Mat 751 Algebraic Topology I - Fall '89

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0. Categories and Covariant Functors

Def 0.1. A category $\mathcal{C}$ consists of
1) A class $\text{Ob}(\mathcal{C})$ whose members are called the objects
   of $\mathcal{C}$.
2) In each ordered pair $(X, Y)$ of objects of $\mathcal{C}$, a set
   $\mathcal{C}(X, Y)$ called the set of morphisms in $\mathcal{C}$ from $X$ to $Y$.
   If $x \in \mathcal{C}(X, Y)$ we call $X$ the domain of $x$ and $Y$ the
   range of $x$. We also write $x : X \to Y$ or $X \xrightarrow{x} Y$
   to denote the fact that $x \in \mathcal{C}(X, Y)$.
3) In each ordered triple $(X, Y, Z)$ of objects of $\mathcal{C}$ a function
   $\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$.
   We denote the image of $(\alpha, \beta)$ under this function by $\beta \circ \alpha$
   and call it the composition of $\alpha$ and $\beta$.
4) We require that whenever $\alpha \in \mathcal{C}(W, X)$, $\beta \in \mathcal{C}(X, Y)$,
   $\gamma \in \mathcal{C}(Y, Z)$, then $\gamma(\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$.
5) We require that for each object $X$ of $\mathcal{C}$ there exist
   a morphism $1_X \in \mathcal{C}(X, X)$ satisfying $\alpha \cdot 1_X = \alpha$
   whenever $\alpha \in \mathcal{C}(W, X)$, and $1_X \circ \beta = \beta$
   whenever $\beta \in \mathcal{C}(Y, X)$ for all objects $Y$ of $\mathcal{C}$.

Example 0.2. The category of sets $\text{Set}$. $\text{Ob}(\text{Set})$
   consists of all sets. If $X, Y \in \text{Ob}(\text{Set})$, $\text{Set}(X, Y)$
   the set of all functions from $X$ to $Y$. The composition
   of morphisms is the usual composition of functions.

Example 0.3. The category of topological spaces $\text{Top}$.
   $\text{Ob}(\text{Top})$ consists of all topological spaces. If $X, Y \in
   \text{Ob}(\text{Top})$, $\text{Top}(X, Y)$ = the set of all continuous
   functions from $X$ to $Y$. The composition of morphisms
   is the usual composition of functions.

Example 0.4. The category of topological pairs $\text{JP}$. $\text{Ob}(\text{JP})$
   consists of all pairs $(X, A)$ where $X$ is a topological
   space and $A$ is a subspace of $X$. $\text{JP}((X, A), (Y, B))$
   the set of all continuous functions from $X$ to $Y$ which
   carry $A$ into $B$. The composition of morphisms
   is the usual composition of functions.
Example 0.5. The category of pointed topological spaces $\text{Top}_0$. $\text{Ob}(\text{Top}_0)$ consists of all ordered pairs $(X, x_0)$ where $X$ is a topological space and $x_0 \in X$.
$\text{Top}_0((X, x_0), (Y, y_0))$ is the set of all continuous functions from $X$ to $Y$ which send $x_0$ to $y_0$. The composition of morphisms is the usual composition of functions.

Example 0.6. The category of groups $\mathcal{U}$. $\text{Ob}(\mathcal{U})$ consists of all groups. $\mathcal{U}(G, H)$ is the set of all group homomorphisms from $G$ into $H$. The composition of morphisms is the usual composition of functions.

Example 0.7. The category of abelian groups $\text{Ab}$. $\text{Ob}(\text{Ab})$ consists of all abelian groups. $\text{Ab}(G, H)$ is the set of all group homomorphisms from $G$ into $H$. The composition of morphisms is the usual composition of functions.

Definition 0.8. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories. A covariant functor $F$ from $\mathcal{C}$ to $\mathcal{D}$ consists of
1) A rule which assigns to each object $X$ of $\mathcal{C}$ an object $F(X)$ (or $F(X)$) of $\mathcal{D}$.
2) A rule which assigns to each morphism $\alpha : X \to Y$ in $\text{Ob}(\mathcal{C})$ a morphism $F\alpha$ (or $F(\alpha)$) in $\text{Ob}(\text{Map}(\mathcal{C}))$ or $\mathcal{D}(F(X), F(Y))$.
3) We require that whenever $\alpha \in \text{Ob}(\mathcal{C})$ and $\beta \in \text{Ob}(\mathcal{C})$, then $F(\beta \circ \alpha) = (F\beta)(F\alpha)$. In terms of diagrams, if

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \\
\downarrow{\beta} & \downarrow{\beta} & \downarrow{\beta} \\
Z & & \\
\end{array}
\]

commutes, then

\[
\begin{array}{ccc}
FX & \xrightarrow{F\alpha} & FY \\
FY & \xrightarrow{F\beta} & FY \\
\end{array}
\]

commutes.
4) We require that for each $X \in \text{Ob}(C)$, $F(1_X) = 1_{FX}$.

If $F$ is a covariant functor from $C$ to $D$ we sometimes write $F : C \to D$ or $C \xrightarrow{F} D$.

Example 0.9. The fundamental group functor $\pi_1$ is a covariant functor from $\text{Top}$ to $\mathbb{G}$.

Example 0.10. The commutator subgroup functor $\Gamma$ is a covariant functor from $\mathbb{G}$ to $\mathbb{G}$. Note that if $f : G \to H$ is a group homomorphism, then $f(\Gamma G) \subseteq \Gamma H$. We take $\Gamma f : \Gamma G \to \Gamma H$ to be the restriction of $f$.

Example 0.11. The abelianization functor from $\mathbb{G}$ to $\text{Ab}$, which assigns to each group $G$ its "abelianization" $G/\Gamma G$. Since, as noted in 0.10, $f(\Gamma G) \subseteq \Gamma H$ for every group homomorphism $f : G \to H$, each such $f$ induces a group homomorphism $G/\Gamma G \to H/\Gamma H$, which we take to be the morphism which the abelianization functor assigns to $f$.

Example 0.12. The "forgetful functor" $\text{Top} \to \text{Top}$, which assigns to each topological pair $(X,A)$ the topological space $X$, and to each morphism $f : (X,A) \to (Y,B)$ in $\text{Top}$, the continuous map $X \to Y$ which underlies $f$. We will denote the latter by $f^\# : X \to Y$.

$f^\#$ is a morphism in the category $\text{Top}$, while $f$ is a morphism in the category $\text{Top}$.

The notation $f^\#$ is not a standard notation. In practice, one often abuses notation and denotes $f$ and $f^\#$ by the same symbol. We have introduced the notation $f^\#$ to emphasize, at least in the beginning, that $f$ and $f^\#$ are morphisms in different categories. Note that it is possible to have $f^\# = g^\#$ with $f \neq g$. For example, if $(X,A)$, $(X,A')$, $(Y,B)$, $(Y,B')$ are topological pairs and $h : X \to Y$ a continuous map such that $h(A) \subseteq B$ and $h(A') \subseteq B'$, then $h$ determines morphisms $f : (X,A) \to (Y,B)$ and
q: (X, A') \to (Y, B')$, which are different if either $A \neq A'$ or $B \neq B'$. But $f^* = g^* = h$.

**Example 3.13.** The restriction functor $\text{JP} \to \text{Top}$ which assigns to each topological pair $(X, A)$ the topological space $A$, and to each morphism $f: (X, A) \to (Y, B)$ in $\text{JP}$ the continuous map $A \to B$ obtained by restriction of $f$. We denote this last morphism by $f^*: A \to B$. (Again, $f^*$ is not a standard notation.)

### 1. Affine Maps, Simplexes, Baricentric Coordinates

**Definition 3.1.** Let $V$ and $W$ be vector spaces over the real numbers $\mathbb{R}$. A function $f: V \to W$ is said to be an affine map if $f$ is the composition of a linear transformation and a translation, i.e., there exists a linear transformation $\lambda: V \to W$ and a point $w_0 \in W$ such that $f(v) = \lambda(v) + w_0$ for all $v \in V$.

It is easy to see that the composition of affine maps is again an affine map. Certainly every real linear transformation is an affine map (in particular, $1_V$ is an affine map for each real vector space $V$), and we obtain the affine category whose objects are real vector spaces, and whose morphisms are the affine maps.

**Definition 3.2.** Let $V$ be a real vector space and $r$ a non-negative integer. An $r$-plane in $V$ is a translate of a linear $r$-dimensional subspace of $V$, i.e., $X$ is an $r$-plane in $V$ iff there exists a linear $r$-dimensional subspace $W$ of $V$ and a point $x_0 \in V$ such that $X = \{w + x_0 | w \in W\}$.
Exercise 1.3. Let $V$ be a real vector space and $X$ a non-empty subset of $V$. Then $X$ is an $r$-plane in $V$ iff for each $x_0 \in X$, $\{x-x_0 \mid x \in X\}$ is an $r$-dimensional linear subspace of $V$. In this case, the linear subspace $\{x-x_0 \mid x \in X\}$ is independent of the choice of $x_0 \in X$.

Note that if $P_0, \ldots, P_r$ $(r > 0)$ are points of a real vector space $V$, then there exists a $k$-plane in $V$ containing all the $P_i$ with $k \leq r$. In fact, if $W$ is the linear subspace spanned by $P_0, P_1, \ldots, P_r$, then $dim\ W \leq r$ and $P_i \in P_0 + W = \{P_0 + w \mid w \in W\}$ for $0 \leq i \leq r$.

Definition 1.4. Let $V$ be a real vector space and $P_0, \ldots, P_r$ $(r \geq 0)$ points in $V$. These points are said to be affinely independent if there does not exist a $k$-plane $X$ in $V$ with $k < r$ such that $P_i \in X$ for $0 \leq i \leq r$.

Thus, every single point $P_0$ is affinely independent, two points $P_0, P_1$ are affinely independent iff $P_0 \neq P_1$, three points $P_0, P_1, P_2$ are affinely independent iff they are not collinear; four points $P_0, P_1, P_2, P_3$ are affinely independent iff they are not coplanar; etc.

Exercise 1.5. Let $V$ be a real vector space and $P_0, \ldots, P_r \in V$. Then $P_0, \ldots, P_r$ are affinely independent iff $P_1 - P_0, \ldots, P_r - P_0$ are linearly independent.

Exercise 1.6. Every non-empty subset of an affinely independent collection of points is affinely independent.

Example 1.7. Let $P_0, \ldots, P_r$ be a linearly independent set of points in a real vector space $V$. Then $0, P_0, \ldots, P_r$ are affinely independent.
Exercise 1.8. Let $V$ be a real $n$-dimensional vector space and suppose $P_0, \ldots, P_n$ are affinely independent points in $V$. Let $W$ be a real vector space and suppose $Q_0, \ldots, Q_n$ are points (not necessarily distinct) in $W$. Then there exists a unique affine map $f: V \rightarrow W$ such that $f(P_i) = Q_i$ for $0 \leq i \leq n$.

Definition 1.9. Let $V$ be a real vector space and suppose $P, Q \in V$. The line segment from $P$ to $Q$ (denoted $PQ$) is $\{ tP + (1-t)Q \mid t \in [0, 1] \}$.

Definition 1.10. Let $V$ be a real vector space and suppose $X \subseteq V$. $X$ is said to be convex if whenever $P, Q \in X$, then $PQ \subseteq X$.

Exercise 1.11. Let $V$ be a real vector space, and suppose $\{ X_a \mid a \in A \}$ (a an index set) is a family of convex subsets of $V$. Then

$$\bigcap_{a \in A} X_a$$

is a convex subset of $V$.

Definition 1.12. Let $V$ be a real vector space, and $X$ a subset of $V$. The convex hull of $X$ is the intersection of all convex subsets of $V$ which contain $X$.

Note that $V$ itself is convex and contains $X$. Thus the convex hull of $X$ is convex, contains $X$, and is contained in every convex subset of $V$ which contains $X$.

Definition 1.13. Let $V$ be a real vector space, and suppose $P_0, \ldots, P_r$ are affinely independent points in $V$. The convex hull of $\{ P_0, \ldots, P_r \}$ is called the $r$-simplex in $V$ with vertices $P_0, \ldots, P_r$. 
Definition 1.14. Let \( P_0, \ldots, P_r \) be affinely independent points in a real vector space \( V \). Any linear combination \( \sum_{i=0}^{r} t_i \cdot P_i \) where \( t_i \geq 0 \) for all \( i \) and \( \sum_{i=0}^{r} t_i = 1 \) is called a convex sum of \( P_0, \ldots, P_r \).

Exercise 1.15. Let \( P_0, \ldots, P_r \) be affinely independent points in a real vector space \( V \). Let \( \tau \) denote the \( r \)-simplex with vertices \( P_0, \ldots, P_r \). Then \( \tau \) consists of all convex sums of \( P_0, \ldots, P_r \).
Moreover, if \( \sum_{i=0}^{r} s_i \cdot P_i = \sum_{i=0}^{r} t_i \cdot P_i \) where both sums are convex sums of \( P_0, \ldots, P_r \), then \( s_i = t_i \) for all \( i \).

Definition 1.16. Let \( P_0, \ldots, P_r \) be affinely independent points in a real vector space \( V \), and let \( \tau \) denote the \( r \)-simplex with vertices \( P_0, \ldots, P_r \). Let \( Q = \sum_{i=0}^{r} \frac{1}{r+1} \cdot P_i \) be a convex sum of \( P_0, \ldots, P_r \).

\((\frac{1}{r+1}, \ldots, \frac{1}{r+1})\) is called the \((r+1)\)-tuple of barycentric coordinates of \( Q \) with respect to \( P_0, \ldots, P_r \).

\( \sum_{i=0}^{r} \frac{1}{r+1} \cdot P_i \) is called the barycenter of \( \tau \).

Exercise 1.17. Let \( P_0, \ldots, P_r \) be affinely independent points in a real vector space \( V \). Let \( \tau \) denote the \( r \)-simplex with vertices \( P_0, \ldots, P_r \). Let \( W \) be a real vector space and \( \phi: V \to W \) an affine map. Suppose \( X \) is a convex subset of \( W \) such that \( \phi(P_i) \in X \) for \( 0 \leq i \leq r \). Then \( \phi(\tau) \subseteq X \).

Let \( \mathbb{R}^\infty \) denote the set of all sequences \( (x_1, x_2, \ldots) \) of real numbers, which are zero after finitely many elements. \( \mathbb{R}^\infty \) is a real vector space. If \( x = (x_1, x_2, \ldots) \), \( y = (y_1, y_2, \ldots) \) are points in \( \mathbb{R}^\infty \), define
df(x,y) = \sqrt{\sum_i (x_i - y_i)^2} \quad d \text{ is a metric on } \mathbb{R}^n.

For \( i \geq 1 \), let \( E_i = (0, \ldots, 0, 1, 0, \ldots) \in \mathbb{R}^n \), and

write \( E_0 = (0,0,\ldots) \in \mathbb{R}^n \). For \( n \geq 1 \), we identify \( \mathbb{R}^n \) with the subspace of \( \mathbb{R}^{n+1} \) spanned by \( E_1, \ldots, E_n \).

The above metric \( d \) on \( \mathbb{R}^n \) restricts to the standard Euclidean metric on \( \mathbb{R}^n \). We also write \( \mathbb{R}^0 = \{E_0\} \). In this way, \( \mathbb{R}^m \) is a subspace of \( \mathbb{R}^n \) whenever \( 0 \leq m \leq n \).

Note that for \( n \geq 0 \), \( E_0, \ldots, E_n \) are affinely independent points in \( \mathbb{R}^n \). We write \( \Delta_n \) in the \( n \)-simplex with vertices \( E_0, \ldots, E_n \).

**Definition 1.18.** \( \Delta_n \) is called the **standard \( n \)-simplex** in \( \mathbb{R}^n \).

\[ \begin{array}{c}
\Delta_0 \\
\Delta_1
\end{array} \]

\( \Delta_n \) is compact, being a closed and bounded subset of \( \mathbb{R}^n \).

**Notation 1.19.** Let \( V \) be a real vector space and \( P_0, \ldots, P_n \) points in \( V \) (not necessarily distinct), \( n \geq 0 \). We denote by \( (P_0, \ldots, P_n) : \Delta_n \rightarrow V \) the restriction to \( \Delta_n \) of the unique affine map \( \mathbb{R}^n \rightarrow V \) which sends \( E_i \) to \( P_i \) for \( 0 \leq i \leq n \).

If \( n \geq 1 \) and \( 0 \leq i \leq n \), we abbreviate \( (P_0, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n) \) by \( (P_0, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n) \).

By 1.17, \( (E_0, \ldots, E_n) : \Delta_n \rightarrow \mathbb{R}^n \) has image contained in \( \Delta_n \).
Definition 1.20. Suppose $n \geq 1$ and $0 \leq i \leq n$. Write $F^n_i : \Delta_{n-1} \to \Delta_{n}$. In the restriction of $(E_0 \cdots \hat{E}_i \cdots E_n)$, $F^n_i$ is called the $i^{th}$ face map into $\Delta_n$.

Each $F^n_i$ is continuous, being the restriction of an affine map. (Affine maps $\mathbb{R}^m \to \mathbb{R}^n$ are clearly continuous).

2. **Singular Chains and Homology**

The geometric idea of homology is to associate with each topological space $X$ and each non-negative integer $n$, objects called $n$-chains. 0-chains can loosely be thought of as finite collections of points in $X$, 1-chains as compact curves in $X$, 2-chains as compact surfaces (possibly with boundary) in $X$, etc. We consider $n$-cycles, i.e. $n$-chains without boundary, and say two $n$-cycles are homologous if together they form the boundary of an $(n+1)$-chain in $X$. The $n^{th}$ homology of $X$ consists of the equivalence classes of $n$-cycles in $X$ under the relation "homologous." Rough intuitive example:

The 1-cycles $a$ and $b$ are homologous since together they form the boundary of the 2-chain $c$. 

![Diagram of 1-cycles and 2-chain](image)
\[ Z \text{ denotes the ring of integers.} \]

**Definition 2.1.** Let \( A \) be any set. The free abelian group on \( A \), temporarily denoted \( F_A \), is the set of all formal expressions \( \sum_{x \in A} n_x x \) where \( n_x \in Z \) and all but finitely many of the \( n_x \) are 0. Addition in \( F_A \) is given by \( \sum_{x \in A} m_x x + \sum_{x \in A} n_x x = \sum_{x \in A} (m_x + n_x)x \).

We identify \( a \in A \) with the element \( \sum_{x \in A} n_x x \) where \( n_x = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases} \). Thus \( A \subset F_A \).

(By convention, if \( A = \emptyset \), then \( F_A = 0 \), the 0-group.)

**Exercise 2.2.** For any set \( A \), \( F_A \) is an abelian group under the addition given in 2.1. Moreover \( F_A \) has the following universal property: Given any abelian group \( G \) and any function \( f: A \rightarrow G \), there exists a unique group homomorphism \( \tilde{f}: F_A \rightarrow G \) which extends the function \( f \).

**Definition 2.3.** Let \( X \) be a topological space and \( n \) a non-negative integer. A singular \( n \)-simplex in \( X \) is a continuous map \( \sigma: \Delta_n \rightarrow X \). We denote by \( Sn(X) \) the free abelian group on the set of all singular \( n \)-simplices in \( X \). \( Sn(X) \) is called the group of singular \( n \)-chains in \( X \).

Let \( f: X \rightarrow Y \) be a continuous map of topological spaces. For each \( n \geq 0 \) there is, by 2.2, a unique group homomorphism \( Sn(f): Sn(X) \rightarrow Sn(Y) \) such that \( Sn(f)(\sigma) = f \circ \sigma \) for each singular \( n \)-simplex \( \sigma \) in \( X \).

**Exercise 2.4.** For each \( n \geq 0 \), \( Sn \) is a covariant functor from \( \text{Top} \) to \( \text{Ab} \).
Definition 2.5. Let \( \varphi : \Delta^n \to X \) be a singular \( n \)-simplex in \( X \), \( n \geq 1 \). Then \( 0 \leq i \leq n \), the \( i \)-th face of \( \varphi \), denoted \( \varphi^{(i)} \), is the singular \((n-1)\)-simplex \( \varphi^{(i)} \in X \). The boundary of \( \varphi \) is the singular \((n-1)\)-chain \( \sum_{i=0}^{n} (-1)^i \varphi^{(i)} \) and denoted \( \partial \varphi \).

Proposition 2.6. Let \( n \geq 1 \) and \( f : X \to Y \) a continuous map. Then the diagram

\[
\begin{array}{ccc}
S_n(X) & \xrightarrow{S_n(f)} & S_n(Y) \\
\partial \downarrow & & \partial \\
S_{n-1}(X) & \xrightarrow{S_{n-1}(f)} & S_{n-1}(Y)
\end{array}
\]

commutes.

(In fancy categorical language, \( \partial \) is a natural transformation from the functor \( S_n \) to the functor \( S_{n-1} \). More briefly, 2.6 is often described by the words \( \partial \) is natural.

Proof. It suffices to check that for each singular \( n \)-simplex \( \varphi : \Delta^n \to X \), \( \partial S_n(f)(\varphi) = S_{n-1}(f) \partial (\varphi) \). Note that for \( 0 \leq i \leq n \),

\[
[S_n(f)(\varphi)]^{(i)} = (f \circ \varphi)^{(i)} F_i^n = f^n(\varphi^{(i)} F_i^n) = f \circ \varphi^{(i)}
\]

\[
= S_{n-1}(f)(\varphi^{(i)}). \quad \text{Thus} \quad S_n(f)(\varphi) = \sum_{i=0}^{n} (-1)^i [S_n(f)(\varphi)]^{(i)}
\]

\[
= \sum_{i=0}^{n} (-1)^i S_{n-1}(f)(\varphi^{(i)}) = S_{n-1}(f)\left( \sum_{i=0}^{n} (-1)^i \varphi^{(i)} \right)
\]

\[
= S_{n-1}(f) \partial (\varphi).
\]

Lemma 2.7. Let \( n \geq 2 \). Then \( F_i^n \circ F_{i-1}^{n-1} = F_i^n \circ F_{i-1}^{n-1} \)

for \( 0 \leq i < i \leq n \).

Proof. It suffices to show
\[(E_0 \cdots E_i \cdots E_n)^{\circ}(E_0 \cdots E_j \cdots E_{n-1}) = (E_0 \cdots E_i \cdots E_n)^{\circ}(E_0 \cdots E_{i-1} \cdots E_{n-1}).\]

Since both compositions are affine maps from \(\mathbb{R}^{n-2}\) to \(\mathbb{R}^n\) and \(E_0, \ldots, En-2\) are affinely independent, it suffices by 1.8 to show that both sides agree on \(E_0, \ldots, En-2\).

A straightforward check shows that both sides send \(E_k\) to

\[
\begin{cases}
E_k & \text{if } k \leq j \\
E_{j+1} & \text{if } k = j \text{ and } j < i-1 \\
E_{j+2} & \text{if } k = j \text{ and } j = i-1 \\
E_{k+1} & \text{if } j < k < i-1 \\
E_{k+2} & \text{if } k \geq i-1.
\end{cases}
\]

Proposition 2.8. For each topological space \(X\) and each \(n \geq 2\), the composition

\[S_n(X) \overset{\sigma}{\to} S_{n-1}(X) \overset{\partial}{\to} S_{n-2}(X)\]

is the \(0\)-homomorphism.

Proof. It suffices to show \(\partial(\partial \sigma) = 0\) for each singular \(n\)-simplex \(\sigma: \Delta^n \to X\). We have

\[\partial(\partial \sigma) = \partial \left( \sum_{i=0}^{n} (-1)^i \sigma(i) \right) = \sum_{i=0}^{n} (-1)^i \partial(\sigma(i))\]

\[= \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{i} (-1)^j \sigma(i) (j) = \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{i+j} \sigma(i) \sigma(j)\]

\[= \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} \sigma_0 \sigma_1 \sigma_2 \sigma_3 + \sum_{0 \leq i \leq j \leq n} (-1)^{i+j} \sigma_0 \sigma_1 \sigma_2 \sigma_3\]

By 2.7, the second sum is

\[\sum_{0 \leq i \leq j \leq n} (-1)^{i+j} \sigma_0 \sigma_1 \sigma_2 \sigma_3 = 0.
\]

Letting \(i' = j\) and \(j' = i-1\), the second sum becomes

\[\sum_{0 \leq i' \leq j' \leq n-1} (-1)^{i'+j'+1} \sigma_0 \sigma_1 \sigma_2 \sigma_3 = -\text{(first sum)}.
\]
Definition 2.9. A chain complex $C$ consists of a sequence of abelian groups $C_0, C_1, C_2, \ldots$ and group homomorphisms $\partial : C_n \rightarrow C_{n-1}, n \geq 1$, such that for all $n \geq 2$, the composition $C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} C_{n-2}$ is the 0-homomorphism.

Example 2.10. For each topological space $X$, the sequence of abelian groups $S_0(X), S_1(X), \ldots$ and homomorphisms $\partial : S_n(X) \rightarrow S_{n-1}(X), n \geq 1$, as defined in 2.3 and 2.5, constitute a chain complex called the singular complex of $X$, and denoted $S(X)$.

Definition 2.11. Suppose $C$ and $D$ are chain complexes. A chain map $f : C \rightarrow D$ consists of a sequence of group homomorphisms $f_n : C_n \rightarrow D_n, n \geq 0$, such that the diagrams

$$
\begin{array}{ccc}
C_n & \xrightarrow{\partial} & C_{n-1} \\
\downarrow f_n & & \downarrow f_{n-1} \\
D_n & \xrightarrow{\partial} & D_{n-1}
\end{array}
$$

commute for all $n \geq 1$.

Example 2.12. Let $f : X \rightarrow Y$ be a continuous map. By 2.6 (naturality of $\partial$), the homomorphisms $S_n(f) : S_n(X) \rightarrow S_n(Y)$ constitute a chain map $S(f) : S(X) \rightarrow S(Y)$.

Exercise 2.13. Suppose $f : C \rightarrow D$ and $g : D \rightarrow E$ are chain maps. Then the sequence of homomorphisms $g \circ f_n : C_n \rightarrow E_n$ constitute a chain map $C \rightarrow E$. We denote this chain map by $gf$ and call it the composition of $f$ and $g$.

Exercise 2.14. If we take chain complexes as objects, chain maps as morphisms, and composition of chain maps as in 2.13, we obtain a category, the category of chain complexes, which we denote by $C\text{C}$. For each chain complex $C$, the morphism $1_C$ consists of
the sequence of identity maps $1_c$, $1_c$, $1_c$, $\ldots$.

**Proposition 2.15.** The rule which assigns to each topological space $X$ its singular complex $S(X)$, and to each continuous map $f : X \to Y$ the chain map $S(f) : S(X) \to S(Y)$ is a covariant functor from Top to $\mathbf{C}$. 

**Proof.** If $f : X \to Y$ and $g : Y \to Z$ are continuous, for each singular n-simplex $\sigma : \Delta_n \to X$ we have

$$S_n(g \circ f)(\sigma) = g \circ (f \sigma) = S_n(g)(f \sigma) = S_n(g)(S_n(f)),$$

so it follows that

$$S_n(g \circ f) = S_n(g) \circ S_n(f).$$

If $f$ is as above, $S_n(1_X)(\sigma) = 1_X \circ \sigma = \sigma$ and so $S_n(1_X) = 1_{S_n(X)}$. Thus $S(1_X) = 1_{S(X)}$.

If $C$ is a chain complex, it is convenient to define $C_n = 0$ for $n < 0$, and $\partial : C_n \to C_{n-1}$ to be the 0-homomorphism in $n \leq 0$.

**Definition 2.16.** Let $C$ be a chain complex. For each $n \in \mathbb{Z}$ we write $Z_n(C)$ for the kernel of $\partial : C_n \to C_{n-1}$ and $B_n(C)$ for the image of $\partial : C_{n+1} \to C_n$. $Z_n(C)$ is called the group of $n$-cycles of $C$, and $B_n(C)$ is called the group of $n$-boundaries of $C$.

The condition $\partial^2 = 0$ says $B_n(C) \subseteq Z_n(C)$ for all $n$.

**Definition 2.17.** Let $C$ be a chain complex. The quotient group $Z_n(C)/B_n(C)$ is called the $n$th homology group of $C$, and denoted $H_n(C)$.

**Proposition 2.18.** Let $f : C \to D$ be a chain map. Then for each $n \in \mathbb{Z}$, $f_n(Z_n(C)) 

\subseteq Z_n(D)$ and $f_n(B_n(C)) \subseteq B_n(D)$. 
Proof. This is immediate from commutativity of the diagram:

\[
\begin{array}{ccc}
C_{n+1} & \rightarrow & C_n \\
\downarrow f_{n+1} & & \downarrow f_n \\
D_{n+1} & \rightarrow & D_n \\
& & \downarrow f_{n-1}
\end{array}
\]

Thus if \( f : C \rightarrow D \), passing to quotients yields a group homomorphism

\[ H_n(f) : H_n(C) \rightarrow H_n(D). \]

Explicitly,

\[ H_n(f)(z + B_n(C)) = f_n(z) + B_n(D) \quad \text{for} \quad z \in Z_n(C). \]

Exercise 2.19. For each \( n \in \mathbb{Z} \), the rule \( H_n \) which assigns to each chain complex \( C \) its \( n \)-th homology group \( H_n(C) \), and to each chain map \( f : C \rightarrow D \) the homomorphism \( H_n(f) : H_n(C) \rightarrow H_n(D) \),

is a covariant functor from \( CB \) to \( Ab \).

Comparing with the singular complex functor \( S : Top \rightarrow CB \), we get for each \( n \in \mathbb{Z} \), a covariant functor \( H_nS : Top \rightarrow Ab \).

Definition 2.20. Let \( X \) be a topological space, \( n \in \mathbb{Z} \). The \( n \)-th singular homology group of \( X \), denoted \( H_n(X) \), is the abelian group \( H_n(S(X)) \).

Thus, abbreviating \( H_nS \) to \( H_n \), each \( H_n \) is a covariant functor from \( Top \) to \( Ab \). If \( f : X \rightarrow Y \) is continuous, we write

\[ H_n(f) = H_nS(f) : H_n(X) \rightarrow H_n(Y). \]
3. Homology of a Point, Augmentation, and $H_0$

The simplest topological space is the empty set $\emptyset$. Since the set of singular $n$-simplices in $\emptyset$ is empty for each $n > 0$, we have $S_n(\emptyset) = 0$ for all $n$. Thus

**Proposition 3.1.** $H_n(\emptyset) = 0$ for all $n \in \mathbb{Z}$.

The next simplest topological space is a one-point space $P$. For each $n > 0$, there is precisely one singular $n$-simplex $\sigma_n$ in $P$.

**Proposition 3.2.** (Dimension Property) If $P$ is a 1-point space, then $H_n(P) = 0$ if $n \neq 0$. $H_0(P) \cong \mathbb{Z}$ with the homology class of the 0-cycle $\sigma_0$ as generator.

**Proof.** For each $n > 0$, $S_n(P) \cong \mathbb{Z}$ with generator $\sigma_n$. For $n > 0$,

$$\partial \sigma_n = \sum_{i=0}^{n} (-1)^i \sigma_n^{(i)}.$$

Now $\sigma_n^{(i)} = \sigma_{n-1}$ for $0 \leq i \leq n$ since $\sigma_{n-1}$ is the only singular $(n-1)$-simplex in $P$. Thus for $n > 0$,

$$\partial \sigma_n = \left( \sum_{i=0}^{n} (-1)^i \right) \sigma_{n-1} = \begin{cases} \sigma_{n-1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Also $\partial \sigma_0 = 0$ since $S_1(P) = 0$. It follows that

$$Z_n(S(P)) = \begin{cases} S_n(P) & \text{if } n \text{ is odd, } n > 0 \\ S_0(P) & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$B_n(S(P)) = \begin{cases} S_n(P) & \text{if } n \text{ is odd, } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

The result follows.
It is convenient to introduce an additional piece of structure on \( S(X) \) called the augmentation.

**Definition 3.3.** Let \( X \) be a topological space. Let \( \epsilon : S_0(X) \to \mathbb{Z} \) denote the unique group homomorphism satisfying \( \epsilon(\sigma) = 1 \) for each singular 0-simplex \( \sigma \) in \( X \). \( \epsilon \) is called the augmentation map.

**Proposition 3.4.** If \( f : X \to Y \) is continuous, then the diagram

\[
\begin{array}{ccc}
S_0(X) & \xrightarrow{S_0(f)} & S_0(Y) \\
\downarrow{\epsilon} & & \downarrow{\epsilon} \\
\mathbb{Z} & & \mathbb{Z}
\end{array}
\]

commutes. (This property is usually paraphrased by the statement "\( \epsilon \) is natural").

**Proof.** It suffices to show that for each singular 0-simplex \( \sigma \) in \( X \), \( \epsilon S_0(f)(\sigma) = \epsilon(\sigma) \). We have \( \epsilon(\sigma) = 1 \) and \( S_0(f)(\sigma) = f \circ \sigma \). Since \( f \circ \sigma \) is a singular 0-simplex in \( Y \), we have \( \epsilon(f \circ \sigma) = 1 \).

**Proposition 3.5.** In each topological space \( X \), the composition

\[
\begin{array}{ccc}
S_1(X) & \xrightarrow{\partial} & S_0(X) & \xrightarrow{\epsilon} & \mathbb{Z}
\end{array}
\]

is the \( 0 \)-homomorphism.

**Proof.** It suffices to show that for each singular 1-simplex \( \sigma \) in \( X \), \( \epsilon \partial(\sigma) = 0 \). We have

\[
\epsilon \partial(\sigma) = \epsilon(\sigma^{(0)} - \sigma^{(1)}) = \epsilon(\sigma^{(0)}) - \epsilon(\sigma^{(1)}) = 1 - 1 = 0.
\]

**Definition 3.6.** An augmented chain complex \( C \) consists of a chain complex together with a group homomorphism \( \epsilon : C_0 \to \mathbb{Z} \) (called the augmentation map) such that the composition

\[
\begin{array}{ccc}
C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\epsilon} & \mathbb{Z}
\end{array}
\]

is the \( 0 \)-homomorphism.

If \( C \) and \( D \) are augmented chain complexes, an
augmented chain map \( f : C \to D \) consists of a chain map such that
\[
\begin{array}{ccc}
C_0 & \xrightarrow{f_0} & D_0 \\
\downarrow \varepsilon & & \downarrow \varepsilon \\
\varepsilon & \to & Z
\end{array}
\]
commutes.

Exercise 3.7. If we take augmented chain complexes as objects, augmented chain maps as morphisms, and composition of augmented chain maps to be the same as composition of chain maps (2.13), we obtain a category, the category of augmented chain complexes, which we denote ACC.

As a consequence of 2.15 and 3.4 we have

Proposition 3.8. \( S \) is a covariant functor from \( \text{Top} \) to \( \text{ACC} \).

Let \( C \) be an augmented chain complex. Then \( Z_0(C) = C_0 \) and \( \varepsilon(B_0(C)) = 0 \). Thus, passing to quotients, \( \varepsilon \) induces a group homomorphism (which, by abuse of notation, is also denoted \( \varepsilon \))
\[
\varepsilon : \text{Ho}(C) \to Z.
\]

Definition 3.7. If \( C \) is an augmented chain complex, the group homomorphism \( \varepsilon : \text{Ho}(C) \to Z \) is called the homology augmentation for \( C \). Explicitly,
\[
\varepsilon(c + B_0(C)) = \varepsilon(c) \quad \text{for all } c \in C_0.
\]

Proposition 3.10. Let \( f : C \to D \) be an augmented chain map. Then the diagram
\[
\begin{array}{ccc}
\text{Ho}(C) & \xrightarrow{\text{Ho}(f)} & \text{Ho}(D) \\
\downarrow \varepsilon & & \downarrow \varepsilon \\
\varepsilon & \to & Z
\end{array}
\]
commutes.
(3.10 is often paraphrased by saying "the homology augmentation is natural.")

Proof. Let \( c \in C_0 \). Then \( \varepsilon \circ Ho(f)(c + B_0(C)) = \varepsilon (f_0(c) + B_0(D)) = \varepsilon f_0(c) = \varepsilon (c) \) (since \( f \) is an augmented chain map) = \( \varepsilon (c + B_0(C)) \).

Definition 3.11. If \( X \) is a topological space, the group homomorphism \( \varepsilon : Ho(X) \to \mathbb{Z} \) induced by the augmentation \( \varepsilon : S_0(X) \to \mathbb{Z} \) is called the homology augmentation for \( X \).

Corollary 3.12. If \( f : X \to Y \) is continuous, then the diagram

\[
\begin{array}{ccc}
Ho(X) & \xrightarrow{Ho(f)} & Ho(Y) \\
\varepsilon \downarrow & & \downarrow \varepsilon \\
\mathbb{Z} & & \mathbb{Z}
\end{array}
\]

commutes (i.e., the homology augmentation for topological spaces is natural.)

3.12 is immediate from 3.8 and 3.10.

Theorem 3.13. Suppose \( X \) is a non-empty topological space. Then \( \varepsilon : Ho(X) \to \mathbb{Z} \) is onto. If \( X \) is path-connected as well, then \( \varepsilon : Ho(X) \to \mathbb{Z} \) is an isomorphism.

Proof. For each \( x \in X \), let \( x \) also denote the singular 0-simplex \( x \) in \( X \) with image \( x \). Each 0-chain in \( X \) is a 0-cycle. For \( x \in X \), write \( [x] \in Ho(X) \) for the homology class of the cycle \( x \) (i.e., \( [x] = x + B_0(X) \)). Then for each \( x \in X \) we have \( \varepsilon ([x]) = \varepsilon (x) = 1 \), and so \( \varepsilon \) is onto since there exists at least one \( x \in X \).

Now suppose \( X \) is path-connected as well. It remains only to show that \( \varepsilon : Ho(X) \to \mathbb{Z} \) is 1-1. Suppose \( \varepsilon (h) = 0 \) for some \( h \in Ho(X) \). We can write \( h = \sum_{x \in X} n_x [x] \) where \( n_x \in \mathbb{Z} \), and all but finitely many
of the \( n_x \) are 0. Then \( 0 = e (x) = \sum_{x \in X} n_x e [x] = \sum_{x \in X} n_x \).

Choose any base point \( x_0 \in X \). Since \( X \) is path-connected, there exists, for each \( x \in X \), a singular 1-simplex \( T_x : \Delta_1 \to X \) such that \( T_x (E_0) = x_0 \) and \( T_x (E_1) = x \). Then \( \partial (T_x) = x - x_0 \), and so \( [x] = [x_0] \) for all \( x \in X \). Thus \( h = \sum_{x \in X} n_x [x] = \left( \sum_{x \in X} n_x \right) [x_0] = 0 [x_0] = 0 \).

**Corollary 3.14.** If \( X \) and \( Y \) are path-connected topological spaces, and \( f : X \to Y \) is continuous, then \( \text{Ho}(f) : \text{Ho}(X) \to \text{Ho}(Y) \) is an isomorphism.

If \( X = Y \), then \( \text{Ho}(f) = 1_{\text{Ho}(X)} \).

**Proof.** By 3.12,

\[
\begin{array}{ccc}
\text{Ho}(X) & \xrightarrow{\text{Ho}(f)} & \text{Ho}(Y) \\
\downarrow & & \downarrow \\
\mathbb{Z} & \xleftarrow{\sim} & \mathbb{Z}
\end{array}
\]

commutes. By 3.13, both \( \varepsilon \)'s are isomorphisms, and so \( \text{Ho}(f) \) is an isomorphism.

If \( X = Y \), then both \( \varepsilon \)'s are the same isomorphism, and so \( \text{Ho}(f) \) is the identity.

We introduce some notation and definitions concerning direct sums of possibly infinite families of abelian groups.

**Definition 3.15.** Let \( \{ G_x \}_{x \in A} \) be an indexed family of abelian groups (written additively) where \( A \) is some index set. The direct sum of this family, written \( \bigoplus_{x \in A} G_x \), consists of all formal expressions \( \sum_{x \in A} g_x \) where \( g_x \in G_x \) and all but a finite number of the \( g_x \) are 0. \( \bigoplus_{x \in A} G_x \) is an abelian group.
under the obvious addition.

To each \( \beta \in A \), we have the obvious inclusion homomorphism

\[
\iota_\beta : G_\beta \to \bigoplus_{\alpha \in A} G_{\alpha},
\]

which we call the canonical inclusion.

**Exercise 3.16.** \( \bigoplus_{\alpha \in A} G_{\alpha} \) has the following universal property: Given any abelian group \( H \) and any family \( f_\alpha : G_\alpha \to H \), \( \alpha \in A \), of group homomorphisms, there exists a unique group homomorphism

\[
\Sigma f_\alpha : \bigoplus_{\alpha \in A} G_{\alpha} \to H
\]

such that for each \( \beta \in A \), the diagram

\[
\begin{array}{ccc}
G_\beta & \xrightarrow{\iota_\beta} & \bigoplus_{\alpha \in A} G_{\alpha} \\
\downarrow{f_\beta} & & \downarrow{\Sigma f_\alpha} \\
H & & \\
\end{array}
\]

commutes.

Explicitly,

\[\left( \sum f_\alpha \right) \left( \sum f_\alpha \right) = \sum_{\alpha \in A} f_\alpha \left( f_\alpha \right) .\]

**Definition 3.17.** Suppose \( \{ G_\alpha \}_{\alpha \in A} \) and \( \{ H_\alpha \}_{\alpha \in A} \) are families of abelian groups, indexed by the same set \( A \). Suppose for each \( \alpha \in A \) we are given a group homomorphism \( f_\alpha : G_\alpha \to H_\alpha \). The direct sum of the \( f_\alpha \), denoted

\[
\bigoplus f_\alpha,
\]

is the unique group homomorphism (from 3.15)

\[
\bigoplus f_\alpha : \bigoplus_{\alpha \in A} G_\alpha \to \bigoplus_{\alpha \in A} H_\alpha
\]

such that for each \( \beta \in A \), the diagram

\[
\begin{array}{ccc}
G_\beta & \xrightarrow{\iota_\beta} & \bigoplus_{\alpha \in A} G_{\alpha} \\
\downarrow{f_\beta} & & \downarrow{\bigoplus f_\alpha} \\
H & & \\
\end{array}
\]

commutes.
commutes (i.e. $\bigoplus f_\alpha = \sum f_\alpha f_\alpha$). Explicitly,

$$\bigoplus f_\alpha (\sum g_\alpha) = \sum f_\alpha (g_\alpha).$$

**Exercise 3.13.** Let \( \{ C^\alpha \}_{\alpha \in A} \) be an indexed family of chain complexes. For each \( \alpha \in A \) and \( n \geq 1 \), write \( d^\alpha_\alpha : C^\alpha_n \to C^\alpha_{n-1} \) as the boundary map in \( C^\alpha \). Then the sequence of abelian groups \( \bigoplus_{\alpha \in A} C^\alpha_n \) for \( n = 0, 1, 2, \ldots \)

together with the group homomorphisms

\[
\bigoplus d_\alpha^\alpha : \bigoplus_{\alpha \in A} C^\alpha_n \to \bigoplus_{\alpha \in A} C^\alpha_{n-1} \quad \text{for} \quad n \geq 1,
\]

consist constitute a chain complex (which we denote by \( \bigoplus C^\alpha \)).

In each \( \alpha \in A \) and \( n \geq 0 \), write \( i^\beta_\alpha : C^\beta_n \to \bigoplus_{\alpha \in A} C^\alpha_n \)

for the canonical inclusion. Then the sequence of homomorphisms \( i^\beta_\alpha \) for \( n = 0, 1, 2, \ldots \) constitute a chain map \( i^\beta : \bigoplus C^\beta \to \bigoplus C^\alpha \). For each \( n \geq 0 \),

\[
\sum H_n (C^\alpha) \cong \bigoplus_{\alpha \in A} H_n (C^\alpha) \to H_n \left( \bigoplus_{\alpha \in A} C^\alpha \right)
\]

is an isomorphism.

If \( D \) is a chain complex and for each \( \alpha \in A \), \( f^\alpha : C^\alpha \to D \) is a chain map, then the sequence of homomorphisms

\[
\sum f^\alpha_n : \bigoplus_{\alpha \in A} C^\alpha_n \to D_n \quad n = 0, 1, 2, \ldots
\]

consists constitute a chain map \( \sum f^\alpha : \bigoplus_{\alpha \in A} C^\alpha \to D \).
Proposition 3.21. Let $X$ be a topological space, and suppose $X = \biguplus_{x \in A} X_x$ as a set, where $\biguplus$ denotes "disjoint union", and suppose each $X_x$ is a union of path-components of $X$ (e.g. if each $X_x$ is itself a path-component of $X$). Write $i_x : X_x \rightarrow X$ for the inclusion map. Then
\[
\sum S(i_x) : \bigoplus_{x \in A} S(X_x) \rightarrow S(X)
\]
is a chain isomorphism.

Proof. Since each $S(i_x)$ is a chain map, $\sum S(i_x)$ is a chain map by 3.18. Let $n \geq 0$. Since $\Delta n$ is path-connected (being a convex set in $\mathbb{R}^n$), each singular $n$-simplex in $X$ has image contained in some path component of $X$, and hence in one of the $X_x$. 

Definition 3.19. Let $C$ and $D$ be chain complexes. A chain map $f : C \rightarrow D$ is called a chain isomorphism if each $f_n : C_n \rightarrow D_n$, $n \geq 0$, is an isomorphism.
Thus, for each singular n-simplex $\xi: \Delta_n \to X$, there exist a unique $x_\xi \in A$ and a unique singular n-simplex $\bar{\xi}: \Delta_n \to \{x_\xi\}$ such that the diagram

$$\begin{array}{ccc}
\Delta_n & \xrightarrow{\bar{\xi}} & \{x_\xi\} \\
\uparrow & & \uparrow \\
\{x_\xi\} & \rightarrow & \{x_\xi\}
\end{array}$$

commutes.

In each $p \in A$, let $j_p^\times: S_n(X_p) \to \bigoplus_{x \in A} S_n(X_x)$ denote the canonical inclusion. Define

$$f_n: S_n(X) \xrightarrow{} \bigoplus_{x \in A} S_n(X_x) \text{ by } f_n(\xi) = j_{x_\xi}^\times(\xi).$$

It is straightforward to check that $\Sigma S_n(i_x)$ and $f_n$ are inverse isomorphisms to one another.

**Corollary 3.22.** Under the hypotheses of 3.21,

$$\Sigma H_n(i_x): \bigoplus_{x \in A} H_n(X_x) \rightarrow H_n(X)$$

is an isomorphism for all $n$.

**Proof.** By 3.18 we have the commutative diagram

$$\begin{array}{ccc}
\bigoplus_{x \in A} H_n(X_x) & \xrightarrow{\Sigma H_n(j_x)} & \Sigma H_n(j_x) \\
\uparrow & & \uparrow \\
\Sigma H_n(j_x) & \rightarrow & H_n(X)
\end{array}$$

where $j_x^\times: S(X_p) \to \bigoplus_{x} S(X_x)$ is the canonical inclusion.

$\Sigma H_n(j_x)$ is an isomorphism by 3.18, and $H_n(\Sigma S(i_x))$ is an isomorphism by 3.21 and 3.20. Thus, $\Sigma H_n(i_x)$ is an isomorphism.
Corollary 3.23. In any topological space $X$, $\text{Ho}(X)$ is isomorphic to a direct sum of copies of $\mathbb{Z}$, with one copy in each path component of $X$.

Proof. Write $X = \bigsqcup_{x \in A} X_x$ where the $X_x$ are the path components of $X$. By 3.22,

$$\text{Ho}(X) \cong \bigoplus_{x \in A} \text{Ho}(X_x),$$

and by 3.13, each $\text{Ho}(X_x)$ is isomorphic to $\mathbb{Z}$.

4. Homology of Pairs and Quotients

Definition 4.1. Let $C$ be a chain complex, $A$ a subcomplex, $D$ of $C$ is a chain complex such that for all $n \in \mathbb{Z}$, $D_n$ is a subgroup of $C_n$, and such that the inclusion maps $i_n: D_n \to C_n$ constitute a chain map $i: D \to C$. (Thus the boundary maps in $D$ are obtained by restriction of the boundary maps in $C$.)

Example 4.2. Let $X$ be a topological space, and $A$ a sub-space of $X$. Let $i: A \to X$ denote the inclusion map. Then for each $n$, $S_n(i): S_n(A) \to S_n(X)$ is 1-1. (For if $\sum \eta_k \varphi_k = 0$ where $\varphi_k \in T_k$ are distinct singular $n$-simplices in $A$ and $\eta_k \in \mathbb{Z}$ for $1 \leq k \leq r$, then $\sum_{k=1}^r \eta_k (i^* \varphi_k) = 0$. Since $i$ is 1-1, $i^* T_1, \ldots, i^* T_r$ are distinct singular $n$-simplices in $X$, and so the $\eta_k$ must all be 0.) We will identify $S_n(i)$ with its image in $S_n(X)$ under the non-singular $S_n(i)$. In this way, $S(A)$ becomes a sub-chain complex of $S(X)$. The inclusion $S(A) \to S(X)$ is the chain map $S(i): S(A) \to S(X)$.

Let $C$ be a chain complex, $D$ a subcomplex of $C$, and let $i: D \to C$ be the inclusion. To each $n \in \mathbb{Z}$ we have the commutative diagram
Thus, passing to quotients, we obtain a group homomorphism
\[ \varphi : \mathbb{C}_n / \mathbb{D}_n \rightarrow \mathbb{C}_{n-1} / \mathbb{D}_{n-1} \]
given by \( \varphi (c + \mathbb{D}_n) = \varphi (c) + \mathbb{D}_{n-1} \).

**Exercise 4.3.** If \( \mathbb{D} \) is a subcomplex of \( \mathbb{C} \), the sequence of abelian groups \( \mathbb{C}_n / \mathbb{D}_n \), \( n = 0, 1, 2, \ldots \), together with the homomorphisms \( \varphi : \mathbb{C}_n / \mathbb{D}_n \rightarrow \mathbb{C}_{n-1} / \mathbb{D}_{n-1} \) constitute a chain complex. (We call this chain complex the quotient complex \( \mathbb{C} \) modulo \( \mathbb{D} \), and denote it \( \mathbb{C} / \mathbb{D} \).)

**Note:** If \( \mathbb{D} \) is a subcomplex of \( \mathbb{C} \) and \( \mathbb{C} \) is augmented, so is \( \mathbb{D} \) by restricting the augmentation \( \varepsilon : \mathbb{C}_0 \rightarrow \mathbb{Z} \) to \( \mathbb{D}_0 \). However, \( \mathbb{D}_0 \) is not necessarily contained in the kernel of \( \varepsilon \), and so we cannot generally pass to quotients to obtain an augmentation \( \varepsilon : \mathbb{C}_0 / \mathbb{D}_0 \rightarrow \mathbb{Z} \).

**Example 4.4.** Let \((X,A)\) be a topological pair. Then by 4.2 and 4.3 we obtain a chain complex \( S(X) / S(A) \).

We denote this quotient complex by \( S(X,A) \) and call it the singular complex of the topological pair \((X,A)\).

Note that if \( A \neq \emptyset \), then \( S(X,A) \) does not inherit an augmentation from \( S(X) \).

**Definition 4.5.** Let \((X,A)\) be a topological pair. The homology group \( H_n(S(X,A)) \) is called the \( n \)th singular homology group of the pair \((X,A)\), and denoted \( H_n(X,A) \).

**Caution:** In general, \( H_n(X,A) \neq H_n(X) / H_n(A) \). In fact, the quotient \( H_n(X) / H_n(A) \) is generally meaningless.
Hn(\cdot): Hn(A) \to Hn(X)

is usually not 1-1, even though
Sn(\cdot): Sn(A) \to Sn(X)

is 1-1.

Note 4.6. Since Sn(\emptyset) = 0 for all n, S(X, \emptyset) is canonically identified with S(X) and Hn(X, \emptyset) is canonically identified with Hn(X) for all n \in \mathbb{Z}.

Let \( f : (X, A) \to (Y, B) \) be a map of topological pairs (see 0.4). From commutativity of

\[
\begin{array}{ccc}
A & \xrightarrow{f^\circ} & B \\
\downarrow & & \downarrow \\
X & \xrightarrow{f^\circ} & Y
\end{array}
\]

(see 0.12, 0.13 in notation) we obtain a commutative diagram of chain maps

\[
\begin{array}{ccc}
S(A) & \xrightarrow{S(f^\circ)} & S(B) \\
\downarrow & & \downarrow \\
S(X) & \xrightarrow{S(f^\circ)} & S(Y)
\end{array}
\]

Thus for each n \in \mathbb{Z} we obtain, by passage to quotients, a group homomorphism

Sn(f): Sn(X, A) \to Sn(Y, B).

Explicitly, Sn(f)(c + Sn(A)) = Sn(f^\circ)(c) + Sn(B).

Exercise 4.7. With notation as above, the homomorphisms Sn(f), n = 0, 1, 2, ... constitute a chain map

S(f): S(X, A) \to S(Y, B).

The rule which assigns to each topological pair (X, A) its singular complex S(X, A), and to each map \( f : (X, A) \to (Y, B) \) the chain map \( S(f): S(X, A) \to S(Y, B) \), is a covariant functor S: TP \to CG.
Note 4.8. By associating to each topological space $X$ the topological pair $(X, \emptyset)$, and to each continuous map $f: X \to Y$ the obvious map of topological pairs $f_\# : (X, \emptyset) \to (Y, \emptyset)$, we can regard $\text{Top}$ as a subcategory of $\text{TP}$. The functor $S: \text{Top} \to \text{Gr}$ in 4.7 is an extension of the earlier functor $S: \text{Top} \to \text{Gr}$ (2.15).

Exercise 4.9. Let $X = \bigsqcup_{x \in A} X_x$ be as in 3.21, and suppose $A \subset X$. For each $x \in A$, let $i_x : (X_x, X_x \cap A) \to (X, A)$ denote the inclusion map of pairs. Then

$$\sum S(i_x) : \bigoplus_{x \in A} S(X_x, X_x \cap A) \to S(X, A)$$

is a chain isomorphism. Thus for each $n \geq 2$,

$$\sum H_n(i_x) : \bigoplus_{x \in A} H_n(X_x, X_x \cap A) \to H_n(X, A)$$

is a group isomorphism.

Exercise 4.10. Let $(X, A)$ be a topological pair with $X$ path-connected and $A \neq \emptyset$. Then $H_0(X, A) = 0$.

Definition 4.11. A 3-term sequence of abelian groups and homomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is said to be exact at $B$ if $\ker g = \text{im} f$.

A sequence of abelian groups and homomorphisms

$$\ldots \to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \ldots$$

(which may or may not terminate in either direction) is an exact sequence of abelian groups if each 3-term segment of it is exact at its middle term.

Exercise 4.12. i) $0 \to A \to 0$ is exact iff $A = 0$.

ii) $A \xrightarrow{f} B \to 0$ is exact iff $f$ is onto.

iii) $0 \to A \xrightarrow{f} B$ is exact iff $f$ is 1-1.
iv) \( 0 \to A \xrightarrow{f} B \to C \) is exact iff \( f \) is an injection and \( C \) is the cokernel of \( f \).

v) Suppose \( A \) is a subgroup of \( B \). Let \( i : A \to B \) denote the inclusion map, and \( p : B \to B/A \) the natural projection (i.e., \( p(b) = b + A \) for all \( b \in B \)). Then

\[
0 \to A \xrightarrow{i} B \xrightarrow{p} B/A \to 0
\]

is exact.

**Definition 4.13.** An exact sequence of the form

\[
0 \to A \to B \to C \to 0
\]

is called a short exact sequence.

Given a short exact sequence as in 4.13, we can identify \( A \) with its image in \( B \), and \( C \) with \( B/A \). Thus every short exact sequence of abelian groups is essentially of the form 4.12(v).

If \( C \) is a chain complex, \( C_0 \xrightarrow{d_1} C_1 \xrightarrow{d_2} C_2 \xrightarrow{d_3} \ldots \)

is exact in general. In fact for \( n > 0 \), exactness occurs at \( C_n \) if and only if \( H_n(C) = 0 \).

Thus the homology groups of a chain complex measure the deviation of the sequence of its boundary maps from being exact.

**Definition 4.14.** A sequence of chain complexes and chain maps

\[
\ldots \to C_k \xrightarrow{f_k} C_{k-1} \xrightarrow{f_{k-1}} C_{k-2} \xrightarrow{f_{k-2}} \ldots
\]

is exact if for each \( n \in \mathbb{Z} \), the sequence of abelian groups and homomorphisms

\[
\ldots \to (C_k)_n \xrightarrow{(f_k)_n} (C_{k-1})_n \xrightarrow{(f_{k-1})_n} (C_{k-2})_n \xrightarrow{(f_{k-2})_n} \ldots
\]

is exact.

Denote the chain complex consisting only of 0 groups by 0. An exact sequence \( 0 \to C \to D \to E \to 0 \) of chain complexes is called a short exact sequence of chain complexes.
Exercise 4.15. Let $D$ be a subcomplex of $C$. For each $n \in \mathbb{Z}$, let $p_n : C_n \to C_n / D_n$ be the natural projection. Then the sequence of group homomorphisms $p_n$, $n = 0, 1, 2, \ldots$, constitute a chain map $p : C \to C / D$. The sequence $0 \to D \xrightarrow{i} C \xrightarrow{p_i} C / D \to 0$, where $i$ is the inclusion, is a short exact sequence of chain complexes.

Example 4.16. Let $(X,A)$ be a topological pair. Let $i : A \to X$ and $j : (X,A) \to (X,A)$ be the inclusion maps (i.e. $j^* = i_X$). Note that $S(i) : S(X) = S(X,A) \to S(X,A)$ as comprised of the natural projections $S_n(X) \to S_n(X)/S_n(A)$. Thus we have a short exact sequence of chain complexes and chain maps

$$0 \to S(A) \xrightarrow{S(i)} S(X) \xrightarrow{S(j)} S(X,A) \to 0.$$ 

Proposition 4.17. Let $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$ be a short exact sequence of chain complexes. Then for each $n \in \mathbb{Z}$, the sequence of abelian groups

$$\begin{array}{c}
H_n(C) \xrightarrow{\text{hom}(f)} H_n(D) \xrightarrow{\text{hom}(g)} H_n(E)
\end{array}$$

is exact.

(Caution: $\text{hom}(f)$ need not be 1-1, $\text{hom}(g)$ need not be onto.)

Proof. Let $a \in H_n(C)$. Say $a = [\gamma]$, $\gamma \in \pi_1(C,
\text{base point})$. Then $H_n(\gamma) = \text{hom}(f)(a) = [\gamma \cdot f] = [\alpha] = 0$ since $\gamma \cdot f = 0$.

Now suppose $a \in \ker H_n(g)$. Say $a = [\gamma]$, $\gamma \in \pi_1(D,
\text{base point})$. We have the commutative diagram with exact rows

$$\begin{array}{ccc}
D_{n+1} & \xrightarrow{g_{n+1}} & E_{n+1} \\
\downarrow & & \downarrow \\
C_n & \xrightarrow{f_n} & D_n & \xrightarrow{g_n} & E_n \\
\downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{c_{n-1}} & C_{n-1} & \xrightarrow{f_{n-1}} & D_{n-1}
\end{array}$$
Then \( C = H_n(g)(a) = [g_n(z)] \) and so \( g_n(z) \in B_n(\mathbb{Z}) \).

Thus there exists \( y \in E_{n+1} \) such that \( 2y = g_n(z) \).

Since \( q_{n+1} \) is onto, there exists \( x \in D_{n+1} \) such that \( q_{n+1}(x) = y \).

By commutativity of (1), \( g_n(z - 2x) = 0 \).

By the exactness of the middle row, there exists \( w \in C_n \)

such that \( f_n(w) = z - 2x \).

We proceed to show that \( w \in Z_n(C) \). Since \( f_{n-1} \) is 1-1, it suffices to show \( f_{n-1}(x) = 0 \).

By commutativity of (2), \( f_{n-1}(w) = \partial f_n(w) \)

\( = \partial (z - 2x) = \partial z - 2 \partial x = 0 \) since \( z \) is a cycle and \( \partial z = 0 \).

Thus we have a homology class \([w] \in H_n(C)\).

We have \( H_n(f)([w]) = [f_n(w)] = [z - 2x] = [z] = a \), and so \( ker H_n(f)^2 \subset im H_n(f) \).

**Corollary 4.18.** For any topological pair \((X,A)\) and each \( n \geq 2 \), the sequence

\[
H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X \cup A)
\]

is exact, where \( i : A \to X \), \( j : (X, \emptyset) \to (X, A) \) are the inclusion maps.

**Definition 4.19.** Suppose \( 0 \to C^1 \xrightarrow{f} C^2 \xrightarrow{g} C^3 \to 0 \)

and \( 0 \to D^1 \xrightarrow{f} D^2 \xrightarrow{g} D^3 \to 0 \) are short exact sequences of chain complexes. A morphism of short exact sequences of chain complexes from the first to the second is a triple of chain maps \( h^i : C_i \to D_i \), \( i = 1, 2, 3 \), such that the diagram of chain maps

\[
\begin{array}{ccc}
C^1 & \xrightarrow{f} & C^2 & \xrightarrow{g} & C^3 \\
\downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 \\
D^1 & \xrightarrow{f} & D^2 & \xrightarrow{g} & D^3
\end{array}
\]

commutes.
Example 4.20. \( \phi: (X,A) \to (Y,B) \) is a map of topological pairs, then \( S(\phi^p), S(\phi^q), S(\phi) \) constitute a map of short exact sequences of chain complexes from 
\[ 0 \to S(A) \to S(X) \to S(X,A) \to 0 \] 
and \[ 0 \to S(B) \to S(Y) \to S(Y,B) \to 0 \]

(Commutativity of the appropriate squares of chain maps follows from commutativity of the squares)

\[
\begin{array}{ccc}
A & \to & X \\
\downarrow f^p & & \downarrow f^p \\
B & \to & Y \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
(X,\emptyset) & \to & (X,A) \\
\downarrow f & & \downarrow f \\
(Y,\emptyset) & \to & (Y,B) \\
\end{array}
\]

Exercise 4.21. With composition of morphisms of short exact sequences of chain complexes defined in the obvious way, taking all short exact sequences of chain complexes as the class of objects, and the collection of all morphisms of short exact sequences of chain complexes as morphisms, we obtain a category, the category of short exact sequences of chain complexes, which we denote \( S\mathbf{CE} \).

Exercise 4.22. The rule which associates to each topological pair \((X,A)\) the short exact sequence of chain complexes \[ 0 \to S(A) \to S(X) \to S(X,A) \to 0 \] , and to each map of topological pairs \( f: (X,A) \to (Y,B) \) the morphism of short exact sequences of chain complexes \( f^p, f^q, f \), as a covariant functor from \( \mathbf{TP} \) to \( S\mathbf{CE} \).

Let \[ 0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0 \] be a short exact sequence of chain complexes. We proceed to construct \( \phi \) for each \( n \in \mathbb{Z} \), a natural homomorphism \( \phi: H_n(E) \to H_{n-1}(C) \). The construction is based on the following lemmas:
Lemma 4.23 (zig-zag lemma). Let \( 0 \to C \to D \to E \to 0 \) be a short exact sequence of chain complexes.
Let \( z \in \text{Z}_n(E) \). Then for each choice of \( y \in \text{D}_n \) such that \( g_n(y) = z \) (such \( y \) exist since \( g_n \) is onto), there exists a unique \( x \in \text{C}_{n-1} \) such that \( f_{n-1}(x) = \partial y \). Moreover, \( x \in \text{Z}_{n-1}(C) \), and the homology class of \( x \), \([x]\in\text{H}_{n-1}(C)\), depends only on \([z]\in\text{H}_n(E)\) (i.e. \([x]\) is independent of the representative cycle \( z \) of \([z]\in\text{H}_n(E)\), and of the choice of pull-back \( y \in \text{D}_n \) of \( z \).

The situation is illustrated in the following diagram:

\[
\begin{array}{cccc}
& y & \to & z \\
& \downarrow & & \downarrow \\
D_n & \xrightarrow{g_n} & E_n & \to 0 \\
& \downarrow & \partial & \\
x & \to & \partial y & \\
0 & \to C_{n-1} & \xrightarrow{f_{n-1}} & \text{D}_{n-1} \\
\end{array}
\]

Proof. The proof is by diagram chasing in the following diagram:

\[
\begin{array}{cccc}
D_{n+1} & \xrightarrow{g_{n+1}} & E_{n+1} & \to 0 \\
\downarrow \circlearrowleft & & \downarrow \circlearrowleft & \\
C_n & \xrightarrow{f_n} & D_n & \xrightarrow{g_n} E_n & \to 0 \\
\downarrow \circlearrowleft & & \downarrow \circlearrowleft & & \downarrow \circlearrowleft \\
C & \to C_{n-1} & \xrightarrow{f_{n-1}} & D_{n-1} & \xrightarrow{g_{n-1}} E_{n-1} \\
\downarrow \circlearrowleft & & \downarrow \circlearrowleft & & \downarrow \circlearrowleft \\
C & \to C_{n-2} & \xrightarrow{f_{n-2}} & D_{n-2} & \\
\end{array}
\]

The above commutes, and the rows are exact. Let \( y \in \text{D}_n \) be such that \( g_n(y) = z \). From commutativity of \( 1 \), \( g_{n-1}(\partial y) = \partial g_n(y) = \partial z = 0 \). Thus \( \partial(y) \in \text{Ker} g_{n-1} \). By exactness, \( \partial(y) \in \text{Im} f_{n-1} \) and so there exists a unique \( x \in \text{C}_{n-1} \) (since \( f_{n-1} \) is 1-1) such that \( f_{n-1}(x) = \partial y \).
From commutativity of (2), $f_{n-2} \partial(x) = \partial f_{n-1}(x) = \partial y = 0$. Thus, since $f_{n-2}$ is 1-1, we have
$\partial x = 0$, i.e. $x \in \mathbb{Z}_{n-1}(C)$.

It remains to show that if $z' \in \mathbb{Z}_n(E)$, $y' \in \mathbb{D}_n$, $x' \in \mathbb{C}_{n-1}$ are such that $[z'] = [z']$, $g_n(y') = z'$, and $f_{n-1}(x') = \partial y'$, (and so $x'$ is necessarily a cycle by the above), then $[x'] = [x']$.

Since $[z'] = [z']$, there exists $a \in \mathbb{E}_{n+1}$ such that $\partial a = z' - z'$. Since $g_{n+1}$ is onto, there exists $b \in \mathbb{D}_{n+1}$ such that $g_{n+1}(b) = a$. Then $g_n(y' - \partial b) = z' - z' - g_n(\partial b) = z' - z' - \partial a = 0$, and so $y' - \partial b \in \ker g_n = \im f_n$. Thus, there exists $c \in \mathbb{C}_n$ such that $f_n(c) = y' - \partial b$. We then have
$f_{n-1}(x' - x') = \partial (y' - \partial b) = \partial (y' - \partial b)$ (since $\partial^2 b = 0$)
$= \partial f_n(c) = f_{n-1} \partial (c)$ by commutativity of (4).
Thus, since $f_{n-1}$ is 1-1, $x - x' = \partial c$ and so $[x] = [x']$.

Thus, given a short exact sequence of chain complexes $0 \rightarrow C \xrightarrow{f} D \xrightarrow{g} E \rightarrow 0$, there is a well-defined function $\partial : H_n(E) \rightarrow H_{n-1}(C)$ (for each $n \in \mathbb{Z}$) given by $\partial [z] = \left[f_{n-1} \partial g_n^{-1}(z)\right]$, $z \in \mathbb{Z}_n(E)$, where $g_n^{-1}(z)$ denotes any $y \in \mathbb{D}_n$ such that $g_n(y) = z$.

**Proposition 4.24.** Let $0 \rightarrow C \xrightarrow{f} D \xrightarrow{g} E \rightarrow 0$ be a short exact sequence of chain complexes. Then, for each $n \in \mathbb{Z}$, the function $\partial : H_n(E) \rightarrow H_{n-1}(C)$ as above is a group homomorphism. Moreover, if $x, \beta, \delta$ is a morphism of short exact sequences of chain complexes from the above to $0 \rightarrow C' \xrightarrow{f'} D' \xrightarrow{g'} E' \rightarrow C$, then the diagram

$$
\begin{array}{ccc}
H_n(E) & \xrightarrow{\partial} & H_{n-1}(C) \\
H_n(E') \downarrow & & \downarrow H_{n-1}(\delta) \\
H_n(E') & \xrightarrow{\partial} & H_{n-1}(C')
\end{array}
$$

commutes. (The last property is paraphrased by saying that "\( \partial \) is natural."
Proof. Let \( z, z' \in \mathbb{Z}^n(E) \). Choose any \( y, y' \in D_n \) such that \( g_n(y) = z \), \( g_n(y') = z' \), and let \( x, x' \in C_{n-1} \) be the unique elements such that \( f_n^{-1}(x) = y \), \( f_n^{-1}(x') = y' \). Then by definition, \( \partial [z] = [x] \), \( \partial [z'] = [x'] \). Since

\[
\partial [z + z'] = [x + x'] = [x] + [x'] = \partial [z] + \partial [z'],
\]

and so \( \partial \) is a group homomorphism.

Let \( \alpha, \beta, \gamma \) be as stated, and \( z \in \mathbb{Z}^n(E) \), \( y \in D_n \), \( x \in C_{n-1} \) as above. We have the commutative diagram

\[
\begin{array}{ccc}
C_{n-1} & \xrightarrow{f_{n-1}} & D_{n-1} \\
\downarrow{\alpha_{n-1}} & & \downarrow{\beta_{n-1}} \\
C'_{n-1} & \xrightarrow{f'_{n-1}} & D'_{n-1}
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{g_n} & \mathbb{E}_n \\
\downarrow{\delta_n} & & \downarrow{\delta_n} \\
\mathbb{Z}_n & \xrightarrow{g'_n} & \mathbb{E}_n'
\end{array}
\]

We have \( \partial [z] = [x] \), \( H_n(\alpha)[x] = [\delta_n(z)] \), \( H_n(\gamma)[x] = [\alpha_{n-1}(x)] \).

By commutativity of the above diagram, \( g'_n \beta_n(y) = \delta_n \delta_n(y) = \delta_n(z) \) and \( f'_{n-1} \alpha_{n-1}(x) = f_{n-1} f_{n-1}(x) = f_{n-1} \partial y = \partial f_n(y) \).

Thus \( \partial [\delta_n(z)] = [\alpha_{n-1}(x)] \). Thus \( \partial H_n(\gamma)[z] = \partial [\delta_n(z)] \)

\( = [\alpha_{n-1}(x)] \), \( H_n(\alpha)[x] = H_n(\gamma)[x] = H_n(\alpha) \partial [z] \), and so

\( \partial H_n(\gamma) = H_n(\alpha) \partial \).

Definition 4.23. The natural map \( \partial : H_n(E) \to H_{n-1}(C) \)

of 4.24 is called the \( n \)th connecting homomorphism for the short exact sequence of chain complexes

\( 0 \to C \to D \xrightarrow{\partial} E \to 0 \). The long sequence

\[
\cdots \to H_n(C) \xrightarrow{H_n(\alpha)} H_n(D) \xrightarrow{H_n(\beta)} H_n(E) \xrightarrow{\partial} H_{n-1}(C) \xrightarrow{H_{n-1}(\gamma)} H_{n-1}(D) \to \cdots
\]

is called the homology sequence of the above short exact sequence of chain complexes.

If \((X, A)\) is a topological pair and \( i : A \to X \), \( j : (X, \emptyset) \to (X, A) \) denote the inclusions, the sequence

\[
\partial \to H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{H_{n-1}(\beta)} H_{n-1}(X) \to \cdots
\]
(homology sequence of $C \rightarrow S(X) \rightarrow S(X,A) \rightarrow C$)

is called the homology sequence of the topological pair $(X,A)$.

By 4.17, the homology sequence of $C \rightarrow C \rightarrow D \rightarrow E \rightarrow 0$ is exact at $H_n(D)$ for each $n \in \mathbb{Z}$. We will see below that it is exact everywhere.

As a consequence of the naturality of $\partial$ (4.24), where $\alpha, \beta, \gamma$ constitute a morphism of short exact sequences of chain complexes from $C \rightarrow C' \rightarrow D' \rightarrow E' \rightarrow 0$ to $C \rightarrow C' \rightarrow D' \rightarrow E' \rightarrow 0$, we obtain the commutative diagram:

$\cdots \rightarrow H_n(C) \xrightarrow{H_n(f)} H_n(D) \xrightarrow{H_n(g)} H_n(E) \xrightarrow{\partial} H_{n-1}(C) \rightarrow H_{n-1}(D) \rightarrow \cdots$

$H_n(f)\downarrow \quad \downarrow H_n(f') \quad \downarrow H_n(g) \quad \downarrow H_{n-1}(f) \quad H_{n-1}(f')\downarrow \quad \downarrow H_{n-1}(g) \quad \downarrow H_{n-1}(g') \quad \downarrow \cdots$

$H_n(C') \rightarrow H_n(D') \rightarrow H_n(E') \rightarrow H_{n-1}(C') \rightarrow H_{n-1}(D') \rightarrow \cdots$

Consequence 4.26. If $f : (X,A) \rightarrow (Y,B)$ is a map of topological pairs, the diagram

$\cdots \rightarrow H_n(A) \xrightarrow{H_n(f)} H_n(X) \xrightarrow{H_n(i)} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} H_{n-1}(X) \rightarrow \cdots$

$H_n(f)\downarrow \quad \downarrow H_n(f') \quad \downarrow H_n(i) \quad \downarrow H_{n-1}(f) \quad \downarrow H_{n-1}(f') \quad \downarrow \cdots$

$H_n(B) \xrightarrow{H_n(i')} H_n(Y) \xrightarrow{H_n(j)} H_n(Y,B) \xrightarrow{\partial} H_{n-1}(B) \rightarrow H_{n-1}(Y) \rightarrow \cdots$

$H_n(i') \quad H_n(j') \quad \downarrow H_{n-1}(i') \quad H_{n-1}(j')$ commutes, where $i, j, i', j'$ are the appropriate inclusion maps.

Proposition 4.27. The homology sequence of a short exact sequence of chain complexes is exact.

Proof. Let $C \rightarrow C' \rightarrow D \rightarrow E \rightarrow 0$ be a short exact sequence of chain complexes. By 4.17, it remains only to check that

$H_{n+1}(E) \xrightarrow{\partial} H_n(C) \xrightarrow{H_n(f)} H_n(D)$

and

$H_n(D) \xrightarrow{H_n(g)} H_n(E) \xrightarrow{\partial} H_{n-1}(C)$ are exact.
Let \( z \in \mathbb{Z}_{n+1}(E) \), choose \( y \in \mathbb{D}_{n+1} \) such that
\[ j_{n+1}(y) = z \]
and let \( x \in \mathbb{C}_{n} \) be such that \( f_{n}(x) = 2y \).
Then \( \partial \mathbb{D} \mathbb{[}x] = [x] \), and so \( \mathbb{H}_{n}(f) \partial \mathbb{D} \mathbb{[}x] = \mathbb{H}_{n}(f) [x] = [f_{n}(x)] = [2y] = 0 \). Thus \( \text{im} \partial \subseteq \ker \mathbb{H}_{n}(f) \).

Let \( x \in \mathbb{Z}_{n}(C) \) be such that \([x] \in \ker \mathbb{H}_{n}(f) \).
Then \( 0 = \mathbb{H}_{n}(f) [x] = [f_{n}(x)] \), and so \( f_{n}(x) = 2y \) for some \( y \in \mathbb{D}_{n+1} \).

From commutativity of
\[
\mathbb{D}_{n+1} \xrightarrow{j_{n+1}} E_{n+1} \\
\partial \downarrow \quad \downarrow \partial \\
\mathbb{D}_{n} \xrightarrow{g_{n}} E_{n}
\]
we have \( \partial j_{n+1}(y) = g_{n} \partial \mathbb{[y]} = g_{n} f_{n}(y) = 0 \), and so \( g_{n+1}(y) \in \mathbb{Z}_{n+1}(E) \).

Then \( \partial \mathbb{D} [j_{n+1}(y)] = [f_{n+1}^{-1} \partial \mathbb{D} g_{n+1}(y)] = [f_{n+1}^{-1} \partial \mathbb{D} f_{n}(x)] = [x] \), and so \([x] \in \text{im} \partial \).

Thus \( \text{ker} \mathbb{H}_{n}(f) \subseteq \text{im} \partial \).

Let \( z \in \mathbb{Z}_{n}(D) \).
Then \( \partial \mathbb{H}_{n}(g)[z] = \partial \mathbb{D} g_{n}(z) = \partial \mathbb{D} f_{n}^{-1} \partial \mathbb{D} g_{n}(z) = \partial \mathbb{D} f_{n}^{-1} \partial \mathbb{D} f_{n}(x) = \partial \mathbb{D} [x] = \partial \mathbb{D} 0 = 0 \).

Thus \( \text{ker} \mathbb{H}_{n}(g) \subseteq \ker \partial \).

Let \( z \in \mathbb{Z}_{n}(E) \) be such that \( \partial \mathbb{D} \mathbb{[z]} \) is \( 0 \). Choose \( y \in \mathbb{D}_{n} \) such that \( j_{n}(y) = z \), and let \( x \in \mathbb{C}_{n-1} \) be such that \( f_{n-1}(x) = 2y \).
Then \([x] = \partial \mathbb{D} \mathbb{[z]} = 0 \), and so \( x = za \) for some \( a \in \mathbb{C}_{n} \).

From commutativity of
\[
\mathbb{C}_{n} \xrightarrow{f_{n}} \mathbb{D}_{n} \\
\partial \downarrow \quad \downarrow \partial \\
\mathbb{C}_{n-1} \xrightarrow{f_{n-1}} \mathbb{D}_{n-1}
\]
we have \( \partial f_{n}(a) = f_{n-1} \partial f_{n}(a) = f_{n-1}(x) = 2y \), and so \( y - f_{n}(a) \in \mathbb{Z}_{n}(D) \).
We have \( \mathbb{H}_{n}(g) [y - f_{n}(a)] = [g_{n}(y) - g_{n+1}(a)] = [j_{n}(y)] \) (since \( g_{n+1} f_{n} = 0 \)) = \([z]\), and so \([z] \in \text{im} \mathbb{H}_{n}(g) \).
Thus \( \ker \partial \subseteq \text{im} \mathbb{H}_{n}(g) \), completing the proof of exactness at \( \mathbb{H}_{n}(E) \).
As a corollary we have

Proposition 4.28. (Exactness Property). For topological pairs $(X, A)$, the connecting homomorphism $\partial: H_n(X, A) \to H_{n-1}(A)$ is natural (i.e., 4.26 holds), and the homology sequence of the pair $(X, A)$ (4.25) is exact.

Corollary 4.29. Let $(X, A)$ be a topological pair with $X$ path-connected and $A \neq \emptyset$. Then $H_0(X, A) = 0$. (See 4.10).

Proof. By 4.28 and 3.12, we have the commutative diagram with exact row

$$
\begin{array}{cccccc}
H_0(A) & \xrightarrow{i_*} & H_0(X) & \xrightarrow{j_*} & H_0(X, A) & \xrightarrow{\partial} & 0 \\
\downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & & \\
\mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \\
\end{array}
$$

where $i, j$ are the inclusions. By 3.13,

$e: H_0(A) \to \mathbb{Z}$ is onto and $e: H_0(X) \to \mathbb{Z}$ is an isomorphism. It follows that $H_0(i_*)$ is onto. Thus by exactness, $H_0(X, A) = 0$.

5. **Homotopy**

Definition 5.1. Let $f, g: (X, A) \to (Y, B)$ be maps of topological pairs. A homotopy from $f$ to $g$ is a map of topological pairs $H: (X \times I, A \times I) \to (Y, B)$ such that for all $x \in X$, $H(x, 0) = f(x)$, $H(x, 1) = g(x)$. We say $f$ is homotopic to $g$ (denoted $f \simeq g$) if there exists a homotopy from $f$ to $g$. We also write $H: f \simeq g$ if $H$ is a homotopy from $f$ to $g$. ($I = [0, 1]$).

Proposition 5.2. $\simeq$ is an equivalence relation on the set of all maps of topological pairs from $(X, A)$ to $(Y, B)$. 
Proof. Let \( f : (X,A) \to (Y,B) \) be a map of topological pairs. Define \( H : (X \times I, A \times I) \to (Y, B) \) by \( H^t(x) = (f(x)) \) for all \((x,t)) \in X \times I.\) Then \( f \cong H \) and \( \sigma \cong h \).

Suppose \( H \cong f \) and \( f \cong g \). Define \( K : (X \times I, A \times I) \to (Y, B) \) by \( K^x(x,t) = \begin{cases} H^x(x,t) & \text{if } 0 \leq t \leq 1/2 \\ f^x(x,t) & \text{if } 1/2 \leq t \leq 1. \end{cases} \) Then \( f \cong h \) and \( \sigma \cong k \) is transitive.

Proposition 5.3. Suppose \( f_1, f_2 : (X,A) \to (Y,B) \) and \( g_1, g_2 : (Y,B) \to (Z,C) \) are maps of topological pairs such that \( f_1 \cong f_2 \) and \( g_1 \cong g_2 \). Then \( g_1 f_1 \cong g_2 f_2 \).

Proof. Say \( f_1 \cong f_2 \) and \( g_1 \cong g_2 \). Then

\[
\begin{align*}
g_1 f_1 &\cong g_1 f_2 \cong g_2 f_2. & \text{Thus } K \cong g_1 f_1 \cong g_2 f_2.
\end{align*}
\]

Exercise 5.4. Suppose \( f, g : (X,A) \to (Y,B) \) are maps of topological pairs such that \( f \cong g \). Then embedding \( J^p \) in \( J^p \) in the canonical way (4.8), we have \( f^p \cong g^p \) and \( f^p \cong g^p \). (Caution: the converse is false.)

Theorem 5.5. (Homotopy Property). If \( f, g : (X,A) \to (Y,B) \) are homotopic, then \( H_n(f) = H_n(g) : H_n(X,A) \to H_n(Y,B) \) for all \( n \in \mathbb{Z} \).

The proof of 5.5 is long (but interesting). We will postpone it till a later section in order not to detract from the overall picture of homotopy theory and show it is used. We proceed to draw some consequences of 5.5, in conjunction with the previous material.
Definition 5.6. Two topological pairs \((X, A)\) and \((Y, B)\) have the same homotopy type (or are homotopy equivalent) if there exist maps of topological pairs \(f : (X, A) \rightarrow (Y, B)\) and \(g : (Y, B) \rightarrow (X, A)\) such that \(gf \simeq 1_{(X, A)}\) and \(fg \simeq 1_{(Y, B)}\). In this case, \(f\) and \(g\) are called homotopy equivalences. (Two topological spaces \(X\) and \(Y\) have the same homotopy type if the topological pairs \((X, \emptyset)\) and \((Y, \emptyset)\) do.)

Proposition 5.7. If \(f : (X, A) \rightarrow (Y, B)\) is a homotopy equivalence, then for all \(n \in \mathbb{Z}\), \(H_n(f) : H_n(X, A) \rightarrow H_n(Y, B)\) is an isomorphism.

Proof. Let \(g : (Y, B) \rightarrow (X, A)\) be a homotopy inverse to \(f\) (i.e., a map of topological pairs such that \(gf \simeq 1_{(X, A)}\) and \(fg \simeq 1_{(Y, B)}\). By the homotopy property 5.5,

\[
H_n(gf) = H_n(1_{(X, A)}), \quad \text{since } H_n \text{ is a covariant functor},
\]

\(H_n(gf) = H_n(g)H_n(f)\) and \(H_n(1_{(X, A)}) = 1_{H_n(X, A)}\). Hence

\[
H_n(g)H_n(f) = 1_{H_n(X, A)}. \quad \text{Similarly } H_n(f)H_n(g) = 1_{H_n(Y, B)}.
\]

Thus \(H_n(f)\) and \(H_n(g)\) are isomorphisms, inverse to one another.

Definition 5.8. Let \((X, A)\) be a topological pair. \(A\) is a retract of \(X\) if there exists a continuous map \(r : X \rightarrow A\) such that \(r(a) = a\) for all \(a \in A\). In this case \(r\) is called a retraction of \(X\) onto \(A\).

\(A\) is called a deformation retract of \(X\) if there exists a retraction \(r : X \rightarrow A\) such that the composition \(X \xrightarrow{r} A \xrightarrow{i} X\) is homotopic to \(1_X\), where \(i\) is the inclusion map. In this case we say \(r\) is a deformation retraction.

Exercise 5.9. For \(n \geq 1\), the \((n-1)\)-sphere \(S^{n-1}\) is a deformation retract of \(\mathbb{R}^n - \{0\}\).

For any topological space \(X\), \(X \times \{0\}\) is a deformation
For $n > 1$, the closed unit $n$-disk $D^n = \{ x \in \mathbb{R}^n \mid \| x \| \leq 1 \}$
is a deformation retract of $\mathbb{R}^n$.

Proposition 5.10. If $r : X \to A$ is a deformation retraction and
$i : A \to X$ the inclusion map, then $r$ and $i$ are
homotopy equivalences. In fact, they are homotopy
inverse of one another.

Proof. By definition, $ir = 1_X$ and $ri = 1_A$.

Definition 5.11. A non-empty topological space $X$ is
counterable if for some $x_0 \in X$, $\{x_0\}$ is a deformation
retract of $X$. In this case we say $X$ is counterable to $x_0$.

Example 5.12. If $X$ is a non-empty convex subset of $\mathbb{R}^n$,
then for each $x_0 \in X$, $X$ is counterable to $x_0$. In fact
if $c : X \to \{x_0\}$ is the unique map and $i : \{x_0\} \to X$ the
inclusion, then $1_X = ci$ where $H : X \times [0,1] \to X$ is given
by $H(x,t) = xt_0 + (1-t)x$.

Thus $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{I}^n$, $\Delta^n$ are all counterable.

Corollary 5.13. If $X$ is counterable, then $\pi_n(X) = 0$ for
all $n \neq 0$, and $E : \pi_0(X) \to \mathbb{Z}$ is an isomorphism.

Proof. Say $X$ is counterable to $x_0$. Let $c : X \to \{x_0\}$ be
the unique map. By 5.10, $c$ is a homotopy equivalence and so by 5.7,
$\pi_0(c) : \pi_0(X) \to \pi_0(\{x_0\})$ is an
isomorphism for all $n \in \mathbb{Z}$. By the Extension Property
(3.2), $\pi_n(\{x_0\}) = 0$ for $n \neq 0$, and so $\pi_n(X) = 0$ for
$n \neq 0$.

By 3.12, the diagram
\[
\begin{array}{ccc}
\pi_0(X) & \xrightarrow{\pi_0(c)} & \pi_0(\{x_0\}) \\
E \downarrow & & \downarrow E \\
\mathbb{Z} & & \mathbb{Z}
\end{array}
\]
commutes.

$E : \pi_0(\{x_0\}) \to \mathbb{Z}$ is an isomorphism. By 3.13, thus
since $\pi_0(c)$ is an isomorphism, $E : \pi_0(X) \to \mathbb{Z}$ is an
isomorphism.
Proposition 5.14. Let \((X, A)\) be a topological pair with \(A\) contractible. Then, for all \(n \geq 0\), \(H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A)\)
is an isomorphism, where \(j: (X, \emptyset) \rightarrow (X, A)\) denotes the inclusion.

Proof. First consider the case \(n > 1\). By the Excision Property (4.28) the sequence

\[
H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A)
\]
is exact. By 5.13, \(H_n(A)\) and \(H_{n-1}(A)\) are both 0. Thus \(H_n(j)\) is an isomorphism by exactness.

We have the commutative diagram with exact rows

\[
\begin{array}{ccc}
H_i(A) & \rightarrow & H_i(X) \\
& \downarrow & \downarrow \\
& H_i(X, A) & \rightarrow H_i(A) \rightarrow H_i(X) \\
\hline
& & \\
& & \\
\end{array}
\]

By 5.13, \(H_i(A) = 0\) and so \(H_i(j)\) is \(1-1\). By 5.13, \(e: H_0(A) \rightarrow \mathbb{Z}\) is an isomorphism, and so \(H_0(i)\) is \(1-1\). It follows, by exactness, that \(H_i(j)\) is onto.

Example 5.15. We calculate the homology groups of the topological pair \((D^1, S^0)\). By 5.13, \(H_0(D^1) = 0\) for \(n \neq 0\) and since \(S^0\) is a discrete space, it follows from 3.22 and 3.2 that \(H_n(S^0) = 0\) for \(n \neq 0\). Thus, from exactness of \(H_n(D^1) \rightarrow H_n(D^1, S^0) \rightarrow H_{n-1}(S^0)\) and the fact that the extreme groups are 0 if \(n > 2\), we have \(H_n(D^1, S^0) = 0\) for \(n \geq 2\). We also know, from 4.29, that \(H_0(D^1, S^0) = 0\). Thus we need only to determine \(H_0(D^1, S^0)\).

\(S^0 = \{-1, 1\}\). We write \([-1]\) and \([1]\) respectively for the homology classes in \(H_0(S^0)\) represented by the singular 0-simplexes with image \(-1\) and \(1\) respectively. It follows from 3.2 and 3.22 that \(H_0(S^0)\) is free abelian on the two generators \([-1]\) and \([1]\). Furthermore, \(e[-1] = e[1] = 1\).

We have the commutative diagram with exact rows

\[
\begin{array}{ccc}
\cdots & \rightarrow & \cdots \\
& \downarrow & \downarrow \\
& \cdots & \rightarrow \cdots \\
\hline
& & \\
& & \\
\end{array}
\]
$\text{Lemma 5.16 ("5-lemma"). Suppose}

\[ A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4 \longrightarrow A_5 \]

\[ f_1 \downarrow \quad f_2 \downarrow \quad f_3 \downarrow \quad f_4 \downarrow \quad f_5 \downarrow \]

\[ B_1 \longrightarrow B_2 \longrightarrow B_3 \longrightarrow B_4 \longrightarrow B_5 \]

\text{is a commutative diagram of abelian groups and homomorphisms, with exact rows. Suppose } f_1, f_2, f_4, f_5 \text{ are isomorphisms. Then } f_3 \text{ is an isomorphism.}

\text{The proof of 5.16 is by diagram chasing, and is left as an exercise.}

\text{Proposition 5.17. Suppose } X \supseteq A \supseteq B \text{ are topological spaces such that the inclusion } i: B \to A \text{ is a homotopy equivalence. Then the inclusion of pairs } j: (X, B) \to (X, A) \text{ induces isomorphisms in homology in all dimensions.}

\text{Proof. We have } j^d = 1_X, j^C = i. \text{ Thus from 4.26 and the Exactness Property (4.28), we have the commutative diagram with exact rows.}
\[ H_n(B) \rightarrow H_n(X) \rightarrow H_n(X, B) \rightarrow H_{n-1}(B) \rightarrow H_{n-1}(X) \]

\[ H_n(l) \downarrow \downarrow H_n(l_x) \downarrow \downarrow H_n(l) \downarrow \downarrow H_n(l_x) \]

\[ H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A) \rightarrow H_{n-1}(X) \]

\[ H_n(l) \text{ and } H_{n-1}(l) \text{ are isomorphisms by 5.7.} \]
\[ H_n(l_x) = 1_{H_n}(x) \Rightarrow H_{n-1}(l_x) = 1_{H_{n-1}}(x) \text{, and hence these maps are isomorphisms. The result now follows by the 5-Lemma.} \]

6. Excision

In this section, the Excision Property for singular homology will be stated, and some consequences will be obtained. The proof (rather long, but interesting) will be postponed till a later section. The Excision Property, in conjunction with the preceding, makes homology computable for a large class of important spaces. Examples of such computations will be given in this section.

Definition 6.1. Let \((X, A)\) be a topological pair and \(U \subseteq A\). We say \(U\) can be \textit{excised} from \((X, A)\) if

\[ \text{for all } n \in \mathbb{Z}, \quad H_n(l): H_n(X - U, A - U) \rightarrow H_n(X, A) \text{ is an isomorphism, where } \iota: (X - U, A - U) \rightarrow (X, A) \text{ is the inclusion map. In this case, } \iota \text{ is called an \textit{excision map.}} \]

Theorem 6.2. (Excision Property). Let \((X, A)\) be a topological pair. Suppose \(U \subseteq X\) is such that \(\overline{U} \subset \text{int } A\), then \(U\) can be excised from \((X, A)\).

We postpone the proof till a later section.
Definition 6.3. A map of topological pairs \( f : (X, A) \to (Y, B) \) is called an expansive relative homeomorphism if:

1) \( f^* \) maps \( X - A \) homeomorphically onto \( Y - B \)
2) There exist subspaces \( A' \subset X, B' \subset Y \) satisfying
   i) \( A \subset \text{int} A' \), \( B \subset \text{int} B' \)
   ii) \( f^* (A' - A) = B' - B \)
   iii) The inclusions \( A \subset A' \), \( B \subset B' \) are homotopy equivalences.

Example 6.4. Let \( n \geq 1 \), then there is an expansive relative homeomorphism \( f : (D^n, S^{n-1}) \to (S^n, \{E_{n+1}\}) \).
(Recall \( E_n = (0, \ldots, 0, 1, 0, \ldots) \).) Explicitly,

\[
f^* (x) = \begin{cases} \left[ \sin \left( \pi \frac{1}{1 \times 11} \right) \right] \frac{x}{11} + \left[ \cos \left( \pi \frac{1}{1 \times 11} \right) \right] E_{n+1} & \text{if } x \neq 0 \\ E_{n+1} & \text{if } x = 0 \end{cases}
\]

A homeomorphism \( g : S^n - \{E_{n+1}\} \to D^n - S^{n-1} \) inverse to the restriction of \( f^* \) is given by

\[
g(y) = \begin{cases} \frac{1}{\pi} \cos^{-1} \left( \langle y, E_{n+1} \rangle \right) \frac{r(y)}{\|r(y)\|} & \text{if } y \neq E_{n+1} \\ 0 & \text{if } y = E_{n+1} \end{cases}
\]

where \( p : \mathbb{R}^{n+1} \to \mathbb{R}^n \) is the projection

\[
p \left( \sum_{i=1}^{n+1} a_i E_i \right) = \sum_{i=1}^{n} a_i E_i \), and \( \langle \rangle \) is the standard inner product in \( \mathbb{R}^{n+1} \).

Taking \( A' = D^n - \{0\} \) and \( B' = S^n - \{E_n\} \), the conditions of 6.3 are met. We leave the details as an exercise.

Proposition 6.5. Let \( f : (X, A) \to (Y, B) \) be an expansive relative homeomorphism. Then for all \( n \geq 2 \),

\( H_n(f) : H_n(X, A) \to H_n(Y, B) \) is an isomorphism.

Proof. Let \( A' \), \( B' \) be as in 6.3. We have the commutative diagram.
Theorem 6.6. Let $n > 1$. Then
\[
\text{H}_i(S^n) \cong \begin{cases} 
Z & \text{if } i = 0 \text{ or } n \\
0 & \text{otherwise}
\end{cases}
\]
and
\[
\text{H}_i(D^n, S^{n-1}) \cong \begin{cases} 
Z & \text{if } i = n \\
0 & \text{otherwise}
\end{cases}
\]

Proof. We proceed by induction on $n$. By 5.15, we have the result for $\text{H}_i(D^1, S^0)$. Since we have an excision relative homeomorphism $(D^1, S^0) \rightarrow (S^1, \{E_2\})$ by 6.4, it follows from 6.5 that
\[
\text{H}_i(S^1, \{E_2\}) \cong \text{H}_i(D^1, S^0) \cong \begin{cases} 
Z & \text{if } i = 1 \\
0 & \text{otherwise}
\end{cases}
\]
By 5.14 we have $\text{H}_i(S^1) \cong \text{H}_i(S^1, \{E_2\})$ for $i > 0$. Since $S^1$ is path-connected, $\text{H}_0(S^1) \cong Z$, and thus the result for $n = 1$ is proved.

Let $n > 1$ and suppose inductively...
\[ H_n(S^{n-1}) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{Z} & \text{if } i = 0 \text{ or } n-1 \\ 0 & \text{otherwise} \end{cases} \]

and
\[ H_i(D^n, S^{n-1}) = \begin{cases} 0 & \text{if } i < n \\ \mathbb{Z} & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases} . \]

For all \( i \leq 2 \) we have the exact sequence
\[ H_i(D^n) \to H_i(D^n, S^{n-1}) \overset{\delta}{\to} H_{i-1}(S^{n-1}) \to H_{i-1}(D^n) . \]

Since \( D^n \) is contractible, we have, for \( i \geq 2 \),
\[ H_i(D^n) = 0 \] and \( H_{i-1}(D^n) = 0 \) by 5.13. Thus, by exactness,
\[ H_i(D^n, S^{n-1}) \cong H_{i-1}(S^{n-1}) \] for \( i \geq 2 \).

Since \( D^n \) is path-connected and \( S^{n-1} \neq \emptyset \), it follows from 4.29 that \( H_0(D^n, S^{n-1}) = 0 \). We have the commutative diagram with exact rows:
\[ H_i(D^n) \to H_i(D^n, S^{n-1}) \overset{\delta}{\to} H_{i-1}(S^{n-1}) \to H_{i-1}(D^n) . \]

Both \( \delta \) 's are isomorphisms, since \( S^{n-1} \) and \( D^n \) are both path-connected for \( n \geq 2 \). Thus \( H_0(D^n) \) is an isomorphism.

Since \( H_i(D^n) = 0 \), it follows by exactness that
\[ H_i(D^n, S^{n-1}) = 0 . \] Thus the result for \( H_i(D^n, S^{n-1}) \) is deduced from the inductive hypothesis.

By 6.4 we have an excision relative homomorphism
\[ (D^n, S^{n-1}) \to (S^n, \{ -1, 1 \}) , \] and so by 6.5
\[ H_i(S^n, \{ -1, 1 \}) \cong H_i(D^n, S^{n-1}) \cong \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases} . \]

By 5.14, \( H_i(S^n) \cong H_i(S^n, \{ -1, 1 \}) \) for \( i > 0 \). Since \( S^n \) is path-connected, \( H_0(S^n) \cong \mathbb{Z} \), completing the proof.

**Definition 6.7.** Let \((X, A)\) be a topological pair. We say \( A \) is a strong deformation retract of \( X \) if there exists a deformation retraction \( r : X \to A \) and a homotopy \( F : X \times I \to X \) from \( r \circ 1_X \) to \( i_X \) (\( i : A \to X \) the inclusion) such that...
\[ F(a,x) = a \text{ for all } a \in A, x \in I. \] In this case, \( r \) is called a strong deformation retraction.

**Proposition 6.8.** All the examples in 5.9 are strong deformation retracts.

\[ S^n - i \text{ is a strong deformation retraction of } D^n - \{0\}. \]

**Proposition 6.9.** Let \((X,A)\) be a topological pair with \( A \) closed in \( X \). Suppose there exists an open subset \( U \) of \( X \) such that \( A \) is a strong deformation retract of \( U \). Let \( X/A \) denote the quotient space obtained from \( X \) by collapsing \( A \) to a point. Write \( \ast \) for the one-point subspace of \( X/A \) which is the image of \( A \) under the quotient map \( X \to X/A \). Then the map of topological pairs \( p : (X,A) \to (X/A, \ast) \), where \( p^\# \) is the quotient map, is an effective relative homeomorphism.

**Proof.** \( p^\# \) clearly maps \( X - A \) continuously and bijectively onto \( X/A - \ast \). Since \( A \) is closed in \( X \) and \( A = (p^\#)^{-1}(\ast) \), it follows that \( \ast \) is closed in \( X/A \) (since \( p^\# \) is a quotient map).

If \( V \) is open in \( X - A \), then \( V \) is also open in \( X \) and \( V = (p^\#)^{-1}(p^\#(V)) \). It follows that \( p^\#(V) \) is open in \( X/A - \ast \), and so the restriction of \( p^\# \) from \( X - A \) to \( X/A - \ast \) is an open map. Thus \( p^\# \) maps \( X - A \) homeomorphically onto \( X/A - \ast \).

Take \( A' = U \), \( B' = p^\#(U) \). Since \( A' \) is a strong deformation retract of \( U \), the inclusion \( A \subset A' \) is a homotopy equivalence. It remains only to show that the inclusion \( \ast \subset p^\#(U) \) is a homotopy equivalence. We will show, in fact, that the constant map \( c : p^\#(U) \to \ast \) is a strong deformation retraction.

Let \( r : U \to A \) be a strong deformation retraction and \( F : U \times I \to U \) a homotopy from \( r \) to \( 1_U \). (\( x : A \to U \) the inclusion) such that \( F(a, r) = a \) for all \( a \in A \) and \( x \in I \).

Let \( p : U \to p^\#(U) \) be the restriction of \( p^\# \). Since \( U \) is open in \( X \), it follows easily that \( p \) is a quotient map. Passing to quotients, \( F \) yields a function \( G : p^\#(U) \times I \to p^\#(U) \) such that the diagram:

\[ \begin{array}{ccc} U & \xrightarrow{r} & A \\
\downarrow & & \downarrow \ \uparrow \\
p^\#(U) \times I & \xrightarrow{G} & p^\#(U) \times I \\
\end{array} \]
\[ U \times I \xrightarrow{F} U \]
\[ \downarrow p \]
\[ p^*(U) \times I \xrightarrow{G} p^*(U) \]

commutes. Since \( I \) is locally compact and Hausdorff, \( p^*1_I \) is a quotient map. Thus, since \( p : F \) is continuous, it follows that \( G \) is continuous. It is easily checked that \( G \) is a homotopy from \( \text{id} \) to \( 1_{p^*(U)} \), where \( j : \ast \to p^*(U) \) is the inclusion, and \( G(\ast, \ast) = \ast \) for all \( \ast \in I \). Thus \( c : p^*(U) \to \ast \) is a strong deformation retraction, completing the proof.

Definition 6.16. Let \( \{(X_\alpha, x_\alpha)\}_{\alpha \in A} \) be a family of disjoint pointed topological spaces over \( A \) (see 6.5). The wedge or 1-point union or bouquet of the above family is the quotient space obtained from \( \coprod_{\alpha \in A} X_\alpha \) (with the disjoint union topology) by collapsing \( \{x_\alpha \mid \alpha \in A\} \) to a point and is denoted \( \vee_{\alpha \in A} (X_\alpha, x_\alpha) \), or more simply \( \vee X_\alpha \) when the base points \( x_\alpha \in X_\alpha \) are understood. In case \( A \) is finite (say \( A = \{1, 2, \ldots, n\} \)) we sometimes write \( X_1 \vee \cdots \vee X_n \) for \( \vee X_\alpha \).

Let \( q : \coprod_{\alpha \in A} X_\alpha \to \vee X_\alpha \) denote the quotient map. In each \( \alpha \in A \), let \( i_\alpha : X_\alpha \to \coprod_{\alpha \in A} X_\alpha \) denote the inclusion, and \( j_\alpha : X_\alpha \to \vee X_\alpha \) the composition \( i_\alpha q \). \( j_\alpha \) is called the inclusion on the \( \alpha \)-th leaf of the wedge.
Proposition 6.11. Let $\{ (X_x, i_x) \mid x \in A \}$ be a family of disjoint pointed topological spaces. Suppose, for each $x \in A$, $i_x$ is a strong deformation retract of some open set in $X_x$ (e.g., if $X_x$ is a manifold). Then for all $n > 0$,

$$
\Sigma H_n(j_x) : \bigoplus_{x \in A} H_n(X_x) \longrightarrow H_n\left( \bigvee_{x \in A} X_x \right)
$$

is an isomorphism, where the $i_x$ are the inclusions on the leaves of the coradge ($\text{Horn}_A i_x$ is a closed in $X_x \forall x \in A$).

Proof: Let $i_x$ be a strong deformation retract of the open set $U_x$ in $X_x$. Then $i_x \times \{ i \}$ is a strong deformation retract of the open set $U_x \times \{ i \}$ in $U_x \times \{ i \}$. Thus, by 6.9 the map $p : \left( \bigsqcup_{x \in A} X_x, \{ i_x \} \times \{ i \} \right) \to \left( \bigsqcup_{x \in A} X_x, \{ i \} \right)$, where $p^x = \delta$, the quotient map, is an epimorphism relative homomorphism. We have the commutative diagram

$$
\begin{array}{ccc}
\Sigma H_n(j_x) & \longrightarrow & H_n\left( \bigvee_{x \in A} X_x \right) \\
\bigoplus_{x \in A} H_n(X_x) & \longrightarrow & H_n\left( \bigsqcup_{x \in A} X_x \right) \\
| & & | \\
\bigoplus_{x \in A} H_n(X_x) & \longrightarrow & H_n\left( \bigsqcup_{x \in A} X_x, \{ i_x \} \times \{ i \} \right)
\end{array}
$$

where $j$, $i$, $i_x$, $i$ are the appropriate inclusions. $H_n(\delta)$ is an isomorphism by 6.5. $\Sigma H_n(i_x)$ and $\Sigma H_n(i)$ are isomorphisms by 3.22 and 4.9, respectively. For $n > 0$, $H_n(\delta)$ and the $H_n(i_x)$ are isomorphisms by 5.14. It follows that $\Sigma H_n(j_x)$ is an isomorphism for $n > 0$. 
The definition of \( X_1 \vee X_2 \vee \cdots \vee X_n \) requires that the \( X_i \) to be

pairwise disjoint. If some of the \( X_i \) are homeomorphic copies of a space \( X \), we sometimes write \( X \) in place

of these \( X_i \) in the above notation. Thus, e.g., \( X \vee X \)

denotes the wedge of two disjoint spaces, each

homeomorphic to \( X \). For example, \( S^1 \vee S^1 \) is a "figure

eight" space.

\[ \begin{array}{c}
\includegraphics[width=0.5\textwidth]{figure-eight.png}
\end{array} \]

By 6.6, \( H_1(S^1) \cong \mathbb{Z} \). We proceed to find convenient

representative cycles for a generator of \( H_1(S^1) \), and compute

the homomorphism \( H_1(x) : H_1(S^1) \to H_1(S^1) \) for certain important

maps \( x : S^1 \to S^1 \).

Recall, from 6.4, that we have an excision relative homomorphism \( \iota : (D^1, S^0) \to (S^1, \{0,1\}) \). In general, the

map \( \iota \) is defined for each \( i \) in place of \( D^1 = [-1, 1] \). Write \( \tilde{\iota} = \{0,1\} \). The affine map

\( \tilde{\iota} : D^1 \to R \) sending \( t \) to \( 2t - 1 \) yields a homomorphism of pairs \( \tilde{\iota} : (I, \tilde{\iota}) \to (D^1, S^0) \) with \( \tilde{\iota}(0) = -1 \), \( \tilde{\iota}(1) = 1 \).

We identify \( S^1 \) with the space of complex numbers of

absolute value 1. Under this identification, \( -e_2 \) is

identified with \( -1 \). Let \( \beta : (I, \tilde{\iota}) \to (S^1, \{1, \bar{1}\}) \) denote the composition

\[ \beta : (I, \tilde{\iota}) \xrightarrow{\tilde{\iota}} (D^1, S^0) \xrightarrow{\iota} (S^1, \{0,1\}) \]

where \( \iota(z) = z \bar{z} \). \( \bar{1} \) is a homomorphism of pairs, and

so \( \beta \) induces an isomorphism in homology in all dimensions. Explicitly, it can be checked that

\[ \beta(z) = e^{2\pi i x} \text{ for all } x \in I. \]

Recall, from 5.15, \( H_1(D^1, S^0) \cong \mathbb{Z} \) with generator

a characterized by \( 2a = [1] - [1] \) where

\( \tilde{\iota} : H_1(D^1, S^0) \to H_0(S^0) \) is the connecting homomorphism for the

pair \( (D^1, S^0) \). From commutativity of
The fact that $H_1(\mathbb{F})$ and $H_0(\mathbb{F})$ are isomorphisms, and $H_0(\mathbb{F})[\mathbb{F}] = [\mathbb{F}]$, $H_0(\mathbb{F})[\mathbb{F}] = [\mathbb{F}]$, the unique $\hat{c} \in H_1(\mathbb{F}, \mathbb{F})$ such that $H_1(\mathbb{F})[\hat{c}] = 0$ is characterized by the fact that $3(\hat{c}) = [\mathbb{F}]$.

$H_1(\mathbb{F}) : H_1(\mathbb{F}, \mathbb{F}) \to H_1(\mathbb{F}, \mathbb{F})$ is an epimorphism (where $j : (\mathbb{F}, \mathbb{F}) \to (\mathbb{F}, \mathbb{F})$ is the inclusion) and so there is a unique $\hat{c} \in H_1(\mathbb{F})$ such that $H_1(\mathbb{F})[\hat{c}] = H_1(\mathbb{F})(\mathbb{F})$.

Theorem 6.12. Let $n$ be a positive integer. For $1 \leq k \leq n$, let $S_k : D_k \to S_k$ be the singular $k$-simplex given by $S_k(x, i) = x$. Let $c_n = \sum_{k=1}^{n} S_n \in H_1(S^n)$. Then $c_n$ is a cycle, and $[c_n] = \hat{c}$, the above generator of $H_1(S^n)$.

Proof. Fix $0 \leq k \leq n$, let $P_k = \sum_{n} S_n \in H_1(S^n)$. Notice that $P_0 = P_n$.

Then, denoting a singular $n$-simplex by the point which determines it, we have $\partial S_{k} = P_k - P_{k-1}$ for $1 \leq k \leq n$.

Thus, $\partial c_n = \sum_{k=1}^{n} (P_k - P_{k-1}) = P_n - P_0 = P_n - P_0 = 0$, and so $c_n$ is a cycle. It suffices to show that there exists a chain $\bar{d} \in S_1(\mathbb{F})$ such that

1) $S_1(j') ([\bar{d}])$ is a cycle in $S_1(\mathbb{F}, \mathbb{F})$, where $j' : (\mathbb{F}, \mathbb{F}) \to (\mathbb{F}, \mathbb{F})$ is the inclusion.

2) $[S_1(j') ([\bar{d}])] = \hat{c}.$

3) $S_1(g') ([\bar{d}]) = c_n.$

In this case, we have such a $\bar{d}$, commutativity of the diagram.
\[
\begin{array}{c}
S_1(I) \xrightarrow{S_1(y^q)} S_1(S^1) \\
S_1(I') \xrightarrow{S_1(j')} \downarrow S_1(j) \\
S_1(I, I') \xrightarrow{S_1(y)} S_1(S^1, \{ij\})
\end{array}
\]

Yields \( H_1(i) [c_0] = [S_1(i)(c_n)] = [S_1(j), S_1(y^q)(d_n)] \)

\[
= [S_1(y) S_1(j')(d_n)] = H_1(g) [S_1(j')(d_n)] = H_1(g)(b),
\]

and so we must have \([c_0] = c\).

The construction of \(dn\) is very easy. For \(1 \leq k \leq n\), let \(\tau_{n,k} : A_1 \to I\) be given by \(\tau_{n,k}(x) = \frac{R-1+x}{n}\),

and \(dn = \sum_{k=1}^{n} \tau_{n,k} \).

It is immediate that \(S_1(y^q)(\tau_{n,k}) = \tau_{n,k} \) for \(1 \leq k \leq n\),

and so 3) holds.

For \(0 \leq k \leq n\), let \(Q_k = \frac{k}{n} \in I\). Then, \(Q_0 = 0\), \(Q_n = 1\).

\(\tau_{n,k}\) is the affine 1-simplex \((Q_{k-1}, Q_k)\), and

\(\partial \tau_{n,k} = (Q_k) - (Q_{k-1})\). Thus

\[
\partial dn = \sum_{k=1}^{n} \{(Q_k) - (Q_{k-1})\} = (Q_n) - (Q_0) = S_0(i')((1) - (0))
\]

where \(i' : I \to I\) is the inclusion. Thus,

\(\partial S_1(j')(d_n) = S_0(i') \partial (d_n) = S_0(i') S_0(i')(1) - (0)) = 0\)

since \(S_0(i') S_0(i') = 0\). Thus, 1) holds.

From the definition of the connecting homomorphism (4.23, 4.24),

\[
\partial [S_1(j')(d_n)] = [S_0(i') \partial S_1(j') d_1 S_1(j')(d_n)]
\]

\[
= [S_0(i') \partial d_n] = [S_0(i') \partial (1) - (0)] = [(1) - (0)]
\]

\[
= [1] - [0], \quad \text{and so} \quad [S_1(j')(d_n)] = 0. \quad \text{Thus, 2) holds,}
\]

completing the proof.

For \(n \in \mathbb{Z}\), let \(\alpha_n : S^1 \to S^1\) be given by \(\alpha_n(z) = z^n\). We proceed to calculate the induced homomorphisms \(H_1(\alpha_n) : H_1(S^1) \to H_1(S^1)\).
Proposition 6.13. Let \( n \geq 0 \), then \( H_1(\mathbb{K}_n) \) is multiplication by \( n \).

Proof. Consider just the case \( n > 0 \). Let \( \delta, f_{n,k}, \) and \( \mathbb{K}_n \) be as in 6.12. Then \( 1 \leq k \leq n \) and all \( t \in D_1 \),

\[
x_n(\delta_{n,k}(e^t)) = x_n(e^{2\pi i (k-1+t)} - e^{2\pi i (k+t)}) = e^{2\pi i t}
\]

and so \( S_i(x_n)(\delta_{n,k}) = \delta_{1,1} \). Hence, \( x_n(\delta_{n,k}) = \delta_{1,1} \) for \( 1 \leq k \leq n \).

By 6.12,

\[
\mathbb{K}_1 = [\mathbb{K}_n] = [\delta_1] \quad \text{and so}
\]

\[
H_1(x_n)(\delta) = H_1(x_n)[\mathbb{K}_n] = [S_i(x_n) \sum_{k=1}^{n} f_{n,k}] = [n \delta_{1,1}] = n [\delta_1]
\]

\[= n \delta_1 \], completing the proof for \( n > 0 \).

We have \( \delta_0 \) is the constant map with value \( 1 \), and so we have a commutative diagram of continuous maps

\[
\begin{array}{ccc}
S^1 & \xrightarrow{\delta_0} & S^1 \\
\downarrow & & \downarrow \\
\{1\} & & \{1\}
\end{array}
\]

Thus

\[
H_1(S^1) \xrightarrow{H_1(\delta_0)} H_1(S^1)
\]

\[
\xrightarrow{H_1(\{1\})}
\]

Since \( H_1(\{1\}) = 0 \), it follows that \( H_1(\delta_0) \) is the zero homomorphism.

Definition 6.14. Let \( \sigma : \Delta_1 \to X \) be any singular 1-simplex. The opposite simplex, \( \sigma : \Delta_1 \to X \), is given by

\[
\sigma(t) = \sigma(1-t) \quad \text{for} \quad 0 \leq t \leq 1.
\]

Lemma 6.15. In any singular 1-simplex \( \sigma : \Delta_1 \to X \), \( \sigma + \sigma \) is a boundary in \( S_1(X) \).

Proof. Let \( \delta_1 : \Delta_1 \to \Delta_1 \) denote the identity map. Note that \( \sigma = S_i(\sigma)(\delta_1) \), \( \sigma = S_i(\sigma)(\delta_1) \) and so \( \sigma + \sigma = S_i(\sigma)(\delta_1 + \delta_1) \). Thus, since \( S_i(\sigma) \) maps boundary to
boundaries, it suffices to show that $\bar{\alpha} + \bar{\beta}$ is a boundary in $S_1(\Delta_1)$. Since $\Delta_1$ is contractible,
$H_1(\Delta_1) = 0$, and so it suffices to show $\bar{\alpha} + \bar{\beta}$ is a cycle. Note that $\bar{\alpha} = (E_0, E_1)$, $\bar{\beta} = (E_1, E_0)$, and so
$\partial(\bar{\alpha} + \bar{\beta}) = (E_1) - (E_0) + (E_0) - (E_1) = 0$, completing the proof.

Theorem 6.16. For all $n \in \mathbb{Z}$, $H_1(x^n)$ is multiplication by $n$, where $x^n : S^1 \to S^1$ is given by $x^n(\pi) = \pi^n$.

Proof. 6.13 gives the result for $n \geq 0$. Since $x^{-n} = x^{-1} \cdot x^n$
for all $n \in \mathbb{Z}$, we have $H_1(x^{-n}) = H_1(x^{-1}) \cdot H_1(x^n)$. Thus,
it remains only to show that $H_1(x^{-1})(\mathbb{Z}) = \mathbb{Z}$.

We have $\overline{f}_{1,1}(x) = e^{2\pi i x}$ for all $x \in \mathbb{Z}$, and
$\overline{f}_{1,1}(1-x) = e^{-2\pi i x}$

$= x^{-1}(e^{2\pi i x}) = x^{-1}\overline{f}_{1,1}(x)$, and so
$\overline{f}_{1,1} = x^{-1} \cdot \overline{f}_{1,1} = S_1(x^{-1}) (\overline{f}_{1,1})$. Thus,
$H_1(x^{-1})(\mathbb{Z}) = [S_1(x^{-1})(c_1)] = [S_1(x^{-1})(\overline{f}_{1,1})] = [\overline{f}_{1,1}]$
$= [c_1]$ (since $\overline{f}_{1,1} + \overline{f}_{1,1}$ is a boundary
by 6.15)
$= [c_1] = \mathbb{Z}.$

Corollary 6.17. If $m$ and $n$ are distinct integers, then $x^m$ is not homotopic to $x^n$.

It can be shown (e.g. by covering spaces) that
every continuous map $f : S^1 \to S^1$ is homotopic to an
$x^n$ for some $n$.

Definition 6.18. A continuous map $f : S^1 \to S^1$ has degree $n$
if $f \simeq x^n$.

More generally, suppose $X$ and $Y$ are homeomorphic
to $S^1$, and let $x \in H_1(X)$, $y \in H_1(Y)$ be generators.
A continuous map $f : X \to Y$ has degree $n$ with respect
to $x$ and $y$ if $H_1(f)(x) = n \cdot y$. 
Example 6.19. The real projective plane $\mathbb{R}P^2$ is the quotient space obtained from $D^2$ by identifying antipodal points on $S^1$, i.e. identify $2\pi/n$ for each $z \in S^1 \subset D^2$.

Write $C$ for the image of $S^1$ under the quotient map $D^2 \to \mathbb{R}P^2$. The quotient map then yields a map of topological pairs $f : (D^2, S^1) \to (\mathbb{R}P^2, C)$, which is easily seen to be an extension of the homeomorphism (taking $A' = D^2 - \{0,2\}$ and $B' = \mathbb{R}P^2 - f(C)$) that satisfies the requirements of 6.3). $C$ is homeomorphic to $S^1$. In fact the map $h : C \to S^1$ given by $h(f(x)) = x^2$ is well-defined, and is a homeomorphism. Let $c_0 \in H_1(S^1)$ be the generator as in 6.12, and let $c \in H_1(C)$ be the unique generator such that $H_1(h)(c) = c$. The composition $S^1 \xrightarrow{f} C \xrightarrow{h} S^1$ is $x^2$, and so $H_1(f^\circ h)(c)$ must be $2x$. (for if $H_1(f^\circ h)(c) = nx$, then $2c = H_1(x^2)(c) = H_1(h)(H_1(f^\circ h)(c)) = H_1(h)(nx) = nc$, and so $n = 2$). Thus $f^\circ h$ has degree 2 with respect to the generators $c, x$.

\[\xymatrix{ S^1 \ar[rr]^{f^\circ h} & & C \ar[ll]_{h} }\]

(Note that $x = [x]$. Note that)
\[ S_1(f^e)(\sigma_{2,n}) = S_1(f^e)(\sigma_{2,2}) = a, \text{ and so } S_1(f^e)(c_2) = 2a. \]

Since \( \mathbb{RP}^2 \) is path-connected, \( H_0(\mathbb{RP}^2) = \mathbb{Z} \).

Let \( n > 2 \). We have the diagram with exact rows:

\[
\begin{array}{c}
\cdots \\
H_n(\mathbb{C}) \longrightarrow H_n(\mathbb{RP}^2) \longrightarrow H_n(\mathbb{RP}^2, \mathbb{C}) \\
\downarrow \\
H_n(f) \\
H_n(D_2, S^1) \\
\end{array}
\]

From 6.6, \( H_n(D_2, S^1) = 0 \) and \( H_n(\mathbb{C}) = 0 \) (since \( \mathbb{C} \cong S^1 \)).

Thus, since \( f \) is an epimorphism relative homomorphism, \( H_n(f) \) is an epimorphism, and so \( H_n(\mathbb{RP}^2, \mathbb{C}) = 0 \). Thus by exactness, \( H_n(\mathbb{RP}^2) = 0 \) for \( n > 2 \).

We have the commutative diagram with exact rows:

\[
\begin{array}{c}
H_2(\mathbb{C}) \longrightarrow H_2(\mathbb{RP}^2) \longrightarrow H_2(\mathbb{RP}^2, \mathbb{C}) \longrightarrow H_1(\mathbb{C}) \longrightarrow H_1(\mathbb{RP}^2) \longrightarrow H_1(\mathbb{RP}^2, \mathbb{C}) \\
\downarrow \quad \downarrow \quad \downarrow \\
H_2(f) \quad H_1(f) \quad H_1(f) \\
H_2(D_2) \longrightarrow H_2(D_2, S^1) \longrightarrow H_1(S^1) \longrightarrow H_1(D_2) \longrightarrow H_1(D_2, S^1) \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \quad \quad \quad 0 \\
\end{array}
\]

The indicated isomorphisms follow from the fact that \( f \) is an epimorphism relative homomorphism. The indicated 0 groups follow from the fact that \( D^2 \) is contractible, \( \mathbb{C} \cong S^1 \), and 6.6. Thus, since \( H_1(f^3) \) is injective with image the subgroup generated by \( 2x \), it follows that \( \partial: H_2(\mathbb{RP}^2, \mathbb{C}) \rightarrow H_1(\mathbb{C}) \) is injective with image generated by \( 2x \). Thus \( H_2(\mathbb{RP}^2) = 0 \), and

\[ H_1(\mathbb{RP}^2) = \frac{\text{free abelian group with generator } x}{\text{subgroup generated by } 2x} = \mathbb{Z}/2\mathbb{Z} \]

is cyclic of order 2. Summarizing:

\[ H_i(\mathbb{RP}^2) = \begin{cases} 2 & \text{if } i = 0 \\ 2/2 = 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \]

Now, \( H_1(\mathbb{C}): H_1(\mathbb{C}) \rightarrow H_1(\mathbb{RP}^2) \) is surjective, where \( \mathbb{C} \) is the conclusion.
Example 6.20. The torus $T$ is the quotient space obtained from $D^2$ by making identifications on $S^1$ as shown:

![Image of torus]

The image of $S^1$ under the quotient map $D^2 \to T$ is homeomorphic to $S^1 \vee S^1$.

![Image of $S^1 \vee S^1$]

The quotient map yields an exact relative homomorphism $f: (D^2, S^1) \to (T, S^1 \vee S^1)$. (As in 6.19, $A' = D^2 - \{0\}$ and $B' = f(A')$ satisfy the requirement of 6.3.) The pictured singular 1-simplices $a$ and $b$ of $S^1 \vee S^1$ are the images, under the maps induced by the inclusions of the leaves, of cycles representing generators of $H_1$ of the respective leaves. Thus by 6.11, $H_1(S^1 \vee S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ with $[a]$ and $[b]$ as generators.
We have
\[ S_1(f^o)(e_4) = S_1(f^o)(e_{a_{1}}) + S_1(f^o)(e_{a_{2}}) + S_1(f^o)(e_{a_{3}}) + S_1(f^o)(e_{a_{4}}) \]
\[ = a + b + \overline{a} + \overline{b} \]
\[ = (a + \overline{a}) + (b + \overline{b}) \in B_{1}(S(S'vS')) \quad \text{by 6.15}. \]
Thus \( H_1(f^o)(e) = 0 \), and so \( H_1(f^o) \) is the \( C \)-homomorphism.
Let \( n \geq 2 \). We have the diagram with exact rows:
\[ H_n(S'vS') \rightarrow H_n(T) \rightarrow H_n(T, S'vS') \]
\[ \uparrow H_n(f) \]
\[ H_n(D^2, S') \]

From 6.6 and 6.11, \( H_n(D^2, S') \) and \( H_n(S'vS') \) are both 0. \( H_n(f) \) is an isomorphism since \( f \) is an effective relative homeomorphism. Thus, by exactness, \( H_n(T) = 0 \) for \( n \geq 2 \).

We have the commutative diagram with exact rows:
\[ H_2(S'vS') \rightarrow H_2(T) \rightarrow H_2(T, S'vS') \rightarrow H_1(S'vS') \rightarrow H_1(T) \rightarrow H_1(T, S'vS') \]
\[ \uparrow \]
\[ \rightarrow H_1(D^2, S') \rightarrow H_1(S') \rightarrow H_1(D^2) \rightarrow H_1(D^2, S') \]

The induced isomorphisms follow from the fact that \( f \) is an effective relative homeomorphism. The indicated 0 groups follow from the fact that \( D^2 \) is contractible, 6.6, and 6.11. Thus, since \( H_1(f^o) \) is the \( C \)-homomorphism, it follows that \( \partial : H_2(T, S'vS') \rightarrow H_1(S'vS') \) is the \( C \)-homomorphism.
Thus, by excision, \( H_2(S) \) and \( H_1(S) \) are isomorphic. Thus, since \( H_2(D^2,S^1) \cong \mathbb{Z} \) by 6.6, we have \( H_2(T) \cong \mathbb{Z} \). We have \( H_1(T) \cong H_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z} \).

Since \( T \) is path-connected, \( H_0(T) \cong \mathbb{Z} \). Summarizing,
\[
H_i(T) = \begin{cases} 
\mathbb{Z} & \text{if } i = 0, 2 \\
\mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 1 \\
0 & \text{otherwise}
\end{cases}
\]

Moreover, \( H_1(S) : H_1(S^1) \cong \mathbb{Z} \to H_1(T) \) is an isomorphism, where \( i \) is the inclusion.

**Example 6.21.** The **Klein bottle** \( K \) is the quotient space obtained from \( D^2 \) by making identifications on \( S^1 \) as shown:

As in the case of the torus, we have an exact relative homeomorphism \( f : (D^2,S^1) \to (K,S^1) \). The first difference between \( K \) and \( T \) is that in the case of \( K \), \( S^1(f^)(C) = a + b + a + b = 2b + (a + \bar{a})c + 2 + B_2(\bar{\bar{S}}(S^1)) \), and so \( H_1(f^)(C) = 2 \mathbb{Z} \). Thus, \( H_1(f^) \) is injective with image the subgroup \( H_1(S^1) \) generated by \( 2 \mathbb{Z} \).

The same argument as in 6.20 yields \( H_n(K) = 0 \) for \( n > 1 \), and \( H_0(K) \cong \mathbb{Z} \). We have the commutative diagram with exact rows:

\[
\begin{array}{cccccccc}
\text{H}_2(S^1 & \to & \text{H}_2(K) & \to & \text{H}_2(K,S^1) & \to & \text{H}_1(S^1) & \to & \text{H}_1(K,S^1)} \\
\downarrow & & & & & & & & \\
\text{H}_2(D^2) & \to & \text{H}_2(D^2,S^1) & \to & \text{H}_1(S^1) & \to & \text{H}_1(D^2) & \to & \text{H}_1(D^2,S^1)}
\end{array}
\]

with indicated isomorphisms and \( 0 \) groups as in 6.20.
Thus, since \( H_1(S^3) \) is injective with image generated by \( 2[0] \), the same is true of \( \partial_1 : H_2(K, S^1(S^3)) \rightarrow H_1(S^3) \). Thus, by exactness, \( H_2(K) = 0 \) and

\[
H_1(K) \equiv \frac{H_1(S^1(S^3))}{\ker (\partial_1 : H_2(K, S^1(S^3)) \rightarrow H_1(S^3))}
\]

\[
= \text{free abelian on generators } [a] \text{ and } [b] \text{, subgroup generated by } 2[0]
\]

Summary:

\[
H_1(K) = \begin{cases} 
\mathbb{Z} & \text{if } n = 0 \\
\mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } n = 1 \\
0 & \text{otherwise}
\end{cases}
\]

7. Application

The preceding will be used to deduce the following:

1) the Brownian Fixed Point Theorem
2) the Fundamental Theorem of Algebra
3) the Jordan-Banach Separation Theorem
4) Brownian's theorem on Invariance of Domain.

All of this is, of course, contingent on the Homotopy and Excision Properties, which we have not yet proved.

It will be convenient to first introduce the notion of reduced homology so as to avoid the necessity of giving separate arguments in dimension 0 using the augmentation.

Definition 7.1. Let \( X \) be a topological space. The reduced homology group \( \tilde{H}_n(X) \), denoted \( \tilde{H}_n(X) \), is defined to be \( H_n(\mathbb{E} : H_0(X) \rightarrow \mathbb{Z}) \). For convenience, we also define \( \tilde{H}_n(X) = H_n(X) \) for \( n \neq 0 \).

Reduced homology is defined only for topological spaces, not for topological pairs.

Exercise 7.2. If \( P \) is a one-point space and \( f : X \rightarrow P \) the unique map, then for all \( n \in \mathbb{Z} \), \( \tilde{H}_n(X) = \ker(H_n(f) : H_n(X) \rightarrow H_n(P)) \).
7.2 gives a new symmetric characterization of the $\tilde{H}_0$ through 7.1. The approach of 7.2 is, in fact, used in generalized homology theories (i.e., theories not satisfying the Dimension Property). In such generalized theories, $\tilde{H}_0$ is generally different from $H_0$ for many values of $n$.

Proposition 7.3. Let $X$ be a non-empty topological space. Then $\tilde{H}_0(X)$ is free abelian of rank one less than the number of path components of $X$. In fact, if $\{x_a\}_{a \in J}$ is the family of path components of $X$ (as an index set), and we choose a point $a_0 \in x_a$ for each $a \in J$, then for each $a \in J$, $\tilde{H}_0(x_a)$ is free abelian with basis $\{ [a_a] - [a_{a_0}] \, | \, a \in J - \{a_0\} \}$.

Proof. Since $\varepsilon ([a_a]) = 1$ for each $a \in J$, it follows from 3.22 and 3.13 that $\tilde{H}_0(x_a)$ is free abelian with basis $\{ [a_a] \, | \, a \in J \}$. Let $S = \{ [a_a] - [a_{a_0}] \, | \, a \in J - \{a_0\} \}$. It is easy to check that $S$ is linearly independent over $\mathbb{Z}$. Denote $E([a_a] - [a_{a_0}]) = 1 - 1 = 0$ for all $a \in J - \{a_0\}$, $S \subseteq \tilde{H}_0(X)$.

Let $h = \sum_{a \in J} n_a [a_a] \in \tilde{H}_0(X)$ where the $n_a \in \mathbb{Z}$ and all but finitely many of the $n_a$ are 0, then $0 = \varepsilon(h) = \sum_{a \in J} n_a$ and so $n_{a_0} = - \sum_{a \in J - \{a_0\}} n_a$. Then

$$h = \sum_{a \in J - \{a_0\}} n_a [a_a] - \left( \sum_{a \in J - \{a_0\}} n_a \right) [a_{a_0}]$$

$$= \sum_{a \in J - \{a_0\}} n_a ([a_a] - [a_{a_0}]) \text{ and so } S \text{ spans } \tilde{H}_0(X) \text{ over } \mathbb{Z}.$$

Thus, $\tilde{H}_0(X)$ is free abelian with basis $S$.

Proposition 7.4. If $f : X \to Y$ is continuous, then $\tilde{H}_0(f)(\tilde{H}_0(X)) \subseteq \tilde{H}_0(Y)$. Write $\tilde{H}_0(f) : \tilde{H}_0(X) \to \tilde{H}_0(Y)$ for the homomorphism obtained by restricting $H_0(f)$. Then $\tilde{H}_0(f)$ is a covariant functor from $Top$ to $Ab$.

Proof. The fact that $H_0(f)(\tilde{H}_0(X)) \subseteq \tilde{H}_0(Y)$ is immediate from
The functionality of $\tilde{H}_n$ is very easy to check, and left as an exercise.

For convenience of notation, we define $\tilde{H}_n = H_n: Z_\geq 0 \rightarrow$ for $n \geq 0$.

Proposition 7.5. Suppose $f, g: X \rightarrow Y$ are continuous maps which are homotopic. Then for all $n \geq 0$, $H_n(f) = H_n(g)$.

Proof. This is trivial since, by the homotopy property, $H_n(f) = H_n(g)$ for all $n$, and $H_n$ on maps is obtained by restriction of $H_0$. 

Proposition 7.6. Let $(X, A)$ be a topological pair. Write $i: A \rightarrow X$, $j: (X, \emptyset) \rightarrow (X, A)$ for the inclusion maps. Then 

$\text{res} \left( \tilde{H}_i(X, A) \rightarrow \tilde{H}_i(A) \right)$ is contained in $\tilde{H}_i(A)$. Write $\tilde{g}: \tilde{H}_i(X, A) \rightarrow \tilde{H}_i(A)$ for the homomorphism obtained from $\tilde{g}$ by restricting the range, and $H_i(j): \tilde{H}_i(X) \rightarrow H_i(X, A)$ for the restriction of $H_i(j): \tilde{H}_i(X) \rightarrow \tilde{H}_i(X, A)$. Then the sequence

$$
\begin{align*}
H_i(X) &\xrightarrow{H_i(i)} H_i(X, A) \xrightarrow{\tilde{g}} \tilde{H}_i(A) \xrightarrow{H_i(j)} \tilde{H}_i(i) \\
H_i(X) &\xrightarrow{H_i(j)} H_i(X, A) \xrightarrow{\tilde{g}} \tilde{H}_i(A) \xrightarrow{H_i(i)} \tilde{H}_i(X) \xrightarrow{H_i(j)} \tilde{H}_i(X, A)
\end{align*}
$$

is exact. If $A \neq \emptyset$, then $H_i(j)$ is onto.

Proof. We have the following commutative diagram with the rows and both diagonals exact.
Thus \( \text{im } \partial = \ker \text{Ho}(\cdot) \subset \ker (\varepsilon : \text{Ho}(A) \to \mathbb{Z}) = \tilde{\text{Ho}}(A) \). The
asserted sequence follows by straightforward diagram chasing.

If \( A \neq \emptyset \), then \( \varepsilon : \text{Ho}(A) \to \mathbb{Z} \) is onto. Let \( x \in \text{Ho}(X,A) \)
and choose any \( y \in \text{Ho}(X) \) such that \( \text{Ho}(j)(y) = x \). Since
\( \varepsilon : \text{Ho}(A) \to \mathbb{Z} \) is onto, we can choose \( z \in \text{Ho}(A) \) such that
\( \varepsilon(z) = \varepsilon(y) \). Then \( \varepsilon(y - \text{Ho}(i)(z)) = \varepsilon(y) - \varepsilon(z) = 0 \) and so
\( y - \text{Ho}(i)(z) \in \text{Ho}(X) \). Moreover, \( \text{Ho}(j)(y - \text{Ho}(i)(z)) = \text{Ho}(j)(y) - \text{Ho}(i)(\text{Ho}(i)(z)) = \text{Ho}(j)(y - \text{Ho}(i)(z)) = x \) since \( \text{Ho}(j) \text{Ho}(i) = 0 \). Thus
\( \text{Ho}(j) \) is onto if \( A \neq \emptyset \).

In notational convenience we write, for \( n > 0 \),
\( \tilde{\partial} = \partial : \text{H}_{n+1}(X,A) \to \tilde{\text{H}}_n(A) \) and \( \text{H}_n(j) = \text{H}_n(j) : \tilde{\text{H}}_n(X) \to \text{H}_n(X,A) \).

Corollary 7.7. In each topological pair \( (X,A) \), the sequence
\[
\begin{array}{cccccccccc}
\text{H}_{n+1}(X,A) & \overset{\tilde{\partial}}{\longrightarrow} & \text{H}_n(A) & \overset{\tilde{\text{H}}_n(i)}{\longrightarrow} & \tilde{\text{H}}_n(X) & \overset{\text{H}_n(j)}{\longrightarrow} & \text{H}_n(X,A) & \to & \cdots & \to & \text{H}_0(X) & \to & \text{H}_0(X,A)
\end{array}
\]

is exact. \( \text{H}_n(j) \) is onto if \( A \neq \emptyset \).

If \( f : (X,A) \to (Y,B) \) is a map of topological pairs, the diagram
\[
\begin{array}{cccccccc}
\text{H}_{n+1}(X,A) & \overset{\tilde{\partial}}{\longrightarrow} & \text{H}_n(A) & \overset{\tilde{\text{H}}_n(i)}{\longrightarrow} & \tilde{\text{H}}_n(X) & \overset{\text{H}_n(j)}{\longrightarrow} & \text{H}_n(X,A) \\
\text{H}_n(f) \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{H}_{n+1}(Y,B) & \overset{\tilde{\partial}}{\longrightarrow} & \text{H}_n(B) & \overset{\tilde{\text{H}}_n(i)}{\longrightarrow} & \tilde{\text{H}}_n(Y) & \overset{\text{H}_n(j)}{\longrightarrow} & \text{H}_n(Y,B)
\end{array}
\]
commutes for all \( n \geq 0 \).
Proof. The above diagram, as obtained by restriction of maps to the corresponding diagrams without the thickened, and the dotted commutes.

Proposition 7.8. For \( n \geq 1 \), \( S^{n-1} \) is not a retract of \( D^n \).

Proof. Suppose \( r : D^n \to S^{n-1} \) is a retract. Then commutativity of

\[
\begin{array}{ccc}
S^{n-1} & \xrightarrow{i} & D^n \\
\downarrow{\tilde{r}} & & \downarrow{r} \\
S^{n-1} & \xrightarrow{\tilde{r}} & D^n \\
\end{array}
\]

where \( i \) is the inclusion, the diagram

\[
\tilde{h}_{n-1}(S^{n-1}) \xrightarrow{i} \tilde{h}_{n-1}(D^n)
\]

commutes. By 6.6, 5.13 and 7.3 , \( \tilde{h}_{n-1}(D^n) = 0 \) and \( \tilde{h}_{n-1}(S^{n-1}) \cong \mathbb{Z} \). Thus commutativity of this last diagram is impossible, and so no such retraction can exist.

As a corollary we have

Theorem 7.9. (The Brouwer Fixed Point Theorem). For every \( n \geq 1 \), every continuous \( f : D^n \to D^n \) has at least one fixed point.

Proof. If a continuous fixed point free \( f : D^n \to D^n \) existed, then would be a retraction \( r : D^n \to S^{n-1} \) given as follows:

\( r(x) = \) retraction with \( S^{n-1} \) of the ray from \( f(x) \) passing through \( x \). Since such a retraction is impossible by 7.8, a fixed point free continuous \( f : D^n \to D^n \) cannot exist.
Theorem 7.10. (The Fundamental Theorem of Algebra.) Let \( n \geq 1 \) and suppose \( p(z) = z^n + q(z) \) where \( q \) is a complex polynomial of degree \( \leq n \). Then, there exists a \( z \in \mathbb{C} \) such that \( p(z) = 0 \).

Proof. Suppose not. If \( q(z) = \sum_{k=0}^{n-1} a_k z^k \) and \( 121 \geq 1 \), then \( |q(z)| \leq \sum_{k=0}^{n-1} |a_k| |z|^k \leq (\max |a_k|) n 121^{n-1} \).

Choose \( R > \max \{1, n \max |a_k|^\frac{1}{2}\} \). Then if \( 121 \geq R \),

\[
\left( \max |a_k| \right) n 2^{n-1} < R 121^{n-1} \leq 121^n, \ i.e. \ |q(z)| < 121^n/121
\]

Define \( f : S^1 \to S^1 \) by \( f(z) = \frac{p(Rz)}{|p(Rz)|} \). Using definition

is possible since \( p(Rz) \neq 0 \) for all \( z \). We will show that \( f \cong \chi_n \) (the \( n \)th power map) and that \( f \) is homotopic to a constant map. By 6.13, this will be a contradiction since any constant map induces the \( 0 \)-homomorphism on \( H_1 \).

Define \( F : S^1 \times I \to S^1 \) by

\[
F(z, t) = \frac{(Rz)^n + t \cdot \frac{p(Rz)}{|p(Rz)|}}{|(Rz)^n + t \cdot \frac{p(Rz)}{|p(Rz)|}|}.
\]

Note that the denominator

is never 0 since \( |Rz|^n > |q(Rz)| / \) for all \( z \in S^1 \). \( F \) is

a homotopy from \( \chi_n \) to \( f \).

Define \( G : S^1 \times I \to S^1 \) by

\[
G(z, t) = \frac{p(t Rz)}{|p(t Rz)|}.
\]

The denominator is never 0

since \( p \) is assumed to have no complex zeros. \( G \) is

a homotopy from a constant map to \( f \).

Lemma 7.11. Let \((X, A)\) be a topological pair, and \( u \in H_n(X, A) \). Then, there exists a compact subset \( C \) of \( X \) and \( w \in H_n(C, C \cdot A) \) such that \( H_n(x)(w) = u \) where \( i : (C, C \cdot A) \to (X, A) \) is the inclusion.

(This lemma is usually paraphrased by the statement
"Singular theoryology has compact supports."

Proof: Say \( u = [z] \) where \( z \in \text{Sn}(X, A) \) is a cycle. Then \( z = \text{Sn}(i)(c) \) for some \( c \in \text{Sn}(X) \) where \( i : (X, \emptyset) \to (X, A) \) is the inclusion. Since \( z \) is a cycle, it follows that \( \exists c \in \text{Sn}_1(A) \subset \text{Sn}_1(X) \).

Write \( c = \sum_{\tau} n_\tau \tau \) where \( \tau \) ranges over the set of singular \( n \)-simplices of \( X \) and all but finitely many of the integers \( n_\tau \) are 0. Let \( C = \bigcup_{n_\tau \neq 0} \tau(A_n) \).

\( C \) is compact, being a finite union of compact spaces. Since each \( \tau \) for which \( n_\tau \neq 0 \) has image in \( C \), \( c \in \text{Sn}(C) \subset \text{Sn}(X) \). Thus \( \exists c \in \text{Sn}_1(C) \cap \text{Sn}_1(A) = \text{Sn}_1(C \cap A) \).

Thus if \( j' : (C, \emptyset) \to (C, C \cap A) \) denotes the inclusion, \( \text{Sn}(j')(c) \) is a cycle. Let \( v = [\text{Sn}(j')(c)] \in H_n(C, C \cap A) \).

The commutative diagram of inclusions

\[
\begin{array}{ccc}
(C, \emptyset) & \xrightarrow{j'} & (C, C \cap A) \\
\downarrow & & \downarrow \iota \\
(X, \emptyset) & \xrightarrow{j} & (X, A)
\end{array}
\]

yields \( \text{Sn}(i)(\text{Sn}(j')(c)) = z \). Thus \( u = [z] = [\text{Sn}(i) \text{Sn}(j')(c)] = H_n(i)[\text{Sn}(j')(c)] = H_n(i)(v) \).

If \( G \) is an abelian group and \( \{ H_x \mid x \in J \} \) a family of subgroups of \( G \) (\( J \) is some index set) we write \( \sum_{x \in J} H_x \) for the subgroup of \( G \) generated by the \( H_x \), \( x \in J \). Explicitly, \( \sum_{x \in J} H_x = \{ \sum_{x \in J} h_x \mid h_x \in H_x \) and all but finitely many of the \( h_x \) are 0. \( \sum_{x \in J} H_x \) is not to be confused with the direct sum \( \bigoplus_{x \in J} H_x \). In each \( \beta \in J \) let \( i_\beta : H_\beta \to \sum_{x \in J} H_x \) denote the inclusion. We
also have the canonical inclusions \( i_\beta : H_\beta \to \bigoplus_{x \in J} H_x \).

In each \( x \in J \), the diagram

\[
\begin{array}{ccc}
\bigoplus_{x \in J} H_x & \xrightarrow{\Sigma j_x} & \sum_{x \in J} H_x \\
\downarrow i_\beta & & \\
H_\beta & \xrightarrow{j_\beta} & H_x
\end{array}
\]

commutes.

If \( X \) is a topological space and \( \{A_x \mid x \in J\} \) is a family of subspaces of \( X \), we have, for each \( x \in J \),

\[\sum_{x \in J} S_n(A_x) \subseteq S_n(X).\]

Since \( S_n(X) \to S_{n-1}(X) \) canonically,

\[S_n(A_x) \to S_{n-1}(A_x)\]

for each \( x \in J \), we obtain a subchain complex \( \sum_{x \in J} S(A_x) \) of \( S(X) \). If \( J \) is finite

(say \( J = \{1, \ldots, k\} \)) we sometimes write \( S(A_1) + \cdots + S(A_k) \)

for \( \sum_{x \in J} S(A_x) \).

**Lemma 7.12.** Let \((X, A)\) be a topological pair and \( U \subseteq A \).

Then \( U \) can be excised from \((X, A)\) (see 6.1) if and only if the inclusion \( j : S(X-U) + S(A) \to S(X) \) induces isomorphisms in homology in all dimensions.

**Proof.** The chain map \( S(X-U) + S(A) \xrightarrow{p} \frac{S(X-U) + S(A)}{S(A)} \)

where \( p \) is the canonical projection, is onto and has kernel \( S(X-U) \cap S(A) = S(A \cap (X-U)) = S(A-U) \). Thus we have a chain isomorphism

\[h : S(X-U, A-U) = \frac{S(X-U)}{S(A-U)} \to \frac{S(X-U) + S(A)}{S(A)}\]

given by \( h(c + S_n(A-U)) = c + S_n(A) \) for all \( c \in S_n(X-U) \).
$$S(X-U, A-U) = \frac{S(X-U)}{S(A-U)} \xrightarrow{h} \frac{S(X-U)+S(A)}{S(A)}$$

Thus, $k$ induces isomorphisms in homology in all dimensions (i.e., $U$ can be excised from $(X,A)$) if and only if $k$ induces isomorphisms in homology in all dimensions.

We have the commutative diagram of chain maps with exact rows:

$$
\begin{array}{cccccc}
0 & \rightarrow & S(A) & \rightarrow & S(X-U)+S(A) & \rightarrow & S(X-U)+S(A) \rightarrow 0 \\
 & & \downarrow^{S(A)} & & \downarrow^{k} & & \downarrow^{S(A)} \\
0 & \rightarrow & S(A) & \rightarrow & S(X) & \rightarrow & S(X)/S(A) \rightarrow 0
\end{array}
$$

This yields the commutative diagram with exact rows:

$$
\begin{array}{cccc}
\cdots & \rightarrow & H_n(\frac{S(X-U)+S(A)}{S(A)}) & \rightarrow & H_n(A) & \rightarrow & H_n(S(X-U)+S(A)) & \rightarrow & H_n(\frac{S(X-U)+S(A)}{S(A)}) & \rightarrow & H_{n-1}(A) \rightarrow H_{n-1}(S(X-U)+S(A)) \\
\downarrow^{H_n(k)} & & \downarrow^1 & & \downarrow^{H_n(1)} & & \downarrow^{H_n(k)} & & \downarrow^1 & & \downarrow^{H_n(1)} \\
H_{n+1}(X,A) & \rightarrow & H_n(A) & \rightarrow & H_n(X) & \rightarrow & H_n(X,A) & \rightarrow & H_{n-1}(A) \rightarrow H_{n-1}(X)
\end{array}
$$

Thus, by the 5-Lemma, $H_n(k)$ is an isomorphism for all $n$ if and only if $H_n(1)$ is an isomorphism for all $n$.

**Corollary 7.13.** Let $A$ and $B$ be subspaces of some topological space. Then the inclusion of pairs $(A, ANB) \rightarrow (A\cup B, B)$ is an excision map if and only if the inclusion of pairs $(B, ANB) \rightarrow (A\cup B, A)$ is an excision map.

**Proof.** By definition (6.1), $(A, ANB) \rightarrow (A\cup B, B)$ is an excision map if and only if $B-A$ can be excised from $(A\cup B, B)$. By 7.12, the latter holds if and only if the
Inclusion \( S((A \cup B) - (B - A)) + S(B) \to S(A \cup B) \) induces isomorphisms in homology, i.e., if and only if the inclusion \( S(A) + S(B) \to S(A \cup B) \) induces isomorphisms in homology. Similarly, the latter is the case if and only if the inclusion of pairs \((B, A \cap B) \to (A \cup B, A)\) is an extension map.

**Definition 7.14.** A topological triad \((X; A, B)\) is an ordered triple consisting of a topological space \(X\) and two subspaces \(A\) and \(B\). If \((X; A, B)\) and \((Y; C, D)\) are topological triads, a *map of triads* \(f: (X; A, B) \to (Y; C, D)\) consists of a continuous map \(f^0: X \to Y\) such that \(f^0(A) \subset C\) and \(f^0(B) \subset D\).

**Definition 7.15.** A topological triad \((X; A, B)\) is *excisive* if \(X = A \cup B\) and one (and hence both) of the inclusions \((A, A \cap B) \to (X, B)\) \((B, A \cap B) \to (X, A)\) is excisive.

**Example 7.16.** If \(X = A \cup B\) and \(A, B\) are both open in \(X\), then \((X; A, B)\) is an excisive triad. For \(B - A = X - A\) is closed in \(X\), \(B\) is open in \(X\), and so \(B - A \subset \text{int} B\). Thus, by the Excision Property 6.2, \(B - A\) can be excised from \((X; B)\), i.e., \((A, A \cap B) \to (X, B)\) is excisive.

**Example 7.17.** Suppose \(X = A \cup B\) and \(A, B\) are both closed in \(X\). Suppose there exists an open set \(U\) in \(X\) such that \(A \cap B\) is a strong deformation retract of both \(U \cap A\) and \(U \cap B\). Then \((X; A, B)\) is an excisive triad, i.e., \((A, A \cap B) \to (X, B)\) is an excisive relative homomorphism under the above conditions.

**Theorem 7.18 (Mayer–Vietoris Sequence).** Let \((X; A, B)\) be an excisive triad. Write \(i_A: A \cap B \to A\), \(i_B: A \cap B \to B\), \(j_A: A \to X\), \(j_B: B \to X\) for the inclusion maps. Then for each \(n\), there exists a natural homomorphism \(\Delta: H_n(X) \to H_{n-1}(A \cap B)\) such that the sequence
Theorem 7.12. Let $\Delta$ be a map of short exact sequences of chain complexes. 

\[ \cdots \rightarrow H_{n+1}(X) \xrightarrow{\Delta} H_n(A \oplus B) \xrightarrow{f} H_n(A) \oplus H_n(B) \xrightarrow{\Delta} H_{n-1}(A \cap B) \rightarrow \cdots \]

where $\Delta$ is a natural transformation. Then the diagram

\[
\begin{array}{c}
H_n(X) \\
\downarrow f^* \\
H_n(Y)
\end{array}
\quad \xrightarrow{\Delta} 
\begin{array}{c}
H_n(A \cap B) \\
\downarrow f^* \\
H_n(C \cap D)
\end{array}
\]

commutes, where $f^* : A \cap B \rightarrow C \cap D$ is the restriction of $f$. 

Proof. We have the short exact sequence of chain complexes

\[ 0 \rightarrow S(A \cap B) \xrightarrow{j} S(A) \oplus S(B) \xrightarrow{j} S(A) + S(B) \rightarrow 0 \]

which yields the long exact sequence of homology groups

\[ \cdots \rightarrow H_n(S(A) + S(B)) \xrightarrow{\partial} H_n(A \cap B) \xrightarrow{\partial} H_n(A) \oplus H_n(B) \xrightarrow{\partial} H_n(S(A) + S(B)) \rightarrow \cdots \]

Let $j : S(A) + S(B) \rightarrow S(X)$ denote the injection. The diagram

\[
\begin{array}{c}
S(A) \oplus S(B) \\
\downarrow j \\
S(X)
\end{array}
\quad \xrightarrow{\partial} 
\begin{array}{c}
S(A) + S(B) \\
\downarrow j \\
S(X)
\end{array}
\]

commutes, and by 7.12, $H_n(j)$ is an isomorphism for all $n$. The asserted exact sequence results with $\Delta$ being the composition

\[ H_n(X) \xrightarrow{\Delta} H_n(S(A) + S(B)) \xrightarrow{\partial} H_n(A \cap B) \xrightarrow{\partial} H_{n-1}(A \cap B). \]

If $f : (X; A, B) \rightarrow (Y; C, D)$ is a map of short exact sequences, we obtain a map of short exact sequences of chain complexes

\[ 0 \rightarrow S(A \cap B) \rightarrow S(A) \oplus S(B) \rightarrow S(A) + S(B) \rightarrow 0 \]

\[ S(f^*) \downarrow \quad \downarrow S(f^*) \quad \downarrow S(f^*) \]

\[ 0 \rightarrow S(C \cap D) \rightarrow S(C) \oplus S(D) \rightarrow S(C) + S(D) \rightarrow 0 \]

where $f_1 : A \rightarrow C$, $f_2 : B \rightarrow D$ are the restrictions of $f^*$, and
\[ S(f^c)| \text{is the restriction of } S(f^c): S(x) \to S(y). \text{ The diagram of chain maps} \]
\[
\begin{align*}
S(A) + S(B) & \xrightarrow{j} S(X) \\
S(f^c) & \downarrow \\
S(C) + S(D) & \xrightarrow{j'} S(Y)
\end{align*}
\]

commute, where \( j' \) is the inclusion. Thus the diagram
\[
\begin{align*}
H_n(X) & \xrightarrow{H_n(j)} H_n(S(A) + S(B)) \xrightarrow{\partial} H_{n-1}(A \cap B) \\
H_n(f^c) & \downarrow \\
H_n(Y) & \xrightarrow{H_n(j')} H_n(S(C) + S(D)) \xrightarrow{\partial} H_{n-1}(C \cap D)
\end{align*}
\]

commutes, which yields naturality of \( \Delta \).

Corollary 7.19. If \((X;A,B)\) is an explosive triple, then
\[ \nu \in (\Delta: H_*(X) \to \tilde{H}_*(A \cap B)) \in \tilde{H}_*(A \cap B). \]

\( \tilde{\Delta} : H_*(X) \to \tilde{H}_*(A \cap B) \) denotes the map obtained by restricting the range of \( \Delta \), then the sequence
\[
\begin{align*}
H_i(X) & \xrightarrow{\tilde{\Delta}} \tilde{H}_i(A \cap B) \xrightarrow{\tilde{\nu}} \tilde{H}_i(A) \oplus \tilde{H}_i(B) \xrightarrow{\tilde{\nu}(A) - \tilde{\nu}(B)} \tilde{H}_i(X)
\end{align*}
\]
is exact. If \( A \cap B \neq \emptyset \), \( \tilde{H}_i(J) \to \tilde{H}_i(J) \) is exact.

Proof. Let \( P \) be a one point space. Then \((P;P,P)\) is an explosive triple. Let \( c: (X;A,B) \to (P;P,P) \) be the unique map of triples. From 7.18 and 7.2 we obtain the commutative diagram
\[
\begin{align*}
\circ & \xrightarrow{c} \circ \\
\vec{H}_i(A \cap B) & \xrightarrow{(\vec{H}_i(A), \vec{H}_i(B))} \vec{H}_i(A) \oplus \vec{H}_i(B) \xrightarrow{\vec{H}_i(A) - \vec{H}_i(B)} \vec{H}_i(X) \\
\downarrow & \downarrow \\
H_i(X) & \xrightarrow{\Delta} H_i(A \cap B) \xrightarrow{(H_i(A), H_i(B))} H_i(A) \oplus H_i(B) \xrightarrow{H_i(A) - H_i(B)} H_i(X) \\
\downarrow & \downarrow \\
H_i(P) & \xrightarrow{\Delta} H_i(P) \xrightarrow{H_i(P) \oplus H_i(P)} H_i(P) \xrightarrow{H_i(P) - H_i(P)} H_i(P)
\end{align*}
\]
where the bottom 2 rows and right 3 columns are all blank. Since $H_1(P) = 0$, commutativity of the left square yields the assertion about $im\Delta$. Existence of the asserted sequence follows by diagram chasing.

If $A \cap B \neq \emptyset$, choose any $x \in A \cap B$. By 7.3, there exists a subset $E \subseteq X$ such that $\{[x] - [x_0] | x_0 \in E\}$ is a basis for $H_0(X)$. If $x \in A \cap E$, then $[x] - [x_0] = (\tilde{H}_0(\tilde{\alpha})) - \tilde{H}_0(\tilde{\beta}))([x] - [x_0]$, $\sigma)$. If $x \in B \cap E$, then $[x] - [x_0] = (\tilde{H}_0(\tilde{\alpha})) - \tilde{H}_0(\tilde{\beta}))([x] - [x_0]$, and so $[x] - [x_0]$ lies in $\text{im}(\tilde{H}_0(\tilde{\alpha})) - \tilde{H}_0(\tilde{\beta}))$ for all $x \in E$.

In uniformity of notation, write $\Delta = \Delta : \tilde{H}_n(X) \rightarrow \tilde{H}_{n-1}(A \cap B)$ for $n \geq 2$.

Corollary 7.20. If $(X; A, B)$ is an spaces tower, the sequence

$$
\begin{align*}
\cdots \rightarrow \tilde{H}_n(X) &\xrightarrow{\tilde{H}_n(\tilde{\alpha})} \tilde{H}_n(A \cap B) \\
&\xrightarrow{\tilde{H}_n(\tilde{\beta})} \tilde{H}_n(A) \oplus \tilde{H}_n(B) \\
&\xrightarrow{\tilde{H}_n(\tilde{\alpha}) \oplus \tilde{H}_n(\tilde{\beta})} \tilde{H}_n(X) \\
&\xrightarrow{\tilde{H}_n(\tilde{\alpha})} \tilde{H}_n(A) &\xrightarrow{\tilde{H}_n(\tilde{\beta})} \tilde{H}_n(B) \\
&\xrightarrow{\tilde{H}_n(\tilde{\alpha} \oplus \tilde{H}_n(\tilde{\beta}))} \tilde{H}_n(X) \\
&\xrightarrow{\tilde{H}_n(\tilde{\alpha}) \oplus \tilde{H}_n(\tilde{\beta})} \tilde{H}_n(A) &\xrightarrow{\tilde{H}_n(\tilde{\beta})} \tilde{H}_n(B) \\
&\xrightarrow{\tilde{H}_n(\tilde{\alpha} \oplus \tilde{H}_n(\tilde{\beta}))} \tilde{H}_n(X) \\
&\xrightarrow{\tilde{H}_n(\tilde{\alpha}) \oplus \tilde{H}_n(\tilde{\beta})} \tilde{H}_n(A) &\xrightarrow{\tilde{H}_n(\tilde{\beta})} \tilde{H}_n(B) \\
&\xrightarrow{\tilde{H}_n(\tilde{\alpha} \oplus \tilde{H}_n(\tilde{\beta}))} \tilde{H}_n(X)
\end{align*}
$$

is exact and natural with respect to maps of spaces towers. If $A \cap B \neq \emptyset$, $\tilde{H}_0(\tilde{\alpha}) - \tilde{H}_0(\tilde{\beta})$ is exact.

The sequence in 7.20 is called the reduced Mayer-Vietoris sequence of the spaces tower $(X; A, B)$.

Exercise 7.21. Let $X$ be a topological space, $A$ a compact subspace of $X$, and $a \in H_n(A)$ such that $H_n(i)(a) = 0$ where $i : A \rightarrow X$ is the inclusion. Then there exists a compact subspace $B$ of $X$ with $A \subset B$ such that $H_n(j)(a) = 0$ where $j : A \rightarrow B$ is the inclusion.

Lemma 7.22. Let $A$ be a subset of $S^n$ which is homeomorphic to $I^n$ for some $r > 0$. Then $\tilde{H}_k(S^n - A) = 0$ for all $k \in \mathbb{Z}$.

Proof. We proceed by induction on $r$. The result is trivial if $r = 0$. If $r > 0$, then $A$ is a one point space and so $S^n - A$ is contractible. Suppose $r > 0$ and assume inductively $\tilde{H}_k(S^n - B) = 0$ if $k \in \mathbb{Z}$ whenever $B \simeq I^{k-1}$. 
Choose a homeomorphism $h: I^r \to A'$. Let $0 \leq i < j < i = 1$ and write $A' = h([t, i] \times I^{r-1})$, $A'' = h([i, j] \times I^{r-1})$. We first show that if $0 \neq u \in \tilde{H}_k(S^n - (A' \cup A''))$, then either $\tilde{H}_k(h''(u)) \neq 0$ or $\tilde{H}_k(h''(u)) \neq 0$, where $h': S^n - (A' \cup A'') \to S^n - A'$, $h'': S^n - (A' \cup A'') \to S^n - A''$ are the inclusion.

Since $A', A''$ are compact, $S^n - A'$ and $S^n - A''$ are open in $S^n$, and hence in $(S^n - A') \cup (S^n - A'') = S^n - (A' \cup A'')$. Thus, by 7.16 $(S^n - (A' \cup A'')) = S^n - (A' \cup A'')$ is an essential trick, and so from the reduced Mayer-Vietoris sequence

$$\tilde{H}_{k+1}(S^n - (A' \cup A'')) \xrightarrow{h''} \tilde{H}_k(S^n - A') \oplus \tilde{H}_k(S^n - A'')$$

is exact. Since $A' \cup A'' = h(\{x\} \times I^{r-1}) \subseteq I^{r-1}$, it follows from the induction hypothesis that $\tilde{H}_{k+1}(S^n - (A' \cup A'')) = 0$. Thus, by exactness, one of $(\tilde{H}_k(h''(u)), \tilde{H}_k(h''(u)))$ is non-zero.

It now follows easily that if $0 < t_0 < t_1 < \cdots < t_k = 1$ is any partition of $I$, and $A_i = h([t_{i-1}, t_i] \times I^{r-1})$ and $j_i: S^n - A_i$ is the inclusion for $1 \leq i \leq k$, then if $0 \neq u \in \tilde{H}_k(S^n - A)$, at least one of the $\tilde{H}_k(j_i)(u)$ is non-zero.

Suppose $0 \neq u \in \tilde{H}_k(S^n - A)$. By compact support (7.11), there exists a compact subset $C$ of $S^n - A$ and $v \in \tilde{H}_k(C)$ such that $u = \tilde{H}_k(i)(v)$ where $i: C \to S^n - A$ is the inclusion. In each $t \in I$, let $B_t = h(\{x\} \times I^{r-1})$. Then $B_t \subseteq I^{r-1}$ and so $\tilde{H}_k(S^n - B_t) = 0$ by the induction hypothesis.

Thus by 7.21 there exists a compact space $C_t$ such that $C \subseteq C_t \subseteq S^n - B_t$ and $\tilde{H}_k(h_t)(v) = 0$ where $h_t: C \to C_t$ is the inclusion. By an easy compactness argument there exists, for each $t \in I$, an open neighborhood $U$ of $t$ in $I$ such that $h(U \times I^{r-1}) \subseteq S^n - C_t$. Thus, by compactness of $I$, there exists a finite partition $0 = t_0 < \cdots < t_k = 1$ of $I$ such that for $1 \leq i \leq k$, there exists a $t \in [t_{i-1}, t_i]$ such that $A_i = h([t_{i-1}, t_i] \times I^{r-1})$ is disjoint from $C_t$. We also have, for some $i$, $\tilde{H}_k(j_i)(u) \neq 0$ where $j_i: S^n - A \to S^n - A_i$ is the inclusion. We have the commutative diagram of maps induced by inclusions.
This yields a contradiction since \( C \neq \tilde{H}_k (L_\epsilon)(\kappa) \)
\( = \tilde{H}_k (L_\epsilon)(\kappa)(\kappa) \) and \( \tilde{H}_k (L_\epsilon)(\kappa) = 0 \).

**Lemma 7.23.** Let \( \mathcal{B} \) be a subset of \( S^n \) which is homeomorphic to \( S^k \), \( 0 \leq k \leq n-1 \). Then

\[
\tilde{H}_k (S^n - \mathcal{B}) \cong \begin{cases} 
\mathbb{Z} & \text{if } \kappa = n-k-1 \\
0 & \text{otherwise}
\end{cases}
\]

**Proof.** We proceed by induction on \( k \). If \( k = 0 \), \( \mathcal{B} \) consists of 2 points, and \( S^n - \mathcal{B} \) has the homotopy type of \( S^{n-1} \), and so

\[
\tilde{H}_0 (S^n - \mathcal{B}) \cong \begin{cases} 
\mathbb{Z} & \text{if } \kappa = n-1 \\
0 & \text{otherwise}
\end{cases}
\]

Then, the result holds for \( k = 0 \).

Let \( k > 0 \) and assume inductively the result is valid for \( k-1 \). We can write \( \mathcal{B} = A_1 \cup A_2 \) where \( A_1, A_2 \) are both homeomorphic to \( S^{k-1} \) and \( A_1 \cap A_2 \) is homeomorphic to \( S^{k-2} \). By compactness of \( A_1 \) and \( A_2 \), \( S^n - A_1 \) and \( S^n - A_2 \) are both open in \( (S^n - A_1) \cup (S^n - A_2) = S^n - (A_1 \cap A_2) \). Then \( (S^n - (A_1 \cap A_2), S^n - A_1, S^n - A_2) \) is an excision triple. Note that \( (S^n - A_1) \cap (S^n - A_2) = S^n - \mathcal{B} \). Thus, the reduced Mayer-Vietoris sequence (7.20) yields the exact sequence

\[
\tilde{H}_q (S^n - A_1) \oplus \tilde{H}_q (S^n - A_2) \rightarrow \tilde{H}_q (S^n - (A_1 \cap A_2)) \rightarrow \tilde{H}_q (S^n - B) \rightarrow \tilde{H}_q (S^n - A_1) \oplus \tilde{H}_q (S^n - A_2)
\]

for all \( q \geq 0 \). By 7.22, the extreme groups are all 0 and so \( \mathcal{B} \) is an isomorphism. By the inductive hypothesis,

\[
\tilde{H}_q (S^n - (A_1 \cap A_2)) \cong \begin{cases} 
\mathbb{Z} & \text{if } q+1 = n-(k-1)-1 \\
0 & \text{otherwise}
\end{cases}
\]

and the result follows.
Theorem 7.24. (Jordan-Brouwer Separation Theorem). Let $B$ be a subset of $S^n$ which is homeomorphic to $S^{n-1}$. Then $S^n - B$ has exactly two components with $B$ as the boundary of each.

Proof. By 7.23, $\text{H}_0(S^n - B) \cong \mathbb{Z}$ and so $S^n - B$ has exactly two path components, $U$ and $V$. Since $B$ is compact, $S^n - B$ is open in $S^n$ and hence $S^n - B$ is locally path-connected (since $S^n$ is). It follows that $U$ and $V$ are the components of $S^n - B$, and both are open in $S^n - B$, and hence open in $S^n$.

We have $\partial U = \overline{U} \cap (S^n - U) \quad$ (by definition) $\overline{U} \cap (S^n - U) \quad$ (since $U$ is open in $S^n$) $\overline{U} - U$.

Similarly $\partial V = \overline{V} - V$.

Since $U \subseteq S^n - V$ and $S^n - V$ is closed in $S^n$, we have $\overline{U} \subseteq S^n - V$. Thus $\partial U \subseteq (S^n - V) - U = B$.

Similarly, $\partial V \subseteq B$. It remains only to prove $B = \overline{U} \cap \overline{V}$.

Let $x \in B$. We must show each open neighborhood $N$ of $x$ in $S^n$ meets both $\overline{U}$ and $\overline{V}$. Then, such an $N$, $B \cap N$ is a neighborhood of $x$ in $B$. Then, since $B \cong S^{n-1}$, $B \cap N$ contains a subset $A$ such that $B - A \cong S^{n-1}$. It suffices to show $A \cap \overline{U}$ and $A \cap \overline{V}$ are both non-empty. By 7.22, $\text{H}_0(S^n - (B - A)) = 0$ and so $S^n - (B - A) = \overline{U} \cup \overline{V}$ is path-connected, and hence connected. As noted above, $\overline{U} \subseteq S^n - V$ and so $\overline{U} \cap \overline{V} = \emptyset$. If $\overline{U} \cap A = \emptyset$, then $\overline{U} \cap (A \cup \overline{V})$ would be empty and so $\overline{U}$ would be both open and closed in $\overline{U} \cup \overline{V}$, contradicting the connectedness of $\overline{U} \cup \overline{V}$. Thus $\overline{U} \cap A \neq \emptyset$. Similarly $\overline{V} \cap A \neq \emptyset$.

Theorem 7.25. (Brouwer's Theorem on Invariance of Domain). Let $U$ be a subset of $S^n$ which is homeomorphic to $R^n$. Then $U$ is open in $S^n$.

Proof. Let $x \in U$. There exists a topological pair $(D, S)$ homeomorphic to $(R^n, S^{n-1})$ such that $x \in D - S \subseteq D \subseteq U$.  

16
It suffices to show that $D - S$ is open in $S^n$. By 7.22, $H_c(S^n - D) = 0$ and so $S^n - D$ is path-connected. By 7.24, $S^n - S$ has exactly 2 components. Now $S^n - S = (S^n - D) \cup (D - S)$. Since $S^n - D$ and $D - S$ are both connected, there must be the components of $S^n - S$. But since $S$ is locally path-connected and $S$ is closed in $S^n$ (since $S$ is compact), the component of $S^n - S$ are open in $S^n$. Thus $D - S$ is open in $S^n$, completing the proof.

Corollary 7.26. Let $M$ be a topological $n$-manifold and $U$ a subspace of $M$ which is also a topological $n$-manifold. Then $U$ is open in $M$.

Proof. Let $x \in U$. Choose an open neighborhood $V$ of $x$ in $M$ which is homeomorphic to $\mathbb{R}^n$. It suffices to show that $V \cap U$ is open in $V$.

Since $V \cap U$ is open in $V$, $V \cap U$ is a topological manifold, and thus a union of subspaces each homeomorphic to $\mathbb{R}^n$. Thus it suffices to show that any subspace $V$ of $U$ with $V \cap U$ open in $V$.

There is a homeomorphism $h$ of $V$ onto an open subset of $S^n$. By 7.25, $h(V)$ is open in $S^n$, and hence open in $h(U)$. Thus, since $h$ is a homeomorphism of $V$ onto $h(U)$, $V$ is open in $U$.

Corollary 7.27. Let $M$ and $N$ be topological $n$-manifolds, with $M$ compact and non-empty, and $N$ connected. Assume $M$ is not homeomorphic to $N$. Then $M$ is not homeomorphic to a subspace of $N$.

Proof. If $X$ were a subspace of $N$ homeomorphic to $M$, then $X$ would be open in $N$ by 7.26. But $X$ is also closed in $N$ since $M$ is compact. This is a contradiction since $N$ is connected and $X \neq \emptyset$ (since $M \neq \emptyset$) and $X \neq N$ (since $M$ is not homeomorphic to $N$).
8. Proof of the Homotopy Property and Acyclic Models

Our strategy is to first introduce a purely algebraic analogue of homotopy between chain maps called chain homotopy. It will be easy to show that chain homotopy between chain maps reduces the same homomorphisms in homotopy. The main work will be to show that homotopy maps between topological pairs induce chain homotopy chain maps between their singular complexes.

The geometric motivation for the notion of chain homotopy is that if $f, g : X \to Y$ are continuous and $F : X \times I \to Y$ is a homotopy from $f$ to $g$, then each singular $(n+1)$-simple $\tilde{f} : \Delta_n \to X$ yields a "singular $(n+1)$-prism" $F_0(\tilde{f} \times 1_I) : \Delta_n \times I \to Y$.

By triangulating the $\Delta_n \times I$ in a specified way, the singular prism $F_0(\tilde{f} \times 1_I)$ yields a singular $(n+1)$-chain $T_n f \in S_{n+1}(Y)$. Geometrically, the boundary of $\Delta_n \times I$ is $(\Delta_n \times 1) \cup (\Delta_n \times \{0\}) \cup (\Delta_n \times \{1\})$, which yields $\partial T_n f = \pm S_n f(\tilde{f}(0)) \pm S_n f(\tilde{f}(1)) \pm T_n(\tilde{f}(0))$. The precise sequence will be given in the formal definition below.

**Definition 8.1.** Let $C, D$ be chain complexes and $f, g : C \to D$ chain maps. A chain homotopy $T : C \to D$ from $f$ to $g$ consists of a sequence of group homomorphisms $T_n : C_n \to D_n$ such that for all $n \geq 0$, $\partial T_n + T_n \circ d = g_n - f_n$. (By convention, $T_n = 0$ for $n < 0$).

If such a chain homotopy exists, we say $f$ is chain homotopic to $g$, denoted $f \sim g$. We also write $f \simeq g$ if $T$ is a chain homotopy from $f$ to $g$.

**Proposition 8.2.** Chain homotopy is an equivalence relation.

**Proof.** To each chain map $f$, $f \simeq f$ where $T_n = 0$ for all $n$. Thus $\sim$ is reflexive.

Suppose $f \simeq g$. Then $g \simeq f$, and so $\sim$ is symmetric.
Suppose \( f \cong g \) and \( g \cong h \). Then \( f \cong g \) and so \( \cong \) is transitive.

**Proposition 8.3.** If \( f, f' : C \to D \) and \( g, g' : D \to E \) are chain maps such that \( f \cong f' \) and \( g \cong g' \), then \( g \circ f \cong g' \circ f' \).

**Proof.** Say \( f \cong f' \) and \( g \cong g' \). Then for all \( n \in \mathbb{Z} \),

\[
(g_n' f'_n - g_n f_n) = (g_n' f'_n - g_n f'_n + g_n f'_n - g_n f_n) = (g_n' - g_n) f'_n + g_n (f'_n - f_n)
\]

\[
= (\partial T_n + T_{n-1} \partial) f'_n + g_n (\partial S_n + S_{n-1} \partial)
\]

\[
= \partial T_n f'_n + T_{n-1} f'_n \partial + g_n \partial S_n + g_n S_{n-1} \partial
\]

\[
= \partial T_n f_n' + T_{n-1} f_n' \partial + \partial g_{n+1} S_n + g_n S_{n-1} \partial \quad (\text{since } f' \text{ and } g \text{ are chain maps})
\]

Thus \( g f \cong g' f' \) where \( U_n = T_n f_n' + g_{n+1} S_n \).

**Proposition 8.4.** Suppose \( f, g : C \to D \) are chain maps with \( f \cong g \). Then \( H_n(f) = H_n(g) \) for all \( n \in \mathbb{Z} \).

**Proof.** Say \( f \cong g \). Let \( z \in Z_n(C) \). Then,

\[
g_n(z) - f_n(z) = (\partial T_n + T_{n-1} \partial)(z) = \partial T_n(z) + 0 \quad (\text{since } \partial^2 = 0)
\]

Thus \( [g_n(z)] = [f_n(z)] \), i.e. \( H_n(f)[z] = H_n(g)[z] \).

Thus to prove the Homotopy Property, it only remains to show that if \( f, g : (X, A) \to (Y, B) \) are homotopic maps of topological pairs, then \( S(f), S(g) : S(X, A) \to S(Y, B) \) are chain homotopic. We shall reduce the problem even further.

**Lemma 8.5.** Suppose \( f \) for each topological pair \( (X, A) \), it were known that \( S(f) \cong S(g) \), where \( i_0, i : (X, A) \to (X \times I, A \times I) \) are given by \( i_0(x) = (x, 0) \),
$i_n^g(x) = (x, i_n^g) \text{ for all } x \in X$. Then the Homotopy Property follows.

Let $f, g : (X, A) \to (Y, B)$ be homotopy maps of pairs. Say $F : (X \times I, A \times I) \to (Y, B)$ is a homotopy from $f$ to $g$. Then $f = F \circ i_0$, $g = F \circ i_1$. Thus

$$S(f) = S(F \circ i_0) = S(F) S(i_0) = S(F) S(i_1) = S(F \circ i_1) = S(g).$$

Thus $H_n(f) = H_n(g)$ for all $n \geq 0$ by 8.4.

Thus to prove the Homotopy Property, it remains only to show $S(i_0) = S(i_1)$ for each topological pair $(X, A)$. We will first establish this in a stronger form for spaces rather than pairs. This stronger form will then enable us to deduce it for pairs.

**Definition 8.6.** Let $C$ be an augmented chain complex. We say $C$ is acyclic if $H_n(C) = 0$ for all $n \neq 0$, and $\varepsilon : H_0(C) \to \mathbb{Z}$ is an isomorphism.

A topological space $X$ is acyclic if $S(X)$ is acyclic.

**Exercise 8.7.** An augmented chain complex $C$ is acyclic if and only if the sequence

$$0 \to Z \xrightarrow{\varepsilon} C_0 \xrightarrow{d} C_1 \xrightarrow{d} C_2 \xrightarrow{d} \cdots$$

is exact.

At this stage, without having yet established the validity of the Homotopy Property, the only space we knew for sure is acyclic is a one point space. The next technical step toward proving the Homotopy Property is to show that $\Delta n$ and $\Delta n \times I$ are acyclic spaces for all $n \geq 0$.

**Theorem 8.8.** Let $X$ be a non-empty, open, convex subspace of $\mathbb{R}^n$. Then $X$ is acyclic.

**Proof.** Since $X$ is non-empty and path-connected,
\( E: H_0(X) \to \mathbb{Z} \) is an isomorphism, say \( 3, 13 \). It remains

only to show \( H_8(X) = 0 \) for all \( q > 0 \).

We will construct a sequence of group homomorphisms

\( T_q: S_q(X) \to S_{q+1}(X) \), \( q > 0 \), such that \( \partial T_q + T_{q-1} = \mathbb{L}_q(X) \)

for all \( q > 0 \). The result will then follow, for if \( z \) is any

cycle in \( S_q(X) \), \( q > 0 \), we would have

\[ z = (\partial T_q + T_{q-1}) (z) = \partial T_q (z) \] and so we would have

\[ [z] = 0. \]

The construction of the \( T_q \) is as follows: Choose

any \( P_0 \in X \). For each singular \( q \)-simplex

\( \tau: \Delta_q \to X \), define a singular \((q+1)\)-simplex

\[ T_q \tau: \Delta_{q+1} \to X \] by

\[(T_q \tau) \left( \sum_{i=0}^{q+1} t_i E_i \right) = \begin{cases} (1-t_0) \left( \sum_{i=0}^{q+1} \frac{t_i}{1-t_0} \Delta \right) E_i + t_0 P_0 & \text{if } t_0 \neq 1, \\
\phantom{=} P_0 & \text{if } t_0 = 1. \end{cases} \]

(Here, \((t_0, \ldots, t_{q+1})\) are barycentric coordinates, i.e. \( t_i \geq 0 \) for all \( i \)

and \( \sum_{i=0}^{q+1} t_i = 1 \)).

\( T_q \tau \) is continuous, for since

\( \Delta_q \) is compact and \( \tau \) is continuous,

\[ \| \tau \left( \sum_{i=0}^{q+1} t_i E_i \right) \| \] is bounded and so

\[ \lim_{t_0 \to 1} \left( 1-t_0 \right) \| \tau \left( \sum_{i=0}^{q+1} \frac{t_i}{1-t_0} \Delta \right) E_i + t_0 P_0 \| = 0. \]

Extend additively to a group homomorphism

\( T_q: S_q(X) \to S_{q+1}(X) \). For each \( q > 0 \) we will show

that for each singular \( q \)-simplex \( \tau: \Delta_q \to X \),
1) \((T_0^a c )^{(0)} = \mathcal{T}\)

2) For \(0 \leq c \leq b\), \(T_{b-1}(\mathcal{T}^{(c)}) = (T_b^a c )^{(c+1)}\).

Assuming this for the moment, the proof is then completed as follows: \(T_0^a c = \mathcal{T}\) as above,

\[
(\partial T_b^a c + T_{b-1}(\partial c ))(\sigma) = \partial (T_b^a c ) + T_{b-1}(\partial c )
\]

\[
= \sum_{i=0}^{b+1} (-1)^i (T_b^a c )^{(i)} + T_{b-1}\left( \sum_{i=0}^{b} (-1)^i c^{(i)} \right)
\]

\[
= (T_b^a c )^{(0)} + \sum_{i=1}^{b+1} (-1)^i (T_b^a c )^{(i)} + \sum_{i=0}^{b} (-1)^i T_{b-1}(c^{(i)})
\]

\[
= \mathcal{T} + \sum_{i=1}^{b+1} (-1)^i (T_b^a c )^{(i)} + \sum_{i=0}^{b} (-1)^i (T_b^a c )^{(i+1)} \quad \text{(by (1) and (2))}
\]

\[
= \mathcal{T}.
\]

We now check 1). For all \(\sum_{i=0}^{b} t_i E_i \in \Delta_b\),

\[
(T_b^a c )^{(0)} \left( \sum_{i=0}^{b} t_i E_i \right) = (T_b^a c ) F_{b+1}^c \left( \sum_{i=0}^{b} t_i E_i \right)
\]

\[
= (T_b^a c ) \left( C^c E_0 + \sum_{i=0}^{b} t_i E_{i+1} \right)
\]

\[
= (1-c) \mathcal{T} \left( \sum_{i=0}^{b} \frac{1}{1-c} t_i E_i \right) + c \mathcal{P}_0 = \mathcal{T} \left( \sum_{i=0}^{b} t_i E_i \right)
\]

and so \((T_b^a c )^{(0)} = \mathcal{T}\).

We now check 2). For all \(\sum_{j=0}^{b} t_j E_j \in \Delta_b\), we have,

For \(0 \leq c \leq b\),

\[
T_{b-1}(\mathcal{T}^{(c)}) \left( \sum_{j=0}^{b} t_j E_j \right) = \begin{cases} (1-t_{c+1}) \mathcal{T}^{(c)} \left( \sum_{j=0}^{b} \frac{1}{1-c} t_{j+1} E_j \right) + \mathcal{P}_0 & \text{if } c+1 \\ \mathcal{P}_0 & \text{if } c = 1 \\ \end{cases}
\]

\[
= \begin{cases} (1-t_{c+1}) \mathcal{T} F_{c+1}^c \left( \sum_{j=0}^{b} \frac{1}{1-c} t_{j+1} E_j \right) + t_0 \mathcal{P}_0 & \text{if } c+1 \\ \mathcal{P}_0 & \text{if } c = 1 \\ \end{cases}
\]

\[
= \begin{cases} (1-t_{c+1}) \mathcal{T} \left( \sum_{j=0}^{b} \frac{1}{1-c} t_{j+1} E_j + \sum_{j=0}^{b} t_{j+1} E_{j+1} \right) + t_0 \mathcal{P}_0 & \text{if } c+1 \\ \mathcal{P}_0 & \text{if } c = 1 \\ \end{cases}
\]
On the other hand, 
\[(T_\delta \mathcal{J})^{(\infty)} (\sum_{j=0}^{\delta} t_j \mathcal{E}_j) = (T_\delta \mathcal{J})^{(\infty)} (\sum_{j=0}^{\delta} t_j \mathcal{E}_j)\]

\[= (T_\delta \mathcal{J}) \left( \sum_{j=0}^{\delta} t_j \mathcal{E}_j + x_{\mathcal{E}_j} + 0 \mathcal{E}_{x_{\mathcal{E}_j}} + \sum_{j=\delta+1}^{\delta} t_j \mathcal{E}_{j+1} \right)\]

\[= \begin{cases} (1-t_0) \mathcal{J} \left( \frac{1}{1-t_0} \left[ \sum_{j=0}^{\delta-1} t_j \mathcal{E}_j + 0 \mathcal{E}_{x_{\mathcal{E}_j}} + \sum_{j=\delta+1}^{\delta} t_j \mathcal{E}_{j+1} \right] \right) + t_0 \mathcal{P}_0 & \text{if } t_0 \neq 1 \\
\mathcal{P}_0 & \text{if } t_0 = 1 \end{cases}\]

and so \((T_\delta \mathcal{J})^{(\infty)} = T_{\delta-1} \mathcal{J}^{(\infty)}\), completing the proof.

Corollary 8.9. In all \(n > 0\), \(A_n\) and \(A_n \times I\) are acyclic.

The geometric content of the proof of the homotopy property is contained in 8.8. The remainder of the proof is algebraic.

Theorem 8.10. There is a rule which assigns to each topological space \(X\) a chain homotopy \(T^X\) from \(S(t_0)\) to \(S(t_1)\) such that \(t_0, t_1: X \to X \times I\) are given by \(t_0(X) = (x, 0)\), \(t_1(X) = (x, 1)\) such that \(t_0\) and \(t_1\) are continuous, \(T\) is commutative, and \(X \to Y\) is continuous, the diagram

\[\begin{array}{ccc}
S_n(X) & \xrightarrow{S_n(t)} & S_n(Y) \\
T^X_n \downarrow & & \downarrow T^Y_n \\
S_{n+1}(X \times I) & \xrightarrow{S_{n+1}(t \times 1)} & S_{n+1}(Y \times I) \\
\end{array}\]

commutes for all \(n\). (We paraphrase this by saying "There is a natural chain homotopy from \(S(t_0)\) to \(S(t_1)\)."")

Proof. \(T^X_n\) will be constructed inductively. Let \(T^X_0 = 0\) for \(n < 0\). We leave the commutative diagrams

\[\begin{array}{ccc}
S_n(X) & \xrightarrow{S_n(t)} & S_n(Y) \\
T^X_n \downarrow & & \downarrow T^Y_n \\
S_{n+1}(X \times I) & \xrightarrow{S_{n+1}(t \times 1)} & S_{n+1}(Y \times I) \\
\end{array}\]
\[
Z \xrightarrow{\varepsilon} S_0(\Delta_0)
\]
\[
1_j \downarrow \quad \downarrow S_0(i_j)
\]
\[
Z \xrightarrow{\varepsilon} S_0(\Delta_0 \times I) \xrightarrow{\partial} S_1(\Delta_0 \times I), \quad j = 0, 1.
\]

The bottom row is exact by \(8.6\) and \(8.7\). Write \(\delta_0 : \Delta_0 \to \Delta_0\) for the identity map. Since
\[
E(S_0(i_0)(\delta_0) - S_0(i_0)(\delta_0)) = E(\delta_0) - E(\delta_0) = 0,
\]
there exists \(d_1 \in S_1(\Delta_0 \times I)\) such that \(E d_1 = S_0(i_0)(\delta_0) - S_0(i_0)(\delta_0)\).

If \(\varphi : \Delta_0 \to X\) is any singular \(0\)-simplex, define
\[
T_0^X(\varphi) = S_1(\varphi \times 1_I)(d_1) \in S_1(X \times I).
\]

Since
\[
\Delta_0 \xrightarrow{\varphi} X
\]
\[
1_j \downarrow \quad \downarrow i_j
\]
\[
\Delta_0 \times I \xrightarrow{\varphi \times 1_I} X \times I
\]

commutes for \(j = 0, 1\),
\[
\partial T_0^X(\varphi) = \partial S_1(\varphi \times 1_I)(d_1) = S_0(\varphi \times 1_I) \partial (d_1)
\]
\[
= S_0(\varphi \times 1_I) \left[ S_0(i_0)(\delta_0) - S_0(i_0)(\delta_0) \right]
\]
\[
= S_0(i_0) S_0(\varphi)(\delta_0) - S_0(i_0) S_0(\varphi)(\delta_0)
\]
\[
= S_0(i_0)(\varphi) - S_0(i_0)(\varphi) = [S_0(i_0) - S_0(i_0)](\varphi).
\]

Thus \(\partial T_0^X + T_0^X \partial = S_0(i_0) - S_0(i_0)\).

We next check that \(T_0\) is natural. Suppose \(f : X \to Y\) is continuous. If \(\varphi : \Delta_0 \to X\) is a singular \(0\)-simplex, then
\[
S_1(f \times 1_I) T_0^X(\varphi) = S_1(f \times 1_I) S_1(\varphi \times 1_I)(d_1)
\]
\[
= S_1(f \varphi \times 1_I)(d_1) = T_0^Y(f \varphi)
\]
\[
= T_0^Y S_0(f)(\varphi)
\]
and so \(S_1(f \times 1_I) T_0^X = T_0^Y S_0(f)\), i.e., \(T_0\) is natural.

Suppose \(n > 0\) and that \(T_k\) has been constructed for \(k < n\) such that each \(T_k\) is natural,
and \( \partial T^n_k + T^n_k \partial = S^n(i_l) - S^n(i_c) \) for each topological space \( X \). We have the commutative diagrams

\[
\begin{array}{ccc}
S_{n-1}(\Delta^n) & \xleftarrow{\partial} & S_n(\Delta^n) \\
S_{n-1}(i_j) & \downarrow & S_n(i_j) \\
S_{n-1}(\Delta^n \times I) & \xleftarrow{\partial} & S_n(\Delta^n \times I) \\
& \xleftarrow{\partial} & S_{n+1}(\Delta^n \times I)
\end{array}
\]

for \( j = 0, 1 \). The bottom row is exact by 8.9 and 8.7.

Write \( \delta_n : \Delta^n \to \Delta^n \) for the identity map. Then

\[
\partial \left[ (S_n(i_l) - S_n(i_c) - T^n_{n-1} \partial) (\delta_n) \right] = \\
(S_{n-1}(i_j) \partial - S_n(i_c) \partial - \partial T^n_{n-1} \partial) (\delta_n) \\
= \left[ (\partial T^n_{n-1} + T^n_{n-2} \partial) \partial - \partial T^n_{n-1} \partial \right] (\delta_n) \\
= (\partial T^n_{n-1} \partial - \partial T^n_{n-1} \partial) (\delta_n) = 0.
\]

Hence there exists \( d_{n+1} \in S_{n+1}(\Delta^n \times I) \) such that

\[
\partial d_{n+1} = (S_n(i_l) - S_n(i_c) - T^n_{n-1} \partial) (\delta_n).
\]

If \( \tau : \Delta^n \to X \) is any singular \( n \)-simplex, define

\[
T^n_X(\tau) = S_{n+1}(\tau \times 1_I)(d_{n+1}) \in S_{n+1}(X \times I).
\]

The proof that \( T^n \) is natural is the same as that given for \( T^0 \).

Since the diagrams

\[
\begin{array}{ccc}
\Delta^n & \xrightarrow{\tau} & X \\
\downarrow & & \downarrow \\
\Delta^n \times I & \xrightarrow{\tau \times 1_I} & X \times I
\end{array}
\]

commute for \( j = 0, 1 \), we have

\[
\partial T^n_X(\tau) = \partial S_{n+1}(\tau \times 1_I)(d_{n+1}) = S_n(\tau \times 1_I) \partial (d_{n+1}) = \\
\]
\[ S_n(\mathcal{F} \times 1_\mathcal{I}) \left( S_n(\mathcal{U}_1) - S_n(\mathcal{U}_0) - T_{n-1}^{\mathcal{A}_n} \right)(\delta_0) \]

\[ = \left[ S_n(\mathcal{U}_1) S_n(\mathcal{F}) - S_n(\mathcal{U}_0) S_n(\mathcal{F}) - S_n(\mathcal{F} \times 1_\mathcal{I}) T_{n-1}^{\mathcal{A}_n} \right](\delta_0) \]

\[ = S_n(\mathcal{U}_1)(\mathcal{F}) - S_n(\mathcal{U}_0)(\mathcal{F}) - T_{n-1}^{\mathcal{X}} S_n(\mathcal{F}) \mathcal{D}(\delta_0) \quad \text{(by naturality of } T_{n-1}^{\mathcal{X}}) \]

\[ = S_n(\mathcal{U}_1)(\mathcal{F}) - S_n(\mathcal{U}_0)(\mathcal{F}) - T_{n-1}^{\mathcal{X}} S_n(\mathcal{F})(\delta_0) \]

\[ = S_n(\mathcal{U}_1)(\mathcal{F}) - S_n(\mathcal{U}_0)(\mathcal{F}) - T_{n-1}^{\mathcal{X}} \mathcal{D}(\delta_0) \quad \text{and so} \]

\[ \partial T_{n+1}^{\mathcal{X}} + T_{n-1}^{\mathcal{X}} \mathcal{D} = S_n(\mathcal{U}_1) - S_n(\mathcal{U}_0), \quad \text{completing the induction.} \]

Corollary 8.11. For topological pairs, there is a natural chain homotopy \( T \) from \( S(A) \) to \( S(A) \), i.e., for each topological pair \((X,A)\), there is a rule which assigns a chain homotopy \( T(X,A) \) from \( S(A) \) to \( S(A) \) whose \( i_0, i_1 : (X,A) \to (X \times \mathcal{I}, A \times \mathcal{I}) \) are given by \( \delta_0(x) = (X,0) \), \( \delta_1(x) = (X,1) \), such that whenever \( f : (X,A) \to (Y,B) \) is a map of topological pairs, the diagram

\[
\begin{array}{ccc}
S_n(X,A) & \xrightarrow{S_n(f)} & S_n(Y,B) \\
T_{n}^{(X,A)} \downarrow & & \downarrow T_{n}^{(Y,B)} \\
S_{n+1}(X \times \mathcal{I}, A \times \mathcal{I}) & \xrightarrow{S_{n+1}(f \times 1_\mathcal{I})} & S_{n+1}(Y \times \mathcal{I}, B \times \mathcal{I})
\end{array}
\]

commutes for all \( n \), where \( (f \times 1_\mathcal{I})^\delta = f^\delta \times 1_\mathcal{I} \).

Proof. By 8.10 we have a natural chain homotopy \( T \) for spaces from \( S(A) \) to \( S(A) \). Thus, given a topological pair \((X,A)\), the diagram

\[
\begin{array}{ccc}
S_n(A) & \xrightarrow{S_n(i)} & S_n(X) \\
T_{n}^{A} \downarrow & & \downarrow T_{n}^{X} \\
S_{n+1}(A \times \mathcal{I}) & \xrightarrow{S_{n+1}(i \times 1_\mathcal{I})} & S_{n+1}(X \times \mathcal{I})
\end{array}
\]

commutes for all \( n \), where \( i : A \to X \) is the inclusion.
Thus, passing to quotients, $T_n^X$ induces a group homomorphism

$$T_n^X : S_n(X^A) \to S_{n+1}(X \times 1, A \times 1)$$

for each $n$. Let $j : (X, \emptyset) \to (X, A)$ denote the inclusion. $T_n(X^A)$ is the unique group homomorphism such that

$$S_n(X) \xrightarrow{S_n(j)} S_n(X^A)$$

commutes. We also have the commutative diagram of chain maps with exact rows

$$
\begin{array}{ccc}
S(X) & \xrightarrow{S(j)} & S(X^A) \\
S(j_k) & \downarrow & \downarrow S(j_k) \\
S(X \times 1) & \xrightarrow{S(j \times 1)} & S(X \times 1, A \times 1)
\end{array}
$$

If $k = 0, 1$, then we can write $a \in S_n(X^A)$, we can write

$$a = S_n(j)(b)$$

for some $b \in S_n(X)$. Thus

$$\partial T_n^{(X^A)} + T_{n-1}^{(X^A)} \partial)(a) = \partial T_n^{(X^A)} S_n(j)(b) + T_{n-1}^{(X^A)} \partial S_n(j)(b)$$

$$= \partial S_n(j \times 1) T_n^X(b) + T_{n-1}^{(X^A)} S_{n-1}(i) \partial(b)$$

$$= S_n(j \times 1) \partial T_n^X(b) + S_n(j \times 1) T_{n-1}^X \partial(b)$$

$$= S_n(j \times 1)(\partial T_n^X + T_{n-1}^X \partial)(b)$$

and so

$$\partial T_n^{(X^A)} + T_{n-1}^{(X^A)} \partial = S_n(j_1) - S_n(j_0)$$

If $f : (X, A) \to (Y, B)$ is a map of topological pairs, consider the diagram
As a consequence of 8.5 and 8.11, the Homotopy Property (Theorem 5.5) is now proved.

The proof of 8.10 given above is a specialization of part of the proof of a rather abstract algebraic theorem called the Acyclic Models Theorem. This abstract theorem will have several later applications, and we proceed to describe it.

Definition 8.12. Let $F$ and $G$ be covariant functors from a category $C$ to a category $D$. A natural transformation $T : F \to G$ is a rule which assigns to each object $X$ in $C$ a morphism $T^X : F(X) \to G(X)$ in $D$ such that for each morphism $\alpha : X \to Y$ in $C$, the diagram

$$
\begin{array}{ccc}
F(X) & \xrightarrow{F(\alpha)} & F(Y) \\
T^X \downarrow & & \downarrow T^Y \\
G(X) & \xrightarrow{G(\alpha)} & G(Y)
\end{array}
$$

commutes.
Example 8.13. Let \( F, G : \mathcal{J}P \to \mathcal{C}C \) (topological pairs to chain complexes) be the functors given by
\[ F = S \] for (singular complexes), \( G(X,A) = S(X \times I, A \times I) \), and if \( f : (X,A) \to (Y,B) \) is a morphism in \( \mathcal{J}P \), then
\[ G(f) = S(f \times I) : S(X \times I, A \times I) \to S(Y \times I, B \times I) \]. The rule which assigns to each topological pair \( (X,A) \) the chain map \( S(i_0) : S(X,A) \to S(X \times I, A \times I) \) is a natural transformation from \( F \) to \( G \).
Similarly, for \( S(i_1) \).

Definition 8.14. Let \( \mathcal{D} \) be an arbitrary category and \( F, G : \mathcal{D} \to \mathcal{G}G \) covariant functors, and \( P, Q : F \to \mathcal{G}G \) natural transformations. A natural chain homotopy \( T \) from \( P \) to \( Q \) is a rule which assigns to each object \( X \) in \( \mathcal{D} \) a chain homotopy \( T_X \) from the chain map \( P_X : F(X) \to G(X) \) to the chain map \( Q_X : F(X) \to G(X) \) such that for each \( n \in \mathbb{Z} \), the rule which assigns to each object \( X \) in \( \mathcal{D} \) the abelian group homomorphism \( T^n_X : F_n(X) \to G_{n+1}(X) \) is a natural transformation from the functor \( F_n \) to the functor \( G_{n+1} \). (\( F_n \) and \( G_{n+1} \) are functors from \( \mathcal{D} \) to \( \text{Ab} \), the category of abelian groups.)

Example 8.15. Take \( F, G \) as in 8.13. Let \( P, Q \) be given by \( P(X,A) = S(i_0) \), and \( Q(X,A) = S(i_1) \). Then, the \( T \) of 8.11 is a natural chain homotopy from \( S(i_0) \) to \( S(i_1) \) in the sense of 8.14.

Definition 8.16. A category with models \( (\mathcal{C}, \mathcal{M}) \) consists simply of a category \( \mathcal{C} \) and a set \( \mathcal{M} \) of some of the objects of \( \mathcal{C} \). When \( \mathcal{M} \) is understood, members of \( \mathcal{M} \) are sometimes called models.

Example 8.17. \( (\text{Top}, \Delta) \) is a category with models where \( \text{Top} \) is the category of topological spaces, and
\[ \Delta = \{ \Delta n | \ n = 0, 1, 2, \ldots \} \].

Definition 8.18. Let \( (\mathcal{C}, \mathcal{M}) \) be a category with models, and \( F : \mathcal{C} \to \text{Ab} \) a covariant functor from \( \mathcal{C} \) to the category of
abelian groups. An $M$-basis $B$ for $F$ is a set satisfying the following:

(1) For each $b \in B$, there is a model $M_b \in M$ such that $b$ is a member of the abelian group $F(M_b)$.

(2) In each object $X$ in $G$, $F(X)$ is a free abelian group with basis \[ \{ F(x)(b) \mid b \in B \}, \quad x \in G(M_b, X). \]

If $F$ admits an $M$-basis, we say $F$ is $M$-free (or free with models $M$).

Example 8.19. In $n \geq 0$, let $S_n : \text{Top} \rightarrow \text{Ab}$ be the $n$th singular chain group functor. Let $\Delta$ be as in 8.17. Then $\{ S_n \}$ is a $\Delta$-basis for $S_n$, where $S_n : \Delta_n \rightarrow \Delta_n$ is the identity map.

Definition 8.20. Let $(D, M)$ be a category with models. Let $F : D \rightarrow \text{AbC}$ be a covariant functor from $D$ to the category of augmented chain complexes. We say $F$ is $M$-acyclic (or acyclic on models $M$) if for each $M \in M$, $F(M)$ is acyclic.

Example 8.21. By 8.9, the singular complex functor $S : \text{Top} \rightarrow \text{AbC}$ and the functor $G : \text{Top} \rightarrow \text{AbC}$ given by $G(X) = S(X \times I)$ for each topological space $X$, and $G(f) = S(f \times 1)$ for each continuous $f : X \rightarrow Y$, are both $\Delta$-acyclic.

Theorem 8.22. (Acyclic Model Theorem) Let $(D, M)$ be a category with models. Let $F, G : D \rightarrow \text{AbC}$ be covariant functors such that

(i) for each $n \in \mathbb{Z}$, $F_n$ is $M$-free;

(ii) $G$ is $M$-acyclic.

Then

(a) (Existence) There exists a natural transformation $P$ from $F$ to $G$;

(b) (Uniqueness) Given any two natural transformations $P, Q$ from $F$ to $G$, there is a natural chain homotopy $T$ from $P$ to $Q$.

The proof of 8.22 is left as an exercise.
Note that 8.10 is a corollary of the Uniqueness part of 8.22 since \( S(x) \) and \( S(u) \) are natural transformations from \( S \to G \) (where \( G \) is as in 8.21), and each \( S_n \) is \( \Delta \)-free, and \( G \) is \( \Delta \)-acyclic.

9. Proof of the Excision Property

If \((X,A)\) is a topological pair and \(U\) is a subset of \(X\), the condition that \(U \subset \text{int} \; A\) is equivalent to the condition that \(X = \text{int} \; (X-U) \cup \text{int} \; A\).

More generally, let \(\mathcal{O}\) be a family of subspaces of \(X\) whose interiors cover \(X\). Write \( S(X,\mathcal{O}) = \sum_{A \in \mathcal{O}} \text{S}(A) \subseteq S(X) \) and let \(j : S(X,\mathcal{O}) \to S(X)\) denote the inclusion map.

By 7.12, the Excision Property (6.2) is a consequence of

Theorem 7.1. (Small Simplicial Theorem). Let \(X\) be a topological space, \(\mathcal{O}\) a family of subspaces of \(X\) whose interiors cover \(X\), and \(j : S(X,\mathcal{O}) \to S(X)\) the inclusion. Then for all \(n \in \mathbb{Z}\), \(H_n(j) : H_n \left( S(X,\mathcal{O}) \right) \to H_n(X)\) is an isomorphism.

The proof of 7.1 proceeds by systematically subdividing singular simplices into chains consisting of smaller singular simplices so as to eventually land in \( S(X,\mathcal{O}) \).

The proof of 9.1 proceeds by systematically subdividing singular simplices into chains consisting of smaller singular simplices so as to eventually land in \( S(X,\mathcal{O}) \).

We proceed to develop this subdivision process.
Definition 9.2. The affine simplicial category $A^A$ has as its objects the standard simplices $\Delta_0, \Delta_1, \Delta_2, \ldots$ and all affine maps between standard simplices as its morphisms. In §1 we defined affine maps between real vector spaces. Here we mean maps $\Delta_m \to \Delta_n$ obtained by restriction of affine maps from $\mathbb{R}^m$ to $\mathbb{R}^n$.) A morphism $\Delta_m \to \Delta_n$ in $A^A$ will sometimes be called an affine $g$-simplex in $\Delta_n$.

If $P_0, P_1, \ldots, P_g \in \Delta_n$, we write $(P_0, P_1, \ldots, P_g)$ for the unique affine $g$-simplex in $\Delta_n$ which sends $E_i$ to $P_i$, $0 \leq i \leq g$.

Definition 9.3. In integers $n \geq 0$ and $g \geq 0$, let $A^A_g(\Delta_n)$ be the subgroup of $S^A_g(\Delta_n)$ spanned by the affine $g$-simplices in $\Delta_n$. $A^A_g(\Delta_n)$ is called the $g$-affine chain group of $\Delta_n$.

Given $f : \Delta_m \to \Delta_n$ is an affine map, then $S^A_g(f) : S^A_g(\Delta_m) \to S^A_g(\Delta_n)$ carries $A^A_g(\Delta_m)$ into $A^A_g(\Delta_n)$ for all $g \geq 0$ since the composition of affine maps is affine. Moreover, if $P_0, \ldots, P_g \in \Delta_n$, $g \geq 0$,

we have $f((P_0, P_1, \ldots, P_g)) = \sum_{i=0}^g (-1)^i (P_0, \ldots, \hat{P_i}, \ldots, P_g) \in A^A_{g-1}(\Delta_n)$.

Thus the $A^A_g(\Delta_n)$, $g \geq 0$, constitute a subchain complex $A^A(\Delta_n)$ of $S(\Delta_n)$. By restricting the singular complex functor $S$, we immediately obtain

Proposition 9.4. $\mathbf{A} : \mathbf{A}^A \to \mathbf{A}^C$ is a covariant functor from the affine simplicial category to the category of augmented chain complexes.

Definition 9.5. Let $\sigma = (P_0, P_1, \ldots, P_g) : \Delta_g \to \Delta_n$ be an affine $g$-simplex in $\Delta_n$. The barycenter of $\sigma$, denoted $B_\sigma$, is the point $\frac{1}{g+1} \sum_{i=0}^g P_i \in \Delta_n$.

Definition 9.6. Let $\sigma = (P_0, P_1, \ldots, P_g) : \Delta_g \to \Delta_n$ be an affine $g$-simplex in $\Delta_n$, and $P$ a point in $\Delta_n$. The germ of
P and \( \sigma \), denoted \( P \star \sigma \), is the affine \((q+1)\)-simplex 
\((P_0, P_1, \ldots, P_q) : \Delta^{q+1} \rightarrow \Delta^n.\)

We also write \( P \star : A_q(\Delta^n) \rightarrow A_{q+1}(\Delta^n) \) in the
unique group homomorphism which sends \((P_0, \ldots, P_q)\)
to \((P_0, P_1, \ldots, P_q)\).

**Definition 9.7.** For each \( q \geq 0 \) and \( n \geq 0 \), we define a
group homomorphism \( \text{sd}_q : A_q(\Delta^n) \rightarrow A_q(\Delta^n) \) (the \( q \)th
stabilization map) by induction on \( q \) as follows:
\( \text{sd}_0 = 1_{A_0(\Delta^n)} \).
If \( q > 0 \) and \( \sigma \) is an affine
\( q \)-simplex in \( \Delta^n \), then \( \text{sd}_q(\sigma) = B_{\sigma} \star \text{sd}_{q-1}(\sigma \sigma) \).

**Example 9.8.** If \( \sigma = (P_0, P_1) \), then \( \text{sd}_1(\sigma) = B_{\sigma} \star \text{sd}_0(\sigma \sigma) \)

\[
= B_{\sigma} \star [(P_1) - (P_0)] = (B_{\sigma}P_1) - (B_{\sigma}P_0).
\]

If \( \sigma = (P_0, P_1, P_2) \), then \( \text{sd}_2(\sigma) = \)

\[
B_{\sigma} \star \text{sd}_1 \left[ (P_1, P_2) - (P_0, P_2) + (P_0, P_1) \right].
\]

\[
= B_{\sigma} \star \left[ (Q_0, P_2) - (Q_0, P_1) - (Q_1, P_2) + (Q_1, P_0) - (Q_2, P_1) - (Q_2, P_0) \right] \]

\[
= (B_{\sigma}Q_0, P_2) - (B_{\sigma}Q_0, P_1) + (B_{\sigma}Q_1, P_0) - (B_{\sigma}Q_2, P_2) - (B_{\sigma}Q_2, P_1). \]

**Lemma 9.9.** If \( n \geq 0 \) and \( c \in A_p(\Delta^n) \), \( q \geq 0 \), and
\( \sigma \in \Delta^n \), then \( \partial (P \star c) = c - P \star (\partial c). \)

If \( c \in A_0(\Delta^n) \), then \( \partial (P \star c) = c - E(c)(P). \)

**Proof.** It suffices to check in affine simplices. Suppose
\( q > 0 \) and \( \sigma = (P_0, P_1, \ldots, P_q). \) Then \( P \star \sigma = (P_0, P_1, \ldots, P_q) \)
and so \( \partial (P \star \sigma) = (P_0, P_1, \ldots, P_q) + \sum_{i=1}^{q+1} (c_i) \star (P_0, P_1, \ldots, P_q) \).
\[
\sigma - P_\ast \left( \sum_{i=1}^{g+1} (-1)^i (P_0 \ldots P_i \ldots P_g) \right) = \sigma - P_\ast (\Delta \sigma).
\]

If \( \sigma = (P_0) \in A_\ast (\Delta n) \), then \( P_\ast \sigma = (P_0 P_0) \) and so \( \partial (P_\ast \sigma) = (P_0) - (P) = \sigma - \varepsilon (\sigma)(P) \).

**Proposition 9.10.** For each \( n \geq 0 \), the sequence of group homomorphisms \( sd_q : A_q (\Delta n) \to A_q (\Delta n), \; q \geq 0 \), constitutes an augmented chain map \( sd : A(\Delta n) \to A(\Delta n) \).

**Proof.** Since \( sd_0 = 1_{\Delta n} \), the diagram

\[
\begin{array}{ccc}
A_\ast (\Delta n) & \xrightarrow{sd_0} & A_\ast (\Delta n) \\
\varepsilon \downarrow & & \downarrow \varepsilon \\
\mathbb{Z} & & \mathbb{Z}
\end{array}
\]


trivially commutes, and so \( sd \) is an augmentation preserving.

If \( \sigma : \Delta_1 \to \Delta_n \) is an affine 1-simplex, then

\[
\partial (sd_1 (\sigma)) = \partial (B_\sigma \ast sd_0 (\Delta \sigma)) = \partial (B_\sigma \ast (\Delta \sigma)) \\
= \partial \sigma - \varepsilon (\Delta \sigma)(B_\sigma) \quad \text{(by 9.9)} \\
= \partial sd_0 (\sigma) \quad \text{(since \( sd_0 = 1_{\Delta_n} \) and \( \varepsilon \Delta = 0 \)).}
\]

Let \( q > 1 \) and assume inducively \( \partial sd_{q-1} = sd_{q-2} \partial \).

Let \( \sigma : \Delta_q \to \Delta_n \) be an affine \( q \)-simplex. Then

\[
\partial sd_q (\sigma) = \partial \left[ B_\sigma \ast sd_{q-1} (\Delta \sigma) \right] \\
= sd_{q-1} (\Delta \sigma) - B_\sigma \ast \partial (sd_{q-1} (\Delta \sigma)) \quad \text{(by 9.9)} \\
= sd_{q-1} \partial (\sigma) - B_\sigma \ast [sd_{q-2} \partial (\Delta \sigma)] \quad \text{(by inductive hypothesis)} \\
= sd_{q-1} \partial (\sigma) \quad \text{(since \( \partial \Delta = 0 \)).}
\]
Lemma 7.11. Let \( f: \Delta m \rightarrow \Delta n \) be a morphism in \( A \mathcal{S} \) (i.e., an affine simplex). Then for each affine \( \mathfrak{g} \)-simplex 

\[
\tau: \Delta^g \rightarrow \Delta m \quad \Rightarrow \quad B_{\mathfrak{g}+g} = f(B_{\mathfrak{g}}).
\]

Proof. If \( \tau = (P_0 \cdots P_\mathfrak{g}) \), then \( f\tau = (f(P_0) \cdots f(P_\mathfrak{g})) \) and so 

\[
B_{\mathfrak{g}+g} = \frac{1}{\mathfrak{g}+1} \sum_{i=0}^{\mathfrak{g}} f(P_i) = f\left( \frac{1}{\mathfrak{g}+1} \sum_{i=0}^{\mathfrak{g}} P_i \right) \quad \text{(since } f \text{ is affine)}
\]

\[
= f(B_{\mathfrak{g}}).
\]

Lemma 9.12. Let \( f: \Delta m \rightarrow \Delta n \) be a morphism in \( A \mathcal{S} \). Let \( P \in \Delta m \). Then the diagram

\[
\begin{array}{ccc}
A_{\mathfrak{g}}(\Delta m) & \xrightarrow{A_{\mathfrak{g}}(f)} & A_{\mathfrak{g}}(\Delta n) \\
\downarrow P^* & & \downarrow \text{f}(P)^* \\
A_{\mathfrak{g}+1}(\Delta m) & \xrightarrow{A_{\mathfrak{g}+1}(f)} & A_{\mathfrak{g}+1}(\Delta n)
\end{array}
\]

commutes for all \( g \geq 0 \),

Proof. It suffices to check on affine \( \mathfrak{g} \)-simplices \( \tau: \Delta^g \rightarrow \Delta m \). Let \( \tau = (P_0 \cdots P_\mathfrak{g}) : \Delta^g \rightarrow \Delta m \). Then

\[
A_{\mathfrak{g}+1}(f)(P^* \tau) = A_{\mathfrak{g}+1}(f)(P \ P_0 \cdots P_\mathfrak{g}) = (f(P) f(P_0) \cdots f(P_\mathfrak{g}))
\]

\[= f(P)^* (f(P_0) \cdots f(P_\mathfrak{g})) = f(P)^* A_{\mathfrak{g}}(f)(\tau).
\]

Proposition 9.13. \( \sigma_d \) is a natural transformation from \( A \) to \( A \).

Proof. We must show that for each morphism \( f: \Delta m \rightarrow \Delta n \) in \( A \mathcal{S} \), the diagram

\[
\begin{array}{ccc}
A_{\mathfrak{g}}(\Delta m) & \xrightarrow{A_{\mathfrak{g}}(f)} & A_{\mathfrak{g}}(\Delta n) \\
\downarrow \text{id}_A & & \downarrow \text{id}_A \\
A_{\mathfrak{g}}(\Delta m) & \xrightarrow{A_{\mathfrak{g}}(f)} & A_{\mathfrak{g}}(\Delta n)
\end{array}
\]

commutes.
commutes for all $t > 0$. We proceed by induction on $t$. The case $t = 0$ is trivial since $s_{d_0}$ is the identity map. Let $t > 0$ and assume, inductively,

$s_{d_{t-1}} A_{t-1} f(t) = A_{t-1} f(t) s_{d_{t-1}}$. Let $T : A_t \to A_n$ be an affine $t$-simplex. Then

\[
A_{t} f(t)(T) = A_{t} f(t) s_{d_{t-1}} T(T) = A_{t} f(t) s_{d_{t-1}} (A_{t-1} f(t) T) = A_{t} f(t) s_{d_{t-1}} A_{t-1} f(t) (T) = A_{t} f(t) (T s_{d_{t-1}} T) = A_{t} f(t) T (s_{d_{t-1}} T) \]

( by 9.10)

( by the inductive hypothesis)

\[
= f(B_{t}) \ast A_{t-1} f(t) s_{d_{t-1}} T(t) = A_{t} f(t) (B_{t} \ast s_{d_{t-1}} T(t)) \]

( by 7.12)

Definition 9.14. Let $X$ be a metric space, $t > 0$, and $c \in S_{d_{t}}(X)$. Say $c = \sum_{\sigma} n_{\sigma} T$ where $T$ ranges over

the singular $t$-simplices of $X$, and all but finitely many of the integers $n_{\sigma}$ are 0. The mesh of $c$, denoted $\text{mesh}(c)$, is $

\max \{ \text{diam} T(A_{t}) \mid n_{\sigma} \neq 0 \}$.

Lemma 9.15. Let $T = (p_0 \ldots p_k) : A_{t} \to A_n$ be an affine $t$-simplex. Then $\text{mesh}(T) = \max_{k, l} \| p_k - p_l \|$. 

Proof. Each point in $T(A_{t})$ is a convex sum of $p_0, \ldots, p_k$. Let

\[
x = \sum_i x_i p_i, \quad y = \sum_j x_j p_j \]

be convex sums.

Then

\[
\| x - y \| = \| x - \sum_j x_j p_j \| = \| \sum_j x_j (x - p_j) \| \quad (\text{since } \sum_j x_j = 1) \]

\[
\leq \sum_j x_j \| x - p_j \| \quad (\text{since the } x_j \geq 0) \]

\[
= \sum_i x_i \| (\sum_i x_i p_i) - p_i \| = \sum_i x_i \| \sum_i x_i (p_i - p_i) \| \quad (\text{since } \sum_i x_i = 1) \]

(\text{by the inductive hypothesis})

\[
\sum_{j} s_{ij} \| P_{k} - P_{j} \| \ (\text{since the } s_{i} > 0)
\]

\[
\sum_{j} s_{ij} \max_{k,e} \| P_{k} - P_{j} \| = \max_{k,e} \| P_{k} - P_{j} \| \ (\text{since } \sum_{i} s_{i} = \sum_{j} s_{ij} = 1).
\]

Proposition 9.16. Let \( n \geq 0 \) and \( c \in A_{\overline{g}}(A_{n}). \)
Then \( \text{mesh}(\text{aff}(c)) \leq \frac{\overline{g}}{\overline{g} + 1} \text{mesh}(c). \)

Proof. We proceed by induction on \( \overline{g}. \) The result is trivial for \( \overline{g} = 0 \) since every \( 0- \) chain has mesh 0.

Let \( \overline{g} > 0 \) and assume, inductively, the lemma holds for affine \((\overline{g} - 1)-\) chains. It suffices to check on affine \( \overline{g}-\) simplices.

Let \( \sigma : A_{\overline{g}} \to A_{n} \) be an affine \( \overline{g}-\) simplex. \( \text{aff}(\sigma) \)

\[ \text{is a linear combination of } \text{simplices of the form} \]

\[ \text{aff}(P_{0} \... P_{\overline{g} - 1}) \]

where \( (P_{0} \... P_{\overline{g} - 1}) \) is an affine \((\overline{g} - 1)-\) simplex occurring in \( \text{aff}(A_{\overline{g}}). \)

By 9.15, it suffices to show that \( \| P_{i} - P_{j} \| \leq \frac{\overline{g}}{\overline{g} + 1} \text{mesh}(\sigma) \) and \( \| B_{\sigma} - P_{i} \| \leq \frac{\overline{g}}{\overline{g} + 1} \text{mesh}(\sigma) \)

for \( 0 \leq i < \overline{g} - 1, \ 0 \leq j < \overline{g} - 1. \)

By the inductive hypothesis, \( \| P_{i} - P_{j} \| \leq \text{mesh}(\text{aff}(\sigma)) \)

\[ \leq \frac{\overline{g} - 1}{\overline{g} + 1} \text{mesh}(\text{aff}(\sigma)). \]

Since \( \text{aff}(\text{aff}(\sigma)) \subset \text{aff}(A_{\overline{g}}) \) for \( 0 \leq k < \overline{g} \), we have \( \text{mesh}(\text{aff}(\sigma)) \leq \text{mesh}(\sigma). \)

Thus since \( \frac{\overline{g} - 1}{\overline{g} + 1} \leq \frac{\overline{g}}{\overline{g} + 1} \), we have \( \| P_{i} - P_{j} \| \leq \frac{\overline{g}}{\overline{g} + 1} \text{mesh}(\sigma). \)

\( \| B_{\sigma} - x \| \leq \frac{\overline{g}}{\overline{g} + 1} \text{mesh}(\sigma) \) for all \( x \in \text{aff}(A_{\overline{g}}). \)

It remains only to show \( \| B_{\sigma} - P_{i} \| \leq \frac{\overline{g}}{\overline{g} + 1} \text{mesh}(\sigma) \)

for \( 0 \leq i < \overline{g} - 1. \) Note that each \( P_{i} \) lies in \( \sigma(A_{\overline{g}}) \); for by an inductive argument, \( P_{i} \) is the barycenter of some affine simplex with vertices in \( \sigma(A_{\overline{g}}). \)

We will, in fact, show \( \| B_{\sigma} - x \| \leq \frac{\overline{g}}{\overline{g} + 1} \text{mesh}(\sigma) \) for all \( x \in \sigma(A_{\overline{g}}). \)

Say \( \sigma = (Q_{0} \... Q_{\overline{g}}) \) and \( x = \sum_{i} \alpha_{i} Q_{i} \) is a convex sum.

Then \( \| B_{\sigma} - x \| = \| \sum_{i} \alpha_{i} (B_{\sigma} - Q_{i}) \| \leq \sum_{i} \alpha_{i} \| B_{\sigma} - Q_{i} \|. \) Let \( x_{0} = B_{\sigma} \).
such that \( \| B_0 - Q_0 \| \leq \| B_0 - Q_c \| \) for \( 0 \leq i \leq \ell \). Then

\[
\| B_0 - x \| \leq \sum x_i \| B_0 - Q_{0c} \| = \| B_0 - Q_0c \|
\]

\[
= \left\| \left( 1 - \frac{x}{\ell + 1} \right) Q_0c \right\| \|
= \left\| \frac{1}{\ell + 1} \sum_{i=0}^{\ell} \left( Q_i - Q_0c \right) \right\|
\]

\[
\leq \frac{1}{\ell + 1} \sum_{i=0}^{\ell} \| Q_i - Q_0c \| \quad \text{(since } Q_i - Q_0c = 0 \text{)}
\]

\[
\leq \frac{1}{\ell + 1} \ell \cdot \max_{\ell} \| Q_i - Q_0c \| \n
\leq \frac{\ell}{\ell + 1} \max_{\ell} (s) \quad \text{(by 9.4)}.
\]

The next step is to carry the subdivision process over to arbitrary singular chains of general spaces to obtain a natural transformation \( Sd : S \rightarrow S \).

**Definition 9.17.** Let \( X \) be an arbitrary topological space. For \( q \geq 0 \), define \( Sd_q : S_q(X) \rightarrow S_{q-1}(X) \) to be the group homomorphism as follows: if \( \sigma : \Delta_q \rightarrow X \) is a singular \( q \)-simplex, then

\[
Sd_q(\sigma) = S_q(\sigma) (sd_q(\delta_q)) \quad \text{where } \delta_q : \Delta_q \rightarrow \Delta_{q-1} \text{ is the identity map.}
\]

**Proposition 9.18.** For each topological space \( X \), the \( Sd_q \) constitute an augmented chain map \( Sd : S(X) \rightarrow S(X) \).

**Proof.** Since \( Sd_0(\delta_0) = \delta_0 \) and \( S_0(\sigma)(\delta_0) = \sigma \) for each \( \sigma : \Delta_0 \rightarrow X \), we have \( Sd_0 = S_0(\sigma) \) and so augmentation is preserved.

Let \( q \geq 0 \) and suppose \( \sigma : \Delta_q \rightarrow X \) is a singular \( q \)-simplex. Then

\[
\partial Sd_q(\sigma) = \partial \left[ S_q(\sigma)(sd_q(\delta_q)) \right]
\]

\[
= S_{q-1}(\sigma) \partial sd_q(\delta_q) \quad \text{(since } S(\sigma) \text{ is a chain map)}
\]

\[
= S_{q-1}(\sigma)sd_{q-1}(\partial(\delta_q)) \quad \text{(by 9.10)}
\]
\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

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\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

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\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

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\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

\[ = S_{q-1}(\tau) \sum_{x} (-1)^{x} A_{\delta-1}(F_{\delta}^{x}) \delta_{\delta-1} \]

Proposition 9.19. Let \( \delta \) be a natural transformation from \( S \) to \( S \).

Proof. We must show that for each continuous \( f : X \to Y \), the diagram

\[
\begin{array}{ccc}
S_{q}(X) & \xrightarrow{S_{q}(f)} & S_{q}(Y) \\
\delta_{q} & \downarrow & \delta_{q} \\
S_{q}(X) & \xrightarrow{S_{q}(f)} & S_{q}(Y)
\end{array}
\]

commutes in all \( q \geq 0 \). Let \( \tau : \Delta_{q} \to X \) be a singular \( q \)-simplex. Then

\[
\delta_{q} S_{q}(f)(\tau) = \delta_{q} (f \tau) = S_{q} (f \tau)(\delta_{q}(\tau))
\]

\[
= S_{q}(f)(S_{q}(\tau)(\delta_{q}(\tau))) \quad \text{(since } S_{q} \text{ is a functor)}
\]

\[
= S_{q}(f) \delta_{q} (\tau).
\]

By iteration, we obtain natural transformations \( \delta^{n}_{q} : S \to S \) for \( n > 0 \), where \( S : \text{Top} \to \text{Ab} \) is the singular complex functor.

Note that \( \delta \) and \( \delta_{q} \) agree on affine chains.

Corollary 9.20. For each singular \( q \)-simplex \( \tau : \Delta_{q} \to X \) and each \( n > 0 \), \( \delta^{n}_{q}(\tau) = S_{q}(\tau)(\delta^{n}_{q}(\tau)) \).
Proposition 9.21. For each \( n \geq 1 \), there exists a natural chain homotopy \( T^{(n)} \) from \( S^n: S \to S \) to the identity transformation \( \mathbf{1}_S: S \to S \) where \( S: \text{Top} \to \text{CEL} \) is the singular complex functor.

Proof. This is immediate from the uniqueness part of the Acyclic Models Theorem (8.22). Hence \( S \) is both \( \Delta \)-acyclic and \( \Delta \)-acyclic, where \( \Delta = \{ \Delta_0, \Delta_1, \Delta_2, \ldots \} \).

We recall the following standard result from general topology:

**Lemma 9.22. (Lebesgue Covering Lemma).** Let \( X \) be a compact metric space and \( \mathcal{O} \) an open covering of \( X \). Then there exists a real \( \delta > 0 \) such that if \( Y \subseteq X \) with \( \text{diam}(Y) < \delta \), then \( Y \) is contained in some member of \( \mathcal{O} \). \( \delta \) is called a Lebesgue number for \( \mathcal{O} \).

**Lemma 9.23.** Let \( X \) be a topological space and \( \mathcal{O} \) a collection of subsets of \( X \) whose interiors cover \( X \). Let \( \Delta^Q \to X \) be a singular \( Q \)-simplex. Then \( \forall n \) sufficiently large, \( S^n(X, \mathcal{O}) \subseteq S^n(X, \mathcal{O}) \).

Proof. Let \( Q = \{ Q^{-1}(\text{int} A) | A \in \mathcal{O} \} \). \( Q \) is an open cover of the compact metric space \( \Delta^Q \). Let \( \delta \) be a Lebesgue number for \( \mathcal{O} \). Choose \( n \) so large that \( \left( \frac{\delta}{n+1} \right)^n \text{diam} \Delta^Q < \delta \).

Since \( \text{mesh} (S^n \delta^Q) \leq \left( \frac{\delta}{n+1} \right)^n \text{diam} \Delta^Q \), by 9.16, each simplex appearing in \( S^n \delta^Q \) has image contained in some member of \( \mathcal{O} \). Since \( S^n(X, \mathcal{O}) = S^n(X, \mathcal{O}) \), each simplex appearing in \( S^n(X, \mathcal{O}) \) has image contained in some member of \( \mathcal{O} \), and so \( S^n(X, \mathcal{O}) \subseteq S^n(X, \mathcal{O}) \).

**Lemma 9.24.** Let \( X \) be a topological space and \( \mathcal{C} \) a covering of \( X \). Let \( T^{(n)} \) be as in 9.21. Let \( c \in S^n(X, \mathcal{O}) \subseteq S^n(X) \). Then \( T^{(n)}(c) \in S_{n+1}(X, \mathcal{O}) \).

Proof. It suffices to check on generators, say \( Y \in \mathcal{O} \) and
\[ \sigma \in S^k_f(Y) \text{ is a singular } f\text{-simplex. By naturality of } T_{f}^{(n)} \text{, the diagram} \]

\[ \begin{array}{ccc}
S^k_f(Y) & \longrightarrow & S^k_f(X) \\
\downarrow T_{f}^{(n)} & & \downarrow T_{f}^{(n)} \\
S^k_{f+1}(Y) & \longrightarrow & S^k_{f+1}(X)
\end{array} \]

commutes, and so \( T_{f}^{(n)}(\sigma) \in S^k_{f+1}(Y) \triangleleft S^k_{f+1}(X, \partial Z) \).

We now prove the Small Simplex Theorem 9.1. We first show \( H^1_f(i) \) is onto.

Let \( h \in H^1_f(X) \). Say \( h = [z] \), \( z \) a cycle in \( S^k_f(X) \).

By 9.23, we can choose \( n \) so large that \( \text{sd}_{f}^n(z) \in S^k_f(X, \partial Z) \).

Since \( \text{sd}_{f}^n \) is a chain map, we have

\[ \text{sd}_{f}^n(z) = \text{sd}_{f}^n \partial z = 0 \]

and so \( \text{sd}_{f}^n(z) \) is a cycle in

\[ S^k_f(X, \partial Z). \]

We have

\[ j(\text{sd}_{f}^n(z)) = \text{sd}_{f}^n(z) \]

\[ = z - \partial T_{f}^{(n)}(z) - T_{f}^{(n)}(z), \]

\[ \text{by 9.21} \]

Thus \( H^1_f(X, \partial Z) \cong [z - \partial T_{f}^{(n)}(z) - T_{f}^{(n)}(z)] \).

We now prove \( H^1_f(i) \) is 1-1. Suppose \( H^1_f(i)(h) = 0 \)

where \( h = [z] \), \( z \) a cycle in \( S^k_f(X, \partial Z) \). Then

\[ 0 = j(z) = \partial c \text{ in some } c \in S^k_{f+1}(X). \]

By 9.23 we can choose \( n \) so large that \( \text{sd}_{f+1}^n(c) \in S^k_{f+1}(X, \partial Z) \).

Then

\[ z - \text{sd}_{f}^n(z) = \partial T_{f}^{(n)}(z) + T_{f}^{(n)}(z) = \partial T_{f}^{(n)}(z). \]

Thus

\[ z = \text{sd}_{f}^n(z) + \partial T_{f}^{(n)}(z) = \text{sd}_{f}^n(\partial c) + \partial T_{f}^{(n)}(z) \]

\[ = \partial \text{sd}_{f+1}^n(c) + \partial T_{f}^{(n)}(z) = \partial \text{sd}_{f+1}^n(c) + T_{f}^{(n)}(z). \]

By 9.24, \( T_{f}^{(n)}(z) \in S^k_{f+1}(X, \partial Z) \). From the above, \( \text{sd}_{f+1}^n(c) \in S^k_{f+1}(X, \partial Z) \).

Thus \( z \) is the boundary of a chain in \( S^k_{f+1}(X, \partial Z) \)

and so \( h = [z] = 0 \), completing the proof of 9.1, and hence also of The Excision Property 6.2.
10. The CW Category

Definition 10.1. A CW complex \( K \) consists of:

1) A topological space \( |K| \) called the underlying space of \( K \).

2) A family \( \text{cell}(K) \) of mutually disjoint subspaces of \( |K| \) whose union is \( |K| \). Members of \( \text{cell}(K) \) are called the cells of \( K \). cell \( (K) \) is a disjoint union of subcollections \( \text{cell}_n(K) \), \( n = 0, 1, 2, \ldots \) (some of the \( \text{cell}_n(K) \) may be empty.) Members of \( \text{cell}_n(K) \) are called the \( n \)-cells of \( K \). A cell in \( \text{cell}_n(K) \) is said to be of dimension \( n \).

We require:

3) For each \( n \)-cell \( A \) of \( K \), \( A = \bar{A} - A \) is a union of finitely many cells of \( K \), each of dimension \( \leq n \). (This property is called closure finiteness. The "C" in "CW" comes from this property.)

4) A subspace \( X \) of \( |K| \) is closed in \( |K| \) if and only if \( X \cap \bar{A} \) is closed in \( \bar{A} \) for each cell \( A \) of \( K \). (This property is paraphrased by saying "\( |K| \) has the weak topology relative to the cells of \( K \). The "W" in "CW" comes from this property.)

5) In each \( n \)-cell \( A \) of \( K \), there exists a relative homeomorphism \( j^A : (D^n, S^{n-1}) \rightarrow (\bar{A}, A) \) such that \( j^A : D^n \rightarrow \bar{A} \) is onto, \( \bar{A} \) is called a characteristic map for the cell \( A \), \( j^A : \partial A \rightarrow \bar{A} \) is called an attaching map for \( A \) (thus for each \( n \)-cell \( A \) of \( K \), \( A \cong D^n - S^{n-1} \cong e^n \), the open unit ball in \( \mathbb{R}^n \).) By convention, \( S^{-1} = \emptyset \), \( D^0 = \{0\} \) and so the \( 0 \)-cells of \( K \) are one-point spaces.

Definition 10.2. Let \( K \) be a CW complex. For \( n \geq 0 \), the \( n \)-skeleton of \( K \), denoted \( K^{(n)} \), is the union of the cells of \( K \) of dimension \( \leq n \).

Remark 10.3. If \( \text{cell}(K) \) is finite, then condition 4 in Definition 10.1 is superfluous.
Example 10.4.

\[ |K| = \]

\[ \text{cell}_0(K) = \{ P_1, P_2, P_3, P_4 \}, \quad \text{cell}_1(K) = \{ E_1, E_2, E_3, E_4, E_5 \}, \]

\[ \text{cell}_2(K) = \{ F_1, F_2 \}, \quad \text{and} \quad \text{cell}_n(K) = \emptyset \text{ for } n > 2. \]

\( K^{(0)} \) is the discrete space consisting of \( P_1, P_2, P_3, P_4 \).

\[ K^{(n)} = \]

\[ \text{and} \quad K^{(m)} = |K| \quad \text{for } n > 2. \]

Example 10.5. (The economical cell decomposition of \( S^n \).)

Let \( n \geq 1 \). \( |K| = S^n \), and \( \text{cell}(K) \) has exactly two

members: \( \text{the } 0 \text{-cell } e_0 = \{(1,0,0,...)\} \) and the

\( n \)-cell \( e_n = S^n - e_0 \). \( \text{Then } \overline{e}_n = S^n \) and \( e_n = e_0 \).

A characteristic map \( \chi_n : (D^n, S^{n-1}) \to (e_n, e_n) = (S^n, e_0) \)

is given by

\[ \chi_n(x) = \begin{cases} (-\cos(\pi \|x\|), \frac{x}{\|x\|} \sin(\pi \|x\|)) & \text{if } x \neq 0 \\ (-1, 0, 0, ... ) & \text{if } x = 0 \end{cases} \]

Example 10.6. (The hemispherical cell decomposition of \( S^n \)).

Let \( n \geq 0 \). \( |K| = S^n \), and \( \text{for } 0 \leq i \leq n, \)

\( \text{cell}_i(K) = \{ e_i^+, e_i^- \} \), where \( e_i^+ = \{(x_0, x_1, ..., x_i, 0, ...) \in S^n \mid x_i > 0 \} \),

\( e_i^- = \{(x_0, x_1, ..., x_i, 0, ...) \in S^n \mid x_i < 0 \} \). \( \overline{e}_i^+ = \{(x_0, x_1, ..., x_i, 0, ...) \in S^n \mid x_i \geq 0 \} \), and so

\( \overline{e}_i^+ = \{(x_0, x_1, ..., x_{i-1}, 0, ...) \in S^{n-1} \} = S^{i-1}. \) Similarly,
\( \partial_i^- = \{ (x_0, x_1, \ldots, x_i, 0, \ldots) \in S^n \mid x_i \leq 0 \} \) and \( \partial_i^+ = S^{i-1} \). \( \forall 0 \leq i \leq n \), characteristic maps \( \kappa_{\partial_i^+} : (D^i, S^{i-1}) \to (\partial_i^+, \partial_i^+) \) and \( \kappa_{\partial_i^-} : (D^i, S^{i-1}) \to (\partial_i^-, \partial_i^-) \)

are given by \( \kappa_{\partial_i^+}^\varphi (x_1, \ldots, x_i, 0, \ldots) = (x_1, \ldots, x_i, \sqrt{1 - \frac{2}{j} x_j^2}, 0, \ldots) \), \( \kappa_{\partial_i^-}^\varphi (x_1, \ldots, x_i, 0, \ldots) = (x_1, \ldots, x_i, -\sqrt{1 - \frac{2}{j} x_j^2}, 0, \ldots) \).

**Example 10.7.**

\( \partial_i \mathcal{K} | = \text{twists} = \)

\[
\begin{array}{c}
\text{cell}_0 (\mathcal{K}) = \{ P_j \}, \quad \text{cell}_1 (\mathcal{K}) = \{ a, b \}, \quad \text{cell}_2 (\mathcal{K}) = \{ e \}, \quad \text{and} \\
\text{cell}_n (\mathcal{K}) = \emptyset \quad \text{for} \quad n > 2.
\end{array}
\]

\( \mathcal{K}^{(0)} = \{ P_j \}, \quad \mathcal{K}^{(1)} = a \equiv S^1 \cup S^1, \)

and \( \mathcal{K}^{(n)} = | \mathcal{K} | = \text{twists} \quad \text{for} \quad n > 2. \)

**Example 10.8.**

\( | \mathcal{K} | = \text{Klein bottle} = \)

\[
\begin{array}{c}
\text{cell}_0 (\mathcal{K}) = \{ P_j \}, \quad \text{cell}_1 (\mathcal{K}) = \{ a, b \}, \quad \text{cell}_2 (\mathcal{K}) = \{ e \}, \quad \text{and} \\
\text{cell}_n (\mathcal{K}) = \emptyset \quad \text{for} \quad n > 2.
\end{array}
\]

\( \mathcal{K}^{(0)} = \{ P_j \} \)
\[ K^{(n)} = \{ K \mid \text{Klein bottle for } n \geq 2 \}. \]

**Example 10.9.** Real projective n-space \( \mathbb{R}P^n \) is the quotient space obtained from \( \mathbb{R}^{n+1} - \{0\} \) by identifying \( x = \lambda x \) whenever \( x \in \mathbb{R}^{n+1} - \{0\} \) and \( \lambda \in \mathbb{R} - \{0\} \). Write \([x] \in \mathbb{R}P^n \) for the image of \( x \in \mathbb{R}^{n+1} - \{0\} \) under the quotient map. For \( 0 \leq i \leq n \), let

\[ e_i = \{ [t_0, \ldots, t_i, 0, \ldots] \in \mathbb{R}P^n \mid t_i \neq 0 \} = \mathbb{R}P^i - \mathbb{R}P^{i-1}. \]

Then \( \mathbb{R}P^n = 1K \) where \( \text{cell}_2(K) = \{ e_i \} \) for \( 0 \leq i \leq n \) and \( \text{cell}_i(K) = \emptyset \) for \( i > n \). For \( 0 \leq i \leq n \), \( \bar{e}_i = \{ [t_0, \ldots, t_i, 0, \ldots] \in \mathbb{R}P^n \} = \mathbb{R}P^i = e_0 \cup e_1 \cup \cdots \cup e_i \), and

\[ e_i = e_0 \cup e_1 \cup \cdots \cup e_{i-1} = \mathbb{R}P^{i-1}. \]

For \( 0 \leq i \leq n \), a characteristic map \( \chi_{e_i} : (D^i, S^{i-1}) \to (\bar{e}_i, e_i) \) is given by

\[ \chi_{e_i}([x_1, \ldots, x_i]) = [x_1, \ldots, x_i, \underbrace{1 - \frac{1}{j}}_{i}x_i, 0, \ldots]. \]

We have \( K^{(n)} = \mathbb{R}P^i \) for \( 0 \leq i \leq n \).

**Example 10.10.** Complex projective n-space \( \mathbb{C}P^n \) is the quotient space obtained from \( \mathbb{C}^{n+1} - \{0\} \) by identifying \( x = \lambda x \) whenever \( x \in \mathbb{C}^{n+1} - \{0\} \) and \( \lambda \in \mathbb{C} - \{0\} \). Write \([x] \in \mathbb{C}P^n \) for the image of \( x \in \mathbb{C}^{n+1} - \{0\} \) under the quotient map. For \( 0 \leq k \leq n \), let \( e_k = \{ [z_0, \ldots, z_k, 0, \ldots] \in \mathbb{C}P^n \mid z_k \neq 0 \} = \mathbb{C}P^k - \mathbb{C}P^{k-1}. \]

Then \( \mathbb{C}P^n = 1K \) where \( \text{cell}_2(K) = \{ e_k \} \) for \( 0 \leq k \leq n \) and \( \text{cell}_i(K) = \emptyset \) if \( i \) is either \( k \) or odd or \( k > 2n \). For \( 0 \leq k \leq n \), \( \bar{e}_k = \{ [z_0, \ldots, z_k, 0, \ldots] \in \mathbb{C}P^n \} = \mathbb{C}P^k = e_0 \cup e_1 \cup \cdots \cup e_k \), and \( e_k = e_0 \cup e_1 \cup \cdots \cup e_{k-1} = \mathbb{C}P^{k-1}. \)

For \( 0 \leq k \leq n \), a characteristic map \( \chi_{e_k} : (D^2, S^{1}) \to (\bar{e}_k, e_k) \) is given as follows: By identifying \( \mathbb{C}P^k \) with \( \mathbb{R}^{2k} \), we can regard \( D^{2k} \) as \( \{ [z_1, \ldots, z_k, 0, \ldots] \in \mathbb{C}P^k \mid \sum |z_j|^2 \leq 1 \} \).

Then \( \chi_{e_k}([z_1, \ldots, z_k, 0, \ldots]) = [z_1, \ldots, z_k, \underbrace{1 - \frac{1}{j}}_{i}z_k, 0, \ldots]. \)

We have \( K^{(2i)} = \mathbb{C}P^i \) for \( 0 \leq i \leq n \).
Definition 10.11. Let \((Y, X)\) be a topological pair. We say 
\(Y\) is obtained from \(X\) by attaching \(n\)-cells if for some 
discrete index set \(J\) there exists a map of topological 
pairs \(h : (D^n \times J, S^{n-1} \times J) \to (Y, X)\) satisfying 
1) \(h^q\) maps \(E^n \times J \) bijectively onto \(Y - X\), \((E^n = D^n - S^{n-1})\).
2) The map \(h_+: X \cup (D^n \times J) \to Y\) given by \(h_+(x) = x\) 
for all \(x \in X\), and \(h_+|D^n \times J = h^q\), is a quotient map.

We call such an \(h\) a cell detection map for \((Y, X)\).

For each \(j \in J\), write \(E_j = h^q(E^n \times \{j\})\).

The \(E_j\) are called the \(n\)-cells of \(Y\) mod \(X\). The map 
\(h_j : (D^n, S^{n-1}) \to (E_j, \bar{E}_j)\) given by \(h_j^q(x) = h^q(x, j)\) is called 
a characteristic map for the \(n\)-cell \(E_j\) of \(Y\) mod \(X\).

Proposition 10.12. Suppose \(Y\) is obtained from \(X\) by 
attaching \(n\)-cells. Let \(\{E_j\}_{j \in J}\) be the set of \(n\)-cells 
of \(Y\) mod \(X\). Then:
1) \(X\) is closed in \(Y\).
2) Each \(E_j\) is open in \(Y\).
3) As a set, \(Y\) is the disjoint union \(X \cup \bigsqcup E_j\).
4) A subset \(C\) of \(Y\) is closed in \(Y\) if and only if 
\(C \cap X\) is closed in \(X\) and \(C \cap \bar{E}_j\) is closed in \(\bar{E}_j\) for each 
\(j \in J\).

Proof. Let \(h_+: X \cup (D^n \times J) \to Y\) be the quotient map arising 
from a cell detection map \(h\) for \((Y, X)\). To show 1), it 
suffices to prove \(h_+^{-1}(X)\) is closed in \(X \cup (D^n \times J)\). We have 
\(h_+^{-1}(X) = X \cup (S^{n-1} \times J)\), which is closed in \(X \cup (D^n \times J)\), proving 1).

Similarly, \(h_+^{-1}(E_j) = E^n \times \{j\}\) which is open in 
\(X \cup (D^n \times J)\) since \(J\) is discrete. Thus \(E_j\) is open in \(Y\) 
for each \(j \in J\), proving 2).

3) is immediate from 1) of 10.11.

Suppose \(C \subseteq Y\) is such that \(C \cap X\) is closed in \(X\) 
and \(C \cap \bar{E}_j\) is closed in \(\bar{E}_j\) for each \(j \in J\). We have 
\(h_+^{-1}(C) = \bigcup_{j \in J} h_j^{-1}(C \cap \bar{E}_j)\)

\[ = (C \cap X) \cup \bigsqcup_{j \in J} (h_j^q)^{-1}(C \cap \bar{E}_j),\]

Since \(C \cap X\) is closed in \(X\) and \((h_j^q)^{-1}(C \cap \bar{E}_j)\) is closed in \(D^n\),
In each $j \in J$, and since $J$ is discrete, it follows that $h_+^{-1}(c)$ is closed in $X \sqcup (D^n \times J)$. Thus $C$ is closed in $Y$. The converse is trivial, and so (4) is proved.

**Proposition 10.13.** Suppose $Y$ is obtained from $X$ by attaching $n$-cells. Let $h: (D^n \times J, S^{n-1} \times J) \to (Y, X)$ be a cell detection map. Then $h$ is an exerese relative homeomorphism.

**Proof.** We first check that $h$ is a relative homeomorphism. By (1) of 10.11, $h^q$ is a continuous bijection of $E^n \times J$ onto $Y \times X$. Thus it remains only to check that $h^q|E^n \times J$ is an open map.

In any subset $U$ of $E^n \times J$, $h_+^{-1}(h^q(U)) = U$. For particular, if $U$ is open in $E^n \times J$, then $U$ is also open in $X \sqcup (D^n \times J)$ since $E^n \times J$ is open in $X \sqcup (D^n \times J)$. Thus $h_+^{-1}(h^q(U))$ is open in $X \sqcup (D^n \times J)$ and so $h^q(U)$ is open in $Y$ (and home in $Y \times X$) since $h_+$ is a quotient map. Thus $h$ is a relative homeomorphism.

Let $A' = (D^n \times \{0\}) \times J$, $B' = X \cup h_+^q(A')$. Then $A'$ is open in $D^n \times J$. $h_+^{-1}(Y-B') = \{0\} \times J$ which is closed in $X \sqcup (D^n \times J)$ and so $B'$ is open in $Y$. Since $S^{n-1} \times J$ is closed in $D^n \times J$ and $X$ is closed in $A'$, 6.3(c) holds. Clearly $h^q(A' - S^{n-1} \times J) = B' - X$; and so 6.3(ii) holds. We complete the proof by showing that $S^{n-1} \times J$ is a deformation retract of $A'$ and $X$ is a deformation retract of $B'$.

Let $i: S^{n-1} \times J \to A'$ denote the inclusion, $r: A' \to S^{n-1} \times J$ be given by $r(x,j) = (\frac{x}{||x||}, j)$, and $f: A' \times I \to A'$ be given by $f((x,j), t) = (\frac{x}{1-t(1-||x||-1)}, j)$. Then $r_i = 1_{S^{n-1} \times J}$, and $f$ is a homotopy from $f_{A'}$ to $r$. If $u, u' \in A'$ are such that $h^q(u) = h^q(u')$, then either $u = u'$ or $u, u'$ both lie in $S^{n-1} \times J$. Since $r$ fixes points of $S^{n-1} \times J$ and $f((x,t)) = u$ for all $u \in S^{n-1} \times J$, these are well-defined functions $F: B' \to X$ and $\tilde{F}: B' \times I \to B'$ given by $F(x) = x = f((x,j))$ for all $x \in X$, and $\tilde{F}(h_+(u)) = h_+(r(u))$, $\tilde{F}(h_+(u), t) = h_+(f((u,j), t))$ for all $u \in A'$. The diagrams
\[
\begin{align*}
X \amalg A' & \xrightarrow{1 \times \nu} X \amalg (S^{n-1} \times J) \\
\downarrow\text{h}_+ & \downarrow & \downarrow\text{h}_+ \\
B' & \xrightarrow{\bar{f}} X \\
\end{align*}
\]

\[
\begin{align*}
(X \amalg A') \times I & \xrightarrow{\pi_1 \times f} X \amalg A' \\
\downarrow\text{h}_+ \times I & \downarrow & \downarrow\text{h}_+ \\
B' \times I & \xrightarrow{\bar{f}} B' \\
\end{align*}
\]

commutes where \( \pi_1 : X \times I \to X \) is projection on the first factor, \( \text{h}_+ : X \amalg A' \to B' \) is a quotient map since \( B' \) is open in \( Y \), \( X \amalg A = \text{h}_+ (B') \) and \( \text{h}_+ \) is a quotient map. Since \( I \) is locally compact and Hausdorff, \( \text{h}_+ | X \times I \) is also a quotient map. Thus since \( 1 \times \nu \), \( \pi_1 \times f \), and the restrictions of \( \text{h}_+ \) are continuous, it follows that \( \bar{f} \) and \( \bar{f} \) are continuous. If \( \text{h} : X \to B' \) denotes the inclusion, then \( \bar{f} \circ \text{h} = 1_x \) and \( \bar{f} \) is a homotopy from \( 1_B' \) to \( \bar{f} \). This completes the proof.

**Proposition 10.14.** Let \( K \) be a CW complex. Then, for each \( n \geq 0 \), \( K^{(n)} \) is obtained from \( K^{(n-1)} \) by attaching \( n \)-cells. The set of \( n \)-cells of \( K^{(n)} \) and \( K^{(n-1)} \) is precisely \( \text{cell}_n (K) \). A cell detection map is given as follows: For each \( A \in \text{cell}_n (K) \), choose a characteristic map \( \chi_A : (D^n, S^{n-1}) \to (\bar{A}, \bar{A}) \). Then the set \( \text{cell}_n (K) \) the discrete topology, and define

\[
\begin{align*}
\text{h} : (D^n \times \text{cell}_n (K), S^{n-1} \times \text{cell}_n (K)) & \to (K^{(n)}, K^{(n-1)}) \\
\text{h}^A(x, A) & = \chi_A(x). \\
\end{align*}
\]

In each \( A \in \text{cell}_n (K) \), \( \bar{c}_A = \text{h}^A (\bar{c}^n \times \lambda_A) = A \), and the characteristic map \( \chi_A \) in the sense of 10.11 coincides with \( \chi_A \).

**Proof.** The only non-immediate detail is the proof that
$h^+ : K^{(n-1)} \sqcup (D^n \times \text{cell}_n(K)) \to K^{(n)}$ is a quotient map.

Let $C$ be a subset of $K^{(n)}$ such that $h^+((C))$ is closed in $K^{(n)} \sqcup (D^n \times \text{cell}_n(K))$. We must show $C$ is closed in $K^{(n)}$. We will, in fact, prove the stronger statement that $C$ is closed in $|K|.$

We have $h^+((C)) = (C \cap K^{(n)}) \sqcup (X_A \times \{\bar{A} \})_{A \in \text{cell}_n(K)}$.

Thus $C \cap K^{(n)}$ is closed in $K^{(n)}$ and for each $A \in \text{cell}_n(K),$ $(X_A \times \{\bar{A} \})_{A \in \text{cell}_n(K)}$ is closed in $D^n$. Thus

$CN \bar{B}$ is closed in $|K|$ for each cell $B$ of $K$ of dimension $< n,$ and since $X_A$ is a quotient map, $CN \bar{A}$ is closed in $A$ (and hence in $|K|$) for all $A \in \text{cell}_n(K),$ i.e., $CN \bar{B}$ is closed in $|K|$ for each cell $B$ of $K$ of dimension $< n.$

Let $m > n$ and assume, inductively, $CN \bar{B}$ is closed in $|K|$ whenever $B$ is a cell of $K$ of dimension $\leq m-1.$

If $B \in \text{cell}_n(K)$, then since $C \subset K^{(n)}$, we have $CN \bar{B} = CN \bar{B}$. By closure finiteness, $B \subset \bigcup_{i=1}^k B_i$ for some finite $k$ where the $B_i$ are cells of $K$ of dimension $< m$. Thus $CN \bar{B} = \bigcup_{i=1}^k CN \bar{B}_i$, and by the inductive hypothesis, each $CN \bar{B}_i$ is closed in $|K|$. Thus, since this is a finite union, $CN \bar{B}$ is closed in $|K|$, completing the induction. Thus since $|K|$ has the weak topology with respect to the cells of $K$, $C$ is closed in $|K|.$

Proposition 10.15. Let $X$ be an arbitrary topological space, $J$ a discrete index set, and $f : S^{n-1} \times J \to X$ a continuous map. Let $X \sqcup (D^n \times J)$ denote the quotient space obtained from $X \sqcup (D^n \times J)$ by identifying $(u, i) \sim f((u, i))$ for all $(u, i) \in S^{n-1}$. Let $g : X \sqcup (D^n \times J) \to X \sqcup (D^n \times J)$ denote the quotient map. Then $g$ maps $X$ homeomorphically onto a subspace of $X \sqcup (D^n \times J)$, which we identify with $X$. $X \sqcup (D^n \times J)$ is obtained from $X$ by attaching $n$-cells.

$h : (D^n \times J, S^{n-1} \times J) \to (X \sqcup (D^n \times J), X)$, obtained by restriction of $g$, is a cell detection map, and $h^+ = g.$
Proof. The only non-immediate detail is the proof that \( g \) maps \( X \) homeomorphically onto a subspace of \( X \times (D^n \times S^{n-1}) \).

\( g \) is clearly surjective, and so it remains only to show that \( g \) is a closed map.

Let \( C \) be a closed subspace of \( X \). Then \( g^{-1}(g(C)) = C \times f^{-1}(C) \). Since \( f \) is continuous, \( f^{-1}(C) \) is closed in \( S^{n-1} \times T \), and hence closed in \( D^n \times T \). Thus \( g^{-1}(g(C)) \) is closed in \( X \times (D^n \times T) \), and so \( g(C) \) is closed in \( X \times (D^n \times T) \) since \( g \) is a quotient map.

Lemma 10.16. Suppose \( Y \) is obtained from \( X \) by attaching \( n \)-cells. Suppose \( X \) is Hausdorff. Let \( h: (D^n \times T, S^{n-1} \times T) \rightarrow (Y, X) \) be a cell detection map. Then for each \( i \neq j \), \( h_i: D^n \rightarrow e_j \) is a closed quotient map.

Proof. We first show \( h_d \) is a quotient map. Let \( C \subseteq e_j \) be such that \( (h_d)^{-1}(C) \) is closed in \( D^n \). We must show \( C \) is closed in \( e_j \) (equivalently, \( C \subseteq e_j \)).

We have \( C \cap X = (C \cap e_j) \) and \( (h_d)^{-1}(C \cap e_j) = (h_d)^{-1}(C) \cap S^{n-1} \). Thus \( (h_d)^{-1}(C \cap e_j) \) is closed in \( S^{n-1} \); and hence compact. Since \( C \cap e_j = h_d((h_d)^{-1}(C \cap e_j)) \) (since \( h_d \) is one-to-one), we have \( C \cap e_j \) is compact, i.e. \( C \cap X \) is compact. Thus, since \( X \) is Hausdorff, \( C \cap X \) is closed in \( X \) (and hence closed in \( Y \)). Hence \( X \) is closed in \( Y \). We have \( h^+_d(C) = (C \times X) \cap (h_d)^{-1}(e_j) \cap (h_d)^{-1}(C \cap e_j) \cap (h_d)^{-1}(C \cap X) \)

\( f^0: S^{n-1} \times T \rightarrow X \), which is closed in \( X \times (D^n \times T) \). Thus, since \( h^d \) is a quotient map, \( C \) is closed in \( Y \).

We next show \( h_d \) is a closed map. Let \( Z \) be closed in \( D^n \). Then \( Z \) is compact. We have \( (h_d)^{-1}(h_d(Z)) = Z \cup (h_d)^{-1}(h_d(Z) \cap X) = Z \cup (h_d)^{-1}(h_d(Z \cap S^{n-1})) \)

\( (h_d: S^{n-1} \times T \rightarrow X) \). \( h_d(Z \cap S^{n-1}) \) is compact, and hence closed in \( X \) (since \( X \) is Hausdorff). Thus \( (h_d)^{-1}(h_d(Z)) \) is closed in \( D^n \) and so \( h_d(Z) \) is closed in \( Y \) (and hence in \( e_j \)) since \( h_d \) is a quotient map.
Lemma 10.17. Suppose $Y$ is obtained from $X$ by attaching $n$-cells. Suppose $X$ is normal. Then $Y$ is normal.

Proof. Choose a cell detection map $h : (D^n \times J, S^{n-1} \times J) \to (Y, X)$. We first show that one point subsets of $Y$ are closed in $Y$.

Let $y \in Y$ and $A \subset Y$. Since $X$ is normal, there is a normal closed $A'$ in $X$ such that $h^{-1}(A') = \gamma$. Then $h^{-1}(A)$ is a one point subspace of $D^n \times I$, and hence closed in $X$.

Now suppose $C_0 \cap C_1$ is a closed subset of $Y$. Since $X$ is normal, there exists (by the Urysohn Lemma) a continuous map $f : X \to I$ such that $f(X \cap C_0) = \gamma_0$, $f(X \cap C_1) = \gamma_1$. Let $f_j : E_j \to \gamma_j$ be a closed quotient map by 10.16. Then, since $D^n$ is normal, it follows by a standard result from general topology that $E_j$ is normal. Let $f_j : E_j \to f(E_j \cap C_0) = \gamma_0$, $f_j(E_j \cap C_1) = \gamma_1$. Then $f_j$ is continuous. Since $f_j : E_j \to \gamma_j$ is a continuous extension, then there exists a continuous extension $\widetilde{f}_j : \overline{E}_j \to \gamma$ of $f_j$. Since $f_j$ agrees with $f$ on $X \cap \overline{E}_j = \overline{E}_j$, in each $j \in J$, there is a well-defined function $F : Y \to I$ whose restriction to $X$ is $f$ and whose restriction to $\overline{E}_j$ is $f_j$. If $C$ is closed in $\gamma$, then $F^{-1}(C) \cap X = f^{-1}(C)$, $F^{-1}(C) \cap \overline{E}_j = \overline{f}_j^{-1}(C)$. Thus, since $f$ and $\overline{f}_j$ are continuous, $F^{-1}(C) \cap X$ is closed in $X$ and $F^{-1}(C) \cap \overline{E}_j$ is closed in $\overline{E}_j$ in each $j \in J$. Thus by 10.12 (4), $F^{-1}(C)$ is closed in $Y$, and so $F$ is continuous. Since $F(C_0) = \gamma_0$ and $F(C_1) = \gamma_1$, $F^{-1}(10^{-1/2}, 1)$ are disjoint open neighborhoods of $C_0$ and $C_1$, respectively, in $Y$.

Corollary 10.18. Let $K$ be a CW complex. Then $K^{(n)}$ is normal for all $n \geq 0$.

Proof. The result is true for $n = 0$ since $K^{(0)}$ is discrete. The result now follows by induction using 10.14 and 10.17.
Exercise 10.19. Let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ be a nested sequence of topological spaces such that $X_n$ is closed in $X_{n+1}$ for all $n \geq 0$. Let $X = \bigcup_n X_n$. Then there is a unique topology on $X$ such that a subset $C$ of $X$ is closed in $X$ if and only if $C \cap X_n$ is closed in $X_n$ for each $n \geq 0$. We call this topology the weak topology on $X$ relative to $\{X_n | n \geq 0\}$. With this topology, each $X_n$ is a closed subspace of $X$. Moreover, if $Y$ is any topological space and $f: X \to Y$ a function, then $f$ is continuous if and only if $f|X_n: X_n \to Y$ is continuous in each $n \geq 0$.

Proposition 10.20. Let $K$ be a CW complex. Then in each $n \geq 0$, $K^{(n)}$ is closed in $K^{(n+1)}$, and the weak topology on $|K|$ coincides with the weak topology relative to $\{K^{(n)} | n \geq 0\}$.

Proof. It is immediate from 10.14 and 10.12 (1) that $K^{(n)}$ is closed in $K^{(n+1)}$ for each $n \geq 0$. Moreover, if $C$ is closed in $|K|$, then $C \cap K^{(n)}$ is closed in $K^{(n)}$ for each $n \geq 0$.

Conversely, suppose $C$ is a subset of $|K|$ such that $C \cap K^{(n)}$ is closed in $K^{(n)}$ for each $n \geq 0$. Let $A \subseteq \text{cell}(K)$. Then since $A \subseteq K^{(n)}$, $C \cap A$ is closed in $A$. Thus, by 10.1 (4), $C$ is closed in $|K|$.

Lemma 10.21. Suppose $Y$ is obtained from $X$ by attaching $n$-cells. Let $C$ be a compact subset of $Y$. Then $C$ meets only a finite number of the cells of $Y$ mod $X$.

Proof. Choose a cell detection map $h: (D^n \times S^{n-1}, S^{n-1}) \to (Y, X)$. Let $S = \{ i \in J \mid X \cap e_i \neq \emptyset \}$. For each $i \in S$, choose a point $x_i \in X \cap e_i$, and let $Z = \{ x_i | i \in S \}$. For each subset $T$ of $S$, let $Z_T = \{ x_i | i \in T \}$. We have $h_+^{-1}(Z_T) \cap D^n \times \{ i \}$.

$$h_+^{-1}(Z_T) \cap (D^n \times \{ i \}) = \begin{cases} \emptyset & \text{if } i \notin T \\ \{(h_+^{-1})(x_i)\} & \text{if } i \in T. \end{cases}$$
Thus, since the one-point subset of \( D^n \times \{1\} \) is closed in \( D^n \times \{1\} \), \( A_+ \) and \( Z_+ \) are closed in \( X = (D^n \times \{1\}) \), and so \( Z_+ \) is closed in \( Y \). Then \( Z = Z_+ \) is a closed, discrete subset of \( Y \). Since \( Z \subseteq C \), \( Z \) is compact and we must have \( S \) finite.

**Proposition 10.22.** Let \( X_0 < X_1 < \cdots \subseteq Y \) be a nested sequence of topological spaces such that \( X_0 \) is discrete and for each \( n > 0 \), \( X_n \) is obtained from \( X_{n-1} \) by attaching \( n \)-cells. Let \( X = U \times \mathbb{N} \) and give \( X \) the weak topology relative to \( \{ \mathbb{N} \times [n] \} \). Then \( X = \langle K \rangle \) where \( K \) is a CW complex with \( K^{(n)} = \mathbb{N} \times [n] \) for all \( n > 0 \). cell \( (K) \) is the collection of one-point subsets of \( X \). In \( n > 0 \), cell \( (K) \) is the collection of \( n \)-cells of \( \mathbb{N} \times \mathbb{N} \) mod \( X_{n-1} \).

**Proof.** Clearly, \( X \) is the disjoint union of the cells of \( K \) as defined above. We proceed to verify condition 10.11(3). Let \( A \in \text{cell}(K) \), then \( A \subseteq X_{n-1} \), and we have

\[
X_{n-1} = \bigcup_{A \in \text{cell}(K)} A \quad \text{since } A \text{ is compact (being the image of } S^{n-1} \text{ under a continuous map), } A \cap X_m \text{ is compact for each } m \leq n-1 \quad \text{since } X_m \text{ is closed in } X \text{ and so by 10.21, } A \cap X_m \text{ meets only finitely many members of cell } (K) \text{. Thus } A \text{ is contained in the union of a finite number of cells of } K \text{, and so 10.11(3) holds.}
\]

We next verify 10.11(4). Trivially, if \( Z \) is closed in \( X \), then \( Z \cap A \) is closed in \( X \) for all \( A \in \text{cell}(K) \). Conversely, suppose \( Z \) is a subspace of \( X \) such that \( Z \cap A \) is closed in \( X \) for each \( A \in \text{cell}(K) \). We prove, by induction on \( n \), that \( Z \cap X_n \) is closed in \( X_n \) for each \( n \) (and hence \( Z \) will be closed in \( X \)). The result is trivial if \( n = 0 \), since \( X_0 \) is discrete. Let \( n > 0 \) and assume, inductively, that \( Z \cap X_{n-1} \) is closed in \( X_{n-1} \). Then, since \( Z \cap A \) is closed in \( X \) for each \( n \)-cell \( A \) of \( X_n \) mod
X_{n-1}, it follows from 10.12 (4) that \( Z \cap X_n \) is closed in \( X_n \), and so 10.1 (4) is satisfied.

By 10.16, the characteristic maps \( f \) for the cells of \( X_n \) and \( X_{n-1} \) relative to a choice of cell decomposition map satisfy the requirements of 10.1 (5), completing the proof.

Example 16.23. We have \( S^0 \subset S^1 \subset S^2 \subset \ldots \). Define \( S^n = \bigcup S^n \), and give \( S^n \) the weak topology relative to \( S^n \cap \{ n \geq 2 \} \). By 10.6, the above sequence satisfies the requirements of 10.22, and so \( S^n \) is the underlying space of a CW complex whose \( n \)-skeleton is \( S^n \cap \{ n \geq 2 \} \).

Example 16.24. We have \( \mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \ldots \). Define \( \mathbb{R}P^n = \bigcup \mathbb{R}P^n \), and give \( \mathbb{R}P^n \) the weak topology relative to \( \mathbb{R}P^n \cap \{ n \geq 2 \} \). By 10.9, the above sequence satisfies the requirements of 10.22, and so \( \mathbb{R}P^n \) is the underlying space of a CW complex whose \( n \)-skeleton is \( \mathbb{R}P^n \cap \{ n \geq 2 \} \).

Example 16.25. We have \( \mathbb{C}P^0 \subset \mathbb{C}P^1 \subset \mathbb{C}P^2 \subset \ldots \). Define \( \mathbb{C}P^n = \bigcup \mathbb{C}P^n \), and give \( \mathbb{C}P^n \) the weak topology relative to \( \mathbb{C}P^n \cap \{ n \geq 2 \} \). By 10.10, the above sequence satisfies the requirements of 10.22, and so \( \mathbb{C}P^n \) is the underlying space of a CW complex whose \( 2n \) and \( 2n+1 \) skeletons are both \( \mathbb{C}P^n \) for \( n \geq 0 \).

Proposition 16.26. Let \( K \) be a CW complex. Then \( |K| \) is normal.

Proof. Let \( C \subset C' \), be disjoint closed subsets of \( |K| \). Then for each \( n \geq 0 \), \( C \cap K^{(n)} \) and \( C' \cap K^{(n)} \) are disjoint closed subsets of \( K^{(n)} \). We inductively construct a sequence of continuous maps \( f_n : K^{(n)} \to I \) satisfying

1) \( f_n(C \cap K^{(n)}) \subset \{ 0, 1 \} \), \( f_n(C' \cap K^{(n)}) \subset \{ 0, 1 \} \);
2) \( f_{n+1}\big| K^{(n)} = f_n \) for all \( n \geq 0 \).

The existence of an \( f : K^{(n)} \to I \) satisfying condition 1) is
trivial since $K^{(0)}$ is discrete. Let $n > 0$ and assume, inductively, $f_{n-1} : K^{(n-1)} \rightarrow I$ has been constructed satisfying 1. Define $g : K^{(n)} \cup (K^{(n)} \cap C_0) \cup (K^{(n)} \cap C_1) \rightarrow I$ by $g(K^{(n)} \cap C_0) = f_{n-1}$, $g(K^{(n)} \cap C_1) \subset I \backslash I_3$. Thus $g$ is well-defined and is continuous since $K^{(n-1)}$, $K^{(n)} \cap C_0$, $K^{(n)} \cap C_1$ are all closed in $K^{(n)}$ and the restrictions of $g$ to these subspaces are continuous. By 10.18, $K^{(n)}$ is normal. Thus, by the Tietze Extension Theorem, a continuous extension $f_n : K^{(n)} \rightarrow I$ of $g$ exists. This completes the inductive construction of the fn.

Define $f : |K| \rightarrow I$ by $f|K^{(n)} = f_n$ for all $n > 0$. $f$ is well-defined by condition 2) above. Since each $f_n$ is continuous, $f$ is continuous by 10.20. Then $f^{-1}(I \cup \mathbb{I})$ and $f^{-1}(\bar{U} \cup \bar{V})$ are disjoint open neighborhoods of $C_0$ and $C_1$, respectively, in $|K|$.

Proposition 10.27. Let $K$ be a CW complex, and $X$ a compact subspace of $|K|$. Then $X$ meets only finitely many of the cells of $K$. In particular, $X \subset K^{(n)}$ for some $n > 0$.

Proof. Let $S$ be the collection of cells of $K$ which meet $X$. For each $A \in S$, choose a point $x_A \in A \cap X$, and let $E = \{ x_A | A \in S \}$. By $T \subset S$, define $Z_T = \{ x_A | A \in T \}$. For each $n > 0$, $X \cap K^{(n)}$ is closed in $X$ (since $K^{(n)}$ is closed in $|K|$) and hence compact. Thus by 10.21, $S \cap cell_n(K)$ is a finite set for each $n > 0$. Thus, for each $n > 0$, $Z_T \cap K^{(n)}$ is a finite set, and hence closed in $K^{(n)}$. Thus by 10.20, $Z_T$ is closed in $|K|$ for every $T \subset S$. Thus $Z$ is a closed discrete subspace of $|K|$. Since $Z \subset X$, $Z$ is compact and so $Z$ must be finite. Hence $S$ is finite.

Definition 10.28. Let $K$ be a CW complex. A subcomplex $L$ of $K$ consists of a subspace $|L|$ of $|K|$ and a subcollection cell $(L)$ of cell $(K)$ such that $|L| = \bigcup A$, and such that every cell $A \in (L)$ that for all $A \in cell(L)$, $\overline{A} \subset |L|$.
Proposition 10.29. Let \( K \) be a CW complex and \( L \) a subcomplex of \( K \). Then \( L \) is a CW complex, and \( L1 \) is closed in \( K1 \).

Proof. Since the cells of \( K \) are mutually disjoint and \( L1 \) is a union of some cells of \( K \), it follows that for each \( A \in \text{cell}(K) \), either \( A \subset L1 \) (and hence \( A \subset L1 \)) or \( A \cap L1 = \emptyset \).

We first show \( L1 \) is closed in \( K1 \). It suffices to show \( LIN A \) is closed in \( K1 \) for each \( A \in \text{cell}(K) \).

If \( A \in \text{cell}(K) \), then \( A \) is compact (being a continuous image of \( D^n \) for some \( n \)), and hence meets only finitely many members of \( \text{cell}(L) \). Say \( B_1, \ldots, B_r \) are the members of \( \text{cell}(L) \) which meet \( A \). Then \( L1 \cap A = \bigcup_i B_i \cap A \). Some \( B_i \subset L1 \) for \( i \in r \), so \( L1 \cap A = \bigcup_i B_i \cap A \), a finite union of closed subsets of \( K1 \) and hence closed in \( K1 \). Thus \( L1 \) is closed in \( K1 \).

Closure finiteness for \( L \) follows immediately from closure finiteness for \( K \). Characteristic maps for the cells of \( L \) exist since they are also cells of \( K \), hence closures in \( L1 \) coincide with their closures in \( K1 \), and \( K \) is a CW complex.

It remains only to check that \( L1 \) has the weak topology with respect to \( \text{cell}(L) \).

Let \( X \) be a subset of \( L1 \) such that \( X \cap A \) is closed in \( L1 \) (and hence in \( K1 \)) for all \( A \in \text{cell}(L) \). Let \( B \) be cell \( (K) \). As above, \( B \) meets only finitely many members of \( \text{cell}(L) \). Say \( A_1, \ldots, A_r \) are the members of \( \text{cell}(L) \) which meet \( B \). Then

\[
X \cap B = \left( \bigcup_{i=1}^{r} X \cap A_i \right) \cap B.
\]

Since each \( X \cap A_i \) is closed in \( K1 \), we have \( X \cap B \) is closed in \( K1 \). Thus, since \( K1 \) has the weak topology with respect to \( \text{cell}(K) \), \( X \) is closed in \( K1 \) (and hence in \( L1 \)), completing the proof.

Example 10.30. Let \( K \) be any CW complex. For \( n \geq 0 \), define \( K_n \) by \( K_0 = K^{(0)} \), \( \text{cell}(K_n) = \bigcup_{m \geq n} \text{cell}(K) \). Then \( K_n \) is a subcomplex of \( K \).
Proposition 10.32. Let \( K \) be a CW complex, and \( L \) a subcomplex of \( K \). Then for each \( n \geq 0 \), \( L \cup K^n \) is obtained from \( L \cup K^{n-1} \) by attaching \( n \)-cells. The collection of cells of \( L \cup K^n \) used \( (L \cup K^{n-1} \cup \text{cells}(K) - \text{cells}(L)) \). Write \( J = \text{cells}(K) - \text{cells}(L) \). If \( \varphi : (D^n \times \text{cells}(K), S^n \times \text{cells}(K)) \to (K^n, K^{n-1}) \) is a cell detection map for \((K^n, K^{n-1})\), then the composition
\[
(h)_{*}^{-1}(Z) = \left( \bigcap_{j \in J} (h_{*})^{-1} (Z \cap e_j) \right) \cup Z.
\]
Thus \( Z \cap L \cup K^{n-1} \) is closed in \( L \cup K^{n-1} \) and \( (h_{*})^{-1}(Z \cap e_j) \) is closed in \( D^n \) in each \( j \in J \). Thus by 10.16, \( Z \cap e_j \) is closed in \( \overline{e_j} \) for each \( j \in J \). Since cell(\( L \cup K^n \)) = cell(\( L \cup K^{n-1} \)) \cup J, \) it follows that \( Z \cap A \) is closed for all \( A \in \text{cell}(L \cup K^n) \), and so \( Z \) is closed in \( L \cup K^n \), completing the proof.

Definition 10.33. Let \( K, L \) be CW complexes. A CW map \( \varphi : K \to L \) consists of a continuous map \( f : K \to L \) such that \( f(K^n) \subseteq L^n \) for all \( n \geq 0 \).
Example 10.34. Let $K$ be a CW complex, and $L$ a subcomplex of $K$. Then $i : L \rightarrow K$ given by $i_! = \text{inclusion map} \quad |L| \rightarrow |K|$ is a CW map.

Definition 10.35. A CW pair $(K, L)$ consists of a CW complex $K$ and a subcomplex $L$ of $K$. A map $f : (K, L) \rightarrow (M, N)$ consists of a CW map $f : K \rightarrow M$ such that $f_! : |L| \rightarrow |N|$. We write $f^0 : L \rightarrow N$ for the CW map whose underlying map $f^0_! : |L| \rightarrow |N|$ is given by restriction of $f_!$.

Composition of CW maps and maps of CW pairs are defined in the obvious way, and it is obvious that we obtain categories.

Definition 10.36. The CW category $\text{CW}$ has as objects all CW complexes and as morphisms all CW maps. The category of CW pairs $\text{CW}P$ has as objects all CW pairs and as morphisms all maps of CW pairs.

We have the empty CW complex, and so we can regard $\text{CW}$ as a subcategory of $\text{CW}P$ in the same way we regard $\text{Top}$ as a subcategory of $\text{SP}$.

By forgetting cell structure and just looking at underlying spaces and pairs and underlying maps of spaces and pairs, we have covariant functors $\iota_1 : \text{CW} \rightarrow \text{Top}$ and $\iota_1 : \text{CW}P \rightarrow \text{SP}$.

11. CW Homology - General Theory

Definition 11.1. A triple (not to be confused with a trident) $(X, A, B)$ consists of three topological spaces with $X \supset A \supset B$. A map of triples $(X, A, B) \rightarrow (X', A', B')$ consists of a continuous map $f : X \rightarrow X'$ such that $f(A) \subset A'$, $f(B) \subset B'$.

If $(X, A, B)$ is a triple, we obtain a short exact
sequence of chain complexes

$$0 \rightarrow S(A)/S(B) \xrightarrow{S(i)} S(X)/S(B) \xrightarrow{S(j)} S(X)/S(A) \rightarrow 0$$

where $i : (A, B) \rightarrow (X, B)$, $j : (X, B) \rightarrow (X, A)$ are the
inclusions. This yields a long exact sequence of
homology groups.

Definition 11.2. If $(X, A, B)$ is a triple, the exact
sequence

$$\cdots \rightarrow H_{n+1}(X, A) \rightarrow H_n(A, B) \rightarrow H_n(X, B) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A, B) \rightarrow \cdots$$

arising from the above short exact sequence of chain
complexes, is called the homology sequence of the
triple $(X, A, B)$.

Proposition 11.3. If $(X, A, B) \rightarrow (X', A', B')$ is a map of
triples with underlying map $f : X \rightarrow X'$, the diagram

$$\cdots \rightarrow H_{n+1}(X, A) \xrightarrow{f_{n+1}} H_{n+1}(X', A') \xrightarrow{f_{n+1}} H_n(A, B) \xrightarrow{f_n} H_n(X, B) \xrightarrow{f_n} H_n(X, A) \xrightarrow{f_{n-1}} H_{n-1}(A, B) \rightarrow \cdots$$

commutes, where the rows are the homology sequences of
the respective triples and $f' : (X, A) \rightarrow (X', A')$, $f' : (A, B) \rightarrow (A', B')$, $f'' : (X, B) \rightarrow (X', B')$ are the maps
of pairs induced by $f$.

Proof. We have the commutative diagram of maps of pairs

$$(A, B) \xrightarrow{f''} (X, B) \xrightarrow{f'} (X, A)$$

$$(A', B') \xrightarrow{f'''} (X', B') \xrightarrow{f'} (X', A')$$
where the horizontal maps are the inclusions. This yields a morphism of short exact sequences of chain complexes

\[ \begin{array}{c}
0 \rightarrow S(A, B) \rightarrow S(X, B) \rightarrow S(X, A) \rightarrow 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 \rightarrow S(A', B') \rightarrow S(X', B') \rightarrow S(X', A') \rightarrow 0
\end{array} \]

which yields the asserted commutative ladder.

**Proposition 11.4.** If \((X, A)\) is a topological pair, the homology sequence of the pair \((X, A)\) coincides with the homology sequence of the triple \((X, A, \emptyset)\).

In any triple \((X, A, B)\), the connecting homomorphism \(\partial : H_n(X, A) \rightarrow H_{n-1}(A, B)\) is the composition

\[ H_n(X, A) \xrightarrow{\partial'} H_{n-1}(A) \xrightarrow{\partial^\prime} H_{n-1}(A, B) \]

where \(\partial'\) is the connecting homomorphism in the homology sequence of the pair \((X, A)\), and \(i : (A, \emptyset) \rightarrow (A, B)\) is the inclusion.

**Proof.** The first statement is immediate.

The inclusion of triples \((X, A, \emptyset) \rightarrow (X, A, B)\) yields, by 11.3, the commutative diagram

\[ \begin{array}{ccc}
H_n(X, A) & \xrightarrow{\partial'} & H_{n-1}(A, \emptyset) \\
\downarrow & & \downarrow H_{n-1}(i) \\
H_n(X, A) & \xrightarrow{\partial} & H_{n-1}(A, B)
\end{array} \]

and thus the second statement holds.

**Definition 11.5.** Let \((K, L)\) be a CW pair. For \(n \geq 0\), define \(C_n(K, L) = H_n(\{L \cup kn\}, \{L \cup kn-1\})\) and \(\partial : C_n(K, L) \rightarrow C_{n-1}(K, L)\) to be the connecting homomorphism \(\partial : H_n(\{L \cup kn\}, \{L \cup kn-1\}) \rightarrow H_{n-1}(\{L \cup kn-1\}, \{L \cup kn-2\})\) in the homology sequence of the triple \(\{L \cup kn\}, \{L \cup kn-1\}, \{L \cup kn-2\}\).
(By convention, $K_n = \emptyset$ for $n < 0$, and $C_n(K,L) = 0$ for $n < 0$). $C_n(K,L)$ is called the $n$th cellular chain group of the CW pair $(K,L)$.

Proposition 11.6. Let $(K,L)$ be a CW pair. Then the $C_n(K,L)$ and $\partial$ as in 11.5 constitute a chain complex $C(K,L)$, which we call the cellular chain complex of the CW pair $(K,L)$.

Proof. Write $X_n = \{LUKn\}$. By 11.4, we have the commutative diagram

$$
\begin{array}{ccc}
C_n(K,L) & \xrightarrow{\partial} & C_{n-1}(K,L) \\
\downarrow & & \downarrow \\
H_n(X_n, X_{n-1}) & \xrightarrow{\partial} & H_{n-1}(X_{n-1}, X_{n-2}) \\
\downarrow & & \downarrow \\
H_{n-1}(X_{n-1}) & \xrightarrow{\partial} & H_{n-2}(X_{n-2})
\end{array}
$$

The composition $\partial \circ H_{n-1}(i)$ is 0, being the composition of two consecutive homomorphisms in the homology sequence of the pair $(X_{n-1}, X_{n-2})$. Thus $\partial \circ \partial = 0$.

Proposition 11.7. Let $(K,L)$ be a CW pair. Then

$$H_n(\{LUKn\}, \{LUKn\}) = \begin{cases} 
0 & \text{if } i \neq n \\
\text{free abelian on a set of generators in} & \text{corresponding cells} \\
\text{1-1 correspondence with} c_n(K) - c_n(L) & \text{if } i = n
\end{cases}
$$

In particular, $C_n(K,L)$ is free abelian on a set of generators in $1-1$ correspondence with $\text{cells}(K) - \text{cells}(L)$.

Proof. Write $J = \text{cells}(K) - \text{cells}(L)$. By 10.32, $\{LUKn\}$ is obtained from $\{LUKn\}$ by attaching $n$-cells, and $J$ is the set of $n$-cells of $\{LUKn\}$ mod $\{LUKn\}$. Let $h : (D^n \times J, S^{n-1} \times J) \to (\{LUKn\}, \{LUKn\})$ be a cell detection map. By 10.13, $h$ is an excision relative homomorphism, and hence induces isomorphisms in homology. Since $(D^n \times J, S^{n-1} \times J) = \bigcup_{j \in J} (D^n \times \{j\}, S^{n-1} \times \{j\})$, the result now follows from 4.9 and 6.6.
If \( f : (K, L) \to (K', L') \) is a map of CW pairs, then for each \( n \), the restriction of \( f^* \) yields a map of triples \((IL^uk, \langle L^ukn_1 \rangle, IL^ukn_2) \to (IL'^uk', \langle L'^uk'n_1 \rangle, IL'^uk'n_2)\). Define \( C_n(f) : C_n(K, L) \to C_n(K', L') \) to be

\[ H_n(f^*) : H_n(IL^uk, \langle L^ukn_1 \rangle, IL^ukn_2) \to H_n(IL'^uk', \langle L'^uk'n_1 \rangle, IL'^uk'n_2) \]

where \( f^* \) is the map of pairs induced by the above map of triples.

By 11.3, the diagram

\[
\begin{array}{ccc}
C_n(K, L) & \to & C_{n-1}(K, L) \\
\downarrow \quad f^* & & \downarrow \quad H_n(f^*) \\
C_n(K', L') & \to & C_{n-1}(K', L')
\end{array}
\]

commutes for all \( n \). Thus

**Proposition 11.8.** If \( f : (K, L) \to (K', L') \) is a map of CW pairs, the homomorphisms \( C_n(f) \) constitute a chain map \( C(f) : C(K, L) \to C(K', L') \).

**Exercise 11.9.** \( C : CW \to GG \) is a covariant functor from the category of CW pairs to the category of chain complexes.

Our aim in this section is to show that for CW pairs \((K, L)\), there is a natural isomorphism from \( H_n(C(K, L)) \) to the singular homology group \( H_n(\langle K, L \rangle) \).

In each CW pair \((K, L)\) there is an isomorphism \( \phi_{K,L} : H_n(C(K, L)) \to H_n(\langle K, L \rangle) \) such that whenever \( f : (K, L) \to (K', L') \) is a map of CW pairs, the diagram

\[
\begin{array}{ccc}
H_n(C(K, L)) & \xrightarrow{\phi_{K,L}} & H_n(\langle K, L \rangle) \\
\downarrow \quad H_n(f^*) & & \downarrow \quad H_n(f) \\
H_n(C(K', L')) & \xrightarrow{\phi_{K',L'}} & H_n(\langle K', L' \rangle)
\end{array}
\]

commutes. Since the chain complex \( C(K, L) \) is small in comparison with the singular complex \( S(\langle K, L \rangle) \) (e.g. finitely generated in dimensions where there are only
finitely many cells), this will greatly facilitate calculation of homology groups.

Our strategy is as follows: Let \((K, L)\) be a CW pair and write \(X_n = (L \cup K_n)\). Consider

\[
H_n(C(K,L)) \xrightarrow{\phi_{K,L}} Z_n(C(K,L)) \leq H_n(X_n, X_{n-1}) \xrightarrow{g_{K,L}} H_n(X_n, L)|L| \xrightarrow{f_{K,L}} H_n(1K_1, L)|L|
\]

where \(\phi_{K,L} : Z_n(C(K,L)) \to Z_n(C(K,L))/\text{Bn}(C(K,L)) = H_n(C(K,L))\)

is projection onto the quotient, and \(f_{K,L}, g_{K,L}\) are induced by the appropriate inclusions of pairs. We will prove:

**Lemma 11.10.** The image of \(g_{K,L}\) is \(Z_n(C(K,L))\).

**Lemma 11.11.** \(g_{K,L}\) is 1-1.

**Lemma 11.12.** \(f_{K,L}\) is onto.

Thus, we will have a well-defined surjective homomorphism \(h_{K,L} : Z_n(C(K,L)) \to H_n(1K_1, L)|L|\) given by \(h_{K,L}(z) = f_{K,L}(g_{K,L}^{-1}(z))\). We will prove:

**Lemma 11.13.** \(\ker h_{K,L} = \text{Bn}(C(K,L))\).

Thus, passage to quotients will yield an isomorphism \(\Theta_{K,L} : H_n(C(K,L)) \to H_n(1K_1, L)|L|\).

Naturality will be immediate, for if \(F : (K,L) \to (K', L')\)

is a map of CW pairs, the diagram

\[
\begin{array}{cccccc}
H_n(C(K,L)) & \xrightarrow{\phi_{K,L}} & Z_n(C(K,L)) & \leq & H_n(X_n, X_{n-1}) & \xrightarrow{g_{K,L}} & H_n(X_n, L)|L| & \xrightarrow{f_{K,L}} & H_n(1K_1, L)|L| \\
\downarrow H_n(C(F)) & & \downarrow C_n(F) & & \downarrow C_n(F) = H_n(F|F) & & \downarrow H_n(F|F) & & \downarrow H_n(F|F) \\
H_n(C(K', L')) & \xrightarrow{\phi_{K',L'}} & Z_n(C(K', L')) & \leq & H_n(X'_n, X'_{n-1}) & \xrightarrow{g_{K',L'}} & H_n(X'_n, L'|L'|) & \xrightarrow{f_{K',L'}} & H_n(1K_1', L'|L'|) \\
\end{array}
\]

commutes, and naturality of \(\Theta_{K,L}\) then follows by an easy diagram chase.
We now proceed to prove the above four lemmas. We will need some sub-lemmas.

Sublemma 11.14. Let \((K,L)\) be a CL pair, and \(X_n = \{L^i | Kn\}\). Then \(H_m(X_n, X_{n-1}) = 0\) whenever \(m > n\) and \(i > 0\).

Proof: We proceed by induction on \(i\). The result is trivial for \(i = 0\). Let \(i > 0\) and assume inductively \(H_m(X_n, X_{n-i+1}) = 0\).

The triple \((X_n, X_{n-i+1}, X_{n-i})\) yields the exact sequence

\[ H_m(X_{n-i+1}, X_{n-i}) \rightarrow H_m(X_n, X_{n-i}) \rightarrow H_m(X_n, X_{n-i+1}), \]

\[ H_m(X_n, X_{n-i+1}) = 0 \] by the induction hypothesis.

\[ H_m(X_{n-i+1}, X_{n-i}) = 0 \] by 11.7, since \(m > n - i + 1\). Thus, by exactness, \(H_m(X_n, X_{n-i}) = 0\), completing the induction.

Proof of Lemma 11.10. The homology sequence of the triple \((X_n, X_{n-1}, X_{n-2})\) yields the exact sequence

\[ H_n(X_n, X_{n-2}) \xrightarrow{H_n(i)} H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-2}, X_{n-1}), \]

and by definition, \(Z_n(C(K,L)) = \ker \partial\). Thus, from the above exactness,

\[ Z_n(C(K,L)) = \text{im} \left( H_n(i) : H_n(X_n, X_{n-2}) \rightarrow H_n(X_n, X_{n-1}) \right). \]

We have the commutative diagram

\[ H_n(X_n, X_{n-1}) \xrightarrow{H_n(i)} H_n(X_n, X_{n-2}) \rightarrow H_{n-1}(X_{n-2}, X_{n-1}). \]

The row is exact, being a portion of the homology sequence of the triple \((X_n, X_{n-2}, X_{n-1})\). By 11.14, \(H_{n-1}(X_{n-2}, X_{n-1}) = 0\) and \(\partial \circ H_n(i) = 0\). Thus,

\[ \text{im} \partial = \text{im} H_n(i) = Z_n(C(K,L)). \]

Proof of Lemma 11.11. The triple \((X_n, X_{n-1}, X_{n-2}, X_{n-1})\) yields the exact sequence \(H_n(X_n, X_{n-1}, X_{n-2}, X_{n-1}) \rightarrow H_n(X_n, X_{n-1}) \rightarrow H_{n-1}(X_{n-2}, X_{n-1}),\) and
$H_n(X_{n-1}, LL) = 0$ by 11.14. Hence $f_{k,L}$ is 1-1.

Sublemma 11.15. Let $(k, L)$ be a CW pair and $X_n = [L \cup K_n]$. Then $H_n(X_{n+i}, X_n) = 0$ for all $i \geq 0$.

Proof: We proceed by induction on $i$. The result is trivial if $i = 0$. Let $i > 0$ and assume inductively $H_n(X_{n+i-1}, X_n) = 0$. The triple $(X_{n+i}, X_{n+i-1}, X_n)$ yields the exact sequence
\[ H_n(X_{n+i-1}, X_n) \to H_n(X_{n+i}, X_n) \to H_n(X_{n+i}, X_{n+i-1}). \]

$H_n(X_{n+i-1}, X_n) = 0$ by the inductive hypothesis, and $H_n(X_{n+i}, X_n) = 0$ by 11.7. Thus, by exactness, $H_n(X_{n+i}, X_{n+i-1}) = 0$, completing the induction.

Sublemma 11.16. Let $(k, L)$ be a CW pair and $X_n = [L \cup K_n]$. Then $H_n(1K_1, X_n) = 0$ for all $n$.

Proof. Let $u \in H_n(1K_1, X_n)$. By compact supports (7.11) there exists a compact subspace $Y$ of $1K_1$ and a $v \in H_n(Y, X_n, Y)$ such that $u = H_n(Y)(v)$, $i: (Y, X_n, Y) \to (1K_1, X_n)$ the inclusion. By 10.27, $Y \subset K^{n+1}$ for $n$ sufficiently large. We then have a commutative diagram of maps induced by inclusion
\[
\begin{array}{ccc}
H_n(Y, X_n, Y) & \xrightarrow{H_n(Y)} & H_n(1K_1, X_n) \\
\downarrow & & \\
H_n(X_{n+i}, X_n) & \xrightarrow{H_n(1K_1)} & H_n(1K_1, X_n)
\end{array}
\]

By 11.15, $H_n(X_{n+i}, X_n) = 0$ and so $u = H_n(Y)(v) = 0$.

Proof of Lemma 11.12. The triple $(1K_1, X_n, LL)$ yields the exact sequence
\[ H_n(X_n, LL) \xrightarrow{f_{k,L}} H_n(1K_1, LL) \to H_n(1K_1, X_n). \]

By 11.16, $H_n(1K_1, X_n) = 0$ and so $f_{k,L}$ is onto.

Proof of Lemma 11.13. The triple $(X_{n+1}, X_n, X_{n-1})$ yields the exact
We have the commutative diagram

\[
\begin{array}{ccc}
H_n(X_n, X_{n-1}) & \xrightarrow{g_{K,L}} & H_n(X_n, 1L) \\
\downarrow & & \downarrow \delta' \\
H_n(X_{n+1}, X_{n-1}) & \xrightarrow{g_{K,L}} & H_n(1K, 1L) \\
\end{array}
\]

where \( \delta' \) is induced by the appropriate inclusions.

The second row and diagonal are exact, being portions of homology sequences of appropriate triples. \( H_{n+1}(1K, X_{n+1}) = 0 \) by 11.16, and \( H_n(X_{n-1}, 1L) = 0 \) by 11.14. Hence \( \delta' \) and \( g' \) are both 1-1. The result now follows by diagram chasing as follows:

Let \( u \in \ker h_{K,L} \). Then \( u = g_{K,L}(v) \) where \( f_{K,L}(v) = 0 \). Thus, since \( f' \) is 1-1, we have \( f'(v) = 0 \) and so \( H_n(v) = 0 \). Hence \( u \in \ker H_n(u) = B_n(C(K,L)) \) and so \( \ker h_{K,L} \subset B_n(C(K,L)) \).

Conversely, let \( u \in B_n(C(K,L)) = \ker H_n(u) \). We have \( h_{K,L}(u) = f_{K,L}(v) \) where \( v \in H_n(X_n, 1L) \) is the unique element such that \( g_{K,L}(v) = u \). Since \( g' \) is 1-1, it follows that \( h'(v) = 0 \) and hence \( f_{K,L}(v) = 0 \), i.e., \( h_{K,L}(u) = 0 \). Thus \( u \in \ker h_{K,L} \) and so \( B_n(C(K,L)) \subset \ker h_{K,L} \), completing the proof.

Thus we have proved

**Theorem 11.17.** There is a natural isomorphism

\[
\Theta_{K,L} : H_n(C(K,L)) \to H_n(1K, 1L)
\]

for CW complexes \((K, L)\).

**Corollary 11.18.** Let \( K \) be a CW complex. Suppose, for some \( n > 0 \), \( K \) has a finite number \( r \) of \( n \)-cells. Then \( H_n(1K) \) is finitely
generated, and has a set of generators with \( \leq r \) elements.

Prop. By 11.7, \( C_n(K) \) is free of rank \( r \). Since \( H_n(C(K)) \) is a quotient of a subgroup of \( C_n(K) \), \( H_n(C(K)) \) is generated by \( \leq r \) elements. The result now follows from 11.17.

Example 11.19. Let \( n \geq 0 \). By 10.10, \( CP^n = |K| \) in the cell
\[ c(K) = \{ c_1 \epsilon c_1, c_2, \ldots, c_{2n} \} \], where \( c_i \epsilon cell_i(K) \). Thus, by 11.7,
\[ c_i(K) \equiv \begin{cases} 2 & \text{if } i \text{ is even and } 0 \leq i \leq 2n \\ 0 & \text{otherwise} \end{cases} \]

Thus, all boundary maps in the chain complex \( C(K) \) are 0, and so \( H_i(C(K)) = C_i(K)/0 = C_i(K) \) for all \( i \). Thus, by 11.17,
\[ H_i(CP^n) \equiv \begin{cases} 2 & \text{if } i \text{ is even and } 0 \leq i \leq 2n \\ 0 & \text{otherwise} \end{cases} \]

Similarly, by 10.25 and 10.22,
\[ H_i(CP^\infty) \equiv \begin{cases} 2 & \text{if } i \text{ is even and } \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

Proposition 11.20. Let \( K \) be a CW complex. Let \( n \geq 0 \) and let \( i : k^m \rightarrow |K| \) be the inclusion. Then \( H_q(i) : H_q(k^m) \rightarrow H_q(|K|) \) is an isomorphism for \( q < n \) and onto \( \{0\} \) for \( q = n \).

Proof. From the homology sequence of the pair \( (|K|, k^m) \),
\[ H_q(k^m) \rightarrow H_q(|K|) \rightarrow H_q(k^m) \rightarrow H_{q-1}(|K|) \rightarrow H_q(k^m) \]
the exact sequence of the statement that \( H_q(|K|, k^m) \cong 0 \) for all \( q \geq n \). By 11.17, \( H_q(|K|, k^m) \cong H_q(C(K, Kn)) \). Since, for \( q \leq n \), every \( q \)-cell of \( K \) is a \( q \)-cell of \( Kn \), we have by 11.7, \( C_q(K, Kn) = 0 \) for \( q \leq n \), and so \( H_q(C(K, Kn)) = 0 \) for \( q \leq n \).

We conclude this section with a treatment of cellular characteristics.
Definition 11.21. A topological space $X$ has finite homology type if $\text{H}_n(X)$ is finitely generated for all $n$, and all but a finite number of the $\text{H}_n(X)$ are 0.

Example 11.22. If $K$ is a finite CW complex (i.e. cell($K$) is a finite collection), then $|K|$ has finite homology type by 11.18.

Recall from algebra that if $G$ is a finitely generated abelian group, then $G \cong \bigoplus \mathbb{Z}^r \oplus \bigoplus \mathbb{Z}_{p_1}^{n_1} \oplus \cdots \oplus \bigoplus \mathbb{Z}_{p_m}^{n_m}$ where $\mathbb{Z}$ is a finite group. The integer $r$ is independent of how $G$ is represented in this way, and we call $r$ the rank of $G$, denoted rank($G$). If $H$ is a subgroup of $G$, then rank($G/H$) = rank($G$) - rank($H$).

Definition 11.23. Let $X$ be a topological space of finite homology type. The Euler characteristic of $X$, denoted $\chi(X)$, is the integer $\sum_{n} (-1)^n \text{rank } \text{H}_n(X)$.

Since singular homology groups are, up to isomorphism, homotopy type invariants, it follows that the Euler characteristic is a homotopy type invariant.

Theorem 11.24. Let $K$ be a finite CW complex. Then $\chi(|K|) = \sum_{n} (-1)^n \text{rank } \text{cell}_n(K)$, where $\text{cell}_n(K)$ = number of $n$-cells in $K$.

Proof. By 11.17, $\text{H}_n(|K|) \cong \text{H}_n(C(K))$, we have $\text{H}_n(C(K)) = \mathbb{Z}_n(C(K))/\text{B}_n(C(K))$ and so $\text{rank } \text{H}_n(|K|) = \text{rank } \text{Z}_n(C(K)) - \text{rank } \text{B}_n(C(K))$. We also have $\text{B}_n(C(K)) \cong \text{C}_n(K)/\text{Z}_n(K)$ and so $\text{rank } \text{B}_n(C(K)) = \text{rank } \text{C}_n(K) - \text{rank } \text{Z}_n(K)$. By 11.7, $\text{rank } \text{C}_n(K) = |\text{cell}_n(K)|$. Thus $\text{rank } \text{H}_n(|K|) = \text{rank } \text{Z}_n(C(K)) - |\text{cell}_n(K)| + \text{rank } \text{Z}_n(C(K))$ for all $n$, and so $\chi(|K|) = \sum_{n} (-1)^n [\text{rank } \text{Z}_n(C(K)) - |\text{cell}_n(K)| + \text{rank } \text{Z}_n(C(K))] = \sum_{n} (-1)^n |\text{cell}_n(K)|$.
\[ \sum_{n} (-1)^{n} \text{rank } \mathbb{Z} \left( C(K) \right) + \sum_{n} (-1)^{n} |\text{cell}_{n}(K)| = \sum_{n} (-1)^{n} \text{rank } \mathbb{Z} \left( C(K) \right) \]

Corollary 11.25. If \( K \) is a finite CW complex, then
\[ \sum_{n} (-1)^{n} |\text{cell}_{n}(K)| \]
depends only on the homotopy type of \( |K| \).

12. **Incidence Numbers and Regular CW Complexes**

**Lemma 12.1.** Let \( K \) be a CW complex and \( A \in \text{cell}_{n}(K) \).
Let \( \tau_{A} : (D^{n}, S^{n-1}) \rightarrow (\bar{A}, \bar{A}) \) be a characteristic map for \( A \).
Then \( \bar{A} \) is obtained from \( \bar{A} \) by attaching one \( n \)-cell,
and the map \( \tau_{A} : (D^{n} \times \{x\}, S^{n-1} \times \{x\}) \rightarrow (\bar{A}, \bar{A}) \) given
by \( \tau_{A}(x,A) = \tau_{A}(x) \) is a cell detection map.

**Proof.** We must check \( (\tau_{A})_{+} : \bar{A} \sqcup (D^{n} \times \{x\}) \rightarrow \bar{A} \) is a quotient map. This is immediate since \( (\tau_{A})_{+} \) is onto,
\( \bar{A} \) is Hausdorff, and \( \bar{A} \sqcup (D^{n} \times \{x\}) \) is compact.

Thus, as a consequence of 10.13, we have

**Corollary 12.2.** If \( K \) is a CW complex, \( A \in \text{cell}_{n}(K) \),
and \( \tau_{A} : (D^{n}, S^{n-1}) \rightarrow (\bar{A}, \bar{A}) \) a characteristic map for \( A \),
then \( \tau_{A} \) is an effective relative homomorphism. In particular,
\[ H_{n}(\bar{A}, \bar{A}) \cong \begin{cases} 0 & \text{if } n \neq 0 \\ Z & \text{if } n = 0 \end{cases} \]

Let \((K, L)\) is a CW pair, write \text{cell}_{n}(K,L) = \text{cell}_{n}(K) - \text{cell}_{n}(L)\ and \text{cell}(K,L) = \text{cell}(K) - \text{cell}(L).

**Definition 12.3.** Let \((K, L)\) be a CW pair and \( A \in \text{cell}_{n}(K,L) \).
Let \( j_{A} : (\bar{A}, \bar{A}) \rightarrow ([L \cup K_{n}], [L \cup K_{n-1}]) \) denote the inclusion map.
If \( x \) is a generator of \( H_{n}(\bar{A}, \bar{A}) \), the element \( H_{n}(j_{A})(x) \in H_{n}(L \cup K_{n}, L \cup K_{n-1}) = \text{cell}(K,L) \) is called an
orientation of the cell \( A =_{\text{in}} (K,L) \).
Proposition 12.4. Let \((K, L)\) be a CW pair and \(n \geq 0\). Then:

1) For each \(A \in \text{cell}_n(K, L)\), the map induced by inclusion \(H_n(j_A) : H_n(A, \overline{A}) \to H_n(\{L \cup kn_1, \ldots, L \cup kn_{n-1}\})\) is 1-1 (and hence \(A\) has exactly two orientations \(\mu_i(K, L) : H_n(j_A)(x) \to -H_n(j_A)(x)\) whenever \(x\) is a generator of \(H_n(A, \overline{A})\)).

2) Suppose, for each \(A \in \text{cell}_n(K, L)\), an orientation \(\mathcal{O}_A \in C_n(K, L)\) for \(A \times \mu_i(K, L)\) has been chosen. Then \(\{\mathcal{O}_A | A \in \text{cell}_n(K, L)\}\) is a free abelian basis for \(C_n(K, L)\).

Proof. For each \(A \in \text{cell}_n(K, L)\), choose a characteristic map \(\alpha_A : (D^n, S^{n-1}) \to (A, \overline{A})\). By 10.14 and 10.32 we have a cell detection map

\[ h : (D^n \times \text{cell}_n(K, L), S^{n-1} \times \text{cell}_n(K, L)) \to (\{L \cup kn_1, \ldots, L \cup kn_{n-1}\}) \]

such that for each \(A \in \text{cell}_n(K, L)\), the diagram

\[
\begin{array}{ccc}
(D^n \times \{A\}, S^{n-1} \times \{A\}) & \xrightarrow{h_A} & (\overline{A}, \overline{A}) \\
\downarrow \alpha_A & & \downarrow \alpha_A \\
(B \in \text{cell}_n(K, L)) & \xrightarrow{\alpha_B} & (D^n \times \{B\}, S^{n-1} \times \{B\})
\end{array}
\]

commutes, where \(\alpha_A \circ h_A = \alpha_B \circ h_B = \alpha_B \). By 10.13, \(H_n(h_A)\) and \(H_n(h_B)\) are isomorphisms. By 4.9,

\[ \sum H_n(\alpha_B) : \bigoplus_{B \in \text{cell}_n(K, L)} H_n(D^n \times \{B\}, S^{n-1} \times \{B\}) \to H_n\left( \bigoplus_{B \in \text{cell}_n(K, L)} (D^n \times \{B\}, S^{n-1} \times \{B\}) \right).\]

is an isomorphism. Thus each \(H_n(\alpha_B)\) is injective. In particular, \(H_n(\alpha_A)\) is injective and hence \(H_n(j_A)\) is injective, proving 1). Moreover, if we choose a generator \(u_B\) of

\[ H_n\left( \bigoplus_{B \in \text{cell}_n(K, L)} (D^n \times \{B\}, S^{n-1} \times \{B\}) \right) \]

it becomes abelian with basis

\[ \{u_B | B \in \text{cell}_n(K, L)\}. \]
\{H_n(x_B)(y_B) | \text{Be cell}_n(K,L)\}$. Thus, since $H_n(h)$ is an isomorphism, $H_n(1\cup K\nu 1,1\cup K\nu n \cup 1)$ is free abelian with basis $\{H_n(h)H_n(x_B)(y_B) | \text{Be cell}_n(K,L)\}$. But for each $A \in \text{cell}_n(K,L)$, commutativity of the above diagram yields $H_n(h)H_n(A)(y_A) = H_n(j_A)H_n(A)(y_A)$. Since $H_n(A)$ is an isomorphism, $H_n(A)(y_A)$ is a generator of $H_n(A,\hat{A})$ and so by definition $H_n(j_A)H_n(A)(y_A)$ is an orientation of $A$ in $(K,L)$. By part 1, $H_n(j_A)H_n(A)(y_A) = \pm \theta_A$, and so 2 follows.

**Proposition 12.5.** Let $(K,L)$ and $(K',L')$ be CW pairs such that $K'$ is a sub-complex of $K$ and $L'$ is a sub-complex of $L$. Let $i:(K,L') \to (K,L)$ be the inclusion. Let $A \in \text{cell}_n(K,L')$, and let $\theta_A$ be an orientation of $A$ in $(K,L')$. Then,

$$C_n(i)(\theta_A) = \begin{cases} \theta_A & \text{if } A \in \text{cell}_n(K,L) \\ 0 & \text{otherwise} \end{cases}$$

**Proof.** We have the commutative diagram of inclusions:

$$\begin{array}{ccc}
\longrightarrow & \longrightarrow & \\
\uparrow & \uparrow & \\
(1\cup K\nu 1,1\cup K\nu n \cup 1) & \longrightarrow & (1\cup K\nu 1,1\cup K\nu n \cup 1) \\
\downarrow & \downarrow & \\
(\hat{A},\hat{A}) & \longrightarrow & (\hat{A}',\hat{A}')
\end{array}$$

Let $x$ be the generator of $H_n(\hat{A},\hat{A})$ such that $H_n(j_A)(x) = \theta_A$. Then $C_n(i)(\theta_A) = H_n(x')(\theta_A) = H_n(x')H_n(j_A)(x) = H_n(x)$. If $A \in \text{cell}_n(K,L)$, then by definition $H_n(j_A)(x)$ is an orientation of $A$ in $(K,L)$. If $A \notin \text{cell}_n(K,L)$, then $\forall A \in \text{cell}_n(L)$ and $\exists A \in \text{cell}_n(L)$ such that $A \notin \text{cell}_n(L)$. Thus, we have a commutative diagram of maps induced by inclusion:

$$\begin{array}{ccc}
H_n(\hat{A},\hat{A}) & \xrightarrow{H_n(j_A)} & H_n(1\cup K\nu 1,1\cup K\nu n \cup 1) \\
\downarrow & & \downarrow \\
H_n(1\cup K\nu n \cup 1,1\cup K\nu n \cup 1) & \xrightarrow{H_n(j_A)} & H_n(1\cup K\nu 1,1\cup K\nu n \cup 1)
\end{array}$$
and so $H_n(j_A)(X) = 0$ since $H_n(1L, K_{n-1}, 1L, K_{n-1}) = 0$.

Let $(K, L)$ be a CW pair. Suppose for each $A \in \text{cell}(K, L)$ an orientation $\sigma_A$ of $A$ in $(K, L)$ has been chosen. Then for each $A \in \text{cell}(K, L)$ we can write

$$\partial(A) = \sum_{B \in \text{cell}(K, L)} [\sigma_A : \sigma_B]\partial_B$$

where $[\sigma_A : \sigma_B] \in \mathbb{Z}$.

Definition 12.6. With notation as above, the integers $[\sigma_A : \sigma_B]$ are called the incidence numbers of $(K, L)$ relative to the orientations $\{\sigma_A | A \in \text{cell}(K, L)\}$.

Thus the calculation of the boundary maps in $C(K, L)$ is equivalent to determining the incidence numbers of $(K, L)$ relative to a choice of orientations. We will see how to do this below from the geometric incidence of the closures of the cells for a special class of CW complexes called regular CW complexes (definition later).

We next observe that for a CW complex $K$, we can introduce an augmentation $\epsilon : C(K) \to \mathbb{Z}$ making $C(K)$ an augmented chain complex.

Definition 12.7. Let $C$ be a CW complex. Define $\epsilon : C_0(K) \to \mathbb{Z}$ to be the homology augmentation $\epsilon : H_0(K_0, K^{-1, 0}) = H_0(K_0) \to \mathbb{Z}$.

Proposition 12.8. Let $K$ be a CW complex. Then $C(K)$, with $\epsilon$ as in 12.7, is an augmented chain complex. Furthermore, $C$ is a covariant functor from the category of CW complexes to the category of augmented chain complexes.

Proof: We must show:

1) The composition $C_1(K) \xrightarrow{\partial} C_0(K) \xrightarrow{\epsilon} \mathbb{Z}$ is the 0-homomorphism.
2) If \( f : K \rightarrow L \) is a CW map, then
\[
\begin{array}{c}
\mathcal{C}_0(K) \\
\mathcal{C}_0(f) \\
\mathcal{C}_0(L)
\end{array} 
\xrightarrow{\mathcal{C}_0(f)} 
\begin{array}{c}
\mathcal{C}_0(K) \\
\mathcal{C}_0(L)
\end{array} 
\xrightarrow{\varepsilon} 
\begin{array}{c}
\mathcal{C}_0(K) \\
\mathcal{C}_0(L)
\end{array} 
\xrightarrow{\varepsilon} 
\begin{array}{c}
\mathcal{C}_0(K) \\
\mathcal{C}_0(L)
\end{array} 
\xrightarrow{\varepsilon}
\]

Since \( K^{(-1)} = \emptyset \), the connecting homomorphism in the homology sequence of the triple \((K^{(1)}, K^{(0)}, K^{(-1)})\) coincides with the connecting homomorphism in the homology sequence of the pair \((K^{(1)}, K^{(0)})\), i.e.
\[
\partial : \mathcal{C}_i(K) \rightarrow \mathcal{C}_{i-1}(K) \quad \text{and} \quad \partial : \mathcal{H}_i(K^{(1)}, K^{(0)}) \rightarrow \mathcal{H}_{i-1}(K^{(0)})
\]
are the homology sequence of the pair \((K^{(1)}, K^{(0)})\). By naturality of the homology augmentation, we have the commutative diagram
\[
\begin{array}{c}
\mathcal{C}_i(K) \\
\mathcal{H}_i(K^{(1)}, K^{(0)})
\end{array} 
\xrightarrow{\partial} 
\begin{array}{c}
\mathcal{C}_{i-1}(K) \\
\mathcal{H}_{i-1}(K^{(0)})
\end{array} 
\xrightarrow{\varepsilon} 
\begin{array}{c}
\mathcal{H}_i(K^{(0)}) \\
\mathcal{H}_i(L^{(0)})
\end{array} 
\xrightarrow{\varepsilon}
\]

where \( i \) is the inclusion. The bottom row is exact, being a portion of the homology sequence of the pair \((K^{(1)}, K^{(0)})\). Thus \( \mathcal{H}_0(K^{(0)}) = 0 \). It follows that the top \( \partial \mathcal{H}_0 = 0 \), and so \( 1) \) holds.

If \( f : K \rightarrow L \) is a CW map, then \( f(\mathcal{C}_0(K^{(1)})) \subset \mathcal{L}^{(0)} \).

Let \( f_0 : K^{(0)} \rightarrow L^{(0)} \) be the restriction of \( f \). Then by naturality of the homology augmentation, the diagram
\[
\begin{array}{c}
\mathcal{C}_0(K) \\
\mathcal{C}_0(f) \\
\mathcal{C}_0(L)
\end{array} 
\xrightarrow{\mathcal{C}_0(f)} 
\begin{array}{c}
\mathcal{H}_0(K^{(0)}) \\
\mathcal{C}_0(f) \\
\mathcal{C}_0(L)
\end{array} 
\xrightarrow{\varepsilon} 
\begin{array}{c}
\mathcal{H}_0(K^{(0)}) \\
\mathcal{H}_0(L^{(0)})
\end{array} 
\xrightarrow{\varepsilon}
\]

commutes, and so \( 2) \) holds.
Proposition 12.9. Let $K$ be a CW complex and $A \in \text{cell}_0(K)$. Let $\sigma_A$ be an orientation of $A$ in $K$. Then $\epsilon(\sigma_A) = \pm 1$.

Proof. $A$ is a one point space, $\overline{A} = A$, and $\hat{A} = \emptyset$. Let $x$ be the generator of $\text{Ho}(A)$ such that $\text{Ho}(\iota_A)(x) = \sigma_A$ where $\iota_A : A \to K^{(\emptyset)}$ is the inclusion. Since $\epsilon : \text{Ho}(A) \to \mathbb{Z}$ is an isomorphism, we have $\epsilon(x) = \pm 1$. By naturality of the homology augmentation,

$$
\begin{array}{ccc}
\text{Ho}(A) & \xrightarrow{\epsilon} & \mathbb{Z} \\
\text{Ho}(\iota_A) & \downarrow & \\
\text{Ho}(K) = \text{Ho}(K^{(\emptyset)}) & \xrightarrow{\epsilon} & \mathbb{Z}
\end{array}
$$

commutes, and so $\epsilon(\sigma_A) = \epsilon(\text{Ho}(\iota_A)(x)) = \epsilon(x) = \pm 1$.

Definition 12.10. A CW complex $K$ is regular if
1) In each $n \geq 0$ and $A \in \text{cell}_n(K)$, the pair $(\overline{A}, A)$ is homeomorphic to $(D^n, S^{n-1})$.
2) In each $A \in \text{cell}(K)$, $A$ is the underlying space of a subcomplex $A_0$ of $K$.

Example 12.11. Example 10.6 (the hemispherical cell decomposition of $S^n$) is a regular CW complex.


Exercise 12.13. The standard $n$-simplex $\Delta_n$ is the underlying space of a regular CW complex $K$ as follows: For $0 \leq k \leq n$, let $S_k = \{ (x_0, x_1, \ldots, x_k) \in \mathbb{R}^{n+1} \mid 0 < x_0 < x_1 < \ldots < x_k \leq 1 \}$. In each $Q = (x_0, \ldots, x_k) \in S_k$, let $A_Q = \{ \sum_i t_i E_{x_i} \in \Delta_n \mid \sum t_i = 1, t_i > 0 \text{ for all } i \}$.

Then $\text{cell}_k(K) = \{ A_Q \mid Q \in S_k \}$.

(Here generally, simplicial complexes, i.e. spaces with
triangulations, are examples of regular CW complexes.

Lemma 12.14. Let $K$ be a regular CW complex such that $|K|$ is a topological $n$-manifold. Let $A \in \text{cell}_n(K)$. Then $|K| - A$ is a strong deformation retract of $|K| - \{x_0\}$ for some $x_0 \in A$.

Proof. We must show that for some $x_0 \in A$ there is a continuous map $F : \left( (|K| - \{x_0\}) \times I \rightarrow |K| - \{x_0\} \right)$ such that

1) $F(x,0) = x$, $F(x,1) \in |K| - A$ for all $x \in |K| - \{x_0\}$,

2) $F(x,t) = x$ for all $x \in |K| - A$, $t \in I$.

Since there is a homeomorphism of pairs $(A, \tilde{A}) \cong (D^n, S^{n-1})$ and since $S^{n-1}$ is a strong deformation retract of $D^n - \{0\}$, we can choose an $x_0 \in A$ such that $\tilde{A}$ is a strong deformation retract of $\tilde{A} - \{x_0\}$. Thus there is a continuous map

$G : (\tilde{A} - \{x_0\}) \times I \rightarrow \tilde{A} - \{x_0\}$ satisfying

1) $G(x,0) = x$, $G(x,1) \in \tilde{A}$ for all $x \in \tilde{A} - \{x_0\}$,

2) $G(x,t) = x$ for all $x \in \tilde{A}$, $t \in I$.

Define $F$ by $F|((\tilde{A} - \{x_0\}) \times I) = G$, and $F(x,t) = x$ for all $x \in |K| - A$, $t \in I$. Then $F$ is well-defined, and $F|((\tilde{A} - \{x_0\}) \times I)$, $F|((|K| - A) \times I)$ are both continuous, $\tilde{A} - \{x_0\}$ is closed in $|K| - \{x_0\}$ and by 7.26 (a corollary to Theorem of Fadell)

$A$ is open in $|K|$, and hence $(|K| - A)$ is closed in $|K| - \{x_0\}$.

Thus $(|K| - A) \times I$ and $(\tilde{A} - \{x_0\}) \times I$ are both closed in $(|K| - \{x_0\}) \times I$, and so $F$ is continuous.

Lemma 12.15. Let $K$ be a regular CW complex such that $|K| = \tilde{A}$ for some $n$-cell $A \in \text{cell}_n(K)$, $n > 0$. Suppose orientations $[\sigma_B : B \in \text{cell}(K)]$ in $K$ of the cells of $K$ have been chosen. Then for each $B \in \text{cell}_{n-1}(K)$, $[\sigma_A : \sigma_B] = \pm 1$.

Proof. Let $B \in \text{cell}_{n-1}(K)$. (Note that cell$_{n-1}(K) \neq \emptyset$.) Otherwise we would have $C_{n-1}(K) = 0$ and so $Z_n(C(K)) = Z_n(K)$. Thus, since $C_n(K) = 0$, we would have $H_n(D^n) \cong H_n(\tilde{A}) = H_n(|K|) \cong H_n(C(K)) \cong C_n(K) = 0$; a contradiction since $n > 0$.)

We just showed that $\tilde{A} - B$ is the underlying space of a subcomplex of $K$. Since $|K| = \tilde{A} = A \cup \tilde{A}$ (as sets),
and $\hat{A}$, as a set, is the disjoint union of the cells of $K$ other than $A$, we have $B = \hat{A}$ and $\hat{A} - B$ is the disjoint union of the cells of $K$ other than $A$ and $B$. We must show that if $C \in \text{cell}(K) - \{A, B\}$, then $C \subset \hat{A} - B$. Since $C \subset \hat{A} - B$, it suffices to show that $\hat{A} - B$ is closed in $|K|$. Since $B \cong \mathbb{R}^n$ and $\hat{A} \cong S^{n-1}$ (since $K$ is regular), $B$ is open in $\hat{A}$, by Lemma 7.25. Thus $\hat{A} - B$ is closed in $\hat{A}$, and hence in $|K|$. Since $\hat{A}$ is closed in $|K|$. We then have a subcomplex $L$ of $K$ with $|L| = \hat{A} - B$, and we have $\text{cell}(L) = \{A, B\}$.

Let $i : (K, \partial) \to (K, L)$ denote the inclusion. It follows from 12.5 that $C_{n}(i)(\partial A) = \partial A$, an orientation of $A$ in $(K, L)$; $C_{n-1}(i)(\partial B) = \partial B$, an orientation of $B$ in $(K, L)$; and $C_{n}(i)(\partial C) = 0$ if $C \in \text{cell}(K) - \{A, B\}$. Hence

$$C_{n}(i) \partial(C_{n}) = C_{n-1}(i) \sum_{B \in \text{cell}(L)} [\partial A : \beta B] \partial B = [\partial A : \beta B] \partial B.$$ 

$C_{n}(K, L)$ is free abelian with generator $\partial A$, and $C_{n-1}(K, L)$ is free abelian with generator $\partial B$. From commutativity of

$$\begin{array}{ccc}
C_n(K) & \xrightarrow{\partial} & C_{n-1}(K) \\
\downarrow & & \downarrow \\
C_n(i) & \xrightarrow{\partial} & C_{n-1}(i)
\end{array}$$

we have $\partial A = \partial C_{n}(i) = C_{n-1}(i) \partial(C_{n}) = [\partial A : \beta B] \partial B$.

Thus it suffices to show $\partial : C_{n}(K, L) \to C_{n-1}(K, L)$ is onto.

Since $\text{cell}(K) = \emptyset$, we have $\mathbb{Z}_{n-1}(C(K, L)) = C_{n-1}(K, L)$.

Thus $H_{n-1}(C(K, L)) = C_{n-1}(K, L) / \text{im} \partial$, and so it suffices to show $H_{n-1}(C(K, L)) = 0$. By 11.17, $H_{n-1}(C(K, L)) \cong H_{n-1}(|K|, L)$.

Since $|K| = \hat{A} \cong \mathbb{D}^n$, $|K|$ is contractible and so $\tilde{H}_i(|K|) = 0$ for all $i$. By 12.14, $|L|$ is a deformation retract of $S^{n-1} - \{P\}$, $P \in S^{n-1}$, and so $\tilde{H}_i(|L|) = 0$ for all $i$; since $S^{n-1} - \{P\}$ is contractible. Since $|L| \not= \emptyset$, it follows from the reduced homology sequence of the pair $(|K|, |L|)$ that $\tilde{H}_i(|K|, |L|) = 0$ for all $i$, completing the proof.
Theorem 12.16. Let $K$ be a regular CW complex. Suppose $\mathbb{E}_A \{ A \in \text{cell}(K) \}$ is a choice of orientations of the cells of $K$. Then, if $n > 0$, $A \in \text{cell}_n(K)$, $B \in \text{cell}_{n-1}(K)$, we have

\[
[\mathbb{E}_A : \mathbb{E}_B] = \begin{cases} 
\pm 1 & \text{if } B \subset A \\
0 & \text{if } B \not\subset A
\end{cases}
\]

Proof. Let $A \in \text{cell}_n(K)$ and let $i : A_0 \to K$ be the inclusion. We have $\text{cell}(A_0) \subset \text{cell}(K)$. By 12.5, for each $E \in \text{cell}(A_0)$ there exists a unique orientation $\mathbb{E}_E$ of $E$ in $A_0$ such that $C_n(i)(\mathbb{E}_E) = \mathbb{E}_E$. Then

\[
C_{n-1}(i) \sum_{B \in \text{cell}_{n-1}(A_0)} [\mathbb{E}_A : \mathbb{E}_B] \mathbb{E}_B = \sum_{B \in \text{cell}_{n-1}(A_0)} [\mathbb{E}_A : \mathbb{E}_B] \mathbb{E}_B.
\]

Thus, if $B \in \text{cell}_{n-1}(K)$ we have

\[
[\mathbb{E}_A : \mathbb{E}_B] = \begin{cases} 
[\mathbb{E}_A : \mathbb{E}_B] & \text{if } B \subset \text{cell}_{n-1}(A_0) \\
0 & \text{otherwise}
\end{cases}
\]

The result now follows since $B \subset \text{cell}_{n-1}(A_0)$ if and only if $B \subset A_0$, and $[\mathbb{E}_A : \mathbb{E}_B] = \pm 1$ for $B \subset \text{cell}_{n-1}(A_0)$ by 12.15.

Let $K$ be a CW complex, and $\mathbb{E}_A \{ A \in \text{cell}(K) \}$ a choice of orientations of the cells of $K$. Then, for $A \in \text{cell}_n(K)$, $n > 0$, we have

\[
0 = \partial \mathbb{E}_A = \partial \left( \sum_{B \in \text{cell}_{n-1}(K)} [\mathbb{E}_A : \mathbb{E}_B] \mathbb{E}_B \right)
\]

\[
= \sum_{B \in \text{cell}_{n-1}(K)} \sum_{C \in \text{cell}_n(K)} [\mathbb{E}_A : \mathbb{E}_B][\mathbb{E}_B : \mathbb{E}_C] \mathbb{E}_B
\]

\[
= \sum_{C \in \text{cell}_n(K)} \left( \sum_{B \in \text{cell}_{n-1}(K)} [\mathbb{E}_A : \mathbb{E}_B][\mathbb{E}_B : \mathbb{E}_C] \right) \mathbb{E}_C
\]
Thus, for each $A \in \text{cell}_n(K)$ and $C \in \text{cell}_{n-2}(K)$ we have
\[
\sum_{B \in \text{cell}_{n-1}(K)} [A:B][B:C] = 0.
\]

Also, by 12.9 we can choose orientations of the $C$-cells such that $E(C_B) = 1$ for all $B \in \text{cell}_0(K)$. Assuming this is done, we have, for each $A \in \text{cell}_1(K)$,
\[
0 = E(A) E \left( \sum_{B \in \text{cell}_1(K)} [A:B][B:C] \right) = \sum_{B \in \text{cell}_1(K)} [A:B][B:C].
\]

**Definition 12.17.** Let $K$ be a regular CW complex. An **orientation system** $[\cdot : \cdot]$ on $K$ is a function
\[
[\cdot : \cdot] : \bigcup_{n \geq 1} \text{cell}_n(K) \times \text{cell}_{n-1}(K) \to \mathbb{Z}
\]
(calling $[A:B]$ to $(A,B)$) satisfying:

1) For all $n \geq 1$ and $A \in \text{cell}_n(K)$, $B \in \text{cell}_{n-1}(K)$,
\[
[A:B] = \begin{cases} 
+1 & \text{if } B \subseteq A \\
0 & \text{otherwise}
\end{cases}
\]

2) If $n \geq 2$, then for all $A \in \text{cell}_n(K)$, $C \in \text{cell}_{n-2}(K)$
we have
\[
\sum_{B \in \text{cell}_{n-1}(K)} [A:B][B:C] = 0.
\]

3) If $A \in \text{cell}_1(K)$, then
\[
\sum_{B \in \text{cell}_0(K)} [A:B] = 0.
\]

**Example 12.18.** If $K$ is a regular CW complex and
\[
\{E_A | A \in \text{cell}(K)\}
\]
is a choice of the orientations of the cells of $K$ such that $E(C_B) = 1$ for each $A \in \text{cell}_1(K)$, then
\[
[\cdot : \cdot] : \bigcup_{n \geq 1} \text{cell}_n(K) \times \text{cell}_{n-1}(K) \to \mathbb{Z}
\]
given by
Example 12.19. Let $K$ be the CW complex pictured below:

![Diagram of a CW complex](image)

\[
\text{cell}_0(K) = \{P_1, P_2\}, \quad \text{cell}_1(K) = \{E_1, E_2\}, \quad \text{cell}_2(K) = \{F\}
\]

and \(\text{cell}_n(K) = \emptyset\) for \(n > 2\).

\(K\) is a regular CW complex.

Our next goal is to show that any incidence system on a regular CW complex \(K\) can be realized as the incidence numbers for a choice of orientation of the cells of \(K\).

**Exercise 12.20.** Let \(K\) be a regular CW complex and suppose \([I : J]\) is an incidence system on \(K\). For \(n \geq 0\), let \(G_n\) be the free abelian group on the set \(\text{cell}_n(K)\). For \(n \geq 1\), define \(\partial : G_n \to G_{n-1}\) by

\[
\partial A = \sum_{B \in \text{cell}_{n-1}(K)} [A : B] B
\]

for each \(A \in \text{cell}_n(K)\), and \(\varepsilon : G_0 \to \mathbb{Z}\) by \(\varepsilon(A) = 1\) for each \(A \in \text{cell}_0(K)\). Then the \(G_n, \partial, \varepsilon\) are above constitute an augmented chain complex.

**Example 12.21.** In Example 12.19 above we have

\[
\partial E_1 = P_2 - P_1, \quad \partial E_2 = P_1 - P_2, \quad \partial F = E_1 + E_2.
\]

Note that

\[
\{P_1, P_2 - P_3\} \text{ is a free abelian basis for } G_1. \quad \text{We have } \text{ker} \varepsilon = G_0, \quad \text{Bker} \varepsilon = \text{subgroup of } G_0 \text{ generated by } P_2 - P_1. \quad \text{Thus}
\]

\[
\text{H}_0(C) = \text{free abelian group on } P_1, \quad \frac{\text{ker} \varepsilon}{\text{Bker} \varepsilon} = \mathbb{Z}.
\]
We have 
\[
\partial(mE_1 + nE_2) = m(P_2 - P_1) + n(P_1 - P_2) = (m-n)(P_2 - P_1),
\]
Thus \( mE_1 + nE_2 \in Z_1(C) \) if and only if \( m = n \). Thus \( Z_1(C) \) is the subcomplex of \( C_1 \) generated by \( E_1 + E_2 \). Since \( E_1 + E_2 = \partial F \), we have \( B_1(C) = Z_1(C) \) and so \( H_1(C) = 0 \).

Note that \( \partial : C_2 \to C_1 \) is injective and so \( Z_2(C) = 0 \). Hence \( H_2(C) = 0 \). For \( n > 2 \), \( C_n = 0 \) and so \( H_n(C) = 0 \) for \( n > 2 \).

Note that this agrees with the singular homology groups of \( |K| \).

**Theorem 12.22.** Let \( K \) be a regular CW complex and \( [\cdot : \cdot] \) an incidence system on \( K \). Then there exist orientations \( \frac{1}{2} E_A = \# \text{cell}(K) \) of the cells of \( K \) such that \([E_A : E_B] = [A : B] \) for all \( A \in \text{cell}_n(K) \), \( B \in \text{cell}_{n+1}(K) \), \( n \geq 1 \), and \( E(E_A) = 1 \) for all \( A \in \text{cell}_0(K) \).

**Proof.** Let \( C \) be the chain complex of 12.20. By 12.9 and 12.4 (1) we can choose, for each \( A \in \text{cell}_0(K) \), a unique orientation \( E_A \) of \( A \) in \( K \) such that \( E(E_A) = 1 \).

Suppose \( A \in \text{cell}_1(K) \). Since \( K \) is regular, there are exactly two \( C \)-cells, \( B_0, B_1 \in \text{cell}_1(K) \) contained in \( A \). Thus for any orientation \( E \) of \( A \) in \( K \) we must have \( \partial E = n_0E_{B_0} + n_1E_{B_1} \), where \( n_0 = \pm 1 \). We have \( 0 = E(\partial E) = n_0E_{B_0} + n_1E_{B_1} \) or \( \partial E = E_{B_1} - E_{B_0} \). Since \([A : B_0] = [A : B_1] = 0 \) and the \([A : B_i] \) are \( \pm 1 \), there is a unique choice \( E_A \) for an orientation of \( A \) in \( K \) such that \( \partial E_A = [A \cdot B_0]E_{B_0} + [A \cdot B_1]E_{B_1} \).

Thus we have orientations \( E_A \) for \( A \in \text{cell}_1(K) \) such that \([E_A : E_B] = [A : B] \) for all \( B \in \text{cell}_0(K) \).

Let \( n > 1 \) and assume, inductively, orientations \( E_A \) for cells \( A \) of \( K \) of dimension \( < n \) have been chosen such that whenever \( 1 \leq i \leq n-1 \) and \( A \in \text{cell}_i(K) \), \( B \in \text{cell}_{i+1}(K) \) we have \([E_A : E_B] = [A : B] \). For \( 0 \leq i \leq n-1 \), define \( f_i : C_i \to C_{i+1}(K) \) by \( f_i(A) = E_A \) for each \( A \in \text{cell}_i(K) \). If \( A \in \text{cell}_{n-1}(K) \), then \( \partial f_{n-1}(A) = \partial E_A = \sum_{B \in \text{cell}_n(K)} [E_A : E_B]E_B \)
the diagram
\[
\begin{array}{ccc}
C_{n-1} & \xrightarrow{f_{n-1}} & C_{n-1}(K) \\
\downarrow & & \downarrow \\
C_{n-2} & \xrightarrow{f_{n-2}} & C_{n-2}(K)
\end{array}
\]

commutes.

Let \( A \in \text{cell}_n(K) \), and let \( j : A \to K \) denote the inclusion. By 12.5 and 12.4 (1), for each \( B \in \text{cell}_{n-1}(A) \) there is a unique orientation \( \tilde{e}_B \) of \( B \) in \( A \) such that \( C_n(1)(\tilde{e}_B) = \tilde{e}_B \). For the chain complex \( C \) we have

\( \partial A = \sum_{B \in \text{cell}_{n-1}(K)} [A:B] B = \sum_{B \in \text{cell}_{n-1}(A)} [A:B] B \) since

\( [A:B] = 0 \) if \( B \neq A \). Thus \( \partial \left( \sum_{B \in \text{cell}_{n-1}(A)} [A:B] B \right) = 0 \)

and so, from commutativity of the above diagram,

\[ \partial \left( \sum_{B \in \text{cell}_{n-1}(A)} [A:B] \tilde{e}_B \right) = \partial f_{n-1} \left( \sum_{B \in \text{cell}_{n-1}(A)} [A:B] B \right) \]

\[ = f_{n-2} \partial \left( \sum_{B \in \text{cell}_{n-1}(A)} [A:B] B \right) = 0, \quad \therefore \quad \sum_{B \in \text{cell}_{n-1}(A)} [A:B] \tilde{e}_B \in Z_{n-1}(C(A)) \).

By regularity,

\[ C_{n-1}(K, [A]) = A \cong D^n \] and so \( H_{n-1}(C(A)) \cong H_{n-1}(D^n) = 0 \).

Thus, \( Z_{n-1}(C(A)) = B_{n-1}(C(A)) \). Since \( C_n(A) \) is a free abelian on one generator, it follows that \( B_{n-1}(C(A)) \) is generated by a single element \( b \). Thus

\[ \sum_{B \in \text{cell}_{n-1}(A)} [A:B] \tilde{e}_B = q \cdot b \] for some \( q \in \mathbb{Z} \). Since

\[ \{ \tilde{e}_B | B \in \text{cell}_{n-1}(A) \} \] is a free abelian basis of \( C_{n-1}(A) \), we
must have $q| [A:B]$ for each $B \in \text{cell}_{n-1}(A)$. Since all these $[A:B]$ are $\pm 1$, we must have $q = \pm 1$. Thus $B_{n-1}(CA)\) is generated by $\sum_{B \in \text{cell}_{n-1}(A)} [A:B] \tilde{e}_B$. But

$$\sum_{B \in \text{cell}_{n-1}(A)} [A:B] \tilde{e}_B = 0,$$

and so we must have $r [A:B] = [\tilde{A}:\tilde{B}]$ for each $B \in \text{cell}_{n-1}(A)$.

Thus, since $[A:B]$ and $[\tilde{A}:\tilde{B}]$ are $\pm 1$, we must have $r = \pm 1$. Thus there is a unique orientation $\tilde{C}_A$ of $A$ in $A$ such that $\tilde{C}_A = \sum_{B \in \text{cell}_{n-1}(A)} [A:B] \tilde{e}_B$.

Define $\tilde{C}_A = C_n(i)(\tilde{C}_A) \in C_n(K)$. Thus,

$$\tilde{C}_A = C_n(i)(\tilde{C}_A) = C_{n-1}(j) \tilde{C}_A \tilde{C}_A$$

$$= C_{n-1}(j) \sum_{B \in \text{cell}_{n-1}(A)} [A:B] \tilde{e}_B = \sum_{B \in \text{cell}_{n-1}(A)} [A:B] \tilde{e}_B$$

$$= \sum_{B \in \text{cell}_{n-1}(K)} [A:B] \tilde{e}_B \quad (\text{since } [A:B] = 0 \forall B \neq A).$$

Thus $[\tilde{C}_A : \tilde{C}_B] = [A:B]$ for all $B \in \text{cell}_{n-1}(K)$. Thus completes the induction.

Exercise 12.23. If $K$ is a regular CW complex and $\tilde{f} : \bigcup_{n \in \text{cell}_{n-1}(K)} \times \text{cell}_{n-1}(K) \to \mathbb{Z}$ satisfies conditions 1) and 2) of Definition 12.17, then condition 2) holds if and only if the composition $C_n \tilde{f} C_{n-1} \tilde{f} C_{n-2}$ are $0$ in $n \geq 2$ where $C_n$, $\tilde{f}$ as in 12.20.
Remark 12.24. As a consequence of 12.22, for any regular CW complex $K$ and incidence system $[\cdot:\cdot]$ on $K$, the chain complex $C$ of 12.20 is chain isomorphic to $C(K)$.

Example 12.25. Let $K$ be the regular CW complex of Example 12.6 (the hemispherical cell decomposition of $S^n$). The following is an incidence system on $K$:

$$[e_i^+: e_i^-] = [e_i^-: e_i^+] = 1 \quad \text{for} \quad 1 \leq i \leq n,$$

$$[e_{i+1}^+: e_i^-] = [e_i^-: e_{i+1}^+] = \begin{cases} 1 & \text{if } 2 \leq i \leq n \text{, } i \text{ even} \\ -1 & \text{if } 1 \leq i \leq n \text{, } i \text{ odd} \end{cases}$$

To check this, conditions 1) and 3) of 12.17 clearly hold. To check 2), let $C$ and $\partial$ be as in 12.20 relative to this $[\cdot:\cdot]$. Note that for $2 \leq 2i \leq n$,

$$\partial e_{2i} = \partial e_{2i}^- = e_{2i-1} + e_{2i-1}^-,$$

and for $1 \leq 2i+1 \leq n$,

$$\partial e_{2i+1} = -\partial e_{2i+1}^- = e_{2i}^+ - e_{2i}^-.$$ Thus

$$\partial \partial e_{2i} = \partial \partial e_{2i}^- = \partial e_{2i-1}^+ + \partial e_{2i-1}^- = 0 \quad \text{for } 2 \leq 2i \leq n,$$

and

$$\partial \partial e_{2i+1} = -\partial \partial e_{2i+1}^- = \partial e_{2i}^+ - \partial e_{2i}^- = 0 \quad \text{for } 3 \leq 2i+1 \leq n.$$ Thus by 12.23, $[\cdot:\cdot]$ is an incidence system on $K$.

Thus by 12.24, $H_i(S^n) \cong H_i(C)$ for all $i$. We leave it as an exercise to calculate the $H_i(C)$ directly to check this last statement.

13. Cellular Quotients of Regular Complexes

Lemma 13.1. Let $K$ and $L$ be CW complexes (not necessarily regular) and $f: K \to L$ a CW map. Suppose $A \in \text{cell}_n(K)$, $B \in \text{cell}_n(L)$ are such that $f|_A$ maps $A$ homeomorphically onto $B$. Let $\sigma_A$ be an orientation of $A$ in $K$. Then $\gamma^n(f)(\sigma_A)$ is an orientation of $B$ in $L$. 
Prof. Since $\bar{A}$ is compact and $|L|$ is Hausdorff, we have $1f1(\bar{A})$ is closed in $|L|$. Thus, since $B \subseteq 1f1(\bar{A})$, we have $\bar{B} \subseteq 1f1(\bar{A})$. On the other hand, $A \subseteq 1f1^{-1}(\bar{B})$ and $1f1^{-1}(\bar{B})$ is closed in $|K|$. Hence $A \subseteq 1f1^{-1}(\bar{B})$, and so $1f1(\bar{A}) \subseteq \bar{B}$. Thus $1f1(\bar{A}) = \bar{B}$.

Let $f': (\bar{A}, \bar{A}) \rightarrow (\bar{B}, \bar{B})$ denote the map of pairs obtained by restriction of $1f1$. If $\chi_A: (D^n, S^{n-1}) \rightarrow (\bar{A}, \bar{A})$ is a characteristic map for $A$, then the composition \[ (D^n, S^{n-1}) \xrightarrow{\chi_A} (\bar{A}, \bar{A}) \xrightarrow{f'} (\bar{B}, \bar{B}) \] is a characteristic map for $B$. For both $\chi_A$ and $f'$ are relative homeomorphisms, and hence so is $f'\chi_A$, and $(f'\chi_A)^q: D^n \rightarrow \bar{B}$ is a quotient map since it is onto, $D^n$ is compact, and $\bar{B}$ is Hausdorff. It follows from 12.1 and 10.13 that both $\chi_A$ and $f'\chi_A$ are excision relative homeomorphisms. Thus $f'$ induces isomorphisms in homology.

We have the commutative diagram:

\[ \begin{array}{ccc}
(\bar{A}, \bar{A}) & \xrightarrow{f'} & (\bar{B}, \bar{B}) \\
\downarrow{\chi_A} & & \downarrow{\chi_B} \\
(k^{(n)}, k^{(n-1)}) & \xrightarrow{f_n} & (L^{(n)}, L^{(n-1)})
\end{array} \]

where $\chi_A, \chi_B$ are the inclusions and $f_n$ is given by restriction of $1f1$. Let $x$ be the generator of $H_n(\bar{A}, \bar{A})$, such that $H_n(\chi_A)(x) = O_A$. Since $H_n(f')$ is an isomorphism, $H_n(f')(x)$ is a generator of $H_n(\bar{B}, \bar{B})$, and so $H_n(\chi_B) H_n(f')(x)$ is an orientation $O_B$ of $B$ in $L$. By commutativity of the above diagram, we have $CH_n(f)(O_A) = H_n(\chi_B) H_n(f')(x) = O_B$, completing the proof.

**Definition 13.2.** A CW map $f: K \rightarrow L$ is called a cellular quotient map if

1) $1f1: |K| \rightarrow |L|$ is onto.

2) For $n \geq 0$ and each $A \in \text{cell}_n(K)$, $1f1$ maps $A$ homeomorphically onto a member of $\text{cell}_n(L)$. 


Example 13.3. Let \( f: K \to L \) be as illustrated:

\[
\begin{array}{c}
\begin{array}{c}
A_1 \\
B_1 \\
P_1 \\
A_2 \\
B_2 \\
P_3 \\
P_4 \\
F \\
\end{array}
\end{array}
\begin{array}{c}
P \\
B \\
F \\
P \\
A \\
P \\
B \\
\end{array}
\xrightarrow{f}
\begin{array}{c}
A \\
B \\
F \\
P \\
A \\
P \\
B \\
\end{array}
\]

1. \( K \) is a closed rectangle; \( 1L \) is a torus.
2. \( \text{cell}(K) = \{ P_1, P_2, P_3, P_4, A_1, A_2, B_1, B_2, F \} \)
3. \( \text{cell}(L) = \{ P, A, B, F \} \)
4. \( f \) is the standard quotient map of \( K \) onto \( 1L \) (see 6.20). \( f \) is a cellular quotient map.

Similarly, by reversing one arrow in the above drawing of \( L \), we obtain a cellular quotient map \( f: K \to L \) with \( K \) as above and \( 1L \) a Klein bottle.

Example 13.4. Let \( K \) be the CW complex of 10.6 (the hemispherical cell decomposition of \( S^n \)) and \( L \) the CW complex of Example 10.9 (\( 1L = \text{IR}^n \)). We then have a cellular quotient map \( f: K \to L \) where \( f|_L: S^n \to \text{IR}^n \) is given by \( f|_L([x_0, t_1, \ldots, t_n, 0, \ldots]) = [x_0, t_1, \ldots, t_n, 0, \ldots] \). For \( 0 \leq i \leq n \), \( f|_L \) maps each of \( e^+ \) and \( e^- \) homeomorphically onto \( e_i \).

To justify the terminology, we prove:

Proposition 13.5. Let \( f: K \to L \) be a cellular quotient map. Then \( f|_L: |K| \to |L| \) is a quotient map.

Proof. Let \( X \subseteq |L| \) be such that \( f|_L^{-1}(X) \) is closed in \( |K| \). To show that \( X \) is closed in \( |L| \), we must show that for each \( B \in \text{cell}(L) \), \( X \cap B \) is closed in \( B \).

Let \( B \in \text{cell}(L) \) and choose any \( A \in \text{cell}(K) \) such that \( f|_L \) maps \( A \) homeomorphically onto \( B \). Then, by the first paragraph of the proof of 13.1, \( f|_L(A) = B \).

Thus, since \( A \) is compact and \( B \) is Hausdorff, the restriction of \( f|_L \) yields a quotient map.
If \( f : K \to L \) is a cellular quotient map and \( A \sim A' \) (say \( |f!(A)| = |f!(A')| = B \in \text{cell}(L) \)), then if \( \sigma_A \) and \( \sigma_A' \) are orientations of \( A \) and \( A' \), respectively, in \( K \), it follows from 13.1 that \( C_n(f)(\sigma_A) \) and \( C_n(f)(\sigma_A') \) are orientations of \( B \) in \( L \), and so \( C_n(f)(\sigma_A) = \pm C_n(f)(\sigma_A') \).

Definition 13.7. Let \( f : K \to L \) be a cellular quotient map, and \( \mathcal{O} = \{ \sigma_A | A \in \text{cell}(K) \} \) a choice of orientations of the cells of \( K \). We say \( \mathcal{O} \) is \( f \)-compatible if whenever \( A, A' \in \text{cell}(K) \) are such that \( A \sim A' \), then \( C_n(f)(\sigma_A) = C_n(f)(\sigma_A') \).

For example, if \( \mathcal{O} = \{ \sigma_B | B \in \text{cell}(L) \} \) is a choice of orientations of the cells of \( L \), then for each \( A \in \text{cell}(K) \), choose \( \sigma_A \) to be the unique orientation of \( A \) in \( K \) such that \( C_n(f)(\sigma_A) = \sigma_{f!(A)}. \) Then \( \mathcal{O} = \{ \sigma_A | A \in \text{cell}(K) \} \) is \( f \)-compatible.

Conversely, if \( \mathcal{O} = \{ \sigma_A | A \in \text{cell}(K) \} \) is a given \( f \)-compatible set of orientations of the cells of \( K \), for each \( B \in \text{cell}(L) \) we obtain a well-defined orientation \( \sigma_B \) of \( B \) in \( L \) by choosing any \( A \in \text{cell}(K) \) such that \( |f!(A)| = B \) and taking \( \sigma_B = C_n(f)(\sigma_A) \). Thus if \( \mathcal{O} = \{ \sigma_A | A \in \text{cell}(K) \} \) is \( f \)-compatible and the boundary homomorphisms in \( C(K) \) are known (i.e. the incidence numbers relative to the orientations in \( \mathcal{O} \) are known), then the boundary homomorphisms in \( C(L) \) are easily deduced; namely if \( A \in \text{cell}(K) \), \( n \geq 1 \), then
\[ e_{k+1}(A) = e_n(t)(\Theta_A) = C_{n-1}(t) \Theta_A = C_{n-1}(t) \sum_{A' \in \text{cell}_{n-1}(k)} [\Theta_A : \Theta_{A'}] \Theta_{A'} \]

Thus it is desirable to have a convenient way to detect when a set of orientations \( \Theta \) is \( f \)-compatible. This will be accomplished for a special class of cellular quotient maps, defined below in 13.9.

**Definition 13.8.** A CW isomorphism is a CW map \( f : K \to L \) such that \( f^* \) is a homeomorphism, and for each \( n \geq 0 \) and \( A \in \text{cell}_n(K) \), \( f^* \) maps \( A \) homeomorphically onto a member of \( \text{cell}_n(L) \).

For example if \( |K| = |L| = [0,1] \) where

\[
K = \begin{array}{ccc}
  & A & \\
P_1 & \rightarrow & P_2
\end{array}
\]

\[
L = \begin{array}{ccc}
  & B & C \\
P_1 & \rightarrow & P_2 & \rightarrow & P_3
\end{array}
\]

\( [0,1] \) is the underlying map of a CW map \( f : K \to L \) with \( f^* \) a homeomorphism, but \( f \) is not a CW isomorphism. Note that a CW isomorphism \( f : K \to L \) establishes a bijection between \( \text{cell}_n(K) \) and \( \text{cell}_n(L) \) for all \( n \geq 0 \).

**Definition 13.9.** A regular cellular quotient map \( f : K \to L \) is a cellular quotient map such that

1) \( |K| \) is a regular CW complex.

2) Whenever \( A, B \in \text{cell}(K) \) are \( f \)-equivalent, there exists a CW isomorphism \( \phi_{A,B} : A \to B \) such that the diagram

\[
\begin{array}{ccc}
  A_K & \xrightarrow{\phi_{A,B}} & B_c \\
  \downarrow j_A & & \downarrow j_B \\
  K & \xrightarrow{f} & L
\end{array}
\]

commutes where \( f_{A,B} \) are the inclusions (i.e. the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f_{A,B}} & B \\
\downarrow & & \downarrow \\
\overline{A} & \xrightarrow{f_{A,B}} & \overline{B}
\end{array}
\]

commutes). Note that \( f_{A,B} \) is uniquely determined since \( f \) maps both \( A \) and \( B \) homeomorphically onto \( f(A) \), and \( A \) is dense in \( \overline{A} \). The \( f_{A,B} \) are called the cell identifications induced by \( f \).

Example 13.10. Example 13.3 is a regular cellular quotient map. \( P_i \mapsto P_i \) for all \( i,j \) and the unique \( \omega \) maps \( (P_i)_c \mapsto (P_j)_c \) are the cell identifications among the \( 0 \)-cells. We have \( A_1 \mapsto A_2, B_1 \mapsto B_2 \). The cell identification \( f_{A_1,A_2} : (A)_c \mapsto (A_2)_c \) sends \( P_i \mapsto P_i \), \( P_1 \mapsto P_3 \), and the cell identification \( f_{B_1,B_2} : (B)_c \mapsto (B_2)_c \) sends \( P_2 \mapsto P_1 \), \( P_3 \mapsto P_4 \).

Notation 13.11. Suppose \( f : K \to L \) is a regular cellular quotient map, suppose \( n > 0 \) and \( A_A', A \in \text{cell}_n(K) \) are such that \( A \mapsto A' \). If \( B, B \in \text{cell}_{n-1}(K) \) are such that \( B \subset A, B \subset A' \), and \( f_{A,A}(B) = B' \) where \( f_{A,A} \) is the cell identification induced by \( f \), we write \( (A,B) \sim (A',B') \).

Thus in Example 13.3 we have \( (A_1,P_1) \sim (A_2,P_1) \), \( (A_1,P_2) \sim (A_2,P_3) \), \( (B_1,P_2) \sim (B_2,P_1) \), \( (B_1,P_3) \sim (B_2,P_4) \).

Theorem 13.12. Let \( f : K \to L \) be a regular cellular quotient map. Let \( O = \{ O_A | A \in \text{cell}(K) \} \) be a choice of orientations of the cells of \( K \) such that:

1) For each \( A \in \text{cell}_0(K) \), \( O(A) = 1 \).

2) Whenever \( (A,B) \sim (A',B') \), we have \( [O_A : O_B] = [O_{A'} : O_{B'}] \).

Then \( O \) is \( f \)-compatible.
Before proving 13.12, we give two examples.

Example 13.13. In Example 13.3, we will choose an $f$-compatible set of orientations of the cells of $K$, and use this to calculate the homology of $1K$, the torus.

To shorten writing, we will abuse notation and denote a cell, and a chosen orientation for it, by the same symbol (e.g., we will write $A_i$ for $O_{A_i}$).

Orient the 0-cells so that $e(P_i) = 1$, $1 \leq i \leq 4$. We may orient $A_1$ (i.e., choose an orientation of $A_1$) so that

$\partial A_1 = P_2 - P_1$.

Then since $(A_1, P_1) \sim_+(A_2, P_2)$ and $(A_1, P_2) \sim_+(A_2, P_3)$, we are forced to orient $A_2$ so that

$[A_2 : P_4] = [A_1 : P_2] = -1$, $[A_2 : P_3] = [A_1 : P_3] = 1$, i.e.

$\partial A_2 = P_3 - P_4$.

Similarly, we may orient $B_1$ so that $\partial B_1 = P_3 - P_2$.

This then forces us to orient $B_2$ so that $\partial B_2 = P_4 - P_3$.

We may orient $F$ so that $[F : A_1] = 1$. Then the condition $\partial F = 0$ forces

$0 = \partial \partial F = \partial \{ [F : A_2] A_2 + [F : B_1] B_1 + [F : B_2] B_2 \}$

$= P_2 - P_1 + [F : A_2] (P_3 - P_4) + [F : B_1] (P_3 - P_2) + [F : B_2] (P_4 - P_1)$

$= (-1 - [F : B_2]) P_1 + (1 - [F : B_1]) P_2 + ([F : A_2] + [F : B_1]) P_3$

$+ (-1 - [F : A_2] + [F : B_2]) P_4$. Thus we must have


$\partial F = A_1 - A_2 + B_1 - B_2$.

The above satisfies the requirements of 13.12. We then can orient the cells of $1K$ by taking

$P = C_0(f)(P_2)$, $1 \leq i \leq 4$,

$A = C_1(f)(A_i)$, $i = 1, 2$,

$B = C_1(f)(B_i)$, $i = 1, 2$,

$F = C_2(f)(F)$.

Then

$\partial A = D_C(f)(A_1) = C_0(f) \partial A_1 = C_0(f)(P_2 - P_1) = P - P = 0$,

$\partial B = D_C(f)(B_1) = C_0(f) \partial B_1 = C_0(f)(P_3 - P_2) = P - P = 0$,

$\partial F = D_C(f)(F) = C_1(f) \partial (F) = C_1(f)(A_1 - A_2 + B_1 - B_2) = A - A + B - B = 0$.

Thus all the boundary homomorphisms in $C(1K)$ are 0, and so

$H_0(1K) \cong C_0(1K) \cong \mathbb{Z}$,

$H_1(1K) \cong C_1(1K) \cong \mathbb{Z} \oplus \mathbb{Z}$,

$H_2(1K) \cong C_2(1K) \cong \mathbb{Z}$,
\[ H_n(\mathbb{L}) \cong C_n(L) = 0 \text{ for } n > 1. \]

**Example 13.14:** We give a similar treatment for the Klein bottle. We have a regular cellular quotient map \( f: K \to L \) as pictured:

This time, \( |\mathbb{L}| = \text{Klein bottle}. \)

Orient the \( P_i \) so that \( v(P_i) = 1, 1 \leq i \leq 4 \). We may orient \( A \) so that \( \partial A_1 = P_2 - P_1 \). This time we have \((A_1, P_1) \neq (A_2, P_2), (A_1, P_2) \neq (A_2, P_4)\) and so we must orient \( A_2 \) so that \( \partial A_2 = P_4 - P_3 \). We may orient \( B_1 \) so that \( \partial B_1 = P_3 - P_2 \). This then forces us to orient \( B_2 \) so that \( \partial B_2 = P_4 - P_1 \).

We may orient \( F \) so that \([F: A_1] = 1\). Then the condition \( \partial \mathcal{F} = 0 \) yields

\[
0 = \partial \mathcal{F} = \partial \left( A_1 + [F:A_1]A_2 + [F:B_1]B_1 + [F:B_2]B_2 \right)
\]

\[
= P_2 - P_1 + [F:A_2](P_4 - P_3) + [F:B_1](P_3 - P_2) + [F:B_2](P_4 - P_1)
\]

\[
= (1 - [F:B_2])P_1 + (1 - [F:B_1])P_2 + ([F:B_1] + [F:B_2])P_3 + ([F:A_2] + [F:B_2])P_4
\]

and so \([F:B_2] = -1, [F:B_1] = 1, [F:A_2] = [F:B_2] = 1\). Thus \( \partial \mathcal{F} = A_1 + A_2 + B_1 - B_2 \).

The above satisfies the requirements of 13.12. Thus, we can orient the cells of \( L \) so that \( P = C_0 ^{(i)}(P_i) \) for \( 1 \leq i \leq 4 \), \( A = C_1 ^{(i)}(A_i) \) for \( i = 1 \) and \( 2 \), \( B = C_2 ^{(i)}(B_i) \) for \( i = 1 \) and \( 2 \), and \( \widetilde{F} = C_2 ^{(i)}(F) \) to satisfy \( \partial \mathcal{F} = \partial \widetilde{F} = \partial A \).

We have \( \partial \widetilde{F} = \partial \mathcal{F} = \partial A = 0 \). We have \( \partial \widetilde{A} = \partial \mathcal{A} = \partial A = 0 \). Thus \( \mathcal{Z}_0(\mathcal{L}) = C_0(\mathcal{L}) \), \( \mathcal{B}_0(\mathcal{L}) = 0 \), and so \( \mathcal{H}_0(\mathcal{L}) \cong C_0(\mathcal{L}) \cong \mathbb{Z} \).

\( \mathcal{Z}_1(\mathcal{L}) = C_1(\mathcal{L}) \) is free abelian on \( \{A_1, B_3\} \), \( \mathcal{B}_1(\mathcal{L}) \) is a subgroup generated by \( 2A \). Thus \( \mathcal{H}_1(\mathcal{L}) \cong \mathcal{H}_1(\mathcal{L}) \cong \text{free abelian on } \{A_1, B_3\} \cong \mathbb{Z} \). \( \text{subgroup generated by } 2A \).

\( \partial : C_2(\mathcal{L}) \to C_1(\mathcal{L}) \) is injective, and so \( \mathcal{H}_2(\mathcal{L}) = 0 \).

For \( n > 2 \), \( C_n(\mathcal{L}) = 0 \) and so \( \mathcal{H}_n(\mathcal{L}) = 0 \).
Proof of 13.12. We must show that whenever $A, A' \in \text{cell}_n(K)$ are such that $A \sim A'$, then $C_n(f)(\overline{\sigma}_A) = C_n(f)(\overline{\sigma}_{A'})$.

For each $n > 0$ and $A \in \text{cell}_n(K)$, let $\overline{\sigma}_A \in C_n(A_c)$ be the unique orientation of $A$ in $A_c$ such that $C_n(j_A)(\overline{\sigma}_A) = \overline{\sigma}_A$ where $j_A : A_c \to K$ is the inclusion.

Suppose $A, A' \in \text{cell}_n(K)$ are such that $A \sim A'$, and let $\phi_{A,A'} : A_c \to A_c'$ be the cell identification induced by $f$. From commutativity of

$$
\begin{array}{ccc}
A_c & \xrightarrow{\phi_{A,A'}} & A_c' \\
\downarrow j_A & & \downarrow j_A' \\
K & \xrightarrow{f} & K \\
\end{array}
$$

we have $C_n(f)(\overline{\sigma}_A) = C_n(f)C_n(j_A)(\overline{\sigma}_A) = C_n(f)C_n(j_A')C_n(\phi_{A,A'})(\overline{\sigma}_{A'})$, while $C_n(f)(\overline{\sigma}_{A'}) = C_n(f)C_n(j_A')(\overline{\sigma}_{A'})$. Thus it suffices to show

1. Whenever $n > 0$ and $A, A' \in \text{cell}_n(K)$ are such that $A \sim A'$, then $C_n(\phi_{A,A'})(\overline{\sigma}_A) = \overline{\sigma}_{A'}$, where $\phi_{A,A'}$ is the cell identification induced by $f$.

Note that for $A, A'$ as above, $C_n(\phi_{A,A'})(\overline{\sigma}_A)$ is an orientation of $A'$ in $A_c'$ since $\phi_{A,A'}$ maps $A$ homeomorphically onto $A'$. Thus $C_n(\phi_{A,A'})(\overline{\sigma}_A) = \lambda_{A,A'}\overline{\sigma}_{A'}$, where $\lambda_{A,A'} = \pm 1$.

We prove (1) by induction on $n$. If $A, A' \in \text{cell}_0(K)$ are such that $A \sim A'$, then by assumption $E(\overline{\sigma}_A) = E(\overline{\sigma}_{A'}) = 1$. By naturality of $E$ we have $E(\overline{\sigma}_A) = ECo(j_A)(\overline{\sigma}_A) = E(\overline{\sigma}_A) = 1$, and similarly $E(\overline{\sigma}_{A'}) = 1$. Thus, by naturality of $E$, $1 = E(\overline{\sigma}_A) = ECo(\phi_{A,A'})(\overline{\sigma}_A) = E'(\lambda_{A,A'} \overline{\sigma}_{A'})$

$= \lambda_{A,A'} E'(\overline{\sigma}_{A'}) = \lambda_{A,A'}$, and so $Co(\phi_{A,A'})(\overline{\sigma}_A) = \overline{\sigma}_{A'}$.

Let $n > 0$ and assume inductively that the result holds for cells of lower dimension.
If $A_c \text{cell}_n(K)$ and $B_c \text{cell}_n(A_c)$, let $B_c \xrightarrow{i_b} A_c$ be the inclusion.

$B_c = \text{cell}_n(i_b) \preceq \text{cell}_n(A_c)$, where $i_b : B_c \to A_c$ is the inclusion.

Then $B_c$ is an orientation of $K$ in $A_c$. We have the commutative diagram $\xymatrix{ B_c \ar[r]^{i_b} \ar[d]^{j_b} & A_c \ar[d]^{j_A} \\ K }$

Hence $C_{n-1}(i(A_c)|i(A_c) - (\tilde{A}_c) = C_{n-1}(i(A_c)|i(A_c) - \sum_{B_c \text{cell}_n(A_c)} \tilde{B}_c \to \tilde{A}_c}$.

On the other hand, we have $C_{n-1}(i(A_c)|i(A_c) - \sum_{\text{cell}_n(A_c)} B_c = \sum_{\text{cell}_n(A_c)} \tilde{B}_c \to \tilde{A}_c}$.

Thus we have $C_{n-1}(i(A_c)|i(A_c) - \sum_{\text{cell}_n(A_c)} B_c = \sum_{\text{cell}_n(A_c)} \tilde{B}_c \to \tilde{A}_c$.

Now suppose $A, A_c \text{cell}_n(K)$ with $A \subset A_c$. Write $\phi = \langle \phi_A, \phi_A \rangle$. By the functorial hypothesis, $C_{n-1}(\phi_A, \phi_A)(\tilde{A}_c) = \tilde{A}_c \phi \tilde{A}_c$.

We also have $B \to \phi(B)$ and $\phi(B) \to \tilde{A}_c$ commutes.

By the functorial hypothesis, $C_{n-1}(\phi_A, \phi_A)(\tilde{A}_c) = \tilde{A}_c \phi \tilde{A}_c$.
Thus, writing $\lambda = \lambda_{A',A'}$, we have

$$\lambda \circ (\overline{\sigma}_{A'}) = \lambda \circ (\lambda \overline{\sigma}_{A'}) = \lambda \circ C_n(\varphi_{A',A'})(\overline{\sigma}_{A}) = C_{n-1}(\varphi_{A,A'} \lambda) \circ (\overline{\sigma}_{A})$$

$$= C_{n-1}(\varphi_{A,A'}) \sum_{B \in \text{cell}_{n-1}(A_c)} [\overline{\sigma}_{A'} : \hat{\sigma}_B] \hat{\sigma}_B = \sum_{B \in \text{cell}_{n-1}(A_c)} [\overline{\sigma}_{A'} : \hat{\sigma}_B] \hat{\sigma}_{\varphi(B)}$$

$$= \sum_{B \in \text{cell}_{n-1}(A_c)} [\overline{\sigma}_{A'} : \hat{\sigma}_{\varphi(B)}] \hat{\sigma}_{\varphi(B)} = \lambda(\overline{\sigma}_{A'}).$$

Thus $$(\lambda - 1) \lambda(\overline{\sigma}_{A'}) = 0.$$ Since $\lambda(\overline{\sigma}_{A'}) \neq 0$ (since $A'$ is regular and $n > 0$) and $C_{n-1}(A_c)$ has no torsion, we must have $\lambda = 1$, completing the induction.

Example 13.15. For $n > 0$, the double covering $\pi: S^n \to \mathbb{RP}^n$ given by $\pi(t_0, t_1, \ldots, t_n, 0, \ldots) = [t_0, t_1, \ldots, t_n, 0, \ldots]$ is the underlying map of a regular cellular quotient map $f: K \to L$ where $K$ is the hemispherical cell decomposition of $S^n$ (Example 10.6) and $L$ is the CW complex of Example 10.9. For $0 \leq i \leq n$, $\pi$ maps each of $e_i^+$ and $e_i^-$ homeomorphically onto $e_i$. The cell identification $\varphi_{e_i^+, e_i^-}: (e_i^+)_{c} \to (e_i^-)_{c}$ is given by $\varphi_{e_i^+, e_i^-}(x) = -x$. Thus for $1 \leq i \leq n$,

$$(e_i^+, e_i^-) \sim (e_i^-, e_i^-) \text{ and } (e_i^+, e_i^-) \sim (e_i^-, e_i^-).$$

Thus by 13.12, the incidence system on $K$ of Example 12.25 is the underlying incidence system of an $f$-compatible choice of orientations of the cells of $K$.

Again, we abuse notation and use the same symbol for a cell and an orientation of that cell. Thus, using the orientations arising from 12.25, we can orient the cells of $L$ so that for $0 \leq i \leq n$,

$$C_i(f)(e_i^+) = C_i(f)(e_i^-) = e_i.$$

Thus, since for $2 \leq i \leq n$ we have $\partial e_{2i} = e_{2i-1} + e_{2i-1}^-$, we have $\partial e_{2i} = \partial C_{2i-1}(f)(e_{2i-1}^+) = C_{2i-1}(f)\partial(e_{2i-1}^+) = C_{2i-1}(f)(e_{2i-1}^+ + e_{2i-1}^-) = e_{2i-1} + e_{2i-1} = 2e_{2i-1}$. Similarly, since $1 \leq 2i+1 \leq n$ we have $\partial e_{2i+1} = e_{2i+1}^+ - e_{2i}$ and so $\partial e_{2i+1} = \partial C_{2i+1}(f)(e_{2i+1}^+) = C_{2i+1}(f)\partial(e_{2i+1}^+)$.
\[ C_2(\mathbb{Z}) (e_2^+ - e_2^-) = e_2^+ - e_2^- = 0. \text{ Thus} \]
\[ Z_0(C(L)) = \text{free abelian on } L, \]
\[ B_0(C(L)) = 0. \]

For \( 1 \leq i \leq n \), \( Z_i(C(L)) = \begin{cases} \text{free abelian on } L_i & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases} \)

\[ B_i(C(L)) = \begin{cases} \text{subgroup of } Z_i(C(L)) \text{ generated by } 2 \, E_i & \text{if } i \text{ is odd and } i < n \\ 0 & \text{otherwise} \end{cases} \]

Thus \( H_i(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/2 & \text{if } 1 \leq i < n-1 \text{ and } i \text{ odd} \\ 0 & \text{if } 2 \leq i \leq n \text{ and } i \text{ even} \\ \mathbb{Z} & \text{if } i = n \text{ and } n \text{ odd} \\ 0 & \text{if } i > n \end{cases} \)

**Corollary 13.16.**

\[ H_i(\mathbb{R}P^\infty) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/2 & \text{if } i > 1 \text{ and } i \text{ odd} \\ 0 & \text{otherwise} \end{cases} \]

13.16 is immediate from 10.24 and 11.20.

14. Tensor Products of Modules and of Chain Complexes

In this section we develop some algebra necessary to deal with the homology of product spaces, and homology with arbitrary coefficients. Throughout this section, \( R \) denotes a commutative ring with unit element \( 1 \neq 0. \)

**Definition 14.1.** An \( R \)-module \( A \) is an additive abelian group together with a rule which assigns
to each $r \in R$, $a \in A$ an element $ra \in A$ satisfying
1. $(r_1 + r_2)a = r_1a + r_2a$ for all $r_1, r_2 \in R$, $a \in A$.
2. $r(a_1 + a_2) = ra_1 + ra_2$ for all $r \in R$, $a_1, a_2 \in A$.
3. $(r_1r_2)a = r_1(r_2a)$ for all $r_1, r_2 \in R$, $a \in A$.
4. $1a = a$ for all $a \in A$.

Example 14.2. Every abelian group $A$ is a $\mathbb{Z}$-module
where $ra$ for $r \in \mathbb{Z}$, $a \in A$, $ra$ has its usual meaning.
If $F$ is a field, the term $F$-module means the same
thing as vector space over $F$.

Definition 14.3. If $A$ and $B$ are $R$-modules, an
$R$-homomorphism $f : A \to B$ is a homomorphism of
abelian groups such that $f(ra) = rf(a)$ for all $r \in R$, $a \in A$.

Example 14.4. Every homomorphism of abelian groups is a
$\mathbb{Z}$-homomorphism.
If $F$ is a field, $F$-homomorphism means the
same thing as linear transformation over $F$.

Exercise 14.5. Taking $R$-modules as objects and
$R$-homomorphisms as morphisms, we obtain a category
$\text{MR}_R$ the category of $R$-modules.

Exercise 14.6. 1) If $f : A \to B$ is an $R$-homomorphism which
is bijective, then $f^{-1} : B \to A$ is also an $R$-homomorphism.
In this case we say $A$ and $B$ are $R$-isomorphic or
isomorphi as $R$-modules $f$ and $f^{-1}$ are called
$R$-isomorphisms.

2) If $A$ is an $R$-module and $B$ an $R$-submodule of $A$ (i.e. a subgroup of $A$ such that $rt \in B$ for all $r \in R$,
$t \in B$), then the quotient group $A/B$ admits a well-defined
$R$-module structure with $R$-action given by $(ra + B) =
ra + B$ for all $r \in R$, $a \in A$. The inclusion map $i : B \to A$
and natural projection $p : A \to A/B$ given by $p(a) = a + B$, are
$R$-homomorphisms.

3) If $f : A \to B$ is an $R$-homomorphism, then $Ker f$
is an $R$-submodule of $A$, and $im f$ is an $R$-submodule
Let $B$. The function $\overline{f} : A/\ker f \to \operatorname{im} f$ given by

$$\overline{f}(a + \ker f) = f(a)$$

is well-defined, and is an $R$-isomorphism.

4) $R$ itself is an $R$-module using the addition in the ring $R$ for the module addition, and the multiplication in the ring $R$ to define the $R$-action.

The trivial group $0$ is an $R$-module.

5) If $\{A_x \mid x \in J\}$ is an indexed family of $R$-modules, then the direct sum of abelian groups

$$\bigoplus_{x \in J} A_x$$

admits a unique $R$-module structure such that for each $x \in J$, the standard inclusion

$$\iota_x : A_x \to \bigoplus_{x \in J} A_x$$

is an $R$-isomorphism.

If $B$ is an $R$-module and if for each $x \in J$ we are given an $R$-homomorphism $f_x : A_x \to B$, then the group homomorphism

$$\Sigma f_x : \bigoplus_{x \in J} A_x \to B$$

is an $R$-homomorphism.

\textbf{Definition 14.7.} An $R$-module $A$ is a \textbf{free} $R$-module if there is a subset $X \subseteq A$ such that each element of $A$ is expressible uniquely in the form $\sum_{x \in X} r_x x$ where $r_x \in R$ and all but finitely many of the $r_x$ are $0$. In this case $X$ is called an $R$-basis for $A$. (By convention, $0$ is a free $R$-module with basis $\emptyset$.)

\textbf{Example 14.8.} If $F$ is a field, every $F$-module is a free $F$-module. The term $F$-basis agrees with the usual linear algebra terminology.

$R$ is a free $R$-module with $R$-basis $e_1$.

\textbf{Exercise 14.9.} 1) If $A$ is a free $R$-module with $R$-basis $X$, then for any $R$-module $B$ and set function $f : X \to B$, there
exists a unique $R$-homomorphism $\tilde{f}: A \to B$ which extends $f$.

2) If \( \{ Ax \mid x \in J \} \) is an indexed family of free $R$-modules such that for each $x \in J$, $Ax$ is an $R$-basis for $Ax$, then $\bigoplus_{x \in J} Ax$ is a free $R$-module with $R$-basis $\bigcup_{x \in J} (x^*)$ where $x^*$ is the inclusion.

3) If $X$ is any set, let $F_R(X)$ denote the set of all formal sums $\sum_{x \in X} r_x X$ where $r_x \in R$ and all but finitely many of the $r_x$ are 0. $F_R(X)$ is an $R$-module in the obvious way. (By convention, $F_R(\emptyset) = 0$.) We regard $X$ as a subset of $F_R(X)$ by identifying $x \in X$ with the formal sum $1 \cdot x \in F_R(X)$. Then $F_R(X)$ is a free $R$-module with $R$-basis $X$. We call $F_R(X)$ the free $R$-module on the set $X$.

For each $x \in X$, let $f_x: R \to F_R(X)$ be the $R$-homomorphism given by $f_x(r) = r x$. Then

$$\sum f_x: \bigoplus_{x \in X} R \longrightarrow F_R(X)$$

is an $R$-isomorphism.

(\( \bigoplus_{x \in X} R \) means $\bigoplus_{x \in X} Ax$ where each $Ax = R$.)

**Definition 14.10.** Let $A$ and $B$ be $R$-modules. Form $F_R(A \times B)$, the free $R$-module on the set $A \times B$. Let $T(A, B)$ be the $R$-submodule of $F_R(A \times B)$ generated by all elements having one of the forms

$$(a_1 + a_2, b_1 + b_2) - (a_1, b_1) - (a_2, b_2),$$

$$(a, b_1 + b_2) - (a, b_1) - (a, b_2),$$

$$r(a, b) - (ra, b),$$

$$r(a, b) - (a, rb)$$

where $a_1, a_2, a \in A$, $b_1, b_2, b \in B$, and $r \in R$. The **tensor product** of $A$ and $B$, denoted $A \otimes_R B$, is the quotient $R$-module $F_R(A \times B)/T(A, B)$.

If $a \in A$, $b \in B$, we write $a \otimes b$ for the image of $(a, b)$ under the natural projection $F_R(A \times B) \to A \otimes_R B$. 


Thus $A \otimes_R B$ consists of all finite linear combinations (with coefficients in $R$) of symbols of the form $a \otimes b$, $a \in A$, $b \in B$, subject to the relations
\[
(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b,
\]
\[
a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2,
\]
\[
r(a \otimes b) = (ra) \otimes b = a \otimes rb
\]
where $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$, $r \in R$.

**Definition 14.11.** Let $A, B, C$ be $R$-modules. A function $f : A \times B \rightarrow C$ is said to be $R$-bilinear if
\[
f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b),
\]
\[
f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2),
\]
\[
f(ra, b) = f(a, rb) = rf(a, b)
\]
for all $a_1, a_2 \in A$, $b, b_1, b_2 \in B$, $r \in R$.

In 14.11, the set $A \times B$ is not to be regarded as an $R$-module, and $f$ is not a morphism of $R$-modules.

**Lemma 14.12.** Let $A$ and $B$ be $R$-modules. The function $\alpha : A \times B \rightarrow A \otimes_R B$ given by $\alpha(a, b) = a \otimes_R b$ is $R$-bilinear. Moreover, if $f : A \times B \rightarrow C$ is any $R$-bilinear function, there exists a unique $R$-homomorphism $g : A \otimes_R B \rightarrow C$ such that
\[
\begin{array}{ccc}
A \times B & \xrightarrow{\alpha} & A \otimes R B \\
\downarrow f & & \downarrow f \\
C & \xleftarrow{g} & C
\end{array}
\]
comutes.

We refer to 14.12 as the universal property for tensor products.

**Lemma 14.13.** Let $f : A \rightarrow A'$, $g : B \rightarrow B'$ be $R$-homomorphisms. Then the function $A \times B \rightarrow A' \otimes_R B'$, which sends $(a, b)$ to $f(a) \otimes g(b)$ is $R$-bilinear. Thus, by the universal property 14.12, there exists a unique $R$-homomorphism $A \otimes_R B \rightarrow A' \otimes_R B'$ which sends $a \otimes b$ to $f(a) \otimes g(b)$ for all $(a, b) \in A \times B$. We denote this $R$-homomorphism...
by \( f \otimes g : A \otimes_R B \to A' \otimes_R B' \), and call it the tensor product (over \( R \)) of \( f \) and \( g \).

Exercise 14.14. 1) If \( A \xrightarrow{f} A' \xrightarrow{f'} A'' \) and \( B \xrightarrow{g} B' \xrightarrow{g'} B'' \) are \( R \)-homomorphisms, then 
\[
(f' \otimes g')(f \otimes g) = (f' \otimes g')(f \otimes g).
\]
2) If \( A \) and \( B \) are \( R \)-modules, then \( 1_A \otimes 1_B = 1_{A \otimes_R B} \).

Let \( \mathcal{M}_R^2 \) denote the category of ordered pairs of \( R \)-modules, i.e., the objects of \( \mathcal{M}_R^2 \) are ordered pairs \((A, B)\) where \( A, B \) are \( R \)-modules, and a morphism \((A, B) \to (A', B')\) consists of an ordered pair \((f, g)\) where \( f : A \to A' \), \( g : B \to B' \) are \( R \)-homomorphisms. Composition is given by \((f, g)(f', g') = (f \circ f', g \circ g')\). It is trivial to check that \( \mathcal{M}_R^2 \) is a category.

The following is an easy consequence of 14.14:

Exercise 14.15. 1) The rule \( \otimes_R : \mathcal{M}_R^2 \to \mathcal{M}_R \) which assigns to each object \((A, B)\) in \( \mathcal{M}_R^2 \) the \( R \)-module \( A \otimes_R B \), and to each morphism \((f, g)\) in \( \mathcal{M}_R^2 \) the \( R \)-homomorphism \( f \otimes g \), is a covariant functor.

2) Let \( A \) be a fixed \( R \)-module. Then the rule \( \otimes_R : \mathcal{M}_R \to \mathcal{M}_R \) which assigns to each \( R \)-module \( B \) the \( R \)-module \( A \otimes_R B \) and to each \( R \)-homomorphism \( f : B \to C \) the \( R \)-homomorphism \( 1_A \otimes f : A \otimes_R B \to A \otimes_R C \) is a covariant functor.

3) Let \( B \) be a fixed \( R \)-module. Then the rule \( \otimes_R^B : \mathcal{M}_R \to \mathcal{M}_R \) which assigns to each \( R \)-module \( A \) the \( R \)-module \( A \otimes_R B \) and to each \( R \)-homomorphism \( f : A \to C \) the \( R \)-homomorphism \( f \otimes 1_B : A \otimes_R B \to C \otimes_R B \) is a covariant functor.

Proposition 14.16. Let \( A \) be an \( R \)-module. Then the functions \( l_A : A \to A \otimes_R R \) and \( r_A : A \to R \otimes_R A \) given by 
\[
l_A(a) = a \otimes 1, \quad r_A(a) = 1 \otimes a
\]
are natural \( R \)-isomorphisms. "Natural" means that if \( f : A \to B \) is an \( R \)-homomorphism, then the diagram
A \otimes_R R \xleftarrow{\lambda_A} A \xrightarrow{r_A} R \otimes_A A

f \otimes 1_R \downarrow \quad \downarrow f \quad \downarrow 1_R \otimes f

B \otimes_R R \xleftarrow{\lambda_B} B \xrightarrow{r_B} R \otimes_R B

commutes.

Proof. For all \( a \in A \), \( r \in R \) we have \( \lambda_A(r a) = (r a) \otimes 1 \)

\( = r (a \otimes 1) = r (a \cdot 1) \), and so \( \lambda_A \) is an \( R \)-homomorphism.

Similarly \( r_A \) is an \( R \)-homomorphism.

The function \( A \times R \to A \) which sends \((a, r)\) to \( r a \) is \( R \)-bilinear, and hence there is an \( R \)-homomorphism \( \chi : A \otimes_R R \to A \) such that \( \chi (a \otimes r) = r a \) for all \( a \in A \), \( r \in R \). We have, for \( a \in A \), \( r \in R \), \( \lambda_A \chi (a \otimes r) = \lambda_A (r a) = (r a) \otimes 1 = r (a \otimes 1) = a \otimes (r 1) = a \otimes r \). Thus since \( \{ a \otimes r \mid a \in A, \ r \in R \} \) generates \( A \otimes_R R \), we must have \( \chi = 1_A \otimes_R \). Thus \( \lambda_A \) and \( \chi \) are \( R \)-isomorphisms, inverse to one another. Similarly for \( r_A \).

If \( f : A \to B \) is an \( R \)-homomorphism we have, for all \( a \in A \), \( \lambda_B f (a) = f(a) \otimes 1 = (f \otimes 1_R) (a \otimes 1) = (f \otimes 1_R) \lambda_A (a) \) and so \( \lambda_B f = (f \otimes 1_R) \lambda_A \). Similarly \( r_B f = (1_R \otimes f) r_A \).

Remark 14.17. Let \( A, B, C \) be \( R \)-modules. Then

1) there is a natural \( R \)-isomorphism \( A \otimes_R B \to B \otimes_R A \)

which sends \( a \otimes b \) to \( b \otimes a \) for all \( a \in A, b \in B \).

2) there is a natural \( R \)-isomorphism

\( A \otimes_R (B \otimes_R C) \to (A \otimes_R B) \otimes_R C \)

which sends \( a \otimes (b \otimes c) \) to \( (a \otimes b) \otimes c \) for all \( a \in A, b \in B, c \in C \).

The proof of 14.17 is straightforward, using the universal property 14.12 for \( \otimes_R \).

Proposition 14.18. Let \( \{ A_x \mid x \in J \} \) and \( \{ B_x \mid x \in K \} \) be indexed families of \( R \)-modules. For each \( x \in J \), \( c \in K \), let \( \lambda_x : A_x \to \bigoplus_{x \in J} A_x \), \( \psi_c : B_c \to \bigoplus_{c \in K} B_c \), denote

\( \bigoplus_{x \in J} A_x \), \( \bigoplus_{c \in K} B_c \), respectively.
the canonical inclusions. Then
\[
\sum_{x, \rho} i_{x} \otimes j_{\rho} : \bigoplus_{(x, \rho) \in J \times K} (A_{x} \otimes_{K} B_{\rho}) \to \bigoplus_{(x, \rho) \in J \times K} A_{x} \otimes_{K} B_{\rho}
\]
is an \(R\)-isomorphism.

Proof. In \((x_0, \rho_0) \in J \times K\), let \(k_{x_0, \rho_0} : A_{x_0} \otimes_{K} B_{\rho_0} \to \bigoplus_{(x, \rho) \in J \times K} (A_{x} \otimes_{K} B_{\rho})\)
denote the canonical inclusion. Let
\[
f : \left( \bigoplus_{x \in J} A_{x} \right) \times \left( \bigoplus_{\rho \in K} B_{\rho} \right) \to \bigoplus_{(x, \rho) \in J \times K} (A_{x} \otimes_{K} B_{\rho})
\]
lie the function given by
\[
f(\sum_{x} i_{x}(a_{x}), \sum_{\rho} j_{\rho}(b_{\rho})) = \sum_{x, \rho} k_{x, \rho}(a_{x} \otimes b_{\rho})
\]
whenever \(a_{x} \in A_{x}, b_{\rho} \in B_{\rho}\), and all but finitely many of these
are \(0\). \(f\) is \(R\)-bilinear and so, by the universal property, there is an \(R\)-isomorphism
\[
\tilde{f} : \left( \bigoplus_{x \in J} A_{x} \right) \otimes_{K} \left( \bigoplus_{\rho \in K} B_{\rho} \right) \to \bigoplus_{(x, \rho) \in J \times K} (A_{x} \otimes_{K} B_{\rho})
\]
such that
\[
\tilde{f}(\sum_{x} i_{x}(a_{x}) \otimes \sum_{\rho} j_{\rho}(b_{\rho})) = \sum_{x, \rho} k_{x, \rho}(a_{x} \otimes b_{\rho})
\]
for \(a_{x}, b_{\rho}\) as above. It is straightforward to check that \(\sum_{x, \rho} i_{x} \otimes j_{\rho}\) and \(\tilde{f}\) are inverses of one
another.

We will often identify \(\left( \bigoplus_{x \in J} A_{x} \right) \otimes_{K} \left( \bigoplus_{\rho \in K} B_{\rho} \right)\) with \(\bigoplus_{(x, \rho) \in J \times K} (A_{x} \otimes_{K} B_{\rho})\)
under the \(R\)-isomorphism \(\tilde{f} \otimes_{K} \rho\). Thus \(\omega_{x} \rho\) for each \((x, \rho) \in J \times K\),
\(A_{x} \otimes_{K} B_{\rho}\) is canonically an \(R\)-submodule of \(\bigoplus_{x \in J} A_{x} \otimes_{K} \left( \bigoplus_{\rho \in K} B_{\rho} \right)\)
by identifying \(A_{x} \otimes_{K} B_{\rho}\) with its image under \(i_{x} \otimes j_{\rho}\).
Caution: If $A$ is an $R$-submodule of $A'$ and $B$ is an $R$-submodule of $B'$, we cannot always identify $A \otimes_R B$ with a submodule of $A' \otimes_R B'$. The tensor product of injective $R$-homomorphisms need not be injective.

Example 14.19. Take $R = \mathbb{Z}$, $A = \mathbb{Z}/2 = A'$, $B = \mathbb{Z}$, $B' = \mathbb{Q}$, the additive group of rational numbers. By 14.16, $\mathbb{Z}/2 \otimes \mathbb{Z} \cong \mathbb{Z}/2$, so $A \otimes_R B \cong \mathbb{Z}/2$.

Let $u$ be the non-zero element of $\mathbb{Z}/2$. Every element of $\mathbb{Z}/2 \otimes \mathbb{Q}$ has the form $u \otimes q$, $q \in \mathbb{Q}$. However, $u \otimes q = u \otimes (2 \cdot \frac{q}{2}) = (2u) \otimes \frac{q}{2} = 0 \otimes \frac{q}{2} = 0$

and so $\mathbb{Z}/2 \otimes \mathbb{Q} = 0$. Thus $A' \otimes_R B' = 0$. Thus if $i: \mathbb{Z} \rightarrow \mathbb{Q}$ denotes the inclusion, $\mathbb{Z}/2 \otimes i$ is not injective.

Proposition 14.20. Let $A$ be a free $R$-module with basis $X$, and $B$ a free $R$-module with basis $Y$. Then $A \otimes_R B$ is a free $R$-module with basis $\{x \otimes y | x \in X, y \in Y\}$.

Proof. Since $F_R(X \times Y)$ is a free $R$-module with basis $X \times Y$, there is a unique $R$-homomorphism $f: F_R(X \times Y) \rightarrow A \otimes_R B$ such that $f(x,y) = x \otimes y$ for all $x \in X, y \in Y$. Since $f(X \times Y) = \{x \otimes y | x \in X, y \in Y\}$, it suffices to prove that $f$ is an $R$-isomorphism.

Let $g: A \otimes B \rightarrow F_R(X \times Y)$ be defined as follows: $g(a) = \sum_{x \in X} r_x x$, $b = \sum_{y \in Y} s_y y$ where $r_x, s_y \in R$ and all but finitely many $r_x$ are 0, then $g(a \otimes b) = \sum_{(x,y) \in X \times Y} r_x s_y (x,y)$. $g$ is well-defined since the above expressions for $a$ and $b$ as linear combinations of elements of $X$ and $Y$, respectively, are unique. It is easy to check that $g$ is $R$-bilinear, and so there is an $R$-homomorphism $\tilde{g}: A \otimes_R B \rightarrow F_R(X \times Y)$ satisfying $\tilde{g}(a \otimes b) = g(a \otimes b)$ for all $a \in A, b \in B$. Note, in particular, $\tilde{g}(x \otimes y) = (x,y)$ for all $x \in X, y \in Y$. It follows immediately that $f$ and $\tilde{g}$ are inverses of one another, and so $f$ is an $R$-isomorphism.
Theorem 14.21. Let \( f : A \to A' \), \( g : B \to B' \) be \( R \)-
homomorphisms which are onto. Then
\[ f \circ g : A \otimes_R B \to A' \otimes_R B' \]
is onto, and \( \ker (f \circ g) \) is the \( R \)-submodule of \( A \otimes_R B \) generated by all elements of
the form \( a \otimes b \) where either \( a \in \ker f \) or \( b \in \ker g \).

Proof. Let \( a' \in A' \), \( b' \in B' \). Since \( f \) and \( g \) are onto,
there exist \( a \in A \), \( b \in B \) such that \( f(a) = a' \), \( g(b) = b' \).
Then \( (f \circ g)(a \otimes b) = a' \otimes b' \) and so \( a' \otimes b' \in \ker (f \circ g) \)
for all \( a' \in A' \), \( b' \in B' \). Since \( \ker (f \circ g) \) generates \( A' \otimes_R B' \), we have \( f \circ g \) is onto.

Let \( D \) be the \( R \)-submodule of \( A \otimes_R B \)
generated by all \( a \otimes b \) where either \( a \in \ker f \) or \( b \in \ker g \).

Clearly \( D \subseteq \ker (f \circ g) \). It remains only to show
\( \ker (f \circ g) \subseteq D \).

Let \( p : A \otimes_R B \to (A \otimes_R B)/D \) denote the canonical
projection. It suffices to show that there exists an
\( R \)-homomorphism \( \overline{h} : A' \otimes_R B' \to (A \otimes_R B)/D \)
such that
\[
\begin{array}{ccc}
A \otimes_R B & \xrightarrow{f \circ g} & A' \otimes_R B' \\
\downarrow p & & \downarrow \overline{h} \\
(A \otimes_R B)/D & & \\
\end{array}
\]
commutes, for then we would have
\( \ker (f \circ g) \subseteq \ker p = D \).

Define \( h : A' \times B' \to (A \otimes_R B)/D \) by
\( h(a', b') = p(a \otimes b) \) where \( a \in f^{-1}(a') \), \( b \in g^{-1}(b') \).

\( h \) is well-defined, for if \( f(a) = a' \), \( g(b) = b' \), then
\[ a \otimes b - a \otimes b = a \otimes b - a \otimes b + a \otimes b - a \otimes b \]
\[ = (a - a) \otimes b + a \otimes (b - b) \in D \]
since \( a - a \in \ker f \), \( b - b \in \ker g \). Thus \( p(a \otimes b) = p(\overline{a} \otimes \overline{b}) \).

It is easily checked that \( h \) is \( R \)-bilinear,
and so there exists a unique \( R \)-homomorphism
\( \overline{h} : A' \otimes_R B' \to (A \otimes_R B)/D \) such that \( \overline{h}(a' \otimes b') = p(a \otimes b) \)
whenever \( a' = f(a) \), \( b' = g(b) \). Thus \( \overline{h}(f \circ g) = p \),
completing the proof.
Example 14.22. Let \( R = \mathbb{Z} \), and \( m \) and \( n \) positive integers. Let \( \rho_m : \mathbb{Z} \rightarrow \mathbb{Z}/m \) and \( \rho_n : \mathbb{Z} \rightarrow \mathbb{Z}/n \) denote reduction mod \( m \) and \( n \), respectively. Since \( \rho_m \) and \( \rho_n \) are onto, it follows from 14.21 that \( \rho_m \circ \rho_n : \mathbb{Z} \times \mathbb{Z} \rightarrow (\mathbb{Z}/m) \times (\mathbb{Z}/n) \) is onto, with kernel the subgroup generated by all \( k \otimes 1 \) where either \( m \mid k \) or \( n \not\mid k \). Since \( \chi : \mathbb{Z} \rightarrow \mathbb{Z} \otimes \mathbb{Z} \)

\[ \chi(x) = x \otimes 1 \]

is an isomorphism, by 14.16, the composition

\[ \mathbb{Z} \xrightarrow{\chi} \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\rho_m \circ \rho_n} (\mathbb{Z}/m) \otimes (\mathbb{Z}/n) \]

is onto with kernel generated by all \( x^{-1}(k \otimes 1) = kl \) where either \( m \mid k \) or \( n \not\mid k \), i.e.,

\[ \ker((\rho_m \circ \rho_n)(x)) \]

is the subgroup of \( \mathbb{Z} \) generated by all \( m x + ny \), \( x, y \in \mathbb{Z} \). The latter is the subgroup of \( \mathbb{Z} \) generated by \( d \), the greatest common divisor of \( m \) and \( n \). Thus \( (\mathbb{Z}/m) \otimes (\mathbb{Z}/n) \cong \mathbb{Z}/d \)

where \( d = \gcd(m, n) \).

14.22. Together with 14.18 and 14.16, enables one to determine \( A \otimes \mathbb{Z} B \) when \( A \) and \( B \) are finitely generated abelian groups.

We have seen in 14.19 that the tensor product of injective \( R \)-modules need not be injective. We need consider some sufficient conditions for the tensor product of \( R \)-modules to be injective. We have already seen one case of this: It follows from 14.18 that if \( A \) is an \( R \)-module direct summand of \( A' \) and \( B \) is an \( R \)-module direct summand of \( B' \), then \( i \otimes j : A \otimes \mathbb{Z} B \to A' \otimes \mathbb{Z} B' \) is injective where \( i \) and \( j \) are the inclusions.

Proposition 14.23. Let \( A \) be a free \( R \)-module and \( f : B \to C \) \( R \)-injective. Then \( 1_A \otimes f : A \otimes \mathbb{Z} B \to A \otimes \mathbb{Z} C \) and \( f \otimes 1_A : \mathbb{Z} B \otimes \mathbb{Z} A \to \mathbb{Z} C \otimes \mathbb{Z} A \) are both injective.
Prof. By 14.16, the diagram

\[ B \longrightarrow C \]

\[ \downarrow g_B \quad \downarrow g_C \]

\[ R \otimes_R B \longrightarrow R \otimes_R C \]

commutes, and \( g_B \), \( g_C \) are \( R \)-isomorphisms. Thus \( 1_R \otimes f \) is injective. Thus if \( R_x \) is an \( R \)-module which is \( R \)-isomorphic to \( R \), it follows that \( 1_R \otimes f : R_x \otimes_R B \longrightarrow R_x \otimes_R C \) is injective. Since \( A \) is a free \( R \)-module, we can write \( A = \bigoplus_{x \in J} R_x \) where each \( R_x \) is \( R \)-isomorphic to \( R \). By 14.18, \( A \otimes_R B \) and \( A \otimes_R C \) are canonically identified with \( \bigoplus_{x \in J} (R_x \otimes_R B) \) and \( \bigoplus_{x \in J} (R_x \otimes_R C) \), respectively, and \( 1_A \otimes f = \bigoplus_{x \in J} (1_R \otimes f) \). Since each \( 1_R \otimes f \) is injective, \( 1_A \otimes f \) is injective. Similarly, \( f \otimes 1_{A} \) is injective.

Lemma 14.24. Suppose \( A \) and \( B \) are \( R \)-modules, and suppose \( a_1, \ldots, a_n \in A \), \( b_1, \ldots, b_n \in B \) are such that

\[ \sum_{i=1}^{n} a_i \otimes b_i = 0 \text{ in } A \otimes_R B. \]

Then there exists a finitely-generated \( R \)-submodule \( B' \) of \( B \) such that \( b_i \in B' \) for \( 1 \leq i \leq n \) and \( \sum_{i=1}^{n} a_i \otimes b_i = 0 \text{ in } A \otimes_R B' \).

Note: One's first impulse is to take \( B' \) to be the \( R \)-submodule generated by \( b_1, \ldots, b_n \). However, this fails. For example, take \( R = \mathbb{Z} \), \( A = \mathbb{Z}/2 \) with generator \( u \), and \( B = \mathbb{Q} \). We know, by 14.17, that \( U \otimes 1 = 0 \text{ in } \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Q} \). However, the \( \mathbb{Z} \)-submodule of \( \mathbb{Q} \) generated by \( 1 \) is \( \mathbb{Z} \), and \( U \otimes 1 \) is not \( 0 \text{ in } \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z} \).
Proof of 14.24. \( \sum_{i=1}^{n} a_i \otimes b_i \) is the image of \( \sum_{i=1}^{n} (a_i, b_i) \in F_{\mathbb{R}}(A \otimes B) \). 

Under the projection, since \( \sum_{i} a_i \otimes b_i = 0 \) in \( A \otimes_{\mathbb{R}} B \), we must have \( \sum_{i}(a_i, b_i) \in T(A, B) \) (see 14.10).

Every member of \( T(A, B) \) is a finite linear combination (with coefficients in \( \mathbb{R} \)) of elements of the form

\[
(x_1 + x_2, y) - (x_1, y) - (x_2, y), \quad (x, y_1 + y_2) - (x, y_1) - (x, y_2), \quad r(x, y) - l(x, y), \quad r(x, y) - l(x, ry) \]

where \( x, x_j \in A \), \( y, y_j \in B \), \( r, e \in \mathbb{R} \); conversely, every \( \mathbb{R} \)-linear combination of elements of the above type lies in \( T(A, B) \). Only a finite number of \( y_j, y \in B \) occur in such a linear combination for \( \sum_{i}(a_i, b_i) \). Take \( B' \) to be the \( \mathbb{R} \)-submodule of \( B \) generated by these \( y_j \), \( y \), and the \( b_i \). Then \( B' \) is finitely-generated, \( b_i \in B' \) for \( 1 \leq i \leq n \), and \( \sum_{i}(a_i, b_i) \in T(A, B) \). Thus \( \sum_{i} a_i \otimes b_i = 0 \) in \( A \otimes_{\mathbb{R}} B \).

Note: In the note preceding the proof, \( (u, 1) = (u, 2 \cdot \frac{1}{2}) \)

\[
= - \left[ 2(u, \frac{1}{2}) - (u, 2 \cdot \frac{1}{2}) \right] + \left[ 2(u, \frac{1}{2}) - (2u, \frac{1}{2}) \right] \\
= \left[ 0(u, \frac{1}{2}) - (0, u, \frac{1}{2}) \right] \quad \text{(since } 2u = 0) \\
\]

and so we can take \( B' \) to be the \( \mathbb{Z} \)-submodule of \( B \) generated by \( \frac{1}{2} \) and \( 1 \); i.e., the \( \mathbb{Z} \)-submodule generated by \( \frac{1}{2} \).

Theorem 14.25. Let \( A \) be a torsion-free abelian group and \( f : B \to C \) an injective \( \mathbb{Z} \)-homomorphism. Then \( f \otimes 1_A : B \otimes_{\mathbb{Z}} A \to C \otimes_{\mathbb{Z}} A \) and \( 1_A \otimes f : A \otimes_{\mathbb{Z}} B \to A \otimes_{\mathbb{Z}} C \) are both injective.

Proof. Let \( x = \sum b_i \otimes a_i \in \ker f \otimes 1_A \). Then \( \sum f(b_i) \otimes a_i = 0 \) in \( C \otimes_{\mathbb{Z}} A \). By 14.24, there exists a finitely-generated \( \mathbb{Z} \)-submodule \( A' \) of \( A \) such that \( a_i \in A' \) for all \( i \) and \( \sum f(b_i) \otimes a_i = 0 \) in \( C \otimes A' \). Write \( \iota : A' \to A \) for the inclusion and let \( y = \sum b_i \otimes a_i \in B \otimes_{\mathbb{Z}} A' \). Then \( x = (1_B \otimes \iota)(y) \) and \( (f \otimes 1_{A'})(y) = 0 \). But since \( A \) is torsion-
free and $A'$ is finitely-generated, $A'$ is a finitely-generated torsion free abelian group, and hence a free abelian group, i.e. a free $\mathbb{Z}$-module. Thus by 14.23, $\Theta 1_A$ is injective, and so $y = 0$. Hence $x = 0$, and so $f \otimes 1_A$ is injective.

The injectivity of $1_A \otimes f$ follows from commutativity of

\[
\begin{array}{ccc}
A \otimes_2 B & \xrightarrow{1_A \otimes f} & A \otimes_2 C \\
\downarrow & & \downarrow \\
B \otimes_2 A & \xrightarrow{f \otimes 1_A} & C \otimes_2 A
\end{array}
\]

where the vertical maps are the natural isomorphisms of 14.17(1).

**Exercise 14.26.** Let $A$ be an $R$-module such that $\otimes 1_A$ preserves injectivity (i.e. whenever $f : B \to C$ is an injective $R$-homomorphism, then $f \otimes 1_A : B \otimes_R A \to C \otimes_R A$ is injective). Then $\otimes 1_A$ preserves exactness, i.e. whenever

\[
\begin{array}{ccc}
B & \xrightarrow{f} & C \\
\downarrow & & \downarrow \\
D & \xrightarrow{g} & D \otimes_R A
\end{array}
\]

is an exact sequence of $R$-homomorphisms, then

\[
\begin{array}{ccc}
B \otimes_R A & \xrightarrow{f \otimes 1_A} & C \otimes_R A \\
\downarrow & & \downarrow \\
D \otimes_R A & \xrightarrow{g \otimes 1_A} & D \otimes_R A
\end{array}
\]

is exact. (It then follows, using 14.17(1), that $1_A \otimes$ also preserves exactness.)

**Definition 14.27.** An $R$-module $A$ is flat if $\otimes 1_A$ (and hence $1_A \otimes$) preserves exactness.

Thus from 14.26, 14.25 and 14.23, any free $R$-module is a flat $R$-module, and any torsion-free abelian group is a flat $\mathbb{Z}$-module.

**Exercise 14.28.** If $A$ is a flat $\mathbb{Z}$-module, then $A$ is torsion-free.
We next consider localization of abelian groups. Although localization will not be used in this course, it plays a central role in modern homotopy theory, and fits in well at this point.

Definition 14.29. Let \( S \) be a set of primes. Let 
\[
\mathbb{Z}_S = \{ \frac{m}{n} \in \mathbb{Q} \mid m, n \in \mathbb{Z}, \text{n relatively prime to each member of } S \}.
\]

\( \mathbb{Z}_S \) is a subgroup (in fact a subring) of \( \mathbb{Q} \), and is called the integers localized at \( S \).

If \( A \) is an arbitrary abelian group, \( A \otimes \mathbb{Z}_S \) is called \( A \) localized at \( S \).

Note that if \( S = \emptyset \), then \( \mathbb{Z}_S = \mathbb{Q} \). At the other extreme, if \( S \) is the set of all primes, then \( \mathbb{Z}_S = \mathbb{Z} \).

If \( S = \{ p \} \), we write \( \mathbb{Z}_p \) for \( \mathbb{Z}_S \). \( \mathbb{Z}_p \) is called the ring (ring) of integers localized at the prime \( p \). If \( A \) is an abelian group, \( A \otimes \mathbb{Z}_p \) is called \( A \) localized at \( p \).

Theorem 14.30. Let \( S \) be a set of primes, and \( A \) an abelian group. Then

1) If each element of \( A \) has order relatively prime to each member of \( S \), then \( A \otimes \mathbb{Z}_S = 0 \).

2) If each element of \( A \) has order a product of primes in \( S \), then the \( \mathbb{Z}_S \)-homomorphism 
\( \alpha : A \rightarrow A \otimes \mathbb{Z}_S \) sending \( a \) to \( a \otimes 1 \) is a \( \mathbb{Z}_S \)-isomorphism.

Proof. 1) Let \( a \in A \), and let \( n \) be the order of \( a \).

Since \( n \) is relatively prime to each member of \( S \), we have, for each \( \frac{m}{n} \in \mathbb{Z}_S \), \( \frac{m}{n} \) is also in \( \mathbb{Z}_S \). Thus 
\[
a \otimes \frac{m}{n} = a \otimes n \cdot \left( \frac{m}{n} \right) = (na) \otimes \left( \frac{m}{n} \right) = 0 \otimes \frac{m}{n} = 0,
\]
proving 1)

2) Let \( \alpha : A \rightarrow A \otimes \mathbb{Z}_S \) be the \( \mathbb{Z}_S \)-homomorphism given by \( \alpha(a) = a \otimes 1 \). If \( n \in \mathbb{Z} \) is relatively prime to each member of \( S \), then the \( \mathbb{Z}_S \)-homomorphism 
\( \alpha : A \rightarrow A \otimes \mathbb{Z}_S \) sending \( a \) to \( na \) is a \( \mathbb{Z}_S \)-isomorphism, and to

for each \( a \in A \) there is a unique element, denoted
\frac{a}{m}, \text{ in } A \text{ such that } n \cdot (\frac{a}{m}) = a.

Define \( f: A \times \mathbb{Z}_n \rightarrow A \) by \( f(a, \frac{m}{n}) = m \cdot (\frac{a}{m}) \)
whenver \( a \in A, m \in \mathbb{Z} \) with \( n \) relatively prime to each member of \( S \). It is easily checked that if \( k \in \mathbb{Z} \)
is relatively prime to each member of \( S \), then
\( m \cdot (\frac{a}{n \cdot k}) = m \cdot (\frac{a}{k}) \), and so \( f \) is well-defined. \( f \) is
easily checked to be \( \mathbb{Z} \)-linear, and so there is a
\( \mathbb{Z} \)-homomorphism \( \tilde{f}: A \otimes_{\mathbb{Z}} \mathbb{Z}_n \rightarrow A \) satisfying
\( \tilde{f}(a \otimes \frac{m}{n}) = m \cdot (\frac{a}{m}) \) for \( a \) and \( \frac{m}{n} \) as above. We have
\( \tilde{f} \circ (a) = \tilde{f}(a \otimes 1) = a \) for all \( a \in A \) and so \( \tilde{f} \circ = 1_{A}. \)

In each \( a \in A \) and \( m, n \in \mathbb{Z} \) with \( n \) relatively prime
to each member of \( S \), we have \( \alpha \bar{f}(a \otimes \frac{m}{n}) = m \cdot (\frac{a}{n \cdot k}) \otimes 1 = \frac{a}{n} \otimes m = \frac{a}{n} \otimes n \cdot (\frac{m}{n}) = n \cdot (\frac{a}{n}) \otimes \frac{m}{n} = a \otimes \frac{m}{n} \)
and so \( \alpha \bar{f} = 1_{A \otimes_{\mathbb{Z}} \mathbb{Z}_n} \). Thus \( \alpha \) and \( \bar{f} \) are
\( \mathbb{Z} \)-isomorphisms, inverse to one another.

Corollary 14.31. \( \mathcal{A} \) is a finite abelian group and \( \mathcal{p} \) a
prime, then \( A \otimes_{\mathbb{Z}} \mathbb{Z}(p) \), the localization of \( A \) at \( \mathcal{p} \), is
isomorphic to the \( \mathcal{p} \)-primary part of \( A \) (i.e. the subfield of \( A \) consisting of all elements having
order a power of \( \mathcal{p} \)).

Definition 14.32. A chain complex of \( \mathbb{R} \)-modules \( C \) is a
chain complex such that each \( C_n \) is an \( \mathbb{R} \)-module and
each \( d: C_n \rightarrow C_{n-1} \) is an \( \mathbb{R} \)-homomorphism.

An \( \mathbb{R} \)-augmented chain complex of \( \mathbb{R} \)-modules \( C \)
consists of a chain complex of \( \mathbb{R} \)-modules, together with
an \( \mathbb{R} \)-homomorphism \( e: C_0 \rightarrow \mathbb{R} \) such that \( e \circ d = 0: C_1 \rightarrow \mathbb{R} \).

Chain maps, augmented chain maps, and chain
homotopies are defined exactly as before, except that
all maps are required to be \( \mathbb{R} \)-homomorphisms.

We form \( \text{Ch}_{\mathbb{R}} \) and \( \text{Ch}_{\mathbb{R}}^\mathbb{R} \), the categories of chain
complexes of \( \mathbb{R} \)-modules, and the category of augmented
chain complexes of \( \mathbb{R} \)-modules.
Definition 14.33. Let $C$ and $C'$ be chain complexes of $R$-modules with boundary maps $\partial$ and $\partial'$, respectively. Define $C \otimes R C'$ as follows: 
$$ (C \otimes_R C')_n = \bigoplus_{i+j=n} (C_i \otimes_R C'_j), $$ and 
$$ \bar{\partial} : (C \otimes_R C')_n \to (C \otimes_R C')_{n-1} $$ is as follows:

$$ \bar{\partial} |_{C_i \otimes_R C'_j} $$ is the composition

$$ C_i \otimes_R C'_j \overset{\Delta}{\longrightarrow} C_i \otimes C'_j \overset{(-1)^i}{\longrightarrow} (C_{i-1} \otimes_R C'_j) \oplus (C_i \otimes_R C'_{j-1}) \in (C \otimes_R C')_{n-1}. $$

Thus if $a \in C_i$, $b \in C'_j$, we have $\bar{\partial}(a \otimes b) = (\partial a) \otimes b + (-1)^i a \otimes (\partial' b)$.

If $C$ and $C'$ are $R$-augmented with augmentations $\varepsilon$ and $\varepsilon'$, respectively, define $\bar{\varepsilon} : (C \otimes_R C') \to R$ to be the composition

$$ C_0 \otimes_R C'_0 \overset{\varepsilon \otimes \varepsilon'}{\longrightarrow} R \otimes_R R \cong R $$

where this last isomorphism is the inverse of either $1_R \otimes 1_R$ (both are given by $r \otimes s \mapsto rs$).

(Thus if $a \in C_0$, $b \in C'_0$, then $\bar{\varepsilon}(a \otimes b) = \varepsilon(a) \varepsilon'(b)$.)

Exercise 14.34. If $C$ and $C'$ are chain complexes of $R$-modules, then $C \otimes_R C'$, as defined in 14.33, is a chain complex of $R$-modules. If each is $R$-augmented with augmentations $\varepsilon$ and $\varepsilon'$, respectively, then $C \otimes_R C'$ is $R$-augmented with augmentation $\bar{\varepsilon}$ as defined in 14.33.

The motivating example for Definition 14.33 is the following:

Example 14.35. Let $K$ and $L$ be finite regular CW complexes. Then $|K| \times |L|$ is the underlying space of a regular CW complex $K \times L$ given as follows:

For $n \geq 0$, cells $n(K \times L) = \{ A \times B \mid A \in \text{cells}_i(K), B \in \text{cells}_j(L), i+j = n \}$. The verification that $K \times L$ is a CW complex with $|K \times L| = |K| \times |L|$ is straightforward. This also works if one of $K$ or $L$ is allowed to be infinite. One must then verify that the product topology on $|K| \times |L|$ coincides with the weak topology relative to the closures of the cells. This fails in general if we allow both $K$ and $L$ to
be infinite.

Let \([ : ]_k\) and \([ : ]_L\) denote the incidence systems on \(K\) and \(L\), respectively, arising from a choice of orientations of the cells where the orientations of the \(0\)-cells are chosen to have augmentation \(+1\).

We shall abuse notation by using the same symbol to denote a cell and the chosen orientation of it.

If \(A \in \text{cell}_m(K)\) and \(B \in \text{cell}_n(L)\), then

\[
A \times B = \overline{A} \times \overline{B} = (A \times B) \cup (\overline{A} \times \overline{B}) \cup (A \times \overline{B}) \cup (\overline{A} \times B)
\]

and so

\[
\text{cell}_{m+n-1}( (A \times B)_c ) = \{ A \times D | D \in \text{cell}_{n-1}(Bc) \} \cup \{ C \times B | C \in \text{cell}_{m-1}(Ac) \}.
\]

Consider \(C(K) \otimes \mathbb{Z} \cong C(L)\). For \(A, B\) as above we have

\[
\overline{\overline{A \otimes B}} = (\overline{\overline{A}}) \otimes \overline{\overline{B}} + (-1)^m A \otimes \overline{\overline{B}}
\]

\[
= \left( \sum_{C \in \text{cell}_{m-n-1}(A_c)} [A : C]]_k C \otimes B \right) + (-1)^m A \otimes \left( \sum_{D \in \text{cell}_{n-1}(Bc)} [B : D]]_L D \right)
\]

\[
= \sum_{C \in \text{cell}_{m-n-1}(A_c)} [A : C]]_k C \otimes B + (-1)^m \sum_{D \in \text{cell}_{n-1}(Bc)} [B : D]]_L A \otimes D.
\]

Thus if we define \([ : ]_{K \times L}\) on \(K \times L\) by

\[
[A \times B : C \times B]_{K \times L} = [A : C]_K, \quad [A \times B : A \times D]_{K \times L} = (-1)^{\text{dim} A} [B : D]]_L,
\]

\([ : ]_{K \times L}\) is an incidence system on \(K \times L\), i.e., the chain complex \(C(K) \otimes \mathbb{Z} \cong C(L)\) is chain isomorphic to \(C(K \times L)\).

If \(X\) and \(Y\) are topological spaces, it is not true that the singular complex \(S(X \times Y)\) is chain isomorphic to \(S(X) \otimes \mathbb{Z} S(Y)\). However, we will see that these chain complexes are chain homotopy equivalent, and so their homology groups will be isomorphic. Thus it is desirable to have a theorem which enables one to calculate \(H_n(C \otimes C')\) in terms of the \(H_n(C)\) and \(H_n(C')\). This is one of our next major projects.
Exercise 14.36. Suppose $C, C', D, D'$ are chain complexes of $R$-modules and $f: C \to C'$, $g: D \to D'$ are chain maps over $R$ (i.e., morphisms in $\text{CER}$). Define $f \circ g: C \otimes R D \to C' \otimes R D'$ by

$$(f \circ g)_n = \bigoplus_{i+j=n} f_{ij} \circ g_j$$

Then $f \circ g$ is a chain map.

If $f$ and $g$ are both $R$-augmented chain maps over $R$, so is $f \circ g$.

Let $\text{CER}^2$ denote the category of ordered pairs of chain complexes of $R$-modules, and $\text{ACER}^2$ the category of ordered pairs of $R$-augmented chain complexes of $R$-modules. Then $\sigma_R: \text{CER}^2 \to \text{CER}$ and $\sigma_R: \text{ACER}^2 \to \text{ACER}$ are covariant functors.

15. A Special Case of the K"unneth Theorem

Let $C$ be a chain complex of $R$-modules. Then since kernels and images of $R$-homomorphisms are $R$-submodules, and quotients of $R$-modules are $R$-modules, $H_n(C)$ is an $R$-module for each $n$.

Exercise 15.1. Let $f: C \to D$ be a chain map over $R$ of chain complexes of $R$-modules. Then for all $n$,

$$(\text{Id})_n(f): H_n(C) \to H_n(D)$$

is an $R$-homomorphism, and $\text{Id}_n: \text{CER} \to \text{CER}$ is a covariant functor.

Lemma 15.2. Let $C$ and $D$ be chain complexes of $R$-modules. Let $z \in Z_i(C)$, $w \in Z_j(D)$. Then $z \otimes w \in Z_{i+j}(C \otimes R D)$ and the homology class $[z \otimes w] \in H_{i+j}(C \otimes R D)$ depends only on $[z] \in H_i(C)$ and $[w] \in H_j(D)$.

Proof. $d(z \otimes w) = (\partial z) \otimes w + (-1)^i z \otimes (\partial w) = 0 \otimes w + (-1)^i z \otimes 0 = 0$, and so $z \otimes w \in Z_{i+j}(C \otimes R D)$.

If $[z] = \overline{z}$ and $[w] = \overline{w}$, then there exist $c \in C_{i+1}$, $d \in D_{j+1}$ such that $dc = \overline{z} - z$, $2d = \overline{w} - w$. Then
\[ \bar{z} \otimes \bar{w} - z \otimes w = (z + \partial c) \otimes (\partial^* + \partial d) - z \otimes w \]
\[ = z \otimes (\partial d) + (\partial c) \otimes w + (\partial c) \otimes (\partial d) \]
\[ = (-1)^i \partial (z \otimes d) + i(c \otimes w) + i(c \otimes d) \in B_{i+j}(C \otimes_R D) \]
and so \[ \bar{z} \otimes \bar{w} = [z \otimes w]. \]

We thus have a well defined function
\[ H_i(C) \times H_j(D) \rightarrow H_{i+j}(C \otimes_R D) \text{ given by } ([z], [w]) \mapsto [z \otimes w]. \]
It is easily checked that this function is \(R\)-bilinear, and thus there is an \(R\)-homomorphism
\[ H_i(C) \otimes_R H_j(D) \rightarrow H_{i+j}(C \otimes_R D) \]
which sends \([z] \otimes [w] \mapsto [z \otimes w]\) whenever \([z] \in H_i(C), [w] \in H_j(D)\).

**Definition 15.3.** Let \(C\) and \(D\) be chain complexes of \(R\)-modules. In each \(n \geq 0\), let \([H(C) \otimes_R H(D)]_n\) denote
\[ \bigoplus_{i+j=n} H_i(C) \otimes_R H_j(D). \]
The \(R\)-homomorphism
\[ \mu : [H(C) \otimes_R H(D)]_n \rightarrow H_n(C \otimes_R D) \text{ given by } [z] \otimes [w] \mapsto [z \otimes w]\] whenever \([z] \in H_i(C), [w] \in H_j(D)\)
\((i+j=n)\) is called the **homology cross product**. We denote \(\mu(a \otimes b)\) by \(a \times b\).

The main aim of this section is to show that under some restrictive hypotheses, the homology cross product map is an \(R\)-isomorphism. This will be the first step towards a theorem which describes \(H_n(C \otimes_R D)\) in terms of the \(H_i(C)\) and \(H_j(D)\). (In the general case, \(\mu\) is not an isomorphism.)

**Exercise 15.4.** The homology cross product is natural, i.e. if \(f : C \rightarrow C', g : D \rightarrow D'\) are chain maps over \(R\), then the diagram
\[ \begin{array}{ccc}
[H(C) \otimes_R H(D)]_n & \xrightarrow{\mu} & H_n(C \otimes_R D) \\
[H(f) \otimes H(g)]_n & \downarrow & \downarrow H_n(f \otimes g) \\
[H(C') \otimes_R H(D')]_n & \xrightarrow{\mu} & H_n(C' \otimes_R D')
\end{array} \]
commutes for all \( n \), where
\[
\left[ H(f) \otimes H(g) \right]_n = \bigoplus_{i+j=n} H_i(f) \otimes H_j(g).
\]

**Exercise 15.5.** If \( \{ D^x \mid x \in J \} \) is an indexed family of chain complexes of \( R \)-modules, then
\[
\bigoplus_{x \in J} D^x \text{ is a chain complex of } R \text{-modules and the canonical inclusions } e^x : D^x \to \bigoplus_{x \in J} D^x \text{ are chain maps over } R. \text{ The isomorphism }
\[
\sum_{x} H_n(i^x) : \bigoplus_{x \in J} H_n(D^x) \to H_n(\bigoplus_{x \in J} D^x)
\]
of 3.18 is an \( R \)-isomorphism.

**Exercise 15.6.** Let \( \{ D^x \mid x \in J \} \) and \( i^x \) be as in 15.5, and let \( C \) be a chain complex of \( R \)-modules. Then
\[
\sum_{x} 1_C \otimes i^x : \bigoplus_{x \in J} (C \otimes_R D^x) \to C \otimes_R (\bigoplus_{x \in J} D^x)
\]
is a chain isomorphism.

**Exercise 15.7.** Let \( \{ D^x \mid x \in J \} \), \( l^x \), and \( C \) be as in 15.6. For each \( b \in J \) let \( j^b : C \otimes_R D^x \to \bigoplus_{x \in J} (C \otimes_R D^x) \) and
\[
P_j : H_j(D^x) \to \bigoplus_{x \in J} H_j(D^x) \text{ denote the canonical inclusions.}
\]
For each \( x \in J \), let \( e^x : \left[ H(C) \otimes_R H(D^x) \right]_n \to H_n(C \otimes_R D^x) \) denote the homology cross product. Then the diagram
Corollary 15.8. Let $C, \{D^x | x \in J\}$, and $M_x$ be as in 15.7. Suppose, in each $x \in J$, the homology cross product $M_x : [H(C) \otimes_R H(D^x)]_n \rightarrow H_n(C \otimes_R D^x)$ is an $R$-isomorphism. Then the homology cross product $M : [H(C) \otimes_R H(\bigoplus_{x \in J} D^x)]_n \rightarrow H_n(C \otimes_R (\bigoplus_{x \in J} D^x))$ is an $R$-isomorphism.

Proof. It suffices to observe that all the vertical maps in the diagram in 15.7 are $R$-isomorphisms in each $i, j$.

$$\sum_x 1_{H_i(c)} \otimes k_j^x : \bigoplus_{x \in J} H_i(c) \otimes_R H_j(D^x) \rightarrow H_i(c) \otimes_R \bigoplus_{x \in J} H_j(D^x)$$

is an $R$-isomorphism by 14.18, and so

$$\bigoplus_{i+j=n} (\sum_x 1_{H_i(c)} \otimes k_j^x)$$

is an $R$-isomorphism.
In each $j$, \[ \sum_{\alpha} H_{j}(i^{\alpha}) \rightarrow \bigoplus_{\alpha \in J} H_{j}(D^{\alpha}) \]
is an $R$-isomorphism by 15.5, and so 
\[ \bigoplus_{i, j = n} H_{i}(c) \otimes \sum_{\alpha} H_{j}(i^{\alpha}) \]
is an $R$-isomorphism.

\[ \sum_{\alpha} H_{n}(i^{\alpha}) \]
is an $R$-isomorphism by 15.5, and 
\[ H_{n}\left( \sum_{\alpha} 1_{c} \otimes i^{\alpha} \right) \]
is an $R$-isomorphism by 15.6.

**Notation 15.9.** Let $A$ be an $R$-module and $q$ a non-negative integer. We denote by $A^{(q)}$ the chain complex given by 
\[ A^{(q)} = \begin{cases} A & \text{if } n = q \\ 0 & \text{otherwise} \end{cases} \]
(and thus $A^{(0)} = 0$ in $A^{(1)}$).

**Lemma 15.10.** Let $A$ be a flat $R$-module, and $C$ any chain complex of $R$-modules. Then, for each $q \geq 0$ and all $n$, the cross product map 
\[ \mu : \left[ H(C) \otimes_{R} H(A^{(q)}) \right]_{n} \rightarrow H_{n}(C \otimes_{R} A^{(q)}) \]
is an $R$-isomorphism.

**Proof.** We have 
\[ H_{i}(A^{(q)}) = \begin{cases} A & \text{if } i = q \\ 0 & \text{otherwise} \end{cases} \]
where we canonically identify $A$ with $H_{q}(A^{(q)})$ under the $R$-isomorphism 
\[ A = Z_{q}(A^{(q)}) \rightarrow H_{q}(A^{(q)}) \]
which sends $a$ to $[a]$. Thus 
\[ \left[ H(C) \otimes_{R} H(A^{(q)}) \right]_{n} = H_{n-q}(C) \otimes_{R} H_{q}(A^{(q)}) = H_{n-q}(C) \otimes_{R} A. \]

We proceed to construct an inverse 
\[ \nu : H_{n}(C \otimes_{R} A^{(q)}) \rightarrow H_{n-q}(C) \otimes_{R} A. \]
We have\( (C \otimes_R A^{(s)})_n = C_{n-g} \otimes_R A \), and the boundary homomorphism \( \partial \) in \( C \otimes_R A^{(s)} \) is \( \partial \otimes 1_A \) where \( \partial \) is the boundary homomorphism in \( C \) (since the boundary map in \( A^{(s)} \) is 0). Since

\[
0 \to Z_{n-g}(C) \xrightarrow{j} C_{n-g} \xrightarrow{\partial} C_{n-g-1} \text{ is exact}
\]

(where \( j \) is the inclusion), and \( A \) is flat, it follows that

\[
0 \to Z_{n-g}(C) \otimes_R A \xrightarrow{j \otimes 1_A} (C \otimes_R A^{(s)})_n \xrightarrow{\overline{\partial}} (C \otimes_R A^{(s)})_{n-1}
\]

is exact, and so \( Z_n(C \otimes_R A^{(s)}) = \text{im} (j \otimes 1_A) \), i.e. \( Z_n(C \otimes_R A^{(s)}) \) is the \( R \)-submodule of \( C_{n-g} \otimes_R A \) generated by all \( z \otimes a \) where \( z \in Z_{n-g}(C) \), \( a \in A \).

We have the commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
0 & \rightarrow & Z_{n-g}(C) \\
\downarrow & & \downarrow j \otimes 1_A \\
B_{n-g}(C) & \rightarrow & C_{n-g} \\
\downarrow & & \downarrow j \\
0 & & 0
\end{array}
\]

where \( j \) is the inclusion and \( \partial' \) is obtained from \( \partial \) by restricting the range. Thus, since \( A \) is flat, we have the commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
C_{n-g+1} \otimes_R A & \rightarrow & Z_{n-g}(C) \otimes_R A \\
\downarrow & & \downarrow j \otimes 1_A \\
B_{n-g}(C) \otimes_R A & \rightarrow & C_{n-g} \otimes_R A \\
\downarrow & & \downarrow j \\
0 & & 0
\end{array}
\]

and so \( B_n(C \otimes_R A^{(s)}) = \text{im} (j \otimes 1_A) \), i.e. \( B_n(C \otimes_R A^{(s)}) \) is
the $R$-submodule of $C_{n-g} \otimes_R A$ generated by all $b \otimes a$ where $b \in B_{n-g}(C)$, $a \in A$.

Let $p: Z_{n-g}(C) \to H_{n-g}(C)$ be the projection $p(z) = \langle z \rangle$. Let
$$f: Z_n(C \otimes_R A^{(k)}) \to H_{n-g}(C) \otimes_R A$$
be the composition
$$Z_n(C \otimes_R A^{(k)}) \cong Z_{n-g}(C) \otimes_R A \xrightarrow{p \otimes 1A} H_{n-g}(C) \otimes_R A.$$

Explicitly, if $z \in Z_{n-g}(C)$ and $a \in A$, then $f(z \otimes a) = \langle z \rangle \otimes a$.

If $b \in B_{n-g}(C)$, we have $f(b \otimes a) = \langle b \rangle \otimes a = 0 \otimes a = 0$, and so $B_{n-g}(C \otimes_R A^{(k)}) \subseteq \ker f$. Thus, passing to quotients, $f$ induces an $R$-homomorphism
$$\nu: H_n(C \otimes_R A^{(k)}) \to H_{n-g}(C) \otimes_R A$$
satisfying
$$\nu([z \otimes a]) = [z] \otimes a \quad \text{wherever } z \in Z_{n-g}(C), a \in A.$$ We have, for all $z \in Z_{n-g}(C)$ and $a \in A$,

$$\mu([z] \otimes a) = \nu(z \otimes a) = [z] \otimes a,$$
and $$\mu([z \otimes a]) = \mu([z] \otimes a) = [z] \otimes a.]$$ Thus, since

$$\{[z] \otimes a \mid z \in Z_{n-g}(C), a \in A\}$$ generates $H_{n-g}(C) \otimes_R A$ and
$$\{[z \otimes a] \mid z \in Z_{n-g}(C), a \in A\}$$ generates $H_n(C \otimes_R A^{(k)})$, we have $\mu = \nu$ and $\mu$ and $\nu$ are the respective identity maps.

**Corollary 15.11.** Suppose $C$ and $D$ are chain complexes of $R$-modules. Suppose each $D_k$ is a flat $R$-module and that $D = 0$ in $D$. Then the homology cross product
$$\mu: \left[ H(C) \otimes_R H(D) \right] \to H_n(C \otimes_R D)$$
is an $R$-isomorphism for all $n$.

**Proof.**

$$D = \bigoplus_{k \geq 0} (D_k)^{(k)}.$$ By 15.10, each $D_k$ is a flat $R$-module. Then the result now follows from 15.8.
Definition 15.12. Let $C$ and $D$ be chain complexes of $R$-modules. We say $C$ and $D$ are chain equivalent over $R$ if there exist chain maps, over $R$, $f: C \rightarrow D$, $g: D \rightarrow C$ such that $fg \simeq 1_D$, $gf \simeq 1_C$ over $R$. In this case, $f$ and $g$ are called chain equivalences over $R$.

Chain equivalence is the analogue, in $C(R)$, of homotopy equivalence in $\text{Top}$.

Proposition 15.13. If $f: C \rightarrow D$ is a chain equivalence over $R$, then $\text{Hom}(f): \text{Hom}(C) \rightarrow \text{Hom}(D)$ is an $R$-isomorphism for all $n$.

Proof. Let $g: D \rightarrow C$ be a chain equivalence over $R$ such that $fg \simeq 1_D$, $gf \simeq 1_C$. By 8.4, $1_{\text{Hom}(D)} = \text{Hom}(fg) = \text{Hom}(f) \text{Hom}(g)$, Similarly $1_{\text{Hom}(C)} = \text{Hom}(gf)$. Thus $\text{Hom}(f)$ and $\text{Hom}(g)$ are isomorphisms, inverse to one another.

Proposition 15.14. Let $f$, $g: C \rightarrow D$ and $f'$, $g': C' \rightarrow D'$ be chain maps over $R$. Suppose $f \simeq g$, $f' \simeq g'$. Then $f \circ f' \simeq g \circ g'$.

Proof. Suppose $S$ is a chain homotopy from $f$ to $g$, and $T$ a chain homotopy from $f'$ to $g'$. Thus $f - g = S_i + S_{i-1} \partial$, $g' - f' = T_j + T_{j-1} \partial$ for all $i$, $j$.

Let $U_n: (C \otimes_R C')_n \rightarrow (D \otimes_R D')_n$ be defined as follows:
For each $i + j = n$, $U_n | C_i \otimes_R C'_j$ is the composition

$C_i \otimes_R C'_j \xrightarrow{\partial_{i} f'_j + (-1)^i g_i \otimes T_j} (D_{i+1} \otimes_R D'_j) \otimes (D_i \otimes_R D'_{j+1}) \rightarrow (D \otimes_R D')_n$.

$\forall x \in C_i$, $y \in C'_j$, where $i + j = n$, we have

$(S_{i} + \sum_{n-i} S_{i-1} \partial)(x \otimes y) = \partial S_{i} (x) \otimes f'_j (y) + (-1)^i S_{i} (x) \otimes T_j (y) + \sum_{n-i} \partial S_{i-1} (x) \otimes \partial (y)$

$= \partial S_{i} (x) \otimes f'_j (y) + (-1)^{i+1} S_{i} (x) \otimes \partial f'_j (y) + (-1)^i [S_{i} (x) \otimes T_j (y) + (-1)^i g_i (x) \otimes \partial T_j (y)]$

$+ [S_{i-1} \otimes f'_j + (-1)^{i-1} g_i \otimes \partial T_j] (x \otimes y) + (-1)^i [S_{i-1} \otimes f'_j + (-1)^{i-1} g_i \otimes \partial T_j (y)] (x \otimes y)$.
\[= \partial S_i(x) \otimes f'_j(y) + (-1)^{i+1} S_i(x) \otimes f'_j(y) + (-1)^i \partial g_j(x) \otimes T_j(y) + \partial a(y) \otimes T_j(y) \]
\[+ S_{i+1}(x) \otimes f'_j(y) + (-1)^{i+1} \partial g_j(x) \otimes T_j(y) + (-1)^i \partial g_j(y) \otimes a(y) + g_j(x) \otimes T_j(y) \]
\[= [(\partial S_i + S_{i+1})(x)] \otimes f'_j(y) + g_j(x) \otimes [(\partial T_j + T_{i+1} + \partial)(y)] +
\[(-1)^{i+1} S_i(x) \otimes f'_j(y) + (-1)^i S_{i+1}(x) \otimes f'_j(y) + (-1)^i \partial g_j(x) \otimes T_j(y) + (-1)^i \partial g_j(y) \otimes a(y) + g_j(x) \otimes T_j(y) \]
\[= [g_j(x) - f'_j(x)] \otimes f'_j(y) + g_j(x) \otimes [g_j(y) - f'_j(y)]
\[= g_j(x) \otimes f'_j(y) - f'_j(x) \otimes f'_j(y) + g_j(x) \otimes g'_j(y) - g_j(x) \otimes f'_j(y)
\[= (g_j \otimes g'_j - f'_j \otimes f'_j)(x \otimes y) \quad \text{and } \bar{\partial} \text{Un + Un} = \overline{\partial} = g \otimes g' - f \otimes f'.
\]

Exercise 15.15. Let \(0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0\) be a short exact sequence of \(R\)-modules and \(R\)-homomorphisms. Then the following conditions are equivalent:
1) \(\ker f\) is an \(R\)-module direct summand of \(B\). (Thus if \(B = \ker f \oplus D\), \(f\) maps \(D\) isomorphically onto \(C\), and so \(B \cong A \oplus C\).
2) There exists an \(R\)-homomorphism \(p : B \rightarrow A\) such that \(p \circ f = 1_A\).
3) There exists an \(R\)-homomorphism \(s : C \rightarrow B\) such that \(s \circ g = 1_c\).

If any, and hence all, of the above hold, we say the above sequence splits.

Example 15.16. \(0 \rightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0\), where \(f(x) = 2x\) for all \(x \in \mathbb{Z}\), does not split.
\(0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0\) does not split.

For any \(R\)-modules \(A\) and \(B\),
\(0 \rightarrow A \xrightarrow{i} A \oplus B \xrightarrow{p} B \rightarrow 0\) splits, where \(i\) is the canonical inclusion and \(p(a, b) = b\).

Proposition 15.17. Let \(0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0\) be a short exact sequence of \(R\)-modules with \(C\) a free \(R\)-module. Then the above sequence splits.
Proof. Let \( X \) be an \( R \)-basis for \( C \). In each \( x \in X \), we can choose a \( b_x \in B \) such that \( g(b_x) = x \) since \( g \) is onto. Let \( s : C \to B \) be the unique \( R \)-homomorphism such that \( s(x) = b_x \) for each \( x \in X \). Then \( g \circ s(x) = g(b_x) = x = 1_c(x) \) for each \( x \in X \). Since \( X \) is a generating set for \( C \), \( g \circ s = 1_c \).

In order to make further progress, we will need to assume that \( R \) is a principal ideal domain (PID), which is equivalent to saying that every \( R \)-submodule of a free \( R \)-module is free. Important examples are \( \mathbb{Z} \), \( \mathbb{Z}_p \) for any set of primes \( S \), and fields.

Lemma 15.18. Let \( R \) be a PID. Let \( C \) be a chain complex of \( R \)-modules such that each \( C_n \) and each \( H_n(C) \) is a free \( R \)-module. Then \( C \) is chain equivalent to a chain complex \( D \) with boundary homomorphisms identically \( 0 \).

Proof. Define \( D_n = H_n(C) \) and \( d = 0 \) in \( D \). We have the short exact sequence \( 0 \to Z_n(C) \to C_n \xrightarrow{d} B_{n-1}(C) \to 0 \). \( B_{n-1}(C) \) is a free \( R \)-module, being a submodule of the free \( R \)-module \( C_{n-1} \). Thus, this exact sequence splits and so \( C_n = Z_n(C) \oplus E_n \) where \( \partial|E_n : E_n \to B_{n-1}(C) \) is an \( R \)-isomorphism. We also have the short exact sequence \( 0 \to E_n \to C_n \xrightarrow{d} B_{n-1}(C) \to 0 \). Since \( H_n(C) \) is a free \( R \)-module by hypothesis, this exact sequence splits. Thus \( Z_n(C) = B_n(C) \oplus F_n \) where \( F_n : F_n \to H_n(C) \) is an \( R \)-isomorphism. Thus, for each \( n \), \( C_n = B_n(C) \oplus F_n \oplus E_n \). In terms of this decomposition, the map \( d : C_n \to C_{n-1} \) is described by the diagram:

\[
\begin{array}{ccc}
C_n = B_n(C) \oplus F_n \oplus E_n & \xrightarrow{d} & C_{n-1} = B_{n-1}(C) \oplus F_{n-1} \oplus E_{n-1} \\
\downarrow & \cong & \downarrow \\
0 & \cong & 0
\end{array}
\]

Before \( f : C \to D \) as follows: For each \( n \), \( f_n \) is described by...
the diagram

\[ C_n = B_n(C) \oplus F_n \oplus E_n \]

\[ f_n \downarrow \quad \cong \quad f_n \downarrow \]

\[ D_n = H_n(C) \]

It is easily checked that \( \partial f_n = 0 = f_{n-1} \) for all \( n \), and so \( f \) is a chain map.

Define \( g : D \to C \) as follows: for each \( n \), \( g_n \) is described by the diagram

\[ D_n = H_n(C) \]

\[ g_n \downarrow \quad \cong \quad (f_n f_n)^{-1} \]

\[ C_n = B_n(C) \oplus F_n \oplus E_n \]

It is easily checked that \( \partial g_n = 0 = g_{n-1} \) and so \( g \) is a chain map.

Note that \( f g = 1_D \). It remains only to show \( g f = 1_C \).

Define a chain homotopy \( T : C \to C \) as follows: for each \( n \), \( T_n \) is described by the diagram

\[ C_n = B_n(C) \oplus F_n \oplus E_n \]

\[ T_n \downarrow \quad \cong \quad (\partial |_{En})^{-1} \]

\[ C_{n+1} = B_{n+1}(C) \oplus F_{n+1} \oplus E_{n+1} \]

\( \forall \, b \in B_n(C), \ (\partial T_n + T_n \partial)(b) = \partial T_n (b) = \partial \partial^{-1}(b) = b \)

\[ = T_n (b) - g_{n+1} f_n (b) \text{ since } f_n (b) = 0. \]

\( \forall \, e \in F_n, \text{ then } (\partial T_n + T_n \partial)(e) = 0 = 1_{cn}(e) - g_n f_n (e) \text{ since } \)

\( g_n f_n (e) = f_n^{-1} f_n (e) = e. \)
If $e \in E_n$, then $(\partial T_n + T_{n-1}) (e) = T_{n-1} \partial (e) = e = 1_{c_n} (e) - g_n f_n (e)$ since $f_n (e) = 0$.

Thus $\partial T_n + T_{n-1} = 1_{c_n} - g_n f_n$, and so $T$ is a chain homotopy from $g f$ to $1_c$.

**Theorem 15.19.** (Important special case of the Künneth Theorem.) Let $R$ be a PID. Suppose $C$, $C'$ are chain complexes of $R$-modules such that for all $n$, $C_n$ and $H_n (C')$ are free $R$-modules. Then the homology cross product $\mu : [H(C) \otimes_R H(C')]_n \rightarrow H_n (C \otimes_R C')$ is an $R$-isomorphism.

**Proof.** By 15.18, there exists a chain equivalence over $R$ $f : C' \rightarrow D$ where $D$ is a chain complex of $R$-modules with $\partial D = 0$. Since $H_n (f) : H_n (C) \rightarrow H_n (D) = D_n$ is an $R$-isomorphism and since the $H_n (C')$ are free $R$-modules, the $D_n$ are free, and hence flat, $R$-modules. Thus by 15.11,

$\mu : [H(C) \otimes_R H(D)]_n \rightarrow H_n (C \otimes_R D)$ is an $R$-isomorphism.

Since $f$ is a chain equivalence, so is $1_c \otimes f : C \otimes_R C' \rightarrow C \otimes_R D$ (for if $g : D \rightarrow C'$ is a chain map such that $g f \cong 1_D$, $g f \cong 1_{C'}$, then by 15.14, $1_c \otimes f (1_c \otimes g) = 1_c \otimes g f \cong 1_c \otimes 1_{C'} = 1_c \otimes f (1_c \otimes 1_{C'})$, and similarly $(1_c \otimes f)(1_c \otimes g) \cong 1_c \otimes f (1_c \otimes D)$. Thus $H_n (1_c \otimes f)$ is an $R$-isomorphism for all $n$. We also have $[H(1_c) \otimes H(f)]_n$ is an $R$-isomorphism since each $H_i (f)$ is. The result now follows from commutativity of the diagram:

$$
\begin{array}{ccc}
[H(C) \otimes_R H(C')]_n & \xrightarrow{\mu} & H_n (C \otimes_R C') \\
[H(1_c) \otimes H(f)]_n & \cong & H_n (1_c \otimes f) \\
[H(C) \otimes_R H(D)]_n & \xrightarrow{\cong} & H_n (C \otimes_R D) \\
\end{array}
$$

15.19, together with 14.35, yields

**Corollary 15.20.** Let $K$ and $L$ be finite regular CW complexes. Suppose $H_n (K)$ is free abelian for all $n$. Then $H_n (K \times L) \cong [H(K) \otimes H(L)]_n = \bigoplus_{i+j=n} H_i (K) \otimes H_j (L)$ for all $n$. 


16. Resolutions, Tor, and the K"unneth Theorem

The fact that the homology cross product is not always an isomorphism stems from the fact that not all \( R \)-modules are flat. (If \( R \) is a field, all \( R \)-modules are free, and hence flat, and so by 15.19, the homology cross product is always an isomorphism in this case.) We proceed to develop the homological algebra necessary to describe the \( H_n(C \otimes C') \) without the assumption that the \( H_1(C') \) are free.

We continue to let \( R \) denote a general commutative ring with \( 1 \neq 0 \). When we need \( R \) to be a PID, we will state so explicitly.

Proposition 16.1. Let \( 0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0 \) be an exact sequence of \( R \)-modules and \( R \)-homomorphisms. (Note: \( f \) is not assumed to be \( 1 \)-1.) Then for any \( R \)-module \( A \), the sequence

\[
0 \to X \otimes_R A \xrightarrow{f \otimes 1_A} Y \otimes_R A \xrightarrow{g \otimes 1_A} Z \otimes_R A \to 0
\]

is exact.

Proof. Exactness at \( Z \otimes_R A \) follows from 14.21. Moreover, by 14.21, \( \ker (g \otimes 1_A) \) is the \( R \)-submodule of \( Y \otimes_R A \) generated by all \( y \otimes a \) where \( y \in \ker g \) (since \( \ker 1_A = 0 \)).

If \( y \in \ker g \), then exactness of the quiver sequence yields \( y = f(x) \) for some \( x \in X \). Thus \( y \otimes a = (f \otimes 1_A)(x \otimes a) \) and so \( y \otimes a \in \im (f \otimes 1_A) \). Thus \( \ker (g \otimes 1_A) \subseteq \im (f \otimes 1_A) \). Since the \( y \otimes a \) as above generate \( \ker (g \otimes 1_A) \).

We have \( g \otimes 1 = 0 \) by exactness of the quiver sequence.

Thus \( (g \otimes 1_A)(f \otimes 1_A) = (g \otimes 1_A) \otimes 1_A = 0 \otimes 1_A = 0 \) and so \( \im (f \otimes 1_A) \subseteq \ker (g \otimes 1_A) \). Thus exactness at \( Y \otimes_R A \) holds.

Recall that \( \otimes 1_A \) does not, in general, preserve injectivity. However, we do have the following:

Exercise 16.2. Let \( 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \) be a short exact sequence of \( R \)-modules which splits. Then for any \( R \)-module \( A \),
\[ 0 \to \bigoplus A \to Y \otimes_R A \to Z \otimes_R A \to 0 \]
is exact and splits.

**Definition 16.3.** Let \( R \) be a PID, and \( A \) an \( R \)-module.
A short free resolution of \( A \) is an exact sequence
\[ 0 \to C_1 \to C_0 \to A \to 0 \]
of \( R \)-modules and \( R \)-homomorphisms such that \( C_0 \) and \( C_1 \) are free \( R \)-modules.

Thus a short free resolution of \( A \) can be thought of as an acyclic \( A \)-augmented chain complex with free chain groups.
A short free resolution of \( A \) is, essentially, a presentation of \( A \) by generators and relations.

**Example 16.4.** Let \( A \) be any \( R \)-module, \( R \) a PID. Let \( \text{Co}(A) = \bigoplus A \), the free \( R \)-module on the set \( A \). To avoid confusion between elements of \( \text{Fr}(A) \) and the \( R \)-module \( A \), for each \( a \in A \) write \( \bar{a} \in \text{Fr}(A) \) for the element \( 1 \cdot a \). (Note: In general, \( a + b \neq 1 \cdot a + b \), \( \bar{r}a \neq r\bar{a} \) in \( \text{Fr}(A) \).) Define \( E_A : \text{Co}(A) \to A \) to be the unique \( R \)-homomorphism such that \( E_A(\bar{a}) = a \) for each \( a \in A \). Define \( C_1(A) = \ker E_A \), and let \( \partial_A : C_1(A) \to \text{Co}(A) \) be the inclusion. \( C_1(A) \) is free, being a submodule of the free \( R \)-module \( \text{Co}(A) \). Thus \( 0 \to C_1(A) \xrightarrow{\partial_A} \text{Co}(A) \xrightarrow{\text{Fr}} A \to 0 \) is a short free resolution of \( A \), the canonical resolution of \( A \).

**Example 16.5.** For each positive integer \( n \),
\[ 0 \to 2 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z}/n \to 0 \]
is a short free resolution of \( \mathbb{Z}/n \) over \( \mathbb{Z} \), where \( \mathbb{Z}/n(x) = nx \).

**Definition 16.6.** Let \( R \) be a PID and suppose \( A, B \) are \( R \)-modules. We define \( \text{Tor}^R(A, B) = \ker \left[ C_1(A) \otimes_R B \xrightarrow{\partial_1} \text{Co}(A) \otimes_R B \right] \).
Thus for any $R$-modules $A$ and $B$ ($R$ a PID), the sequence

$$0 \to \text{Tor}_R^1(A, B) \to C_1(A) \otimes_R B \xrightarrow{\partial_A \otimes 1_B} C_0(A) \otimes_R B \xrightarrow{\varepsilon_A \otimes 1_B} A \otimes_R B \to 0$$

is exact.

Let $C(A)$ denote the chain complex given by

$$C(A)_i = \begin{cases} C_i(A) & \text{if } i \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

with $\partial_A : C_1(A) \to C_0(A)$ the non-trivial boundary homomorphism. Write $C(A) \otimes_R B$ for the chain complex $C(A) \otimes_R B^0$ (see 15.9). Thus $\varepsilon_A \otimes 1_B$ induces an $R$-isomorphism $H_0(C(A) \otimes_R B) \cong A \otimes_R B$, and $H_i(C(A) \otimes_R B) = \text{Tor}_R^i(A, B)$, and $H_i(C(A) \otimes_R B) = 0$ for $i \neq 0$.

If $R$ is not a PID, short free resolutions are not always possible (long resolutions 

$$\cdots \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to A \to 0$$

always are) and one obtains higher non-trivial $H_i$.

Suppose $f : A \to A'$ is an $R$-homomorphism ($R$ a PID). Let $C_0(f) : C_0(A) \to C_0(A')$ be the unique $R$-homomorphism such that $C_0(f)(a) = f(a)$ for each $a \in A$. Then

$$\begin{array}{ccc}
C_0(A) & \xrightarrow{\varepsilon_A} & A \\
\downarrow{C_0(f)} & & \downarrow{f} \\
C_0(A') & \xrightarrow{\varepsilon_{A'}} & A'
\end{array}$$

commutes; and so $C_0(f)$ carries $C_i(A)$ into $C_i(A')$. Thus, by restriction, we obtain an $R$-homomorphism

$$C_1(f) : C_1(A) \to C_1(A').$$

Proposition 16.7. With notation as above, $C_0(f)$ and $C_1(f)$ constitute a chain map $C(f) : C_0(A) \to C_0(A')$. The rule which assigns to each $R$-module $A$ the chain complex
C(A), and to each R-homomorphism \( f \) the chain map\( C(f) \), is a covariant functor \( C: \mathcal{M}_R \to \mathcal{C}(R) \) (\( R \) a PID).

**Definition 16.8.** Let \( R \) be a PID and \( f: A \to A', g: B \to B' \) R-homomorphisms. Let \( g^{(o)}: B^{(o)} \to \left(B'\right)^{o} \) denote the chain map given by \( (g^{(o)})_o = g \). Thus we get a chain map \( C(f) \otimes g^{(o)}: C(A) \otimes R B \to C(A') \otimes R B' \). Define \( \text{Tor}(f, g): \text{Tor}^R(A, B) \to \text{Tor}^R(A', B') \) to be the R-homomorphism \( H_1(C(f) \otimes g^{(o)}) \).

Thus \( \text{Tor}(f, g) \) is the unique R-homomorphism such that

\[
\begin{array}{ccc}
\text{Tor}^R(A, B) & \longrightarrow & C_1(A) \otimes R B \\
\text{Tor}(f, g) \downarrow & & \downarrow \text{C}_1(f) \otimes g \\
\text{Tor}^R(A', B') & \longrightarrow & C_1(A') \otimes R B'
\end{array}
\]

commutes.

**Lemma 16.9.** Let \( R \) be a PID. The rule which assigns to each ordered pair \((A, B)\) of \( R \)-modules the \( R \)-module \( \text{Tor}^R(A, B) \), and to each ordered pair of \( R \)-homomorphisms \((f, g)\) the \( R \)-homomorphism \( \text{Tor}(f, g) \), is a covariant functor \( \text{Tor}: \mathcal{M}_R \to \mathcal{M}_R \).

**Lemma 16.10.** Let \( R \) be a PID. If \( A \) and \( B \) are \( R \)-modules, let \( E_{A, B} : H_0(C(A) \otimes R B) \to A \otimes R B \) denote the \( R \)-isomorphism induced by \( E_{A, B} : C_0(A) \otimes R B \to A \otimes R B \). Suppose \( f: A \to A', g: B \to B' \) are \( R \)-homomorphisms. Then the diagram

\[
\begin{array}{ccc}
H_0(C(A) \otimes R B) & \xrightarrow{E_{A, B}} & A \otimes R B \\
\downarrow H_0(C(f) \otimes g^{(o)}) & & \downarrow f \otimes g \\
H_0(C(A') \otimes R B') & \xrightarrow{E_{A', B'}} & A' \otimes R B'
\end{array}
\]

commutes.
Since, by definition, \( A_{IB} \) preserves exactness when \( B \) is flat, we have

**Proposition 16.11.** Let \( R \) be a PID, and \( B \) a flat \( R \)-module. Then for any \( R \)-module \( A \), \( Tor^R(A, B) = 0 \).

Tor is not easily computable from the above definition since the canonical resolutions are too long. The canonical resolutions serve primarily to establish Tor as a functor. The following theorem enables us to get away from using the canonical resolutions.

**Theorem 16.12.** (Fundamental Theorem of Homological Algebra for PID's). Let \( R \) be a PID. Suppose \( A, B \) are \( R \)-modules, and suppose \( 0 \to C_1 \xrightarrow{b} C_0 \xrightarrow{e} A \to 0 \), \( 0 \to D_1 \xrightarrow{d} D_0 \xrightarrow{e} B \to 0 \) are arbitrary short free resolutions over \( R \), and suppose \( f : A \to B \) is an \( R \)-homomorphism. Then

1) (Existence) there exists a chain map \( g : C \to D \) such that

\[
\begin{array}{ccc}
C_0 & \xrightarrow{e} & A \\
\downarrow{g_0} & & \downarrow{f} \\
D_0 & \xrightarrow{e} & B 
\end{array}
\]

commutes.

2) (Uniqueness) Any two chain maps as in (1) are chain homotopic.

**Proof.** 1) Let \( X \) be an \( R \)-basis for \( C_0 \). For each \( x \in X \), choose any \( dx \in D_0 \) such that \( e(dx) = f(x) \). This is possible since \( e : D_0 \to B \) is onto. Let \( g_0 : C_0 \to D_0 \) be the unique \( R \)-homomorphism which satisfies \( g_0(x) = dx \) for each \( x \in X \). Then the required diagram commutes. Thus, \( g_0 \) carries \( \ker (\partial : C \to C_0) \) into \( \ker (\partial : D \to D_0) \). Since both \( \partial \)'s are injective, there exists a unique \( R \)-homomorphism \( g_1 : C_1 \to D_1 \) such that

\[
\begin{array}{ccc}
C_1 & \xrightarrow{b} & C_0 \\
\downarrow{g_1} & & \downarrow{\partial} \\
D_1 & \xrightarrow{d} & D_0 
\end{array}
\]
commutes. Then \( g_0, g_1 \) constitute a chain map \( g : C \to D \) satisfying the requirements of \( D \).

2) Suppose \( g, h : C \to D \) are chain maps satisfying \( D \). Let \( X \) be as above. For each \( x \in X \) we have
\[
\varepsilon(h_0(x) - g_0(x)) = h \varepsilon(x) - g \varepsilon(x) = 0.
\]
Thus, by exactness of \( D \), there exists \( e_x \in D_1 \) such that
\[
g_0(x) = h_0(x) - g_0(x).
\]
Set \( T_0 : C \to D \), be the unique \( R \)-homomorphism such that \( T_0(x) = e_x \) for each \( x \in X \). Define \( T_i : C_i \to D_{i+1} \) to be 0 for \( i \not= 0 \). Then
\[
\partial T_0 + T_1 \partial = \partial T_0 = h_0 - g_0.
\]
We have
\[
\partial(\partial T_1 + T_0) = \partial \varepsilon(h_1 - g_1).
\]
Thus since \( \partial : D_1 \to D_0 \) is injective, we have \( \partial T_1 + T_0 \partial = h_1 - g_1 \).

Thus \( T \) is a chain homotopy from \( g \) to \( h \).

Note the similarity between 16.12 and the Cech's Model's Theorem.

**Corollary 16.13.** Let \( R \) be a PID and \( A \) an \( R \)-module. Let \( C \) and \( D \) be short free resolutions of \( A \) over \( R \). Then \( C \) and \( D \) are chain equivalent. In fact, any augmentation-preserving chain map \( f : C \to D \) (i.e., a chain map such that
\[
\begin{array}{ccc}
C_0 & \to & D_0 \\
\varepsilon & & \varepsilon \\
\downarrow & & \downarrow \\
A & \to & E
\end{array}
\]
commutes) is a chain equivalence. Moreover, any two augmentation-preserving chain equivalences from \( C \) to \( D \) are chain homotopic.

Thus if \( A \) is an \( R \)-module (\( R \) a PID) and \( C \) is any short free resolution of \( A \), then exists an augmentation-
preserving chain equivalence \( h : C \to C(A) \) which is unique up to chain homotopy. To any \( R \)-module \( B \), \( h \otimes 1_B : C \otimes_R B \to C(A) \otimes_R B \) is a chain equivalence (we write \( C \otimes_R B = C \otimes_R B^{(1)} \)) and hence

\[
H_1(h \otimes 1_B) : H_1(C \otimes_R B) \to H_1(C(A) \otimes_R B) = \mathbb{Z}_R^R(A, B)
\]

is an \( R \)-isomorphism. Since \( h \) is unique up to chain homotopy, \( H_1(h \otimes 1_B) \) does not depend on the choice of augmentation-preserving chain equivalence \( h \).

Identifying \( \ker (C \otimes_R B \otimes 1_B \to C \otimes_R B) = H_1(C \otimes_R B) \) with \( \mathbb{Z}_R^R(A, B) \) and this isomorphism, we get an exact sequence

\[
0 \to \mathbb{Z}_R^R(A, B) \to C \otimes_R B \otimes 1_B \to C \otimes_R B \otimes 1_B \to A \otimes_R B \to 0
\]

for any short free resolution \( C \) of \( A \).

**Proposition 16.14.** Let \( R \) be a PID. Suppose \( f : A \to A' \), \( g : B \to B' \) are \( R \)-homomorphisms. Let \( C, C' \) be short free resolutions over \( R \) of \( A \) and \( A' \), respectively. Suppose \( h : C \to C' \) is a chain map such that

\[
C_0 \xrightarrow{\varepsilon} A \\
\downarrow h_0 \quad \downarrow f \\
C_0' \xrightarrow{\varepsilon} A'
\]

commutes. Then the diagram

\[
0 \to \mathbb{Z}_R^R(A, B) \to C \otimes_R B \otimes 1_B \to C \otimes_R B \otimes 1_B \to A \otimes_R B \to 0
\]

\[
\mathbb{Z}_R^{f, g} \left\vert \mathbb{Z}_R^{h, g} \right. \mathbb{Z}_R^{h, g} \downarrow \mathbb{Z}_R^{h, g} \downarrow h \circ g \\
0 \to \mathbb{Z}_R^R(A', B') \to C \otimes_R B' \otimes 1_{B'} \to C \otimes_R B' \otimes 1_{B'} \to A' \otimes_R B' \to 0
\]

commutes.

**Proof.** The only non-trivial part is commutativity of the left-hand square. Let \( R : C \to C(A) \), \( l : C' \to C(A') \) be any augmentation-preserving
chain equivalences. Consider the diagram

\[
\begin{array}{ccc}
\text{Tor}^R(A, B) & \leftarrow & H_1(k \otimes_B 1_B) \\
\downarrow H_1(C(\mathcal{f}) \otimes g) & \cong & H_1(C \otimes_R B) & \rightarrow & C \otimes_R B \\
\text{Tor}^R(A', B') & \leftarrow & H_1(l \otimes_B 1_B) \\
\end{array}
\]

It remains only to check commutativity of the left hand square. We have \( H_1(C(\mathcal{f}) \otimes g) = H_1(C(\mathcal{f}) k \otimes g), \)
\( H_1(l \otimes_B 1_B) = H_1(l h \otimes g). \) Thus it suffices to show
\( C(\mathcal{f}) k \cong l h. \) Now \( C(\mathcal{f}) k, l h : C \rightarrow C(A') \) are chain maps such that

\[
\begin{array}{ccc}
C_0 & \xrightarrow{\varepsilon} & A & \xleftarrow{\varepsilon} & C_0 \\
C_0(\mathcal{f}) k & \downarrow & l h & \downarrow & l h \\
C_0(A') & \xrightarrow{\varepsilon_{A'}} & A' & \xleftarrow{\varepsilon_{A'}} & C_0
\end{array}
\]

commutes. Thus, by the Fundamental Theorem of Homological Algebra, \( C(\mathcal{f}) k \cong l h. \)

**Proposition 16.15.** Let \( R \) be a PID. Suppose \( A, B \) are \( R \)-modules such that either \( A \) or \( B \) is free. Then
\( \text{Tor}^R(A, B) = 0. \)

**Proof.** If \( B \) is free, then \( B \) is flat and the result follows from 16.11. If \( A \) is free, then \( 0 \rightarrow 0 \rightarrow A \rightarrow 0 \) is a short free resolution of \( A \), and so

\[
0 \rightarrow \text{Tor}^R(A, B) \rightarrow C \otimes_R B \rightarrow A \otimes_R B \rightarrow A \otimes_R B \rightarrow 0
\]

is exact, and the result follows.

**Example 16.16.** We calculate \( \text{Tor}^2(\mathbb{Z}/m, \mathbb{Z}/n) \) where \( m, n \) are positive integers.

\[
0 \rightarrow \mathbb{Z} \xrightarrow{fm} \mathbb{Z} \xrightarrow{p_m} \mathbb{Z}/m \rightarrow 0
\]
is a short free resolution of $\mathbb{Z}/m$, where $f_m(x) = mx$. Thus

$$0 \to \text{Tor}_2^\mathbb{Z}(\mathbb{Z}/m, \mathbb{Z}/n) \to \mathbb{Z}^\oplus \mathbb{Z}/n \xrightarrow{f_m \otimes 1_{\mathbb{Z}/n}} \mathbb{Z}^\oplus \mathbb{Z}/n \xrightarrow{0} \mathbb{Z}/m \otimes \mathbb{Z}/n \to 0$$

is exact. Thus $\text{Tor}_2^\mathbb{Z}(\mathbb{Z}/m, \mathbb{Z}/n)$ is cyclic, being a subgroup of $\mathbb{Z}^\oplus \mathbb{Z}/n \cong \mathbb{Z}/n$. We write $|G|$ in the order of $G$, by exactness,

$$|\text{Tor}_2^\mathbb{Z}(\mathbb{Z}/m, \mathbb{Z}/n)| = \left| \ker (f_m \otimes 1_{\mathbb{Z}/n}) \right| = n - \left| \text{im} (f_m \otimes 1_{\mathbb{Z}/n}) \right|$$

$$= n - \left| \ker (f_m \otimes 1_{\mathbb{Z}/n}) \right| = \left| \text{im} (f_m \otimes 1_{\mathbb{Z}/n}) \right| = \left| \mathbb{Z}/m \otimes \mathbb{Z}/n \right|$$

$$= \gcd(m, n).$$

Thus $\text{Tor}_2^\mathbb{Z}(\mathbb{Z}/m, \mathbb{Z}/n) \cong \mathbb{Z}/d$ where

$$d = \gcd(m, n).$$

**Proposition 16.17.** Let $R$ be a PID, $\{A_\alpha | \alpha \in J\}$ an indexed family of $R$-modules, and $B$ an $R$-module. For each $\beta \in J$, let $i_\beta : A_\beta \to \bigoplus_{\alpha \in J} A_\alpha$ denote the canonical inclusion. Then

1) $\sum \text{Tor}(i_\beta, 1_B) : \bigoplus_{\alpha \in J} \text{Tor}^R(A_\alpha, B) \to \text{Tor}^R(\bigoplus_{\alpha \in J} A_\alpha, B)$

is an $R$-isomorphism.

2) $\sum \text{Tor}(1_B, i_\alpha) : \bigoplus_{\alpha \in J} \text{Tor}^R(B, A_\alpha) \to \text{Tor}^R(B, \bigoplus_{\alpha \in J} A_\alpha)$

is an $R$-isomorphism.

**Proof.** $0 \to \bigoplus_{\alpha \in J} \text{C}i(A_\alpha) \xrightarrow{\bigoplus_{\alpha \in J} \text{d}_{A_\alpha}} \bigoplus_{\alpha \in J} \text{C}o(A_\alpha) \xrightarrow{\bigoplus_{\alpha \in J} \text{f}_{A_\alpha}} \bigoplus_{\alpha \in J} A_\alpha \to 0$

is a short free resolution of $\bigoplus_{\alpha \in J} A_\alpha$. Thus by 16.14 we have the commutative diagram with exact rows.
\[
0 \rightarrow \mathcal{J} \mathbb{R}(A_\alpha, B) \rightarrow C_i(A_\alpha) \otimes B \xrightarrow{\partial_i} C_0(A_\alpha) \otimes B
\]

Thus we have the commutative diagram with exact rows

\[
0 \rightarrow \bigoplus_{\alpha \in J} \mathcal{J} \mathbb{R}(A_\alpha, B) \rightarrow \bigoplus_{\alpha \in J} (C_i(A_\alpha) \otimes B) \xrightarrow{\bigoplus \partial_i} \bigoplus_{\alpha \in J} (C_0(A_\alpha) \otimes B)
\]

Note that \( C_i(i_\alpha) : C_i(A_\alpha) \rightarrow \bigoplus_{\alpha \in J} C_i(A_\alpha) \) is the canonical inclusion for \( i = 0, 1 \). Thus the \( \Sigma C_i(i_\alpha) \otimes 1_B, i = 0, 1 \), are \( R \)-isomorphisms by 14.18. Thus \( \Sigma \mathcal{J} \mathbb{R}(i_\alpha, 1_B) \) is an \( R \)-isomorphism by the 5-lemma, proving 1).

For each \( \beta \in J \) we have, by 16.14, the commutative diagram with exact rows

\[
0 \rightarrow \mathcal{J} \mathbb{R}(B, A_\beta) \rightarrow C_1(B) \otimes A_\beta \xrightarrow{\partial_1 \otimes 1_B} C_0(B) \otimes A_\beta
\]

Thus we have the commutative diagram with exact rows.
\[ 0 \to \bigoplus_{x \in J} \mathcal{I}_R^x(B, A_x) \to \bigoplus_{x \in J} (C(B) \otimes_R A_x) \to \bigoplus_{x \in J} (C_0(B) \otimes_R A_x) \]

The two right vertical maps are \( R \)-isomorphisms by 14.18, and so \( \Sigma \mathcal{I}_R^x(B, i_x) \) is an \( R \)-isomorphism by the S-lemma, proving 2).

Exercise 16.18. Let \( R \) be arbitrary, and \( C \) and \( D \) chain complexes of \( R \)-modules. Then there exists a natural chain morphism \( \tau : C \otimes_R D \to D \otimes_R C \) characterized by \( \tau(a \otimes b) = (-1)^{pq} b \otimes a \), whenever \( a \in C_p \), \( b \in D_q \). If \( C \) and \( D \) are \( R \)-augmented, \( \tau \) preserves augmentation. Moreover,

\[
\begin{array}{ccc}
\mathsf{H}_n(C \otimes_R D) & \to & \mathsf{H}_n(D \otimes_R C) \\
\mu & & \\
\downarrow \quad \tau & & \\
\mathsf{H}_n(C) \otimes_R \mathsf{H}_n(D) & \to & \mathsf{H}_n(D) \otimes_R \mathsf{H}_n(C)
\end{array}
\]

commutes, where \( \tau_n \) is the \( R \)-isomorphism characterized by \( \tau_n(x \otimes y) = (-1)^{pq} y \otimes x \), whenever \( x \in \mathsf{H}_p(C) \), \( y \in \mathsf{H}_q(D) \).

Corollary 16.19. Let \( R,C \), and \( D \) as in 16.18,

\[ \mu : [\mathsf{H}(C) \otimes_R \mathsf{H}(D)]_n \to \mathsf{H}_n(C \otimes_R D) \]

is an \( R \)-isomorphism if and only if \( \mu : [\mathsf{H}(D) \otimes_R \mathsf{H}(C)]_n \to \mathsf{H}_n(D \otimes_R C) \) is an \( R \)-isomorphism.

Let \( R \) be a PID and \( C,D \) chain complexes of \( R \)-modules. For each integer \( n \) we write

\[ [\mathcal{I}_R^x(H(C), H(D))]_n = \bigoplus_{i+j=n} \mathcal{I}_R^x(H_i(C), H_j(D)) \]
Theorem 16.20. (Künneth Theorem). Let $R$ be a PID. Suppose $C$ and $D$ are chain complexes of $R$-modules such that each $C_n$ is a free $R$-module. Then there is a natural exact sequence

$$0 \to [\text{Hom}_R(C, R)^{\otimes} \otimes R \text{Hom}_R(D, R)]_n \to \text{Hom}_R(C \otimes_R D) \to \text{Hom}^R(\text{Hom}_R(C, R), \text{Hom}_R(D, R))]_{n-1} \to 0$$

where $\otimes$ is the homology cross product.

Moreover, if $Z_n(D)$ is a direct summand of $D_n$ for each $n$, the above sequence splits (but not naturally).

Remark 16.21. A "natural splitting" would mean that there is an $R$-homomorphism $\nu: \text{Hom}_R(C \otimes_R D) \to [\text{Hom}_R(C, R)^{\otimes} \otimes R \text{Hom}_R(D, R)]_n$ such that $\nu 1 = 1$ and such that $\nu$ is natural, i.e. whenever $(C', D')$ is another pair of chain complexes satisfying the hypotheses of 16.20 and $f: C \to C'$, $g: D \to D'$ are chain maps, then the diagram

$$
\begin{array}{ccc}
[\text{Hom}_R(C, R)^{\otimes} \otimes R \text{Hom}_R(D, R)]_n & \leftarrow & \text{Hom}_R(C \otimes_R D) \\
\downarrow & & \downarrow \text{Hom}(f \otimes g) \\
[\text{Hom}_R(C', R)^{\otimes} \otimes R \text{Hom}_R(D', R)]_n & \leftarrow & \text{Hom}_R(C' \otimes_R D')
\end{array}
$$

commutes. It is known that no such natural $\nu$ exists.

Remark 16.22. Splitting of the sequence in 16.20 will occur in either of the following situations:

1) If each $D_n$ is free, then $0 \to Z_n(D) \to D_n \to B_{n-1}(D) \to 0$ is exact and splits since each $B_{n-1}(D)$ is free, being an $R$-submodule of the free $R$-module $D_n$.

2) If $Z = 0$ in $D$, then $Z_n(D) = D_n$.

Proof of 16.20. Let $Z$ denote the chain complex with $Z_n = Z_n(C)$, $Z = 0$. Let $B$ denote the chain complex with $B_n = B_n - (C)$, $B = 0$. The inclusion $i: Z \to C$, and the map $j: C \to B$ obtained by restricting the ranges of the $j: C_n \to C_{n-1}$, are chain maps, and the sequence $0 \to Z \xrightarrow{i} C \xrightarrow{j} B \to 0$ is a short exact
sequence of chain complexes which splits in each dimension (since \( B_n \) is free, being an \( R \)-submodule of the free \( R \)-module \( C_{n-1} \)). (Note, however, that the splitting is not necessarily a splitting as chain complexes, i.e. the splitting maps \( Z_n \leftarrow C_n \) and \( C_n \leftarrow B_n \) for each \( n \) need not constitute chain maps \( Z \leftarrow C \) and \( C \leftarrow B \).) Thus, because of the above splitting, the sequence of chain maps

\[
0 \to Z \otimes_R D \xrightarrow{i \circ 1_D} C \otimes_R D \xrightarrow{\partial \otimes 1_D} B \otimes_R D \to 0
\]

is exact. Thus for each \( n \) we have an exact sequence of \( R \)-modules

\[
H_{n+1}(B \otimes_R D) \xrightarrow{\bar{j}} H_n(Z \otimes_R D) \xrightarrow{H_n(i \otimes 1_D)} H_n(C \otimes_R D) \xrightarrow{H_n(\partial \otimes 1_D)} H_n(B \otimes_R D) \xrightarrow{\bar{j}} H_{n-1}(Z \otimes_R D)
\]

where \( \bar{j} \) is the connecting homomorphism arising from the above short exact sequence of \( R \)-modules.

In particular we have \( H_{p+1}(B) = B_{p+1} = B_p(C) \), \( H_p(Z) = Z_p = Z_p(C) \). Write \( j_p : H_{p+1}(B) \to H_p(Z) \) for the inclusion \( B_p(C) \to Z_p(C) \), and

\[
(j \otimes 1)_n = \bigoplus_{p+q=n} (j_p \otimes 1_{H_q(D)}) : [H(B) \otimes_R H(D)]_{n+1} \to [H(Z) \otimes_R H(D)]_n.
\]

We claim the diagram

\[
\begin{array}{ccc}
H_{n+1}(B \otimes_R D) & \xrightarrow{\bar{j}} & H_n(Z \otimes_R D) \\
\downarrow M_B & & \downarrow M_Z \\
H_{n+1}(Z \otimes_R D) & \xrightarrow{\bar{j}} & H_n(C \otimes_R D)
\end{array}
\]

commutes where \( M_B, M_Z \) are the homology cross products. In fact if \( b \in B_p(C) \) and \( z \in Z_q(D) \), we have

\[
M_B(j_p \otimes 1)(b \otimes [z]) = M_Z([b] \otimes [z]) = [b \otimes z], \quad \bar{j} M_B([b] \otimes [z]) = \bar{j} [b \otimes z].
\]

Consider the zig-zag diagram defining \( \bar{j} \):

\[
(C \otimes_R D)_{n+1} \xrightarrow{(i \otimes 1)_n} (Z \otimes_R D)_n \xrightarrow{\bar{j}} (C \otimes_R D)_n
\]
Let \( cp \) be such that \( c = b \). Then \( c \otimes z \) is a pull-back of \( b \otimes z \) under \( (\otimes 1)_{n+1} \). We have \( \partial (c \otimes z) = (\partial c) \otimes z + c \otimes (\partial z) \) \( = (\partial c) \otimes z = b \otimes z \), and \( (\otimes 1)_{n} (b \otimes z) = b \otimes z \). Thus \( \partial [b \otimes z] = [b \otimes z] \), proving the claim.

Note that the projection \( Zp(c) \rightarrow H_p(c) \) is the homology homomorphism \( H_p(i) : H_p(Z) \rightarrow H_p(C) \). Thus

\[ 0 \rightarrow H_{p+1}(B) \xrightarrow{\partial_p} H_p(Z) \xrightarrow{\partial_p(i)} H_p(C) \rightarrow 0 \]

is a short free resolution of \( H_p(c) \). Thus for each \( q \) we get a natural exact sequence

\[ 0 \rightarrow [\partial^R (H_p(c), H_q(D))]_{n+1} \rightarrow [H(B) \otimes_R H(D)]_{n+1} \rightarrow [H(Z) \otimes_R H(D)]_n \rightarrow [H(C) \otimes_R H(D)]_n \rightarrow 0. \]

Summing over \( p+q = n \) we obtain a natural exact sequence

\[ 0 \rightarrow [\partial^R (H(C), H(D))]_n \rightarrow [H(B) \otimes_R H(D)]_n \rightarrow [H(Z) \otimes_R H(D)]_n \rightarrow [H(C) \otimes_R H(D)]_n \rightarrow 0. \]

Thus we obtain the commutative diagram with exact rows

\[
\begin{array}{ccccccc}
& & H_{n+1}(B \otimes_R D) & \xrightarrow{3} & H_n(Z \otimes_R D) & \xrightarrow{H_n(\otimes 1)} & H_n(C \otimes_R D) & \xrightarrow{H_n(\otimes 1)} & H_{n-1}(Z \otimes_R D) \\
& & \downarrow \mu_3 & & \downarrow \mu & & \downarrow \mu & & \downarrow \mu \\
H_{n+1}(B \otimes_R D) & \xrightarrow{3} & H_n(Z \otimes_R D) & \xrightarrow{H_n(\otimes 1)} & H_n(C \otimes_R D) & \xrightarrow{H_n(\otimes 1)} & H_{n-1}(Z \otimes_R D) \\
\end{array}
\]

The \( \mu_3 \) and \( \mu_2 \) are \( R \)-isomorphisms by 15.11 and 16.17. The desired exact sequence now follows by diagram chasing in the above diagram. Naturality of \( \otimes \) follows easily by naturality of this diagram and 16.14.

It remains to prove splitting under the hypothesis that \( Z_n(D) \) is a direct summand of \( D_n \) for each \( n \). We proceed to construct an \( R \)-homomorphism \( \gamma : H_n(C \otimes_R D) \rightarrow [H(C) \otimes_R H(D)]_n \) such that \( \gamma \mu = 1 \).

Since each \( C_p \) is free, it follows that \( Z_p(c) \) is a direct summand of \( C_p \) for each \( p \). By hypothesis, \( Z_p(D) \) is a direct
As a corollary we have

\[ D_{122}^{[2]} = 1 \]

and so the diagonal map

\[ f = 1 \]

Zero. Hence, passing to $\text{Gr}^j$ at the origin of $x$, $y$, and $z$, we have

\[ 0 \]

and so

\[ (0) \]
Proposition 16.23. Let \( R \) be a PID. Then there is a natural \( R \)-isomorphism \( \mathcal{I}_R(A, B) \cong \mathcal{I}_R(B, A) \) for \( R \)-modules \( A, B \).

\[ \begin{align*}
& \text{Proof. Let } C(A) \text{ and } C(B) \text{ be the canonical short free resolutions of } A \text{ and } B, \text{ respectively. Then, } \\
& H_n(C(A)), H_n(C(B)) \text{ are } 0 \text{ for } n > 0 \text{ and the augmentation residue natural isomorphisms } \\
& H_0(C(A)) \cong A, \text{ and } H_0(C(B)) \cong B. \\
& \text{By 16.18 we have a natural } R \text{-isomorphism } \\
& H_1(C(A) \otimes_R C(B)) \cong H_1(C(B) \otimes_R C(A)). \\
& \text{The Eilenberg-Zilber Theorem (16.20) yields the natural exact sequence } \\
& 0 \to [H(C(A)) \otimes_R H(C(B))] \to H_1(C(A) \otimes_R C(B)) \to \mathcal{I}_R(H(C(A)), H(C(B))) \to 0. \\
& \text{We have } \\
& [H(C(A)) \otimes_R H(C(B))], = H_1(C(A)) \otimes_R H_0(C(B)) \oplus H_0(C(A)) \otimes_R H_1(C(B)) \\
& \cong 0 \otimes_R B \oplus A \otimes 0 = 0, \\
& \mathcal{I}_R(H(C(A)), H(C(B))) \cong \mathcal{I}_R(H_0(C(A)), H_0(C(B))), \text{ which is } \\
& \text{naturally } R \text{-isomorphic to } \mathcal{I}_R(A, B). \text{ Thus, } \\
& H_1(C(A) \otimes_R C(B)) \text{ is naturally } R \text{-isomorphic to } \mathcal{I}_R(A, B). \\
& \text{Similarly, } H_1(C(B) \otimes_R C(A)) \text{ is naturally } R \text{-isomorphic to } \\
& \mathcal{I}_R(B, A).
\]

\[ 17. \text{ Homology with Coefficients and the Universal Coefficient Theorem} \]

Exercise 17.1. Let \( A \) be an abelian group, and \( M \) an \( R \)-module (\( R \) arbitrary). Then,

1. \( A \otimes_R M \) has an \( R \)-module structure characterized as follows: In all \( a \in A, \ m \in M, \ r \in R, \ r(a \otimes m) = a \otimes rm. \)
2. If \( f : A \to B \) is a \( Z \)-homomorphism and \( g : M \to N \) an \( R \)-homomorphism, then \( f \otimes g : A \otimes_R M \to B \otimes_R N \) is an \( R \)-homomorphism.
3. If \( A \) is an abelian group and \( M, N \) \( R \)-modules, there is a natural \( R \)-isomorphism
   \[ A \otimes_R (M \otimes_R N) \to (A \otimes_R M) \otimes_R N \]
   which sends \( a \otimes (m \otimes n) \) to \( (a \otimes m) \otimes n \) for all \( a \in A, \ m \in M, \ n \in N. \)
4. If \( A \) is the abelian with basis \( X \), then \( A \otimes_R R \) is a \( R \)\(-module with basis \( \{x \otimes 1 \mid x \in X \}. \)
Definition 17.2. Let \((X,A)\) be a topological pair, and \(G\) an abelian group. We write \(S(X,A;G)\) for the chain complex \(S(X,A) \otimes G\), and \(\tilde{H}_n(X,A;G) = H_n(S(X,A);G)\). \(\tilde{H}_n(X,A;G)\) is called the \(n\)th singular homology group of the pair \((X,A)\) with coefficients in \(G\).

Let \(f:(X,A) \to (Y,B)\) be a map of topological pairs, let \(\tilde{H}_n(f;G) : \tilde{H}_n(X,A;G) \to \tilde{H}_n(Y,B;G)\) denote the homomorphism \(H_n(S(f) \otimes 1_G)\).

Note that if \(G\) is an \(R\)-module, then, by 17.1, \(S(X,A) \otimes G\) becomes a chain complex of \(R\)-modules, and so each \(\tilde{H}_n(X,A;G)\) has an \(R\)-module structure. Moreover, by 17.1, if \(f:(X,A) \to (Y,B)\) is a map of topological pairs, \(S(f) \otimes 1_G : S(X,A) \otimes G \to S(Y,B) \otimes G\) is a map of chain complexes of \(R\)-modules, and so \(\tilde{H}_n(f;G) : \tilde{H}_n(X,A;G) \to \tilde{H}_n(Y,B;G)\) is an \(R\)-homomorphism.

Exercise 17.3. Let \(G\) be an \(R\)-module. Then the rule which assigns to each topological pair \((X,A)\) the chain complex \(S(X,A) \otimes G\), together with the map \(f:(X,A) \to (Y,B)\) the chain complex \(S(f) \otimes 1_G : S(X,A) \otimes G \to S(Y,B) \otimes G\), is a covariant functor \(S(\_;G) : J_T \to \mathcal{G}_R\).

Thus, for each \(n\), \(\tilde{H}_n(\_;G) : J_T \to \mathcal{M}_R\) is a covariant functor.

Theorem 17.4. (Universal Coefficient Theorem)

1. Let \(G\) be an abelian group. Then for topological pairs \((X,A)\) there is a natural exact sequence

\[
0 \to \tilde{H}_n(X,A;G) \otimes \mathbb{Z} \xrightarrow{\partial_n} \tilde{H}_n(X,A;\mathbb{Z}) \otimes G \to \tilde{H}_n(X,A;G) \to 0
\]

which splits (splitting not natural).

2. If \(R\) is a PID and \(G\) an \(R\)-module, then for topological pairs \((X,A)\) there is a natural exact sequence of \(R\)-homomorphisms

\[
0 \to \tilde{H}_n(X,A;R) \otimes R \xrightarrow{\partial_n R} \tilde{H}_n(X,A;\mathbb{Z}) \otimes R \to \tilde{H}_n(X,A;R) \to 0
\]

which splits (splitting not natural).
Theorem 17.5. Homology with coefficients in $G$ satisfies the "Eilenberg-Steenrod axioms" for homology, i.e. each $H^n(U, G)$ is a covariant functor from the category of topological pairs to the category of abelian groups (as $R$-modules if $G$ is an $R$-module) and satisfies
1) (Dimension Axiom) If $P$ is a point, then
$$H^n(P, G) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$
2) (Excision Axiom) If $(X, A)$ is a topological pair, then for each $n$ there is a natural isomorphism $(R$-isomorphism if $G$ is an $R$-module) $i: H^n(X, A; G) \to H^n_1(A, G)$ such that the sequence
$$\cdots \to H^n(A; G) \xrightarrow{H^n(i, G)} H^n(X, G) \xrightarrow{H^n(j, G)} H^n(X, A; G) \xrightarrow{H^n(i, G)} H^n_1(A, G) \xrightarrow{H^n(j, G)} H^n(X, A) \to \cdots$$

is exact where $i: A \to X$, $j: (X, G) \to (X, A)$ are the inclusions.
3) (Homotopy Axiom) If $f, g: (X, A) \to (Y, B)$ are homotopy maps of pairs, then $H^n(f, G) = H^n(g, G): H^n(X, A; G) \to H^n(Y, B; G)$ for all $n$.
4) (Excision Axiom) If $(X, A)$ is a topological pair and $\overline{U} \subseteq \text{int } A$, then $H^n(i, G): H^n(X-U, A-U; G) \to H^n(X, A; G)$ is an isomorphism for all $n$, where $i: (X-U, A-U) \to (X, A)$ is the inclusion.

Proof: 1) is immediate from 3, 2 and 17.4.

Since each $S_n(X, A)$ is free abelian,
$$0 \to S_n(A) \xrightarrow{S_n(i)} S_n(X) \xrightarrow{S_n(j)} S_n(X, A) \to 0$$
is exact
and splits for each $n$. Thus
$$0 \to S(A; G) \xrightarrow{S(i) \otimes 1_G} S(X; G) \xrightarrow{S(i) \otimes 1_G} S(X, A; G) \to 0$$
is a short exact sequence of chain complexes (of $R$-modules if $G$ is an $R$-module). It now follows from 4.24 and 4.27.

Suppose $f \cong g$. It follows from 8.11 that $S(f) \cong S(g)$.

Hence by 15.14, $S(f) \otimes I_G \cong S(g) \otimes I_G$. Thus

$H_n(f; G) = H_n(S(f) \otimes I_G) = H_n(S(g) \otimes I_G) = H_n(g; G)$, (proving 3).

Suppose $(X, A), U, i$ are as in 4). By 17.4 we have the commutative diagram with exact rows

$$
0 \to H_n(X-U; A-U) \otimes_2 G \xrightarrow{\mu} H_n(X-U; A-U; G) \xrightarrow{\beta} \operatorname{Tor}_2(H_n(X-U; A-U) G) \to 0
$$

$$
0 \to H_n(X; A) \otimes_2 G \xrightarrow{\mu} H_n(X; A; G) \xrightarrow{\beta} \operatorname{Tor}_2(H_n(X; A) G) \to 0
$$

$H_n(i)$ and $H_{n-1}(i)$ are isomorphisms by the ordinary Excision Property 6.2. Thus $H_n(i) \otimes_2 I_G$ and $\operatorname{Tor}_2(H_n(i); G)$ are isomorphisms. Thus by the 5-Lemma, $H_n(i; G)$ is an isomorphism, proving 4).

**Example 17.6.** Let $G$ be a torsion-free abelian group. By 14.25 and 14.26, $G$ is a flat $\mathbb{Z}$-module. Thus $\operatorname{Tor}_2(H; G) = 0$ for all abelian groups $H$. Thus, by the Universal Coefficient Theorem (17.4),

$$
\mu: H_n(X; A) \otimes_2 G \xrightarrow{\sim} H_n(X; A; G)
$$

is an isomorphism for each topological pair $(X, A)$.

**Example 17.7.** Let $S$ be a set of primes. By 14.30,

$$(\mathbb{Z}/2) \otimes_2 \mathbb{Z}_S \cong \begin{cases} 
\mathbb{Z}/2 & \text{if } 2 \in S \\
0 & \text{if } 2 \notin S
\end{cases}
$$

Thus, by 17.6 and 13.15,

$$
H_i(\mathbb{R}P^n; \mathbb{Z}_S) \cong \begin{cases} 
\mathbb{Z}_S & \text{if } i = 0 \\
\mathbb{Z}_S & \text{if } i = n, n \text{ odd} \\
\mathbb{Z}/2 & \text{if } 1 \leq i \leq n-1, i \text{ odd, and } 2 \in S \\
0 & \text{otherwise}
\end{cases}
$$
Example 17.8. Let $p$ be a prime. By 14.22 and 16.16,
\[(\mathbb{Z}/2) \otimes_{\mathbb{Z}} (\mathbb{Z}/p) \cong \mathbb{Z}/p \otimes (\mathbb{Z}/2, \mathbb{Z}/p) \cong \begin{cases} 
\mathbb{Z}/2 & \text{if } p = 2 \\
0 & \text{otherwise}
\end{cases}
\]
By 14.16, \( \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/p) \cong \mathbb{Z}/p \), and by 16.5,
\[\operatorname{Tor}^2(\mathbb{Z}, \mathbb{Z}/p) = 0. \]
It follows from 13.15 and 17.4 that
\[H_i(\mathbb{R}P^n; \mathbb{Z}/2) \cong \begin{cases} 
\mathbb{Z}/2 & \text{if } 0 \leq i \leq n \\
0 & \text{otherwise}
\end{cases}
\]
and if \( p \) is odd,
\[H_i(\mathbb{R}P^n; \mathbb{Z}/p) \cong \begin{cases} 
\mathbb{Z}/p & \text{if } i = 0 \\
\mathbb{Z}/p & \text{if } i = n, n \text{ odd} \\
0 & \text{otherwise}
\end{cases}
\]

18. The Eilenberg-Zilber Theorem and the Homology of Products

Lemma 18.1. Let \( C \) and \( D \) be \( \mathbb{Z} \)-augmented chain complexes of free abelian groups. Suppose \( \varepsilon : \mathcal{H}_0(C) \to \mathbb{Z} \) and \( \varepsilon : \mathcal{H}_0(D) \to \mathbb{Z} \) are both isomorphisms. Then \( \varepsilon : \mathcal{H}_0(C \otimes_{\mathbb{Z}} D) \to \mathbb{Z} \) is an isomorphism.

Proof. Recall from 14.33 and 14.34 that
\[\varepsilon : Co \otimes_{\mathbb{Z}} D_0 = (Co \otimes_{\mathbb{Z}} D)_0 \to \mathbb{Z}\] is the composition
\[Co \otimes_{\mathbb{Z}} D_0 \xrightarrow{\varepsilon \otimes \varepsilon} Z \otimes Z \xrightarrow{\sim} Z\]
where the last isomorphism is the canonical isomorphism. It follows easily that the diagram
\[
\begin{array}{ccc}
\mathcal{H}_0(C \otimes_{\mathbb{Z}} D) \\
\downarrow \varepsilon \otimes \varepsilon \\
Z \otimes Z \\
\downarrow \cong \\
\mathcal{H}_0(D) \\
\mathcal{H}_0(C \otimes_{\mathbb{Z}} D) \\
\end{array}
\]
commutes. By the Kunneth Theorem (16.20), it is an
isomorphism. Thus, since $E \otimes E$ is an isomorphism, the result follows.

**Theorem 18.2 (Eckeland-Heilbronn Theorem).** For topological spaces $X$ and $Y$, there exists a natural augmentation-preserving chain equivalence $E : S(X) \otimes S(Y) \rightarrow S(X \times Y)$, and any two such are naturally chain homotopic.

(We will call such $\Theta$ an **Eckeland-Heilbronn map**.)

Combining 18.2 with the K"unneth Theorem (16.20) we have

**Corollary 18.3.** For topological spaces $X$ and $Y$, there exists a natural exact sequence

$$0 \rightarrow \left[ H(X) \otimes \mathbb{Z} H(Y) \right]_n \rightarrow H_n(X \times Y) \rightarrow \left[ \text{Tor}^\mathbb{Z}_n(H(X), H(Y)) \right] \rightarrow 0$$

which splits (splitting not natural).

**Proof of 18.2.** The proof is by acyclic models. Let $\text{Top}^2$ denote the category of ordered pairs of topological spaces and $M = \{ (\Delta m, \Delta n) \mid m \geq 0, n \geq 0 \}$. We have covariant functors $T, U : \text{Top}^2 \rightarrow \text{Abb}$ given by

$$T(X, Y) = S(X) \otimes S(Y), \quad U(X, Y) = S(X \times Y).$$

By the Acyclic Models Theorem (18.22), it suffices to show that $T_n$ and $U_n$ are $M$-acyclic for all $n$, and $T, U$ are both $M$-acyclic.

$$T(\Delta m, \Delta n) = S(\Delta m) \otimes S(\Delta n).$$

Since the $\Delta i$ are contractible, $S(\Delta m)$ and $S(\Delta n)$ are both acyclic. It follows from the K"unneth Theorem (16.20) and 18.1 that $S(\Delta m) \otimes S(\Delta n)$ is acyclic, and so $T$ is $M$-acyclic.

$$U(\Delta m, \Delta n) = S(\Delta m \times \Delta n)$$

which is acyclic since $\Delta m \times \Delta n$ is contractible. Thus $U$ is $M$-acyclic.

For $n \geq 0$, $\delta^n \otimes \delta^n |_{\delta^n \otimes \delta^n = n^3}$, where $\delta : \Delta i \rightarrow \Delta i$ is the identity map, is an $M$-basis for $T$, and so $T$ is $M$-acyclic. Let $\Delta n : \Delta n \rightarrow \Delta m \times \Delta n$ denote the diagonal map, i.e., $\Delta n(x) = (x, x)$. Then $\Delta n^3$ is an $M$-basis for $U$ and so $U$ is $M$-acyclic, completing the proof.
Definition 18.4. Let \((X, A)\) and \((Y, B)\) be topological pairs. We define \((X, A) \times (Y, B)\) to be the topological pair \((X \times Y, A \times Y \cup X \times B)\).

Note that if \(f: (X, A) \rightarrow (X', A')\) and \(g: (Y, B) \rightarrow (Y', B')\) are maps of topological pairs, then 
\((f \times g)(A \times Y \cup X \times B) \subset (A' \times Y' \cup X' \times B')\) and thus we obtain a map of topological pairs 
\(f \times g: (X, A) \times (Y, B) \rightarrow (X', A') \times (Y', B')\).

Exercise 18.5. The rule which assigns to each ordered pair of topological pairs \((X, A), (Y, B)\) the topological pair 
\((X, A) \times (Y, B)\), and to each ordered pair of maps of topological pairs \((f, g)\) the map of topological pairs \(f \times g\), as a covariant functor from \(\text{Top}^2\) to \(\text{Top}\).

We wish to extend the Eilenberg-Zilber theorem to get a natural chain map 
\(S(X, A) \bigotimes_{\mathbb{Z}} S(Y, B) \rightarrow S((X, A) \times (Y, B))\) which induces isomorphisms in homology for suitable pairs of pairs.

Proposition 18.6. Any Eilenberg-Zilber map \(\Theta\) extends to a natural chain map 
\(\Theta: S(X, A) \bigotimes_{\mathbb{Z}} S(Y, B) \rightarrow S((X, A) \times (Y, B))\) for pairs of pairs. Moreover, if the triad \((A \times Y \cup X \times B; A \times Y, X \times B)\) is excisive (see 7.15),
\(H_n(\Theta): H_n(S(X, A) \bigotimes_{\mathbb{Z}} S(Y, B)) \rightarrow H_n((X, A) \times (Y, B))\)
is an isomorphism for all \(n\), and conversely.

Corollary 18.7. For pairs of topological pairs \((X, A), (Y, B)\) such that \((A \times Y \cup X \times B; A \times Y, X \times B)\) is excisive, then there is a natural exact sequence 
\(0 \rightarrow \left[ H(X, A) \bigotimes_{\mathbb{Z}} H(Y, B) \right]_n \rightarrow H_n((X, A) \times (Y, B)) \rightarrow \mathbb{Z}^2(H(X, A), H(Y, B))_{n-1} \rightarrow 0\)
which splits (splitting not natural).
Note 18.8. Each of the following is a sufficient condition for existences in 18.6:
1) If A is open in X and B is open in Y.
2) If \( A = \emptyset \) and B arbitrary (or \( B = \emptyset \) and A arbitrary).

Proof of 18.6. By naturality of \( \Theta \), we have the commutative diagram

\[
\begin{array}{ccc}
S(A) \otimes S(Y) & \xrightarrow{\Theta_{X,Y}} & S(A \times Y) \\
S(i) \otimes 1_{S(Y)} & \downarrow & S(1 \times 1_Y) \\
S(X) \otimes S(Y) & \xrightarrow{\Theta_{X,Y}} & S(X \times Y) \\
1_{S(X)} \otimes S(Y) & \uparrow & S(1 \times 1_Y) \\
S(X) \otimes S(B) & \xrightarrow{\Theta_{X,B}} & S(X \times B)
\end{array}
\]

where \( i : A \rightarrow X \), \( j : B \rightarrow Y \) are the inclusions, \( S(i) \otimes 1_{S(Y)} \) and \( 1_{S(X)} \otimes S(Y) \) are both injective since for each \( n \), \( S_n(A) \) is a direct summand of \( S_n(X) \) and \( S_n(B) \) is a direct summand of \( S_n(Y) \). We regard \( S(A) \otimes S(Y) \) and \( S(X) \otimes S(B) \) as sub-complexes of \( S(X) \otimes S(Y) \) and these inclusions. Similarly, \( S(A \times Y) \) and \( S(X \times B) \) are regarded as sub-complexes of \( S(X \times Y) \). Thus we have the commutative diagram

\[
\begin{array}{ccc}
S(X) \otimes S(Y) & \xrightarrow{\Theta_{X,Y}} & S(X \times Y) \\
& \uparrow & \uparrow \\
S(A) \otimes S(Y) + S(X) \otimes S(B) & \xrightarrow{\Theta} & S(A \times Y) + S(X \times B) \\
& \xrightarrow{\kappa} & S(A \times Y + X \times B)
\end{array}
\]

Passing to quotients, \( \Theta_{X,Y} \) induces a natural chain map

\[
\Theta : \frac{S(X) \otimes S(Y)}{S(A) \otimes S(Y) + S(X) \otimes S(B)} \rightarrow \frac{S(X \times Y)}{S(A \times Y + X \times B)} = S((X_1, A) \times (Y_2, B)).
\]

The complex on the right is naturally chain isomorphic to \( S((X_1, A) \times (Y_2, B)) \) since...
\[ 0 \to S(A) \otimes S(B) \to S(A) \otimes S(Y) \to S(X) \otimes S(Y) \to 0 \]
is exact. Note that when \( A = \emptyset = B, \Theta = \Theta^{X,Y} \).

By a 5-lemma argument similar to that given in the proof of 7.12, this relative \( \Theta \) will induce isomorphisms in homology if and only if \( k \circ (\Theta) \) does. This can be done if we show \( \Theta \) induces isomorphisms in homology since if \( k \) induces isomorphisms in homology if and only if
\[(A \times Y, \emptyset \times B), (A \times Y, X \times B) \text{ is acyclic.}\]

In general, if \( M \) and \( N \) are subgroups of a larger abelian group, we have an exact sequence
\[ 0 \to M \cap N \xrightarrow{x} M \oplus N \xrightarrow{\beta} M + N \to 0 \]
where \( x(x) = (x, -x), \beta(x, y) = x + y \). We thus obtain the commutative diagram with exact rows
\[ 0 \to S(A) \otimes S(B) \to S(A) \otimes S(Y) \oplus S(X) \otimes S(B) \to S(A) \otimes S(Y) + S(X) \otimes S(B) \to 0 \]
\[ \downarrow \Theta^{A,B} \quad \downarrow \Theta^{A,Y} \oplus \Theta^{X,B} \quad \downarrow \Theta \]
\[ 0 \to S(A \times B) \to S(A \times Y) \oplus S(X \times B) \to S(A \times Y) + S(X \times B) \to 0 \]

Since \( \Theta^{A,B}, \Theta^{A,Y}, \Theta^{X,B} \) are chain equivalences, the left 2 vertical maps induce isomorphisms in homology. By another 5-lemma argument analogous to that in 7.12, \( \Theta \) induces isomorphisms in homology, completing the proof.
1. Calculate the homology groups of the double torus

2. Let $X$ be a topological space. The suspension of $X$, denoted $SX$, is the quotient space obtained from $X \times I$ by identifying $X \times \{0\}$ to a point $P$ and $X \times \{1\}$ to a point $Q$ ($Q \neq P$).

proved: For all $n \geq 2$, $H_n(SX) \cong H_{n-1}(X)$.

3. Let $G$ be a finitely generated abelian group. Prove: For each integer $n \geq 1$, there exists a topological space $X_n$ such that $H_n(X_n) \cong G$. 
4. Do Exercise 7.21 in the notes.

5. Let \((X,A)\) be a topological pair and suppose \(r:X \to A\) is a retraction (not necessarily a deformation retraction).
   Prove: For each \(n \in \mathbb{Z}\), the homomorphism \(\alpha:H_n(X) \to H_n(A) \oplus H_n(X,A)\) given by \(\alpha(u) = (H_n(r)(u), H_n(j)(u))\) is an isomorphism where \(j:(X,\emptyset) \to (X,A)\) is the inclusion.

6. Let \(M\) be the closed Möbius strip, and let \(B\) denote its boundary.

(Note: \(B\) is homeomorphic to \(S^1\).)
Prove: \(B\) is not a retract of \(M\).
7. Prove the Acyclic Models Theorem.
8. a) Describe a regular CW complex $K$ such that $|K| \cong S^1 \times S^2$.
   b) Compute the homology groups of $S^1 \times S^2$ by calculating the homology groups of the chain complex $C(K)$. 
9. Let \( n \geq 1 \) and let \( \alpha : S^n \to S^n \) denote the antipodal map. Determine the homomorphism \( H_n(\alpha) : H_n(S^n) \to H_n(S^n) \).

10. We describe points in \( \mathbb{R}^3 \) via spherical coordinates \((\rho, \theta, \phi)\). \( \rho \) = distance from the origin; \( \theta \) = "longitude", a real number mod \( 2\pi \); \( \phi \) = "co-latitude", \( 0 \leq \phi \leq \pi \). (See picture below). Let \( X \) be the quotient space obtained from \( \mathbb{R}^3 \) by making identifications on \( S^2 \) as follows: \((1, \theta, \phi) \sim (1, \theta + \frac{2}{3} \pi, \pi - \phi)\) for \( 0 \leq \phi \leq \frac{1}{2} \pi \). (Thus no identifications are made in the open ball \( E^3 \), each point in \( e_2^+ \) is identified with exactly one point in \( e_2^- \), and each point on the equator is identified with exactly two other points on the equator.) Compute the homology groups of \( X \).

12. Let $p$ be a prime, and let $R = \mathbb{Z}(p)$, the integers localized at $p$.
   
   a) Prove: $R$ is a PID.
   
   b) Prove: For each positive integer $n$, $\mathbb{Z}/p^n$ admits a unique $R$-module structure.
   
   c) Find $\mathbb{Z}/p^m \otimes_R \mathbb{Z}/p^n$ for arbitrary positive integers $m,n$.
   
   d) Find $\text{Tor}^R(\mathbb{Z}/p^m, \mathbb{Z}/p^n)$ for arbitrary positive integers $m,n$. 
1. Let $X$ and $Y$ be topological spaces of finite homology type. 
Prove: $E(X \times Y) = E(X)E(Y)$ where $E$ denotes the Euler Characteristic.

2. Suppose $X$ and $Y$ are topological spaces such that $H_n(X)$ and $H_n(Y)$ are as given below:

<table>
<thead>
<tr>
<th>$H_n(X)$</th>
<th>$H_n(Y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}$ if $n = 0$</td>
<td>$\mathbb{Z}$ if $n = 0$</td>
</tr>
<tr>
<td>$\mathbb{Z}/12$ if $n = 1$</td>
<td>$\mathbb{Z}$ if $n = 1$</td>
</tr>
<tr>
<td>$\mathbb{Z}/20$ if $n = 2$, $\mathbb{Z}/15$ if $n = 2$</td>
<td>$\mathbb{Z}/8$ if $n = 3$</td>
</tr>
<tr>
<td>$0$ otherwise</td>
<td>$0$ otherwise</td>
</tr>
</tbody>
</table>

Find $H_n(X \times Y)$ for all $n$. Express each group as a direct sum of cyclic groups.

3. Let $X$ and $Y$ be topological spaces, and let $T : X \times Y \to Y \times X$ be given by $T(x,y) = (y,x)$. Let $\tau : S(X) \otimes S(Y) \to S(Y) \otimes S(X)$ be as in 16.18. Prove: The diagram

$$
\begin{array}{ccc}
S(X) \otimes S(Y) & \xrightarrow{\tau} & S(Y) \otimes S(X) \\
\downarrow \phi & & \downarrow \phi \\
S(X \times Y) & \xrightarrow{S(T)} & S(Y \times X)
\end{array}
$$

commutes up to chain homotopy, where $\phi$ is an Eilenberg-Zilber map.

4. Let $0 \to G \to H \to J \to 0$ be a short exact sequence of abelian groups. Prove: For topological pairs $(X,A)$, there is a natural long exact sequence

$$
\cdots \to H_{n+1}(X,A;J) \to H_n(X,A;G) \to H_n(X,A;H) \to H_n(X,A;J) \to H_{n-1}(X,A;G) \to \cdots
$$
5. Let \( f : (X,A) \rightarrow (Y,B) \) be a map of topological pairs such that 
\( H_n(f) : H_n(X,A) \rightarrow H_n(Y,B) \) is an isomorphism for all \( n \).

Prove: For each abelian group \( G \), \( H_n(f;G) : H_n(X,A;G) \rightarrow H_n(Y,B;G) \) is an isomorphism for all \( n \).

6. Let \( X \) be a topological space such that \( H_n(X) \neq 0 \) for some \( n > 0 \).

Prove: The map \( T : X \times X \rightarrow X \times X \) given by \( T(x,y) = (y,x) \) is not homotopic to the identity map on \( X \times X \).

7. Let \( \mathbb{RP}^3/\mathbb{RP}^2 \) denote the quotient space obtained from \( \mathbb{RP}^3 \) by collapsing \( \mathbb{RP}^2 \) to a point, and let \( f : \mathbb{RP}^3 \rightarrow \mathbb{RP}^3/\mathbb{RP}^2 \) denote the quotient map.

Prove: \( f \) does not extend to a continuous map \( g : \mathbb{RP}^4 \rightarrow \mathbb{RP}^3/\mathbb{RP}^2 \).

8. Let \( n \) be a non-negative integer.

a) Prove: If \( f : S^{2n} \rightarrow S^{2n} \) is continuous, then there exists at least one \( x \in S^{2n} \) such that either \( f(x) = x \) or \( f(x) = -x \).

b) Prove: If \( g : \mathbb{RP}^{2n} \rightarrow \mathbb{RP}^{2n} \) is continuous, then \( g \) has at least one fixed point.
Hom_R(A, B) : R-module

Functional properties:
\[ A' \rightarrow A \xrightarrow{\alpha} B' \rightarrow B' \]

\[ \text{Hom}_R(f, g) : \text{Hom}_R(A, B) \rightarrow \text{Hom}_R(A', B') \quad R \text{-hom.} \]
\[ \alpha \rightarrow \partial \times \delta \]

Cech complexes of R-modules \((C, \delta)\):

\[ C^0 \overset{\delta}{\rightarrow} C^1 \overset{\delta}{\rightarrow} C^2 \overset{\delta}{\rightarrow} \cdots \]

Zer of a cochain complex of R-modules \(B^n(C) = \ker \delta : C^n \rightarrow C^{n+1} = n \text{th cocycles of } C \)

Bd of a cochain complex of R-modules \(B^n(C) = C^n \rightarrow C^n = n \text{th coboundaries of } C \)

\[ H^n(C) = \frac{B^n(C)}{C^n} = n \text{ th cohomology of } C \]

\[ H^n : \text{Cech}^* \rightarrow \text{R}^* \text{ is a covariant functor.} \]

\[ \text{If } C \text{ a cech complex of } R\text{-modules, } A \text{ an } \]

R-module \( \text{Hom}_R(C, A) \) is cech complex geri as follows

\[ \text{Hom}_R(C, A)^n = \text{Hom}_R(C^n, A) \]

\[ \partial^n = (-1)^n \text{Hom}_R(\mathbb{Z}, 1) : \text{Hom}_R(C^n, A) \rightarrow \text{Hom}_R(C^{n+1}, A) \]

\[ \text{Hom}(-)A : \text{Cech}^* \rightarrow \text{Cech}^* \text{ is a covariant functor.} \]

\[ \text{A top. pair: } (X, A) \quad C \text{ a cech complex of abel. qps, } \]

A abel. qps \( B \) R-module \( \text{Hom}_R(A, B) \) is an \( R \)-module \( \text{Hom}_R(A, B) \equiv \text{Hom}_R(A \otimes_R B, B) \)

\[ (X, A) \text{ top. pair, } G \text{ an } \]

R-module \( \text{Cech}^* \quad S^*_{(X, A; G)} = \text{Hom}_2(S(X, A); G) \equiv \text{Hom}_R(S(X, A; R), G) \)

\[ H^n(X, A, G) = H^n(S^*_{(X, A; G)}) = n \text{ th cech complex of } (X, A) \]

with coeff. in G.
\[ A \to B \to C \to 0 \quad \text{exact seq. of } R \quad \text{mod.}\]

\[ 0 \to \text{Hom}_R(C, G) \to \text{Hom}_R(B, G) \to \text{Hom}_R(A, G) \to 0 \]

\[ A, B \ R \text{-modules}, \quad R \text{ a PID.} \]

Get \( R \)-celestial complex \( \text{Hom}_R(C(A), B) \).

Define \( \text{Ext}_R(A, B) = H^1(\text{Hom}_R(C(A), B)) \).

Act

\[ 0 \to \text{Hom}_R(A, B) \to \text{Hom}_R(C(A), B) \to \text{Hom}_R(C(A), B) \to \text{Ext}_R(A, B) \to 0 \]

\[ \text{Ext}_R \text{ is a functor of 2 variables, contravariant in } 1st, \]
\[ \text{covariant in } 2nd. \]

U.C.T. \( R \) a PID, \( C \) chain ring of \( R \)-mod. \( \text{Ext}_R \) exact seq.

\[ 0 \to \text{Ext}_R(\text{Hom}_R(C, A)) \to H^n(\text{Hom}_R(C, A)) \to \text{Hom}_R(\text{Hom}_R(C, A)) \to 0 \]

which splits (not natural).

Calculations of \( \text{Ext} \):

1) \( \text{Ext}_R(A, B) \to C \frac{\text{mod.}}{B} \to R \) and \( B \).

2) \( \text{Ext}_R(\mathbb{Z}/p \mathbb{Z}, B) \cong (\mathbb{Z}/p \mathbb{Z}) \otimes_B B \quad \text{and } B \).

\[ \text{Note: } \text{Ext} \text{ is not symmetric in } A \text{ and } B. \]

\[ \otimes \]

\[ \text{Hom}_R, \text{Hom}_R \text{ cross prod.: } (R \times R) \to s \otimes s \]

\[ H_p(X, R) \otimes_R H_q(Y, R) \to H_pq(s(X, F)) \otimes_R s(Y, G) \]

\[ \text{H}_{p+1}(X \times Y, R) \to H_{p+1}q(s(X, F)) \otimes_R s(Y, G) \]

\[ H_{p+1}(X \times Y, R) \]
Can also do this via cohomology.

\[
S^*(X; \mathbb{R}) @>>> S^*(Y; \mathbb{R})
\]

\[
\Rightarrow \quad H^\alpha_{\mathbb{R}}(S(X; \mathbb{R}) \otimes \mathbb{R} H^\beta_{\mathbb{R}}(S(Y; \mathbb{R}))) @>>> 0
\]

\[
H^\alpha_{\mathbb{R}}(S(X) \otimes S(Y; \mathbb{R})) @>>> H^\alpha_{\mathbb{R}}(S(X \times Y; \mathbb{R}))
\]

\[
\Rightarrow H^\alpha_{\mathbb{R}}(S(X); \mathbb{R}) = S^*(X; \mathbb{R})
\]

\[
\Rightarrow \quad S^*(X; \mathbb{R}) \otimes \mathbb{R} S^*(X; \mathbb{R}) \xrightarrow{\text{cup prod}} S^*(X \times X; \mathbb{R}) \xrightarrow{\text{cup prod}} S^*(X; \mathbb{R})
\]

\[
\Rightarrow \quad H^\alpha(X; \mathbb{R}) \otimes \mathbb{R} H^\delta(X; \mathbb{R}) \rightarrow H^{\alpha+\delta}(X; \mathbb{R})
\]

Don't work with homology.

\[
H^\alpha(X; \mathbb{R}) \text{ becomes a graded ring under cup prod.}
\]

\[
1 \in H^0(X; \mathbb{R}), \quad u \cup u = (-1)^{\deg u} u \cup u,
\]

in general.

\[
\text{if } f : X \rightarrow Y, \quad f^* : H^\alpha(Y) \rightarrow H^\alpha(X) \text{ is a graded homomorphism.}
\]

**Example:**

\[
H^\alpha(RP^n; \mathbb{Z}/2) = \mathbb{Z}/2[u]/(u^{n+1}) \cup u \in H^\alpha(RP^n; \mathbb{Z}/2)
\]

\[
H^\alpha(S^n; \mathbb{Z}/2) \hookrightarrow \cdots \rightarrow RP^n \text{ embed, } i^*(u_m) = u_m
\]

\[
\text{for } m < n, \quad RP^n \text{ is not a retract of } RP^n \text{ unless } m = n
\]

In some cases, this is not deducible from homology or cohomology.
groups above. Cup products are needed.

Homotopy Groups

\((X, x)\) a pointed space. \(\Pi_n(X, x) = \pi_n\text{ of pointed homotopy}
\]
class of maps \(S^n \to X\).
\(\Pi_n(X, x)\) is a group under \(\cdot\). \([x]\), \([y]\) \in \Pi_n(X, x)
\[
S^n \overset{\pi_0}{\longrightarrow} S^n \vee S^n \overset{x \vee y}{\longrightarrow} X \vee X \overset{h \vee d}{\longrightarrow} X
\]
Can show: \(\Pi_n\) is abelian for \(n \geq 1\). \(\pi_0(X, x)\) is a set
\(\pi_0\text{ of path components\(\)} X\).

Homotopy cup \(h_\#: \Pi_n(X, x) \to \Pi_0(X)\):
Choose \(x \in \Pi_n(S^m)\).
\(h_\# [x] = \Pi_0(x) (\in)\).
\(h_\#\) is a group homomorphism.

\([X, X_1, \ldots, X_n]\) is \(n\)-connected if \(\Pi_i(X, x) = 0\) for \(i \leq n\).

Homotopy Then: \(X\) \(n-1\)-connected, then
\(h_\#: \Pi_i(X) \to \Pi_0(X)\) is 1-1, and for \(h_\#\) is connected, subgroup
\(\Pi_i(X)\).

Representability of \(H^n(\quad; G)\):

\(G\) an abelian group, \(n \geq 0\). Then \(F\) a complex \(K(\sigma, n)\) (unique up to homotopy type), such that
and an element \(c_n \in H^n(K(\sigma, n); G)\) \(c\) in an
\([^X, K(\sigma, n)] \longrightarrow H^n(X; G)\) is a bijection.

\([x] \longmapsto c^*([x])\)

In particular, \(X = \text{S}^m\): \(H^n(\text{S}^m; G) \cong \begin{cases} 0 & \text{if } m \neq n \\ G & \text{if } m = n \end{cases}\)

So \(\Pi_m(K(\sigma, n)) \cong \begin{cases} G & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}\)
\[ S^1 = K(\mathbb{Z}, 1) \]
\[ \mathbb{R}P^n = K(\mathbb{Z}/2, 1) \]
\[ \mathbb{C}P^n = K(\mathbb{Z}, 2) \]

\[ \text{Obstruction theory} \]

\[ \text{Extension problem:} \]
\[ \text{Does } f: A \to Y \text{ extend to a cont. map } g: X \to Y? \]

Let \( K \) be a CW complex and suppose (for simplicity)
\( Y \) a 0-connected space. (In this case, all different base pts
\( y_1, y_2, \quad \Pi_n(Y, y_1) \) and \( \Pi_n(Y, y_2) \) can be canonically identified).

\[ \text{Suppose } f: K^{(n-1)} \to Y \text{ be continuous. Does } f \text{ extend to } K^n \text{?} \]

\[ \text{Let } E \to \text{ a cell of } K, \quad c_{E}: D^n \to \{0, 1\} \text{ a char. map, } \]
\[ \text{extends to } E \in Y \Rightarrow S^{n-1} \to E \text{ extends to } D^n. \]

\[ \text{In general, this map defines a element } c \in \Pi_{n-1}(Y). \]

\[ C_n(K) \to \Pi_{n-1}(Y), \quad \text{ i.e. a cochain } c \in \text{Hom}(C_n(K), \Pi_{n-1}(Y)) \]
\[ \Theta \quad c_{E} \]

\[ \text{(excision)} \]

\[ \text{Then } c \text{ is a cocycle. } [c] \in H^n(K; \Pi_{n-1}(Y)) \]
\[ \text{is } 0 \Leftrightarrow \text{ every cell } K^{(n)} \text{ extends to } K^n. \]

\[ \text{Cohomology operation} \]
\[ \text{A cohomology operation } d \text{ of type } (m, A; n, B) \]
\[ (A, B \text{ chain groups}) \text{ is a nate which associates to each } \]
\[ \text{map } f: X \to Y \text{ a function (not necessarily linear)} \]
\[ \alpha_x: H^m(X; A) \to H^m(Y; A) \to H^n(X; B) \]
\[ \text{commutes,} \]
Observation Theory

Basic fact 1. A map \( f : S^{n-1} \rightarrow Y \) is null-homotopic \( \iff \) \( f \) extends to a cont. map \( \tilde{f} : D^n \rightarrow Y \).

Given \( K \) a CW complex, \( Y \) a top. space, \( L \) a subcomplex of \( K \), \( f : L \rightarrow Y \). Let \( A \in \text{cells}(K,L) \) be such that \( A \subset L \). Then \( h : (D^n, S^{n-1}) \rightarrow (A, \partial A) \) be a choice for \( f | A \). Then \( f \) extends to \( L \cup \text{int} A \) \( \iff \) the augmented

\[ S^{n-1} \rightarrow A \rightarrow Y \]

is null-homotopic.

Basic fact 2. If \( Y \) is 1-connected, then \( \Pi_k(Y, y_0) \) and \( \Pi_k(Y, y_1) \)
are can. equivalently, for \( Y \) as pointed homotopy \( \equiv \) this as homotopy.

In this case, can instead apply result \( \Pi_k(Y) \).

Choose gen. \( c_n \in H^n_k(D^n, S^{n-1}) \). Let \( (K, L) \) be
a CW pair and suppose \( f : L \rightarrow Y \).

Does \( \tilde{f} \) extend to \( L \cup \text{int} K \)?

Take each \( A \in \text{cells}(K,L) \) choose choice map \( \eta_A : (D^n, S^{n-1}) \rightarrow (A, \partial A) \) and extend \( \eta_A : (A, \partial A) \rightarrow (L \cup \text{int} K, \partial L \cup \text{int} K) \)

into \( K \). Then \( \exists \eta_A : (A, \partial A) \rightarrow (L \cup \text{int} K, \partial L \cup \text{int} K) \)

for each \( A \in \text{cells}(K,L) \) the map

\[ C^n \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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Thus, e.g., if \( H^n(1K_1, 1L; \pi_{m-1}(Y)) = 0 \),

such that \( f : 1L \to Y \), which extends to \( 1L \cup K^{m-1} \)

also extends to \( 1L \cup K^m \).

**Reparametrization of Cohomology**

Then (Euler characteristic): Let \( n \) be a pos. integer, and \( \pi \)

an abelian group. Then \( \text{Hom}(K(\pi, n), \pi) \)

and an element \( u_n \in H^n(K(\pi, n); \pi) \) such that for every

CW complex \( \mathcal{X} \),

\[
\begin{align*}
[\text{map}(1L, K(\pi, n))] & \to H^n(1L, \pi) \\
[f] & \mapsto f^*(u_n)
\end{align*}
\]

is a bijection.

\( K(\pi, n) \) is called **Eilenberg-Mac Lane space** \( \text{CE}^n \).

Since the \( \pi \) and \( n \), \( K(\pi, n) \) can be taken to be a

CW complex, and it is unique up to homotopy type.

\[
\text{Note: } \pi_k (K(\pi, n)) = [S^k, K(\pi, n)] = H^n(S^k, \pi)
\]

\[
\cong \begin{cases} 0 & k \neq n \\ \pi_n & k = n \end{cases}
\]

\( \text{Eilenberg-MacLane space of type } (\pi, n) \).

\( Y \) a space such that \( \pi_k (Y) = \begin{cases} \pi_k & k = n \\ 0 & \text{otherwise} \end{cases} \)

then \( Y \) is an **Eilenberg-Mac Lane space of type** \((\pi, n)\).

**Examples**

- \( S^1 \) is a typ. \((Z, 1)\)
- \( CP^\infty \) is a typ. \((Z, 2)\)
- \( RP^\infty \) is a typ. \((Z/2, 1)\).

\( K(\pi, n) \)’s have many homotopy groups, but complicated

homotopy and cohomology groups. The \( K(\pi, n) \)’s act

basic building blocks for spaces in general (up to

homotopy type).
Cohomology operators

$\tilde{H}, \tilde{G}$ abel groups, $m, n \geq 0$ integers. A cohomology operator $X$ of type $(m, \tilde{H}, n, \tilde{G})$ is a rule which assigns to each space $X$ a function $a_X : \tilde{H}^m(X; \tilde{H}) \to \tilde{H}^n(X; \tilde{G})$ such that

for each map $f : X \to Y$ a commutative diagram

\[
\begin{array}{ccc}
\tilde{H}^m(Y; \tilde{H}) & \xrightarrow{a_Y} & \tilde{H}^n(Y; \tilde{G}) \\
\downarrow & & \downarrow \\
\tilde{H}^m(X; \tilde{H}) & \xrightarrow{a_X} & \tilde{H}^n(X; \tilde{G})
\end{array}
\]

Examples:

1) Let $R$ be a PID. For each pos. $n \in \mathbb{N}$, define $a_X : \tilde{H}^m(X; R) \to \tilde{H}^n(X; R) \text{ by } a_X(u) = u^n$. Col. op. of type $(m, R; n, R)$.

2) (Eulerian) $\nabla : 0 \to A \to B \to C \to 0$ is an exact seq. of abel. groups, we get (analogous to every pair of abel groups) a natural exact seq.

$\tilde{H}^m(X, A) \to \tilde{H}^m(X, B) \to \tilde{H}^m(X, C) \xrightarrow{\delta} \tilde{H}^{m+1}(X, A)$

Let $\tilde{H}^m(X, o)$ col. op. of type $(n, C; n, o, A)$, (called the Eulerian operator, assoc. with $0 \to A \to B \to C \to 0$).

Recall: functoriality of a space $S_X$. Hence for $\Sigma : \tilde{H}^i(X; \tilde{H}) \to \tilde{H}^{i+1}(X; \tilde{A})$ a $i \geq 0$, $\Sigma$ must be constant.
Let (continued): In each $i > 0$ and $n > 0$, $E$ coh. eq.

1. $f_i^{n+1}$ type $(n, \frac{n}{2}; n+1, \frac{n+1}{2})$ rel.

2. In each $x$, $f_i^n: H^n(x, \frac{n}{2}) \to H^{n+1}(x, \frac{n+1}{2})$ as a group hom.

3. $f_i^0$ is the Bott-seeve map with $x \to \frac{x}{2} \to \frac{x}{2} \to \frac{x}{2}$.

4. $f_i^n: H^n(x, \frac{n}{2}) \to H^n(x, \frac{n}{2})$ is zero.

5. If $i > n$, $f_i^n: H^n(x, \frac{n}{2}) \to H^{n+i}(x, \frac{n+i}{2})$ is 0.

6. In each $n \geq 1$, the diagram

$$
\begin{array}{ccc}
H^n(x, \frac{n}{2}) & \xrightarrow{f_i^n} & H^{n+i}(x, \frac{n+i}{2}) \\
\downarrow & & \downarrow \\
H^{n+i}(x, \frac{n+i}{2}) & \xrightarrow{f_i^n} & H^{n+i+i}(x, \frac{n+i+i}{2})
\end{array}
$$

commutes.

7. (Continued) Let $\alpha \in H^n(x, \frac{n}{2})$, $\beta \in H^n(x, \frac{n}{2})$,

then $f_i^n(\alpha \cdot \beta) = \sum_{j=0}^i (f_i^j \alpha) \cdot (f_i^{i-j} \beta)$.

Have seen: $\mathbb{CP}^1$ is not a retract of $\mathbb{CP}^2$, using cup products.

Question: Is $\mathbb{CP}^1$ a retract of $\mathbb{CP}^2$?

<table>
<thead>
<tr>
<th>$H^0(\mathbb{CP}^1)$</th>
<th>$H^0(\mathbb{CP}^2)$</th>
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Cup product 1 per dim. all are 0 for dimensional reasons. However, we can use $f_i^n$ to show such a retract is not split as follows:
Write \( u = \text{gen of } H^2(\mathbb{C}P^2, \mathbb{Z}_2) \).

\[ u = \text{gen of } H^2(\mathbb{C}P^2, \mathbb{Z}_2). \]

Then \( u^2 \neq 0. \) Note: \( u^2 = e_0^2 \cdot u \). \( e_0^2(\Sigma u) = 2 \cdot e_0 \cdot u \neq 0. \)

\( r: \mathbb{C}P^2 \to \mathbb{C}P^1 \) exact; \( r \) a complement.

\[ \mathbb{C}P^1 \longrightarrow \mathbb{C}P^2 \]

\[ 1 \]

\[ \mathbb{C}P^1 \]

\[ \mathbb{C}P^1 \]

\[ r \]

Thus \( r^*(\Sigma u) = \Sigma v. \)

\( c \neq e_0^2(\Sigma v) = e_0^2 r^*(\Sigma u) = r^* e_0^2(\Sigma w) = 0, \) cont.

**Attaching cells:**

Let \( a: S^{n-1} \to X \) cont. \( a \) on \( X \cup D^n \), sphere

attaching \( a \) on \( X \cup D^n \), attaching an \( n \)-cell with attaching map \( a \).

\( \text{Pick: } a^*(x) = 1 \), then \( X \cup D^n \approx X \cup \alpha D^n \).

2) Similarly, \( S^n \) with \( S(S^{n-1}) \). Then \( x: S^{n-1} \to X \),

\[ \text{get } Sx: S^n \to SX. \]

Then \( (SX) U_{\alpha} D^{n+1} \approx S(X \cup D^n) \).

As a consequence of 1), \( a^*(x) = 1 \), \( X \cup D^n \approx X \cup S^n. \)

**Recall:** \( CP^1 \approx S^2 \), \( CP^2 \) is obtained from \( CP^1 \) by

attaching a 4-cell. Let \( x: S^3 \to S^2 \) denote the attaching map:

\( \text{Claim: } x \neq * \). For it were, we would have

\( CP^2 \approx CP^1 \cup S^4 \) and \( CP^1 \) would then be a

retract of \( CP^2 \), which we know is false. Thus

\( \Pi_3(S^2) \neq 0 \), and \([x] \neq 0 \).

\( \text{Claim: } \Sigma x: S^4 \to S^3 \) is not \( \Sigma \). For if it were

we would have \( \Sigma(CP^2) \approx \Sigma(CP^1) \cup S^5 \), which we know

is false. Thus \( \Pi_4(S^3) \neq 0. \)
Cal. \phi + E \rightarrow \text{map}

\nu_m \in H^m(K(\Pi, m); \Pi)
to map.

\phi \rightarrow \phi(\nu_m) \text{ is a 1-1 map. Define set}

A set \Phi \text{ of type } (m, \Pi, n, G) \text{ and } H^n(K(\Pi, m); G).
Vector Bundles, Theory, and Cohomology

Eq: Real n-plane bundle $\alpha$ to base $\gamma$

1. A continuous map $\pi: E(\alpha) \to B(\gamma)$
2. For each $x \in B(\gamma)$, a real $n$-plane, and a point $p \in \pi^{-1}(x)$

$$n: \pi^{-1}(x) \to \mathbb{R}^n$$

3. For each $x \in B(\gamma)$, an open subset $U$ of $x$ and a bundle

$$p_\gamma^{-1}(U) \to \mathbb{R}^n$$

$$\pi \big|_U$$

A subbundle $h$ is known as fibers.

$\pi: \alpha$ is a real $n$-plane bundle

If $\alpha$, $\beta$ are real $n$-plane bundles, a map $\gamma$ of $n$-plane bundles from $\alpha$ to $\beta$ and of a pair of $n$-plane bundles $\gamma: E(\alpha) \to E(\beta)$, $f: \alpha \to \beta$, $E(\alpha) \to E(\beta)$.

If $\alpha$ is fibers from auto fibers, and

$$E(\alpha) \to E(\beta)$$

$$\pi \big|_U$$

$\beta(\alpha) = \beta(\beta)$ are $\alpha$, $\beta \equiv \gamma^2$.

Any $n$-plane bundles $\alpha: X \to e_\alpha$ and $\beta: X \to e_\beta$.

$\pi_\alpha = \pi_\beta$ and $f = \pi^{-1}(\gamma)$.

$\pi_\alpha$ is an open rel an subset of real $n$-plane bundle.

If $f: X \to Y$ a top. space, with $\text{Vect}_{\alpha}(X) = \text{vect}_\alpha(\gamma) = \text{closed}$.

If $\eta_k(\alpha)$ with $\beta(\alpha) = X$, $\beta(\eta_k(\alpha)$, $\eta_k(\alpha) = X$.

$\alpha$ a $n$, $\gamma$ as $\alpha$.

$\pi^{-1}(\gamma) \to \mathbb{R}^n$

$\pi^{-1}(\gamma)$ an $\alpha$ as $\pi$ follows $E(\alpha) \to \mathbb{R}^n$.

If $f: Y \to X$ a $\pi^{-1}(\gamma)$.

$\pi_\alpha$ as $\pi$ with $\beta(\alpha) = X$.
\[ x \in U \Rightarrow f^* x \subseteq \mathcal{E}^*_\mu. \] 

**Prop:** \( \text{Vect}_n \) is a contr. functor from \( X \) to \( \text{vect} \) of real \( n \)-plane bundles.

Then: Suppose \( X \) param. p-space (e.g. \( \mathbb{C} \)-space), and \( f \approx g : X \to Y \). Then

\[ f^* = g^* : \text{Vect}_n(Y) \to \text{Vect}_n(X) \]

(i.e. \( f^* x = g^* x \forall x \in \text{Vect}_n(Y) \)).

**Example:** \( \text{Homom. of } \mathbb{Z} \), \( \text{Homom. of } \mathbb{R} \)

\( \text{Homom. of } \mathbb{C} \), \( \text{Homom. of } \mathbb{R}^n \)

\[ \delta_n(A, \nu) = A, \delta_n(A, \nu) = \mathbb{R}^n \]

\[ i(X \times \text{top space}, \delta_n^X) \text{ is functorial} \]

\[ x \in X, \quad \delta_n^X : [X, \text{Gn}(\mathbb{R}^\infty)] \to \text{Vect}_n(X) \]

\[ [x] \mapsto [x^* \delta_n^X] \]

Then (obvious): \( i \delta_n^X \) param. p-space, \( i \delta_n^X \) is bijective.

\[ [x^* \delta_n^X] = \delta_n^X(x) = [x^* \delta_n^X] \]

\[ x \in \text{param. p-space, } x \in \text{Top} \]

Hence: \( x \in \text{param. p-space, } x \in \text{Top} \)

\[ f^* : \text{Gn}(\mathbb{R}^\infty) \to \text{Gn}(\mathbb{R}^\infty) \]

\[ f^* : \text{Gn}(\mathbb{R}^\infty) \to \text{Gn}(\mathbb{R}^\infty) \]

\[ H^i(\text{Gn}(\mathbb{R}^\infty), Z/2) = \]

\[ 2/2 [\omega_1, \omega_2, \ldots, \omega_n] \quad \omega_i \in H^i(\text{Gn}(\mathbb{R}^\infty), Z/2) \]

\[ \omega_i = \tau^i \text{ remainder } \text{Stiefel-Whitney class} \]

\[ \tau^i : x \overline{\text{stiefel-whitney class}} \]

\[ \tau^i = \tau^i \text{ stiefel-whitney class of } x \]
For a function $f: X \to Y$ and $x \in X$, let

$$w_i(f^*)(x) = f^*w_i(x).$$

Ex. $n=1$: Recall $RP^\infty = K(\mathbb{Z}/2, 1)$. Then, for CW complexes,

$$\pi_1(\mathbb{Z}/2) = \pi_1(RP^\infty) = \pi_1(K(\mathbb{Z}/2, 1)) \cong H^1(X, \mathbb{Z}/2),$$

where $\pi_1(X)$ is the fundamental group of $X$.

Embedding of $X$ in $\mathbb{R}^n$ space.

Let $F(X, 2) = X \times X - \Delta$, where $\Delta$ is the diagonal of $X \times X$, and

$$F'(X, 2) = \text{quot. space obtained from } F(X, 2) \text{ by identification } (x, y) \sim (y, x).$$

Here and pl. denote $\lambda_x$ on $F(X, 2)$ as follows:

$$E(\lambda_x) = \text{quot. space obtained from } F(X, 2) \times \mathbb{R}$$

by sending

$$(x, y, r) \sim (y, x, -r).$$

Then $[x, y, r] \mapsto [x, y]$.

Also, have canonical Lie bundle $\lambda_n$ on $RP^n$ as follows:

$E(\lambda_n) = \text{quot. space obtained from } S^n \times \mathbb{R} \text{ by sending } (x, r) \sim (-x, -r),$ with

$[x, y, r] \mapsto [x, y].$

Let $X \subset \mathbb{R}^n$, get map $f$ and Lie bundle

$$f: \lambda_x \mapsto \lambda_{n-1} \text{ as follows.}$$

Then

$$F(X, 2) \xrightarrow{f^*} \mathbb{R}P^n$$

and

$$E(\lambda_x) \xrightarrow{f^*} E(\lambda_{n-1})$$

in particular, $w_i(\lambda_x) = f^*_B(w_i(\lambda_{n-1})).$ Now $w_i(\lambda_{n-1})^n = 0,$
and so if \( X \subset \mathbb{R}^n \), cannot be \( u_1(\lambda x)^n = 0 \), then

\[ u_1(\lambda x)^n \neq 0 \] 

then \( X \) is not homeomorphic to a subspace of \( \mathbb{R}^n \).

In this way, non-embedding theorems can be proved. E.g. if \( n = 2 \), then \( \mathbb{R}P^n \neq \mathbb{R}^{2n-1} \).

**Whitney Product Formula**

Given \( x, y \in \mathcal{X} \), if \( x \) and \( y \) are \( X \). Can for Whitney say

\[ x \oplus y, \quad \text{in } \mathcal{X} \oplus \mathcal{X}, \quad \text{in } \mathcal{X} \]

\[ \mathcal{E}_x(x \oplus y) = \mathcal{E}_x(x) \oplus \mathcal{E}_y(y) \quad \forall x \in X. \]

If \( \alpha \) is a \( n \cdot \) plan. b. def: \( u_\alpha(x) = 1 \in \Gamma^0(X, \mathcal{X}) \)

\[ u_\alpha(x) = 0 \quad \forall \alpha > n. \]

**Whitney Product Thm.** If \( \alpha \) and \( \beta \) are \( X \). Then \( \forall \alpha > 0 \)

\[ u_{k}(x \oplus y) = \sum_{i=0}^{k} u_i(x)(y) u_{k-i}(y). \]

One formally defines total stuff. Whitney def: \( u(x) = 1 + u_1(x) + u_2(x) + \cdots \)

Whitney prod. b. The fns: \( u(x \oplus y) = u(x)(y) u_1(y) \).

**Note.** \( \mathcal{E}_n = \text{tang. } n \cdot \) plan. b. \( X \), \( u(\mathcal{E}_n) = 1. \)

Implye \( x, \beta \) are \( \mathcal{X} \). Then \( x \oplus \beta = \mathcal{E}_{n+1} \)

Then \( u(x)(y) = 1 \). Thus \( u(x), u(y) \) algebra depend on the fns. \( u(x) \oplus u(y) = u(x, y) \).

If \( M \) smooth \( n \cdot \) -folded, \( f: M \rightarrow \mathbb{R}^{n+m} \) an immersion. \( \forall v \in \mathcal{T}_M \oplus \mathcal{V} = \mathcal{E}_{n+m} \), we \( u(v) \)

\[ u(v) = \overline{u(T_m)} \quad \forall \mathcal{V} \in \mathcal{E}_{n+m} \]

\[ \overline{u(T_m)} = 0 \quad \forall \mathcal{V} \in \mathcal{E}_m. \]

**Else**
Thm: \( \overline{\omega}(M^n) \neq 0 \), then \( M^n \) does not
survive in \( \mathbb{R}^{n+1} \).

Ex: If \( n = 2^r \), \( RP^n \) does not survive \( \mathbb{R}^{2n-2} \).