\|} \tilde{x}$$

and observe that $\nabla h(\bar{x})\hat{x} = 0$. Moreover, it follows from (4.7) and the PMFCQ that

$$\langle x^*, \hat{x} \rangle = \langle x^*, x_0 + \frac{c}{\|x^*\| \cdot \|\hat{x}\|} \hat{x} \rangle \geq 2c + \frac{c}{\|x^*\| \cdot \|\hat{x}\|} \langle x^*, \hat{x} \rangle \geq 2c - c = c \quad \text{and} \quad (4.8)$$

$$\langle \nabla g_t(\bar{x}), \hat{x} \rangle = \langle \nabla g_t(\bar{x}), x_0 \rangle + \frac{c}{\|x^*\| \cdot \|\hat{x}\|} \langle \nabla g_t(\bar{x}), \hat{x} \rangle \leq -\frac{\delta c}{\|x^*\| \cdot \|\hat{x}\|} = -\tilde{\delta} \quad (4.9)$$

for all $t \in T_\varepsilon(\bar{x})$ with $\tilde{\delta} := \frac{\delta c}{\|x^*\| \cdot \|\hat{x}\|} > 0$. Observing that $\hat{x} \neq 0$ by (4.9), suppose without loss of generality that $\|\hat{x}\| = 1$. Furthermore, we get from definition of the limiting normal cone that there are sequences $\varepsilon_n \downarrow 0$, $\eta_n \downarrow 0$, $x_n \xrightarrow{\Omega} \bar{x}$, and $x_n^* \xrightarrow{w^*} x^*$ as $n \rightarrow \infty$ with

$$\langle x_n^*, x - x_n \rangle \leq \varepsilon_n \|x - x_n\| \quad \text{for all } x \in \mathcal{B}_{\eta_n}(x_n) \cap \Omega, \quad n \in \mathbb{N}. \quad (4.10)$$

Since the mapping h is strictly differentiable at \bar{x} with the surjective derivative $\nabla h(\bar{x})$, it follows from the Lyusternik-Graves theorem (see, e.g., [22, Theorem 1.57]) that h is *metrically regular* around \bar{x} , i.e., there are neighborhoods U of \bar{x} and V of $0 = h(\bar{x})$ and a constant $\mu > 0$ such that

$$\text{dist}(x; h^{-1}(y)) \leq \mu \|y - h(x)\| \quad \text{for any } x \in U \text{ and } y \in V. \quad (4.11)$$

Since $h(x_n) = 0$ and $\nabla h(\bar{x})\hat{x} = 0$, we have

$$\|h(x_n + t\hat{x})\| = \|h(x_n + t\hat{x}) - h(x_n) - \nabla h(\bar{x})(t\hat{x})\| = o(t) \quad \text{for each small } t > 0.$$

Thus the metric regularity (4.11) implies that for any small $t > 0$ there is $x_t \in h^{-1}(0)$ with $\|x_n + t\hat{x} - x_t\| = o(t)$ when $x_n \in U$. This allows us to find $\tilde{\eta}_n < \eta_n$ and $\tilde{x}_n := x_{\tilde{\eta}_n} \in h^{-1}(0)$ satisfying $\tilde{\eta}_n + o(\tilde{\eta}_n) \leq \eta_n$ and $\|x_n + \tilde{\eta}_n \hat{x} - \tilde{x}_n\| = o(\tilde{\eta}_n)$. Note that

$$\|x_n - \tilde{x}_n\| \leq \tilde{\eta}_n \|\hat{x}\| + \|x_n + \tilde{\eta}_n \hat{x} - \tilde{x}_n\| = \tilde{\eta}_n + o(\tilde{\eta}_n) \leq \eta_n,$$

i.e., $\tilde{x}_n \in \mathcal{B}_{\eta_n}(x_n)$. Observe further that

$$\|x_n - \tilde{x}_n\| \geq \tilde{\eta}_n \|\hat{x}\| - \|x_n + \tilde{\eta}_n \hat{x} - \tilde{x}_n\| = \tilde{\eta}_n - o(\tilde{\eta}_n).$$

By the classical uniform boundedness principle there is a constant M such that $M > \|x_n^*\|$ for all $n \in \mathbb{N}$ due to $x_n^* \xrightarrow{w^*} x^*$ as $n \rightarrow \infty$. It follows from (4.8) that $\langle x_n^*, \hat{x} \rangle > 0$ for $n \in \mathbb{N}$ sufficiently large. Then we have

$$\begin{aligned} \frac{\langle x_n^*, \tilde{x}_n - x_n \rangle}{\|\tilde{x}_n - x_n\|} &= \frac{\langle x_n^*, \tilde{x}_n - \tilde{\eta}_n \hat{x} - x_n \rangle}{\|\tilde{x}_n - x_n\|} + \frac{\langle x_n^*, \tilde{\eta}_n \hat{x} \rangle}{\|\tilde{x}_n - x_n\|} \\ &\geq -M \frac{\|\tilde{x}_n - \tilde{\eta}_n \hat{x} - x_n\|}{\|\tilde{x}_n - x_n\|} + \tilde{\eta}_n \frac{\langle x_n^*, \hat{x} \rangle}{\|\tilde{x}_n - x_n\|} \\ &\geq -M \frac{o(\tilde{\eta}_n)}{\tilde{\eta}_n - o(\tilde{\eta}_n)} + \frac{\tilde{\eta}_n}{\tilde{\eta}_n + o(\tilde{\eta}_n)} \langle x_n^*, \hat{x} \rangle. \end{aligned}$$

Since $o(\tilde{\eta}_n)/\tilde{\eta}_n \rightarrow 0$ when $n \rightarrow \infty$, the latter inequalities yield that

$$\liminf_{n \rightarrow \infty} \frac{\langle x_n^*, \tilde{x}_n - x_n \rangle}{\|\tilde{x}_n - x_n\|} \geq \langle x^*, \hat{x} \rangle.$$

Combining this with (4.8) and (4.10) gives us that $\tilde{x}_n \notin \Omega$ for all large $n \in \mathbb{N}$.

Now define $u_n := x_n + \tilde{\eta}_n \hat{x} - \tilde{x}_n$ and get $\|u_n\| = o(\tilde{\eta}_n)$ and $\|\tilde{x}_n + u_n - x_n\| = \tilde{\eta}_n$ by the arguments above. It follows from our standing assumptions (SA), condition (3.3), and inequality (4.9) that for each $t \in T_\varepsilon(\bar{x})$ we have

$$\begin{aligned}
-\tilde{\delta} &\geq \frac{\langle \nabla g_t(\bar{x}), \tilde{\eta}_n \hat{x} \rangle}{\tilde{\eta}_n} = \frac{\langle \nabla g_t(\bar{x}), \tilde{x}_n + u_n - x_n \rangle}{\|\tilde{x}_n + u_n - x_n\|} = \frac{\langle \nabla g_t(\bar{x}), \tilde{x}_n - x_n \rangle}{\|\tilde{x}_n + u_n - x_n\|} + \frac{\langle \nabla g_t(\bar{x}), u_n \rangle}{\|\tilde{x}_n + u_n - x_n\|} \\
&\geq \frac{\langle \nabla g_t(\bar{x}), \tilde{x}_n - x_n \rangle}{\|\tilde{x}_n - x_n\|} \frac{\|\tilde{x}_n - x_n\|}{\|\tilde{x}_n + u_n - x_n\|} + \frac{\langle \nabla g_t(\bar{x}), u_n \rangle}{\|\tilde{x}_n + u_n - x_n\|} \\
&\geq \left(\frac{g_t(\tilde{x}_n) - g_t(x_n)}{\|\tilde{x}_n - x_n\|} - r(\hat{\eta}_n) \right) \frac{\|\tilde{x}_n - x_n\|}{\|\tilde{x}_n + u_n - x_n\|} - \sup_{\tau \in T_\varepsilon(\bar{x})} \|\nabla g_\tau(\bar{x})\| \frac{o(\tilde{\eta}_n)}{\tilde{\eta}_n} \\
&\geq \left(\frac{g_t(\tilde{x}_n)}{\|\tilde{x}_n - x_n\|} - r(\hat{\eta}_n) \right) \frac{\|\tilde{x}_n - x_n\|}{\|\tilde{x}_n + u_n - x_n\|} - \sup_{\tau \in T} \|\nabla g_\tau(\bar{x})\| \frac{o(\tilde{\eta}_n)}{\tilde{\eta}_n},
\end{aligned}$$

where $\hat{\eta}_n := \max\{\|x_n - \bar{x}\| \text{ and } \|\tilde{x}_n - \bar{x}\|\} \rightarrow 0$ as $n \rightarrow \infty$. Note that

$$\frac{\tilde{\eta}_n - o(\tilde{\eta}_n)}{\tilde{\eta}_n} \leq \frac{\|\tilde{x}_n - x_n\|}{\|\tilde{x}_n + u_n - x_n\|} \leq \frac{\tilde{\eta}_n + o(\tilde{\eta}_n)}{\tilde{\eta}_n},$$

which implies that $\frac{\|\tilde{x}_n - x_n\|}{\|\tilde{x}_n + u_n - x_n\|} \rightarrow 1$ as $n \rightarrow \infty$. Furthermore, since $r(\hat{\eta}_n) \rightarrow 0$ and $\frac{o(\tilde{\eta}_n)}{\tilde{\eta}_n} \rightarrow 0$ as $n \rightarrow \infty$, we have $g_t(\tilde{x}_n) \leq -\frac{\tilde{\delta}}{2} \|\tilde{x}_n - x_n\| \leq 0$ for each $t \in T_\varepsilon(\bar{x})$ when $n \in \mathbb{N}$ is sufficiently large. Indeed, assuming otherwise that $t \notin T_\varepsilon(\bar{x})$ gives us

$$\begin{aligned}
g_t(\tilde{x}_n) &\leq g_t(\bar{x}) + \langle \nabla g_t(\bar{x}), x_n - \bar{x} \rangle + \|x_n - \bar{x}\| r(\hat{\eta}_n) \\
&\leq -\varepsilon + \sup_{\tau \in T} \|\nabla g_\tau(\bar{x})\| \hat{\eta}_n + \hat{\eta}_n r(\hat{\eta}_n) \leq 0 \text{ for all large } n \in \mathbb{N}.
\end{aligned}$$

Thus $g_t(\tilde{x}_n) \leq 0$ for all $t \in T$ and also $h(\tilde{x}_n) = 0$ when $n \in \mathbb{N}$ is sufficiently large, i.e., $\tilde{x}_n \in \Omega$, a contradiction. Hence we conclude that $N(\bar{x}; \Omega) \subset A_\varepsilon$ for all $\varepsilon \in (0, \tilde{\varepsilon})$, which implies the inclusion ‘‘C’’ in (4.2) and completes the proof of the theorem. \triangle

Let us show now that the PMFCQ condition is essential for the validity of both normal cone representations in (4.1) and (4.2); moreover, this condition cannot be replaced by its weaker EMFCQ version.

Example 4.3 (violation of the normal cone representations with no PMFCQ). Consider the infinite inequality system in \mathbb{R}^2 given in Example 3.6. It is shown therein that the EMFCQ holds at $\bar{x} = (-1, 0)$ while the PMFCQ does not. It is easy to check that in this case $\hat{N}(\bar{x}; \Omega) = N(\bar{x}; \Omega) = \mathbb{R}_+ \times \mathbb{R}_-$ while

$$\text{cl cone}\{\nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x})\} = \text{cl cone}\{(1, 0) \cup \{(t, 0) \mid t \in (0, \varepsilon)\}\} \subset \mathbb{R}_+ \times \{0\}.$$

i.e., the inclusions ‘‘C’’ in (4.1) and (4.2) are violated.

The next example shows that the perturbed active index set $T_\varepsilon(\bar{x})$ cannot be replaced by its unperturbed counterpart $T(\bar{x})$ in the normal cone representations (4.1) and (4.2).

Example 4.4 (perturbation of the active index set is essential for the normal cone representations). Let us reconsider the nonlinear infinite system in problem (3.7):

$$\begin{cases} g_1(x) = x_1 + 1 \leq 0, \\ g_n(x) = \frac{1}{3n}x_1^3 - x_2 \leq 0, \quad n \in \mathbb{N} \setminus \{1\}, \end{cases}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $T := \mathbb{N}$. It is easy to check this inequality system satisfies our standing assumptions and that the functions g_t are uniformly strictly differentiable at $\bar{x} = (-1, 0)$. Observe further that $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq -1, x_2 \geq 0\}$ and hence $N(\bar{x}; \Omega) = \mathbb{R}_+ \times \mathbb{R}_-$. As shown above, both PMFCQ and EMFCQ conditions hold at \bar{x} . However, we have $T(\bar{x}) = \{1\}$ and

$$N(\bar{x}; \Omega) \neq \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T(\bar{x}) \} = \text{cone} \{ \nabla g_1(\bar{x}) \} = \text{cone} \{(1, 0)\} = \mathbb{R}_+ \times \{0\},$$

which shows the violation of the unperturbed counterparts of (4.1) and (4.2). Observe that

$$\begin{aligned} \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \} &= \text{cone} \left\{ (1, 0) \cup \left\{ \left(\frac{1}{n}, -1 \right) \mid n \in \mathbb{N} \setminus \{1\}, n \geq \frac{1}{\varepsilon} \right\} \right\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 < 0\}, \end{aligned}$$

which is not a closed subset. On the other hand, we have

$$N(\bar{x}; \Omega) = \bigcap_{\varepsilon > 0} \text{cl cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \},$$

which illustrates the validity of the normal cone representations in Theorem 4.2.

Now we derive several consequences of Theorem 4.2, which are of their independent interest. The first one concerns the case when the $\{ \nabla g_t(\bar{x}) \mid t \in T \}$ may not be bounded in X^* as in our standing assumptions. It follows that the latter case can be reduced to the basic case of Theorem 4.2 with some modifications.

Corollary 4.5 (normal cone representation for infinite systems with unbounded gradients). *Considering the constraint system (3.1), assume the following:*

- (a) *The functions g_t , $t \in T$, are Fréchet differentiable at the point \bar{x} with $\| \nabla g_t(\bar{x}) \| > 0$ for all $t \in T$ and the mapping h is strictly differentiable at \bar{x} .*
- (b) *We have that $\lim_{\eta \downarrow 0} \tilde{r}(\eta) = 0$, where $\tilde{r}(\eta)$ is defined by*

$$\tilde{r}(\eta) := \sup_{t \in T} \sup_{\substack{x, x' \in \mathbb{B}_\eta(\bar{x}) \\ x \neq x'}} \frac{|g_t(x) - g_t(x') - \langle \nabla g_t(\bar{x}), x - x' \rangle|}{\| \nabla g_t(\bar{x}) \| \cdot \| x - x' \|} \quad \text{for all } \eta > 0. \quad (4.12)$$

- (c) *The operator $\nabla h(\bar{x}): X \rightarrow Y$ is surjective and for some $\varepsilon > 0$ there are $\tilde{x} \in X$ and $\sigma > 0$ such that $\nabla h(\bar{x})\tilde{x} = 0$ and that*

$$\langle \nabla g_t(\bar{x}), \tilde{x} + x \rangle \leq 0 \quad \text{whenever } \| x \| \leq \sigma \quad (4.13)$$

for each $t \in \tilde{T}_\varepsilon(\bar{x}) := \{ t \in T \mid g_t(\bar{x}) \geq -\varepsilon \| \nabla g_t(\bar{x}) \| \}$. Then the limiting normal cone to Ω at \bar{x} is computed by formula (4.2).

Proof. Define $\tilde{g}_t(x) := g_t(x) \|\nabla g_t(\bar{x})\|^{-1}$ for all $x \in X$ and $t \in T$ and observe that the feasible set Ω from (1.2) admits the representation

$$\Omega = \{x \in X \mid \tilde{g}_t(x) \leq 0, h(x) = 0\}.$$

Replacing g_t by \tilde{g}_t in Theorem 4.2, we have that the functions $\{\tilde{g}_t\}$ and h satisfy the standing assumptions (SA) as well as condition (3.3) with the function (4.12) instead of $r(\eta)$. Furthermore, it follows from (4.13) that for some $\varepsilon > 0$ there are $\tilde{x} \in X$ and $\sigma > 0$ satisfying $\nabla h(\tilde{x})\tilde{x} = 0$ and such that

$$\langle \nabla \tilde{g}_t(\tilde{x}), \tilde{x} \rangle \leq - \sup_{x \in \mathcal{B}_\sigma(\tilde{x})} \langle \nabla \tilde{g}_t(\tilde{x}), x \rangle = -\sigma \|\nabla \tilde{g}_t(\tilde{x})\| \text{ whenever } t \in \tilde{T}_\varepsilon(\tilde{x}),$$

which turns into $\langle \nabla \tilde{g}_t(\tilde{x}), \tilde{x} \rangle \leq -\sigma$ for all $t \in \tilde{T}_\varepsilon(\tilde{x}) = \{t \in T \mid \tilde{g}_t(\tilde{x}) \geq -\varepsilon\}$. Hence the PMFCQ condition holds for the functions \tilde{g}_t and h at \tilde{x} . It follows from Theorem 4.2 that

$$\begin{aligned} N(\tilde{x}; \Omega) &= \bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ \nabla \tilde{g}_t(\tilde{x}) \mid t \in \tilde{T}_\varepsilon(\tilde{x}) \} + \nabla h(\tilde{x})^*(Y^*) \\ &= \bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ \nabla g_t(\tilde{x}) \|\nabla g_t(\tilde{x})\|^{-1} \mid t \in \tilde{T}_\varepsilon(\tilde{x}) \} + \nabla h(\tilde{x})^*(Y^*) \\ &= \bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ \nabla g_t(\tilde{x}) \mid t \in \tilde{T}_\varepsilon(\tilde{x}) \} + \nabla h(\tilde{x})^*(Y^*), \end{aligned}$$

which gives (4.2) and completes the proof of the corollary. \triangle

Now we compare the result of Corollary 4.5 with the recent one obtained in [26, Theorem 3.1 and Corollary 4.1] for inequality constraint systems, i.e., with $h = 0$ in (3.1). The latter result is given by the inclusion form

$$N(\bar{x}; \Omega) \subset \bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \}$$

in the case of $\|\nabla g_t(\bar{x})\| = 1$ for all $t \in T$ under the Fréchet differentiability of g_t around \bar{x} (in (as) we need it merely *at* \bar{x}) and the replacement of (b) of Corollary 4.5 by the following *equicontinuity* requirement on g_t at \bar{x} : for each $\gamma > 0$ there is $\eta > 0$ such that

$$\|\nabla g_t(x) - \nabla g_t(\bar{x})\| \leq \gamma \text{ for all } x \in \mathcal{B}_\eta(\bar{x}), t \in T. \quad (4.14)$$

Let us check that the latter assumption together with the Fréchet differentiability of g_t around \bar{x} imply (b) in Corollary 4.5. Indeed, suppose that (4.14) holds and then pick any $x, x' \in \mathcal{B}_\eta(\bar{x})$. Employing the classical Mean Value Theorem, find $\hat{x} \in [x, x'] \subset \mathcal{B}_\eta(\bar{x})$ such that $g_t(x) - g_t(x') = \langle \nabla g_t(\hat{x}), x - x' \rangle$. This gives

$$\begin{aligned} \frac{|g_t(x) - g_t(x') - \langle \nabla g_t(\bar{x}), x - x' \rangle|}{\|\nabla g_t(\bar{x})\| \cdot \|x - x'\|} &= \frac{|\langle \nabla g_t(\hat{x}), x - x' \rangle - \langle \nabla g_t(\bar{x}), x - x' \rangle|}{\|x - x'\|} \\ &\leq \frac{|\langle \nabla g_t(\hat{x}) - \nabla g_t(\bar{x}), x - x' \rangle|}{\|x - x'\|} \\ &\leq \|\nabla g_t(\hat{x}) - \nabla g_t(\bar{x})\| \leq \gamma \end{aligned}$$

and yields $\lim_{\eta \downarrow 0} \tilde{r}(\eta) \leq \gamma$ for all $\gamma > 0$, which ensures the validity of (b) in Corollary 4.5.

The next consequence of Theorem 4.2 concerns problems of semi-infinite programming and presents sufficient conditions for the fulfillment of simplified representations of the normal cones to feasible constraints with no closure operations in (4.1) and (4.2) and with the replacement of the perturbed index set $T_\varepsilon(\bar{x})$ by that of active constraints $T(\bar{x})$.

Corollary 4.6 (normal cones for semi-infinite constraints). *Let X and Y be finite-dimensional spaces with $\dim Y < \dim X$. Assume that T is a compact metric space, that the function $t \in T \mapsto g_t(\bar{x})$ is u.s.c., and the mapping $t \in T \mapsto \nabla g_t(\bar{x})$ is continuous. Suppose further that system (3.1) satisfies the PMFCQ at \bar{x} . Then we have*

$$\tilde{N}(\bar{x}; \Omega) = \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T(\bar{x}) \} + \nabla h(\bar{x})^*(Y^*), \quad (4.15)$$

where $\tilde{N}(\bar{x}; \Omega) = \hat{N}(\bar{x}; \Omega)$ when the functions g_t are uniformly Fréchet differentiable at \bar{x} and $\tilde{N}(\bar{x}; \Omega) = N(\bar{x}; \Omega)$ when g_t are uniformly strictly differentiable at \bar{x} .

In particular, if we assume in addition that both $t \in T \mapsto g_t(\bar{x})$ and $(x, t) \in X \times T \mapsto \nabla g_t(x)$ are continuous, then we also have (4.15) for $\tilde{N}(\bar{x}; \Omega) = N(\bar{x}; \Omega)$ provided that merely the EMFCQ condition holds at \bar{x} .

Proof. Let $X = \mathbb{R}^d$ for some $d \in \mathbb{N}$. It follows from Proposition 3.1 that g_t , $t \in T$, and h satisfy our standing assumptions (SA). Since system (3.1) satisfies the PMFCQ at \bar{x} , there are $\tilde{\varepsilon} > 0$, $\delta > 0$, and $\tilde{x} \in X$ such that $\langle \nabla g_t(\bar{x}), \tilde{x} \rangle < -\delta$ for all $t \in T_\varepsilon(\bar{x})$ and $\varepsilon \in (0, \tilde{\varepsilon})$. Observe that the perturbed active index set $T_\varepsilon(\bar{x})$ is compact in T for all $\varepsilon > 0$ due to the u.s.c. assumption on $t \in T \mapsto g_t(\bar{x})$. It follows from the continuity of $t \in T \mapsto \nabla g_t(\bar{x})$ that $\{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \}$ is a compact subset of \mathbb{R}^d .

We now claim that $0 \notin \text{co} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \}$. Indeed, it follows for any $\lambda \in \tilde{\mathbb{R}}_+^{T_\varepsilon(\bar{x})}$ with $\sum_{t \in T_\varepsilon(\bar{x})} \lambda_t = 1$ that

$$\sum_{t \in T_\varepsilon(\bar{x})} \lambda_t \langle \nabla g_t(\bar{x}), \tilde{x} \rangle \leq - \sum_{t \in T_\varepsilon(\bar{x})} \lambda_t \delta = -\delta < 0,$$

which yields that $0 \neq \sum_{t \in T_\varepsilon(\bar{x})} \lambda_t \nabla g_t(\bar{x})$, i.e., $0 \notin \text{co} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \}$.

Hence it follows from [16, Proposition 1.4.7] that the conic hull $\text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \}$ is closed in \mathbb{R}^d . Combining this with Theorem 4.2, it suffices to show that

$$\bigcap_{\varepsilon > 0} \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \} = \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T(\bar{x}) \}. \quad (4.16)$$

Observe that the inclusion “ \supset ” in (4.16) is obvious due to $T(\bar{x}) \subset T_\varepsilon(\bar{x})$ as $\varepsilon > 0$. To justify the converse inclusion, pick an arbitrary element x^* from the set on the left-hand side of (4.16). By the classical Carathéodory theorem, for all large $n \in \mathbb{N}$ we find $\lambda_n \in \mathbb{R}_+^{d+1}$ and

$$\nabla g_{t_{n_1}}(\bar{x}), \dots, \nabla g_{t_{n_{d+1}}}(\bar{x}) \in \{ \nabla g_t(\bar{x}) \mid t \in T_{\frac{1}{n}}(\bar{x}) \} \subset \mathbb{R}^d$$

satisfying the relationship

$$x^* = \sum_{k=1}^{d+1} \lambda_{n_k} \nabla g_{t_{n_k}}(\bar{x}), \quad (4.17)$$

which implies in turn that

$$\langle x^*, \tilde{x} \rangle = \sum_{k=1}^{d+1} \lambda_{n_k} \langle \nabla g_{t_{n_k}}(\bar{x}), \tilde{x} \rangle \leq - \sum_{k=1}^{d+1} \lambda_{n_k} \delta.$$

Hence the sequence $\{\lambda_n\}$ is bounded in \mathbb{R}^{d+1} , and so is

$$\{ \lambda_n \times (\nabla g_{t_{n_1}}(\bar{x}), \dots, \nabla g_{t_{n_{d+1}}}(\bar{x})) \} \subset \mathbb{R}^{d+1} \times \mathbb{R}^{d(d+1)}.$$

By the classical Bolzano-Weierstrass theorem and the compactness of T , we assume without loss of generality that the sequence $\{t_{n_k}\}$ converges to some $\bar{t}_k \in T$ for each $1 \leq k \leq d+1$ and that $\{\lambda_n\}$ converges to some $\bar{\lambda} \in \mathbb{R}^{d+1}$ as $n \rightarrow \infty$. Note that $0 \geq g_{t_{n_k}}(\bar{x}) \geq -\frac{1}{n}$ for all $n \in \mathbb{N}$ sufficiently large, which gives us

$$0 \geq g_{\bar{t}_k}(\bar{x}) \geq \limsup_{n \rightarrow \infty} g_{t_{n_k}}(\bar{x}) \geq \limsup_{n \rightarrow \infty} -\frac{1}{n} = 0$$

for all $1 \leq k \leq d+1$. Combining the latter with (4.17) ensures that

$$x^* = \sum_{k=1}^{d+1} \bar{\lambda}_k \nabla g_{\bar{t}_k}(\bar{x}) \in \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T(\bar{x}) \},$$

which yields the inclusion “ \subset ” in (4.16). Thus we arrive at formula (4.15).

The second part of the corollary follows from the first part, Proposition 3.1, and Proposition 3.5. This completes the proof of the claimed result. \triangle

The results obtained in Corollary 4.6 can be compared with [7, Theorem 3.4], where “ \subset ” in (4.15) was obtained for $h = 0$ under the following conditions: T is scattered compact (meaning that every subset $S \subset T$ has an isolated point), g_t are Fréchet differentiable for all $t \in T$, the mappings $(x, t) \in X \times T \mapsto g_t(x)$ and $(x, t) \in X \times T \mapsto \nabla g_t(x)$ are continuous, and the EMFCQ condition holds at \bar{x} . We can see that these assumptions are significantly stronger than those Corollary 4.6. Note, in particular, that the scattering compactness requirement on the index set T is not different in applications from T being finite.

The next question we address in this section is about the possibility to obtain normal cone representations of the “unperturbed” type as in Corollary 4.6 while in infinite programming settings with no finite dimensionality, compactness, and continuity assumptions made above. The following theorem shows that this can be done when the PMFCQ is accompanied by the NFMFCQ condition of Definition 3.8.

Theorem 4.7 (unperturbed representations of normal cones for infinite constraint systems). *Let the functions g_t , $t \in T$, be uniformly Fréchet differentiable at \bar{x} , and let that system (3.1) satisfy the PMFCQ and NFMFCQ conditions at \bar{x} . Then*

$$\widehat{N}(\bar{x}; \Omega) = \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T(\bar{x}) \} + \nabla h(\bar{x})^*(Y^*). \quad (4.18)$$

If in addition the functions g_t , $t \in T$, are uniformly strictly differentiable at \bar{x} , then

$$N(\bar{x}; \Omega) = \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T(\bar{x}) \} + \nabla h(\bar{x})^*(Y^*). \quad (4.19)$$

Proof. First we claim that the set $\bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \}$ belongs to the set

$$\left\{ x^* \in X^* \mid (x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^* \text{cone} \{ (\nabla g_t(\bar{x}), \langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})) \mid t \in T \} \right\}. \quad (4.20)$$

Indeed, it follows from the PMFCQ for (3.1) at \bar{x} that $\nabla h(\bar{x})$ is surjective and there are $\tilde{\varepsilon} > 0$, $\delta > 0$, and $\tilde{x} \in X$ such that $\nabla h(\bar{x})\tilde{x} = 0$ and that $\langle \nabla g_t(\bar{x}), \tilde{x} \rangle < -\delta$ for all $\varepsilon \leq \tilde{\varepsilon}$ and $t \in T_\varepsilon(\bar{x})$. To justify the claimed inclusion to (4.20), pick an arbitrary element $x^* \in \bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \}$ and for any $\varepsilon \in (0, \tilde{\varepsilon})$ find a net $(\lambda_\nu)_{\nu \in \mathcal{N}} \subset \mathbb{R}_+^T$ with

$$x^* = w^* - \lim_{\nu} \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{t\nu} \nabla g_t(\bar{x}). \quad (4.21)$$

This implies the relationships

$$\langle x^*, \tilde{x} \rangle = \lim_{\nu} \sum_{t \in T_{\varepsilon}(\bar{x})} \lambda_{t\nu} \langle \nabla g_t(\bar{x}), \tilde{x} \rangle \leq -\delta \limsup_{\nu} \sum_{t \in T_{\varepsilon}(\bar{x})} \lambda_{t\nu} \quad \text{and} \quad (4.22)$$

$$\langle x^*, \bar{x} \rangle = \lim_{\nu} \sum_{t \in T_{\varepsilon}(\bar{x})} \lambda_{t\nu} \langle \nabla g_t(\bar{x}), \bar{x} \rangle = \lim_{\nu} \sum_{t \in T_{\varepsilon}(\bar{x})} \lambda_{t\nu} (\langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x}) + g_t(\bar{x})).$$

The later equality together with (4.22) give us that

$$0 \geq \langle x^*, \bar{x} \rangle - \limsup_{\nu} \sum_{t \in T_{\varepsilon}(\bar{x})} \lambda_{t\nu} (\langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})) \geq \liminf_{\nu} \sum_{t \in T_{\varepsilon}(\bar{x})} \lambda_{t\nu} g_t(\bar{x}) \geq \frac{\varepsilon}{\delta} \langle x^*, \tilde{x} \rangle.$$

By passing to a subnet and combining this with (4.21), we get

$$(x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^* \text{cone} \{ (\nabla g_t(\bar{x}), \langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})) \mid t \in T \} + \{0\} \times [\frac{\varepsilon}{\delta} \langle x^*, \tilde{x} \rangle, 0]$$

for all $\varepsilon \in (0, \tilde{\varepsilon})$, which implies that x^* belongs to the set in (4.20) by taking $\varepsilon \downarrow 0$.

Involving further the NFMCC condition, we claim the equality

$$\bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_{\varepsilon}(\bar{x}) \} = \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T(\bar{x}) \}. \quad (4.23)$$

The inclusion “ \supset ” in (4.23) is obvious since $T(\bar{x}) \subset T_{\varepsilon}(\bar{x})$ for all $\varepsilon > 0$. To justify the converse inclusion, pick any x^* belonging to the left-hand side of (4.23). By the NFMCC condition, it follows from (4.20) that there is $\lambda \in \tilde{\mathcal{R}}_+^T$ such that

$$(x^*, \langle x^*, \bar{x} \rangle) = \sum_{t \in T} \lambda_t (\nabla g_t(\bar{x}), \langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})), \quad (4.24)$$

which readily yields the equalities

$$0 = \sum_{t \in T} \lambda_t \langle \nabla g_t(\bar{x}), \bar{x} \rangle - \sum_{t \in T} \lambda_t (\langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})) = \sum_{t \in T} \lambda_t g_t(\bar{x}).$$

Since $g_t(\bar{x}) \leq 0$, we get $\lambda_t g_t(\bar{x}) = 0$ for all $t \in T$. Combining this with (4.24) gives us

$$x^* \in \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T(\bar{x}) \},$$

which implies the inclusion “ \subset ” in (4.23). To complete the proof of the theorem, we combine the obtained equality (4.23) with finally Theorem 4.2. \triangle

Observe from Proposition 3.11 that formula (4.18) holds under our standing assumptions (SA) and the MFCQ condition at \bar{x} when T is a finite index set. Furthermore, the formula for the limiting normal cones (4.19) is also satisfied if all the functions g_t are strictly differentiable at \bar{x} . It follows from Proposition 3.11 that Corollary 4.6 can be derived from a semi-infinite version of Theorem 4.7 in addition to the assumptions of this corollary we suppose that the function $t \in T \mapsto g_t(\bar{x})$ is continuous in T .

The next example shows that the PMFCQ condition cannot be replaced by the EMFCQ one in Theorem 4.7 to ensure the unperturbed normal cone representations (4.18) and (4.19) in the presence of the NFMCC.

Example 4.8 (EMFCQ combined with NFMFCQ does not ensure the unperturbed normal cone representations). We revisit the semi-infinite inequality constraint system in Example 3.3. It is shown there that this system satisfied the EMFCQ but not PMFCQ at $\bar{x} = (-1, 0)$. It is easy to check that the set

$$\begin{aligned} \text{cone} \{ (\nabla g_t(\bar{x}), \langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})) \mid t \in T \} &= \text{cone} \left((1, 0, -1) \cup \{(t, 0, 0) \mid t \in (0, 1]\} \right) \\ &= \{ x \in \mathbb{R}^3 \mid x_1 + x_3 \geq 0, x_1 \geq 0 \geq x_3, x_2 = 0 \} \end{aligned}$$

is closed in \mathbb{R}^3 , i.e., the NFMFCQ condition holds at \bar{x} . Observe however that both representations (4.18) and (4.19) are not satisfied for this system since we have

$$\widehat{N}(\bar{x}; \Omega) = N(\bar{x}; \Omega) \neq \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T(\bar{x}) \} = \text{cone} \{(1, 0)\} = \mathbb{R}_+ \times \{0\}.$$

Now we present a consequence of Theorem 4.7 with the corresponding discussions.

Corollary 4.9 (normal cone for infinite convex systems). *Assume that all the functions g_t , $t \in T$, in (3.1) are convex and uniformly Fréchet differentiable and that $h = A$ is a surjective continuous linear operator. Suppose further that system (3.1) satisfies the PMFCQ (equivalently the SSC) at $\bar{x} \in \Omega$. Then the normal cone to Ω at \bar{x} in sense of convex analysis is computed by*

$$N(\bar{x}; \Omega) = \bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \} + A^*(Y^*).$$

If in addition the NFMFCQ holds at \bar{x} , then we have

$$N(\bar{x}; \Omega) = \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T(\bar{x}) \} + A^*(Y^*). \quad (4.25)$$

Proof. It follows directly from Proposition 3.7 and Theorem 4.7. \triangle

For $h = 0$ in (3.1) the equality in (4.25) can be deduced from [11, Corollary 3.6] under another *Farkas-Minkowski Constraint Qualification* (FMCQ) defined as follows:

(FMCQ) *The conic hull $\text{cone}\{\text{epi } g_t^* \mid t \in T\}$ is weak* closed in $X^* \times \mathbb{R}$ under the additional assumption that the functions g_t are l.s.c., where*

$$\varphi^*(x^*) := \sup \{ \langle x^*, x \rangle - \varphi(x) \mid x \in X \}, \quad x^* \in X^*,$$

stands for the Fenchel conjugate of a convex function.

It is worth noting that the above FMCQ condition is a *global* property, and hence formula (4.25) holds at every $\bar{x} \in \Omega$. By the contrary, our new NFMFCQ condition (3.11) is constructed at a fixed point $\bar{x} \in \Omega$. The next example shows that the combination of the PMFCQ (or the SSC) and the NFMFCQ conditions for infinite convex inequality systems is not stronger than the FMCQ one.

Example 4.10 (PMFCQ combined with NFMFCQ does not imply FMCQ for convex inequality systems). Define a function $g_t : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $g_t(x_1, x_2) := tx_1^2 - x_2$ for all $(x_1, x_2) \in \mathbb{R}^2$ and $t \in T := (0, 1)$, and let $\bar{x} = (0, 0) \in \mathbb{R}^2$. It is easy to see that all the functions g_t , $t \in T$, are convex and differentiable and that the standing assumptions are satisfied. For each $t \in T$ we have

$$g_t^*(a_1, a_2) = \sup_{(x_1, x_2) \in \mathbb{R}^2} \{ a_1 x_1 + a_2 x_2 - tx_1^2 + x_2 \} = \begin{cases} \frac{a_1^2}{4t} & \text{if } a_2 = -1, \\ \infty & \text{otherwise.} \end{cases}$$

This implies that $\text{epi } g_t^* = \{(a, -1, \frac{a^2}{4t} + r) \mid a \in \mathbb{R}, r \geq 0\}$, which yields in turn that

$$C := \text{cone} \{ \text{epi } g_t^* \mid t \in T \} = \text{cone} \left\{ (a, -1, \frac{a^2}{4} + r) \mid a \in \mathbb{R}, r \geq 0 \right\}.$$

The latter set is not closed in \mathbb{R}^3 since $\{0\} \times \{0\} \times \mathbb{R}_+ \not\subset C$ while $\{0\} \times \{0\} \times \mathbb{R}_+ \subset \text{cl}C$. Moreover, we see that $\nabla g_t(\bar{x}) = (0, -1)$ for all $t \in T$, and then the PMFCQ is satisfied. Furthermore, it follows that the set

$$\text{cone} \{ (\nabla g_t(\bar{x}), \langle \nabla g_t(\bar{x}), \bar{x} \rangle - g_t(\bar{x})) \mid t \in T \} = \text{cone} \{(0, -1, 0)\} = \{0\} \times \mathbb{R}_- \times \{0\}$$

is closed in \mathbb{R}^3 . Hence the PMFCQ and NFMFCQ conditions hold but the FMCQ does not.

Finally in this section, we give specifications of obtained normal cone representations in the case linear infinite systems.

Proposition 4.11 (normal cone representations for linear infinite constraint systems). *Consider the constraint system (3.1) with $g_t(x) = \langle a_t^*, x \rangle - b_t$ for all $t \in T$, and let $h = A : X \rightarrow Y$. Assume that A is a surjective continuous linear operator and that the coefficient set $\{a_t^* \mid t \in T\}$ is bounded in X^* . If the SSC condition holds at \bar{x} , then*

$$N(\bar{x}; \Omega) = \bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ a_t^* \mid t \in T_\varepsilon(\bar{x}) \} + A^*(Y^*)$$

for the feasible set $\Omega := \{x \in X \mid Ax = 0, \langle a_t^*, x \rangle - b_t \leq 0, t \in T\}$. On the other hand, assuming the weak* closedness of $\text{cone} \{ (a_t^*, b_t) \mid t \in T \}$ in $X^* \times \mathbb{R}$ and that $h = 0$ gives us

$$N(\bar{x}; \Omega) = \text{cone} \{ a_t^* \mid t \in T(\bar{x}) \}.$$

Proof. The first statement is a specification of Corollary 4.9. The second one follows from the proofs given in [5, Proposition 3.1] and [6, Theorem 3.2] by using the classical Farkas Lemma for linear infinite systems. \triangle

5 Optimality Conditions in Nonlinear Infinite Programming

In this section we employ general principles in optimization and the calculus results on computing the normal cones to the infinite constraint sets in Section 4 to deriving necessary optimality conditions for problems of infinite and semi-infinite programming. We confine ourselves to optimality conditions of the “lower” subdifferential type conventional in minimization. Condition of the other (“upper” or superdifferential) type can be derived from the calculus results of Section 4 using an approach developed in [22, Chapter 5]; see also the recent paper [6] for the implementation of the latter approach in the case of semi-infinite and infinite programs with linear constraints.

Our first theorem in this section concerns infinite programs of type (1.1) in arbitrary Banach spaces involving Fréchet differentiable cost functions.

Theorem 5.1 (necessary optimality conditions for differentiable infinite programs in general Banach spaces). *Let \bar{x} be a local minimizer of the infinite program (1.1) under the PMFCQ condition imposed on the constraints at \bar{x} . Suppose further that*

the inequality constraint functions g_t , $t \in T$, are uniformly Fréchet differentiable at \bar{x} and the cost function f is Fréchet differentiable at this point. Then we have the inclusion

$$0 \in \nabla f(\bar{x}) + \bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \} + \nabla h(\bar{x})^*(Y^*). \quad (5.1)$$

If in addition the NFMFCQ holds at \bar{x} , then there exist multipliers $\lambda \in \widetilde{\mathbb{R}}_+^T$ and $y^* \in Y^*$ satisfying the differential KKT condition

$$0 = \nabla f(\bar{x}) + \sum_{t \in T(\bar{x})} \lambda_t \nabla g_t(\bar{x}) + \nabla h(\bar{x})^* y^*. \quad (5.2)$$

Proof. It is clear that \bar{x} is a local optimal solution to the following unconstrained optimization problem with the *infinite penalty*:

$$\text{minimize } f(x) + \delta(x; \Omega), \quad (5.3)$$

where Ω is the feasible constraint set (1.2). Applying the generalized Fermat rule to the latter problem (see, e.g., [22, Proposition 1.114]), we have

$$0 \in \widehat{\partial}(f + \delta(\cdot; \Omega))(\bar{x}). \quad (5.4)$$

Since f is Fréchet differentiable at \bar{x} , it follows from the sum rule of [22, Theorem 1.107] applied to (5.4) and from the first relationship in (2.4) that

$$0 \in \nabla f(\bar{x}) + \widehat{\partial}\delta(\bar{x}; \Omega)(\bar{x}) = \nabla f(\bar{x}) + \widehat{N}(\bar{x}; \Omega). \quad (5.5)$$

Now using the Fréchet normal cone representation of Theorem 4.2 in (5.5), we arrive at (5.1). The second part (5.2) of this theorem readily follows from Theorem 4.7. \triangle

The next theorem establishes necessary conditions for local minimizers of infinite programs (1.1) with general nonsmooth cost functions in the framework of Asplund spaces.

Theorem 5.2 (necessary optimality conditions for nonconvex infinite programs defined on Asplund spaces, I). *Let \bar{x} be a local minimizer of problem (1.1), where the domain space X is Asplund while the image space Y is arbitrary Banach. Suppose that the constraint functions g_t , $t \in T$, are uniformly strictly differentiable at \bar{x} , that the cost function f is l.s.c. around \bar{x} and SNEC at this point, and that the qualification condition*

$$\partial^\infty f(\bar{x}) \cap \left[- \bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \} - \nabla h(\bar{x})^*(Y^*) \right] = \{0\} \quad (5.6)$$

is fulfilled; the latter two assumptions are automatic when f is locally Lipschitzian around \bar{x} . If the PMFCQ condition holds at \bar{x} , then

$$0 \in \partial f(\bar{x}) + \bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \} + \nabla h(\bar{x})^*(Y^*). \quad (5.7)$$

If in addition we assume that the NFMFCQ holds at \bar{x} and replace (5.6) by

$$\partial^\infty f(\bar{x}) \cap \left[- \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T(\bar{x}) \} - \nabla h(\bar{x})^*(Y^*) \right] = \{0\}, \quad (5.8)$$

then there exist multipliers $\lambda \in \widetilde{\mathbb{R}}_+^T$ and $y^ \in Y^*$ such that the following subdifferential KKT condition is satisfied:*

$$0 \in \partial f(\bar{x}) + \sum_{t \in T(\bar{x})} \lambda_t \nabla g_t(\bar{x}) + \nabla h(\bar{x})^* y^*. \quad (5.9)$$

Proof. Observe first that the feasible set Ω is locally closed around \bar{x} . Indeed, it follows from (3.3) that there are $\gamma > 0$ and $\eta > 0$ sufficiently small such that

$$\|h(x) - h(x')\| \leq (\|\nabla h(\bar{x})\| + \gamma)\|x - x'\| \quad \text{and} \quad \|g_t(x) - g_t(x')\| \leq \sup_{\tau \in T} (\|\nabla g_\tau(\bar{x})\| + \gamma)\|x - x'\|$$

for all $x, x' \in \mathcal{B}_\eta(\bar{x})$ and $t \in T$. Picking any sequence $\{x_n\} \subset \Omega \cap \mathcal{B}_\eta(\bar{x})$ converging to some x_0 as $n \rightarrow \infty$, we have

$$\|h(x_0)\| \leq (\|\nabla h(\bar{x})\| + \gamma)\|x_n - x_0\| \quad \text{and} \quad g_t(x_0) \leq \sup_{\tau \in T} (\|\nabla g_\tau(\bar{x})\| + \gamma)\|x_n - x_0\| + g_t(x_n)$$

for each $t \in T$ and $n \in \mathbb{N}$. By passing to the limit as $n \rightarrow \infty$, the latter yields that $h(x_0) = 0$ and $g_t(x_0) \leq 0$ for all $t \in T$, i.e., $x_0 \in \Omega \cap \mathcal{B}_\eta(\bar{x})$, which justifies the local closedness of the feasible set Ω around \bar{x} .

Employing now the generalized Fermat rule to the solution \bar{x} of (5.3) with the closed set Ω and using [22, Theorem 3.36] on the sum rule for basic/limiting subgradients in Asplund spaces when f is SNEC at \bar{x} yield that

$$0 \in \partial(f + \delta(\cdot; \Omega))(\bar{x}) \subset \partial f(\bar{x}) + \partial \delta(\bar{x}; \Omega) = \partial f(\bar{x}) + N(\bar{x}; \Omega) \quad (5.10)$$

provided that $\partial^\infty f(\bar{x}) \cap (-N(\bar{x}; \Omega)) = \{0\}$. We apply further to both latter conditions the limiting normal cone representation of Theorem 4.2. This gives us the optimality condition (5.7) under the fulfillment of (5.6) and the PMFCQ at \bar{x} . Applying finally Theorem 4.7 instead of Theorem 4.2 in the setting above, we arrive at the KKT condition (5.9) under the assumed NFMFCQ at \bar{x} and (5.8), which completes the proof of the theorem. \triangle

An important ingredient in the proof of Theorem 5.2 is applying the subdifferential sum rule from [22, Theorem 3.36] to the sum $f + \delta(\cdot; \Omega)$, which requires that either f is SNEC at \bar{x} or Ω is SNC at this point. While the first possibility was used above, now we are going to explore the second alternative. The next proposition presents verifiable conditions ensuring the SNC property of the feasible set Ω at \bar{x} .

Proposition 5.3 (SNC property of feasible sets in infinite programming). *Let X be an Asplund space, and let $\dim Y < \infty$ in the framework of (1.1). Assume that all the functions g_t , $t \in T$, are Fréchet differentiable around some $\bar{x} \in \Omega$ and that the corresponding derivative family $\{\nabla g_t\}_{t \in T}$ is equicontinuous around this point, i.e., there exists $\varepsilon > 0$ such that for each $x \in \mathcal{B}_\varepsilon(\bar{x})$ and each $\gamma > 0$ there is $0 < \tilde{\varepsilon} < \varepsilon$ with the property*

$$\|\nabla g_t(x') - \nabla g_t(x)\| \leq \gamma \quad \text{whenever} \quad x' \in \mathcal{B}_{\tilde{\varepsilon}}(x) \cap \Omega \quad \text{and} \quad t \in T. \quad (5.11)$$

Then the feasible set Ω in (1.2) is locally closed around \bar{x} and SNC at this point provided that the PMFCQ condition holds at \bar{x} .

Proof. Consider first the set $\Omega_1 := \{x \in X \mid g_t(x) \leq 0, t \in T\}$. By using arguments similar to the proof of Theorem 5.2, we justify the local closedness of Ω_1 around \bar{x} . Now let us prove that Ω_1 is SNC at this point. To proceed, pick any sequence $(x_n, x_n^*) \in \Omega_1 \times X^*$, $n \in \mathbb{N}$, satisfying

$$x_n \xrightarrow{\Omega_1} \bar{x}, \quad x_n^* \in \widehat{N}(x_n; \Omega_1) \quad \text{and} \quad x_n^* \xrightarrow{w^*} 0 \quad \text{as} \quad n \rightarrow \infty.$$

Taking (5.11) into account, we see that the functions g_t , $t \in T$ satisfy the standing assumptions (SA) at x_n for all $n \in \mathbb{N}$ sufficiently large. Moreover, the proof showing

that assumption (3.3) holds at x_n follows from the discussions right after Corollary 4.5. Since the PMFCQ condition holds at \bar{x} , there exist $\delta > 0$, $\varepsilon > 0$, and $\tilde{x} \in X$ such that $\langle \nabla g_t(\tilde{x}), \tilde{x} \rangle \leq -2\delta$ for all $t \in T_{2\varepsilon}(\bar{x})$. Observe that $T_\varepsilon(x_n) \subset T_{2\varepsilon}(\bar{x})$ for all large $n \in \mathbb{N}$. Indeed, whenever $t \in T_\varepsilon(x_k)$ we have

$$\begin{aligned} 0 \geq g_t(\bar{x}) &\geq g_t(x_n) - \langle \nabla g_t(\bar{x}), x_n - \bar{x} \rangle - \|x_n - \bar{x}\|s(\|x_k - \bar{x}\|) \\ &\geq -\varepsilon - \sup_{\tau \in T} \|\nabla g_\tau(\bar{x})\| \cdot \|x_n - \bar{x}\| - \|\tilde{x}_n - \bar{x}\|s(\|x_n - \bar{x}\|) \geq -2\varepsilon \end{aligned}$$

for all large $n \in \mathbb{N}$, where $s(\cdot)$ is defined in (3.2). Further, it follows from (5.11) that

$$\langle \nabla g_t(x_n), \tilde{x} \rangle \leq \langle \nabla g_t(\bar{x}), \tilde{x} \rangle + \|\nabla g_t(x_n) - \nabla g_t(\bar{x})\| \cdot \|\tilde{x}\| \leq -2\delta + \|\nabla g_t(x_n) - \nabla g_t(\bar{x})\| \cdot \|\tilde{x}\| \leq -\delta$$

when $n \in \mathbb{N}$ is sufficiently large. Hence we suppose without loss of generality that

$$T_\varepsilon(x_n) \subset T_{2\varepsilon}(\bar{x}) \text{ and } \sup_{t \in T_\varepsilon(x_n)} \langle \nabla g_t(x_n), \tilde{x} \rangle \leq -\delta \text{ whenever } n \in \mathbb{N}. \quad (5.12)$$

Applying now Theorem 4.2 in this setting, we have that for each $n \in \mathbb{N}$ there exists a net $\{\lambda_{n\nu}\}_{\nu \in \mathcal{N}} \subset \tilde{\mathbb{R}}_+^{T_\varepsilon(x_n)}$ such that

$$x_n^* = w^* - \lim_{\nu} \sum_{t \in T_\varepsilon(x_n)} \lambda_{tn\nu} \nabla g_t(x_n).$$

Combining this with (5.12) yields that

$$\langle x_n^*, \tilde{x} \rangle = \lim_{\nu} \sum_{t \in T_\varepsilon(x_n)} \lambda_{tn\nu} \langle \nabla g_t(x_n), \tilde{x} \rangle \leq -\delta \liminf_{\nu} \sum_{t \in T_\varepsilon(x_n)} \lambda_{tn\nu}.$$

Furthermore, for each $x \in X$ we get the relationships

$$\begin{aligned} \langle x_n^*, x \rangle &= \liminf_{\nu} \sum_{t \in T_\varepsilon(x_k)} \lambda_{tn\nu} \langle \nabla g_t(x_n), x \rangle \leq \liminf_{\nu} \sum_{t \in T_\varepsilon(x_n)} \lambda_{tn\nu} \sup_{\tau \in T} \|\nabla g_\tau(x_n)\| \cdot \|x\| \\ &\leq -\frac{\langle x_n^*, \tilde{x} \rangle}{\delta} \sup_{\tau \in T} \|\nabla g_\tau(x_n)\| \cdot \|x\|, \end{aligned}$$

which imply that $\|x_n^*\| \leq -\frac{\langle x_n^*, \tilde{x} \rangle}{\delta} \sup_{\tau \in T} \|\nabla g_\tau(x_n)\|$ for all $n \in \mathbb{N}$. Since $x_n^* \xrightarrow{w^*} 0$, it follows from the latter that $\|x_n^*\| \rightarrow 0$ as $n \rightarrow \infty$ and thus the set Ω_1 is SNC at \bar{x} .

Consider now the set $\Omega_2 := \{x \in X \mid h(x) = 0\}$, which is obviously closed around \bar{x} . It follows from [22, Theorem 1.22] and finite dimensionality of Y that Ω_2 is SNC at \bar{x} . Moreover, we get from [22, Theorem 1.17] that $N(\bar{x}; \Omega_2) = \nabla h(\bar{x})^*(Y^*)$. Thus for any $x^* \in N(\bar{x}; \Omega_1) \cap (-N(\bar{x}; \Omega_2))$ there is $y^* \in Y^*$ such that $x^* + \nabla h(\bar{x})^* y^* = 0$, and then

$$\langle x^*, \tilde{x} \rangle = -\langle \nabla h(\bar{x})^* y^*, \tilde{x} \rangle = -\langle y^*, \nabla h(\bar{x}) \tilde{x} \rangle = 0.$$

Since $x^* \in N(\bar{x}; \Omega_1)$, we find by Theorem 4.2 such a net $\{\lambda_\nu\}_{\nu \in \mathcal{N}} \in \tilde{\mathbb{R}}_+^T$ that

$$x^* = w^* - \lim_{\nu} \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{t\nu} \nabla g_t(\bar{x}),$$

which yields in turn that

$$0 = \langle x^*, \tilde{x} \rangle = \lim_{\nu} \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{t\nu} \langle \nabla g_t(\bar{x}), \tilde{x} \rangle \leq -2\delta \liminf_{\nu} \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{t\nu}.$$

This ensures the relationships

$$\langle x^*, x \rangle = \liminf_{\nu} \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{t\nu} \langle \nabla g_t(\bar{x}), x \rangle \leq \liminf_{\nu} \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{t\nu} \sup_{\tau \in T} \|\langle \nabla g_\tau(\bar{x}) \| \|x\| = 0$$

for all $x \in X$. Hence we have $x^* = 0$, and so $N(\bar{x}; \Omega_1) \cap (-N(\bar{x}; \Omega_2)) = \{0\}$. It finally follows from [22, Corollary 3.81] that the intersection $\Omega = \Omega_1 \cap \Omega_2$ is SNC at \bar{x} , which thus completes the proof of the proposition. \triangle

Observe that the assumption $\dim Y < \infty$ is essential in Proposition 5.3. To illustrate this, consider a particular case of (1.1) when $T = \emptyset$. It follows from [22, Theorem 1.22] that the inverse image $\Omega = h^{-1}(0)$ is SNC at $\bar{x} \in \Omega$ if and only if the set $\{0\}$ is SNC at $0 \in Y$. Since $N(0; \{0\}) = Y^*$, the latter holds if and only if the weak* topology in Y^* agrees with the norm topology in Y^* , which is only the case of $\dim Y < \infty$ by the classical Josefson-Nissenzweig theorem from theory of Banach spaces.

Now we are ready to derive an aforementioned alternative counterpart of Theorem 5.2.

Theorem 5.4 (necessary optimality conditions for nonconvex infinite programs defined on Asplund spaces, II). *Let \bar{x} be a local minimizer of infinite program (3.1) under the assumptions of Proposition 5.3. Suppose also that f is l.s.c. around \bar{x} and that the qualification condition (5.6) is satisfied. Then we have the optimality condition (5.7). If in addition we assume that the NFMCCQ holds at \bar{x} and replace (5.6) by (5.8), then there exist multipliers $\lambda \in \tilde{\mathbb{R}}_+^T$ and $y^* \in Y^*$ such that the subdifferential KKT condition (5.9).*

Proof. It is similar to the proof of Theorem 5.2 with applying Proposition 5.3 on the SNC and closedness property of Ω in the sum rule (5.10) of [22, Theorem 3.36]. \triangle

The next result provides necessary and sufficient optimality conditions for convex problems of infinite programming in general Banach spaces.

Theorem 5.5 (necessary and optimality conditions for convex infinite programs). *Let both spaces X and Y be Banach. Assume that all the functions g_t , $t \in T$, are convex and uniformly Fréchet differentiable and that $h = A$ is a surjective continuous linear operator. Suppose further that the cost function f is convex and continuous at some point in Ω . If the PMFCQ condition (equivalently the SSC condition) holds at \bar{x} , then \bar{x} is a global minimizer of problem (1.1) if and only if*

$$0 \in \partial f(\bar{x}) + \bigcap_{\varepsilon > 0} \text{cl}^* \text{cone} \{ \nabla g_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \} + A^*(Y^*).$$

If in addition the NFMCCQ condition holds, then \bar{x} is a global minimizer of problem (1.1) if and only if there exist $\lambda \in \tilde{\mathbb{R}}_+^T$ and $y^ \in Y^*$ such that*

$$0 \in \partial f(\bar{x}) + \sum_{t \in T(\bar{x})} \lambda_t \nabla g_t(\bar{x}) + A^* y^*. \quad (5.13)$$

Proof. Observe that \bar{x} is a global minimizer of problem (1.1) if and only if it is a global minimizer of the convex unconstrained problem (5.3), which is equivalent to the fact that

$$0 \in \partial(f + \delta(\cdot; \Omega))(\bar{x}).$$

Applying the convex subdifferential sum rule to the latter inclusion, we conclude that \bar{x} is a global minimizer of problem (1.1) if and only if

$$0 \in \partial f(\bar{x}) + \partial \delta(\bar{x}; \Omega) = \partial f(\bar{x}) + N(\bar{x}; \Omega).$$

The rest of the proof follows from Corollary 4.9. △

Note that some versions of necessary optimality condition of the KKT type (5.13) were derived in [6, Theorems 3.1 and 3.2] for infinite problems with linear constraints but possibly nonconvex cost functions under the SSC and the linear counterpart of the FMCQ; see Example 4.10 and the corresponding discussions above.

Observe also that the results of Theorem 5.4 and Theorem 5.5 are formulated with no change in the case of semi-infinite programs, while in Theorem 5.1 we just drop the SNEC assumption on f , which holds automatically when X is finite-dimensional.

In conclusion we present a consequence of our results for the classical framework of semi-infinite programming while involving nonsmooth cost functions.

Corollary 5.6 (necessary optimality conditions for semi-infinite programs with compact index sets). *Let \bar{x} be a local minimizer of program (1.1), where both spaces X and Y are finite-dimensional with $\dim Y < \dim X$. Assume that the index set T in (1.1) is a compact metric space, that the mappings $(x, t) : X \times T \mapsto g_t(x)$ and $(x, t) : X \times T \mapsto \nabla g_t(x)$ are continuous, and that the cost function f is l.s.c. around \bar{x} with the fulfillment of (5.8). If in addition the EMFCQ holds at \bar{x} , then there exist multipliers $\lambda \in \tilde{\mathbb{R}}_+^T$ and $y^* \in Y^*$ satisfying the subdifferential KKT condition (5.9).*

Proof. By Proposition 3.9 we have that the NFMFCQ condition holds at \bar{x} under the assumptions made. Then this corollary follows directly from Theorem 5.2. △

When f is smooth around \bar{x} , assumption (5.8) holds automatically while (5.9) reduced to the differential KKT condition (5.2). Then Corollary 5.6 reduces to a well-known result in semi-infinite programming that can be found, e.g., in [15, Theorem 3.3] and [21, Theorem 2].

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