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Math 752 Algebraic Topology II - Winter '84

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Math 752
Algebraic Topology II
Winter '84, WSU
Let $R$ be a comm. ring with unit $1 \neq 0$. If $A$, $B$ are $R$-modules, write $\text{Hom}_R(A, B)$ = set of all $R$-homomorphisms from $A$ into $B$. $\text{Hom}_R(A, B)$ has the structure of an $R$-module with operations as follows: if $f, g \in \text{Hom}_R(A, B)$ and $r \in R$, define $f + g$ and $r \cdot f$ by $(f + g)(a) = f(a) + g(a)$, $(r \cdot f)(a) = r \cdot f(a)$.

**Example:** If $R$ is a field, $\text{Hom}_R(A, R) = A^*$ = dual space of $A$.

**Theorem:** $f: A' \to A$, $g: B \to B'$ are $R$-homomorphisms. Define $\text{Hom}(f, g): \text{Hom}_R(A, B) \to \text{Hom}_R(A', B')$ by

$$\text{Hom}(f, g)(a) = g \circ f.$$

**Note:** For fixed $A$, the assignment $B \mapsto \text{Hom}_R(A, B)$

$$\begin{array}{c}
B \mapsto \text{Hom}_R(A, B) \\
B \mapsto \text{Hom}_R(A, B)
\end{array}$$

is a covariant functor from cat. of $R$-modules to itself.

**In field $B$?**

Of $C$, $D$ categories. A covariant functor $T: C \to D$ assigns to each object $X \in C$ an object $T(X)$ in $D$, and to each morphism $f: X \to Y$ in $C$, a morphism $T(f): T(Y) \to T(X)$ in $D$, satisfying

1) $T(1_X) = 1_{T(X)}$
2) if $X \rightarrow Y \rightarrow Z$ are morphisms in $C$, then

$$T(gf) = T(f)T(g).$$
Example: In field $B$, the morphism
\[ A \longrightarrow \text{Hom}_R(A, B) \]
\[ A' \rightarrow A \longrightarrow \text{Hom}(f, 1_B) \]
is a contravariant functor from cat. of $R$-modules to itself.

Note: $\text{Hom}_R(R, A) \cong A$ for $R$-module $A$.

\[ f \longmapsto f(1) \]

Def: A cochain complex of $R$-modules $C$ consists of a seq. of $R$-modules $C^0, C^1, C^2, \ldots$ and $R$-homomorphisms
\[ \delta^n: C^n \rightarrow C^{n+1}, \ n \geq 0, \ \delta^n \circ \delta^{n+1} = 0 \ \forall n. \]

A cochain complex $C$ is coaugmented over $R$ if there is given an $R$-hom. $\eta: R \rightarrow C^0$, $\eta^2 = 0$.

If $C$ and $D$ are cochain complexes, a cochain map
\[ f: C \rightarrow D \]
is a seq. of $R$-homomorphisms $f^n: C^n \rightarrow D^n$
\[ C^n \overset{f^n}{\longrightarrow} D^n \]
A cochain map $f$ is coaugmented if
\[ R \overset{\eta}{\longrightarrow} C^0 \overset{f^0}{\longrightarrow} D^0 \]
\[ \eta \circ f^0 = 0 \]

Can form cat. of cochain complexes of $R$-modules + cat. of coaugmented cochain complexes of $R$-modules.

Example: Let $C$ be a cochain complex of $R$-modules, $A$ an $R$-module. Define a cochain complex
\[ \text{Hom}_R(C, A) \]
as follows: $\text{Hom}_R(C, A)^n = \text{Hom}_R(C^n, A)$, and $\delta^n = (-1)^n \text{Hom}(\delta_{n+1}, 1_A): \text{Hom}_R(C, A)^n \rightarrow \text{Hom}_R(C, A)^{n+1}$. 
Thus if $\alpha \in \text{Hom}_R (C; A)^n, \ x \in C^{n+1}$,
\[(\delta x)(x) = (-1)^n \alpha (\delta x).\]

Note: Some treatments omit the sign; others use the negative of this. If functions are consistently written on the left and their arguments on the right, our convention seems to be the most natural one.

If $f : C \to D$ is a chain map, the homomorphisms
$\text{Hom}(f_n, 1_A) : \text{Hom}_R (D, A)^n \to \text{Hom}_R (C, A)^n$ constitute a chain map
$(\text{Hom}(f), 1_A) : \text{Hom}_R (D, A) \to \text{Hom}_R (C, A)$.

To find $A$, the assignment $C \mapsto \text{Hom}_R (C, A)$,
\[ f \mapsto \text{Hom}(f, 1_A) \]
is a contravariant functor from cat. of chain complexes of $R$-modules to cat. of cochain complexes of $R$-modules.

In case $A = R$, write $C^* = \text{Hom}_R (C, R)$ and
\[ f^* = \text{Hom}(f, 1_R). \]

If $C$ is $R$-augmented with any, $\varepsilon : C_0 \to R$,
the composition $R \leftarrow \text{Hom}_R (R, R) \to \text{Hom}_R (C_0, R) = C^* \circ \text{Hom}(\varepsilon, 1_A)$
is an $R$-coaugmentation of $C^*$. The assignment
$C \mapsto C^*, f \mapsto f^*$ is a contravariant functor from cat.
of chain complexes of $R$-modules to cat. of $R$-coaugmented
cochain complexes of $R$-modules.

Def: Let $C$ be a cochain complex of $R$-modules. Define
$Z^n(C) = \ker (\delta^n) : C^n \to C^{n+1}$ = module of $n$-cycles of $C$,
$B^n(C) = \text{im} (\delta^n) : C^{n-1} \to C^n =$ " $n$-coboundaries of $C$,
$\text{Hom}_R (B^n(C), C) \subset Z^n(C)$. Define $H^n (C) = Z^n(C)/B^n(C) =
\text{n-th cohomology group of } C$.

If $f : C \to D$ is a chain map, $f$ maps cycles to cycles, coboundaries to coboundaries, and hence induces a map
$H^n(f) : H^n(C) \to H^n(D)$. For each $n$, $H^n$ is a covariant functor from cat. of cochain complexes of $R$-modules to cat. of $R$-modules.
Note: In any cochain complex \( C \), \( B^0(C) = 0 \) and so \( H^0(C) = Z^0(C) \). If \( C \) is \( R \)-coaugmented with coaug. \( \gamma : R \rightarrow C^0 \), then \( \gamma \in Z^0(C) \) and so we get a coaug. \( \gamma : R \rightarrow H^0(C) \).

If \( f : C \rightarrow D \) is a coaug. cochain map, then

\[
\begin{array}{ccc}
R & \xrightarrow{f} & H^0(f) \\
\gamma & \downarrow & \downarrow \\
& \rightarrow & H^0(D)
\end{array}
\]

commutes.

Def. \( (X,A) \) a top. pair; \( G \) an abel. group. Let \( \mathcal{Q}^\ast(X,A;G) = \text{Hom}_G(\mathcal{Q}(X,A), G) \) be a cochain complex of \( (X,A) \) with coefficients in \( G \).

Define \( H^n(X,A;G) = H^n(\mathcal{Q}^\ast(X,A;G)) \) as singular cohomology group of \( (X,A) \) with coefficients in \( G \).

If \( f : (X,A) \rightarrow (Y,B) \) is a map of top. pairs, then \( f^* : H^n(Y,B;G) \rightarrow H^n(X,A;G) \) for \( n \geq 0 \), \( H^n(\text{Hom}(G,f), G) \).

\( H^n(\cdot;G) \) is a contravariant functor from cat. of top. pairs to cat. of abel. groups.

Note: If \( A \) an abel. group, \( B \) an \( R \)-module, then \( \text{Hom}_R(A,B) \) has the structure of an \( R \)-module with \( (r \cdot f)(a) = r \cdot (f(a)) \). Moreover, if \( f : A \rightarrow A \) is a \( Z \)-homomorphism, \( g : B \rightarrow B \) an \( R \)-homomorphism, then \( \text{Hom}(f,g) \) is an \( R \)-hom.

Prop. \( A \) an abel. group, \( B \) an \( R \)-module. Then \( 3 \) nat. \( R \)-maps:

\[
\text{Hom}_R(A,B) \cong \text{Hom}_R(A \otimes R, B).
\]

Proof: \( f : A \rightarrow B \) is a \( Z \)-hom., define \( \overline{f} : A \otimes R \rightarrow B \)

by \( \overline{f}(a \otimes r) = r \cdot f(a) \). Can check \( \overline{f} \) is well-defined, and \( \overline{f} \rightarrow f \) is a natural \( R \)-map.
Page: R arbitrary, \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{0} \) an exact seq of \( R \)-modules. Then for any \( R \)-module \( D \),

\[
0 \to \text{Hom}_R(C,D) \xrightarrow{\text{Hom}_R(f,D)} \text{Hom}_R(B,D) \xrightarrow{\text{Hom}_R(g,D)} \text{Hom}_R(A,D)
\]

is exact, \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{0} \) is a split seq of \( R \)-modules, then

\[
0 \to \text{Hom}_R(C,D) \xrightarrow{\text{Hom}_R(f,D)} \text{Hom}_R(B,D) \xrightarrow{\text{Hom}_R(g,D)} \text{Hom}_R(A,D) \to 0
\]

is a split seq.

Proof: Exercise.

Now let \( R \) be a PID, \( A \) an \( R \)-module, and

\[
0 \to C(A) \xrightarrow{\partial_A} C_0(A) \xrightarrow{\varepsilon_A} A \to 0 \text{ the canonical seq of } A.
\]

For any \( R \)-module \( B \), can form the cochain complex \( \text{Hom}_R(C(A),B) \).

Note: \( H^n(\text{Hom}_R(C(A),B)) = 0 \) for \( n \neq 0,1 \). We have

\[
H^0(\text{Hom}_R(C(A),B)) = Z^0(\text{Hom}_R(C(A),B)) = \ker \text{Hom}(\partial_A,1_B).
\]

By exactness of

\[
0 \to \text{Hom}_R(A,B) \xrightarrow{\text{Hom}(\partial_A,1_B)} \text{Hom}_R(C_0(A),B) \xrightarrow{\text{Hom}(\partial_A,1_B)} \text{Hom}_R(C(A),B),
\]

we have \( H^0(\text{Hom}_R(C(A),B)) \cong \ker \text{Hom}_R(A,B) \).

Def. \( \mathfrak{E}_R(A,B) = H^1(\text{Hom}_R(C(A),B)) \).

Since \( Z^1(\text{Hom}_R(C(A),B)) = \ker \text{Hom}_R(C_1(A),B) \),

we have \( \mathfrak{E}_R(A,B) = \frac{\text{Hom}_R(C_1(A),B)}{\ker \text{Hom}(\partial_A,1_B)} \).
and so we obtain a natural exact sequence

\[ 0 \to \text{Hom}_R(A, B) \xrightarrow{\text{Hom}(f, i_B)} \text{Hom}_R(C_1(A), B) \xrightarrow{\text{Hom}(g, i_B)} \text{Hom}_R(C_1(A), B) \to E^1_{tr}(A, B) \to 0 \]

where \( f : A' \to A, \quad g : B \to B' \) are \( R \)-homomorphisms, and thus obtain a chain map \( C(f) : C(A') \to C(A) \), and the \( \text{Hom}_R(C(f), g) \) constitute a cochain map \( \text{Hom}_R(C(f), g) : \text{Hom}_R(C(A), B) \to \text{Hom}_R(C(A'), B') \). Define

\[ E^1(f, g) = H'(\text{Hom}_R(C(f), g)) : E^1_{tr}(A, B) \to E^1_{tr}(A', B'). \]

In fact, \( E^1(f, g) : \text{Hom}_R(C(f), g) \) is a contravariant functor from \( \text{cat. of } R \)-modules to itself, and for fixed \( A, B \)

\[ B \mapsto E^1_{tr}(A, B) \]

is a covariant functor from \( \text{cat. of } R \)-modules to itself.

Recall that if \( C \) is any s.f.n. of \( A \), there is an \( A \)-augmented chain equivalence \( f : C \to C(A) \), unique up to chain homotopy. Thus, for any \( R \)-module \( B \),

\[ \text{Hom}(f, i_B) : \text{Hom}_R(C(A), B) \to \text{Hom}_R(C, B) \]

is a cochain equivalence, and hence we have an isomorphism

\[ f^* : E^1_{tr}(A, B) \to H'(\text{Hom}_R(C, B)) \]

which is independent of the choice of \( A \)-augmented chain equiv. \( f \).

Just as in the case of \( L \omega \), it follows that if \( C \) is a s.f.n. of \( A', C' \) a s.f.n. of \( A' \), \( f : A' \to A, \quad g : B \to B' \)

\( R \)-homomorphisms, \( h : C' \to C \) a chain map \( h \)

\[ C_0 \xrightarrow{e} \quad C_0' \xrightarrow{e'} \quad A' \]

commutes, then

\[ 0 \to \text{Hom}_R(A, B) \to \text{Hom}_R(C_0, B) \to \text{Hom}_R(C_1(B), B) \to E^1_{tr}(A, B) \to 0 \]

\[ \downarrow \text{Hom}(f, i_B) \quad \downarrow \text{Hom}(h, i_B) \quad \downarrow \text{Hom}(d, i_B) \quad \downarrow E^1(f, g) \]

\[ 0 \to \text{Hom}_R(A', B') \to \text{Hom}_R(C_0', B') \to \text{Hom}_R(C_1(B'), B') \to E^1_{tr}(A', B') \to 0 \]

commutes.
Prop. In any $R$, $\operatorname{Hom}_R (A, \prod B_x) \cong \prod_{x \in \lambda} \operatorname{Hom}_R (A, B_x)$,

$\operatorname{Hom}_R (\bigoplus_{x \in \lambda} A_x, B) \cong \prod_{x \in \lambda} \operatorname{Hom}_R (A_x, B)$.

Case 1: Exercise.

(i) If $R$ is a PID, then

1) $\operatorname{Ext}_R (A, \prod B_x) \cong \prod_{x \in \lambda} \operatorname{Ext}_R (A, B_x)$.

2) $\operatorname{Ext}_R (\bigoplus_{x \in \lambda} A_x, B) \cong \prod_{x \in \lambda} \operatorname{Ext}_R (A_x, B)$.

Proof: 1) In each $\lambda$,

$0 \to \operatorname{Hom}_R (A, B_x) \to \operatorname{Hom}_R (C_\lambda (A), B_x) \to \operatorname{Hom}_R (C_\lambda (A), B_x) \to \operatorname{Ext}_R (A, B_x) \to 0$

is exact. Hence a direct prod. of short sequences is exact, we obtain the commutative diagram with exact rows

$0 \to \prod_{x \in \lambda} \operatorname{Hom}_R (A, B_x) \to \prod_{x \in \lambda} \operatorname{Hom}_R (C_\lambda (A), B_x) \to \prod_{x \in \lambda} \operatorname{Hom}_R (C_\lambda (A), B_x) \to \prod_{x \in \lambda} \operatorname{Ext}_R (A, B_x) \to 0$

yielding the nat. iso. 1).

2) $\bigoplus_{x \in \lambda} C_\lambda (A_x)$ is a s. f. of $\bigoplus_{x \in \lambda} A_x$. Thus, obtain comm. diagram with exact rows

$0 \to \operatorname{Hom}_R (\bigoplus_{x \in \lambda} A_x, B) \to \operatorname{Hom}_R (\bigoplus_{x \in \lambda} C_\lambda (A_x), B) \to \operatorname{Hom}_R (\bigoplus_{x \in \lambda} C_\lambda (A_x), B) \to \operatorname{Ext}_R (\bigoplus_{x \in \lambda} A_x, B) \to 0$

yielding the nat. iso. 2).
Prop: If $R$ is a PID, $A$ a free $R$-module. Then $\text{Ext}_R(A, B) = 0$ for all $B$.

Proof: $0 \to 0 \to A \xrightarrow{i_A} A \to 0$ is a short exact sequence.

In general, if $B$ is free it does not follow that $\text{Ext}_R(A, B) = 0$. (Hom-into a free module does not always preserve exactness, in contrast to tensoring with a free module.)

Prop: For $n > 1$, $\text{Ext}_2(\mathbb{Z}/n, A) \cong A/nA \cong \mathbb{Z}/n \otimes A$ for all abelian groups $A$.

Proof: $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n \to 0$ is a short exact sequence.

\[
\begin{array}{ccc}
\text{Hom}_\mathbb{Z}(\mathbb{Z}, A) & \xrightarrow{\text{Hom}(\mathbb{Z}, \pi)} & \text{Hom}_\mathbb{Z}(\mathbb{Z}/n, A) \\
A & \xrightarrow{n} & A
\end{array}
\]

This exact sequence occurs in $\text{Ext}_2(\mathbb{Z}/n, A)$, so $\text{Ext}_2(\mathbb{Z}/n, A) \cong A/nA$.

Now $\mathbb{A} \cong \mathbb{Z} \otimes A \xrightarrow{\pi \otimes 1_A} \mathbb{Z}/n \otimes A$ is exact

with kernel $nA$, so $A/nA \cong \mathbb{Z}/n \otimes A$.

Let $R$ be a ring, $C$ a chain complex of $R$-modules, $A$ an $R$-module. Let $\mathcal{A} \in \text{Hom}_R(C, A) = \text{Hom}_R(C, A)^n$ be a cycle.

Write $[\mathcal{A}] = \text{cohomology class of } \mathcal{A}$.

Claim: In any cycle $\gamma \in C_n$, $\mathcal{A}(z) \in A$ depends only on $[\mathcal{A}]$ and on $\mathcal{A}$.

\[
\mathcal{A}(z) + (\delta x)(z) + \mathcal{A}(z) = \mathcal{A}(z) + \mathcal{A}(z) + (\delta x)(z) + (\delta x)(z) + (\delta x)(z) = \mathcal{A}(z) + \delta x(z).
\]

Thus, each cohomology class $[\mathcal{A}] \in H^n(\text{Hom}_R(C, A))$ yields a function $\text{Hom}(C) \to A$, which is easily seen to be an $R$-homomorphism.

Thus, we obtain a function $V : H^n(\text{Hom}_R(C, A)) \to \text{Hom}_R(\text{Hom}(C), A)$ given by $V([\mathcal{A}])([\mathcal{A}]) = \mathcal{A}(z)$, which is easily seen to be a natural $R$-homomorphism.
Theorem (Universal Coefficients in Cohomology): Let \( R \) be a PID, \( C \) a chain complex of free \( R \)-modules, \( A \) an \( R \)-module. Then \( \exists \) an exact seq.
\[
0 \to \text{Ext}_R^i(H_{n-1}(C), A) \to H^n(Hom_R(C, A)) \to \text{Hom}_R(H_n(C), A) \to 0
\]
which splits (splitting not natural).

Proof: We have a s.e.s. of chain complexes
\[
0 \to \mathbb{Z} \to C \to \overline{C} \to 0
\]
where \( Z_n = Z_n(C), \overline{Z}_n = B_n = B_{n-1} = \mathbb{Z}_n, \overline{A}_k = 0. \)
and \( \overline{\delta}_n = \partial_n |_{\overline{B}_{n-1}}. \)
Since each \( \overline{B}_n \) is free, above s.e.s. splits in each degree, and so we get a s.e.s. of cochain complexes
\[
\text{Hom}(\overline{\delta}, \mathbb{Z}_n) \to \text{Hom}(\delta, \mathbb{Z}_n) \to \text{Hom}(\delta, \mathbb{Z}_n) \to 0,
\]
due \( \delta_0 = 0\) and \( \delta_2 = 0\) we have
\[
H^n(\text{Hom}(\mathbb{Z}, A)) = \text{Hom}(Z, A), \quad \text{and } H^n(\text{Hom}(\overline{B}, A)) = \text{Hom}(\overline{B}, A).
\]
Let \( j_n : B_n \to Z_n \) denote the inclusion.

Claim 1: The connecting homomorphism \( \delta : H^n(\text{Hom}(\mathbb{Z}, A)) \to H^n(\text{Hom}(\overline{B}, A)) \) arising from (x) is \((-1)^n\text{Hom}(j_n, \mathbb{I}_A) ; H^n(\text{Hom}(Z, A)) \to H^n(\text{Hom}(B, A)).
\]
\[
\text{Hom}(\delta, \mathbb{Z}_n) \to \text{Hom}(\delta, \mathbb{Z}_n) \to \text{Hom}(\delta, \mathbb{Z}_n) \to 0
\]
due \( \delta_0 = 0\) and \( \delta_2 = 0\) we have
\[
H^n(\text{Hom}(\mathbb{Z}, A)) = \text{Hom}(Z, A), \quad \text{and } H^n(\text{Hom}(\overline{B}, A)) = \text{Hom}(\overline{B}, A).
\]
Let \( f \in \text{Hom}(Z, A) \) and \( g \in \text{Hom}(Z, A) \) s.t. \( f = \text{Hom}(j_n, \mathbb{I}_A)(g) \), \( x \in \mathbb{Z}, \delta f = (-1)^n g \partial \). On the other hand, \( \text{Hom}(\delta, \mathbb{Z}_n)((-1)^n \text{Hom}(j_n, \mathbb{I}_A)(f)) = (-1)^n f j_n \delta = (-1)^n \delta f \delta = (-1)^n \delta^2 f \), proving Claim 1.

Claim 2: The diagram
\[
\begin{array}{ccc}
\text{Hom}(\overline{\delta}, \mathbb{I}_A) & \to & \text{Hom}(\overline{\delta}, \mathbb{I}_A) \\
n & \downarrow & \downarrow \\
H^n(\text{Hom}(Z, A)) & \to & H^n(\text{Hom}(Z, A))
\end{array}
\]
commutes.
\[
H_n \left( \mathbb{R}, (C, A) \right) \cong \mathbb{Z} \quad \text{for } n \geq 0,
\]

where \( \mathbb{R} \) is the real numbers, \( C \) is a cone, and \( A \) is an abelian group.

The sequence is split by the projection \( \pi : \mathbb{R} \to \mathbb{Z} \), where \( \pi(x) = \lfloor x \rfloor \).

\[\pi(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}\]

Thus, \( \mathbb{R} \) is the universal cover of \( \mathbb{Z} \).

**Claim:** \( \pi \) is a splitting of the sequence.

For any \( x, y \in \mathbb{R} \), we have
\[
\pi(x + y) = \pi(x) + \pi(y).
\]

Thus, \( \pi \) is a splitting of the sequence.
Ca: \((X,A)\) top. pair, \(G\) abel. grp. \(I\) nat. exact seq.

\[ 0 \rightarrow \mathcal{H}_\mathcal{E}^2(H_{n-1}(X,A),G) \rightarrow H^n(X,A;G) \rightarrow \mathcal{H}_\mathcal{R}^n(H_n(X,A),G) \rightarrow 0 \]

which splits non-naturally.

\(R\) a PID and \(G\) an \(R\)-module, \(I\) nat. exact seq.

\[ 0 \rightarrow \mathcal{H}_\mathcal{E}^2(H_{n-1}(X,A;R),G) \rightarrow H^n(X,A;G) \rightarrow \mathcal{H}_\mathcal{R}^n(H_n(X,A;R),G) \rightarrow 0 \]

which splits non-naturally.

For second part, use the nat. resim of cochain complexes \(\mathcal{H}^n_{\mathcal{R}}(Q(X,A);G) \cong \mathcal{H}^n_{\mathcal{R}}(Q(X,A) \otimes \mathcal{R},G)\).

Ca: \(\forall \, F\) a field, \(V: H^n(X,A;F) \rightarrow \mathcal{H}_{\mathcal{F}}^n(H_n(X,A;F),F)\) is an invm.

Then: Supp. all with coeff. in \(G\) satisfies the "Eilenberg-Steenrod axioms" in cobology, i.e. for each \(n \geq 0\), \(H^n(\_;G)\) is a cochain functor from top. pairs to abel. grps. (\(R\)-module, \(G\) is an \(R\)-module) sol.

1) (Descent, axiom) \(\forall \, P\) a point, \(H^i(P;G) = \{0\} \forall \, i \geq 0\).

2) (Eilenberg: \((X,A)\) a top. pair, \(I\) nat. exact seq.

\[ \xymatrix{ H^n(X,A;G) \ar[r]^-j & H^n(X,G) \ar[r]^-\sim & H^n(A;G) \ar[r]^-\delta & H^{n+1}(X,A;G) \ar[l]_i } \]

\(i: A \rightarrow X\), \(j: (X,\emptyset) \rightarrow (X,A)\) continuus.

3) (Hmtn. topo) \(\forall \, f \sim g: (X,A) \rightarrow (Y,B)\), then

\[ f^* = g^*: H^n(Y,B;G) \rightarrow H^n(X,A;G) \]

4) (Epim) \(\forall \, U \subseteq \text{int} \, A\), \((X,A)\) top. pair.

\(i: (X-U,A-U) \rightarrow (X,A)\) continuus, then

\[ i^*: H^n(X,A;G) \rightarrow H^n(X-U,A-U;G)\] is an invm.

Proof: Similar to proving \(H^n(\_;G)\) soln. for the E-S axioms.

Use E-S axioms for \(H^n(\_;\mathcal{Z})\), together with U.C.T.
Cohomology Cross Product

$R$ a PID. Write

$$[H^*(X;A;R) \otimes H^*(Y;B;R)]^n =$$

$$\bigoplus_{p+q=n} H^p(X;A;R) \otimes H^q(Y;B;R).$$

Def: A top. pair $(X,A)$ is acyclic if $A$ is open in $X$.

The cohomology cross product

$$\bar{\mu}: [H^*(X;A;R) \otimes H^*(Y;B;R)]^n \rightarrow H^*(((X,A) \times (Y,B);R)$$

is a natural $R$-hom which will be defined for acyclic modules pairs $(X,A)$, $(Y,B)$. It will be defined as the composition of 3 natural maps, 2 of which are purely algebraic. 3rd involves $E = 3$ morph.

1st map: $C,D$ cochain complexes of $R$-modules. Tensor product $C \otimes D$ is defined just as for chain complexes, and as in the chain complex case we get a natural $R$-hom.

$$\mu: \left[ H^*(C) \otimes H^*(D) \right]^n \rightarrow H^*(C \otimes D)$$

$$\left[ [x] \otimes [y] \right] \mapsto [x \otimes y]$$

$$\left( \left[ H^*(C) \otimes H^*(D) \right]^n = \bigoplus_{p+q=n} H^p(C) \otimes H^q(D) \right).$$

Thus for any top. pair we get natural map

$$\bar{\mu}: \left[ H^*(X;A;R) \otimes H^*(Y;B;R) \right]^n \rightarrow H^*((X,A;R) \otimes (Y,B;R)).$$

2nd map: $C,D$ chain complexes of abel. groups. 1st not map of cochain complexes of $R$-modules

$$\eta: \text{Hom}_{\mathcal{Z}_2}(C,R) \otimes \text{Hom}_{\mathcal{Z}_2}(D,R) \rightarrow \text{Hom}_{\mathcal{Z}_2}(C \otimes D,R)$$

given as follows: $\eta: f \in \text{Hom}_{\mathcal{Z}_2}(C,R)$, $g \in \text{Hom}_{\mathcal{Z}_2}(D,R)$, $x \in C_n$, $y \in D_s$, then

$$\eta(f \otimes g)(x \otimes y) = \begin{cases} (-1)^{p+q} f(x) g(y) & \text{if } p = r \text{ and } q = s \\ 0 & \text{otherwise} \end{cases}$$
Straightforward to check: $\varphi$ is a natural map of cochain complexes of $R$-modules. Thus, passing to cohomology, $\varphi$ induces a nat. $R$-map
$$
\varphi^* : H^n(R\varphi_0(\mathcal{C}, R) \otimes R\varphi_0(\mathcal{D}, R)) \to H^n(R\varphi_0(\mathcal{C} \otimes R \mathcal{D}, R)).
$$
Thus, for any top. pairs $(X, A), (Y, B)$, get nat. $R$-maps,
$$
\varphi^* : H^n(Q(X, A; R) \otimes Q(Y, B; R)) \to H^n(Q((X, A) \otimes R \mathcal{D}, R))
$$

3rd map: Recall: for admissible pairs $(X, A), (Y, B)$ we have a natural Eilenberg-Zilber chain map
$$
\Theta : Q(X, A) \otimes Q(Y, B) \to Q((X, A) \times (Y, B))
$$
which induces $\cong$ in homology. On the absolute case, we have proved $\Theta$ is a chain equivalence, but in the relative case we have not proved $\Theta$ is a chain equivalence since in the proof of Theorem, we did not prove that $Q(X, A) \to Q(X)$ is a chain equivalence, but only that it induces $\cong$ in homology).

Since, for admissible pairs $(X, A)$ and $(Y, B)$, $\varphi$ induces nat. $\cong$ in homology for all $n$, $\varphi_0(\Theta, R)$ induces a nat. $\cong$,
$$
\varphi^* : H^n((X, A) \times (Y, B); R) \to H^n(Q((X, A) \otimes R \mathcal{D}, R))
$$
(cannot call these $\cong$).

The 3rd map is $(\Theta^* )^{-1}$.

$\tilde{\varphi}$ is defined to be the composition
$$
\tilde{\varphi} = (\Theta^* )^{-1} \varphi^* : [H^*(X, A; R) \otimes R H^*(Y, B; R)]^n \to H^n((X, A) \times (Y, B); R)
$$

Let $a \in H^p(X, A; R), b \in H^q(Y, B; R)$, write,
$$
a x b = \tilde{\varphi}(a \otimes R b) \in H^{p+q}((X, A) \times (Y, B); R).
$$
ax b is called the cross-product of $a$ and $b$.

Prop: if $(X, A') \to (X, A), (Y, B') \to (Y, B)$ are nat. maps of admissible pairs, $a \in H^p(X, A; R), b \in H^q(Y, B; R)$, then
$$
(f \times g)^*(a \otimes b) = (f \times g)^* \tilde{\varphi}(a \otimes b) = \tilde{\varphi} (f^* \otimes g^*)(a \otimes b).
$$

Proof: This is a restatement of naturality of $\tilde{\varphi}$. In
$$
(f \times g)^*(a \otimes b) = (f \times g)^* \tilde{\varphi}(a \otimes b) = \tilde{\varphi} (f^* \otimes g^*)(a \otimes b)
$$
$$
= \tilde{\varphi} (f^* a \otimes g^* b) = f^* a \times g^* b.
$$
\[
\begin{align*}
\text{Claim:} & \quad \text{If } \phi \in H^*(X;R), \text{ then } ax(\phi(x)) = c(x, \phi(x)) \in H^*(X \times Y; R) \\
\text{Proof:} & \quad \text{Support the following comes to certain heterotopy.}
\end{align*}
\]
The proof proceeds by applying the Horn rule to the diagram of chain maps:

\[
\begin{align*}
Q(X) \otimes Q(Y) \otimes Q(Z) \xrightarrow{1 \otimes \Theta} Q(X) \otimes Q(Y \times Z) \\
\Theta \otimes 1 \downarrow & & \downarrow \Theta \\
Q(X \times Y) \otimes Q(Z) \xrightarrow{\Theta} Q(X \times Y \times Z),
\end{align*}
\]

and it suffices to show this last diagram commutes up to chain homotopy.

Recall, if natural chain equivalences \( F : \Delta \to Q \) and \( G : Q \to \Delta \) satisfy the contractive chain homotopy \( GF = 1_{\Delta} \), \( FG = 1_{Q} \). A cyclic model yields a natural augmented chain equivalence

\[
\Delta(X) \otimes \Delta(Y) \xrightarrow{\phi} \Delta(X \times Y), \text{ unique up to chain homotopy}, \text{ and } \Theta \text{ was taken to be the composition}
\]

\[
Q(X) \otimes Q(Y) \xrightarrow{G \otimes G} \Delta(X) \otimes \Delta(Y) \xrightarrow{\phi} \Delta(X \times Y) \xrightarrow{F} Q(X \times Y).
\]

Conclude, diagram
(1) and (2) commute up to chain homotopy by def. of $\Theta$ and fact that $F_0 \simeq 1_Q$.

(3) + (4) commute up to chain homotopy since

$$\Theta \circ (F \circ F) = F \circ \Phi \circ (G \circ G) \circ (F \circ F) \simeq F \circ \Phi \circ GF \simeq 1_A.$$ 

(5) commutes up to chain homotopy by acyclic models: Take any $C$ of ordered triples of top. spaces, $M = \{(\Delta^p, \Delta^q, \Delta^r)\}$, all proper (no degenerate cones). The functors $S, T : C \to \text{Cat}$ of any ch. complex, given by $S(X, Y, Z) = \Delta(X) \otimes \Delta(Y) \otimes \Delta(Z)$, $T(X, Y, Z) = \Delta(X \times Y \times Z)$ are both $M/\mathbb{D}$-free and acyclic on models.

$\Phi(\mathbb{I} \otimes \Phi)$ and $\Phi(\Phi \otimes \mathbb{I}) : S(X, Y, Z) \to T(X, Y, Z)$ are natural augmented chain maps, and hence, by the acyclic models theorem, chain homotopy.

Proof: $X, Y$ top. spaces, $T : Y \times X \to X \times Y$ the twist map $T(y, x) = (x, y)$. Then for $a \in H^n(X; R), b \in H^m(Y; R)$,

$$T^*([a, b]) = (-1)^{mp} [b, a].$$

Proof: For chain complexes $C, D$, let $T : C \otimes D \to D \otimes C$ denote the natural augmented chain map $T(C \otimes D) = (-1)^{|c||d|} D \otimes C$, and similarly $T$ for chain complexes. Suff. to show following commutes up to chain homotopy:

$$Q^*(X; R) \otimes Q^*(Y; R) \xrightarrow{\tau^*} \text{Hom}_2(G(Y) \otimes G(X); R) \xleftarrow{\Theta^*} Q^*(X \times Y; R)$$

$$Q^*(Y; R) \otimes Q^*(X; R) \xrightarrow{\tau^*} \text{Hom}_2(G(X) \otimes G(Y); R) \xleftarrow{\Theta^*} Q^*(Y \times X; R)$$

(1) commutes by direct check. (2) is the dual of

$$Q(X) \otimes Q(Y) \xrightarrow{\Theta} Q(X \times Y)$$

$$\tau^* \uparrow \quad \uparrow \Theta^*$$

$$Q(Y) \otimes Q(X) \xrightarrow{\Theta} Q(Y \times X)$$

so suffices to check the bent diagonal commutes up to chain homotopy.
By an argument similar to that of previous prop., suffices to show some diagram with $Q$ replaced by $\Delta$ commutes up to chain homotopy. This follows by acyclic models.

Recall: in any top. space $X$, $Q^*(X; R)$ has an opposing $\pi: R \to Q^*(X; R)$, which yields a natural opposing $\pi^*: R \to H^0(X; R)$, as clear to $\varepsilon: H_0(X; R) \to R$ which is auto if $X \neq \emptyset$.

If $X \neq \emptyset$, let $\gamma(1) \in H^0(X; R)$ be denoted by $1$.

By contradiction, if $f: X \to Y$ and $X, Y \neq \emptyset$, then $f^*(1) = 1$.

Prop: if $X, Y$ non-empty top. spaces, $\pi_1: X \times Y \to X$ proj. on 1st factor, then for each $a \in H^n(X; R)$, $\pi_1^*(a) = a \times 1$.

Proof: let $P$ be a point, $p: Y \to P$ the constant map. Suppose we knew $\pi_1^*(a) = a \times 1$ where $\pi_1: X \times P \to X$ is proj. on 1st factor. Then by commutativity of

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{\pi_1} & X \\
1 \times P & \downarrow & \\
X \times P & \xrightarrow{\pi_1} & X
\end{array}
\]

and naturality of cross products,

$\pi_1^*(a) = (1 \times P)^* \pi_1^*(a) = (1 \times P)^*(a \times 1) = 1^*(a) \times P^*(1) = a \times 1$.

Hence, suffices to prove prop. in $Y = P$. Suff. to show following commutes up to chain homotopy:

\[
\begin{array}{ccc}
Q^*(X; R) \otimes Q^*(P; R) & \xrightarrow{\phi} & H_{op}^*(Q(X) \otimes Q(P); R) \\
\downarrow{\phi \text{ (invers)}} & & \downarrow{\phi} \\
\text{Hom}(Q(X), R) \otimes \text{Hom}(Q(P), R) & \xrightarrow{0^*} & Q^*(X \times P; R)
\end{array}
\]

where $\phi(x) = x \otimes \phi(1)$ for $x, \phi(1) \in Q^*$.

and $K$ is the composition

\[
\begin{array}{ccc}
Q_n(X) \otimes Q_n(P) & \xrightarrow{\phi_n} & Q_n(K) \\
\phi_n \otimes 1 & \xrightarrow{1 \circ} & u
\end{array}
\]

(Recall: $Q_n(P) = 1, n \neq 0, \phi_0(P) \cong \mathbb{D}$, gen. $1_0$).
Exactly checked: 1. commutes, and \( k \) is a natural map.

\[
\begin{align*}
Q(X) & \xrightarrow{k} Q(X) \otimes Q(P) \\
& \xrightarrow{\alpha) e} Q(X \times P).
\end{align*}
\]

Proof is completed in same way as previous props. Can replace \( \alpha \) by \( \Delta \), and use any other models.

Similarly, \( \alpha \) as a consequence of commutativity.

Prop: \( X, Y \) top. spaces \( \neq \emptyset \), \( \xi \in H^q(X, \mathbb{R}) \). Then \( \pi_{q+2} = 1 \times b^q \).

Extension of properties of cross prod. to relative case

\[ (X, A) \) a top. pair. Consider \( (X, UCA, CA) \), \( CA \) comm. on \( A \).

By assumption and def. restriction, the conclusion is: \( (X, A) \rightarrow (X, UCA, CA) \) induces \( \xi \) in cohomology. Note: \( (X, A) \) admits

\[ (X, UCA, CA) \) admits.

Lemma: \( (X, A), (Y, B) \) admissible pairs. Then the map induced by restriction

\[ H^p((X, UCA, CA) \times (Y, B)) \rightarrow H^p((X, UCA) \times (Y, B)) \]

is \( 1-1 \) (relative coeff).

Proof: The sequel of chain completes

\[ 0 \rightarrow Q(CA) \otimes Q(Y, B) \rightarrow Q(X, UCA) \otimes Q(Y, B) \rightarrow Q(X, UCA, CA) \otimes Q(Y, B) \rightarrow 0, \]

together with \( \otimes \) in prod, yields the exact seq

\[ H^{p+1}(X, UCA) \times (Y, B)) \rightarrow H^{p+1}(CA \times (Y, B)) \rightarrow H^p((X, UCA, CA) \times (Y, B)) \rightarrow H^p((X, UCA) \times (Y, B)) \rightarrow 0. \]

Hence, suff to show \( H^{p+1}(X, UCA, CA) \times (Y, B)) \rightarrow H^{p+1}(CA \times (Y, B)) \) is equi.

Write \( \tilde{x} \) a vertex of \( CA \). Since \( CA \) is contractible, the inclusion \( \tilde{x} \times (Y, B) \subseteq CA \times (Y, B) \) is a homotopy equivalence, and hence
induced in cohomology. Hence, suff. to show the map induced by restriction 
$H^p((X \cup A) \times (Y, B)) \to H^{p-1}(X \times (Y, B))$ is into. 
But $X \times (Y, B)$ is a retract of $X \cup A \times (Y, B)$, so each induced 
by soul is into.

**Prop:** $(X, A)$ admissible, $Y \neq \emptyset$, $R \simeq \mathbb{Z}$, $\pi: (X, A) \times Y \to (X, A)$ 
pre in first pair. Then, for any $a \in H^p((X, A), R)$, $\pi^*(a) = a \times 1$.

**Proof:** Here comm. diagram

\[
\begin{array}{ccc}
H^p(X, A) & \xrightarrow{\pi^*} & H^p((X, A) \times Y) \\
\uparrow i^* & & \uparrow (i \times 1)^* \\
H^p(X \cup A, \Delta) & \xrightarrow{\pi^*} & H^p((X \cup A, \Delta) \times Y) \\
\downarrow j^* & & \downarrow (j \times 1)^* \\
H^p(X \cup A) & \xrightarrow{\pi^*} & H^p((X \cup A) \times Y) \\
\downarrow a^* & & \downarrow a^* \times 1 \\
& & \\
\end{array}
\]

We know $(j \times 1)^*(a^* \times 1) = j^* a^* \times 1^* = a^* \times 1$ by calc. of cross prod. 
Since $(j \times 1)^*$ is 1-1 (by lemma) and bottom square commutes, 
bottom line $\pi^*(a^*) = a^* \times 1$. Then 
$\pi^* a = \pi^* j^* (a^*) = (j \times 1)^* \pi^* (a^*) = (j \times 1)^*(a^* \times 1) = a^*(a^*) \times 1^* = a^* \times 1$.

**Prop:** $(X, A), (Y, B)$ admissible pairs, $R \simeq \mathbb{Z}$, $\pi: (X, A) \times (Y, B) \to (X, A) \times (Y, B)$ interchanges pairs.

**Proof:** Similar to proof of preceding using diagram.
The diagram represents a sequence of morphisms in an algebraic context, involving the exterior product of cohomology groups. The notation and structure suggest a proof or derivation in algebraic topology, possibly related to the Künneth formula or a similar concept.

The diagram includes:
- A sequence of cohomology groups and maps:
  - $H^p(X,A) \otimes H^q(Y,B) \xrightarrow{\alpha \otimes \beta} H^{p+q}(X,A) \times (Y,B)$
  - $\xrightarrow{T} H^{p+q}((Y,B) \times (X,A))$

The text suggests the following points:
- Proposition: $(X,A), (Y,B), (Z,C)$ admissible pairs, $R = PID$, $\alpha \in H^p(X,A; R), \beta \in H^q(Y,B; R), \gamma \in H^r(Z,C; R)$.
- Lemma: $a \times (b \times c) = (a \times b) \times c$.

Proof: Similar to above, using the Künneth formula.

The sequence and maps are depicted with arrows indicating the direction of the mappings and the relationships between the cohomology groups and their products.
Cup Products

Def. (Absolute case): $X$ a top. space, $R$ a PID. Let $d: X \to X \times X$ denote the diagonal map, $d(x) = (x,x)$. Let $a \in H^p(X; R)$, $b \in H^q(X; R)$. The cup product $a \smile b \in H^{p+q}(X; R)$ is defined by

$$a \smile b = d^*(a \times b).$$

Note: A cup product exists in homology, but a product analogous to cup product doesn't exist in homology since homology is additive. Let

$$[H(X; R) \otimes H(X; R)] \to H_n(X \times X; R) \xrightarrow{d_+} H_n(X; R).$$

Let $(X; A, B)$ a triad and $d: X \to X \times X$ the diagonal map,

$$d^{-1}(X \times B - A \times X) = A \cup B.$$ Thus, get cup of tri. pairs

$$d: (X, A \cup B) \to (X, A) \times (Y, B).$$

Def. (Rel. cup prod): $(X; A, B)$ an admissible triad (i.e. $A, B$ pairs of $X$),

$R$ a PID. Let $a \in H^p(X; A, R)$, $b \in H^q(X; B, R)$. The cup prod

of $a$ and $b$ is $a \cup b = d^*(a \times b) \in H^{p+q}(X; A \cup B, R).$

Def: A map of admissible triads $f: (X'; A', B') \to (X, A, B)$

consists of a conr map $X' \to X$ which carries $A'$ into $A$, $B'$ into $B'$.

By restriction, such an $f$ gives maps of admissible pairs

$$f_{A'B'}: (X', A' \cup B') \to (X, A \cup B), \quad f_A: (X', A') \to (X, A), \quad f_B: (X', B') \to (X, B).$$

Prop: Let $f: (X'; A', B') \to (X, A, B)$ be a map of admissible triads,

$a \in H^p(X; A; R)$, $b \in H^q(X; B, R)$, then

$$d^+_B(a \cup b) = f_A^*(a) \cup f_B^*(b).$$

Proof:

$$\begin{array}{ccc}
(X', A' \cup B') & \xrightarrow{d'} & (X', A') \times (X', B') \\
\downarrow f_{A'B'} & & \downarrow f_A \times f_B \\
(X; A \cup B) & \xrightarrow{d} & (X, A) \times (X, B)
\end{array}$$

Consider.

Then, $f^*_{A'B'}(a \cup b) = f^*_{A'B'}d^*(a \times b) = (d'f_{A'B'})^*(a \times b) = (f_A \times f_B)^*(f_A^*(a) \times f_B^*(b))$

$$= (d')^*(f_A \times f_B)^*(a \times b) = (d')^*(f_A^*a \times f_B^*b) = f_A^*(a) \cup f_B^*(b).$$
Prop. a) Cup prod. is additive, i.e., if \( A, B, C \) open in \( X \), \( R \subset \mathbb{P}^D \),
\( \omega \in H^p(X, A; R) \), \( \mu \in H^q(X, B, R) \), \( c \in H^1(X, C; R) \), then
\[ a \cup (b \cup c) = (a \cup b) \cup c. \]
b) Cup prod. is commutative, i.e., if \( A, B, C \) open in \( X \), \( \omega \in H^p(X, A; R) \),
\( \mu \in H^q(X, B; R) \), then \( a \cup b = (-1)^{pq} b \cup a \).
c) If \( X \neq \emptyset \), \( a \in H^0(X, A; R) \), then \( 1 \omega = a \cup 1 = a. \)
d) \( a \cup (b+c) = a \cup b + a \cup c \), \( (ra) \cup b = a \cup (rb) = r(a \cup b), r \in \mathbb{R} \).

**Proof:**

**a)** \(\xrightarrow{d} (X, A) \times (X, B) \times (X, C) \xrightarrow{d \times 1} (X, A) \times (X, B) \times (X, C) \) commutes.

Hence \( a \cup (b \cup c) = d^* (a \times d^* (b \times c)) = d^* (1 \times d) (a \times (b \times c)) \)
\[= d^* (d \times 1)^* ((a \times b) \times c) = (a \cup b) \cup c. \]

**b)** \(\xrightarrow{d} (X, A) \times (X, B) \xrightarrow{T} (X, B) \times (X, A) \) commutes.

Hence \( a \cup b = d^* (a \times b) = d^* T^* (a \times b) = d^* [(-1)^p (b \times a)] = (-1)^p b \cup a. \)

**c)** \(\xrightarrow{d} (X, A) \times X \xrightarrow{\pi} (X, A) \) commutes.

Hence \( a = 1^* a = d^* \pi^* a = d^* (a \times 1) = a \cup 1. \)

Similarly, \( 1 \cup a = a. \)

**d)** Cross prod. is \( R \)-bilinear.
Let \( (X, A, A') \), \( (Y, B, B') \) be schemes. Let:
\[
a \in H^p(X, A, R), \ b \in H^q(Y, B, R), \ \alpha \in H^p(X, A', R), \ l \in H^q(Y, B', R).
\]
Then \((a \times b) \cup (a' \times b') = (-1)^{p+q} (a \cup a') \times (b \cup b') \).

\[
\text{Proof:} \quad (X, A \cup A') \times (Y, B \cup B') \xrightarrow{d} (X, A) \times (Y, B) \times (X, A') \times (Y, B')
\]

\[
d_x \times d_Y
\]

\[
(1 \times 1 \times 1)
\]

\[
(X, A) \times (X, A') \times (Y, B) \times (Y, B')
\]

Hence \((a \times b) \cup (a' \times b') = d^* ((a \times b) \times (a' \times b')) = (d_x \times d_Y)^* (1 \times 1 \times 1)^* (a \times (b \times a') \times (b' \times b')) = (-1)^{p+q} (a \cup a') \times (b \cup b').
\]

Corollary: \((X, A), (Y, B)\) schemes. Let:
\[
a \in H^p(X, A, R), \ b \in H^q(Y, B, R).
\]
Then \(a \times b = \pi_1^* (a) \cup \pi_2^* (b)\)

where \(\pi_1 : (X, A) \times Y \to (X, A), \ \pi_2 : X \times (Y, B) \to (Y, B)\) are projections.

Proof: \(\pi_1^* a \cup \pi_2^* b = (a \times 1) \cup (1 \times b) = (\alpha \cup 1) \times (1 \cup b) = a \times b\).

Then: Suppose \(X = A \cup B\), \(A, B\) open in \(X\) if \(A, B\) both connected.
\(R\) is a PID. \(a \in H^p(X, R), \ b \in H^q(X, R), \ p > 0, q > 0\). Then \(a \cup b = 0\).

Proof: Here common diagram
\[
\begin{align*}
H^p(X) \otimes H^q(X) & \xrightarrow{0} H^{p+q}(X) \\
0 \uparrow & \uparrow 1
\end{align*}
\]

\[
H^p(X, A) \otimes H^q(X, B) \xrightarrow{0} H^{p+q}(X, A \cup B).
\]

Assume \(A, B\) both connected and \(p, q > 0\); \(i^!, \) and \(i^*\) are isomorphic.
Let $A \cup B = X$ and $\text{H}^{p+q}(X, A \cup B) = 0$. Hence, $\text{H}^{p}(X) \otimes \text{H}^{q}(X) \rightarrow \text{H}^{p+q}(X)$ is 0.

This proof generalizes to yield the following:

Theorem. If $X = \bigcup_{i=1}^{n} \text{open, contractible subspaces}$, then, all $n$-fold cup products of non-degenerate elements in $H^{*}(X; R)$ vanishes.

Cor. If $X$ is the suspension of some space, then all cup products of non-degenerate elements in $H^{*}(X; R)$ vanishes.

A cohomology Kernels Theorem.

Let $R$ be a PID, $C$, $\mathcal{D}$, be chain complexes of $R$-modules with each $C^{n}$ free. Then just as in the homology case we get a natural exact sequence

$$0 \rightarrow \left[ \text{H}^{*}(C) \otimes \text{H}^{*}(\mathcal{D}) \right] \rightarrow \text{H}^{*}(C \otimes \mathcal{D}) \rightarrow \text{Ext}_{R}^{1}(H^{*}(C), H^{*}(\mathcal{D})) \rightarrow 0$$

Def. A chain complex $C$ of $R$-modules is of finite type if each $C_{n}$ is finitely generated.

Recall. If $C$ a free chain complex and each $H_{n}(C)$ is free, then $C$ is chain equivalent to a chain complex $C'$ with $H_{n}(C') = 0$. Then if each $H_{n}(C)$ is finitely generated, then $C'$ will be of finite type.

Prop. If $C$ a free chain complex of abelian groups of finite type, $D$ any chain complex of abelian groups. Then

$$\gamma: \text{Hom}_{R}(C, R) \otimes \text{Hom}_{R}(D, R) \rightarrow \text{Hom}_{R}(C \otimes D, R)$$

is an isomorphism of chain complexes.

Proof. Immediate from def. of $\gamma$ and following fact, left as an exercise: If $A$, $B$ abelian groups with $A$ finitely generated and free, then $\gamma: \text{Hom}_{R}(A, R) \otimes \text{Hom}_{R}(B, R) \rightarrow \text{Hom}_{R}(A \otimes B, R)$ is a monomorphism. Similarly $\gamma(f \otimes g)(a \otimes b) = f(a)g(b)$ is an isomorphism.
Then \((x, A), (y, B)\) admit pairs \(R\) a P.I.D. Suppose \(H^n(x, A; R)\)
\(\text{is \textit{finite}}\) \(\text{gen} + \text{free \ for \ all } n\). Then the \textit{core prop. map}

\[
\overline{M} : [H^*(x, A; R) \otimes H^*(y, B; R)]^n \rightarrow H^n(x, A) \times (y, B; R)
\]
\(\text{as \ an \ isom.}

Proof: \(\overline{M} = (\theta^*)^{-1} \gamma^* M\). Since \((\theta^*)^{-1} \gamma^* M\) \text{ is an isom.}, \text{ will be}
\text{ done \ if \ we \ show \ \gamma^* M \text{ is \ an \ isom.}

Since \(H^n(x, A)\) \text{ is \textit{finite}} \text{ gen. and \textit{free gen.} \ for \ all } n, \exists \ \text{free \ chain complex \ of \ finite \ type} C, \\
\text{ and \ a \ chain \ equiv.} \ f : C \rightarrow Q(X, A). \ \text{Then} \ (H_0^*(C, R) \text{ is \ a \ \textit{complex \ of \ free} R-module \ since \ each \ \text{C}_n \ \text{is \ \textit{finite}} \ \text{gen.} \ \text{free} \ \text{ideal}. \ \text{Not \ true \ that} \ Q^*(x, A; R) \ \text{is \ \textit{free}.}

\(f \ : \ H^n(x, A) \rightarrow H^*(Q, x, A; R)\) \ \text{since} \ \text{the} \ H^n(x, A) \ \text{are \ \textit{free} \ \textit{ideal} \ \textit{finite}} \ \text{gen.,} \ \text{by \ the \ \textit{L.C.T.} \ all \ the} \ H^n(C, R) \ \text{are \ \textit{free} \ \textit{R-module}.

Then \(M : [H^*(Q, x, A; R) \otimes H^*(y, B, R)]^n \rightarrow H^n(Q, x, A; R) \otimes \gamma^*(y, B, R)^n
\)
\(\text{is \ an \ isom. (since \ all \ \textit{Tor} \ \textit{terms} \ are \ \textit{0.)}

\text{Following \ commutes:}

\[
\begin{array}{ccc}
H^*(Q, x, A; R) \otimes H^*(y, B, R) & \rightarrow & H^*(Q, x, A; R) \otimes H^*(y, B, R) \\
\downarrow (H^* \otimes I)^n & & \downarrow (H^* \otimes I)^n \\
H^n(Q, x, A; R) \otimes \gamma^*(y, B, R) & \rightarrow & H^n(Q, x, A; R) \otimes H^*(y, B, R)
\end{array}
\]

\text{Virt. maps as \ \textit{equiv} + \text{ is \ a \ \textit{chain} \ \textit{equiv.} \ \textit{Bottom} \ \textit{maps} \ \textit{are} \ \textit{equiv},

\text{so \ \textit{top} \ \gamma^* M \ \text{is} \ \textit{equiv.}\n
\text{Example:} \ X = S^2 \times S^3, \ Y = \Sigma (S^1 \times S^2 \times S^4).

X \ and \ Y \ \text{have \ ret. \ homology \ groups}
\(\text{as (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0...))}. \ \text{Since} \ Y \ \text{is \ a \ \textit{suspension},} \ \text{any \ part \ \textit{of} \ \textit{ret.} \ \textit{dim.} \ \text{elements} \ \textit{are} \ \textit{0} \ \text{as} \ H^*(Y). \ \text{Let} \ a, b \ \text{denote} \ \textit{gen. of} \ H^2(S^2, Z), \ H^3(S^3, Z), \ \text{resp.}
\text{By \ above \ \textit{Kem} \ \text{then,} \ \text{a} \ \text{b} \ \text{is \ \textit{J} \ \textit{of} \ \textit{J} \ \textit{a} \ \textit{b} \ \neq 0.} \ \text{Hence} \ \text{a} \ \text{b} \ \text{is \ \textit{J} \ \textit{of} \ \textit{J} \ \textit{a} \ \textit{b} \ \neq 0.} \ \text{Hence} \ \text{H^*(X, Z) \ \text{is \ \textit{J} \ \textit{of} \ \textit{J} \ \textit{H^*(Y, Z) \ \text{as \ a \ \textit{graded} \ \textit{ring}.} \ \text{Hence} \ X \ \text{and} Y \ \text{\textit{cannot} \ \textit{have} \ \textit{the} \ \textit{same} \ \textit{homotopy} \ \textit{type}.}
Compatibility of cup products with connecting homomorphisms

Lemma: Let $0 \to C \to D \to E \to 0$ be a s.e.s. of cochain complexes of $R$-modules which splits in each degree, and let $F$ another cochain complex of $R$-modules. Then $0 \to C \otimes_R F \to D \otimes_R F \to E \otimes_R F \to 0$ is also exact.

Then $H^p(E) \otimes H^q(F) \xrightarrow{\delta \otimes 1} H^{p+q}(C) \otimes H^q(F)$

$\mu \downarrow$

$H_{p+q}(E \otimes F) \xrightarrow{\delta} H_{p+q+1}(C \otimes F)$

commutes, where $\delta$ and $\overline{\delta}$ are the appropriate connecting homomorphisms.

Proof: Let $a = [z] \in H^p(E)$, $b = [w] \in H^q(F)$.

Result: $\mu(a \circ b) = [z \circ w]$.

To obtain $\delta a$:

\[ X \rightarrow 2 \]
\[ D \rightarrow E \]
\[ S \]
\[ C \rightarrow D \]
\[ \delta \]
\[ \delta a = [z] \]

Then $\mu((\delta a) \circ b) = [y \circ w]$.

To obtain $\overline{\delta} \mu(a \circ b)$:

\[ X \otimes W \rightarrow 2 \otimes W \]
\[ (D \otimes F) \rightarrow (E \otimes F) \]
\[ \delta \]
\[ (C \otimes F) \rightarrow (D \otimes F) \]
\[ (\delta x) \circ w \pm x \circ \delta w \]
\[ (\varepsilon x) \circ w \quad \text{since } \delta w = 0 \]
\[ \delta \mu(a \circ b) = [y \circ w] = \mu((\delta \circ 1)(a \circ b)) \]
Lemma: \((X,A)\) admissible, \(Y\) a tp space. \(R\) a PID then
\[
H^p(A;R) \otimes_R H^q(Y;R) \xrightarrow{\delta \otimes 1} H^{p+1}(X,A;R) \otimes_R H^q(Y;R)
\]
\[
\xrightarrow{\mu} H^{p+q}(X,Y;R)
\]

commutes where \(\delta\) is the coun. kern. of coh. seq. of \((X,A)\),
\(R\) coun. kern. of coh. seq. of \((X \times Y, A \times Y)\).

Proof: Have diagram (add arrows to \(R\))
\[
\begin{array}{ccc}
H^p(A) \otimes_R H^q(Y) & \xrightarrow{\delta \otimes 1} & H^{p+q}(X,Y;R) \\
\uparrow \mu & & \uparrow \mu \\
H^p(A) \otimes H^q(Y) & \xrightarrow{\delta \otimes 1} & H^{p+q}(X,Y;R)
\end{array}
\]

\(\delta\) is comm. hom. arising from \(\delta\).

\(\text{I:}\) \(0 \to Q^*(X,A;R) \to Q^*(X;R) \to Q^*(A;R) \to 0\)
\(\delta\) arises from

\(\text{II:}\) \(0 \to \text{Hom}(Q(X,A) \otimes Q(Y);R) \to \text{Hom}(Q(X) \otimes Q(Y);R) \to 0\)
\(\delta\) arises from

\(\text{III:}\) \(0 \to \text{Hom}(Q(X) \otimes Q(Y);R) \to \text{Hom}(Q(X) \otimes Q(Y),R) \to 0\)
\(\delta\) arises from

\(\text{IV:}\) \(0 \to Q^*(X,A \times Y;R) \to Q^*(X \times Y;R) \to Q^*(A \times Y;R) \to 0\)

\(\text{I+II}:\) commute by def. of \(\mu\), \(\text{III}\): commute by lemma.

\(\text{IV}\): commute since \(Q\) is a complex from \((\text{II})\) to \((\text{III})\).

\(\text{V}\): commute since \(\text{Hom}(0,1)\) is a complex from \((\text{IV})\) to \((\text{III})\).
Note. In terms of elements, $(Sa) \times b = \overline{S}(a \times b)$.

**Key: Mayer-Vietoris Sequence**

Suppose $G_1$, $G_2$ are subgroups of an abelian group $G$, $H_i$ a subgroup of $G_i$, $i=1,2$. Consider homomorphisms

$$\frac{G_1}{H_1} \oplus \frac{G_2}{H_2} \xrightarrow{p} \frac{G_1 + G_2}{H_1 + H_2}$$

given by

$$p(g_1 + H_1, g_2 + H_2) = g_1 + g_2 + (H_1 + H_2).$$

$p$ is clearly onto.

**Key:**

- **Key:**

Let $\tilde{x} : G_1 \cap G_2 \to \frac{G_1}{H_1} \oplus \frac{G_2}{H_2}$ be given by

$$\tilde{x}(g) = (g + H_1, -g + H_2).$$

Then $\tilde{x}$ is well defined.

**Key:**

- **Key:**

Then $\tilde{x} : G_1 \cap G_2 \to H_1 \cap H_2$. Then, passing to quotients,

$\tilde{x}$ induces a monomorphism

$$\hat{x} : \frac{G_1 \cap G_2}{H_1 \cap H_2} \to \frac{G_1}{H_1} \oplus \frac{G_2}{H_2}$$

and we have an exact seq

$$0 \to \frac{G_1 \cap G_2}{H_1 \cap H_2} \xrightarrow{\hat{x}} \frac{G_1}{H_1} \oplus \frac{G_2}{H_2} \xrightarrow{p} \frac{G_1 + G_2}{H_1 + H_2} \to 0$$

where $[g] = ([g], [-g])$, $p([g], [g]) = [g + g]$.

Now suppose $X = X_1 \cup X_2$, $A = A_1 \cup A_2$, $A_i \subset X_i$, $X_i, A_i$ all open in $X$. Above yields a s.e.s. of chain complexes

$$0 \to \frac{Q(X_1 \cap X_2)}{Q(A_1 \cap A_2)} \to \frac{Q(X_1)}{Q(A_1)} \oplus \frac{Q(X_2)}{Q(A_2)} \to \frac{Q(X_1) + Q(X_2)}{Q(A_1 \cup A_2)} \to 0$$

**Note:** Complex on right is free in each degree, since $Q(A_1) + Q(A_2)$ has a basis consisting of a partition of a union of $Q(X_1) + Q(X_2)$. Hence, also so a splitting in each degree.

By the main lemma in proving the expansion property for homology (see all cube closure), the conclusion
\[ Q(X_1) + Q(X_2) \to Q(X), \quad Q(A_1) + Q(A_2) \to Q(A) \]

hence, passing to quotients, \( Q = \frac{Q(X)}{Q(A)} \)

hence follows in homology, and hence in cohomology by

U.C.T. Thus, for abelian coefficients, we obtain a natural short exact seq.

\[ 0 \to H^n(XA) \to H^n(X_1A_1) \oplus H^n(X_2A_2) \to H^{n+1}(X,A) \to 0 \]

called the relative Mayer-Vietoris sequence of the admissible triad

\[ ((X,A), (X_1A_1), (X_2A_2)) \]

**Lemma:** The M.V. seq. of \( ((X,A), (X,\emptyset), (A,A)) \) is the col. seq. of the pair \( (X,A) \).

\[ (X \supset A \supset B, \text{ the M.V. seq. of } ((X,A), (X,B), (A,A))) \]

is the col. seq. of the triad \( (X,A,B) \).

**Lemma:** \( (X_1, X_2) \) an admissible triad. Then, the connecting homomorphism

\[ \Delta: H^n(X \cap X_2) \to H^{n+1}(X) \text{ in the M.V. seq. of above triad} \]

as the comp.

\[ H^n(X_1 \cap X_2) \xrightarrow{\delta} H^{n+1}(X_1, X_2) \xrightarrow{j^*} H^{n+1}(X, X_2) \xrightarrow{i^*} H^{n+1}(X) \]

where \( \delta \) is the inclusion homomorphism \( j^* \) is the restriction and \( i^* \) is the corestriction (Lamberta's comm. hom. in the col. seq. of the pair \( (X_1, X_2) \)).

**Proof:** Write \( X_{12} = X_1 \cap X_2 \). The admissible triad

\[ ((X_1, X_2, X_{12}) \]

\[ (X, X_1, X_2) \]

\[ U \]

\[ (X_1, X_2) \]

\[ ((X_1, X_{12}), (X_1, \emptyset), (X_{12}, X_2)) \]

and rest. of M.V. seq. yields the corestriction.
Theorem: \((X, Y, B)\) an admissible pair, \((Y, B)\) an admissible pair. Then for any PID \(R\),

\[ H^n(X_1 \cap X_2) \xrightarrow{\Delta} H^{n+1}(X_1) \]

\[ 1 \uparrow \]

\[ H^n(X_1 \cap X_2) \xrightarrow{\Delta} H^{n+1}(X_1, X_2) \]

\[ \downarrow \]

\[ H^n(X_1 \cap X_2) \xrightarrow{\Sigma} H^{n+1}(X_1, X_1, X_2). \]

Proof: Case 1: \(B = \varnothing\). Consider diagram:

\[ H^p(X_1) \otimes H^q(Y) \xrightarrow{\Delta \otimes 1} H^p(X) \otimes H^q(Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p(X_1, X_2) \otimes H^q(Y) \xrightarrow{\Delta \otimes 1} H^p(X_1, X_2) \otimes H^q(Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2) \otimes H^q(Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2) \otimes H^q(Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2) \otimes H^q(Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2) \otimes H^q(Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2) \otimes H^q(Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2) \otimes H^q(Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \]

\[ \xrightarrow{\Delta \otimes 1} \]

\[ H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \xrightarrow{\Delta \otimes 1} H^p \otimes (X_1, X_2, X_1, X_2, X_1, X_2, Y) \]
Case 2: Second case. Let \( i : (Y, B) \rightarrow (Y \cap B, cB) \),
\( j : (Y \cap B, \phi) \rightarrow (Y \cap B, cB) \) denote the inclusions.
Consider diagram

\[
\begin{array}{ccc}
H^p(X_2) \otimes H^q(Y, B) & \xrightarrow{\otimes 1} & H^{p+q}(X) \otimes H^q(Y, B) \\
\downarrow \phi & & \downarrow \phi \\
H^p(X_2) \otimes H^q(Y \cap B, cB) & \xrightarrow{\otimes 1} & H^{p+q}(X) \otimes H^q(Y \cap B, cB) \\
\downarrow \phi & & \downarrow \phi \\
H^p(X_2, X \cap (Y, B)) & \xrightarrow{\phi} & H^{p+q}(X, (Y \cap B, cB)) \\
\downarrow \phi & & \downarrow \phi \\
H^p(X_2, (Y \cap B, cB)) & \xrightarrow{\phi} & H^{p+q}(X, (Y \cap B, cB)) \\
\end{array}
\]

(1), (2), (3), (4) commute by adjunction.
(5), (6) commute by direct check.
(7), (8) commute by pullback of \( \overline{\alpha} \) via \( M, V, x \).
(9) commutes by case (1).

(10): \((X, X_1, X_2)\) an admissible triple, \( A \) open in \( X \). Write
\( X_2 = X_1 \cap X_2, \quad A_2 = X_2 \cap A, \quad i = (i_2) \),
\( A_1 = A \cap A_2 = X_1 \cap A \).
Let PID, \( a \in H^q(X, \mathbb{R}; R) \). Then following commutes (well in \( R \))

\[
\begin{array}{ccc}
H^p(X_2) & \xrightarrow{\Delta} & H^{p+q}(X) \\
\downarrow \phi & & \downarrow \phi \\
H^p(X_1) \oplus H^{p+1}(X_2) & \xrightarrow{\Delta} & H^{p+q}(X_2) \\
\downarrow \phi & & \downarrow \phi \\
H^p(X_2, X_1) \oplus H^{p+1}(X_2, A_2) & \xrightarrow{\Delta} & H^{p+q}(X_2, A_2) \\
\end{array}
\]
Let $R$-orientation do not always exist. If $E \to P$ is a principal $G$-bundle, then $E = (E, E^0) \to (E, E^0)$ is an $R$-orientation.

Consider a real $n$-plane bundle $P \to B$. Fix a choice $A \in C^*$.

Now consider a real $n$-plane bundle $P \to B$. Fix a choice $A \in C^*$.

Let $E = (E, E^0)$ be a choice $A \in C^*$.

For any $V$, one has the $G$-action of a real $n$-plane bundle $P \to B$. Fix a choice $A \in C^*$.

Let $E = (E, E^0) = (E, E^0)$ be a choice $A \in C^*$.
and so $E \to P$ is $R$-orientable.

**Def.** $E \to P$, $E' \to P'$ $n$-plane bundles. A morphism $f : E \to E'$ $P \to P'$ consists of a pair of cont. maps $f_0 : B \to B'$ and $f_0 : B \to B'$ such that

\[
\begin{array}{ccc}
E & \xrightarrow{f_0} & E' \\
\downarrow f & \quad & \downarrow f' \\
B & \quad & B'
\end{array}
\]

1) For each $x \in B$, $f_0$ maps $E_x$ isomorphically onto $E'_x$.

Can form cat. of real $n$-plane bundles with morphisms as above.

(Not: more general morphisms of vector bundles are sometimes useful, e.g. in differential geometry).

If $f$ is a morphism from $E \to P$ to $E' \to P'$, then $f_0 \in (E')^0$ and so we get a map of top. pairs

\[
\tilde{f} : (E, E^0) \to (E', E'^0). \quad \text{Moreover, for each } x \in B, \tilde{f} \text{ yields a home of pairs } \tilde{f}_x : (E_x, E_x^0) \to (E'_x, E'_x^0).
\]

(\text{conjugates})

**Prop:** Let $f$ be a morphism of $n$-plane bundles from $E \to B$ to $E' \to B'$. Suppose $U$ is an $R$-orientation of $E' \to B'$. Then $\tilde{f}^*(U)$ is an $R$-orient.

(3.5) $E \to B$.

**Proof:** In each $x \in B$, $i_x^* \tilde{f}^*(U) = \tilde{f}_x^* (i_{x_0}^* \tilde{U}(U))$ by conjugativity of above. $i_{x_0}^* \tilde{U}(U)$ is an $R$-orient of $E'_0$ since $U$ is an $R$-orient of $E' \to B'$. Hence, since $\tilde{f}_x$ is a home of pairs,
\[ \overline{f^*} (x^*) (U) \text{ is an R-orient. of } E_x. \]

Cor: Any twisted n-plane bundle admits an R-orient. for any R.

Prop: \( U \rightarrow E \) trivial, \( E \) any bundle of n-plane bundles

\[
\begin{align*}
&: E \\
\downarrow^p & \\
B \\
\end{align*}
\]

\[ 0 \rightarrow B \times \mathbb{R}^n \\
\downarrow^q \\
B \\
\]

We have a complex of \( n \)-plane bundles

\[ \mathbb{R}^n \]

\[ \downarrow^\pi \]

\[ \text{Result now follows since} \]

\[ \mathbb{R}^n \]

\[ \rightarrow \text{admits an R-orient.} \]

\[ \downarrow^p \]

\[ \text{Note: } U \rightarrow E \] any \( n \)-plane bundle and \( X \subset B \), then the \n
\[ \begin{array}{c}
\cdot \\
\downarrow \pi \end{array} \]

\[ \text{constitute a complex of } n \text{-plane bundles.} \]

\[ \text{In particular, if } U \text{ is an R-orient. of } E \]

\[ \text{and } i: X \rightarrow B \text{ the inclusion, } i^* (U) \text{ is an} \]

\[ \text{R-orient. } \]

\[ \pi (X) = \pi (X) \subset E \] the R-orient. induced by \( U \).
If $k$ is a positive integer, a vector space $E$ is a $k$-dimensional vector space over a field $F$. Let $E$ be a vector space over a field $F$. Then $E$ is a $k$-dimensional vector space over $F$.

Let $E$ be a vector space over a field $F$. Then $E$ is a $k$-dimensional vector space over $F$.

Let $E$ be a vector space over a field $F$. Then $E$ is a $k$-dimensional vector space over $F$.


\[
\begin{array}{cccc}
H^i(B) & \overset{\pi_1^*}{\longrightarrow} & \bar{\psi} & \\
\downarrow \psi & & \downarrow \vec{\psi} & \\
H^i(E) & \overset{\nu}{\longrightarrow} & H^{n+i}(E,\mathbb{E}^0) & \\
\end{array}
\]

commutes, \(\psi \circ \pi_1^*(x) = \psi(a \times 1) = (a \times 1) \circ U\)
\[
= (a \times 1) \circ (1 \times \bar{U}) = a \times \bar{U} = \bar{\psi}(a).
\]

Since \(\pi_1\) is a homotopy equivalence, \(\pi_1^*\) is an isomorphism. \(\bar{\psi}\) is an isomorphism. Proof case 1.

**General 1-presentable case.**

Write \(B = \bigcup_{x} B_{x}\) \(B_{x}\) the path-components of \(B\).

Let \(i_{x}: B_{x} \times V \rightarrow B \times V\), \(i_{x} : B_{x} \times (V, V^0) \rightarrow B \times (V, V^0)\)
denote the inclusions. Write \(U_{x} = i_{x}^*(U)\). Then, \(U_{x}\) is an \(R\)-invariant of \(B \times V \rightarrow B_{x} \times V\), see 1.\(i\) case 1.

\(\bar{\psi}_{x} : H^i(B_{x} \times V) \rightarrow H^{n+i}(B_{x} \times (V, V^0))\)
\(\bar{\psi}_{x}\) an isomorphism.

% Following commutes: \(H^i(B \times V) \overset{\psi}{\longrightarrow} H^{n+i}(B \times (V, V^0))\)

% \(\downarrow (i_{x}^*) \)
% \(\downarrow \bar{\psi}_{x} \)
% \(\bigwedge_{\alpha} H^i(B_{x} \times V) \bigwedge_{\alpha} \bar{\psi}_{x} \)
% \(\bigwedge_{\alpha} H^{n+i}(B_{x} \times (V, V^0))\)

\(\bar{\psi}_{x} \bigwedge_{\alpha} i_{x}^*(x) = i_{x}^*(x \cup U)\)
\(= i_{x}^* \cup \bar{\psi}_{x} \bigwedge_{\alpha} U\)
(\(\cup\) of admissibly \(x\)-finite maps\)
\(\cup (B_{x} \times V; \Phi, B_{x} \times V^0) \rightarrow (B \times V; \Phi, B \times V^0)\)
\(= i_{x}^* \cup U_{x} \bigwedge_{\alpha} = \bar{\psi}_{x} i_{x}^*(x)\).

Bottom commutes. End proof 1.
Let $a = 0$. Then $x = 0$ if and only if $\exists (x, e, f) \in H$ such that $a + x = 0$. Hence, $x = 0$.

For the remainder, then $x \times x = 0$.

Let $G = \mathbb{R}$ with the operation $+$, where $G = \{a \times b | a, b \in \mathbb{R}\}$.

Choose $e \in G$ such that $e \times a = a \times e = a$ for all $a \in \mathbb{R}$. Then $e = 1$.

Choose $b \in G$ such that $a \times b = b \times a = 0$ for all $a \in \mathbb{R}$. Then $b = 0$.
an R-module, must have $r = 0$, so $a = 0$.

Second case: Let $E$ be the proper complement of $B$, $E = E|B$, $j_x : (E, E^o) \rightarrow (E_E, E^o)$ is an inclusion. By hyp., $\exists x \in B$ for each $i$. Let $x^i(a) = 0$. Here comm. diag:

$$
\begin{align*}
&\xymatrix{ & (E, E^o) \ar[dl]_{j_x^i} \ar[d] \ar[dr] \ar[d] & \\
& (E_E, E^o) \ar[d] \ar[dr] & \\
& (E, E^o) } \end{align*}
$$

and so $x^i j_x^i a = 0$. By case 1, $i_x^o a = 0$. For each $i$.

Hence:

$$
\begin{align*}
&\xymatrix{ (i_x^o)_*: H^n(E, E^o, R) \ar[d] & \ar[r] & \prod H^n(E_x, E^o_x, R) \ar[d] & \\
& H^n(E, E^o, R) & \ar[r] & H^n(E_x, E^o_x, R) } \\
&\xymatrix{ i_x^o & \\
& a & }
\end{align*}
$$

Then: $E$ an n-pl. hole over $B$, fun. pres., $\exists B = \{E|B\}$ a finite spec.
cover of $B$ (R. P. D.), $\exists x \in B$ a compatible set of R-oriets.
of \{E|B\}. Thm 3 unique R-oriets. $U$ of $E$; $U_i$ is the R-ori eti.

Proof: Uniqueness: If $U, V$ are 2 such R-oriets, consider $i_x^o U = i_x^o V \forall x \in B$, so $U = V$ by comm.

Existence: Consider $x \in K$ of elts. in cover. Case of $K = 1$.

Assume $K > 1$ and then holds $\exists$ cover with $\leq K$ elts.

While $B = A_1 \cup A_2$, where $A_1 = B_1 \supset A_2 = B_2 \cup \ldots B_K$.

By hyp. $\exists$ unique R-oriets. $U'$ of $E|A_2$. $U_i$'s

unions $U_i$ for $2 \leq i \leq K$. $U_i$ and $U_i'$ are compatible, $\forall i$ for $x \in A_1 \cap A_2$, any $x \in B_1 \cap B_2$. 

$$
\begin{align*}
&\xymatrix{ & (E|B, U, (E|B, U)) \ar[d] \ar[dr] & \\
& (E, E^o) \ar[d] & \ar[r] & (E|A_2, (E|A_2, U')) } \end{align*}
$$

$U_i = i_x^o U_i' = i_x^o U_i = i_x^o U_i$ for the $U_i$ compatible.

While $E_i = E|A_i, i = 1, 2$, $E_{12} = E|A_2 A_2$. Here $U, V$ deg. with coeffs.
in $R$. 


\[ H^n(E, E^c) \rightarrow H^n(E_1, E_1^c) \oplus H^n(E_2, E_2^c) \rightarrow H^n(E_{12}, E_{12}^c) \]

\( i_1^* U_1, i_2^* U_2 \) are \( R \)-orient. of \( E \mid A_1, A_2 \) restricting to same \( R \)-orient. on each plane side \( U_1, U_2 \) as compatible. Hence by lemma, \( i_1^* U_1 - i_2^* U_2 = 0 \). By exactness, \( \exists U \in H^n(E, E^c) \) such that \( j_1^* U = U_1 \) \( j_2^* U = U_2 \). Easily checked: \( U \) is an \( R \)-orient. on each plane side \( U_1, U_2 \) and hence extending \( U_1, \ldots, U_n \) (since \( U_2 \) induces \( U_{12}, \ldots, U_n \)).

**Cor:** \( E \) a fin. pres. \( n \)-plane bundle. Then \( E \) has a unique \( \mathbb{Z}/2 \) orient.

**Proof:** \( H^n(E, E^c; \mathbb{Z}/2) \) has only one generator, so any collection of \( \mathbb{Z}/2 \) orient. rel. to any open cover of base space is compatible. If \( \{ B_i \} \) is a finite open cover of base space, carrying to an atlas, then \( E \mid B_i \) has a \( \mathbb{Z}/2 \) orient. for each \( i \). Hence, by this, \( E \) has a \( \mathbb{Z}/2 \) orient. (Two choices: immediate from lemma.

**Prop:** \( E \) the underlying real \( n \)-plane bundle of a fin. pres. comp. \( n \)-plane bundle. Then \( E \) is \( R \)-orientable for every \( R \).

**Proof:** Let \( \{ \alpha_x : p^{-1}(B_i) = E_i \rightarrow B_i \times \mathbb{C}^n \} \) be a finite atlas for above comp. \( n \)-plane bundle. Let \( u \in H^n(C^n, C^n; R) \) be any \( R \)-module generator, and let \( U_i = h_i^x(1 \times u) \in H^n(E_i, E_i^c; R) \). \( (h_i : (E_i, E_i^c) \rightarrow B_i \times (\mathbb{C}^n, (\mathbb{C}^n)^c)) \). For each \( x \in B_i \), let \( f_i^x \) denote the composition

\[
E_x \xrightarrow{\tilde{\alpha}_i^x} E_i \xrightarrow{h_i^x} B_i \times C^n \xrightarrow{\pi_i} C^n.
\]

Then \( f_i^x \) is an isom. of complex vector spaces, and we have

\[
\tilde{f}_i^x(U_i) = f_i^x(h_i^x(1 \times u)) = \tilde{h}_i^x f_i^x(U_i) = f_i^x(U_i) = \text{orient. of } E_x \text{ since } \tilde{f}_i \text{ is a linear of ramps.}
\]

\( U_i \) is an \( R \)-orient. of \( E \mid B_i \).

Suff. to show: \( E \mid U_i \) is compatible.

Suppose \( x \in B_i \cap B_j \). Suff. to show composition

\[
(C^n, (C^n)^c) \xrightarrow{f_i^x} E_x \xrightarrow{f_j^x} (C^n, (C^n)^c) \cong 1 \), for then
\[ \bar{x}_{ij}^+(u_i) = \bar{g}_{ij}(u) = \bar{g}_{ij}^+ \left( \bar{g}_{ij} \bar{g}_{ij}^{-1} \right)^x(u) = (\bar{g}_{ij} \bar{g}_{ij}^{-1} \bar{g}_{ij}^+) (u) = \bar{g}_{ij}^+(u) = \bar{x}_{ij}(u_j). \]

Now \( \bar{g}_{ij} \bar{g}_{ij}^{-1} \in GL(n, C) \).  

To show \( GL(n, C) \) is path-connected, let then \( t \mapsto x(t) \) be a path from \( I_n \) (identity matrix) to \( \bar{g}_{ij} \bar{g}_{ij}^{-1} \), then \[
\begin{align*}
(C^n, C^\infty, \{0\}) & \xrightarrow{(y_1, t)} (C^n, C^\infty, \{0\}) \\
\alpha(t)(y) & \xrightarrow{(y_1, t)} \alpha(t)(y)
\end{align*}
\]
which would be a homotopy from \( I_n \) to \( \bar{g}_{ij} \bar{g}_{ij}^{-1} \).

Let \( A \in GL(n, C) \).  
\[ \exists T \in GL(n, C) : TAT^{-1} \text{ is upper triangular} \] (since \( C \) is algebraically closed).  

To show: \( \exists \) path \( \alpha : [0, 1] \rightarrow GL(n, C) \) s.t. \( \alpha(0) = I_n \), \( \alpha(1) = TAT^{-1} \). 

(For then \( t \mapsto T^{-1}x(t)T \) would be a path from \( I_n \) to \( A \) in \( GL(n, C) \).) 

Can write \( TAT^{-1} = \text{diag} (\lambda_1, \ldots, \lambda_n) + B \) where \( B \) is upper triangular with \( C \) as diagonal, and each \( \lambda_i \in C \setminus \{0\} \), since \( C \setminus \{0\} \) is path-connected.  

\[ \exists \text{ path } \beta : [0, 1] \rightarrow C \setminus \{0\} \] s.t. \( \beta(0) = 1, \beta(1) = \lambda_i \). 

Define \( \alpha(t) = \text{diag} \left( (\beta(t))_1, \ldots, (\beta(t))_n \right) + tB \).

**Projective spaces**

Let \( F = R \) or \( C \), and let \( V \) be a finite-dimensional vector space over \( F \).  
The \( F \)-projective space of \( V \), \( P_F(V) \), is the quotient space obtained from \( V \) by identifying \( U \sim \lambda U \) whenever \( U \in V^\circ \), \( \lambda \in F \setminus \{0\} \).  
Write \([\omega] \in P_F(V)\) for the coset of \( \omega \in V^\circ \) under this quotient map \( V^\circ \rightarrow P_F(V) \).

Can identify \([\omega] \) with the 1-dimensional \( F \)-subspace of \( V \) spanned by \( \omega \), and will identify \( \omega \) with \([\omega] \).  

Let \( P_F(R^{n+1}) = R^{n+1} \), \( P_C(C^{n+1}) = CP^n \).

If \( f : V \rightarrow W \) is an isomorphism of \( F \)-vector spaces, \( f \) induces a homomorphism \( P_F(f) : P_F(V) \rightarrow P_F(W) \).  
\[ [\omega] \mapsto [f(\omega)] \]

Thus, if \( \dim_F V = n+1 \), by HW prob 3,
Recall (last semester): \[ H^i(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & 0 \leq i \leq 2n, \\
0 & \text{otherwise,} \end{cases} \]

By U.C.T., follows that
\[ H^i(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & 0 \leq i \leq 2n \\
0 & \text{otherwise,} \end{cases} \]

and some will be true if \( \mathbb{P}^n \) is replaced by \( P_F(V) \) when \( \dim V = n+1 \).

We have an \( F \)-line bundle \( p : L_F(V) \to P(F) \) as follows.
\[ L_F(V) = \{ ([v], u) \in P(F) \times V \mid u \in [v] \}, \]
\[ p([v], u) = [v]. \]
Vector space structure on \( L_F(V)_{[v]} \) is given by
\[ ([v], u_1) + ([v], u_2) = ([v], u_1 + u_2), \quad ([v], u) \cdot ([v], w) = ([v], uw). \]

Clearly, \( \{(v), [v]\} \) is an \( F \)-basis for \( L_F(V)_{[v]} \).

Total triviality: for each \( v \in V^0 \), let \( B_v = \{ u \in P(F) \mid \langle u, v \rangle > 0 \} \).
Then \( \{B_v\}_{v \in V^0} \) is an open cover of \( P(F) \).
Define \( h_v : P^{-1}(B_v) \to B_v \times F \) by
\[ h_v([v], x) = ([v], \langle v, x \rangle). \]
Then \( \{h_v\}_{v \in V^0} \) constitute an atlas for above as an \( F \)-line bundle. Above is called the canonical \( F \)-line bundle over \( P(F) \).

Note: the map \( L_F(V)^0 \to V^0 \) is a homeomorphism:
\[ ([v], u) \mapsto u. \]

Thus, if \( \dim V = n+1 \), \( L_F(V)^0 \) has the homotopy type of
\[ \{ \begin{cases} S^n & F = \mathbb{R} \\
S^{2n+1} & F = \mathbb{C} \end{cases} \]
Thus for any coefficient \( j^* : H^i(\mathrm{L}_F(V), L_F(V)^\circ) \to H^i(\mathrm{L}_F(V)) \) we have:

\[
\begin{cases}
\operatorname{min} \{ m \in \mathbb{N} \mid i \leq n - 1 \} , \quad 1 \leq i < n \\ \operatorname{min} \{ m \in \mathbb{N} \mid i \leq 2n \} , \quad i = n \text{ or } F = \mathbb{C}.
\end{cases}
\]

**Proof:**

**Suff.** To show \( j^* \) is a generator of \( H^i(\mathrm{L}_F(V); \mathbb{Z}/2) \), it suffices to show \( j^* \) generates \( H^i(\mathrm{L}_F(V); \mathbb{Z}/2) \).

Let \( \theta : H^i(\mathrm{L}_F(V), L_F(V)^\circ; \mathbb{Z}/2) \to H^i(\mathrm{L}_F(V); \mathbb{Z}/2) \) denote the unique \( \mathbb{Z}/2 \)-linear orientation \( \mathrm{L}_F(V) \xrightarrow{\theta} \mathrm{L}_F(V) \). Since \( \theta : \mathrm{L}_F(V) \to \mathrm{L}_F(V) \) is a homotopy equivalence, \( j^*(u) \) is a generator of \( H^i(\mathrm{L}_F(V); \mathbb{Z}/2) \). By the above, for any \( a \in H^i(\mathrm{L}_F(V); \mathbb{Z}/2) \), we have \( j^*(a) \) is a generator of \( H^i(\mathrm{L}_F(V); \mathbb{Z}/2) \).
Cor: \( \mathbb{Z}, 1 \leq m < n \), \( \mathbb{R}P^n \) is not a retract of \( \mathbb{R}P^m \), and \( \mathbb{C}P^n \) is not a retract of \( \mathbb{C}P^m \).

\textbf{Proof:} Let \( r: \mathbb{R}P^n \to \mathbb{R}P^m \) be a retraction. Then

\[
\begin{tikzcd}
\mathbb{R}P^n & \mathbb{R}P^m \\
\mathbb{R}P^n \ar[swap]{u}{r} & \mathbb{R}P^m \ar{u}{1}
\end{tikzcd}
\]

Let \( \alpha \in H^i(\mathbb{R}P^n, \mathbb{Z}/2) \) and \( \beta \in H^i(\mathbb{R}P^n, \mathbb{Z}/2) \) be the generators. Then \( \alpha = r^*(\alpha) = r^* \circ \beta \). We must have \( r^*(\alpha) = \beta \).

Since \( \alpha = 0 \), we have \( 0 = r^*(\alpha) = (r^* \circ \beta)(\alpha) = \beta \neq 0 \), contradiction.

Proof that \( \mathbb{C}P^n \) is not a retract of \( \mathbb{C}P^m \) is similar.
Homotopy Theory

Category of pointed top. spaces

Objects: Pair (X, x0), X a top. space, x0 in the base point of (X, x0) (or more briefly, the base pt. of X).

Morphisms: Cont. maps sending base pt. to base pt.

Def: f, g: X → Y pointed maps, x0 = base pt. of X, y0 = base pt. of Y.
A pointed homotopy from f to g is a homotopy h: X×I → Y from f to g, i.e. h(x0, t) = y0 for all t ∈ I.

Write f ≅ g if h is a pointed homotopy from f to g

x0 is an equiv. rel. Moreover, if f1 ≅ f2 : X → Y,
and g1 ≅ g2 : Y → Z, then g1f1 ≅ g2f2.

Write [X, Y] = set of pointed homotopy classes of maps
from X to Y.

Basic problem: Determine [X, Y], i.e. find its cardinality
and a complete set of representative maps.

A number of classification problems can be reduced to
homotopy classification problems.

Examples: 1) T an abel. group, F space K(T, n), n ≥ 0, 2.
For any CW space X, H^n(X, T) is an actual l-c
coalg. with [X^+, K(T, n)] (X^+ = X w. empty base pt.)
2) F space Bo(n) = set of norm. classes of real n-pl.
subspaces in a pointed CW space X w. in 1-1 corres. with
[X, Bo(n)].
3) Obstruction theory

\begin{array}{c}
    X \\
    \downarrow \\
    Z \rightarrow Y \\
    \text{lifting problem}
\end{array}
\begin{array}{c}
    X \\
    \downarrow \\
    Y \longrightarrow Z \\
    \text{extension problem}
\end{array}

Homotopy groups (i.e. [S^n, X]) play an important role.

Problem of computing [X, Y] in general is difficult, has
been solved in special cases.
Best chance for success: When $X \times Y$ (or both) possess some additional structure which gives rise to some sort of algebraic structure on $[X,Y]$ (e.g. group).

$Y(x_0, y_0)$ are pointed spaces, regard their wedge $X \vee Y$ as $\{ (x,y) \in X \times Y \mid \text{either } x = x_0 \text{ or } y = y_0 \} \subset X \times Y$.

$(x, y_0)$ will serve as base point for both $X \vee Y$ and $X \times Y$.

If $(Y, y_0)$ is a pointed space, we have a contr. map $\varphi: X \vee Y \to Y$ given by $\varphi(x_0, y) = \varphi(y, y_0) = y$. $\varphi$ is called the folding map.

Def. An $H^*$-space consists of a pointed space $(Y, y_0)$ together with pointed maps $m: Y \times Y \to Y$, $\eta: Y \to Y$ satisfying

\[ Y \vee Y \xrightarrow{i} Y \times Y \xrightarrow{\mu} Y \] commutes up to pointed homotopy,

\[ Y \times Y \xrightarrow{i \times 1} Y \times Y \xrightarrow{1 \times \mu} Y \] commutes up to pointed homotopy,

\[ Y \xrightarrow{d} Y \times Y \xrightarrow{1 \times \eta} Y \times Y \xrightarrow{\mu} Y \] commutes up to pointed homotopy, where $d = \text{diagonal}$, $* = \text{counit}$, $\eta = \text{counit}$, $\mu$ is called the ass. map, $\eta$ the minim. map.

Example: If $Y$ is a top. group wth unit $e$, and $\eta$ is given by $\eta(y) = y^{-1}$, then $Y$ is an $H^*$-space ($e = \text{unit cell of } Y$).

In fact, i), ii), iii) actually commute in this case. i) expresses the fact that $y_0$ is a 2-sided unit element. ii) expresses the
assert of small, ii) answers, \( y^{-1}y^{-1} = y \).

The terminology \( H^* \)-space is not standard. A pointed space \( Y \) together with a pointed map \( \xi: Y \times 0 \rightarrow Y \) such that \( \xi \) is called an \( H \)-space (terminology \( H \)-space is standard).

**Important example:** \((X, x_0)\) a pointed space. Let

\[
\Omega X = \{ \alpha: I \rightarrow X \mid \alpha(0) = \alpha(1) = x_0 \},
\]
topologized as a subspace of \( C(I, X) \) with the compact-open topology.

\( \Omega X \) is called the loop space of \( X \).

Define \( \varepsilon: \Omega X \times \Omega X \rightarrow \Omega X \), \( \eta: \Omega X \rightarrow \Omega X \) by

\[
\varepsilon(x, (\cdot)) (t) = \begin{cases} 
\alpha(2t) & 0 \leq t \leq \frac{1}{2} \\
\alpha(2t-1) & \frac{1}{2} \leq t \leq 1
\end{cases}, \quad \eta(x)(t) = x(1-t).
\]

It is not hard to show, using properties of \( C \cdot 0 \) topo., that \( \varepsilon, \eta \) are cont.

Let \( * \in \Omega X \) denote the constant path at \( x_0 \).

\((\Omega X, *)\) is a pointed space.

Claim: \( \Omega X \) is an \( H^* \)-space with null \( \varepsilon \), inversion \( \eta \).

**Proof:**

1. \( \varepsilon \) x \( \epsilon \in \Omega X \),

\[
\varepsilon(i(x, *))(t) = \varepsilon(i(x, x))(t) = \begin{cases} 
\alpha(2t) & 0 \leq t \leq \frac{1}{2} \\
x_0 & \frac{1}{2} < t \leq 1
\end{cases},
\]

\[
\varepsilon(i(*, x))(t) = \varepsilon(i(x, x))(t) = \begin{cases} 
x_0 & 0 \leq t \leq \frac{1}{2} \\
\alpha(2t-1) & \frac{1}{2} < t \leq 1
\end{cases}.
\]

Define \( h: (\Omega X \times \Omega X) \times I \rightarrow \Omega X \) by

\[
h((x, x), s)(t) = \begin{cases} 
\alpha \left( \frac{2t + s + 1}{s + 1} \right) & 0 \leq t \leq \frac{1}{2} (s + 1) \\
x_0 & \frac{1}{2} (s + 1) \leq t \leq 1
\end{cases},
\]

\[
h((x, x), s)(t) = \begin{cases} 
x_0 & 0 \leq t \leq \frac{1}{2} (1-s) \\
\alpha \left( \frac{2t + s - 1}{s + 1} \right) & \frac{1}{2} (1-s) \leq t \leq 1
\end{cases}.
\]

\( h \) is well-defined and is a pointed homotopy from \( \varepsilon \) to \( \eta \).
ii) Let \( \alpha, \beta, \gamma \in \mathcal{A}_X \) it can be checked that

\[
\mathcal{M}(1\times 1)(\alpha, \beta, \gamma)(t) = \begin{cases} 
\alpha(4t) & 0 \leq t \leq \frac{1}{4} \\
\beta(4t-1) & \frac{1}{4} \leq t \leq \frac{1}{2} \\
\gamma(2t-1) & \frac{1}{2} \leq t \leq 1
\end{cases}
\]

\[
\mathcal{M}(1\times \mathcal{A})(\alpha, \beta, \gamma)(t) = \begin{cases} 
\alpha(2t) & 0 \leq t \leq \frac{1}{2} \\
\beta(4t-1) & \frac{1}{2} \leq t \leq \frac{3}{4} \\
\gamma(4t-3) & \frac{3}{4} \leq t \leq 1
\end{cases}
\]

Define

\[ h : (\mathcal{A}_X \times \mathcal{A}_X \times \mathcal{A}_X \times \mathcal{A}_X) \times 1 \rightarrow \mathcal{A}_X \text{ by} \]

\[
h(\alpha, \beta, \gamma, \delta)(t) = \begin{cases} 
\alpha(\frac{4t}{1+s}) & 0 \leq t \leq \frac{1}{4}(1+s) \\
\beta(4t-1-s) & \frac{1}{4}(1+s) \leq t \leq \frac{1}{4}(2+s) \\
\gamma(\frac{4t-s-2}{2-s}) & \frac{1}{4}(2+s) \leq t \leq 1
\end{cases}
\]

\[ h \text{ is well-defined and is a pointed homotopy from } \mathcal{M}(1\times 1) \text{ to } \mathcal{M}(1\times \mathcal{A}). \]

iii) Let \( \alpha \in \mathcal{A}_X \)

\[
\mathcal{M}(1\times \mathcal{A}) \delta(\alpha)(t) = \begin{cases} 
\alpha(2t) & 0 \leq t \leq \frac{1}{2} \\
\alpha(2-2t) & \frac{1}{2} \leq t \leq 1
\end{cases}
\]

Define \( h : (\mathcal{A}_X) \times 1 \rightarrow \mathcal{A}_X \text{ by} \]

\[
h(\alpha, \delta)(t) = \begin{cases} 
\alpha(2t(1-s)) & 0 \leq t \leq \frac{1}{2} \\
\alpha((2-2t)(1-s)) & \frac{1}{2} \leq t \leq 1
\end{cases}
\]

\[ h \text{ is a pointed homotopy from } \mathcal{M}(1\times \mathcal{A}) \delta \text{ to } \alpha. \]

Similarly \( \mathcal{M}(1\times 1) \delta \xrightarrow{\sim} * \).

If \( Y \) an \( H^+ \)-space with unit \( \mathcal{u} \), and \( f, g : X \rightarrow Y \)

are pointed maps, define \( f \circ g \) to be the composition:

\[
X \xrightarrow{f} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow{\mu} Y
\]
Note: The pointed homotopy class of $fg$, $[fg]$, depends only on $[f]$ and $[g]$, i.e. $[fg]$ is independent of the choice of representatives. Hence, we can define a product via $[X,Y] 	imes [Y,Z] 	o [X,Z]$.

Prop: $[X,Y]$ is a group under above prod. Unit elt is $[\ast]$, $[\ast]^{-1} = [\ast f]$.

Proof: Unit elt: $[X][\ast] = [\ast] \circ f$. Hence $\ast f \simeq f$, and so $[X][\ast] = [\ast f]$.

Similarly, $[f][\ast] = [\ast f]$.

Divergence: $[f][\ast f] = [f \circ \ast f]$. $f \circ \ast f$ is the composition $X \xrightarrow{d} X \times X \xrightarrow{f \times \ast f} Y \times Y \xrightarrow{\mu} Y$.

We have the comm. Diag.

Since $\mu(1 \times \gamma) \nabla \simeq \ast \times \gamma$, it follows that $f \circ \gamma \simeq \ast \times \gamma$, so $[f][\gamma] = [\ast]$. Similarly, $[\gamma f][f] = [\ast]$. 

Associativity:

- $d_1(1 \times \gamma) \nabla \simeq \ast \times \gamma$, so $[1 \times \gamma] = [\ast]$. 
- Similarly, $\gamma f(1 \times f) \nabla \simeq \ast \times \gamma$, so $[\gamma f][f] = [\ast]$. 
- Again, $\gamma f(1 \times f) \nabla \simeq \ast \times \gamma$, so $[\gamma f][f] = [\ast]$. 

Thus, $[X,Y]$ is a group under above prod.
Let \( [X,Y] \) be a functor in each argument from pointed spaces to pointed sets; continuous in 1st variable, commutative in 2nd variable, w.r.t composition of maps.

If \( f \colon X' \to X \) a pointed map, \( f^* \colon [X,Y] \to [X',Y] \) is defined by \( f^*([g]) = [gf] \).

If \( g \colon Y \to Y' \) a pointed map, \( g_* : [X,Y] \to [X,Y'] \) is defined by \( g_*([f]) = [gf] \).

Prop: \( Y \) an \( H^* \)-space, \( f : X' \to X \) a pointed map, then \( f^* : [X,Y] \to [X',Y] \) is a morphism of groups.

Proof: Have comm. diag.

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\downarrow{d} & & \downarrow{d} \\
X \times X & \xrightarrow{x \times h} & Y \times Y \\
\downarrow{g \times h f} & & \downarrow{g \circ \circ h f} \\
X' \times X' & \xrightarrow{g^* \circ h f} & Y' \\
\end{array}
\]

and so \( f^*([g][h]) \overset{\text{def}}{=} [(g \circ h)f] = [g \circ h f] = [g][h] \).

Thus if \( Y \) is an \( H^* \)-space, \( [ , Y] \) is a contr. functor from pointed spaces to groups.
HW: Exercise: If $Y$ is a pointed space such that $[X,Y]$ has a natural group structure for each pointed space $X$, then this natural group structure comes from an $H^*$-structure on $Y$.

Def: $X$, $Y$ pointed spaces, the smash product of $X$ and $Y$, denoted $X \wedge Y$, is $X \times Y / X \vee Y$.

$X \wedge Y$ is again pointed with base pt. The point to which $X \vee Y$ is identified. Write $X \wedge y = \text{image of } (x,y)$ under the quotient map $X \times Y \to X \wedge Y$. Thus $X \wedge y = x \times x = x$.

If $f : X \to Y$, $f' : X' \to Y'$ are pointed maps, then $f \wedge f'(X \vee X') \subseteq (Y \vee Y')$. Passing to quotients, we obtain a pointed map $f \wedge f' : X \wedge X' \to Y \wedge Y'$.

Following properties are easily checked: $(f_1 \wedge f_2)(y_1 \wedge y_2) = f_1(y_1) \wedge f_2(y_2)$, $f \wedge * = * \wedge y$, $f \times_0 f' \wedge g \times_0 g' = (f \wedge g) \times_0 (f' \wedge g')$.

We have natural homomorphisms

$$(X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z),$$

$$(X \wedge y) \wedge Z \to (X \wedge y) \wedge (X \wedge Z).$$

Note: $\wedge$ and $\vee$ in the pointed category are in many respects analogous to $\times$ and $+$ in the unpointed cat.

If $(X, A)$ a top. pair (not pointed), then $X/A$ is a pointed space with base pt. the point to which $A$ is identified. Construction: $X/A = \text{disjoint union of } X$ with a new point $*$, taken to be the base point of $X/A$.

Suppose $(X, A)$, $(Y, B)$ top. pairs. Let $p_A : X \to X/A$, $p_B : Y \to Y/B$ denote the quotient maps. The composition

$$p_A \times \text{id} : X \times Y \to (X/A) \times (Y/B)$$

is a map of pointed spaces.
\[
X \times Y \xrightarrow{\text{quot.}} (X/A) \times (Y/B) \xrightarrow{\text{quot.}} (X/A) \wedge (Y/B)
\]
is a quotient map, and sends \((X \times B) \sqcup (A \times Y)\) to \(*\). Thus, passing to quotients, get a quotient map

\[f: \frac{X \times Y}{(X \times B) \sqcup (A \times Y)} \rightarrow \frac{(X/A) \wedge (Y/B)}{\ast},\]
easily seen to be a bijection.

\[
\begin{array}{c}
\text{Y} \\
\downarrow \\
\text{A} \\
\downarrow \\
\text{B} \\
\downarrow \\
\text{X}
\end{array}
\]

*Example*: 1) Regard \(S^n\) as \(I^n / I^n\), \(n \geq 0\). Then
\[
S^m \wedge S^n = \frac{I^m \times I^n}{(I^m \times I^n) \cup (I^m \times I^n)} = \frac{I^{m+n}}{I^{m+n}} = S^{m+n}.
\]
In particular, if \(m \geq 1\),
\[
S^m \cong S^{m-1} \wedge S^1.
\]

2) In any pointed \(X\), the space \(X \wedge S^1\) is called the reduced suspension of \(X\). Note: \(X \equiv X/\ast\), \(S^1 = I/\{0,1\}\),
\[
X \wedge S^1 \equiv \frac{X \times I}{(X \times \{0,1\}) \cup (\ast \times I)} = \text{quot. space obtained from cylinder on } X \text{ by collapsing top and bottom lines, and attaching this line except } \ast \text{ to a point}.
\]

3) \(X \wedge S^0\) is canonically homeomorphic to \(X\) for any pointed \(X\).
Def: A $\omega$-$H^*$ space consists of a pointed space $(X, x_0)$ together with pointed maps $m': X \to X \vee X$ (the comultiplication) and $\eta': X \to X$ (the counit). Following commutes up to pointed homotopy:

1) $X \times X \xrightarrow{i} X \vee X \xleftarrow{m'} X$  
2) $X \xrightarrow{m'} X \vee X \xrightarrow{1 \vee m'} X \vee X \vee X$  

(Notation: if $f: X \to Y$, $g: Z \to W$ pointed maps, then $f \circ g: X \vee Z \to Y \vee W$ is the pointed map given by $(f \circ g)(x, z) = (f(x), g(z))$.)

Example: Let $X$ be any pointed space. Claim: $X \wedge S^1$ is a $\omega$-$H^*$ space with comultiplication $m': X \wedge S^1 \to (X \wedge S^1) \vee (X \wedge S^1)$ and counit $\eta': X \wedge S^1 \to X \wedge S^1$ given as follows: 

Considering $S^1$ as $I/\{0, 1\}$, we fix $\langle t \rangle \in S^1$ for any $t \in I$ under the quotient map $I \to S^1$. Define 

$$m'(x \wedge \langle t \rangle) = \begin{cases} \langle x \wedge \langle 2t \rangle, * \rangle & 0 \leq t \leq \frac{1}{2} \\ \langle x \wedge \langle 2t-1 \rangle, * \rangle & \frac{1}{2} \leq t \leq 1 \end{cases}$$
Let $X$ be a co-$H^*$ space with connect. $x'$, conn. $y'$, and $Y$ a pointed space. If $f, g : X \to Y$ are pointed maps, define $f \circ_{s} g$ to be the composition

\[ X \xrightarrow{x'} X \vee X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{c_{p}} Y. \]
Lema: [f x g] depends only on [f] and [g]. Thus, [f o g] depends only on [f] and [g]. Hence, can define a prod in [X X Y] by [f][g] = [f o g].

Prop: X a c-o-H* space with c-o-mul at 'e', c-o-mul 'n'. Then for any pointed Y, [X X Y] is a group under the above prod. Unit elt is [x] and group inverse: \([f]^{-1} = [f^*++]\). Moreover, [X, ] is a covariant functor from cat of pointed spaces to cat of groups.

Proof: Ht sepere (essentially dual to proof of corresp prop for H* space).

Exercise: X pointed => [X, ] factors through the cat of groups. Then X is a c-o-H* space with the c-o-H* structure on [X X Y] coming from the c-o-H structure on X via the □ construction.

Notation: Let \((Y, y_0)\) be pointed. For \(n \geq 0\), \([S^n, Y]\) is denoted \(\Theta_n(Y, y_0)\) (or more briefly \(\Theta(Y)\) if there is no danger of confusion). For \(n \geq 1\), \(S^n\) is a reduced simplicial, hence a c-o-H* space, and \(\Theta_n(Y, y_0)\) is a group, the \(n\)th homotopy group of \((Y, y_0)\). \(\Theta(Y, y_0)\) is usually called the fundamental group of \(Y\). \(\Theta(Y, y_0)\) is just a pointed set in general (an 1-1 coming with the path-components of \(Y\)).

Prop: Suppose X is a c-o-H space, Y an H* space. Then the group structure on [X X Y] coming from the c-o-H* structure on \(X\) commutes with the group structure coming from the H* structure on \(Y\). Moreover, in this case, the group \([X X Y]\) is abelian.

Proof: Let \(f, g : X \rightarrow Y\) be pointed maps. To show the 2 group structures coincide, must show \(f \circ g \simeq f \circ \overline{g}\).
Rectangle commutes. Side triangles are commutative by props. Thus
\[ f \circ g = \mu(f \circ g) = \alpha(f \circ g) \alpha' = f \circ g. \]

To show \([X,Y]\) abel., suffice to show \(f \circ g \simeq g' \circ f\).

\[\begin{array}{c}
\xymatrix{ 
X 
\ar[r]^f \ar[d]_g & Y \\
X \times X 
\ar[u]^i \\
X \times X 
\ar[r]_{f \circ g} & Y \times Y \\
Y \times Y 
\ar[u]_i \\
Y 
\ar[u]^f \\
}
\end{array}\]

Here \(1 \circ f \) or \(f \circ 1 \) commute. \((1, (2, 3)) \) commute by (i.) for \(H^+ \) and \(H^* \) spaces, resp.

\((2) \circ Y \) an \(H^* \) space, \(\Pi_n(Y, Y)\) is abel. for \(n \geq 1\).
(In particular, \(\Pi_1(Y, Y)\) is abel.).

Recall: facts about the \(-\infty\) topology: \(\| X, Y \) are top. spaces (unpointed), write \(C(X, Y)\), space of cont. maps from \(X \to Y\) with \(C(0, Y) = Y\). If \(X\) is loc. compact and Hausdorff, then \(\psi : C(X, Y) \to C(Z, C(X, Y))\) quasi. by \(\psi(f)(z)(x) = f(z, x)\) is a natural isom.

Corresp. prop. for pointed spaces, \(\| X, Y \) are pointed, write \(C_0(X, Y)\) = space of pointed cont. maps from \(X \to Y\) with \(C_0(0, Y) = Y\). If \(X\) is loc. compact Hausdorff, then for every pointed \(Z, Y, Y_0 : C_0(Z \times X, Y) \to C_0(Z, C_0(X, Y))\) quasi. by \(Y_0(f)(z)(x) = f(z, x)\) is a natural isom.

Easily checked: \(f \simeq g \iff Y_0(f) \simeq Y_0(g)\). Thus \(Y_0\) induces a natural bijection \(Y_0 : [Z \times X, Y] \to [Z, C_0(X, Y)]\) whenever \(X\) is loc. compact and Hausdorff.
Note: \( C_0(S^1, Y) \) is naturally homotopic with \( \Omega Y \). Thus taking \( X = S^1 \) we obtain a natural bijection

\[ \Psi_0 : [S^1, Y] \rightarrow [S^1, \Omega Y] . \]

### Prop. \( \Psi_0 \) is a natural group isomorphism.

**Proof:** Remains only to show that \( \Psi_0 \) is a group isomorphism. Straightforward to check: \( \Psi_0(g \cdot h) = (\Psi_0(g)) \cdot (\Psi_0(h)) \) using the explicit def. of the comultip. in \( S^1 \) and the unit in \( \Omega Y \).

\( \mathrm{Co} \): For all \( n \geq 1 \) and all pointed \( (Y, *Y) \) we have a natural group action. \( \Psi_0 : \Pi_n(Y, *Y) \rightarrow \Pi_{n-1}(\Omega Y, *) \).

\( \mathrm{Co} : \) For \( n \geq 2 \), \( \Pi_n(Y, *Y) \) is abelian for any pointed \( Y \).

**Proof:** \( \Pi_n(Y, *Y) \cong \Pi_{n-1}(\Omega Y, *) \cong [S^{n-1}, \Omega Y] \) which is abel. since \( S^{n-1} \) is a co-H\(^*\)-space (since \( n-1 \geq 1 \)) and \( \Omega Y \) is an H\(^*\)-space.

Regard \( I \) as a pointed space with base pt. \( 0 \).

**Def.** If \( X \) is pointed, the reduced cone on \( X \) is \( X \wedge I \).

Thus \( X \wedge I = \frac{X \times I}{(X \times 0) \cup (x \times I)} = \) quotient space obtained from the unreduced cone \( \frac{X \times I}{X \times \{0\}} \) by collapsing \( x \times I \) to a point.

**Note:** \( X \wedge I \) is contractible in the pointed sense, i.e. \( 1 \wedge I \cong * \).

For \( 1 \wedge I = \wedge 1 \). Since \( 1 \cong * \), \( 1 \wedge I \cong 1 \wedge * = X \).

**Note:** We have a canonical embedding \( X \subset X \wedge I \) given by \( x \mapsto x \times 1 \).

Regard \( X \) as a subspace of \( X \wedge I \) in this way. Then

\[ \frac{X \wedge I}{X} \cong X \wedge S^1 . \]
Relative Homotopy Sets

A pointed top. pair consists of a top. pair \((X,A)\) with a base point \(* \in A\). If \(\{(X,A), (Y,B)\}\) are pointed top. pairs, a map of pointed pairs \(f : (X,A) \to (Y,B)\) consists of a pointed map \(f : X \to Y\) s.t. \(f(A) \subset B\).

If \(f, g : (X,A) \to (Y,B)\) are maps of pointed pairs, a pointed homotopy \(h : (X,A) \times I \to (Y,B)\) from \(f\) to \(g\) is a homotopy from \(f\) to \(g\) s.t. \(h(x,t) = * \quad \forall x \in X\). Write \(f \simeq g\) in this case. \(\simeq\) is an equip. rel.

Write \([[(X,A),(Y,B)]]\) = set of pointed homotopy classes of pointed maps from \((X,A)\) to \((Y,B)\). \([\_,\_]\) is a functor of 2 variables from pointed pairs to pointed sets, continuous in 1st, covariant in 2nd.

Def: \(\Pi_{n}(Y,B)\) a pointed pair and \(n \geq 1\), define \(\Pi_{n}(Y,B) = [[(S^{n},\ast) : \ast) \times I \to (Y,B)] = \Pi_{n}^{h}\) homotopy set of \((Y,B)\).

Def: \((Y,B)\) a pointed pair. Define \(P(Y,\ast, B) = \{ \lambda \in C_{0}(I,Y) | \lambda(1) \in B \}\). (Ums. \(P(Y, \ast, \ast) = 2Y\).)

\[\text{Let } C_{0}((X,A),(Y,B)) = \text{set of } C_{0}(X,Y) \text{ connecting } \text{of those maps } f \circ \tau : f(A) \subset B.\]

Recall. Homot. \(\Psi_{0} : C_{0}(X^{I}, Y) \to C_{0}(X, C_{0}(I,Y))\).

Can check \(\Psi_{0}\) carries \(\Pi_{n}((X^{I}, X), (Y,B))\) into \(\Pi_{n}(X, P(Y, \ast, B))\).

Moreover, \(f \simeq g \iff \Psi_{0}(f) \simeq \Psi_{0}(g)\), so get natural bijection \([([X^{I}, X], (Y,B)] \cong [[X, P(Y, \ast, B)]].\) In particular, has natural bijection \(\Pi_{n}(Y,B) \cong \Pi_{n-1}(P(Y, \ast, B), \ast)\).

Thus, \(\Pi_{n}(Y,B)\) has a natural group structure for \(n > 2\), and is abel. for \(n > 3\).

When \(B = \ast\), above reduces to the usual map \(\Pi_{n}(Y, \ast) \cong \Pi_{n-1}(2Y, \ast)\) for \(n > 1\).

Note: \(\Pi_{0}(Y,B)\) is undefined, except for case \(B = \ast\).
If \( n \geq 1 \) and \( f: (S^n \wedge I, S^n) \to (Y, B) \) a pointed map, then \( f|_{S^n}: S^n \to B \) is a pointed map. Moreover, \([f|_{S^n}]\) depends only on \([f]\). Thus, get a natural isomorphism of pointed sets,

\[ \Theta: \pi_0(Y, B) \to \pi_{n-1}(B, *) . \]

Let \( p: P(Y, *, B) \to B \) be given by \( p(A) = \lambda(1) \). \( p \) is continuous (by properties of \(-O\) top.) and pointed.

Prop. \((Y, B)\) a pointed pair, \( n \geq 1 \). Then,

\[ \pi_n(Y, B) \xrightarrow{\Theta} \pi_{n-1}(B, *) \]

\[ \Psi \downarrow \]

\[ \pi_{n-1}(P(Y, *, B), *) \text{ commutes.} \]

Proof: \( Y \leftarrow f: (S^n \wedge I, S^n) \to (Y, B) \) pointed map,

\[ \Psi(f)(x) = \Psi(f)(x)(1) = f(x1) = f(x) \] (recall: \( X \) is identified with subspace \( X \wedge I \) with \( x \mapsto x1 \)).

Cor. \( \pi_n \) \( n \geq 2 \), \( \Theta: \pi_n(Y, B) \to \pi_{n-1}(B, *) \) is a group homomorphism.

Note: \( \exists f: (X, A) \to (Y, B) \) a map of pointed pairs, get cont. \( Pf: P(X, *, A) \to P(Y, *, B) \) given by \( Pf(A) = fX \).

\( P \) is a covariant functor from pointed pairs to pointed spaces.

In special case \( A, B \) are both \(* \), write \( \Omega f = Pf: \Omega X \to \Omega Y \).

Naturality of \( \Psi \):

\[ \pi_n(X, A) \xrightarrow{\Theta} \pi_n(Y, B) \]

\[ \Psi \downarrow \]

\[ \pi_{n-1}(P(X, *, A), *) \xrightarrow{(Pf)_*} \pi_{n-1}(P(Y, *, B), *) \text{ commutes. (Easy check). In particular } \Theta \text{ is a group homomorphism for } n \geq 2 . \]

Given a pointed pair \((Y, B)\), let \( i: (B, *) \to (Y, *) \) and \( j: (Y, *) \to (Y, B) \) denote the inclusions. We obtain a sequence of isomorphisms of pointed sets (mostly group homomorphisms).
The homotopy sequence of a pointed pair \((Y, B)\) is exact.

Proof: First, at \(\pi_n(Y, y_0)\), \(n \geq 1\).

Have commutative diagram

\[
\begin{array}{ccccccccc}
\pi_n(B, y_0) & \xrightarrow{j_*} & \pi_n(Y, y_0) & \xrightarrow{j_+} & \pi_n(Y, B)
\end{array}
\]

\[
\begin{array}{ccccccccc}
\Psi_0 \downarrow \quad \quad \quad \quad \quad \quad \Psi_1 \downarrow \quad \quad \quad \quad \quad \quad \Psi_2 \downarrow \\
\pi_{n-1}((\Omega B, *)^{(P_c)_*}) & \xrightarrow{(P_c)_*} & \pi_{n-1}((\Omega Y, *)^{(P_j)_*}) & \xrightarrow{(P_j)_*} & \pi_{n-1}(P(Y, *, B), *)
\end{array}
\]

So suffice to check bottom row exact.

Let \(f : S^{n-1} \to \Omega B\) be a pointed map.

Then \((P_j)(P_c)f(x)(*) = j_+ f(x)(*) = f(x)(*)\).

Define \(h : S^{n-1} \times I \to P(Y, *, B)\),

\[h(x, s)(t) = f(x)(st)\] \(h\) is a pointed homotopy from \(x \cdot (P_j)(P_c)f\) and so

\[\lim (P_c)_* = (P_j)_*^{-1}(*).\]
Now suppose \( f : S^{n-1} \to \Omega Y \) is a pointed map \( \in \).

(\( P_j \)) \( \ast [\ell] = \ast \). Every pointed homotopy

\[
h : S^{n-1} \times I \to \Omega (Y, \ast (B)) \n\]

\[
h(x,0)(t) = f(x)(t) \]

\[
h(x,1)(t) = \ast.
\]

Define \( g : S^{n-1} \to \Omega B \) by

\[
g(x)(t) = h(x,1-t)(1).
\]

Suff to show \( (P_i)_g \sim_k f \).

Define \( k : S^{n-1} \times I \to \Omega Y \) by

\[
k(x,s)(t) = h(x, \min \{s, 1-t\}) \left( \min \left\{ \frac{t}{1-s} \right\} \right)
\]

interpreted as \( 1 \) if \( s = 1 \).

Continuity easily checked, using

\[
h(x,1)(t) = \ast.
\]

Thus \( f \sim_k (P_i)_g \).

(Alternately, let \( g(x)(t) = \begin{cases} y_0 & 0 \leq t \leq \frac{1}{2} \\ h(x,2-t)(1) & \frac{1}{2} \leq t \leq 1. \end{cases} \)

\( g : S^1 \to \Omega B \).

Define \( k : S^{n-1} \times I \to \Omega Y \) by

\[
k(x,s)(t) = \begin{cases} h(x,s)(\frac{t}{1-s}) & 0 \leq t \leq 1 - \frac{s}{2} \\ h(x,2-t)(1) & 1 - \frac{s}{2} \leq t \leq 1. \end{cases}
\]

Then \( f \sim_k (P_i)_g \).
Lemma at \( \Pi_n(Y, B), n \geq 1 \):

Here commutative diagram

\[
\begin{array}{ccc}
\Pi_n(Y, y_0) & \xrightarrow{\psi} & \Pi_0(Y, B) & \xrightarrow{\partial} & \Pi_{n-1}(B, y_0) \\
\psi_0 & \xrightarrow{=} & \psi & \xrightarrow{=} & 1 \\
\Pi_{n-1}(\Omega Y, *) & \xrightarrow{P_j} & \Pi_{n-1}(P(Y_j, B), *) & \xrightarrow{\pi} & \Pi_{n-1}(B, y_0)
\end{array}
\]

so suff to show bottom row is exact,

\[
p(P_j) = *, \quad \text{so} \quad \text{im}(P_j)_* = \pi^{-1}(*) .
\]

Suppose \( f : S^{n-1} \to P(Y, *) \) a pointed map.

Define \( g : S^{n-1} \to \Omega Y \) by

\[
g(x)(t) = \begin{cases} 
  f(x)(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\
  h(x, 2t-1) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

Suff to show: \( f \simeq (P_j)_* g \).

Define

\[
k : S^{n-1} \times I \to P(Y_j, B) \) by

\[
k(x, s)(t) = \begin{cases} 
  f(x)(\frac{2t}{2-s}) & \text{if } 0 \leq t \leq 1 - \frac{s}{2} \\
  h(x, 2t+s-2) & \text{if } 1 - \frac{s}{2} \leq t \leq 1.
\end{cases}
\]

Then \( f \simeq k \circ (P_j)_* g \).
Fibrations

Def. A cont. map \( p : E \to B \) (not necessarily pointed) has the homotopy lifting property for \( X \) if

\[
\begin{align*}
X \times 0 & \xrightarrow{f} E \\
\downarrow h & \\
X \times I & \xrightarrow{p} B
\end{align*}
\]

gives, continuous \( f \), \( h \), rectangle commutes, \( \exists \) contains \( h \), both triangle commutes.

\( p \) is a \textbf{fibration} if it has the HLP in all \( X \) (sometimes called a \textit{Hurewicz} fibration).
Examples: 1) \( B \times F \) is a \textit{fibration}, let \( \pi \) a quotient map

\[
X \times 0 \rightarrow B \times F
\]

\[
\exists \tilde{h} : X \times I \rightarrow B \times F \text{ by}\]

\[
\tilde{h}(x,t) = (h(x,t), \pi_2 f(x,0))
\]

\( X \times I \rightarrow B \)

2) \( p : E \rightarrow B \) is a \textit{fiber bundle projection} with \textit{fiber} \( F \)

If \( \forall x \in B \), \( \exists \) an \textit{alb} \( U \cap x \subseteq B \) and a \textit{homeo}\( \)

\[
U \times F \overset{\theta}{\rightarrow} p^{-1}(U) > U \times F \overset{\pi}{\rightarrow} p^{-1}(U)
\]

\( \pi_i \quad \pi_j \quad \pi \quad \pi \quad \pi \quad \pi \)

Can be proved: If \( p : E \rightarrow B \) is a \textit{fiber bundle projection} with \( B \) \textit{paracompact}, then \( p \) is a \textit{fibration}.

References: in proof: Steenrod, Top of Fibre Bundles, p. 47-54

Spanier, p. 92-96.

3) \( \text{Let} \ (Y, A, B) \ \text{be a top. triad}. \ \text{Write}\)

\[
P(Y, A, B) = \{ \lambda \in E(I, Y) \mid \lambda(0) \in A, \lambda(1) \in B \} \]

\( \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda 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4) If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are fibrations, easily proved

$qf : X \to Z$ is a fibration.

Thus, combining examples 3) and 1),

$P(Y, A, B) \to A$, $P(Y, A, B) \to B$ are fibrations,

$\Lambda \xrightarrow{\lambda(0)} \Lambda(1)$

In particular, $P(Y, \ast, B) \to B$ is a fibration.

The (Cellular homotopy lifting extension theorem, CHLET):

Let $p : E \to B$ be a fibration and $(K, L)$ a CW pair.

Suppose given continuous maps $h, f$ such that

the following rectangle commutes:

$$
\begin{array}{c}
1K \times I \\
\downarrow \\
1K \times I
\end{array}
\xrightarrow{f}
\begin{array}{c}
E \\
\downarrow p
\end{array}

\begin{array}{c}
\Lambda(0) \\
\downarrow h
\end{array}
\begin{array}{c}
\Lambda(1)
\end{array}

Then $f$ continuous $h : 1K \times I \to E \Rightarrow$ both triangles commute.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure}
\caption{Cellular homotopy lifting extension theorem diagram}
\end{figure}

Idea of proof: Extend one cell at a time.

Lemmas: If $n \geq 1$, $f$ homotopic to map

$(D^n \times I, D^n \times 0) \to (D^n \times I, D^n \times 0 \cup S^{n-1} \times I)$.
\[ f(I, t) \rightarrow \begin{cases} 
\left( \left( \frac{2-t}{1+t} \right) x + \frac{1-t}{2} \right), & \text{if } 0 \leq \|x\| \leq \frac{1+t}{2} \\
\left( \left( \frac{1-t}{2} \right) \frac{x}{\|x\|} + \left( \frac{1+t}{2} - 1 \right) \right), & \text{if } \frac{1+t}{2} \leq \|x\| \leq 1
\end{cases} \]

as the desired hence.

**Case:** Let \( p : E \rightarrow B \) be a fibration. Suppose quasi continuous \( h \), \( f \) \( \mathbb{R} \)-rectable.

\[
\begin{align*}
D^n \times \Omega \cup S^n \times I & \xrightarrow{f} E \\
\downarrow & \\
D^n \times I & \xrightarrow{h} B
\end{align*}
\]

Then \( f \) continues \( h \) \( \Rightarrow \) both triangles commute.

**Proof of CHLET:** Write \( K_n^* = (1k \times 0) \cup (K^n \cup 11) \times I \).

Proof proceeds by inductively constructing a sequence of maps \( \tilde{h}_n : K_n^* \rightarrow E \) satisfying:

1) \( \tilde{h}_n - \tilde{h}_{n-1} = f \)
2) \( \tilde{h}_n | K_{n-1}^* = \tilde{h}_{n-1} \)
3) \( p \tilde{h}_n = h | K_n^* \).

We would then have a well-defined function \( \tilde{h} : 1k \times 11 \rightarrow E \) making the required triangles commute \( \Rightarrow \tilde{h} | K_n^* = \tilde{h}_n \) \( \tilde{h} \) would be continuous, since each \( \tilde{h}_n \) is and \( K_n^* \subset K^n \).
Construction of $\tilde{h}^0$: In each 0-cell $E_x^0$ of $K$ not in $L$,

$$\begin{align*}
E^0_x \times 0 & \xrightarrow{\tilde{h}^0 \times 0} E \\
\bigcap & \downarrow \tilde{h}^0 \times 1 \\
E^0_x \times I & \xrightarrow{h \mid E^0_x \times I} B
\end{align*}$$

If contains $h^0 \Rightarrow$ both triangles commute.

Define $\tilde{h}^0$ by $\tilde{h}^0 | (1 \cap 1 \times 0) \cup (1 \cap 1 \times I) = f$, $\tilde{h}^0 | E^0_x \times I = h^0$.

Suppose inductively $n > 0$ and $\tilde{h}^{n-1}$ constructed.

Let $E^n_x$ be an $n$-cell of $K$ not in $L$, and $\tilde{h}^n_x : D^n_x \to E^n_x$ the chain map in $E^n_x$. Note that

$E^n_x \cap K_{n-1} = \tilde{E}^n_x = E^n_x - E^n_x$. Consider

$$\begin{align*}
(D^n_x \times 0) \cup (S^{n-1}_x \times I) \xrightarrow{(\tilde{h}^n_x \times I)} (\tilde{E}^n_x \times 0) \cup (\tilde{E}^n_x \times I) \xrightarrow{\tilde{h}^{n-1}} E \\
\bigcap & \downarrow \tilde{h}^n_x \times 1 \\
D^n_x \times I & \xrightarrow{\eta^*_n \times I} \tilde{E}^n_x \times I \xrightarrow{h \mid} B
\end{align*}$$

By construction, $E$ cont. $K^n_x$ making both triangles commute. Passing to quotients, $\tilde{h}^n_x$ yields a continuous $h^n_x$ making

$$\begin{align*}
(\tilde{E}_x \times 0) \cup (\tilde{E}_x \times I) & \xrightarrow{h_{n-1}} E \\
\bigcap & \downarrow \tilde{h}^n \\
\tilde{E}^n_x \times I & \xrightarrow{h \mid} B
\end{align*}$$

Define $\tilde{h}^n$ by $\tilde{h}^n | K^n_x = \tilde{h}^{n-1}$, $\tilde{h}^n | E^n_x \times I = h^n_x$ for each $n$-cell $E^n_x$ of $K$ not in $L$. 

Theorem: Let \( p : E \to B \) be a pointed fibration (i.e., a pointed map and a fibration). Write \( F = p^*(\ast) \). Then, commuting
\[
p : (E,F) \to (B,\ast)
\]
is the map of (top.) pairs which \( p \) yields, \( p_n : \mathcal{P}_n(E,F) \to \mathcal{P}_n(B,\ast) \) is a bijection for all \( n \geq 1 \).

Proof: \( p \) is onto. Suppose \( f : (S^{n-1} \wedge I, S^{n-1}) \to (B,\ast) \) is a pointed map. Consider
\[
\begin{array}{ccc}
S^{n-1} \times O \cup (x \times I) & \xrightarrow{*} & E \\
\downarrow \mathcal{h} & & \downarrow \mathcal{p} \\
S^{n-1} \times I & \xrightarrow{f \times 1} & S^{n-1} \wedge I & \xrightarrow{f} & B \\
\end{array}
\]
Diagram commutes, since \( (S^{n-1},\ast) \) is the underlying space
pair of a CW pair, by CHLET, \( \mathcal{h} \) continues \( \mathcal{h} \).
Both triangles commute. Since \( \mathcal{h} : (S^{n-1} \times O \cup O \times I) \to \ast \), passage
to quotients yields a pointed map \( \mathcal{h} : S^{n-1} \wedge I \to E \)
\[
\begin{array}{ccc}
S^{n-1} \wedge I & \xrightarrow{f} & B \\
\end{array}
\]
commutes.

Since \( \mathcal{h}(x \wedge 1) = \ast \), we have \( \mathcal{h}(S^{n-1}) = \mathcal{h}(S^{n-1} \wedge 1) \subseteq p^*(\ast) = F \)
and \( \mathcal{h} \) yields a pointed map of pairs
\[
\begin{array}{ccc}
\mathcal{h} : (S^{n-1} \wedge I, S^{n-1}) & \xrightarrow{f \times 1} & (E,F) \\
\end{array}
\]
\( \mathcal{p} \mathcal{h} = f \). Then,
\[
[f] = [\mathcal{h}] \quad \text{and so, } \mathcal{h}, \text{ is onto.}
\]

\( f \) is \( 1-1 \): Suppose \( p \cdot [f] = p \cdot [g] \) where \( f, g : (S^{n-1} \wedge I, S^{n-1}) \to (E,F) \)
are pointed. \( F \) pointed homotopy
\[
\begin{array}{ccc}
h : (S^{n-1} \wedge I, S^{n-1}) \times I & \to & (B,\ast) \\
\end{array}
\]
from \( p f \) to \( p g \).
Consider
\[
\begin{array}{ccc}
(S^{n-1} \times I) \times O \cup (S^{n-1} \times I (O \times I) \times I) & \xrightarrow{r} & E \\
\downarrow \mathcal{h} & & \downarrow \mathcal{p} \\
S^{n-1} \times I \times I & \xrightarrow{f} & (S^{n-1} \times I) \times I & \xrightarrow{h} & B \\
\end{array}
\]
where \( q(x, s, t) = (x \wedge t, s) \),

\[
\begin{align*}
    r(x, s, 0) &= x, \\
    r(x, 0, t) &= f(x \wedge t), \\
    r(x, 1, t) &= g(x \wedge t), \\
    r(x, s, t) &= x.
\end{align*}
\]

Straightforward to check \( r \) is continuous and deforms commutes.

\((S^{n-1} \times I, S^{n-1} \times 0 \cup S^{n-1} \times 1 \cup x \times I)\) is the underlying space pair of a CW pair. Hence by CHLET, \( E \) continues to make both the triangles commute. Since \( \tilde{r}(x, s, 0) = r(x, s, 0) = x \) and \( \tilde{r}(x, s, t) = r(x, s, t) = x \), passage to quotients yields a continuous \( \tilde{h} : (S^{n-1} \times I) \xrightarrow{\text{quotient}} E \xrightarrow{p} B \)

\[\begin{array}{ccc}
(S^{n-1} \times I) \times 0 \cup (S^{n-1} \times 0 \cup S^{n-1} \times 1 \cup x \times I) \times I & \xrightarrow{r} & E \\
\downarrow \quad \tilde{h} & & \downarrow \quad p \\
S^{n-1} \times I \times I & \xrightarrow{\text{quotient}} & (S^{n-1} \times I) \times I & \xrightarrow{h} & B
\end{array}\]

commutes.

Since \( \tilde{h}(S^{n-1} \times I) = x \), we have \( \tilde{h}(S^{n-1} \times I) \subseteq p^{-1}(x) = F \).

We have \( \tilde{h}(x, 1, t) = \tilde{h} q(x, 1, t) = r(x, 1, t) = x \). Thus \( \tilde{h} : (S^{n-1} \times I, S^{n-1} \times I) \xrightarrow{\text{quotient}} (E, F) \) is a pointed homotopy. We have \( \tilde{h}(x, 1, t) = \tilde{h} q(x, 1, t) = r(x, 1, t) = \tilde{g}(x \wedge t) \).

Thus \( p_* \equiv 1 - 1 \).

Now for a pointed fibration \( E \xrightarrow{p} B \) with fibre \( F \equiv p^{-1}(x) \), we have

\[
\begin{array}{cccccc}
\pi_n(F) & \xrightarrow{q_*} & \pi_n(E) & \xrightarrow{p_*} & \pi_n(E,F) & \xrightarrow{q_*} & \pi_{n-1}(F) \\
\downarrow \quad r_* & & \downarrow \quad p_* & & \downarrow \quad r_* & & \downarrow \quad p_* \end{array}
\]

and so we obtain the exact sequence

\[
\begin{array}{cccccccc}
\pi_n(F) & \xrightarrow{q_*} & \pi_n(E) & \xrightarrow{p_*} & \pi_n(B) & \xrightarrow{q_*} & \pi_{n-1}(E) & \xrightarrow{p_*} & \cdots & \pi_0(B) & \xrightarrow{q_*} & \pi_0(F) & \xrightarrow{p_*} & \pi_0(E)
\end{array}
\]

called the homotopy sequence of the pointed fibration \((E, p, B, F)\).
This exact sequence is natural w.r.t. morphisms of pointed fibrations, i.e., if

$$E_1 \xrightarrow{f_X} E_2$$
$$\downarrow f_{E_1} \quad \downarrow f_{E_2}$$
$$B_1 \xrightarrow{f_{B_2}} B_2$$

commutes where $p_1$, $p_2$ are pointed fibrations, $f_X$, $f_{B_2}$ pointed maps, and $F_2 = f_{E_1}^{-1}(*)$, then

$$\Pi_n(F_1) \xrightarrow{f_{E_1}} \Pi_n(E_1) \xrightarrow{f_{E_2}} \Pi_n(B_1) \xrightarrow{f_{B_2}} \Pi_{n-1}(F_2)$$

commutes where $f_\pi = \text{restriction of } f_{E_1}$.

Comments:

1. (1), (2) commute since $\Pi_0$ is a functor.

Consistency of (3):

\[
\begin{array}{rcl}
\Pi_n(B_1) & \xrightarrow{f_{B_2}} & \Pi_n(B_2) \\
\Pi_n(E_1, F_1) & \xrightarrow{f_{E_2}} & \Pi_n(E_2, F_2) \\
\Pi_n(B_1) & \xrightarrow{f_{B_2}} & \Pi_n(B_2) \\
\Pi_n(E_1, F_1) & \xrightarrow{f_{E_2}} & \Pi_n(E_2, F_2) \\
\Pi_n(F_1) & \xrightarrow{f_{B_2}} & \Pi_n(F_2) \\
\end{array}
\]

Example: $R^n$ is a fibre bundle with base $\mathbb{I}$.

$s^1 = R/\mathbb{Z}$

$R$ is contractible, so $\Pi_n(R) = 0$ for all $n$. Thus $\Pi_n(\mathbb{I}) = 0$. For all $n > 2$, $\Pi_n(S^1) \xrightarrow{\beta} \Pi_{n-1}(\mathbb{Z})$. $\mathbb{I}$ is contractible, so $\Pi_n(\mathbb{I}) = 0$ for $n > 2$. Thus $\Pi_n(S^1) = 0$ for all $n > 2$. 


Example: Regard $S^1$ as $\{ z \in \mathbb{C} \mid |z| = 1 \}$, and $S^{2n+1}$ as $\{ (z_1, \ldots, z_{2n+1}) \in \mathbb{C}^{2n+1} \mid \sum_{i=1}^{2n+1} |z_i|^2 = 1 \}$. Then

$$S^{2n+1} \xrightarrow{p} \mathbb{C}P^n \xrightarrow{[z_1, \ldots, z_{2n+1}]}$$

is a fiber bundle projection with fiber $S^1$.

Thus $p_\ast : \pi_n(S^{2n+1}) \to \pi_n(\mathbb{C}P^n)$ is an isomorphism for all $n \geq 3$.

Special case $n = 1$: $\mathbb{C}P^1 \cong S^2$, let $f$ be a bundle $S^3 \to S^2$ with fiber $S^1$ (one of the Hopf fibrations). Thus

$$\pi_n(S^3) \xrightarrow{f} \pi_n(S^2) \text{ for } n \geq 3.$$

In particular, $\pi_3(S^2) \cong \pi_3(S^3)$, since $1 : S^3 \to S^3$ is not $\ast$ (from homology), $\pi_3(S^3) \neq 0$. Thus, $\pi_3(S^2) \neq 0$.

Homotopy sequence of a pointed triple

Let $(Y, A, B)$ be a pointed triple, i.e., $* \in B \subset A \subset Y$. Let $i : (A, B) \subset (Y, B)$, $j : (Y, B) \subset (Y, A)$, $r : (A, *) \subset (A, B)$ denote the inclusions. Define $\tilde{\partial} : \pi_n(Y, A) \to \pi_{n-1}(A, B)$, where $\partial$ is from the homotopy seq. of $(Y, A)$. Thus, we get a sequence

$$\to \pi_n(A, B) \xrightarrow{i_*} \pi_n(Y, B) \xrightarrow{j_*} \pi_n(Y, A) \xrightarrow{\tilde{\partial}} \pi_{n-1}(A, B) \to \pi_n(Y, B) \xrightarrow{j_*} \pi_n(Y, A) \xrightarrow{\tilde{\partial}} \pi_{n-1}(A, B) \to \cdots \to \pi_1(A, B) \to \pi_0(Y, B) \xrightarrow{p_1} \pi_0(Y, A)$$

called the homotopy sequence of the pointed triple. $(Y, B = *, A, *)$ alone is homotopy seq. of pointed pair $(Y, A)$.

HW Exercise: Homotopy seq. of pointed triple is exact and natural.

Note: If $p : E \to B$ a fibration, $B_0 \subset B$, then $p^{-1}(B_0) \to B_0$ is a fibration.

Prop: $p : E \to B$ a pointed fibration with fiber $F$. Let $* \in B_0 \subset B$, and $E_0 = p^{-1}(B_0)$. Then $p_\ast : \pi_n(E, E_0) \to \pi_n(B, B_0)$ is an isomorphism for $n \geq 2$. 
In a certain theory, we consider the following:

\[ f(x) = \frac{1}{x} \] for \( x \neq 0 \).

Using this function, we define a sequence \( \{a_n\} \) as follows:

\[ a_n = f(a_{n-1}) \]

for \( n \geq 1 \). We are interested in finding the behavior of this sequence for different initial values of \( a_1 \).

In particular, we are interested in the case where \( a_1 = 1 \). We observe that:

\[ a_1 = f(a_0) \]

For the initial value \( a_0 = 1 \), we have:

\[ a_1 = f(1) = \frac{1}{1} = 1 \]

and

\[ a_2 = f(a_1) = f(1) = \frac{1}{1} = 1 \]

We conjecture that for any \( a_0 = 1 \), the sequence \( \{a_n\} \) converges to 1.

To prove this, we use induction on \( n \).

For the base case, when \( n = 1 \), we have already shown that \( a_1 = 1 \).

Assume that for some \( k \geq 1 \), we have \( a_k = 1 \).

We need to show that \( a_{k+1} = 1 \).

\[ a_{k+1} = f(a_k) = f(1) = \frac{1}{1} = 1 \]

Thus, by induction, the sequence \( \{a_n\} \) converges to 1 for any initial value \( a_0 = 1 \).
Note: If $B = \ast$, the identification

$$[S^n, Y] \cong [(S^n \wedge \Sigma, S^n^{-1}), (Y, \ast)]$$

is $[f] \sim_\sim [g \circ j]$ for every pointed map $j: (S^n, \ast) \rightarrow (Y, \ast)$. Thus, $h_n \circ [g] = (j \circ \pi_n) \circ (\lambda_n) = j \circ \pi_n \circ (\lambda_n) = j \circ \pi_n (k_n)$.

When $B = \ast$, also define $h_0: \Pi_0 (Y, \ast) \rightarrow \Pi_0 (Y, \ast)$ by $h_0 [f] = f_\ast (k_0)$, $f: (S^0, \ast) \rightarrow (Y, \ast)$.

Prop. $h_n$ is natural for all $n$. Moreover, if $(Y, B)$ is a pointed pair,

$$\begin{bmatrix} \Pi_n (Y, B) & \to & \Pi_{n-1} (B, \ast) \\ h_n & \downarrow & h_{n-1} \\ H_n (Y, B) & \to & H_{n-1} (B, \ast) \end{bmatrix}$$

commutes for all $n \geq 1$,

where top $\partial$ is homotopy seq. of $(Y, B)$, bottom $\partial$ from homology seq. of triple $(Y, B, \ast)$.

Proof: Suppose $n \geq 1$ and $g: (X, A) \rightarrow (Y, B)$ a pointed map of top pairs. If $f: (S^n \wedge I, S^n^{-1}) \rightarrow (X, A)$ is pointed, we have $g_{\ast} [f] = [g \circ f]$ and $h_n = g_{\ast} h_n \circ (\lambda_n) = g_{\ast} f_\ast (\lambda_n) = h_n [g \circ f]$. So $h_n$ is nat. Similarly $h_0$ is natural.

Let $n \geq 1$ and $f: (S^n \wedge I, S^n^{-1}) \rightarrow (Y, B)$ a pointed map.

Then $p_{\ast} [f]$ is again, by $f: (S^n \wedge I, S^n^{-1}, \ast) \rightarrow (B, \ast)$, $f$ yields a map of triples $(S^n \wedge I, S^n^{-1}, \ast) \rightarrow (Y, B, \ast)$, and by unit. of homology seq. of a triple $\partial f \circ \lambda_n = (f \mid S^n^{-1}) \circ \lambda_n$.

Hence, $h_{n-1} \circ [f] = h_{n-1} \circ [f] = (f \mid S^n^{-1}) \circ (\lambda_n) = h_{n-1} [f]$.

Next: Show when $\Pi_n$ is group-valued, $h_n$ is a group hom.

Lemma: Let $n \geq 1$ and $\lambda': (S^n, \ast) \rightarrow (S^n \wedge S^n, \ast)$ the counit. Let $i_1, i_2: (S^n, \ast) \rightarrow (S^n \wedge S^n, \ast)$ denote the inclusions on the 1st and 2nd leaves, resp. Then $\lambda'(k_n) = i_1 \ast k_n + i_2 \ast k_n$. 


Proof: \( H_n(S^n \vee S^n, \ast) \) is free abelian with basis \( i_{x_1}^n \ast_{K_n}, i_{x_2}^n \ast_{K_n} \).

Hence \( i_{x_1}^n(K_n) = a i_{x_1}^n \ast_{K_n} + b i_{x_2}^n \ast_{K_n} \) for unique \( a, b \in \mathbb{Z} \).

Let \( \pi_i : (S^n \vee S^n, \ast) \to (S^n, \ast) \) denote projection on \( i \)-th factor, \( i = 1, 2 \),

and \( \sigma_i : (S^n \times S^n, \ast) \to (S^n, \ast) \) projection on \( i \)-th factor, \( i = 1, 2 \).

Following commutes up to homotopy:

\[
\begin{array}{ccc}
  (S^n \vee S^n, \ast) & \xrightarrow{\pi_1} & (S^n, \ast) \\
  \downarrow & & \downarrow \\
  (S^n \times S^n, \ast) & \xrightarrow{\sigma_1} & (S^n, \ast)
\end{array}
\]

Moreover, \( \pi_i \ast_1 = \pi_2 \ast_2 = 1_{S^n} \), \( \pi_1 \ast_2 = \pi_2 \ast_1 = \ast \),

\( \sigma_1 \ast = \sigma_2 \ast = 1_{S^n} \). Hence \( \pi_i \ast_1 = 1 \), \( \pi_1 \ast_2 = 0 \), \( \sigma_1 \ast \sigma_2 = 1 \).

and \( \ast \ast_1 K_n = \pi_1 \ast_1 K_n = \pi_2 \ast_1 K_n \ast_2 \ast_1 K_n \ast_2 = \ast K_n \ast_2 \) and so \( \ast \ast_1 = 1 \). Similarly, \( \ast \ast_2 = 1 \).

Prop: Let \( n \geq 1 \). Then \( H_n : \Pi_n(Y, \ast) \to H_n(Y, \ast) \) is a group hom.

Proof: Let \( f, g : (S^n, \ast) \to (Y, \ast) \). Then

\( [f] + [g] = [f \circ g] \) when \( f \circ g \) is the composition

\[
\begin{array}{ccc}
  (S^n, \ast) & \xrightarrow{\mu'} & (S^n \vee S^n, \ast) \\
  \downarrow & & \downarrow \\
  (S^n, \ast) & \xrightarrow{f \circ g} & (Y \vee Y, \ast) \xrightarrow{\varphi} (Y, \ast)
\end{array}
\]

Following commutes:

\[
\begin{array}{ccc}
  (S^n, \ast) & \xrightarrow{f} & (Y, \ast) \\
  \downarrow & & \downarrow \\
  (S^n \vee S^n, \ast) & \xrightarrow{f \circ g} & (Y \vee Y, \ast) \xrightarrow{\varphi} (Y, \ast)
\end{array}
\]

Hence \( h_n([f] + [g]) = h_n([f \circ g \ast]) = (f \circ g \ast)(K_n) = \ast_2 (f \circ g \ast)(K_n) \ast_1 \)

\( = \ast_2 (f \circ g \ast)(i_{x_1}^n \ast_{K_n} + i_{x_2}^n \ast_{K_n}) = f \ast_{K_n} + g \ast_{K_n} \)

\( = h_n([f]) + h_n([g]) \).
Note: For any pointed \( Y \), \( P(Y; *, Y) \) is contractible in the pointed sense. In fact \( h: P(Y; *, Y) \times I \to P(Y; *, Y) \) given by \( h(\lambda, s)(x) = A(st) \) is a pointed homotopy from \(*\) to \(*\).

**Proof:** \( In n \geq 2, \ h_n : \pi_n (Y, B) \to H_n (Y, B) \) is a group homomorphism.

**Proof:** Have pointed fibration \( p : P(Y; *, Y) \to Y \) and \( p^{-1}(B) = P(Y; *, B) \). Have commutative diagram

\[
\begin{array}{ccc}
\pi_n (P(Y; *, Y), P(Y; *, B)) & \xrightarrow{h_n} & H_n (P(Y; *, Y), P(Y; *, B)) \\
\downarrow \cong & & \downarrow \cong \\
\pi_n (Y, B) & \xrightarrow{h_n} & H_n (Y, B)
\end{array}
\]

Here, some vertical maps are group homomorphisms to show that \( h_n \) is a group homomorphism.

Have commutative diagram

\[
\begin{array}{ccc}
\pi_n (P(Y; *, Y), P(Y; *, B)) & \xrightarrow{\tilde{h}_n} & \pi_{n-1} (P(Y; *, B), *) \\
\downarrow h_n & & \downarrow h_{n-1} \\
H_n (P(Y; *, Y), P(Y; *, B)) & \xrightarrow{\cong} & H_{n-1} (P(Y; *, B), *)
\end{array}
\]

with both \( \tilde{h} \)'s isomorphism since \( P(Y; *, Y) \) is contractible. \( h_{n-1} \) is a group homomorphism by absolute case, and so \( h_n \) is also.

The little Hurewicz Theorem

Claim: Suppose \( X \) is a path-connected pointed space. Then \( h_1 : \pi_1 (X, *) \to H_1 (X, *) \) is onto, and the kernel of \( h_1 \) in the commutative subgroup of \( \pi_1 (X, *) \).
Path composition: Suppose \( \sigma, \tau : I \to X \) are singular 1-cubes. \( \gamma \in \mathbb{R} \): \( \gamma(1) = \tau(0) \). We can form a singular 1-cube \( \sigma \ast \tau : I \to X \)

Given \( \gamma \):

\[
(\sigma \ast \tau)(t) = \begin{cases} 
\sigma(2t) & 0 \leq t \leq \frac{1}{2} \\
\tau(2t-1) & \frac{1}{2} \leq t \leq 1
\end{cases}
\]

Notation:

\( c_i, \ldots, c_n \in \mathbb{Q}^n(X) \), write \( c_i \equiv c_j - \exists \in \mathbb{Q}^n(X) \).

Then \( \mathbb{P}_i : \mathbb{Q}^n(X) \to \mathbb{Q}^n(X) \) is the projection and \( \mathbb{P}_i(c_i) \) are cycles, \( i = 1, 2 \), \( c_i \equiv c_j \Rightarrow \mathbb{P}_i(c_i) \sim \mathbb{P}_i(c_j) \)

and hence \( \{ \mathbb{P}_i(c_i) \} = \{ \mathbb{P}_i(c_j) \} \). (Write \( \mathbb{P}_i \) for homology class).

Note:

\( \mathbb{P}_i : n \geq 1 \), \( \mathbb{Q}_n(X \ast X) = \mathbb{Q}_n(X) \) some \( \mathbb{Q}_n(X) = 0 \) for \( n \geq 1 \).

Then \( \mathbb{Q}_n(X \ast X) = \mathbb{Q}_n(X) / \mathbb{D}_n(X) \) for \( n \geq 1 \). Easily checked.

Exercises:

1) Let \( \sigma, \tau : I \to X \) be any, 1-cubes \( : \sigma(1) = \tau(0) \).

Then \( \sigma \ast \tau = \tau + \sigma \).

2) There \( \sigma : I \to X \) a sing. 1-cube, let \( \sigma : I \to X \) be the sing. 1-cube given by \( \tilde{\sigma}(t) = \sigma(1-t) \). Then \( \tilde{\sigma} \equiv -\sigma \).

**Proof:**

1) \( \tilde{\sigma} \)

2) \( \tilde{\sigma}(t) = \sigma(1-t) \) when \( \mathbb{D}(\sigma(0)) = \mathbb{P}_i \).

\( \tilde{\sigma} \) are such \( \mathbb{D}(\sigma(0)) \in \mathbb{D}_i(X) \).

Then \( \partial \tilde{\sigma} = \tilde{\sigma} + \tau - \sigma \ast \tau - C_{\tau(0)} \) when \( C_{\tau(0)} = \text{contour map with value } \tau(0) \).

Define \( \mathbb{P}_i : I^2 \to X \)

\( \mathbb{P}(s,t) = \begin{cases} 
\sigma \left( \frac{2s}{2-t} \right) & 0 \leq s \leq 1 - \frac{t}{2} \\
\tau \left( t + 2s - 2 \right) & 1 - \frac{t}{2} \leq s \leq 1.
\end{cases} 
\]

Then \( \partial \mathbb{P} = \tilde{\sigma} + \tau - \tilde{\sigma} \ast \tau - C_{\tau(0)} \) and \( \mathbb{P} \equiv -\tilde{\sigma} \).

Then \( \partial \mathbb{P} = \tilde{\sigma} + \tau - \tilde{\sigma} \ast \tau - C_{\tau(0)} \) and \( \tilde{\sigma} \equiv -\tilde{\sigma} \). Then
Note: Let \( \sigma : I \to X \) denote the projection. The \( k_1 = \{ F_0(\delta) \} \) and \( p : Q_1^u(S^1) \to Q_1(S^1, *) \). Then \( F : S^1 \to X \) is a pointed map, \( h_1[F] = F_*(k_1) = F_\ast \{ p_0(\delta) \} = \{ p_0(F(\delta)) \} \).

Note: If \( \sigma : I \to X \) is a singular 1-cube and \( \sigma(0) = \sigma(1) = * \), then \( p_0(\sigma) \) is a cycle in \( Q_1(X, *) \) and \( \{ p_0(\sigma) \} \in H_1(X, *) \). Since \( h_1 \) is a quotient, \( h_1[F] = \{ p_0(F(\delta)) \} = \{ p_0(\sigma) \} \).

Proof that \( h_1 \) is exact: In each \( x \in X \), choose a path \( p_x : I \to X \) with \( p_x(0) = x \) and \( p_x(1) = * \), for every \( x \in X \), let \( p_x : I \to X \) denote the 0-cube with 1-cube \( \sigma : I \to X \) and \( \Theta : Q_0(X) \to Q_1(X) \) denote the unique homomorphism.

Let \( \Theta : Q_0(X) \to Q_1(X) \) denote the unique homomorphism.

3. \( \Theta(c_x) = \overline{c_x} \).

In each sing, 1-cube \( \sigma : I \to X \), let \( \tilde{\sigma} = (p_\sigma(\delta) \uparrow \sigma) \star \overline{p_\sigma(\delta)} \).

Then \( \tilde{\sigma} = \tilde{p_\sigma(\delta)} + \tilde{\delta} + \overline{\tilde{p_\sigma(\delta)}} \) and \( \{ p_\sigma(\tilde{\delta}) \} \in \text{im} \ h_1 \).

Let \( x \in H_1(X, *) \). Then \( x = \{ p_\sigma(\tilde{p_\sigma(\delta)}) \} \), \( x \in Z \), \( X_\sigma : I \to X \) a 1-cube collection of 1-cubes. Hence

\[
\tilde{\sigma} = (p_\sigma(\delta) \uparrow \sigma) \star \overline{p_\sigma(\delta)}.
\]

Thus \( \tilde{\sigma} = \overline{p_\sigma(\delta)} + p_\sigma(\delta) \star \overline{p_\sigma(\delta)} \).

Hence \( e\Theta = 0 \) and \( \Theta : Q_0(X) \to Z \) is an isomorphism, we must have \( \Theta(\tilde{p_\sigma(\delta)}) = 0 \). Thus \( 0 = \sum n_\sigma (c_{\sigma.0} + \overline{c_{\sigma.0}}) \), and so

\[
0 = \Theta(\sigma) = \sum \sigma_\sigma (p_{\sigma.0} - \overline{p_{\sigma.0}}).
\]

Thus \( \sum n_\sigma \sigma_\sigma = \sum \sigma_\sigma (\overline{p_{\sigma.0}} + p_{\sigma.0}) \equiv \sum \sigma_\sigma (\sigma_\sigma + \overline{p_{\sigma.0}} + p_{\sigma.0}) \equiv \sum n_\sigma \sigma_\sigma \) and so \( x = \{ p_\sigma(\tilde{p_\sigma(\delta)}) \} = \sum \sigma_\sigma \{ p_\sigma(\delta) \} \in \text{im} \ h_1 \).

Since each \( \{ p_\sigma(\delta) \} \in \text{im} \ h_1 \) and \( h_1 \) is a group homomorphism.
Write $\Gamma = \text{commutator subgroup of } \pi_1(X,*)$. Since $\pi_1(X,*)$ is abelian, $\Gamma \leq \text{ker } \pi$. Let $\pi: \pi_1(X,*) \to \pi_1(X,*)/\Gamma$ be the projection. To complete the proof of the Lefschetz fixed-point theorem, it suffices to show if $\Gamma$ group homomorphism

$$ x: \pi_1(X,*) \to \pi_1(X,*)/\Gamma $$

then

$$ \pi_1(X,*) \xrightarrow{\pi} \pi_1(X,*)/\Gamma $$

is commutative (in some $k$, $\pi = \Gamma$, it would then follow that $\text{ker } \pi = \Gamma$).

Path-continuity and coincidences.

Let $\sigma, \tau: I \to X$ be any 1-curves, $\tau(0) = \tau(1)$, $\sigma(0) = \tau(0)$, $\sigma(1) = \tau(1)$; we say $\sigma$ is homotopic to $\tau$ rel. endpoints (written $\sigma \simeq_0 \tau$) if $\exists$ homotopy $h: I \times I \to X$ s.t.

$$ h(t,0) = \tau(t), \quad h(t,1) = \tau(t), \quad h(0,s) = \tau(0), \quad h(1,s) = \sigma(1) = \tau(1) $$

for all $t, s$. Write $\sigma \simeq_0 \tau$ in this case.

Some proof that $\simeq$ is an equivalence relation yields $\simeq_0$ is an equivalence relation.

Homotopy respecting of path composition

Theorem: 1) $\eta, \sigma, \tau: I \to X$ are any 1-curves, $\tau(0) = \tau(1)$, then $\tau \simeq_0 \tau_1 \simeq_0 \tau_2 \simeq_0 \tau_1 \tau_2$.

2) $\eta, \sigma, \mu: I \to X$ satisfy $\eta(0) = \tau(0), \tau(1) = \mu(0)$, then $(\eta \simeq_0 \tau) \simeq_0 (\tau \simeq_0 \mu)$.

3) $\sigma \simeq_0 \tau$ then $C_\sigma \simeq_0 C_\tau$.

4) $C_{\sigma(0)} \simeq_0 C_{\tau(0)}, C_{\sigma(1)} \simeq_0 C_{\tau(1)}$.

Proof: 1) Say $\tau_1 \simeq_0 \tau_2$ then $\eta \simeq_0 \eta \simeq_0 \tau_2$. Define

$$ \lambda: I \times I \to X \text{ by } (t,s) \mapsto \begin{cases} h(t,\tau(0)) & 0 \leq t \leq \frac{1}{2} \\ k(t, \tau(1)) & \frac{1}{2} \leq t \leq 1. \end{cases} $$

Then $\tau_1 \times_0 \tau_2 \simeq_0 \tau_1 \times_0 \tau_2$.
2) Define \( h : \mathbb{I} \times \mathbb{I} \to X \) by

\[
h(t, s) = \begin{cases} 
\sigma \left( \frac{4t}{1+s} \right) & \text{if } 0 \leq t \leq \frac{1}{4} (1+s) \\
\tau \left( 4t-1-s \right) & \text{if } \frac{1}{4} (1+s) \leq t \leq \frac{1}{4} (2+s) \\
\varepsilon \left( \frac{4t-5-2}{2-s} \right) & \text{if } \frac{1}{4} (2+s) \leq t \leq 1.
\end{cases}
\]

Then \((\sigma \times \tau) \ast h \xrightarrow{\sim} \sigma \ast (\tau \times \varepsilon)\).

3) Define \( h : \mathbb{I} \times \mathbb{I} \to X \) by

\[
h(t, s) = \begin{cases} 
\tau \left( 2t (1-s) \right) & \text{if } 0 \leq t \leq \frac{1}{2} \\
\overline{\tau} \left( 2t-1+s (2-t) \right) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

Then \( \overline{\sigma} \ast \overline{\tau} \xrightarrow{\sim} \overline{\sigma} \ast \overline{\tau} \). Similarly \( \overline{\sigma} \xrightarrow{\sim} \overline{\tau} \ast \varepsilon_{0(1)} \).

4) Define \( h : \mathbb{I} \times \mathbb{I} \to X \) by

\[
h(t, s) = \begin{cases} 
\delta (1) & \text{if } 0 \leq t \leq \frac{5}{2} \\
\sigma \left( \frac{2t-5}{2-s} \right) & \text{if } \frac{5}{2} \leq t \leq 1.
\end{cases}
\]

Then \( \overline{\sigma} \xrightarrow{\sim} \overline{\sigma} \ast \overline{\tau} \). Similarly \( \overline{\sigma} \xrightarrow{\sim} \overline{\tau} \ast \varepsilon_{0(1)} \).

If \( \sigma : \mathbb{I} \to X \) is a path, write \( <\sigma> = \text{path homotopy class rel. end points of } \sigma \).

Suppose \( \sigma_1, \ldots, \sigma_n : \mathbb{I} \to X \) are paths \( \Rightarrow \sigma_n (1) = \sigma_{n-1} (0) \), \( 1 \leq n \leq n-1 \). Then because \( \epsilon \) is convex, \( \ast = \overline{\sigma} \).

\( <\sigma_1 \ast \cdots \ast \sigma_n> \) is unambiguous, indep. of parenthesization.

If \( \sigma : \mathbb{I} \to X \) satisfies \( \delta (0) = \delta (1) = x \), then \( \overline{\sigma} \) factors as

\[
\begin{array}{ccc}
\mathbb{I} & \xrightarrow{\delta} & \mathbb{I}/\partial \mathbb{I} = \mathbb{I}' \\
\overline{\sigma} & \searrow & \swarrow \overline{\tau} \\
\times & \nearrow & X
\end{array}
\]

where \( \overline{\sigma} \) is a pointed map. Note that \( [\overline{\sigma}] \in \mathbb{M}_1 (X, x) \).
depends only on \( \langle \tau \rangle \), and if \( \tau_1, \ldots, \tau_n : I \to X \) satisfy
\[
\tau_i(0) = \ast, \quad \tau_i(1) = \tau_{i+1}(0), \quad 1 \leq i \leq n-1, \quad \text{and} \quad \tau_n(1) = \ast,
\]
then
\[
\left[ \tau_1 \ast \cdots \ast \tau_n \right] \in \pi_1(X, \ast) \text{ is unambiguously defined}.
\]

(indep. of parenth.) If \( \sigma, \tau : I \to X \) both send \( 0, 1 \) to \( \ast \),
it is immediate that \( \sigma \ast \tau = \sigma \sqcup \tau \) and so
\[
\left[ \sigma \ast \tau \right] = \left[ \sigma \right] \left[ \tau \right].
\]

Note also \( \overline{\sigma} = \sigma \gamma \) and so
\[
\left[ \overline{\sigma} \right] = \left[ \sigma \right]^{-1}. \quad \text{Also} \quad \left[ \overline{\ast} \right] = \text{null elt of } \pi_1(X, \ast) \).
\]

Note:
\[
\overline{\sigma \ast \tau} = \overline{\sigma} \ast \overline{\tau}, \quad \overline{\sigma} = \overline{\ast}.
\]

In this notation, \( H_1 \left[ \tau \right] = \{ P_1(\tau) \} \), since \( \tau = \overline{\ast \gamma} \).

Construction of \( \lambda : H_1(X, \ast) \to \pi_1(X, \ast)/\Gamma \):

As before, for each \( x \in X \), choose a path \( \tilde{\rho} : I \to X \)
from \( \ast \) to \( x \). Define a homomorphism \( \beta : \tilde{\rho} \gamma(\tilde{\rho}) \to \pi_1(X, \ast)/\Gamma \)
as follows: For each sing. 1-cube \( \tau : I \to X \), define
\[
\beta(\tau) = \left[ \tilde{\rho}_{\tau(0)} \ast \overline{\tau} \ast \tilde{\rho}_{\tau(1)} \right] \Gamma.
\]

(OK since \( \pi_1(X, \ast)/\Gamma \)
is abel. and the sing. 1-cubes form a free abel. basis of \( Q_1(X) \)).

If \( \tau \) is degenerate, then \( \tau = \tilde{\rho}_x \) for some \( x \in X \), and
\[
\langle \langle \tilde{\rho}_x \ast \tilde{\rho}_x \rangle \rangle = \langle \langle \tilde{\rho}_x \rangle \rangle = \langle \langle \ast \rangle \rangle \text{ and hence}
\]
\[
\beta(\tilde{\rho}_x) = \left[ \ast \right] \Gamma = 1 \in \pi_1(X, \ast)/\Gamma.
\]

Thus, passing to
\[
y : Q_1(X) = Q_1(X, \ast) \to \pi_1(X, \ast)/\Gamma.
\]

Let
\[
\overline{\delta} = y|Z_1(Q(X)) : Z_1(Q(X)) \to \pi_1(X, \ast)/\Gamma.
\]

Claim: \( \overline{\delta} (B_1(Q(X^1))) = 1 \in \pi_1(X, \ast)/\Gamma \).

In supposing \( \mu : I^2 \to X \) is a singular 2-cube.
Suppose to show \( \beta(\Theta \mu) = 1 \).
We have
\[
C(\partial \mu) = C(\mu^{(1)} - \mu^{(2)} - \mu^{(3)} + \mu^{(4)})
\]
\[
= C(\mu^{(1)})^{-1} C(\mu^{(2)})^{-1} C(\mu^{(3)})^{-1} C(\mu^{(4)})^{-1}
\]
where \(\mu(0, 0) = A, \mu(0, 1) = B, \mu(1, 0) = C, \mu(1, 1) = D.
\]

Then
\[
C(\partial \mu) = [P_A \times \mu^{(2)} \times \overline{P_B}] [P_B \times \mu^{(1)} \times \overline{P_C}] [P_D \times \mu^{(3)} \times \overline{P_0}] [P_A \times \mu^{(4)} \times \overline{P_D}]^{-1}
\]
\[
= \left[ P_A \times \mu^{(2)} \times \overline{P_B} \times \mu^{(1)} \times \overline{P_C} \times \mu^{(3)} \times \overline{P_D} \times \mu^{(4)} \times \overline{P_A} \right]^{-1}
\]
\[
= \left[ P_A \times \mu^{(2)} \times \overline{P_B} \times \mu^{(1)} \times \overline{P_C} \times \mu^{(3)} \times \overline{P_D} \times \mu^{(4)} \times \overline{P_A} \right]^{-1}
\]
\[
= \left[ P_A \times (\mu^{(2)} \times \overline{\mu^{(1)}}) \times (\mu^{(3)} \times \overline{\mu^{(4)}}) \times \overline{P_A} \right]^{-1}
\]
\[
\text{But } \mu^{(2)} \times \overline{\mu^{(1)}} = \mu^{(3)} \times \overline{\mu^{(4)}}.
\]
Thus, writing \(\Sigma = \mu^{(1)} \times \mu^{(2)}
\)
\[
C(\partial \mu) = [P_A \times \Sigma \times \overline{P} \times \overline{P_A}]^{-1} = [\Sigma \times \overline{P}]^{-1}
\]
Thus, passing to quotients, \(\Sigma
\)

Therefore, a group homomorphism
\[
\alpha : H_1(X, \pi) \to \Pi_1(X, \pi)/\Gamma \quad \text{given by } \alpha \{p, c\} = \beta(c)
\]
\[
\text{We have } \alpha \cdot H_1[\Sigma] = \alpha \{\pi(1)\} = \beta(\overline{\pi})
\]
\[
= [P_A \times \overline{\Sigma} \times \overline{P}]^{-1} \Gamma = [\Sigma \times \overline{P}]^{-1} \Gamma = p[\Sigma]
\]
completing the proof.
Def: \( Y \) a pointed space, \( n \geq 0 \). \( Y \) is \( n \)-connected if \( \pi_i(Y, \ast) = 0 \) for \( 0 \leq i \leq n \).

Then \( 0 \)-connected = path-connected.

1-connected is also called simply-connected.

Def: \((Y, B)\) a pointed pair, \( n \geq 1 \). \((Y, B)\) is \( n \)-connected if \( Y \) is 0-connected and \( \pi_i(Y, B) = 0 \) for \( 1 \leq i \leq n \).

**The Hurewicz Theorem**

Then (Hurewicz): \((Y, B)\) a pointed pair with both \( Y \) and \( B \) simply-connected. Suppose, for some \( n \geq 1 \), \((Y, B)\) is \( n \)-connected. Then \( h_i: \pi_i(Y, B) \rightarrow H_i(Y, B) \) is a bijection for \( 1 \leq i \leq n+1 \), \( \text{i.e.} H_i(Y, B) = 0 \) for \( 1 \leq i \leq n \) and \( h_{n+1}: \pi_{n+1}(Y, B) \rightarrow H_{n+1}(Y, B) \) is an isomorphism.

Cor: If \( Y \) is simply-connected and \( H_i(Y, \ast) = 0 \) for \( i \leq n \), then \( \pi_i(Y, \ast) = 0 \) for \( i \leq n \) and \( h_n: \pi_{n}(Y, \ast) \rightarrow H_{n}(Y, \ast) \) is an isomorphism.

Example: \( S^n, n \geq 2 \), is simply-connected by Van Kampen's theorem. \( H_i(S^n, \ast) = 0 \) for \( i < n-1 \). Hence \( \pi_n(S^n, \ast) = 0 \) for \( i \leq n \) and \( h_n: \pi_{n}(S^n, \ast) \rightarrow H_{n}(S^n, \ast) \cong \mathbb{Z} \) is an isomorphism.

Start of proof of Hurewicz theorem: Induction on \( n \). Case \( n = 1 \):

By definition Hurewicz then, \( h_1: \pi_1(Y, \ast) \rightarrow H_1(Y, \ast) \) and \( h_1: \pi_1(B, \ast) \rightarrow H_1(B, \ast) \) are both onto, so \( H_1(Y, \ast) = 0 = H_1(B, \ast) \). From homology seq of pairs \((Y, B, \ast)\), follows easily that \( H_2(Y, B) = 0 \) for \( i \leq 1 \). Have comm. diagram

\[
\begin{array}{ccc}
\pi_2(Y, B) & \xrightarrow{\partial} & \pi_2(\mathbb{P}(Y, \ast), Y, \mathbb{P}(Y, \ast, B)) \\
\downarrow h_2 & & \downarrow h_2 \\
H_2(Y, B) & \xrightarrow{\partial} & H_2(\mathbb{P}(Y, \ast), Y, \mathbb{P}(Y, \ast, B))
\end{array}
\]

From exactness of \( \pi_2(Y, \ast) \rightarrow \pi_2(Y, B) \rightarrow \pi_1(B, \ast) \) and fact that \( \pi_2(Y, \ast) \) is abelian, follows that \( \pi_2(Y, B) \) is abelian.
Hence \( \prod_i (P(Y^*, B), *) \) is abelian. From the definition

\[
\mathcal{O} Y \rightarrow P(Y^*, B)
\]

\[
\prod_0 (\mathcal{O} Y, *) \rightarrow \prod_0 (P(Y^*, B), *) \rightarrow \prod_0 (B, *) \quad \text{exact}
\]

\[
\prod_1 (Y, *) \quad 0
\]

It follows that \( P(Y^*, B) \) is path-connected.

Hence, by the Hurewicz theorem, \( \pi_1 \) is an isomorphism.

Hence, a proof for \( n=1 \) could be completed by

**Lemma (proof defined):** If \((Y, B)\), \(Y, B\) are 1-connected, then \( \pi_1 : H_2 (P(Y^*, Y), P(Y^*, B)) \rightarrow H_2 (Y, B) \) is an isomorphism.

Now suppose \( n > 1 \) and the Hurewicz theorem is true in lower dimensions. Then \( H_n : \prod_n (Y, B) \rightarrow H_n (Y, B) \) is an isomorphism (so \( H_n (Y, B) = 0 \) for \( 1 \leq i \leq n \)). Hence, consider the following diagram:

\[
\begin{array}{ccc}
\prod_{n+1} (Y, B) & \xrightarrow{p_{n+1}} & \prod_{n+1} (P(Y^*, Y), P(Y^*, B)) \\
\downarrow h_{n+1} & & \downarrow h_{n+1} \\
H_{n+1} (Y, B) & \xrightarrow{p_n} & H_{n+1} (P(Y^*, Y), P(Y^*, B))
\end{array}
\]

Since \( \pi_n (P(Y^*, B), *) \cong \prod_{n+1} (Y, B) \) for \( n \geq 1 \) and (as before) \( P(Y^*, B) \) is path-connected, it follows that \( P(Y^*, B) \) is \( n-1 \)-connected (hence at least 1-connected). Hence, by hypothesis, \( \pi_n \) is an isomorphism. Hence, a proof will be completed by

**Lemma (proof defined):** Suppose \( Y, B \) are 1-connected and for some \( n \geq 1 \), \( H_n (Y, B) = 0 \) for \( n \leq n \). Then \( \pi_n : H_{n+1} (P(Y^*, Y), P(Y^*, B)) \rightarrow H_{n+1} (Y, B) \) is an isomorphism.

**Note:** 2nd defined Lemma includes 1st.
Theorem (J. H. Whitehead): \( X, Y \) simply-connected, \( f : (X, x_0) \to (Y, y_0) \) a pointed map. Let \( n \geq 1 \). Then following are equivalent:

i) \( f_* : \pi_k(X, x_0) \to \pi_k(Y, y_0) \) is an isomorphism for \( k \leq n \) and onto for \( k = n+1 \).

ii) \( f_* : \text{Hom}(X) \to \text{Hom}(Y) \) is an isomorphism for \( k \leq n \) and onto for \( k = n+1 \).

Proof: Let \( \text{Cyl}(f) = \) mapping cylinder of \( f = \) quotient space obtained from \( (X \times I) \sqcup Y \) by identifying \((x, 0) \in X \times I \) with \( f(x) \in Y \) and \((x_0, t) \in X \times I \) with \( y_0 \) for all \( t \in I \). Let \( \pi : (X \times I) \sqcup Y \to \text{Cyl}(f) \) be the quotient map, \( \text{Cyl}(f) \) is pointed with basepoint \( \pi(y_0) \) (which is \( \pi(x_0, t) \) for all \( t \in I \)).

We have pointed inclusions

\[ i : X \to \text{Cyl}(f) \]
\[ x \mapsto \pi(x, 1) \]

and \( j : Y \to \text{Cyl}(f) \) and a pointed map

\[ r : \text{Cyl}(f) \to Y \text{ given by } r(\pi(y)) = y , \quad r(\pi(x, t)) = f(x) \]

\( i \) and \( r \) are pointed homotopy equivalences, unique to each other. In fact, \( r \circ i = 1_Y \) and \( i \circ r \sim \text{homotopy} \) where

\[ h : \text{Cyl}(f) \times I \to \text{Cyl}(f) \text{ is given by } h(\pi(y), t) = \pi(y) , \]
\[ h(\pi(x, s), t) = \pi(x, s \cdot t) . \]

Moreover, the diagram

\[ \begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow i & & \downarrow r \\
\text{Cyl}(f) & \xrightarrow{c} & \text{Cyl}(f)
\end{array} \]

commutes.

Identify \( X \) as a subspace of \( \text{Cyl}(f) \) with \( Y \). (Thus, base pt. of \( \text{Cyl}(f) \) is identified with \( x_0 \).) Thus, since \( r \) induces \( \sim \) in homology and homotopy, \( (c) \) is equiv to

\[ (c') \quad i_* : \pi_k(X, x_0) \to \pi_k(\text{Cyl}(f), x_0) \text{ is an isomorphism for } k \leq n \]
and onto for \( k = n+1 \).
and (ii) is equivalent to

\[(ii)' \quad i^*: H_k(X) \rightarrow H_k(Cyl(F)) \text{ is an isomorphism for } k \leq n\] and is an isomorphism for \( k = n+1. \)

Then homotopy seq. of \((Cyl(F), X), (ii)\) is equal to

\[(ii)'': \quad \text{Tor} (H_k(Cyl(F), X)) = 0 \quad \text{for } k \leq n+1.\]

Then homotopy seq. of \((Cyl(F), X), (ii)\) is equal to

\[(ii)''': \quad H_k(Cyl(F), X) = 0 \quad \text{for } k \leq n+1.\]

By Hensel's theorem, \((i)'' \implies (ii)'''.\)

Motivation for spectral sequences
Consider the case of a product bundle
\(T = B \times F \rightarrow B.\) By the Künneth theorem and the U.C.T.,

\[H_n(T) \cong \bigoplus_{p+q=n} H_p(B; H_q(F)) = \bigoplus_{p+q=n} E_{p,q}.\]

In the case of a general fibration \(F \rightarrow T \rightarrow B,\)

it is not true that \(H_n(T) \cong \bigoplus_{p+q=n} E_{p,q}.\) However, \(\bigoplus_{p+q=n} E_{p,q}\)

is a first approximation to \(H_n(T).\) It turns out that

\[E = \bigoplus_{p+q=n} E_{p,q}\]

has a differential \(d: E \rightarrow E\) (which lowers degree by 1), and the homology

\[\text{ker} d = E' = \bigoplus_{p+q=n} E'_{p,q}\]

is a better approximation to \(H_n(T).\) (In case of a product bundle, \(E'\) is the cone of a differential.) \(E'' = \text{ker} d''\) as an even better approximation to \(H_n(T),\) etc.
**Spectral Sequences**

Let \( R \) be a commutative ring with \( 1 \neq 0 \).

**Def:** A **bi-graded** \( R \)-module \( E \) is a family \( \{ E_{p,q} \} \) of \( R \)-modules, where \( p, q \in \mathbb{Z} \).

If \( E = \{ E_{p,q} \} \) is another bi-graded \( R \)-module, a *morphism* of bi-graded \( R \)-modules \( f: E \to F \) consists of a family \( f_{p,q}: E_{p,q} \to F_{p,q} \) of \( R \)-homomorphisms.

Can form the *cot of bi-graded* \( R \)-modules.

If \( E = \{ E_{p,q} \} \) a bi-graded \( R \)-module, can associate with \( E \) the *singly-graded* \( R \)-module \( \{ E_{n} \} \), where \( E_{n} = \bigoplus_{p+q=n} E_{p,q} \).

Elements of \( E_{p,q} \) are said to have *bi-degree* \((p,q)\) and *total degree* \( p+q \).

**Def:** If \( E = \{ E_{p,q} \} \) a *differential bi-degree* \((-r,-r-1)\) is a family of homomorphisms \( d: E_{p,q} \to E_{p-r,q+r-1} \) such that \( d^2 = 0 \), i.e. the compositions

\[
E_{p,q} \xrightarrow{d} E_{p-r,q+r-1} \xrightarrow{d} E_{p-2r,q+2r-2} \quad \text{are all 0.}
\]

Note: Such a \( d \) becomes total degree \( -r+1 \). \( (E,d) \) will be called a \((-r,-r-1)\) differential bi-graded module.

**Def:** If \( (E,d) \), \( (F,d) \) are \((-r,-r-1)\) diff. bi-graded modules, a *morphism* of \((-r,-r-1)\) diff. bi-graded modules \( f: (E,d) \to (F,d) \) consists of a morphism of bi-graded \( R \)-modules:

\[
\begin{array}{ccc}
E_{p,q} & \xrightarrow{f_{p,q}} & F_{p,q} \\
\downarrow d & & \downarrow d \\
E_{p-r,q+r-1} & \xrightarrow{f_{p-r,q+r-1}} & F_{p-r,q+r-1} \\
\end{array}
\]

commutes for all \( p,q \).
Can form out of homology spectral sequences.

Def. A spectral sequence $E$ is called a first quadrant spectral sequence if $E_{p,q} = 0$ if $p < 0$ or $q < 0$. 

Def. A homology spectral sequence $E = \{ (E^r, d^r) \}$ consists of a sequence $\{ (E^r, d^r), (E^{r+1}, d^{r+1}), (E^{r+2}, d^{r+2}), \ldots \}$ such that

1) $(E^r, d^r)$ is a $(r, r-1)$-diff. Augmented module, $r = 2, 3, \ldots$
2) For each $r \geq 2$, there is given an isomorphism $\Theta^r: H(E^r, d^r) \cong E^{r+1}$ of Augmented $R$-modules.

Thus $(E^r, d^r)$ determines the Augmented $R$-module $E^{r+1}$, but does not determine $d^{r+1}$.

Def. If $E = \{ (E^r, d^r) \}$ and $\{ \tilde{E}^r, \tilde{d}^r \}$ are homology spectral sequences, a morphism of spectral sequences $\tau: E \to \tilde{E}$ consists of a sequence $\tau^r: (E^r, d^r) \to (\tilde{E}^r, \tilde{d}^r)$ of $(r, r+1)$-diff. Augmented $R$-module homomorphisms, $r = 2, 3, \ldots$ such that

$$H(E^r, d^r) \xrightarrow{H(\tau^r)} H(\tilde{E}^r, \tilde{d}^r)$$

commutes.
Note: if $E_{p,q} = 0$ for $p < 0$ or $q < 0$, then $E$ is a 1-st quadrant spectral sequence. More generally, if $E_{p,q} = 0$ for some $p, q, r$, then $E_{p+r, q} = 0$ for $s > r$.

It is useful to picture a bigraded module $E$ by associating with each lattice point $(p, q)$ in the plane the module $E_{p,q}$. Then if $E$ is a spectral sequence, the differentials are pictured as follows.

![Diagram of a spectral sequence]

Let $E$ be a 1-st quadrant homology spectral sequence. Note that if $r > p$, then $E_{p+r, q+r+1} = 0$ since $p-r < 0$, and so $\ker \left[ d^r: E_{p+r, q+r+1} \to E_{p+r+1, q+r} \right] = E_{p+r, q+r+1}$.

Also if $r > q+1$, then $E_{p+r, q-r+1} = 0$ since $q-r+1 < 0$ and so $\ker \left[ d^r: E_{p+r, q-r+1} \to E_{p+r, q-r+1} \right] = 0$.

Hence $H_{p+q} (E^r, d^r) \cong E_{p+r, q+r} \cong E_{p+q}^r$ if $r > p, q+1$.

Hence $E_{p+q}^r \cong E_{p+q}^r \cong E_{p+q}^r \cong E_{p+q}^r \cong \cdots$ if $r > p, q+1$.

Define $E_{p+q} = E_{p+q}^r$, $r > \max \{ p, q+1 \}$, and $E^\infty = \{ E_{p+q} \}$; $E^\infty$ is a bigraded $R$-module.

If $f: E \to \overline{E}$ is a morphism of 1-st quadrant homology spectral sequences, it induces $f^\infty: E^\infty \to \overline{E}^\infty$, a morphism of bigraded modules, defined by
\[ f_{p, g}^r = f_{p, g}^r : E_{p, g}^r \to E_{p, g}^r, \quad r > \text{max } \{p, q+1\} \]

It does not matter which \( r > \text{max } \{p, q+1\} \) we choose since

\[ E_{p, g}^r \xrightarrow{f_{p, g}^r} E_{p, g}^r \]

\[ E_{p, g}^{r+1} \xrightarrow{f_{p, g}^{r+1}} E_{p, g}^{r+1} \]

counters.

\( E \to E^\infty \) is a covariant functor from cmt. of 1st grad.

homology spectral sequence to cmt. of bigraded modules.

\textbf{Then (Dwyer–Kan)} (1st version): Let \( F \xrightrightarrows T \to B \) be a pointed

fibration with 1-connected base \( B \). Let \( B' \) be any subspace

of \( B \), and \( T' = p'(B') \). (\( B' \) need not contain the base point of \( B \).

In particular, allow \( B' = \emptyset \). Let \( R \) be any commutative ring

with \( 1 \neq 0 \). Then \( 1 \) st grad. homology spectral sequence

\( E \) of \( R \)-modules, and \( J_n \) for each \( n \neq 0 \) a suited family

of \( R \)-modules,

\[ 0 = J_{-1, m+1} \subset J_{0, m} \subset \cdots \subset J_{n-1, 1} \subset J_{n, 0} = H^0(T, T'; R) \]

such that

\[ E_2^{p, q} \cong H_p(B, B'; H_q(F; R)) \quad \text{and} \quad E^\infty_{p, q} \cong J_{p, q}/J_{p-1, q+1}. \]

Proof will be given later.

\textbf{Application:} \( \Sigma Y \) a pointed space, write \( P \Sigma Y = P(Y, *, Y) \).

We have a pointed fibration \( \Sigma Y \to P \Sigma Y \to Y \) with \( P \Sigma Y \)

contractible.

Let \( n \geq 2 \) and consider \( \Sigma S^n \to P \Sigma S^n \to S^n \).

\( S^n \) is 1-connected, so \( 1 \) st grad. homology spectral seq. \( E \): \( E_2^{p, q} = H_p(S^n; H_q(\Sigma S^n)) \), \( E^\infty_{p, q} = J_{p, q}/J_{p-1, q+1} \) where

\[ 0 = J_{-1, m+1} \subset J_{0, m} \subset \cdots \subset J_{m, 0} = H^0(P \Sigma S^n). \]

If \( m > 0 \), \( H^0(PS^n) = 0 \) since \( PS^n \) is contractible.

Hence \( J_{-1, m+1} = 0 \) if \( p+q > 0 \), and so \( E^\infty_{p, q} = 0 \) if

\( (p, q) \neq (0, 0) \).
By the universal coeff. theorem,

\[ H_p(S^n; H_q(S^n)) = \begin{cases} H_b(S^n) & p = 0 \text{ or } n \\ 0 & p \neq 0, n \end{cases} \]

Hence \( E_{p,q}^2 = 0 \) unless \( p = 0 \) or \( p = n \), and

\[ E_{p+1, q} \cong H_b(S^n), \quad E_{p-1, q} \cong H_b(S^n). \]

Thus \( E_{p,q}^r = 0 \) unless \( p = 0 \) or \( n \).

Note that if \( r \neq n \), every \( d_{p,q}^r \) must be 0 since one of \( E_{p,q}^r \), \( E_{p-1,q+1}^r \) is 0. Hence \( E_{p,q}^{r+1} \cong E_{p,q}^{r+2} \cong \ldots \cong E_{p,q}^\infty \) for all \( p, q \).

Then for \( (p, q) \neq (0, 0) \), \( E_{p,q}^{\infty} = 0 \).

For each \( b \in \mathbb{Z} \),

\[ E_{p+1, q+n-1}^{n+1} \cong E_{p, q+n}^{n+1} \text{ and } d_{p,q}^n \]

so

\[ E_{p,q}^n \Rightarrow E_{p,q+n+1}^n \text{ is an isomorphism.} \]

Since \( E_{p,q}^{n+1} \cong E_{p,q}^{n+2} \cong H_b(S^n) \) and \( E_{p,q+n-1}^{n+1} \cong E_{p, q+n-1}^2 \cong H_b(S^n) \)

we have \( H_{p+1-n}(S^n) \cong H_b(S^n) \) whenever \( q > 1-n \).

Since \( \Pi_0(S^n) = \Pi_1(S^n) = 0 \), \( S^n \) is path-connected

and so \( H_k(S^n) = 0 \). Follows that

\[ H_k(S^n) \cong \begin{cases} \mathbb{Z} & k = m(n-1) \text{, } m \geq 0 \\ 0 & \text{otherwise} \end{cases} \]
Let $E$ be a $1^\text{st}$ quadrant homology spectral sequence. For $p > 0$, assume $[\partial^r: E_{p+r-1} \to E_{p+1}] = 0$ and $r+1 < 0$. Hence

$$H_p(E') = \ker [\partial^r: E_{p+1} \to E_{p+r-1}] \subset E_{p+1}$$

and so we have monomorphisms $E_{p+1}^{r+1} \cong H_p(E') \subset E_{p+1}$. Thus we have a sequence of monomorphisms

$$E^\infty_{p+1} = E_{p+1} \hookrightarrow E_{p+1}^p \hookrightarrow E_{p+1}^{p-1} \hookrightarrow \cdots \hookrightarrow E_{p+1}^{3} \hookrightarrow E_{p+1}^2.$$ 

Denote this composite monomorphism by $i_B: E^\infty_{p+1} \hookrightarrow E_{p+1}^2$. $i_B$ will be called the base edge monomorphism.

Easily checked: $i_B$ is natural, i.e., if $f: E \to E'$ is a morphism of $1^\text{st}$ quadrant spectral sequences, then

$$
\begin{array}{ccc}
E^\infty_{p+1} & \xrightarrow{i_B} & E^2_{p+1} \\
\downarrow f_{p+1} & & \downarrow f_{p+1} \\
E^\infty_{p+1} & \xrightarrow{i_B} & E^2_{p+1}
\end{array}
$$

Base-Edge Theorem: Let $F \overset{i}{\to} T \overset{p}{\to} B$ be a pointed fibration with $B$ 1-connected, $F$ 0-connected. Let $B' \subset B$, $T' = p^{-1}(B')$. Let $E$ be the long-exact spectral sequence (coeff. in $R$) of the above. Then following commutes:

$$
\begin{array}{ccc}
H_p(T',T';R) & \overset{p_*}{\to} & H_p(B,B';R) \\
\downarrow p_* & & \downarrow p_* \\
E_p^{\infty} & \xrightarrow{i_B} & E_p^{2}
\end{array}
$$

where $p_*$ is the composition

$$H_p(T',T';R) \xrightarrow{i_B} J_{p+1}^{\infty} = J_{p+1}/J_{p+1}^{\infty} \cong E^\infty_{p+1}.$$
Let \( Y; B \) be a pointed pair with \( Y \) 1-connected. Suppose, for some \( n \geq 1 \), \( H_i(Y; B) = 0 \) for \( i \leq n \). Then \( p_* : H_{n+1}(P(Y, x, Y), P(Y, x, B)) \to H_{n+1}(Y, B) \) is an isomorphism.

Proof: Since \( Y; B \) is 1-connected, \( SY \) is 0-connected. We have a fibration \( SY \to P(Y, x, Y) \to Y \) with \( p^{-1}(B) = P(Y, x, B) \). Let \( E \) be the Leray-Serre spectral sequence of the above. From commutativity of

\[
\begin{array}{ccc}
H_{n+1}(P(Y, x, Y), P(Y, x, B)) & \rightarrow & H_{n+1}(Y, B) \\
\downarrow & & \\
E_{p,q} & \rightarrow & E_{p,q+1}
\end{array}
\]

\[
E_{p,q} \rightarrow E_{p,q+1} \rightarrow E_{p+1,0}
\]

it suffices to show \( p_* \) and \( i_* \) are isomorphisms.

By U.C.T., \( H_p(Y; B; H_q(SY)) = 0 \) for \( p \leq n \), i.e., \( E_{p,q} = 0 \) for \( p \leq n \). Hence for all \( 2 \leq r \leq \infty \),

\( E_{p,q}^r = 0 \) for \( p \leq n \). Thus \( H_{n+1,0} \) is 0 for all \( r \geq 2 \) and so the conclusion \( E_{n+1,0} \rightarrow E_{n+1,0} \) is an isomorphism for all \( r \geq 2 \), i.e., \( i_* \) is an isomorphism. Also, since \( E_{p,q} = 0 \) for \( p \leq n \),

\[
\begin{array}{c}
E_{p,q}^r = 0 \\
\hline
n+1
\end{array}
\]
It follows that $O = \Sigma_{1,n+2} = \Sigma_{0,n+1} = \Sigma_{1,n} = \ldots = \Sigma_{n,1}$.

Thus $\pi_\bullet: \Sigma_{n+1,0} \to \Sigma_{n+1,0}/\Sigma_{n,1}$ is an isomorphism.

**Fundamental Suspension Theorem**

Let $X$ be any pointed space. Write $SX = X \wedge S^1$ = reduced suspension of $X$. If $f: X \to Y$ is pointed, write $[f] = f \wedge 1_S: SX \to SY$. Note that $[Sf]$ depends only on $[f]$. In particular, if $g: S^n \to X$ is pointed, $Sg: S^n \wedge S^1 = S^{n+1} \to SX$ and we obtain a well-defined function $S: \Pi_n(X,*) \to \Pi_{n+1}(SX,*)$, called the suspension map. $S$ is natural: For if $f: X \to Y$ is pointed, consider

$$
\begin{array}{ccc}
\Pi_n(X,*) & \xrightarrow{S} & \Pi_{n+1}(SX,*) \\
\downarrow f_* & & \downarrow (Sf)_* \\
\Pi_n(Y,*) & \to & \Pi_{n+1}(SY,*)
\end{array}
$$

If $g: S^h \to X$ is pointed, then

$$(Sf)_* [g] = (Sf)_* [g \wedge 1_S] = [(f \wedge 1_S) (g \wedge 1_S)] = [f \wedge 1_S] = S[f] = S [g]$$

so above rectangle commutes.

$\Phi: \Pi_n(X,*) \to \Pi_{n+1}(SX,*)$ is a group homomorphism.

If $f: SX \to Y$ is pointed, let

$\hat{f}: X \to \Omega Y$ be quasi loop $\hat{f}(x)(t) = f(x \wedge \langle t, t \rangle)$

Recall: We have a natural group isomorphism

$$\psi_0: \Pi_{n+1}(Y,*) \to \Pi_n(\Omega Y,*) \text{ for all } n \geq 0.$$

$$[\hat{f}] \longrightarrow \psi_0([f])$$
Let \( i = \tilde{1}_{SX} : X \to \Omega SX \).

\[ i \text{ is a pointed inclusion map.} \]

**Lemma:** For any pointed \( X \),

\[ \pi_n \left( \Omega (SX) \right) \xrightarrow{\Sigma} \pi_{n+1} \left( SX, * \right) \]

\[ \xrightarrow{i_*} \]

\[ \pi_n \left( \Omega SX, * \right) \]

commutes for all \( n \).

**Proof:** \( \psi_* \Sigma [f] = \psi_* [f \wedge 1_\ast] = [\tilde{f} \wedge 1_\ast] \),

\[ i_* [f] = [f \wedge \tilde{1}_\ast] \quad \text{and it is easily verified that} \]

\[ \tilde{f} \wedge 1_\ast = f \circ \tilde{1}_\ast. \]

**Cor:** \( S : \pi_n \left( X, * \right) \to \pi_{n+1} \left( SX, * \right) \) is a group homomorphism for \( n \geq 1 \).

**Example:** Let \( n \geq 1 \). By the Homotopy Theorem,

\[ i_n : \pi_n \left( S^n, * \right) \to \pi_n \left( S^n, * \right) \]

is an isomorphism. (This follows from the fact that \( S^1 \) is an \( H^\ast \)-space.)

Note that \( i_n \left[ 1_{S^n} \right] = \left( 1_{S^n} \right)_* \left( k_n \right) = k_n = \text{gen. of } \pi_n \left( S^n, * \right) \cong \mathbb{Z} \),

and so \( \pi_n \left( S^n, * \right) \cong \mathbb{Z} \) with generator \( [1_{S^n}] \). Since \( S \left( 4S^n \right) = \Sigma^{n+1} \), \( S : \pi_n \left( S^n, * \right) \to \pi_{n+1} \left( S^{n+1}, * \right) \) is an isomorphism.

Thus, \( i_* : \pi_n \left( S^n, * \right) \to \pi_n \left( \Omega S^{n+1}, * \right) \) is an isomorphism.
Thus (Freudenthal Suspension Theorem). Let \( n \geq 2 \). Then 
\[
S^n \to \mathbb{R} \mathbb{P}^{n+1} \to S^{n+1}
\]
is an isomorphism if \( k \leq 2n - 2 \) and onto if \( k = 2n - 1 \).

Case: For any \( k \geq 0 \), the geometric class of \( \pi_{n+k}(S^n, *) \)
is independent of \( n \) for \( n \geq k+2 \).

**Proof:** By lemma, we wish to show \( \mathcal{I}_k : \pi_k(S^n, *) \to \pi_k(\mathbb{R}S^{n+1}, *) \)
is an isomorphism if \( k \leq 2n - 2 \) and onto if \( k = 2n - 1 \). Since \( n \geq 2 \), \( S^n \) and \( \mathbb{R}S^{n+1} \) are both 
\( (n-1) \)-connected. Hence, by the Whitehead theorem, sufficient to show:

\[ \mathcal{I}_k : \pi_k(S^n, *) \to \pi_k(\mathbb{R}S^{n+1}, *) \]
is an isomorphism if \( k \leq 2n - 2 \) and onto if \( k = 2n - 1 \).

Then \( k = 2n - 1 \) we have 
\[ \pi_k(S^n, *) \cong \pi_k(\mathbb{R}S^{n+1}, *) \cong \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k \neq 0 \end{cases} \]

Thus, we need only to show \( \mathcal{I}_k : \pi_k(S^n, *) \to \pi_k(\mathbb{R}S^{n+1}, *) \)
is an isomorphism. We have the commutative diagram:

\[
\begin{array}{ccc}
\pi_k(S^n, *) & \xrightarrow{\mathcal{I}_k} & \pi_k(\mathbb{R}S^{n+1}, *) \\
\downarrow \pi_k(S^n, *) & & \downarrow \pi_k(\mathbb{R}S^{n+1}, *) \\
\pi_k(S^n, *) & \xrightarrow{\mathcal{I}_k} & \pi_k(\mathbb{R}S^{n+1}, *)
\end{array}
\]

\( S^n \) and \( \mathbb{R}S^{n+1} \) are both \( (n-1) \)-connected, and so by the \( H \)-connecting theorem, both \( \pi_k \)s are isomorphic. Already noted \( \mathcal{I}_k \) is an isomorphism. Hence, bottom \( \mathcal{I}_k \) is an isomorphism.

---

**Fibered edge suspension**

Let \( E \) be a \( (2n) \)-quad. Homology spectral sequence. In each \( q \geq 0 \) we have the composite suspension

\[
E_{0,q} \xrightarrow{E_{0,q}} E_{0,q} / \lim [d^r : E_{0,q-r+1} \to E_{0,q}] = H_{0,q}(E^r) E_{0,q} = E_{0,q}^{r+1}
\]

Let \( \mathcal{F} : E_{0,q} \to E_{0,q} \) denote the composition

\[
E_{0,q} \xrightarrow{\mathcal{F}} E_{0,q} \xrightarrow{\mathcal{F}} \cdots \xrightarrow{\mathcal{F}} E_{0,q} = E_{0,q}.
\]
It is called the fibre edge epimorphism.

Easily checked: $p_F$ is natural, i.e. if $f: E \to E$ is a morphism of 1st quadrant spectral sequences, then

$$
\begin{align*}
E_{1,1}^2 & \xrightarrow{p_F} E_{1,1}^\infty \\
E_{2,1}^2 & \xrightarrow{p_F} E_{2,1}^\infty
\end{align*}
$$

commutes.

Fibre-Edge Lemma: Let $F \xrightarrow{i} T \xrightarrow{p} B$ be a pointed filtration with $B$ 1-connected. Let $E$ be the Leray-Serre spectral sequence of this filtration ($\mathbb{R}$-coeff). Then the following commutes for all $q$:

$$
\begin{align*}
H_q(F; R) & \xrightarrow{i_*} H_q(T; R) \\
\phi & \Rightarrow \\
E_{1,1}^2 & \xrightarrow{p_F} E_{1,1}^\infty
\end{align*}
$$

where $i_*$ is the monomorphism $E_{0,q} = J_{0,q} \subset J_{1,q} = H_q(T; R)$, and $\phi$ is the composite isomorphism

$$
\begin{align*}
H_q(F; R) & \xrightarrow{\cong} \mathbb{Z} \otimes H_q(F; R) \\
& \xrightarrow{\varepsilon \otimes 1} H_0(B; \mathbb{Z}) \otimes H_q(F; R) \\
& \xrightarrow{\mu} (\text{u.c. T.}) \\
& H_0(B; H_q(F; R)) \\
& \cong E_{0,1}^2
\end{align*}
$$

(Proof deferred.)
Spectral Sequence of a Filtered Chain Complex

Def. A filtered chain complex $(C, F)$ of $R$-modules consists of a chain complex $C$ of $R$-modules, and a nested family of sub-chain complexes

$$0 = E_0 C = F_0 C \subset F_1 C \subset \ldots \subset C$$

Example: Let $X$ be a filtered top. space, i.e. we are given a nest of subspaces $\emptyset = X_0 \subset X_1 \subset \ldots \subset X$. Let $Q(X)$ be the singular complex of $X$, $F_p Q(X) = Q(X_p)$. Then $(Q(X), F)$ is a filtered chain complex.

(The filtered chain complex yielding the Leray–Serre spectral sequence will be different).

Def. A morphism of filtered chain complexes $\phi : (C, F) \to (\tilde{C}, \tilde{F})$ consists of a chain map $\phi : C \to \tilde{C}$ so:

$$\phi(F_p C) \subset \tilde{F}_p \tilde{C} \forall p$$

Can form cat. of filtered chain complexes of $R$-modules.

Convention: $F_0 C = 0 \forall p < 0$.

We proceed to define a covariant functor from the cat. of filtered chain complexes of $R$-modules to the cat. of spectral sequences of $R$-modules.

Def. Let $(C, F)$ be a filtered chain complex. For $r \geq 1$, define

$$E_{p,q}^r (C,F) = \ker \left[ H_{p+q} (F_pC/F_{p-r}C) \rightarrow H_{p+q} (F_{p+r-1}C/F_{p-r}C) \right]$$

Note: step of $r$ between numerator $+$ denominator. $1^{st}$ quotient involves filtrations $\leq p$, $2^{nd}$ quotient factors out filtrations $< p$.

The inclusion of triples $$(F_p C, F_{p-r} C, F_{p-2r} C)$$

yields the commutative diagram.
$$\begin{align*}
\text{H}_{p+q}(F_p C/F_{p-1} C) & \xrightarrow{i_*} \text{H}_{p+q}(F_{p+r-1} C/F_{p-1} C) \Rightarrow E^r_{p+q}(C,F) = \lim i_* \\
\text{H}_{p+q-1}(F_p C/F_{p-1} C) & \xrightarrow{i_*} \text{H}_{p+q-1}(F_{p+r-1} C/F_{p-1} C) \Rightarrow E^r_{p+q-1}(C,F) = \lim i_*
\end{align*}$$

where the vertical maps are the connecting homomorphisms from the homology sequences of the above tuples. Thus

$$\partial: E^r_{p+q}(C,F) \to E^r_{p+q-1}(C,F).$$ Define

$$d^r: E^r_{p+q}(C,F) \to E^r_{p+q+r-1}(C,F)$$ to be the restriction of $\partial$.

Thus

$$\begin{align*}
\text{H}_{p+q}(F_{p+r-1} C/F_{p-1} C) & \Rightarrow E^r_{p+q}(C,F) \\
\partial & \Rightarrow d^r \\
\text{H}_{p+q-1}(F_p C/F_{p-1} C) & \Rightarrow E^r_{p+q-1}(C,F)
\end{align*}$$
comutes.

Thus: $\{ E^r_{p+q}(C,F), d^r \}$ is a spectral sequence of $R$-modules.

1) $E^{r\prime}_{p+q}(C,F) = \text{H}_{p+q}(F_p C/F_{p-1} C)$, and

$$d^{r\prime}: E^{r\prime}_{p+q}(C,F) \to E^{r\prime}_{p+q-1}(C,F)$$ is the connecting homomorphism.

$$\partial: \text{H}_{p+q}(F_p C/F_{p-1} C) \to \text{H}_{p+q-1}(F_{p+r-1} C/F_{p-1} C)$$ is the homology seq. of the tuple $(F_p C, F_{p-1} C, F_{p-2} C)$.

2) The isomorphisms $\Theta^r: H(E^r, d^r) \to E^{r+1}$ are such that

$$\begin{align*}
\text{H}_{p+q}(E^{r}(C,F), d^{r\prime}) & \xrightarrow{\Theta^r} E^{r+1}_{p+q}(C,F) \\
\text{Ker} [d^r: E^r_{p+q}(C,F) \to E^r_{p+q+r-1}(C,F)] & \xrightarrow{\text{H}_{p+q}(F_{p+r-1} C/F_{p-1} C)} \text{H}_{p+q}(F_{p+r-1} C/F_{p-1} C)
\end{align*}$$

commutes, where $j: (F_{p+r-1} C, F_{p-1} C) \to (F_{p+r} C, F_{p-1} C)$ is the inclusion.
3) \((C,F) \mapsto \{ E_{p,q}^{r}(C,F), d_{r}^{}\}\) is a covariant functor from the cat. of filtered chain complexes of \(R\)-modules to the category of spectral sequences.

**Proof:**

Write \(F_{p} = F_{p} C\), \(E_{p,q}^{r} = E_{p,q}^{r}(C,F)\).

Prove that \(d_{r}^{r}d_{r}^{r} = 0\):

Following commutes:

\[
\begin{array}{ccc}
H_{p+\ell}(F_{p+r-1}/F_{p-1}) & \xrightarrow{d} & E_{p,q}^{r} \\
H_{p+q-1}(F_{r-1}/F_{p-r-1}) & \xrightarrow{d} & E_{p-1,q+r-1}^{r} \\
H_{p+r-2}(F_{p-1}/F_{r-1}) & \xrightarrow{d} & E_{p-2,r+2r-2}^{r}
\end{array}
\]

and enclosed composition is 0 from homology seq. of \(F_{p-1}/F_{r-1}\).

Thus \(d_{r}^{r}d_{r}^{r} = 0\).

Write \(Z_{p,q}^{r} = \ker \left[ E_{p,q}^{r} \xrightarrow{d_{r}^{r}} E_{p-1,q+r-1}^{r} \right]\),

\[
B_{p,q}^{r} = \text{im} \left[ E_{p+r-1,q+r-1}^{r} \xrightarrow{d_{r}^{r}} E_{p,q}^{r} \right].
\]

Then \(H_{p,q}^{r} (E^{r},d^{r}) = Z_{p,q}^{r} / B_{p,q}^{r}\). To prove \(\{E^{r},d^{r}\}\) is a spectral sequence and to establish (2), it suffices to show

a) \(j_{*}\left( Z_{p,q}^{r} \right) = E_{p,q}^{r+1}\)

b) \(\ker \left( j_{*} | Z_{p,q}^{r} \right) = B_{p,q}^{r}\)

where \(j_{*}\) is as in (2).

**Proof (a):** Have commutative diagram.
Excited exactness is from homology sequences of appropriate triples. \( Z^r_{p,q} = \text{im } \partial \cap \text{ker } \partial \)

Note \( (\partial^\ast)^r \circ \partial = 0 \) since it factors through \( H_{p+q} (F_{p-1}/F_{p-1}) = 0 \).

a) now follows by diagram chasing an above diagram.

Proof of b): Will prove the stronger result:

\[ \text{ker } \left( \partial^\ast \left| E^r_{p,q} \right. \right) = B^r_{p+q} \]

This follows from diagram chasing in following diagram, with indicated exactness from homology sequences of appropriate triples.
1) is immediate.
2) is straightforward.

**Note:** \( E_{p,q}^r (C,F) = 0 \) \( \forall p < 0 \). In \( H_{p+q} (F_p C/F_{p-1} C) = 0 \) since \( F_p C = 0 \) \( \forall p < 0 \).

**Def.** A filtered chain complex \( (C,F) \) is canonically bounded if, \( \forall p \), \( (F_p C)_i = C_i \) \( \forall i \leq p \).

(Filtration of \( \mathcal{A}(X) \) arising from a filtered top. space \( X \) is not usually canonically bounded).

**Note:** If \( (C,F) \) is canonically bounded, then \( E(C,F) \) is a 1st grad. spectral sequence, \( \forall k \), \( q < 0 \),

\[ (F_p C)_{p+q} = C_{p+q} = (F_{p-1} C)_{p+q} \] \( \forall p+q \leq p-1 \), and so

\[ E_{p,q}^1 (C,F) = H_{p+q} (F_p C/F_{p-1} C) = 0. \]

In any filtered chain complex \( (C,F) \), define

\[ J_{p,q} (C,F) = \text{in} \left[ H_{p+q} (F_p C) \xrightarrow{\cdot q} H_{p+q} (C) \right]. \]

It is immediate that \( J_{p,q} \) is natural w.r.t. morphisms of filtered chain complexes. From commutativity of

\[
\begin{array}{ccc}
H_{p+q} (F_{p-1} C) & \xrightarrow{\cdot q} & H_{p+q} (C) \\
\downarrow & & \downarrow \\
H_{p+q} (F_p C) & \xrightarrow{\cdot q} & H_{p+q} (C)
\end{array}
\]

it is immediate that \( J_{p-1,q+1} (C,F) \subset J_{p,q} (C,F) \).

Since \( F_1 C = 0 \), \( J_{-1,n+1} (C,F) = 0 \) \( \forall n \). Thus we have a natural filtration

\[ 0 = J_{-1,n+1} (C,F) \subset J_{0,n} (C,F) \subset J_{1,n-1} (C,F) \subset \ldots \subset H_n (C). \]
Thus, $\text{surjective } (C,F) \text{ is canonically bounded. Then, }$ \nabla_{n,0}(C,F) = H_n(C)$, and there is a natural isomorphism

$$ \phi: J_{p,q}^r (C,F) \rightarrow E_{p,q}^\infty (C,F) $$

such that for all $r > \max \{ p, q+1 \}$, the diagram

$$ \begin{array}{c}
\text{H}_{p+q}^r (F_p C) \xrightarrow{i^*} \text{H}_{p+q}^r (F_{p+r-1} C) \xrightarrow{\cong} \text{H}_{p+q}^r (F_p C) \rightarrow \text{H}_{p+q}^r (C) \\
\downarrow \quad (\text{can. ext.)} \\
\text{H}_{p+q}^r (F_p C) / \text{H}_{p+q}^r (F_{p+r-1} C) \rightarrow \text{J}_{p,q}^r (C,F) \\
\end{array} $$

commutes.

**Proof:** By canonical boundedness, $(F_n C)_n = C_n$ and $\text{H}_n (F_n C) \rightarrow \text{H}_n (C)$ is onto. Hence $\text{J}_{n,0}(C,F) = H_n (C)$.

Let $r > \max \{ p, q+1 \}$, and let $\phi^r_{p,q}: J_{p,q}^r (C,F) \rightarrow E_{p,q}^r (C,F)$ denote the composition of the maps in the diagram moving counterclockwise. $\phi^r_{p,q}$ is natural. Note that $\phi^r_{p,q}$ is onto since

$$ \text{H}_{p+q}^r (F_p C) \xrightarrow{i^*} \text{H}_{p+q}^r (F_{p+r-1} C) $$

commutes.

Must find $\phi^r_{p,q}$

1. ker $\phi^r_{p,q} = \text{J}_{p-1, q+1} (C,F)$, \( \phi^r_{p,q} \) would then induce a natural isomorphism $\phi^r: J_{p,q}^r (C,F) / \text{J}_{p-1, q+1} (C,F) \rightarrow E_{p,q}^r (C,F)$.

Must then check
2) For \( r > \max \{ p, q+1 \} \),

\[
J_{p,q} \to E_{p,q}^r (C, F)
\]

\[
J_{p+1,q+1} \to E_{p+1,q+1}^r (C, F)
\]

\[
\phi^r \to E_{p,q}^r (C, F)
\]

\[
\phi^{r+1} \to E_{p,q}^{r+1} (C, F)
\]

**Corollary.**

Proof (i): Write \( F_p = F_p(C) \), \( J_{p,q} = J_{p,q}^r (C, F) \). We have the commutative diagrams in the left-hand column. Recall:

\[
\begin{array}{ccc}
H_{p+q} (F_{p-1}) & \to & H_{p+q} (F_p) \\
\downarrow & & \downarrow \\
H_{p+q} (F_{p+1}) & \to & H_{p+q} (C)
\end{array}
\]

\[
\begin{array}{ccc}
J_{p+q} & \to & J_{p+q}
\end{array}
\]

Note that \( \ker f_{p,q}^r = \ker \left( J_{p+q}^r (F_{p-1}) \left| J_{p+q}^r \right. \right) \) now follows easily from above diagram.

Proof (ii): Write \( E_{p,q}^r = E_{p,q}^r (C, F) \) and let \( Z_{p,q}^r, B_{p,q}^r \) denote \( \ker d^r \) and \( \text{coker} \ W_{p,q}^r \), resp., in \( E_{p,q}^r \).

Recall: \( \rho \) is the isomorphism \( \Theta : H_{p+q}^r (C, F) \to E_{p,q}^{r+1} \) as given as follows: \( \rho : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \), write \( [x] = x + E_{p,q}^r \). Then \( \rho ([x]) = J_{p+q} [x] \) where \( J_{p+q} : H_{p+q}^r (F_{p-1}/F_{p-1}) \to H_{p+q}^r (F_{p+1}/F_{p+1}) \). In particular, for \( r > \max \{ p, q+1 \} \), the cohomology of \( E_{p,q}^r \) with \( E_{p,q}^{r+1} \) is such that

\[
\begin{array}{ccc}
E_{p,q}^r & \to & E_{p,q}^{r+1} \\
\downarrow & & \downarrow \\
H_{p+q}^r (F_{p+1}/F_{p-1}) & \to & H_{p+q}^r (F_{p+1}/F_{p-1})
\end{array}
\]

commutes.

Let \( y \in J_{p,q}^r \), and write \( [y] = y + J_{p-1,q+1} \). Then by def.

\[
\Phi^r [y] = J_{p+q}^r (y) \quad \text{we have the commutative diagram}
\]
The result is now immediate.
The category of map pairs

Def: A map pair \( \pi : (T, T_0) \to (B, B_0) \) is a map of topological pairs such that \( T_0 = \pi^{-1}(B_0) \).

\( \pi \) is not necessarily pointed. In particular, allow \( T_0 = B_0 = \emptyset \).

Def: A morphism of map pairs \( f \) from \( \pi : (T, T_0) \to (B, B_0) \) to \( \pi' : (T', T'_0) \to (B', B'_0) \) consists of a pair of maps of topological pairs \( f_T : (T, T_0) \to (T', T'_0) \), \( f_B : (B, B_0) \to (B', B'_0) \) such that

\[
\begin{array}{ccc}
(T, T_0) & \xrightarrow{f_T} & (T', T'_0) \\
\pi & \downarrow & \downarrow \pi' \\
(B, B_0) & \xrightarrow{f_B} & (B', B'_0)
\end{array}
\]

commutes.

Form the category of map pairs.

The Serre filtration

Let \( \pi : T \to B \) be a map (not necessarily a fibration), and \( \sigma : I^n \to T \) a non-degenerate singular n-cube. The weight of \( \sigma \) (with respect to \( \pi \)), \( \omega(\sigma) \), is the smallest non-negative \( p \leq n \) such that \( \pi \delta^p (\xi_1, \ldots, \xi_p, \xi_{p+1}, \ldots, \xi_n) \) is independent of \( \xi_{p+1}, \ldots, \xi_n \).

Thus \( \omega(\sigma) = 0 \) if and only if \( \pi \delta^p \) is constant, \( \omega(\sigma) = n \) if and only if \( \pi \delta^p \) is non-degenerate.

Prop: Let \( \pi : T \to B \) be a map. Let \( \sigma : I^n \to T \) be a non-degenerate singular n-cube, \( n \geq 1 \). Suppose \( \omega(\sigma) = p \). Then for each \( 1 \leq i \leq n \) and \( \varepsilon = 0, 1 \), either \( \sigma_{\varepsilon}^{(i)} \) is degenerate, or

\[
\omega(\sigma_{\varepsilon}^{(i)}) = \begin{cases} p & \text{if } 1 \leq i \leq n \\ p & \text{if } 1 \leq i \leq p \end{cases}
\]

The proof is straightforward.
Thus for fixed $p$ and any commutative ring with unit $R$, the non-degenerate singular cubes of weight $\leq p$ span a subcomplex $F_p^T Q(T; R)$ of $Q(T; R)$.

Convention: $F_p^T Q(T; R) = 0$ for $p < 0$.

Note: $\left[ F_p^T Q(T; R) \right]_i = Q_i^T(T; R)$ if $i \leq p$. Thus $(Q(T; R), F^T)$ is a canonically bounded filtered chain complex.

If $\pi: (T, T_0) \to (B, B_0)$ is a map pair, set

$$F_p^T Q(T, T_0; R) = \frac{F_p^T Q(T; R) + Q(T_0; R)}{Q(T_0; R)} = Q(T; R) = Q(T, T_0; R).$$

This yields a canonically bounded filtered chain complex $(Q(T, T_0; R), F^T)$.

It is clear that if $f$ is a morphism of map pairs, for each non-degenerate $T: I^n \to T$, either $f_T$ is degenerate or $w(f_T, T) \leq w(T)$. Hence $f$ induces a morphism of filtered chain complexes, and it is straightforward to check that $\pi \mapsto (Q(T, T_0; R), F^T)$ is a covariant functor from the category of map pairs to the category of canonically bounded filtered chain complexes of $R$-modules.

The Leray–Serre Spectral Sequence of a map pair

Composing the above functor from map pairs to canonically bounded filtered chain complexes with the earlier algebraic functor from canonically bounded filtered chain complexes to 1st quadrant spectral sequences, we obtain a covariant functor $\pi \mapsto E(\pi)$ from the category of map pairs to the category of 1st quadrant spectral sequences of $R$-modules. The following is immediate from the earlier algebra on spectral sequences of filtered chain complexes:
Prop: Let \( \pi : (T, T_0) \to (B, B_0) \) be a map pair. Then for each \( n \geq 0 \) there exists a natural filtration
\[
0 = J_{-n,n+1}(\pi) \subset J_{0,n}(\pi) \subset \ldots \subset J_{n,n}(\pi) = H_n(T, T_0; R)
\]
and natural isomorphisms
\[
E_{p, q}^\infty(\pi) \cong J_{p, q}(\pi)/J_{p-1, q+1}(\pi)
\]
for all \( (p, q) \).

To prove the Leray-Serre theorem, it remains to prove: If \( \pi : (T, T_0) \to (B, B_0) \) is a map pair for which \( F \to T \xrightarrow{\pi} B \) is a pointed fibration with \( B \) 1-connected, then there are natural isomorphisms
\[
E_{p, q}^2(\pi) \cong H_p(B, B_0; H_q(F; R)).
\]
To show this, we investigate \( E_1(\pi) \) and show that the chain complex \( (E_1^*, q(\pi), d^1) \) is naturally chain-isomorphic to \( Q(B, B_0) \otimes H_q(F; R) \).

Special Case 1: \( 1 : (B, B_0) \to (B, B_0) \). For any \( \sigma : I^n \to B \), \( \omega(\sigma) < n \) if and only if \( \sigma \) is degenerate. Thus
\[
F_p^1 Q_n(B, B_0; R) = \begin{cases} 0 & \text{if } p < n \\ Q_n(B, B_0; R) & \text{if } p \geq n \end{cases},
\]
and so
\[
\left( \frac{F_p^1 Q(B, B_0; R)}{F_{p-1}^1 Q(B, B_0; R)} \right) \cong \begin{cases} 0 & \text{if } \leq p \\ Q_p(B, B_0; R) & \text{if } \geq p \end{cases}.
\]
Thus
\[
E_{p, q}^1(1) = \begin{cases} 0 & \text{if } q \neq 0 \\ Q_p(B, B_0; R) & \text{if } q = 0 \end{cases}.
\]
From the diagram
it follows that the connecting homomorphism

\[ H_p \left( \frac{F_p}{F_{p-1}} \right) \longrightarrow H_{p-1} \left( \frac{F_{p-1}}{F_{p-2}} \right) \]

is \( \partial : Q_p \left( B, B_{sc} ; R \right) \longrightarrow Q_{p-1} \left( B, B_{sc} ; R \right) \).

Thus the chain complex \( (E'^p_q(1), \partial'^q) \) is \( Q \left( B, B_{sc} ; R \right) \). It follows that \( E^2_{p,q}(1) \) is naturally isomorphic to

\[ \begin{cases} 0 & \text{if } q \neq 0 \\ H_p \left( B, B_{sc} ; R \right) & \text{if } q = 0. \end{cases} \]

In this case, the spectral sequence is concentrated on the base edge, and \( E^r_{p,q}(1) \cong E^{r+1}_{p,q}(1) \) for all \( r \geq 1 \).

**Claim:**

\[ H_p \left( B, B_{sc} ; R \right) \]

\[ \xrightarrow{F_p} \]

\[ E^\infty_{p,0}(1) \]

\[ \xrightarrow{\partial^0_{B}} \]

\[ E^2_{p,0}(1) \]

commutes.

**Proof of claim:** Write \( E^r_{p,q} = E^r_{p,q}(1) \), \( F_p = F^1_p Q \left( B, B_{sc} ; R \right) \), \( Q = Q \left( B, B_{sc} ; R \right) \).

Let \( z \in Z_p Q \). Since

\[ H_{p,0} \left( E'^1_q, d'^1 \right) \]

\[ \xrightarrow{\partial'^1} \]

\[ E^2_{p,0} \]

\[ \downarrow \text{onto} \]

\[ Z_p Q = Z^1_{p,0} \]

\[ \downarrow \]

\[ Q_p = H_p \left( F_p / F_{p-1} \right) \]

\[ \xrightarrow{\partial'} \]

\[ H_p \left( F_p / F_{p-1} \right) \]

commutes, \( \partial' \left( z + B_p Q \right) = \left( z + F_{p-1} \right) + B_p \left( F_{p+1} / F_{p-1} \right) \in E^2_{p,0} \subset H_p \left( F_{p+1} / F_{p-1} \right) \).
Let \( r > \max \{ p, 1 \} \). From commutativity of

\[
Hp(Q) = \tau_{r,0} = \text{im} \left[ \tau_P(F_P) \rightarrow Hp(Q) \right]
\]

and the fact that \((F_{p+r-1})_i = Q_i \) for \( i \leq p + r - 1 \), it follows

that \( \mu \in \mathbb{Z}_p(F_{p+r-1}) \). Since \( \mu(z + B_p(F_{p+r-1})) = z + B_p Q \), it follows that

\[
\tau_B(z + B_p Q) = \tau(z) + B_p(F_{p+r-1}/F_p) = (z + F_p) + B_p(F_{p+r-1}/F_p)
\]

Thus

\[
E_{\tau,0} \supset E_{\tau,0}^{r+1} \supset H_p(F_{p+r-1}/F_{p-1})
\]

for \( r > \max \{ p, 1 \} \) (which follows from definition of the isomorphisms \( \tau_P : H_p(F^r, \delta') \rightarrow E_{\tau,0}^{r+1} \)), and the fact that \((F_{p+1})_i = Q_i \) for \( i \leq p + 1 \), \( z + F_p \in \mathbb{Z}_p(F_{p+r-1}/F_{p-1}) \). Thus

since \( \tau(z + F_{p-1}) = z + F_{p-1} \in (F_{p+r-1}/F_{p-1})_p \),

\[
\tau_B(z + F_{p-1} + B_p(F_{p+r-1}/F_{p-1})) = (z + F_{p-1}) + B_p(F_{p+r-1}/F_{p-1})
\]

Thus

\[
\tau_B(B_p Q) = (z + F_{p-1}) + B_p(F_{p+r-1}/F_{p-1}) = \Theta(z + B_p Q)
\]

proving the claim.
Special Case 2: \( \pi: X \to P \), \( P \) a point. Then for each non-degenerate \( \sigma: I^n \to X \), \( \pi \sigma \) is constant and so \( \omega(\sigma) = 0 \) for all \( \sigma \). Thus \( F_\sigma^* \omega(X; R) = \omega(X; R) \), and so
\[
\frac{F_\sigma^* \omega(X; R)}{F_{\sigma^{-1}}^* \omega(X; R)} = \begin{cases} 
0 & \text{if } p \neq 0 \\
\omega(X; R) & \text{if } p = 0.
\end{cases}
\]
Thus \( E_{p, q}(\pi) = \begin{cases} 
0 & \text{if } p \neq 0 \\
\omega(X; R) & \text{if } p = 0.
\end{cases} \)

In this case the spectral sequence is concentrated on the fibre edge. All the differentials (including \( d^1 \)) are thus 0.

In fact, for \( r \geq 1 \),
\[
E_{r, q}^r(\pi) = \text{im} \left[ H_q(F_{\sigma}^r) \to H_q(F_{\sigma^{-1}}^r) \right] = H_q(X; R)
\]
and from commutativity of
\[
\begin{array}{ccc}
H_{0, q}(E_r^r) & \xrightarrow{\theta^r} & E_{0, q}^{r+1}(\pi) \\
\uparrow & = & \uparrow \\
\| & \downarrow & \| \\
H_q(F_{\sigma^{-1}}^r) & \xrightarrow{J_q} & H_q(F_{\sigma}^r) \\
\| & & \| \\
H_q(X; R) & \xrightarrow{1} & H_q(X; R)
\end{array}
\]
it follows that each \( \theta^r \) is the identity map on \( H_q(X; R) \).

It follows easily that both \( p_F: E_{0, q}^2(\pi) \to E_{0, q}^{\infty}(\pi) \) and \( \iota_F: E_{0, q}^{\infty}(\pi) \to H_q(X; R) \) (the composition
\[
E_{0, q}^\infty(\pi) \xrightarrow{\iota_F} H_q(X; R)
\]
is the identity map on \( H_q(X; R) \).
The \( (B, B_0) \)-decomposition of \( E'_{(\pi)} \)

Let \( \pi : (T, T_0) \rightarrow (B, B_0) \) be a map pair. Recall: \( E'_{p, q}(\pi) = H^{p+q}_q \left( \left( F_p^{(T, T_0; R)} / F_{p-1}^{(T, T_0; R)} \right) \right) \).

Write \( F_p^{(T; R)} = F_p^{(T, T_0; R)} + Q(T_0; R) \subset Q(T; R) \).

By definition, \( F_p^{(T, T_0; R)} = F_p^{(T; R)} / Q(T_0; R) \).

Thus, \( F_p^{(T, T_0; R)} / F_{p-1}^{(T, T_0; R)} = F_p^{(T; R)} / Q(T_0; R) = F_{p-1}^{(T; R)} / Q(T_0; R) \),

and so we have a natural isomorphism \( E'_{p, q}(\pi) \cong H^{p+q}_q \left( F_p^{(T; R)}/F_{p-1}^{(T; R)} \right) \).

Moreover, from the map of triples

\[
\left( F_p^{(T; R)}, F_{p-1}^{(T; R)}, F_{p-2}^{(T; R)} \right) \rightarrow \left( F_p^{(T; R)} / Q(T_0; R), F_{p-1}^{(T; R)} / Q(T_0; R), F_{p-2}^{(T; R)} / Q(T_0; R) \right)
\]

it follows that \( \chi^1 : E'_{p, q}(\pi) \rightarrow E'_{p-1, q}(\pi) \) is identified with the connecting homomorphism \( H^{p+q}_q \left( F_p^{(T; R)} / F_{p-1}^{(T; R)} \right) \rightarrow H^{p+q-1}_q \left( F_{p-1}^{(T; R)} / F_{p-2}^{(T; R)} \right) \).

in the homology sequence of the triple \( (F_p^{(T; R)}, F_{p-1}^{(T; R)}, F_{p-2}^{(T; R)}) \).

\( (F_p^{(T; R)} / F_{p-1}^{(T; R)})_n \) is the free \( R \)-module on the non-degenerate singular \( n \)-cubes \( \sigma \) of \( T \) such that \( \omega(\sigma) = p \) and \( \sigma(I^n) \neq T_0 \).

If \( \sigma : I^n \rightarrow T \) is non-degenerate and \( \omega(\sigma) \leq p \), write \( \langle \sigma \rangle \in (F_p^{(T; R)} / F_{p-1}^{(T; R)})_n \) for the image of \( \sigma \).

Thus \( \langle \sigma \rangle = 0 \) if either \( \omega(\sigma) \leq p-1 \) or \( \sigma(I^n) \subset T_0 \).

Note: \( (F_p^{(T; R)} / F_{p-1}^{(T; R)})_n = 0 \) for \( n < p \).

Let \( \sigma : I^n \rightarrow T \) be non-degenerate. Say \( \omega(\sigma) = p \).

Define \( \sigma_B : I^p \rightarrow B \) by \( \sigma_B(x_1, \ldots, x_p) = \pi \sigma(x_1, \ldots, x_p, x_{p+1}, \ldots, x_n) \).

The last \( n-p \) coordinates in this last expression can be arbitrary since \( \omega(\sigma) = p \). Note that \( \sigma_B \) is a non-degenerate singular \( p \)-cube in \( B \) (if it were degenerate, \( \pi \sigma \) would be independent.
of the last $n-p+1$ coordinates, and so we would have $w(x) \leq p-1$, contradiction.)

It is easily checked that for $p \leq i \leq n$ and $e=0,1$,

$$(\sigma_{e}^{(i)})_{B} = \sigma_{B}^{(i)} \text{ if } \sigma_{e}^{(i)} \text{ is non-degenerate}.$$ 

Recall: $w(\langle \sigma_{e}^{(i)} \rangle) = \{ p \text{ if } p \leq i \leq n \}$ when $\sigma_{e}^{(i)}$ is non-degenerate. Thus $\langle \sigma_{e}^{(i)} \rangle = 0$ for $1 \leq i \leq p$. Thus, since

$$\Theta = \sum_{i=1}^{n} (-1)^{i} \{ \sigma_{0}^{(i)} - \sigma_{e}^{(i)} \} > 0 \text{ it follows that}$$ 

$$\Theta = \sum_{i=p+1}^{n} (-1)^{i} \{ \langle \sigma_{0}^{(i)} \rangle - \langle \sigma_{e}^{(i)} \rangle \}$$ 

(Convention: $\langle \sigma_{e}^{(i)} \rangle = 0$ if $\sigma_{e}^{(i)}$ is degenerate).

Thus, for a fixed non-degenerate $\tau : I^{p} \to B$, \{ $\langle \sigma \rangle \mid \sigma_{B} = \tau$ \} span a subcomplex $C^{e}(\pi)$ of $F_{p}/F_{p-1}$.

If $\tau : I^{p} \to B$ is degenerate or has image in $B_{0}$, define $C^{e}(\pi) = 0$.

$C^{e}(\pi)$ is natural with respect to morphisms of map pairs, i.e., if $f : \pi \to \pi'$ is a morphism of map pairs and $\tau : I^{p} \to B$, then the chain map induced by $f$,

$$f_{\#} : F_{p}/F_{p-1}^{\pi} \to F_{p}/F_{p-1}^{\pi'}$$

carries $C^{e}(\pi)$ into $C^{e}(\pi')$. (If $f_{\#} \tau$ is degenerate, then $w(\langle f_{\#} \tau \rangle) = 0$ for any $\tau$ such that $\sigma_{B} = \tau$, and so $f_{\#} \langle \tau \rangle = \langle f_{\#} \tau \rangle = 0$.)

For a map pair $\pi : (T, T_{0}) \to (B, B_{0})$, let $N_{p}(\pi) =$ set of all non-degenerate $\tau : I^{p} \to B$ such that $\tau(I^{p}) \not\subset B_{0}$. For each non-degenerate $\tau : I^{p} \to T$ such that $w(\sigma) = p$ and $\sigma(I^{p}) \not\subset T_{0}$, there exists a unique $\tau \in N_{p}(\pi)$ such that $\sigma_{B} = \tau$ (since $T_{0} = \tau^{-1}(B_{0})$). Thus

$$F_{p}^{\pi}/F_{p-1}^{\pi} = \bigoplus_{\tau \in N_{p}(\pi)} C^{e}(\pi) \text{, and so}$$ 

$$E_{p+1}^{1}(\pi) = \bigoplus_{\tau \in N_{p}(\pi)} H_{p+1}(C^{e}(\pi)).$$
We must now describe \( d_1 \) in terms of this decomposition.

**Def:** Let \( C, \overline{C} \) be chain complexes. A chain map of degree \( r \)
\( f: C \to \overline{C} \) consists of a sequence of homomorphisms
\( f_n: C_n \to C_{n+r} \) such that for all \( n \),

\[
\begin{align*}
C_n & \xrightarrow{f_n} C_{n+r} \\
\delta & \downarrow \quad \downarrow (-1)^r \delta \\
C_{n-1} & \xrightarrow{f_{n-1}} C_{n+r-1}
\end{align*}
\]

commutes.

Just as for ordinary chain maps (i.e., chain maps of degree 0),
such an \( f \) sends cycles to cycles, boundaries to boundaries,
and hence induces \( f_*: H_n(C) \to H_{n+r}(\overline{C}) \). Moreover if
\( f: C \to \overline{C}, g: \overline{C} \to \overline{C} \) are chain maps of degrees \( r \) and \( s \),
resp., then \( g \circ f \) is a chain map of degree \( r+s \) and
\( (g \circ f)_* = g_* f_* \).

**Terminology.** If \( \sigma: I^n \to T, \tau: I^p \to B \) are such that
\( \overline{\sigma}_{B} = \tau \), we will say \( \sigma \) covers \( \tau \).

**Note:** If \( \sigma \) covers \( \tau: I^p \to B \), it is easily checked that
for \( 1 \leq i \leq p, \varepsilon = 0,1 \), \( \sigma^{(i)} \) covers \( \tau^{(i)} \) (provided \( \sigma^{(i)} \)
is non-degenerate).

For each \( \tau \in N_p(T), \) and \( 1 \leq i \leq p, \varepsilon = 0,1 \),
define \( \partial_{E, \varepsilon} ^{(i)}: C^e(T) \to C^{e+i}(T) \) by

\[
\partial_{E, \varepsilon} ^{(i)} \langle \tau \rangle = \langle \tau^{(i)} \rangle.
\]

**Prop:** The \( \partial_{E, \varepsilon} ^{(i)} \), \( 1 \leq i \leq p, \varepsilon = 0,1 \), are chain maps
of degree \(-1\).

**Proof:** Recall: If \( i < j \), then \( (\sigma^{(i)})^{(j)}(\varepsilon) = (\sigma^{(j)})^{(i-1)}(\varepsilon) \),
\( i \neq j \). Thus if \( \sigma: I^{p+q} \to T \) is non-degenerate and covers \( \tau \),
\( \partial E_{E, \varepsilon} ^{(i)} \langle \tau \rangle = \partial \langle \tau^{(i)} \rangle = \sum_{j=p}^{p+q-1} (-1)^j \{ \langle (\sigma^{(i)})^{(j)}(\varepsilon) \rangle - \langle (\sigma^{(i)})^{(j)}(\varepsilon) \rangle \} \)
\[
\begin{align*}
= \sum_{j=p+1}^{p+t} (-1)^j \left\{ \left\langle \sigma^{(i)}_\varepsilon \right\rangle_{(j-1)} - \left\langle \sigma^{(i)}_\varepsilon \right\rangle_1 \right\} \\
= - \sum_{j=p+1}^{p+t} (-1)^j \left\{ \left\langle \sigma^{(i)}_\varepsilon \right\rangle_{(j)} - \left\langle \sigma^{(i)}_\varepsilon \right\rangle_1 \right\} \\
= - \mathcal{C}^{\varepsilon}_{(i)} \sum_{j=p+1}^{p+t} (-1)^j \left\{ \left\langle \sigma^{(i)}_\varepsilon \right\rangle - \left\langle \sigma^{(i)}_1 \right\rangle \right\} \\
= - \mathcal{C}^{\varepsilon}_{(i)} \mathcal{T} < \sigma >.
\end{align*}
\]

The \( \mathcal{C}^{\varepsilon}_{(i)} \) are natural. If \( f : \pi \to \pi' \) is a morphism of map pairs, it is easily verified that

\[
\begin{array}{ccc}
\mathcal{C}^{\varepsilon}_{(i)} & \xrightarrow{\mathcal{C}^{\varepsilon}_{(i)}} & \mathcal{C}^{\varepsilon}_{(i)} \\
\downarrow & & \downarrow \\
\mathcal{C}^{\varepsilon}_{(i)} & \xrightarrow{\mathcal{C}^{\varepsilon}_{(i)}} & \mathcal{C}^{\varepsilon}_{(i)}
\end{array}
\]

commutes.

**Theorem:** The differential

\[
\begin{array}{ccc}
d^! : & \bigoplus_{\varepsilon \in H_p(\pi)} H_{p+\varepsilon} \left( C^\varepsilon(\pi) \right) & \longrightarrow \bigoplus_{\varepsilon \in H_{p+1}(\pi)} H_{p+\varepsilon-1} \left( C^\varepsilon(\pi) \right) \\
& \mathcal{C}^{\varepsilon}_{(i)} & \xrightarrow{\mathcal{C}^{\varepsilon}_{(i)}} \\
& \mathcal{C}^{\varepsilon}_{(i)} & \xrightarrow{\mathcal{C}^{\varepsilon}_{(i)}}
\end{array}
\]

is

\[
\begin{align*}
\sum_{i=1}^{p} (-1)^i \left( \mathcal{C}^{\varepsilon}_{(i)} - \mathcal{C}^{\varepsilon}_{(i)} \right)
\end{align*}
\]

**Proof:** Let \( \alpha \in H_{p+\varepsilon} \left( C^\varepsilon(\pi) \right) \). Say

\[
\sum_{\sigma} r_{\sigma} \left\langle \sigma \right\rangle \in C^\varepsilon(\pi) \subseteq \overline{F}_p^{\pi} / F_{p-1}^{\pi}
\]

is a cycle representing \( \alpha \), where the \( r_{\sigma} \in \mathbb{R} \) and the \( \sigma \) are non-degenerate of weight \( p \).

Then \( \sum_{\sigma} r_{\sigma} \sigma \) is a chain in \( \overline{F}_p^{\pi} \) such that

\[
\partial \left( \sum_{\sigma} r_{\sigma} \sigma \right) \in F_{p-1}^{\pi}. \text{ By definition of the connecting}
\]
homomorphism in the homology sequence at a triple, $d'\alpha$ is the homology class of the image of

$$\partial \left( \sum r_\sigma \sigma \right) \in \mathbb{F}_p \cap \mathbb{F}^n_{p-1}.$$

We have

$$\partial \left( \sum r_\sigma \sigma \right) = \sum r_\sigma \left( \sum_{i=1}^{p+1} (-1)^i \left\{ \sigma^{(i)} - \sigma^{(i)}_1 \right\} \right) + \sum_{i=p+1}^{p+1} (-1)^i \left\{ \sigma^{(i)}_c - \sigma^{(i)}_1 \right\}.$$

Now for $i > p$ each $\sigma^{(i)}_c$ has weight $p$, while for $i \leq p$ each $\sigma^{(i)}_c$ has weight $< p$. Since singular cubes of different weights are independent and since

$$\partial \left( \sum r_\sigma \sigma \right) \in \mathbb{F}_p \cap \mathbb{F}^n_{p-1},$$

we must have

$$\sum r_\sigma \left( \sum_{i=1}^{p+1} (-1)^i \left\{ \sigma^{(i)} - \sigma^{(i)}_1 \right\} \right) \in \mathbb{Q}(T_0, R).$$

Thus

$$\partial \left( \sum r_\sigma \sigma \right) = \sum r_\sigma \left( \sum_{i=1}^{p} (-1)^i \left\{ \sigma^{(i)} - \sigma^{(i)}_1 \right\} \right) \mod \mathbb{Q}(T_0, R)$$

and hence $d'\alpha = \text{homology class of}$

$$\sum r_\sigma \left( \sum_{i=1}^{p} (-1)^i \left\{ \left\langle \sigma^{(i)} \right\rangle - \left\langle \sigma^{(i)}_1 \right\rangle \right\} \right)$$

$$= \sum r_\sigma \left( \sum_{i=1}^{p} (-1)^i \left\{ \sigma^{(i)} - \sigma^{(i)}_1 \right\} \right) \left\langle \sigma \right\rangle$$

$$= \sum_{i=1}^{p} (-1)^i \left\{ \sigma^{(i)} - \sigma^{(i)}_1 \right\} \left( \sum r_\sigma \left\langle \sigma \right\rangle \right)$$

and so

$$d'\alpha = \sum_{i=1}^{p} (-1)^i \left\{ \sigma^{(i)} - \sigma^{(i)}_1 \right\} \left( \sum r_\sigma \right).$$
Special Case 1: \( I : (B, B_0) \rightarrow (B, B_0) \).
Each \( \tau \in N_p(1) \) has a unique non-degenerate lift, namely \( \tau \).
Thus \( C^p(1)_n = \begin{cases} 0 & \text{if } n \neq p \\ \text{free } R\text{-module on } \langle \tau \rangle & \text{if } n = p \end{cases} \).
Thus \( C^p(1) \) has trivial boundary operator, and \( H_p(C^p(1)) = C^p(1)_p = \text{free } R\text{-module on } \langle \tau \rangle \). Since \( \delta \langle \tau \rangle = 0 \), we have \( \delta' \langle \tau \rangle = \sum_{i=1}^{n} (-1)^i \langle \tau^{(i)} \rangle \). 

Special Case 2: \( \pi : X \rightarrow P \), \( P \) a point.
Let \( \tau_0 : I^0 \rightarrow P \) denote the unique map \( \tau_0 \) is the only non-degenerate singular cube in \( P \), and every non-degenerate singular cube in \( X \) covers \( \tau_0 \). Thus \( C^{0}(\pi) = \Omega(X, R) \), and \( E_{0,1}^{1}(\pi) = H_p(C^{0}(\pi)) = H_p(X, R) \). \( \tau_0 \) has no faces, and as noted earlier, \( \delta' = 0 \).

The function space decomposition of \( E^1(\pi) \)

Let \( \pi : (T, T_0) \rightarrow (B, B_0) \) be a map pair, and \( \tau \in N_p(\pi) \).
Let \( X^\pi(\pi) = \{ \mu : I^p \rightarrow T | \mu \text{ covers } \tau \} \) with the compact-open topology.
If \( \sigma : I^p \rightarrow T \) is non-degenerate and covers \( \tau \), define \( \text{ad}^\pi \sigma : I^p \rightarrow X^\pi(\pi) \) by
\[
(\text{ad}^\pi \sigma)(u, t_1, \ldots, t_p) = \pi( u, \tau(u_1), \ldots, \tau(u_p), t_1, \ldots, t_p ) .
\]
Notice that \( (\text{ad}^\pi \sigma)(t_1, \ldots, t_p) \in X^\pi(\pi) \), for \( \pi( u_1, \ldots, u_p, t_1, \ldots, t_p) = \tau(u_1, \ldots, u_p) \) since \( \pi_{\tau_0} = \tau \).
Also, since \( \sigma(t_1, \ldots, t_p) \) is not independent of \( t_1, \ldots, t_p \) (since \( \sigma \) is non-degenerate), \( \text{ad}^\pi \sigma \) is non-degenerate.
Conversely, suppose \( \mu : I^p \rightarrow X^\pi(\pi) \) is non-degenerate.
Define \( \text{ad}_\mu : I^p \rightarrow T \) by
\[
(\text{ad}_\mu)(u, t_1, \ldots, t_p) = \mu(u_1, \ldots, u_p, t_1, \ldots, t_p) .
\]
Clearly, \( \text{ad}_\mu \) is non-degenerate. Since \( \mu(u_1, \ldots, u_p, t_1, \ldots, t_p) \) covers \( \tau \), \( \pi \text{ad}_\mu(u_1, \ldots, u_p, t_1, \ldots, t_p) = \pi \mu(u_1, \ldots, u_p, t_1, \ldots, t_p) = \tau(u_1, \ldots, u_p) \) and so \( \text{ad}_\mu \mu \) covers \( \tau \).

It is easily checked that \( \text{ad}^\pi \text{ad}_\mu = \mu \), \( \pi \text{ad}_\mu = \mu \), and so \( \text{ad}_\mu \) is a bijection from the set of non-degenerate singular \( p \)-cubes of \( X^\pi(\pi) \) to the...
set of non-degenerate singular $p+q$-cubos of $T$ which cover $c$. We thus obtain an $R$-isomorphism which raises degree by $p$

$$\alpha_T : Q(\chi^e(\pi); R) \rightarrow C^e(\pi)$$

given by $\alpha_T(\mu) = \langle \text{ad}_T \mu \rangle$, $\mu : \mathcal{I}^e \rightarrow \chi^e(\pi)$ non-degenerate.

We wish to show $\alpha_T$ is a chain map of degree $p$.

**Lemma:** Let $c \in \mathcal{N}_p(\pi)$, and suppose $\mu : \mathcal{I}^e \rightarrow \chi^e(\pi)$ is non-degenerate. Then for $1 \leq e \leq q$ and $\epsilon = 0, 1$,

$$\text{ad}_T(\mu^{(e)}_{\epsilon}) = (\text{ad}_T \mu)^{(p+\epsilon)}_{\epsilon}$$

(It is understood that $\text{ad}_T(\mu^{(e)}_{\epsilon}) = 0$ if $\mu^{(e)}_{\epsilon}$ is degenerate.)

**Proof:** $\text{ad}_T(\mu^{(e)}_{\epsilon})(u_{1}, \ldots, u_{p}, x_{1}, \ldots, x_{p-1}) =$

$$\mu^{(e)}_{\epsilon}(x_{1}, \ldots, x_{p-1})(u_{1}, \ldots, u_{p}) = \mu(x_{1}, \ldots, \epsilon, \ldots, x_{p-1})(u_{1}, \ldots, u_{p})$$

$$(\text{ad}_T \mu)^{(p+\epsilon)}_{\epsilon}(u_{1}, \ldots, u_{p}, x_{1}, \ldots, x_{p-1}) = (\text{ad}_T \mu)^{(p+\epsilon)}_{\epsilon}(u_{1}, \ldots, u_{p}, x_{1}, \ldots, x_{p-1})$$

$$= \mu(x_{1}, \ldots, \epsilon, \ldots, x_{p-1})(u_{1}, \ldots, u_{p}).$$

**Theorem:** $\alpha_T$ is a chain isomorphism of degree $p$ for each $c \in \mathcal{N}_p(\pi)$.

**Proof:** It remains only to show $\partial \alpha_T = (-1)^p \alpha_T \partial$.

Let $\mu : \mathcal{I}^e \rightarrow \chi^e(\pi)$ be non-degenerate. Then

$$\alpha_T \partial(\mu) = \alpha_T \sum_{i=1}^{q} (-1)^{i} \{ \mu^{(e)}_{0} - \mu^{(e)}_{i} \}$$

$$= \sum_{i=1}^{q} (-1)^{i} \{ \langle \text{ad}_T(\mu^{(e)}_{0}) \rangle - \langle \text{ad}_T(\mu^{(e)}_{i}) \rangle \},$$

$$\partial \alpha_T(\mu) = \partial \langle \text{ad}_T \mu \rangle = \sum_{i=1}^{q} (-1)^{i} \{ \langle (\text{ad}_T \mu)^{(e)}_{0} \rangle - \langle (\text{ad}_T \mu)^{(e)}_{i} \rangle \}$$

$$= \sum_{i=1}^{q} (-1)^{p+\epsilon+i} \{ \langle (\text{ad}_T \mu)^{(p+\epsilon)}_{0} \rangle - \langle (\text{ad}_T \mu)^{(p+\epsilon)}_{i} \rangle \}.$$
When \( \tau \in \mathbb{N} \) and \( \mu \in \mathbb{N} \), let \( \bar{X}_2^{(\mu)} \) denote the \( \mu \)-adic completion of \( \bar{X}_2 \), and let \( \bar{X}_2^{(\mu)} \) denote the \( \mu \)-adic completion of \( \bar{X}_2 \) with respect to the \( \mathbb{Z}_p \)-adic topology. Then for \( \mu \leq p \), we have:

\[
\tau_n \to \tau \quad \text{in} \quad \bar{X}_2^{(\mu)}.
\]

By the \( \mu \)-adic completeness of \( \bar{X}_2^{(\mu)} \), we conclude that:

\[
\tau_n \to \tau \quad \text{in} \quad \bar{X}_2^{(\mu)}.
\]

Thus, the isomorphism:

\[
(\mu, \bar{X}_2^{(\mu)}) \cong (\mu, \bar{X}_2^{(\mu)})
\]

is an isomorphism of \( \mu \)-adic completions.

In the other case, let \( \tau \) be an element of \( \bar{X}_2 \) such that \( \tau \notin \bar{X}_2 \). Define:

\[
\gamma \in \bar{X}_2^{(\mu)}
\]

Then:

\[
\gamma \in \bar{X}_2^{(\mu)}.
\]

By the \( \mu \)-adic completeness of \( \bar{X}_2^{(\mu)} \), we conclude that:

\[
\gamma \in \bar{X}_2^{(\mu)}.
\]

We now deal with the case of \( \tau \in \mathbb{Z}_p \). Let:

\[
\tau = \sum_{n=1}^{\infty} \frac{a_n}{p^n},
\]

where \( a_n \) are integers. Define:

\[
\bar{X}_2^{(\mu)} = \left\{ (\tau) \in \bar{X}_2^{(\mu)} : \tau \neq \mu \right\}.
\]

Then:

\[
\bar{X}_2^{(\mu)} = \left\{ (\tau) \in \bar{X}_2^{(\mu)} : \tau \neq \mu \right\}.
\]

Finally, we have:

\[
\left\{ \left( \sum_{n=1}^{\infty} \frac{a_n}{p^n} \right) \in \bar{X}_2^{(\mu)} : \tau \neq \mu \right\} = \left\{ \left( \sum_{n=1}^{\infty} \frac{a_n}{p^n} \right) : \tau \neq \mu \right\}.
\]

We claim that:

\[
\tau \to \sum_{n=1}^{\infty} \frac{a_n}{p^n},
\]

defines a morphism of \( \mathbb{Z}_p \)-algebras, and is an isomorphism of \( \mathbb{Z}_p \)-algebras.
\( p_{\varepsilon}(\cdot) : Q(X^e(\pi)) \to Q(X^{e\varepsilon}(\pi)) \) to be

\[
\begin{align*}
& \{ \varepsilon (\cdot) \} \text{ if } e \varepsilon (\cdot) \in N_{\varepsilon \varepsilon}(\pi) \\
& 0 \text{ if either } e \varepsilon (\cdot) \text{ is degenerate or } e \varepsilon (\cdot)(T^{-1}) \subset B_0.
\end{align*}
\]

**Lemma:** Let \( e \varepsilon \in N_{\varepsilon \varepsilon}(\pi) \). Then for \( i \leq i \leq p \) and \( e = 0, 1 \),

\[
\begin{align*}
\alpha_{\varepsilon} : Q(X^e(\pi), R) & \xrightarrow{e \varepsilon(\cdot)} Q(X^{e\varepsilon}(\pi), R) \\
\alpha_{\varepsilon}^* : C^* e(\pi) & \xrightarrow{e \varepsilon(\cdot)} C^* e\varepsilon(\pi)
\end{align*}
\]

commutes.

**Proof:** The result is trivial if \( e \varepsilon (\cdot) \notin N_{\varepsilon \varepsilon}(\pi) \). Assume \( e \varepsilon (\cdot) \in N_{\varepsilon \varepsilon}(\pi) \). Let \( \lambda : T \to X^e(\pi) \) be non-degenerate. Then

\[
\begin{align*}
\lambda_{\varepsilon}^* (\cdot) = x_{\varepsilon}^* (\cdot) = x_{\varepsilon}^* (\cdot) = \langle ad_{e \varepsilon} (e \varepsilon(\cdot)), \mu \rangle = \langle ad_e (e(\cdot)), \langle \mu \rangle \rangle \rangle.
\end{align*}
\]

Thus we must check \( ad_{e \varepsilon} (e \varepsilon(\cdot)), \mu \rangle = \langle ad_e (e(\cdot)), \langle \mu \rangle \rangle \rangle \).

We have \( ad_{e \varepsilon} (e \varepsilon(\cdot)), \mu \rangle \)

\[
\begin{align*}
& = \langle e(\cdot), \mu \rangle (a, \ldots, a, \ldots, a) \\
& = \langle e(\cdot), \mu \rangle (a, \ldots, a, \ldots, a) \\
& = \langle e(\cdot), \mu \rangle (a, \ldots, a, \ldots, a) \\
& = \langle e(\cdot), \mu \rangle (a, \ldots, a, \ldots, a) \\
& = \langle e(\cdot), \mu \rangle (a, \ldots, a, \ldots, a)
\end{align*}
\]

**Theorem:**

\[
\begin{align*}
\sum_{\varepsilon \in N_{\varepsilon \varepsilon}(\pi)} H^e_e(\chi^e(\pi), R) & \xrightarrow{\sum_{\varepsilon \in N_{\varepsilon \varepsilon}(\pi)}} H^e_e(\chi^e(\pi), R)
\end{align*}
\]

is given by

\[
\begin{align*}
\sum_{\varepsilon \in N_{\varepsilon \varepsilon}(\pi)} (-1)^i \sum_{\varepsilon \in N_{\varepsilon \varepsilon}(\pi)} \langle p_{\varepsilon}(\cdot), \mu \rangle = \sum_{\varepsilon \in N_{\varepsilon \varepsilon}(\pi)} \langle p_{\varepsilon}(\cdot), \mu \rangle
\end{align*}
\]
Proof: By the lemmas,

\[ \bigoplus_{\tau \in N_{p}(1)} \text{H}_{p+\delta}(C^{\tau}(\pi)) \xrightarrow{\mathbb{F}_{\delta}} \bigoplus_{\tau \in N_{p-1}(1)} \text{H}_{p-\delta-1}(C^{\tau}(\pi)) \]

\[ \bigoplus_{\tau \in N_{p}(1)} \text{H}_{\delta}(X^{\tau}(\pi); R) \xrightarrow{\mathbb{F}_{\delta}} \bigoplus_{\tau \in N_{p-1}(1)} \text{H}_{\delta}(X^{\tau}(\pi); R) \]

commutes.

Special Case 1: \( I: (B, B_0) \to (B, B_0) \), Since each \( \tau \in N_p(1) \) has \( \tau \) as its unique lift, the space \( X^{\tau}(1) \) consists of the single element \( \tau \). Let \( \mu_{\tau}: I^\circ \to X^\tau(1) \) denote the unique map. We have \( \text{ad}_{\tau}(\mu_{\tau})(u_1, \ldots, u_p) = \mu_{\tau}(\tau(u_1, \ldots, u_p)) = \tau(u_1, \ldots, u_p) \) and so \( \text{ad}_{\tau}(\mu_{\tau}) = \tau \). Thus \( \chi_{\tau}: \text{Q}(X^{\tau}(1); R) \to C^\tau(1) \) is the chain isomorphism of degree \( p \) given by \( \chi_{\tau}(\mu_{\tau}) = <\tau> \). Write \( \{\mu_{\tau}\} \in \text{H}_0(X^\tau(1); R) \) for the homology class of \( \mu_{\tau} \). We have

\[ E_{p,0}(1) \cong \bigoplus_{\tau \in N_{p}(1)} \text{H}_0(X^\tau(1); R) \]

For each \( 1 \leq i \leq p \), \( \varepsilon = 0, 1 \) for which \( \tau^{(i)}_{\varepsilon} \in N_{p-1}(1) \),

\( \check{\rho}_{\varepsilon}^{(i)}: X^{\tau^1}(1) \to X^{\tau^{(i)}_{\varepsilon}}(1) \) is the unique map of 1-point spaces. Since \( \{\mu_{\tau^1}\} \) and \( \{\mu_{\tau^{(i)}_{\varepsilon}}\} \) both have augmentation 1, it follows that \( \check{\rho}_{\varepsilon}^{(i)} \{\mu_{\tau^1}\} = \{\mu_{\tau^{(i)}_{\varepsilon}}\} \). Thus

\[ d^\varepsilon \{\mu_{\tau^1}\} = \sum_{i=1}^{p} (-1)^{\varepsilon} \left( \{\mu_{\tau^1}\} - \{\mu_{\tau^{(i)}_{\varepsilon}}\} \right) \]

Special Case 2: \( \pi: X \to P \), \( P \) a point. Let \( T_0: I^\circ \to P \) denote the unique non-degenerate singular cube in \( P \). For each \( x \in X \), let \( T_x: I^\circ \to X \) denote the singular 0-cube with image \( x \). Then \( X^{T_0}(\pi) = \{ T_x | x \in X \} \), and the function \( h: X^{T_0}(\pi) \to X \) given by \( h(T_x) = x = T_x(1) \) is a homeomorphism. For each non-degenerate \( \mu: I^\delta \to X^{T_0}(\pi) \), \( \text{ad}_{T_0}(\mu)(x_1, \ldots, x_p) = \mu(\varepsilon_1, \ldots, x_p)(0) = h\mu(\varepsilon_1, \ldots, x_p) \) and so
\[ \text{ad}_{\varphi}(u) = h(u). \text{ Thus } X_{\varphi} : Q(X, R) \to C^v(\varphi) = Q(X, R) \]
is the chain map \( Q(h) \), and \( (X_{\varphi})^{-1} = (h^{-1})_* : H_q(X, R) \to H_q(X, R) \).

**The Fibration Lemma**

Let \( \pi : (T, T_0) \to (B, B_0) \) be a map pair. For each \( x \in B \), write \( F_x = \pi^{-1}(x) \). Let \( \tau \in \pi_0(\pi) \). For each \( u \in \pi_0^p \), define

\[
\varepsilon_u^\pi : X^\pi(\tau) \to F_{\varepsilon(u)} \text{ by } \varepsilon_u^\pi(\tau) = \sigma(u).
\]

Each \( \varepsilon_u^\pi \) is continuous. The object of this section is to prove the following:

**Fibration Lemma**: Suppose \( \pi : T \to B \) is a fibration. Then for each \( \tau \in \pi_0(\pi) \) and \( u \in \pi_0^p \), \( \varepsilon_u^\pi : X^\pi(\tau) \to F_{\varepsilon(u)} \) is a homotopy equivalence.

The proof involves some basics about fibrations, which we proceed to describe.

Suppose \( \pi : T \to B \) is a fibration, and \( f : X \to B \) a continuous map. Define \( f^*(T) = \{(x,y) \in X \times T \mid f(x) = \pi(y)\} \), and \( \tilde{f} : f^*(T) \to T \) by \( \tilde{f}(x,y) = x \), \( \tilde{f}(x,y) = y \). Then

\[
\begin{array}{ccc}
X & \xrightarrow{\tilde{f}} & T \\
\downarrow & & \downarrow \pi \\
X & \xrightarrow{f} & B
\end{array}
\]

commutes.

Claim: \( \tilde{f} : f^*(T) \to X \) is a fibration.

For suppose given a commutative diagram of continuous maps

\[
\begin{array}{ccc}
Y \times 0 & \xrightarrow{g} & f^*(T) \\
\downarrow & & \downarrow \tilde{f} \\
Y \times I & \xrightarrow{h} & X
\end{array}
\]
Then the outer rectangle

\[
\begin{array}{ccc}
Y \times 0 & \overset{f} \to & T \\
\downarrow & & \downarrow \pi \\
Y \times I & \overset{f} \to & B
\end{array}
\]

commutes, and so

since \( \pi \) is a fibration, a continuous \( F \) exists so that both triangles commute. Define \( H: Y \times I \to f^*(T) \) by \( H(z) = (G(z), F(z)) \). Then \( H \) is continuous, and both triangles in

\[
\begin{array}{ccc}
Y \times 0 & \overset{g} \to & f^*T \\
\downarrow & & \downarrow \rho \\
Y \times I & \overset{G} \to & X
\end{array}
\]

commute.

\( C: f^*T \to X \) is called the fibration induced from \( \pi \) by \( f \).

Note that for each \( x \in X \), \( f \) carries \( \pi^{-1}(x) \) homeomorphically onto \( \pi^{-1}(f(x)) \).

Let \( T, u \) be as in the statement of the fibration lemma. Form the induced fibration \( C: \tau^*(T) \to I^p \). Write \( F_u = \tau^*(u) \). Then \( \tau \) maps \( F_u \) homeomorphically onto \( F_{\tau(u)} \).

Let \( \Gamma^*(T) = \text{space of sections of } C \)

\[ = \{ \sigma: I^p \to \tau^*(T) \mid \text{}\sigma = \int_{\tau^p} \} \]

with the compact-open topology. Define \( f: \Gamma^*(T) \to \chi^c(\pi) \), \( g: \chi^c(\pi) \to \Gamma^*(T) \) by

\[ f(\Gamma) = \tau \sigma(\eta) \]

\[ g(\mu)(\eta) = (\eta, \mu(\eta)) \]

\( f \) and \( g \) are continuous, and it is easily checked that they are homeomorphisms, inverse to one another.

For each \( u \in I^p \) define \( e_u: \Gamma^*(T) \to F_u \) by \( e_u(\eta) = \tau(\eta) \).

Then the diagram

\[
\begin{array}{ccc}
F_u & \overset{\tau}{\to} & F_{\tau(u)} \\
\downarrow e_u & & \downarrow e_{\tau(u)} \\\n\Gamma^*(T) & \overset{f}{\to} & \chi^c(\pi)
\end{array}
\]

commutes.

Since \( f \) and \( \tau \mid_{F_u} \) are homeomorphisms, the proof of the Fibration Lemma will be completed by showing \( e_u \) is a homotopy equivalence.
Def: Two fibrations $\pi_1 : T_1 \to B$, $\pi_2 : T_2 \to B$ are said to be fibre-homotopy equivalent if there exist fibre-preserving maps $\alpha : T_1 \to T_2$, $\beta : T_2 \to T_1$.

(i.e. $\pi_1 \xrightarrow{\alpha} T_2 \xrightarrow{\beta} \pi_2 \xrightarrow{\pi_1}$ both commute)

Such that $\alpha \beta$ is fibrewise homotopic to $1_{T_2}$ and $\beta \alpha$ is fibrewise homotopic to $1_{T_1}$ (i.e. there exist homotopies $h_i : T_i \times I \to T_i$, $i = 1, 2$ such that $\beta \alpha \cong 1_{T_1}$, $\alpha \beta \cong 1_{T_2}$, and $\pi_i \cdot h_i(x, t) = \pi_i(x)$ for all $(x, t) \in T_i \times I$, $i = 1, 2$.)

Prop: Let $\pi : T \to B$ be a fibration with $B$ contractible. Then this fibration is fibre-homotopy equivalent to the product bundle $\pi_i : B \times F \to B$ for any $i \in B$.

The proof of this proposition can be found in Spanier, pp. 100-102 (it is corollary 15 on p. 102).

The proof of the Fibration Lemma is now completed as follows: Since $I^0$ is contractible, there exist fibre-homotopy equivalences $\tau^*(T) \xrightarrow{\alpha} I^0 \times F \xrightarrow{\beta} \tau^*(T)$ and a fibrewise homotopy $h : \tau^*(T) \times I \to \tau^*(T)$ from $\beta \alpha$ to $1_{\tau^*(T)}$. Define $g : F \to \Gamma(\mathbb{P})$ by $g(y)(x) = \beta(x, \pi_2 \alpha(y))$ where $\pi_2 : I^0 \times F \to F$ is projection on the 2nd factor. $g(y) \in \Gamma(\mathbb{P})$ for each $y \in F$, for $g(y)(x) = \beta(x, \pi_2 \alpha(y)) = \pi_1(x, \pi_2 \alpha(y))$ (since $\beta$ is fibre-preserving) $= x$. We proceed to show $E u g \cong 1_{F \mathbb{P}}$ and $g E u \cong 1_{\mathbb{P}(\mathbb{P})}$.

For all $y \in F \mathbb{P}$, $\pi_1 \alpha(y) = u$ since $\alpha$ is fibre-preserving.
Thus \( e_u g(y) = g(y)(u) = \beta(u, \pi_2 \alpha(y)) = \beta(\pi_1 \alpha(y), \pi_2 \alpha(y)) \)
\( = \beta \alpha(y) \) and so \( e_u g = \beta \alpha | \text{Fe} : \text{Fe} \to \text{Fe} \). Since \( h \) is a fibrewise homotopy, \( h | \text{Fe} \times \text{I} \) is a homotopy from \( \beta \alpha | \text{Fe} \) to \( 1_{\text{Fe}} \). Thus \( e_u g = 1_{\text{Fe}} \).

For each \( \tau \in \Gamma(\text{p}) \), \( \theta \circ e_u(\tau)(x) = \beta(\tau(u), \pi_2 \alpha(\tau(x))) \).
Let \( k : \text{I}^0 \times \text{I} \to \text{I}^0 \) be a homotopy from the constant map with value \( u \) to \( 1_{\text{I}^0} \) (it exists since \( \text{I}^0 \) is contractible). For each \( \tau \in \text{I} \), the map \( \omega_\tau : \text{I}^0 \to \text{I}^0(\text{I}) \) given by \( \omega_\tau(x) = \beta(x, \pi_2 \alpha(\tau(x))) \) is a section, for \( \beta \circ (x, \pi_2 \alpha(\tau(x))) = \pi_1(\tau(x)) \). We have \( \omega_\tau(x) = \beta(x, \pi_2 \alpha(\tau(x))) \) since \( \pi_1 \alpha = \beta \). Thus the map \( \Gamma(\text{p}) \times \text{I} \to \Gamma(\text{p}) \) given by \( (\tau, x) \mapsto \omega_\tau(x) \) is a homotopy from \( \theta \circ e_u \) to the map \( \lambda \) given by \( \lambda(\tau) = \beta \alpha(\tau) \). Finally, using the fibrewise homotopy \( h \) from \( \beta \alpha \) to \( 1_{\text{I}^0(\text{I})} \), the map \( L : \Gamma(\text{p}) \times \text{I} \to \Gamma(\text{p}) \) given by \( L(\tau, x) = h(\tau(x), x) \) is a homotopy from \( \lambda \) to \( 1_{\Gamma(\text{p})} \), completing the proof of the Fibration Lemma.

**Special Case 1:** \( I : (B, B_0) \to (B, B_0) \). As noted earlier, \( X^C(1) \)

is a 1-point space for each \( \tau \in N_p(1) \). Each fibre is also

a 1-point space, and so each \( e_u \) is a homeomorphism of

1-point spaces.

**Special Case 2:** \( \pi : X \to P \), \( P \) a point. There is only one

fibre, namely \( F_P = X \), and \( e_P^c : X^c(\pi) \to X \) is the

homeomorphism A described earlier.

The twisted \( \mathcal{A}^c(B, B_0) \otimes H(F) \) description of \( E^I \)

for a fibration pair

Let \( \pi : (E, E_0) \to (B, B_0) \) be a fibration pair, i.e. a

map pair with \( \pi : T \to B \) a fibration.

Write \( O = (0, \ldots, 0) \in \text{I}^p \). If \( \tau \in N_p(\pi) \), write

\( F_\tau = F_{\tau(0)} \). The isomorphisms \( (E^c)_\tau : H_F(X^c(\pi) \otimes R) \to H_F(F_{\tau(0)} \otimes R) \),
together with the function space description of \( E^I \), yield
a natural isomorphism \( E^l_{\pi\mu}(\pi) \cong \bigoplus_{\tau \in \pi(\pi)} H_{\tau}(F; R) \).

Note: This last direct sum is indexed by the \( \tau \in \pi(\pi) \).

If it happens that \( \tau \neq \tau' \) but \( \tau(\circ) = \tau'(\circ) \), then \( F_{\tau} = F_{\tau'} \), but \( H_{\tau}(F; R) \) and \( H_{\tau'}(F; R) \) occur as separate summands. In order to keep track of summands, write \( \tau \times H_{\tau}(F; R) \) as an isomorphic copy of \( H_{\tau}(F; R) \).

Thus if \( \tau \neq \tau' \) but \( F_{\tau} = F_{\tau'} = F \), \( \tau \times H_{\tau}(F; R) \) is notationally different from \( \tau' \times H_{\tau'}(F; R) \).

We will write \( \tilde{\pi}_0 : H_{\tau}(X^\tau(\pi); R) \to \tau \times H_{\tau}(F; R) \) for the map

\[
\tilde{\pi}_0(a) = (\tau, (e^\tau)_* (a)).
\]

Thus the \( \tilde{\pi}_0 \) yield a natural isomorphism

\[
E^l_{\pi\mu}(\pi) \cong \bigoplus_{\tau \in \pi(\pi)} \tau \times H_{\tau}(F; R).
\]

It is convenient to define \( \tau \times H_{\tau}(F; R) = 0 \) if \( \tau \) is either degenerate or has image in \( B_0 \).

Note: If it were possible to naturally identify the various \( H_{\tau}(F; R) \) with \( H_{\tau'}(F; R) \) for a single \( F \), then the map

\[
Q_{\pi}(B; B_0) \otimes H_{\tau}(F; R) \to \bigoplus_{\tau \in \pi(\pi)} \tau \times H_{\tau}(F; R)
\]

sending \( \tau \otimes a \mapsto (\tau, a) \) would be an isomorphism.

An additional hypothesis (orientability) will later be required to achieve this.

We must now identify \( d' \) in terms of this last description of \( E^l(\pi) \).

Let \( \pi : (T, T_0) \to (B, B_0) \) be a fibration pair. Write \( \pi_a : T \to B \) for its underlying absolute pair.

Let \( \lambda : I \to B \) be a path from \( x \) to \( y \). If \( \lambda \) is not constant, then \( \lambda \in N_1(\pi_a) \) and by the fibration lemma we have homotopy equivalences.
Thus we have an isomorphism

$$h_\lambda \overset{\text{det}}{=} (e_0^\lambda)_*(e_1^\lambda)^{-1} : H^*_0(F_{i_0}; R) \to H^*_0(F_{i_1}; R).$$

If $\lambda$ is the constant map at $x$, define

$$h_\lambda = 1_{H^*_0(F_{i_0}; R)}.$$

Let $\tau \in N_p(\pi)$, $p \geq 1$. For $1 \leq i \leq p$,

$$\tau^{(i)}(0) = \tau(0),$$

and so $F_{i_0} = F_{\tau^{(i)}}$.

We have

$$\tau^{(i)}(0) = (0, \ldots, 1-t, \ldots, 0).$$

Let $\lambda^{(i)} : I \to I^p$ be given by $\lambda^{(i)}(t) = (0, \ldots, 1-t, \ldots, 0)$.

Then $\tau \lambda^{(i)}$ is a path in $B$ from $\tau^{(i)}(0)$ to $\tau(0)$.

Lemma: Let $\tau \in N_p(\pi)$, $p \geq 1$. Then for $1 \leq i \leq p$,

$$X^{\tau}(\pi) \xrightarrow{e_0^\tau} F_{\tau_0} \xrightarrow{1} F_{\tau_0^{(i)}} \xleftarrow{\xi_0^{(i)}} X^{\tau^{(i)}}(\pi) \xrightarrow{e_0^{(i)}} F_{\tau_0^{(i)}},$$

commutes (provided $\tau^{(i)} \in N_{p-1}(\pi)$), and

$$H^*_0(X^{\tau}(\pi); R) \xrightarrow{(e_0^\tau)_*} H^*_0(F_{\tau_0}; R) \xrightarrow{e_0^{(i)}_*} H^*_0(F_{\tau_0^{(i)}}; R) \xrightarrow{(e_0^{(i)})_*} H^*_0(X^{\tau^{(i)}}(\pi); R).$$

commutes (provided $\tau^{(i)} \in N_{p-1}(\pi)$).

Proof: Commutativity of the first diagram is immediate.

Define $\alpha : X^{\tau}(\pi) \to X^{\tau \lambda^{(i)}}(\pi) \pi$ by

$$\alpha((\tau)(t)) = \tau(0, \ldots, 1-t, \ldots, 0).$$

The latter does lie in $X^{\tau \lambda^{(i)}}(\pi)$,
for $\pi_0(0, \ldots, 1-t, \ldots, 0) = \pi_1(0, \ldots, 1-t, \ldots, 0) = \pi \lambda_{\gamma}(t)$. It is easily checked that the following commutes:

The commutativity of the 2nd diagram now follows.

**Theorem:** Let $\pi$ be a fibration pair. Then for $p \geq 1$,

$$d^i : \bigoplus_{\tau \in N_p(\pi)} \tau \times H_b^i(F_{\pi}; \mathbb{R}) \to \bigoplus_{\tau \in N_{p-1}(\pi)} \tau \times H_b^i(F_{\pi}; \mathbb{R})$$

is given as follows: For $\tau \in N_p(\pi)$, $a \in H_b^i(F_{\pi}; \mathbb{R})$, $d^i(\tau, a) = \sum_{i=1}^{p} (-1)^i \left\{ (\tau_{\gamma_0}^i, a) - (\tau_{\gamma_i}^i, h_{\pi \lambda_{\gamma_i}}(a)) \right\}$.

**Proof:** Write $d^i : \bigoplus_{\tau \in N_p(\pi)} H_b^i(X^\pi(\pi); \mathbb{R}) \to \bigoplus_{\tau \in N_{p-1}(\pi)} H_b^i(X^\pi(\pi); \mathbb{R})$ for $d^i$ using the function space description of $E^i$. We have

The commutative diagram.
Given $\sigma \in N_p(\pi)$, $\alpha \in H_f(F_{\sigma};R)$, we can write $(\sigma, \alpha) = \overline{\epsilon}_{\sigma}(b)$ for some $b \in H_f(X^\sigma(\pi);R)$. Thus $\alpha = (\epsilon_{\sigma})_*(b)$. Then

\[ d'(\sigma, \alpha) = d'(\overline{\epsilon}_{\sigma}(b)) = \left( \bigoplus_{\tau \in N_{p-1}(\pi)} \overline{\epsilon}_{\tau} \right) \left( d'(b) \right) \]

\[ = \left( \bigoplus_{\tau \in N_{p-1}(\pi)} \overline{\epsilon}_{\tau} \right) \sum_{i=1}^{p} (-1)^i \left\{ P^{(i)}_{\sigma \lambda} (b) - P^{(i)}_{\lambda \sigma} (b) \right\} \]

\[ = \sum_{i=1}^{p} (-1)^i \left\{ \overline{\epsilon}_{\sigma} \circ \sigma_{\lambda}^{-1} \circ P^{(i)}_{\sigma \lambda} (b) - \overline{\epsilon}_{\lambda} \circ \sigma_{\lambda}^{-1} \circ P^{(i)}_{\lambda \sigma} (b) \right\} \]

\[ = \sum_{i=1}^{p} (-1)^i \left\{ (\sigma_{\lambda}) (\sigma \circ \lambda) \circ \sigma^{-1} \circ P^{(i)}_{\lambda \sigma} (b) - (\sigma_{\lambda}) \circ \sigma^{-1} \circ P^{(i)}_{\lambda \sigma} (b) \right\} \]

\[ = \sum_{i=1}^{p} (-1)^i \left\{ (\sigma_{\lambda}) (\sigma \circ \lambda) \circ \sigma^{-1} - (\sigma_{\lambda}) \circ \sigma^{-1} \circ \sigma_{\lambda}^{-1} \right\} \] (by Lemma a)

\[ = \sum_{i=1}^{p} (-1)^i \left\{ (\sigma_{\lambda}) (\sigma \circ \lambda) - (\sigma_{\lambda}^{-1}) \circ \sigma \circ \lambda \right\} (\sigma_{\lambda}^{-1} \circ \sigma \circ \lambda) (\alpha) \]
Special Case 1: \( I : (B, B_0) \rightarrow (B, B_0) \). Each \( e_\xi : X^\xi \rightarrow F_\xi \) is a homeomorphism of 1-point spaces. For each \( x \in B \), let \( e_\xi \in H_0(F_\xi; R) \) be the element of augmentation 1. Let \( \lambda : I \rightarrow B \) be a path in \( B \) from \( x \) to \( y \). Since \( e_0^0 \) and \( e_1^0 \) are augmentation-preserving, so is \( h_\lambda : H_0(F_\xi; R) \rightarrow H_0(F_\xi; R) \) and so \( h_\lambda(a) = a \). If we write \( \tilde{a}_\xi = (c, a_{\xi(o)}) \in \mathbb{C} \times H_0(F_\xi; R) \),

\[ \bigoplus_{c \in \mathbb{C}} \mathbb{C} \times H_0(F_\xi; R) \] is the free \( R \)-module with basis \( \{ \tilde{a}_\xi \mid c \in \mathbb{C} \} \).

If \( p > 0 \) and \( 1 \leq i \leq p \) we have \( h_\lambda a = a \) and we obtain

\[ d^1(\tilde{a}_\xi) = \sum_{i=1}^{p} (-1)^i (\tilde{a}_\xi - \tilde{a}_{\xi_{i-1}}) \].

Special Case 2: \( \pi : X \rightarrow P \), \( P \) a point. If \( \tau_0 : I^p \rightarrow P \) is the unique \( \mathcal{O} \)-cube in \( P \), \( F_{\tau_0} = X \) and

\[ \tilde{e}_\tau : H_0^p(X^\tau; R) \rightarrow \mathbb{C} \times H_0(X; R) \] is given by

\[ \tilde{e}_\tau(a) = (\tau_0, h_\tau(a)) \] where \( h : X^\tau \rightarrow X \) is the homeomorphism described earlier.

Properties of the isomorphisms \( h_\lambda \)

**Lemma:** Let \( \pi : (I, I_0) \rightarrow (B, B_0) \) be a fibration pair. Let \( \lambda, \mu : I \rightarrow B \) be paths such that \( \lambda(0) = \mu(0) \). Then \( h_{\lambda \mu} = h_\lambda h_\mu \).

**Proof:** We have continuous maps \( \alpha : X^{\lambda \mu}(\pi) \rightarrow X^{\lambda}(\pi) \),

\[ \beta : X^{\lambda \mu}(\pi) \rightarrow X^{\mu}(\pi) \] given by

\[ \alpha(x) = \pi \left( \frac{1 + x}{2} \right), \quad \beta(x) = \pi \left( \frac{1 + x}{2} \right). \]

Write \( x = \lambda(0), \ y = \lambda(1) = \mu(0), \ z = \lambda(1) \). The following diagram commutes:
Thus \( h_{\lambda} = (e^{\lambda})_*(e_1^{\mu})^{-1} = (e_0^{\lambda+\mu})_*(e_1^{\lambda+\mu})^{-1} \).
\( h_{\mu} = (e_0^{\mu})_*(e_1^{\mu})^{-1} = (e_0^{\lambda+\mu})_*(e_1^{\lambda+\mu})^{-1} \)

and so \( h_{\lambda} h_{\mu} = (e_0^{\lambda+\mu})_*(e_1^{\lambda+\mu})^{-1} (e_{\frac{1}{2}}^{\lambda+\mu})_*(e_1^{\lambda+\mu})^{-1} = (e_0^{\lambda+\mu})_*(e_{\frac{1}{2}}^{\lambda+\mu})^{-1} \).

**Lemma 2:** Let \( f : \pi \to \pi' \) be a morphism of fibration pairs. Let \( \lambda : I \to \mathcal{B} \) be a path from \( x \) to \( y \). Then

\[
\begin{align*}
H_t (F_y ; R) &\xrightarrow{h_{\lambda}} H_t (F_x ; R) \\
(f_y)_* &\downarrow && (f_x)_* \\
H_b (F_{f(y)} ; R) &\xrightarrow{h_{\lambda}^{-1}} H_b (F_{f(x)} ; R)
\end{align*}
\]

commutes where \( f_x : F_x \to F_{f(x)} \), \( f_y : F_y \to F_{f(y)} \) are the restrictions of \( f \) to the fibres.

**Proof:** We have the commutative diagram
Lemma 3: Let $\pi : (T,T_o) \to (B,B_o)$ be a fibration pair. Suppose $\lambda, \mu : I \to B$ are paths which are homotopic rel endpoints. Then $h_\lambda = h_\mu$.

Proof: Let $H : I \times I \to B$ be a homotopy rel endpoints from $\lambda$ to $\mu$. Say $\lambda(0) = \mu(0) = x$, $\lambda(1) = \mu(1) = y$.

Note that $\lambda = H_0^{(2)}$, $\mu = H_1^{(2)}$ and we have the maps $p_0^{(2)} : X^H(\pi) \to X^{\lambda}(\pi)$, $p_1^{(2)} : X^H(\pi) \to X^{\mu}(\pi)$.

The following diagram commutes:

Since $H(0,x) = x$ for all $x \in I$, we obtain a homotopy $G : X^H(\pi) \times I \to F_x$ from $e_0^H$ to $e_0^H$ given by
\[ G(\tau, t) = e^{H}_{(\omega, \tau)}(\tau) \]  
Hence \[ e^{H}_{(\omega, \tau)}* = e^{H}_{(\omega, \tau)}* \]. Similarly
\[ e^{H}_{(1, 0)}* = e^{H}_{(1, 0)}*. \]  
Hence
\[
\begin{align*}
\lambda &= (e^A_0)^* (e^A_1)^{-1} = \\
&= (e^H_{(\omega, \tau)})^* (e^H_{(\omega, \tau)})^{-1} (e^H_{(\omega, \tau)})^* (e^H_{(\omega, \tau)})^{-1} = H_{\lambda u}.
\end{align*}
\]

**Orientability**

**Definition:** Let \( \pi : T \to B \) be a fibration with \( B \) path-connected. \( \pi \) is said to be R-orientable if whenever \( x, y \in B \) and \( \lambda, \mu : I \to B \) are paths from \( x \) to \( y \), then
\[ h_{\lambda} = h_{\mu} : H_0(F_x; R) \to H_0(F_y; R) \]  
for all \( g \).

**Proposition:** Let \( \pi : T \to B \) be a fibration with \( B \) simply-connected. Then \( \pi \) is R-orientable for every \( R \).

**Proof:** This is immediate from Lemma 3 of the preceding section since the simple-connectivity of \( B \) implies that any two paths from \( x \) to \( y \) are homotopic rel end points.

**Proposition:** Let \( \pi : T \to B \) be an R-orientable fibration. Then for any continuous map \( f : X \to B \) with \( X \) path-connected, the induced fibration \( \tilde{f} : f^*T \to X \) is R-orientable.

**Proof:** The maps \( \tilde{f} : f^*T \to T \), \( f : X \to B \) constitute a morphism of fibrations \( \tilde{f} \to \pi \), and \( \tilde{f}_x : F_x \to F_{f(x)} \) are homeomorphisms. Hence \( \tilde{f}_x* \) and \( \tilde{f}_y* \) are isomorphisms for any \( x, y \in X \).

Let \( \lambda, \mu : I \to X \) be paths from \( x \) to \( y \). By Lemma 2 of the previous section,
\[
\begin{align*}
\lambda &= (\tilde{f}_x*)^{-1} \lambda (\tilde{f}_y*) \quad &\lambda (\tilde{f}_x*)^{-1} = \\
\mu &= (\tilde{f}_x*)^{-1} \mu (\tilde{f}_y*) \quad &\mu (\tilde{f}_x*)^{-1} = \\
\end{align*}
\]
But since \( \pi \) is R-orientable and \( \lambda, \mu \) are paths in \( B \) from \( f(x) \) to \( f(y) \), we have
\[ h_{\lambda} = h_{\mu} \]  
Hence \( h_{\lambda} = h_{\mu} \).

**Note:** If \( \pi : T \to B \) is a fibration such that \( F_x = \pi^{-1}(x) \) is path-connected for every \( x \in B \), and if \( \lambda : I \to B \) is a path
From $x$ to $y$, then from commutativity of

$$H_0(F_y;R) \xrightarrow{h_{x\lambda}} H_0(F_x;R)$$

it follows that $h_{x\lambda}$ is independent of $\lambda$ in dimension 0. Thus if $B$ is path-connected and if for some $x \in B$, $F_x$ is path-connected and $H_i(F_x;R) = 0$ for $i > 0$, $\pi$ will be $R$-orientable.

In particular if $B$ is path-connected, $1: B \to B$ is $R$-orientable for every $R$.

**Notation:** If $\pi : T \to B$ is an $R$-orientable fibration and $x, y \in B$, write $h_{x,y} : H_0(F_y;R) \to H_0(F_x;R)$ for $h_{x\lambda}$ where $\lambda$ is any path in $B$ from $x$ to $y$.

**Proposition:** Let $\pi : T \to B$ be an $R$-orientable fibration. Then whenever $x, y, z \in B$, $h_{x,y} h_{y,z} = h_{x,z}$.

**Proof:** This follows immediately from Lemma 1 of the previous section.

**Proposition:** Let $f : \pi \to \pi'$ be a morphism of $R$-orientable fibrations. Then for any $x, y \in B$, the diagram

$$
\begin{array}{ccc}
H_0(F_y;R) & \xrightarrow{h_{x,y}} & H_0(F_x;R) \\
(F_y)_* \downarrow & & \downarrow (F_x)_* \\
H_0(F_{\pi(y)};R) & \xrightarrow{h_{(x),(y)}} & H_0(F_{\pi(x)};R)
\end{array}
$$

commutes.

**Proof:** This follows immediately from Lemma 2 of the previous section.
The Leray - Serre Theorem

Theorem: Let \( R \) be a commutative ring with unit \( 1 \neq 0 \). Let \( \mathcal{F}_R \) denote the category of map pairs \( \Im : (T, T_0) \to (B, B_0) \) such that \( B \) is path-connected and \( F \to T \to B \) is an \( R \)-orientable pointed fibration. Then there is a covariant functor \( \pi : E^2(\pi) \to E^\infty(\pi) \) from \( \mathcal{F}_R \) to the category of \( \mathbb{Z}^+ \) quadrant spectral sequences of \( R \)-modules satisfying

1) there are natural \( R \)-isomorphisms

\[
\Psi : H_p(B, B_0; H_q(F, R)) \cong E^2_{p,q}(\pi)
\]

2) for each \( n \) there is a natural filtration

\[
O = J_{-1,n+1}(\pi) \subset J_{0,n}(\pi) \subset \cdots \subset J_{n,0}(\pi) = H_n(T, T_0; R)
\]

and natural \( R \)-isomorphisms

\[
\alpha_p : J_{p,q}(\pi)/J_{p-1,q+1}(\pi) \cong E^\infty_{p,q}(\pi)
\]

Proof: Write \( F = \pi^{-1}(x_0) \), \( x_0 \) the base-point of \( B \). All that remains in the proof is to produce natural isomorphisms

\[
\gamma : \bigoplus \mathcal{T} \times H_q(F_\mathcal{T}; R) \longrightarrow \mathcal{Q}_p(B, B_0) \otimes H_q(F; R)
\]

which commute with the boundary maps. The boundary map on the left is \( d' \), and that on the right is \( d \otimes 1 \).

Define \( \gamma \) by

\[
\gamma(\mathcal{T}, a) = \mathcal{T} \otimes h_{x_0, \pi(0)}(a)
\]

whenever \( \mathcal{T} \in \mathcal{N}_p(\pi) \), \( a \in H_q(F; R) \). Since the \( \mathcal{A}_{xy} \) are isomorphisms and \( \mathcal{Q}_p(B, B_0) \) is free abelian on \( \mathcal{N}_p(\pi) \), it is immediate that \( \gamma \) is an isomorphism. Naturality of \( \gamma \) follows easily from the naturality property of the \( \mathcal{A}_{xy} \). It remains only to show

\[
\delta d'(\mathcal{T}, a) = (d \otimes 1) \gamma(\mathcal{T}, a)
\]

whenever \( \mathcal{T} \in \mathcal{N}_p(\pi) \), \( a \in H_q(F; R) \). We have

\[
\begin{align*}
\delta d'(\mathcal{T}, a) &= \gamma \left( \sum_{i=1}^p (-1)^i \left\{ (\mathcal{T}^{(i)}, a) - (\mathcal{T}^{(i)}, h_{x_0, \pi(0)}(a)) \right\} \right) \\
&= \gamma \left( \sum_{i=1}^p (-1)^i \left\{ (\mathcal{T}^{(i)}_0, a) - (\mathcal{T}^{(i)}_1, h_{x_0, \pi(0)}(a)) \right\} \right) \\
&= \sum_{i=1}^p (-1)^i \left\{ \mathcal{T}^{(i)}_0 \otimes h_{x_0, \pi(0)}(a) - \mathcal{T}^{(i)}_1 \otimes h_{x_0, \pi(0)}(a) \right\}
\end{align*}
\]
\[
\begin{align*}
&= \sum_{i=1}^{p} (-1)^i \left\{ \tau_0^{(i)} \otimes h_{x_0, \tau(c)}(a) - \tau_1^{(i)} \otimes h_{x_0, \tau(c)}(a) \right\} \\
&\quad \text{(since } \tau_0^{(i)}(o) = \tau(c) \text{, and } h_{x,y} h_{y,z} = h_{x,z}) \\
&= \left( \sum_{i=1}^{p} (-1)^i \{ \tau_0^{(i)} - \tau_1^{(i)} \} \right) \otimes h_{x_0, \tau(c)}(a) \\
&= \left( \exists \tau \right) \otimes h_{x_0, \tau(c)}(a) = (\exists \otimes 1)(\tau \otimes h_{x_0, \tau(c)}(a)) \\
&= (\exists \otimes 1) \chi(\tau_a), \text{ completing the proof.}
\end{align*}
\]

Note: Naturality of \( \psi \) means the following: By definition, if \( \pi : (T, T_0) \to (B, B_0) \) is an object in \( \mathcal{F} \), then \( B \) is pointed. For any morphism \( f \) in \( \mathcal{F} \), \( f_B \) is a pointed map. Let \( f : \pi \to \pi' \) be a morphism in \( \mathcal{F} \). Write \( F = \pi^{-1}(x_0) \), \( F' = (\pi')^{-1}(x_0') \) where \( x_0, x_0' \) are the base points in \( B \) and \( B' \), respectively. By restriction of \( f_T \) we get a map \( f_F : F \to F' \), and thus a chain map

\[
\chi(f_B) \otimes H(f_T) : \chi(B, B_0) \otimes H_\mu(F; R) \to \chi(B', B_0') \otimes H_\mu(F'; R).
\]

Naturality of \( \psi \) means that

\[
\begin{array}{ccc}
H_\mu(B, B_0; H_\mu(F; R)) & \xrightarrow{[\chi(f_B) \otimes H(f_T)]_\phi} & H_\mu(B', B_0'; H_\mu(F'; R)) \\
\psi(\pi) & \mapsto & \psi(\pi') \\
E^2_{f_T}(\pi) & \xrightarrow{E^2_{f_T}(f)} & E^2_{f_T}(\pi')
\end{array}
\]

commutes.

Note: \( \chi(f_B) \otimes H(f_T) = (\chi(f_B) \otimes 1)(1 \otimes H(f_T)) = (1 \otimes H(f_T))(\chi(f_B) \otimes 1) \).

Thus we obtain the commutative diagram.
The Base - Edge Theorem

Special Case 1: \((B, B_0) \rightarrow (B, B_0), B\) path-connected. We investigate the chain isomorphism \(\gamma\) in the proof of the Leray-Serre Theorem in this case. Choose a base point \(x_0 \in B\), and write \(\alpha_0 = \alpha_{x_0} \in H_0(F_{x_0}; R) = H_0(\{x_0\}; R)\), using the earlier notation (p. 25). Recall that for any \(x \in B\), \(h_{x, x}(\alpha_{x_0}) = \alpha_{x_0}\). Thus for any \(\tau \in \pi_1 (1)\), \(\gamma(\tau) = \gamma(\tau, \alpha_{x_0}) = \tau \otimes \alpha_{x_0}\). We now look at the composition of all the natural chain isomorphisms in the preceding from the original \(E^1_{\gamma, x_0}(1)\) to \(Q(B, B_0) \otimes H_0(\{x_0\}; R)\).

Original \[E^1_{\gamma, x_0}(1) = Q_p(B, B_0; R) = Q_p(B, B_0) \otimes \frac{R}{2} \rightarrow \tau \otimes 1\]

\[Q(B, B_0)\) decomposition \[\bigoplus_{\tau \in \pi_1 (1)} H_p(C^\tau(1); R) = \bigoplus_{\tau \in \pi_1 (1)} C^\tau(1) \bigoplus_{\tau \in \pi_1 (1)} \bigoplus_{x \in X_0} \phi x \bigoplus_{\tau \in \pi_1 (1)} \leftarrow \bigoplus_{\tau \in \pi_1 (1)} X^\tau(1); R \bigoplus_{\tau \in \pi_1 (1)} H_0(F_{x_0}; R) \bigoplus_{\tau \in \pi_1 (1)} Q_p(B, B_0) \otimes H_0(\{x_0\}; R) \bigoplus_{\tau \in \pi_1 (1)} \tau \otimes \alpha_{x_0} \bigoplus_{\tau \in \pi_1 (1)} \left(\text{since } \varepsilon(\alpha_{x_0}) = 1\right) \]

Function space decomposition \[\bigoplus_{\tau \in \pi_1 (1)} H_0(X^\tau(1); R) \bigoplus_{\tau \in \pi_1 (1)} \tau \otimes H_0(F_{x_0}; R) \bigoplus_{\tau \in \pi_1 (1)} Q_p(B, B_0) \otimes H_0(\{x_0\}; R) \bigoplus_{\tau \in \pi_1 (1)} \tau \otimes \alpha_{x_0} \]
Thus, since all the vertical maps are isomorphisms and $\varepsilon(\alpha_0) = 1$, the composition of the inverses of the above is the chain map

$$1 \otimes \varepsilon : Q(B, B_0) \otimes H_0(\mathbf{T} \otimes R) \longrightarrow Q(B, B_0) \otimes R = E^1_{\infty, 0}(1).$$

Thus the natural isomorphism

$$\Psi : H_p(B, B_0; H_0(\mathbf{T} \otimes R)) \longrightarrow E^2_{p, 0}(1)$$

is the composition

$$H_p(B, B_0; H_0(\mathbf{T} \otimes R)) \xrightarrow{\varepsilon^*} H_p(\mathcal{E}'(1), d') \xrightarrow{\Theta^1} E^2_{p, 0}(1).$$

Recall (p. 4): $\Theta^1$ is the composition

$$H_p(B, B_0; \mathcal{R}) \xrightarrow{P_B} E^\infty_{p, 0}(1) \xrightarrow{\iota_B} E^2_{p, 0}(1), \text{ i.e.}$$

$$H_p(B, B_0; H_0(\mathbf{T} \otimes R)) \xrightarrow{\Psi^*} E^2_{p, 0}(1)$$

$$\cong \begin{array}{c}
\varepsilon^* \\
\downarrow \\
(\varepsilon^*)^* \\
\uparrow \\
\Pi^* \\
\end{array}$$

$H_p(B, B_0; \mathcal{R}) \xrightarrow{P_B} E^\infty_{p, 0}(1)$

commutes.

Recall: For a general $\pi : (T, T_0) \to (B, B_0)$ in $\mathcal{R}$,

$P_B : H_p(T, T_0; \mathcal{R}) \to E^\infty_{p, 0}(\pi)$ is the composition

$$H_p(T, T_0; \mathcal{R}) = J_{p, 0}(\pi) \to J_{p, 0}(\pi)/J_{p+1, 1}(\pi) \xrightarrow{\Theta} E^\infty_{p, 0}(\pi).$$

Since $\Theta$ and $J$ are natural with respect to morphisms in $\mathcal{R}$, so is $P_B$.

Recall: $\iota_B : E^\infty_{p, 0} \to E^2_{p, 0}$ is defined for any $1^{st}$ quadrant spectral sequence, and is natural with respect to morphisms of $1^{st}$ quadrant spectral sequences. Hence $\iota_B : E^\infty_{p, 0}(\pi) \to E^d_{p, 0}(\pi)$ is natural with respect to morphisms in $\mathcal{R}$. 
Theorem: Let \( \pi : (T, T_0) \to (B, B_0) \) be an object in \( \mathcal{F}_R \). Let \( x_0 \) be the base-point of \( B \), and assume \( F = \pi^{-1}(x_0) \) is path-connected. Then

\[
\begin{array}{cccccc}
H_p(T, T_0; R) & \xrightarrow{\pi_*} & H_p(B, B_0; R) \\
\downarrow{\pi_B} & & \uparrow{\cong} \\
E_{p, \infty}(\pi) & \xrightarrow{\iota_B} & E_{p, 0}(\pi) & \xrightarrow{\cong} & H_p(B, B_0; H_0(F; R)) \\
\end{array}
\]

commutes.

Proof: Since \( B \) and \( F \) are path-connected, so is \( T \). (For choose a

base point \( y \in \pi^{-1}(x_0) \). If \( y \in T \), there exists a path \( \lambda : I \to B \)

from \( \pi(y) \) to \( x_0 \). By the homotopy lifting property there exists a lift

\( \tilde{\lambda} : I \to T \) of \( \lambda \) such that \( \tilde{\lambda}(0) = y \). Then \( \tilde{\lambda}(1) \in F = \pi^{-1}(x_0) \) and since

\( F \) is path-connected, there exists a path \( \mu \) in \( F \) from \( \tilde{\lambda}(1) \) to \( x_0 \). Then \( \tilde{\lambda} \mu \) is a path in \( T \) from \( y \) to \( x_0 \). Thus \( 1 : (T, T_0) \to (T, T_0) \)

is an object in \( \mathcal{F}_R \).

We have a morphism \( f : 1 \to \pi \) in \( \mathcal{F}_R \) as follows:

\[
\begin{array}{cccccc}
(T, T_0) & \xrightarrow{1 = f_T} & (T, T_0) \\
\downarrow{1} & & \downarrow{\pi} \\
(T, T_0) & \xrightarrow{\pi = f_B} & (B, B_0) \\
\end{array}
\]

By naturality of \( f_B \), \( \iota_B \), and \( \psi \) and commutativity of

\[
\begin{array}{cccccc}
H_0(\{H_0\} \times R) & \xrightarrow{H(f_T)} & H_0(F \times R) \\
\downarrow{\cong} & & \downarrow{\cong} \\
R & \xrightarrow{\cong} & R \\
\end{array}
\]

we have the commutative diagram.
By Special Case i (p. 33) the composition of the vertical maps on the left is the identity map on $H_p(T_0; R)$. The theorem now follows.
The Fibre Edge Theorem

Special Case 2: \( \pi : X \to P \), \( P \) a point. \( P \) is simply-connected, so \( \pi \) is an object in \( \mathcal{F}_R \). Write \( \tau_c : I \to P \) for the unique map. Since \( h_{\pi,0} = 1 \), it follows that

\[
\chi : \tau_c \times H_{\mathbb{R}}(X_j R) \to Q_0(P) \otimes H_{\mathbb{R}}(X_j R) \quad \text{is given by}
\]

\[
y(\tau_c \circ \alpha) = \tau_c \otimes \alpha \quad \text{.}
\]

We now look at the composition of all the natural chain isomorphisms in the preceding from the original \( E_{0,\mathbb{R}}(\pi) \) to \( Q_0(P) \otimes H_{\mathbb{R}}(X_j R) \) in special case 2:

**Original** \( E_{0,\mathbb{R}}(\pi) = H_{\mathbb{R}}(X_j R) \)

\[
\begin{align*}
\begin{array}{c}
\text{Q}(B,B_0) \text{ decomposition} \\
H_{\mathbb{R}}(C^{\tau_c}(\pi)) = H_{\mathbb{R}}(X_j R)
\end{array}
\end{align*}
\]

\[
\chi : \begin{array}{c}
X^{\tau_c}(\pi) \to X \\
\text{homeo.}
\end{array}
\]

\[
(\tau_c)_* (a)
\]

**Function space decomposition**

\[
H_{\mathbb{R}}(X^{\tau_c}(\pi), R) \quad (\tau_c)_* (a)
\]

**Twisted \( Q(B,B_0) \otimes H(F) \) description**

\[
\begin{array}{c}
\tau_c \times H_{\mathbb{R}}(X_j R) \\
\text{isom.}
\end{array}
\]

\[
\begin{array}{c}
\chi \circ (\tau_c \otimes \alpha)
\end{array}
\]

Thus the composition of the inverses of the above isomorphisms is the composition

\[
Q_0(P) \otimes H_{\mathbb{R}}(X_j R) \xrightarrow{\text{iso.}} Y \otimes H_{\mathbb{R}}(X_j R) \xrightarrow{\text{iso.}} H_{\mathbb{R}}(X_j R).
\]
Call this last composition $\alpha$. $\alpha$ is a map of chain complexes with trivial boundary.

Recall that $d_t^r = 0$ for $r \geq 1$ in $E^2(\pi)$ and so we can canonically identify $E^r_{o^r} (\pi) = E^r_{o^r} (\pi) = H^r_{o^r} (X_j R)$ for all $r \geq 1$. Each $d_t^r : E^r_{o^r} (\pi) \to E^{r+1}_{o^{r+1}} (\pi) = H^r_{o^r} (X_j R)$ is the identity map. Thus the natural isomorphism

$\Psi : H_0 (P; H^r_{o^r} (X_j R)) \to E^{2}_{o^r} (\pi) = H^r_{o^r} (X_j R)$

is $\alpha$. It is easily checked that $\alpha$ (and hence $\Psi$) is the composition

$H_0 (P; H^r_{o^r} (X_j R)) \xrightarrow{\alpha} H_0 (P) \otimes H^r_{o^r} (X_j R) \xrightarrow{\otimes 1} \mathbb{Z} \otimes H^r_{o^r} (X_j R) \xrightarrow{\text{nat. isom.}} H^r_{o^r} (X_j R)$

where $\mu : H_0 (P) \otimes H^r_{o^r} (X_j R) \to H_0 (P; H^r_{o^r} (X_j R))$ is the map in the universal coefficient theorem ($\mu \{ \text{c} \} \otimes \{ \text{c} \} = \{ \text{c} \}$).

Recall: (p. 16) In special case 2, $P^c : E^{2}_{o^r} (\pi) \to E^{\infty}_{o^r} (\pi)$ and $\mu^c : E^{\infty}_{o^r} (\pi) \to H^r_{o^r} (X_j R)$ are both the identity map on $H^r_{o^r} (X_j R)$.

Theorem: Let $F \xrightarrow{\iota} T \xrightarrow{\pi} B$ be an $R$-orientable fibration. Then the diagram

$$
\begin{array}{ccc}
H^r_{o^r} (F_j R) & \xrightarrow{\alpha} & H_0 (T_j R) \\
\downarrow{\cong} & & \uparrow{\cong} \\
H_0 (B; H^r_{o^r} (F_j R)) & \xrightarrow{\cong} & E^{2}_{o^r} (\pi) \xrightarrow{p^c} E^{\infty}_{o^r} (\pi)
\end{array}
$$

commutes, where $\cong$ is the composition

$H^r_{o^r} (F_j R) \xrightarrow{\cong} \mathbb{Z} \otimes H^r_{o^r} (F_j R) \xrightarrow{\otimes 1} H_0 (B) \otimes H^r_{o^r} (F_j R) \xrightarrow{\cong} H_0 (B; H^r_{o^r} (F_j R))$.

Recall: Implicitly, $B$ is path-connected if $\pi$ is in $\mathcal{F}_R$.

Proof: Let $x_0$ be the base point of $B$ (thus $F = \pi^{-1} (x_0)$). We have the special case 2 fibration $P : F \to \{ x_0 \}$ and a morphism $\iota : P \to \pi$ in $\mathcal{F}_R$ as follows:
By naturality of $\xi_F$, $\Pi_F$ and $\Psi$, the diagram:

$$H_8(F; R) \xrightarrow{\xi_{\cdot}} H_8(T; R)$$

$$E_{\cdot, \cdot}^\infty (\cdot) \xrightarrow{E_{\cdot, \cdot}^\infty (\cdot)} E_{\cdot, \cdot}^\infty (\cdot)$$

$$\Psi(\cdot) \xrightarrow{\Psi(\cdot)} \Psi(\cdot)$$

$$H_0(\{x\cdot\}, H_q(F; R)) \xrightarrow{\xi_{\cdot}} H_0(B, H_q(F; R))$$

commutes.

Since $\xi_F(\cdot)$ and $\Pi_F(\cdot)$ are both the identity on $H_8(F; R)$, the result will follow from the above diagram if we show $\xi_{\cdot}^* \Psi(\cdot)^{-1} = \xi_{\cdot}$.

This is immediate from commutativity of

$$H_8(F; R) \xrightarrow{\text{nat. isom.}} H_8(F; R)$$

$$\Psi(\cdot) \xrightarrow{\text{nat. isom.}} \Psi(\cdot)$$

$$\Psi(\cdot) \xrightarrow{\text{nat. of } \Psi(\cdot)} \Psi(\cdot)$$

$$H_0(\{x\cdot\}, H_q(F; R)) \xrightarrow{\text{nat. of } \mu} H_0(\{x\cdot\}, H_q(F; R))$$

$$H_0(\{x\cdot\}, H_q(F; R)) \xrightarrow{\text{nat. of } \mu} H_0(\{x\cdot\}, H_q(F; R))$$

$$H_0(\{x\cdot\}, H_q(F; R)) \xrightarrow{\text{nat. of } \mu} H_0(\{x\cdot\}, H_q(F; R))$$

$$H_0(\{x\cdot\}, H_q(F; R)) \xrightarrow{\text{nat. of } \mu} H_0(\{x\cdot\}, H_q(F; R))$$

$$H_0(\{x\cdot\}, H_q(F; R)) \xrightarrow{\text{nat. of } \mu} H_0(\{x\cdot\}, H_q(F; R))$$

$$H_0(\{x\cdot\}, H_q(F; R)) \xrightarrow{\text{nat. of } \mu} H_0(\{x\cdot\}, H_q(F; R))$$

$$H_0(\{x\cdot\}, H_q(F; R)) \xrightarrow{\text{nat. of } \mu} H_0(\{x\cdot\}, H_q(F; R))$$
Connecting a map to a fibration

Prop. \( f : X \to Y \) a pointed map. \( i : X \to E_f \) and a fibration \( p : E_f \to Y \) is

\[
X \xrightarrow{i} E_f \\
\xrightarrow{f} Y
\]

commutes.

Proof: Let \( E_f = \{ (x, \lambda) \in X \times C(\mathbb{I}, Y) \mid f(x) = \lambda(0) \} \),

\[
i^*_f : X \to E_f
\]

\[
\begin{array}{c}
\lambda \in C(\mathbb{I}, Y) \\
\end{array}
\xrightarrow{\pi_1} (x, \lambda(t)) = \text{const. point} \quad \text{for } \lambda(t) = f(x),
\]

\[
p f : E_f \to Y
\]

\[
\begin{array}{c}
(x, \lambda) \\
\end{array}
\xrightarrow{\lambda(1)} Y
\]

Then above diagram commutes.

Proof that \( i^*_f \) is a pointed homotopy equivalence:

Let \( \pi_1 : E_f \to X \) be given by \( \pi_1 (x, \lambda) = x \). Then \( \pi_1 \circ i^*_f = 1_x \).

\[
\lambda \in C(\mathbb{I}, Y), \quad t \in \mathbb{I}
\]

define \( \lambda_t \in C(\mathbb{I}, Y) \)

\[
\lambda(t) = \lambda(0)
\]

Then \( \lambda_0 = \lambda_{(0)} \), \( \lambda_1 = \lambda \), and \( \lambda_t(0) = \lambda(0) \) \( \forall t \in \mathbb{I} \).

Define \( h : E_f \times \mathbb{I} \to E_f \) by \( h((x, \lambda), t) = (x, \lambda_t) \).

\( h \) is a homotopy from \( i^*_f \) to \( 1_{E_f} \).

Let \( h((x, \lambda), t) = (x, \lambda(m(t))) \) \( \forall t \in \mathbb{I} \) \( \forall x \in X \).

Thus \( h \) is a pointed homotopy \( \forall \) choice of base point \( s \) in \( X \).

Proof that \( p f : E_f \to Y \) is a fibration:

Suppose given

\[
\begin{array}{ccc}
2 \times 0 & \xrightarrow{g} & E_f \\
\downarrow & & \downarrow p f \\
2 \times \mathbb{I} & \xrightarrow{h} & Y
\end{array}
\]

Write \( g(z, 0) = (k(z), l(z)) \) when \( k : Z \to X, \quad l : Z \to C(\mathbb{I}, Y) \).

Thus \( l(z)(0) = t(k(z)) \) and \( l(z)(1) = p f \circ g(z, 0) = G(z, 0) \).
Define \( H \) by
\[
H(t; s) = \begin{cases} 
1(t)(\frac{2s}{2-s}) & \text{if } 0 \leq s \leq 1-\frac{t}{2} \\
G(t; z(s)+2-2s) & \text{if } 1-\frac{t}{2} \leq s \leq 1.
\end{cases}
\]

Then \( H \) is continuous and both triangles commute.

Let \( X \) and \( Y \) be \( 0 \)-connected and \( f: X \to Y \) a pointed map such that \( f^*: \pi_n(X) \to \pi_n(Y) \) is an isomorphism for all \( n \). Then \( f_*: \pi_n(X) \to \pi_n(Y) \) is an isomorphism for all \( n \).

Note: Here, previously, see that \( X, Y \) are both \( 1 \)-connected (Whitehead theorem).

Part 1: We set \( F = \text{fibre of } \varphi \) and \( \pi_f: \pi_g \to Y \). From connectedness of
\[
X \xrightarrow{i^*} E_f \xrightarrow{\varphi} Y
\]
and fact that \( i^* \) is a homotopy equivalence, it follows that suffices to show \( (\pi_f)_* : \pi_f(E_f) \to \pi_f(Y) \) is an isomorphism for all \( n \).

Since \( i^* : \pi_n(X) \to \pi_n(E_f) \) is an isomorphism, it follows that \( \pi_{f*} : \pi_n(E_f) \to \pi_n(Y) \) is an isomorphism for all \( n \). Hence, from the homotopy sequence of the fibration \( F \to E_f \to Y \), it follows that \( \pi_n(F) = 0 \) for all \( n \). Hence, by the Hurewicz theorem, \( H_n(F) = 0 \) for all \( n > 0 \). For each \( n \), \( \pi_n \) is connected (see \( \pi_n(F) = 0 \)), so by earlier remark, the fibration \( E_f \to Y \) is \( Z \)-orientable. \( \pi_{1,0}(E_f) = 0 \) for \( \delta \neq 0 \), from which it follows easily that \( \pi_f \) is an isomorphism. Hence, by the Hurewicz theorem, each \( (\pi_f)_* : \pi_f(E_f) \to \pi_f(Y) \) is an isomorphism.
Co-homology Spectral Sequences

Def: A cohomology spectral sequence of $R$-modules $E$ consists of a sequence of degenerate differ. $(E_r, d_r)$-modules $(E_r, d_r)$ (i.e. $d_r : E_r^{p, q} \rightarrow E_r^{p+1, q-1}$) together with isomorphism

$$H^p(E_{r+1}) \cong E_r^{0, p+1}$$


\[\text{lim} \left[ E_r^{p, q+r-1} \rightarrow E_r^{p, q} \right] \]

Def: A cohomology spectral sequence of $R$-modules $E$ is multiplicative if there are given $R$-homomorphisms

$$m_r : E_r^{p, q} \otimes E_r^{p', q'} \rightarrow E_r^{p+p', q+q'}$$

satisfying

1) $m_r ((a, b)) = (d_r a) \cdot b + (-1)^{p+q} a \cdot (d_r b)$ whenever $a \in E_r^{p, q}$.

(Note that as a consequence of 1), if $a \in E_r^{p, q}$, $b \in E_r^{p', q'}$ are in the kernel of $d_r$, so is $a \cdot b$, and $\{a \otimes b \in H^{p, q} (E_r, d_r) \}$ depends only on $\{a \otimes b \in H^{p, q} (E_r, d_r) \}$ and $\{b \in H^{p', q'} (E_r, d_r) \}$. We thus obtain an $R$-homomorphism

$$H(d_r) : H^p (E_r, d_r) \otimes H^{p', q'} (E_r, d_r) \rightarrow H^{p+p', q+q'} (E_r, d_r)$$

\[\{a \otimes b\} \rightarrow \{a \otimes b\} \]

2) For each $r$ and all $p, q, p', q'$, the diagram

$$H^p (E_r, d_r) \otimes H^{p', q'} (E_r, d_r) \rightarrow H^{p+p', q+q'} (E_r, d_r)$$

comutes.

Note: If $E$ is a 1st quadrant cohomology spectral sequence (i.e. $E_r^{p, q} = 0$ for $p < a$ or $q < 0$), then $E_{\infty}$ is defined just as in the homology case. Hence $E_{\infty}$ has properties

$$m_\infty : E_{\infty} \otimes E_{\infty} \rightarrow E_{\infty}^{p+p', q+q'},$$

namely $d_r$ for $r$ sufficiently large.
For $i$-th graded multiplication cohomology spectral sequence $R$-modules.

Note: If $H_r$ is associative for some $r$, so is $H(d_r)$ and consequently, so is $H(d_r)$ for all $r > r$.

Similarly, for graded-commutativity $(a \cdot b = (-1)^{|a||b|} b \cdot a$ where $|a| = p + q$ if $a \in E_r^{p,q}$).

Eq. Products with Coefficient

Let $R$ be a PID. Suppose $A, B, C$ are $R$-modules and we are given an $R$-homomorphism

$$\alpha : A \otimes B \rightarrow C.$$ 

Then for any top. group there is given an $R$-hom.

$$\beta \alpha : H^0(X; A) \otimes_R H^0(X; B) \rightarrow H^0(X; C)$$

satisfying

1) $\beta \alpha$ is natural with respect to continuous covers, i.e.

$$\begin{array}{ccc}
H^0(X; A) \otimes_R H^0(X; B) & \xrightarrow{\alpha} & H^0(X; C) \\
\downarrow f^* & & \downarrow f^*
\end{array}$$

2) $\beta \alpha$ is natural with respect to $\alpha$, i.e. if we are given another $R$-homomorphism

$$\alpha' : A' \otimes B' \rightarrow C'$$

and $R$-homomorphisms $f : A \rightarrow A'$, $g : B \rightarrow B'$, $h : C \rightarrow C'$ such that

$$\begin{array}{ccc}
A \otimes B & \xrightarrow{\alpha} & C \\
\downarrow f & & \downarrow h \\
A' \otimes B' & \xrightarrow{\alpha'} & C'
\end{array}$$

commutes.
for each each $X$,

$$H^0(X; A) \otimes_R H^8(X; B) \xrightarrow{\partial y} H^{p+q}(X; C)$$

$$\begin{array}{c}
\downarrow \gamma_{\alpha} \\
H^0(X; A') \otimes_R H^8(X; B') \xrightarrow{\alpha \otimes g} H^{p+q}(X; C')
\end{array}$$

3) $\varphi: \alpha: R \otimes_R R \rightarrow R$ is the canonical restriction,

$\alpha \otimes 1 \rightarrow \alpha$

then $\alpha \otimes: H^0(X; R) \otimes_R H^8(X; R) \rightarrow H^{p+q}(X; R)$ is the cap product

This previously studied.

$\alpha \times$ is the composition

$$H^0(X; A) \otimes H^8(X; B) \xrightarrow{\alpha \otimes g} H^{p+q}(\alpha^*(X; A) \otimes \delta^*(X; B))$$

$$\begin{array}{c}
\downarrow \gamma^*_{\alpha} \\
H^{p+q}(\text{Hom}_R(\alpha^*(X), C))
\end{array}$$

$\chi: H^{p+q}(\text{Hom}_R(\alpha^*(X), C)) \rightarrow H^{p+q}(X; C)$

1) Leray–Serre Theorem (Mult. Cohom.: $R \rightarrow \text{PID}$, $F \rightarrow T \rightarrow B$

an $R$-morphism. Then there is a natural 1st quad. mult.

coh. spectral sequence of $R$-modules $E(\Pi)$ satisfying

$$\Psi: E^{p+q}_2(\Pi) \rightarrow H^p(B; H^q(F; R))$$

such that the diagram.
\[ E_{\pi}^{\prime,p} (\pi) \otimes_{R} E_{\pi}^{\prime,p'} (\pi') \xrightarrow{\mu} E_{\pi}^{\prime+p',1+1'} (\pi') \]

\[ \Phi \otimes \Phi \]

\[ \text{where } \alpha_F : H^F (F, R) \otimes H^8 (F, R) \to H^8 (F, R) \text{ is the cup product pairing.} \]

2) There is a natural filtration

\[ H^n (\pi) = E_{\pi}^{n,0} (\pi) \supset E_{\pi}^{n-1,1} (\pi) \supset \ldots \supset E_{\pi}^{n-k,k-1} (\pi) = 0 \]

and natural isomorphisms \[ \Phi : E_{\pi}^{n} (\pi) \to E_{\pi}^{n,0} (\pi) / E_{\pi}^{n+1,1} (\pi). \]

Moreover, \( u \in E_{\pi}^{n} (\pi), v \in E_{\pi}^{n+1,1} (\pi) \Rightarrow u \circ v \in E_{\pi}^{n+p',1+1'} (\pi) \)

and the diagram

\[ E_{\pi}^{n} (\pi) \otimes E_{\pi}^{n+1,1} (\pi) \xrightarrow{\Phi \otimes \Phi} E_{\pi}^{n+p',1+1'} (\pi) \]

commutes.
Let $F \to T \to B$ be an $R$-module filtration and suppose $F$ is $0$-connected. We have the commutative diagram

$$
\begin{array}{ccc}
\bigoplus_r H^0(F_r) & \to & H^0(F) \\
\bigotimes_r H^0(F_r) & \to & H^0(F) \\
\end{array}
$$

where $\bigotimes$ is the coaugmentation. Thus by naturality of $\underline{\text{colim}}$ with respect to $X$, the diagram

$$
\begin{array}{ccc}
\bigoplus_r H^0(B_r) & \to & H^0(B) \\
\bigotimes_r H^0(B_r) & \to & H^0(B) \\
\end{array}
$$

commutes. Thus $E_2^{\cdot,0}(\pi) \approx H^*(B)$ as a graded ring.

In any $R$-module $A$ we have a coaugmentation

$$
\eta_A : A \to H^0(X, A)
$$

which is an isomorphism if $X$ is $0$-connected. $\eta_A$ is induced by

$$
A \cong \text{Hom}_Z(\mathbb{Z}, A) \to \text{Hom}_Z(\mathbb{Q}_0(X), A) = \mathbb{Q}^0(X, A)
$$

Define $\Delta : A \otimes R B \to C$ as an $R$-homomorphism,

$$
\begin{array}{ccc}
H^0(X, A) \otimes_R H^0(X, B) & \to & H^0(X, C) \\
\bigotimes_r H^0(X_r) & \to & H^0(X) \\
\end{array}
$$

Thus $\eta_A \otimes \eta_B$ : $A \otimes R B \to C$ commutes.

Let $F \to T \to B$ as above.
\[ H^0(B; H^e(F, R)) \otimes_R H^0(B; H^{e'}(F, R)) \xrightarrow{\mu_X} H^0(B; H^{e+e'}(F, R)) \]

\[ \cong \quad \xrightarrow{\cong} \quad \cong \quad \xrightarrow{\cong} \]

\[ H^e(F, R) \otimes_R H^{e'}(F, R) \xrightarrow{\cup} H^{e+e'}(F, R) \]

commutes, and so \[ E^o_{2,0}(\pi) \cong H^e(F, R) \text{ as a graded ring.} \]

Suppose \[ \pi : A \otimes R B_i \to C_i \text{ are } R \text{-homomorphisms,} \]

\[ 1 \leq i \leq n. \]

Write \[ \chi = \bigoplus \chi_i : A \otimes \bigoplus B_i \to \bigoplus C_i. \]

For any \( X, \)

\[ H^p(X; A) \otimes_R H^{p'}(X; \bigoplus B_i) \xrightarrow{\mu_X} H^{p+p'}(X; \bigoplus C_i) \]

\[ \cong \quad \xrightarrow{\cong} \quad \cong \quad \xrightarrow{\cong} \]

\[ \bigoplus_i H^p(X; A) \otimes_R H^{p'}(X; B_i) \to \bigoplus_i H^{p+p'}(X; C_i) \]

commutes. In particular, if \( p, p' \) each \( \geq 0 \), then \( \mu_X \) is an isomorphism.

In particular, suppose \( X \) is \( 0 \)-connected. Then

\[ H^p(X; R) \otimes_R H^0(X; R) \xrightarrow{\cup} H^p(X; R) \text{ as an isom.} \]

Thus if \( A \) is a finitely generated free \( R \)-module and \( \pi : R \otimes_R A \to A \) the canonical homomorphism,

\[ H^p(X; R) \otimes_R H^0(X; A) \xrightarrow{\mu_X} H^p(X; A) \text{ is an isom.} \]

Thus, if \( F \to T \xrightarrow{\pi} B \) is an \( R \)-orientable fibration with \( F \) \( 0 \)-connected, and if each \( H^e(F, R) \) is free and finitely generated as an \( R \)-module, then

completeness of
\[
\begin{align*}
H^0(F; R) \otimes H^8(F; R) & \xrightarrow{\gamma \otimes 1} H^8(F; R) \\
\gamma \otimes 1 & \xrightarrow{=} \uparrow 1 \\
R \otimes H^8(F; R) & \xrightarrow{\text{cohom.}} H^8(F; R)
\end{align*}
\]

Examples:
\[
\begin{align*}
H^0(B; H^0(F; R)) \otimes H^0(B; H^8(F; R)) & \xrightarrow{\mu X F} H^0(B; H^0(F; R)) \\
\mu \otimes 1 & \xrightarrow{=} \uparrow 1 \\
H^0(B; R) \otimes H^0(B; H^8(F; R)) & \xrightarrow{=} H^0(B; H^8(F; R))
\end{align*}
\]

commutes, and so top \( H^8 \) is an isomorphism. Thus, if each \( H^8(F; R) \) is a free and finitely generated \( R \)-module, and \( F \) is \( \sigma \)-compact, \( E_n^{p, q}(n) \otimes E_2^{p, q}(n) \xrightarrow{H_2} E_2^{p, q}(n) \) is an isomorphism.

Example: \( S^\infty = \bigcup_n S^n \) with the weak topology,
\[
\begin{array}{c}
\mathbb{C}P^\infty = \bigcup \mathbb{C}P^n \\
\end{array}
\]
with \( \mathbb{C}P^n \) \( \to \mathbb{C}P^n+1 \) having fibers \( S^1 \); such that \( S^{2n+1} \to S^{2n+3} \to \mathbb{C}P^n \to \mathbb{C}P^n+1 \) commutes.

This yields a fibre bundle \( S^\infty \to \mathbb{C}P^\infty \) with fibre \( S^1 \).

Claim: \( \pi_n(S^\infty) = 0 \) for all \( n \). To see this, if \( f: S^n \to S^\infty \) is a pointed continuous map, then since \( S^n \) is compact and \( S^\infty \) has the weak topology, \( f \) \( \in N > n \) and a fortiori
We have \( S^n \to S^\infty \) and hence \([\varphi] = \iota \cdot [\varphi] \). But

\[ [\varphi] = 0 \text{ since } H_n(S^n) = 0 \text{ for } n < N. \]

Thus, by the Hurewicz theorem, \( H_n(S^\infty) = 0 \) for \( n \neq 0 \).

It follows from the homotopy sequence of the above fibration that

\[ H_n(\mathbb{C}P^\infty) = \begin{cases} 0 & \text{if } n \neq 2 \\ \mathbb{Z} & \text{if } n = 2. \end{cases} \]

In particular, \( \mathbb{C}P^\infty \) is 2-connected, or, above fibrations are \( \mathbb{Z} \)-orientable.

Let \( E \) denote the cohomology spectral sequence of this fibration with \( \mathbb{Z} \) coefficient. Since \( H^q(S^1) = 0 \) for \( q \neq 0,1 \) we have \( E_2^{p,q} = 0 \) for \( q \neq 0,1 \). Thus,

\[ E_2^{p,1} = E_3^{p,1}. \]

Since \( H^n(S^\infty) = 0 \) for \( n \neq 0 \), \( E_\infty^{0,1} = 0 \) for \( (p,q) \neq (0,0) \). Thus, \( E_3^{p,1} = 0 \) for \( (p,q) \neq (0,0) \).

From induction on \( p \),

\[ 0 \to E_3^{p,1} \to E_2^{p,1} \xrightarrow{d_2} E_2^{p+2,0} \to E_3^{p+2,0} \to 0. \]

It follows that \( d_2 : E_2^{p,1} \to E_2^{p+2,0} \) is an isomorphism for \( p \neq -2 \).

Write \( x = \varphi^* \) of \( E_2^{0,1} = H^0(\mathbb{C}P^\infty; H^1(S^1)) \approx \mathbb{Z} \).

Since \( 0 = \partial_{2,2}^{1,0} \partial_{2,0}^{1,0} = H^1(\mathbb{C}P^\infty; H^1(S^1)) \)

we have \( H^1(\mathbb{C}P^\infty) = 0 \). Thus \( E_2^{1,0} \to E_2^{1,1} \to E_2^{1,1} \)

is an isomorphism, \( E_2^{1,1} = 0 \). Thus \( E_2^{1,0} \to E_2^{2,0} = H^2(\mathbb{C}P^\infty) \).

By induction, \( H^{2,0}(\mathbb{C}P^\infty) = 0 \).

\[ E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} = H^2(\mathbb{C}P^\infty), \text{ We have } y = d_2 x. \]

Thus \( H^2(\mathbb{C}P^\infty) \approx \mathbb{Z} \), \( \exists y \). We have \( dy = 0 \).
Since $E_2^{2,1} \otimes E_2^{0,1} \longrightarrow E_2^{2,1}$, and

$E_2^{2,1} \cong \mathbb{Z}$, gen. $y \cdot x$. We have $d_2: E_2^{2,1} \longrightarrow E_2^{4,0}$, gen. $d_2(y \cdot x)$. But

$d_2(y \cdot x) = (d_2y) \cdot x + y \cdot d_2x = 0 \cdot x + y \cdot y = y^2$.

Thus, $H^4(\mathbb{C}P^\infty) \cong \mathbb{Z}$, gen. $y^2$.

Assume unitarily, $H^{2n}(\mathbb{C}P^\infty) \cong \mathbb{Z}$, gen. $y^n$.

Since $E_2^{2n,0} \otimes E_2^{0,1} \longrightarrow E_2^{2n+1,0}$ is an isom.,

$E_2^{2n+1,0} \cong \mathbb{Z}$, gen. $y^n \cdot x$. We have $d_2(y^n) = 0$.

Since $d_2: E_2^{2n+1,0} \longrightarrow E_2^{2n+2,0}$ is an isom.,

$E_2^{2n+2,0} \cong \mathbb{Z}$, gen. $d_2(y^n \cdot x)$. But $d_2(y^n \cdot x) = d_2(y^n) \cdot x + y^n \cdot d_2x = 0 \cdot x + y^n \cdot y = y^{n+1}$.

Thus, we obtain

$H^*(\mathbb{C}P^\infty) = \mathbb{Z}[y]$, $y \in H^2(\mathbb{C}P^\infty)$. 
Some's $G$-theory

Def: A class $C$ of abelian groups is called a dense class if all $O$ groups are in $C$, and whenever $0 \to A \to B \to C \to 0$ is exact, $B \in C \Rightarrow A \in C$ and $C \in C$.

Examples: 1) all $O$ groups
2) all abelian groups
3) all finitely-generated abelian groups
4) all finite abelian groups
5) all finite abelian $p$-groups, $p$ a fixed prime
6) let $S$ be a set of primes. All finite abelian groups of order a prod. of primes in $S$. (This generalizes 5).

Prop: $C$ a dense class. Then
i) \( \forall A \subseteq B, \text{ then } B \in C \Rightarrow A \in C \text{ and } B/A \in C. \)
ii) \( \forall A \cong B, \text{ then } B \in C \Rightarrow A \in C \)
iii) \( \forall A \to B \to C \text{ exact and } A \in C, C \in C, \text{ then } B \in C. \)
iv) \( A \oplus B \in C \Rightarrow A \in C \text{ and } B \in C. \)

Proofs are all trivial.

Def: Let $C$ be a dense class, \( f : A \to B \) a homomorphism of abelian groups. \( (A, B \text{ not nec. in } C) \). \( f \) is called a $C$-isomorphism if \( \ker f \cap C, \text{ a } C$-epimorphism

Proof: $C$ a dense class, \( f : A \to B \) a $C$-isomorphism. \( \text{Then } A \in C \Rightarrow B \in C. \)

Proof is trivial.

Prop: $C$ a dense class. Suppose \( f : A \to B, g : B \to C \) are $C$-isomorphisms. Then \( gf : A \to C \) is a $C$-isomorphism.

Proof left as exercise.

Main goal is to obtain the following:
Then (Mod C hereafter thus): Let C be a strongly homologically complete base class. Let X be a 1-connected pointed space. Suppose for some \( n \geq 1 \), \( H_i(X;^*) \in C \) for \( 1 \leq i \leq n \). Then \( H_i(X;^*) \in C \) for \( 1 \leq i \leq n \) and \( \delta_{n+1} : H_{n+1}(X;^*) \rightarrow H_n(X;^*) \) is a \( C \)-isomorphism.

(With this def. of strongly homologically complete base. Turns out all C examples above are strongly homologically complete.)

Cor: If C is strongly homologically complete and X is a 1-connected pointed space, then \( H_i(X;^*) \in C \) for all \( i \). Hence \( H_i(X;^*) \in C \) for all \( i \).

In particular, if X is 1-connected, \( H_i(X;^*) \) is finitely generated for all \( i \).

Lemma: E a homology spectral sequence of abelian groups, C a base class. Suppose for some \( r, p, q \), \( E_{r,p,q} \in C \). Then \( E_{s,p,q} \in C \) for all \( s \geq r \).

Proof: \( E_{r+1} \) is quotient of a subgroup of \( E_{r,p,q} \), and hence is in C. Result now follows by induction.

Def: A base class C is homologically complete if \( A \in C \), \( B \in C \Rightarrow A \otimes B \in C \) and \( \text{Tor}^\mathbb{Z}_2(A;B) \in C \).

Example: It is immediate that examples 1-6 above are homologically complete.

Lemma: Let C be a homologically complete base class. Suppose X is a 1-connected pointed space such that for some \( n \geq 2 \), \( H_i(X) \in C \) for \( 1 \leq i \leq n \). Then \( H_i(X) \in C \) for \( 1 \leq i \leq n-1 \).

Proof: The fibration \( \Omega X \rightarrow PX \rightarrow X \) is 2-orientable since X is 1-connected. Let E be the Leray-Serre spectral sequence of this fibration with \( C \)-coefficients. Since \( PX \in C \) is contractible, \( E_{1,0} = 0 \) for \((p, q) = (0, 0)\). In particular, \( E_{0,1} = E_{2,0} = 0 \).
and so \( d^2 : E^2 \rightarrow E^0 \) must be an isomorphism. Now
\[
E^2_{p,0} = H_0(X; H_0(\varnothing X)) \cong H_0(X) \in C, \quad \text{so } E^2_{\cdot,0} \in C. \quad \text{But}
\]
\[
E^2_{0,i} = H_0(X; H_i(\varnothing X)) \cong H_i(\varnothing X), \quad \text{so } H_i(\varnothing X) \in C.
\]

Suppose \( 1 \leq q \leq n-1 \) and that, inductively,
\[
H_q(\varnothing X) \in C \quad \text{for } 1 \leq r \leq q-1. \quad \text{Remains only to show:}
\]
\[
H^1_q(\varnothing X) \in C. \quad \text{Since } E^2_{p,0} = H_0(X; H_0(\varnothing X)) \cong H_0(X) \in C, \quad \text{remains only to show } E^2_{0,\cdot} \in C.
\]

By the U.C.T. and homological completeness of \( C \),
\[
E^2_{p,i} = H_p(X; H_i(\varnothing X)) \in C \quad \text{for } 0 \leq p \leq n \text{ and } 1 \leq i \leq q.
\]
Thus, \( E^2_{p,i} \in C \quad \text{for } 0 \leq p \leq n, 1 \leq i \leq q \), and \( 2 \leq r \leq \infty \).

Also, \( \text{for } 1 \leq p \leq n, \quad E^2_{p,0} = H_p(X; H_0(\varnothing X)) \cong H_p(X) \in C, \quad \text{so } E^p_{p,0} \in C \quad \text{for } 1 \leq p \leq n \text{ and } 2 \leq r \leq \infty. \)

The proof will be completed by proving, by decreasing induction on \( r \), that
\[
E_{0,q} \in C \quad \text{for all } r.
\]
We have \( C = E_{0,\cdot} = E_{0,0} \),
so certainly \( E_{0,0} \in C. \)

Suppose that \( \text{for some } r \in 
\]
the range \( 2 \leq r \leq q+2 \), \( E_{0,r} \in C. \)
We have the exact sequence
\[
E^r \rightarrow E^{r-1} \rightarrow E^{r-2} \rightarrow \cdots \rightarrow E^1 \rightarrow E^0 \rightarrow \text{coker } d^1.
\]
Note that \( 1 \leq r-1 \leq q+1 \leq n, q-r+2 \leq q. \quad \text{Hence by } 1^r
\]
inductive assumption, \( E^{r-1} \in C. \quad \text{By } 2^r \text{th inductive assumption, } E^r \in C. \quad \text{Hence, } E_{0,r} \in C. \quad \text{Completing the inductive proof.}

Killing Homotopy groups.

Thus, let \( X \) be a \( 0 \)-connected pointed space which admits a reduced covering space (e.g., if \( X \) is 1-connected).
Then for each \( n \geq 1 \), there exists a space \( X(0, \ldots, n) \) and an inclusion \( i : X \rightarrow X(0, \ldots, n) \)
such that \( i^* : \pi_q(X) \rightarrow \pi_q(X(0, \ldots, n)) \) is an isomorphism for \( q \leq n \) and \( \pi_q(X(0, \ldots, n)) \cong \mathbb{Z} \) for \( q > n. \)
Moreover, \( X(0, \ldots, n) \) can be obtained by first attaching a family of \( n+2 \)-cells to \( X \), then a family of \( n+3 \)-cells to this.
new space, then a family of \( n+1 \)-cells to this new space, etc.

Idea of construction: To each \( x \in \Pi_{n+1}(X) \), attach an \( n+2 \)-cell to \( X \) with attaching map representing \( x \). Call this new space \( K_{n+2} \). Can prove: \( X \to K_{n+2} \) induces \( \cong \) in \( \Pi_k \) for \( k \leq n \), and \( \Pi_{n+1}(K_{n+2}) = 0 \). To each \( \beta \in \Pi_{n+2}(K_{n+2}) \), attach an \( n+3 \)-cell to \( K_{n+2} \) with attaching map representing \( \beta \). Call this new space \( K_{n+3} \). Can prove: \( K_{n+2} \to K_{n+3} \) induces \( \cong \) in \( \Pi_k \) for \( k \leq n+1 \), and \( \Pi_{n+2}(K_{n+3}) = 0 \), etc. Take \( X(0, \ldots, n) = \cup K_n \) with the weak topology.

Def: A pointed space \( X \) is said to be of type \( (\Pi, n) \), if \( n \geq 1 \), \( \Pi \) a group (abelian if \( n > 1 \)) if

\[
\Pi_k(X) = \begin{cases} 
\Pi_k & k = n \\
0 & k \neq n 
\end{cases}
\]

Examples: 1) \( S^1 \) is of type \( (\Pi, n) \) if \( n \geq 1 \).
2) \( CP^\infty \) is of type \( (\Pi, n) \)
3) \( RP^\infty \) is of type \( (\Pi, n) \) (base fibration \( S^0 \to S^\infty \to RP^\infty \))
4) \( HX \) is of type \( (\Pi, n) \), \( n > 1 \) then \( 2X \) is of type \( (\Pi, n-1) \).

Spaces of type \( (\Pi, n) \) in some \( \Pi, n \) are called bordant.
Let $k^n$ be obtained from $k^m$ by attaching, for each $x \in R$, an $n+1$-cell with attaching map representing $x$. Can prove: $k^n$ is $n$-connected, and $\pi_0(k^n) = 1$. Take $|k| = k^{n+1}(0, \ldots, n).

Thus a $Y$ of type $(\Pi, n)$, define $f_n : k^n \to Y$ as follows. Let $p : F \to F/R = \Pi_0(Y)$ be the projection.

In the wedge summand $S^n_x$ in $k^n$ corresponding to $x \in F$, define $f_n|S^n_x = \pi_0(p(x)) \in \Pi_0(Y)$. Each $f_n$ (attaching map for an $n+1$ cell) is homotopically trivial, so $f_n$ extends to a map $f_{n+1} : k^{n+1} \to Y$.

Since $\pi_0(Y) = 0$ for $n > 0$, can successively extend over all skeleta.

Def. A class $C$ is strongly homologically complete if it is homologically complete, and if $\lambda$ denotes the class $C$, and $X$ is a space of type $(\Pi, i)$, then $H_i(X) \in C$ for all $i > 0$.

Then let $C$ be a homologically complete class consisting of finitely-generated abelian groups. Then $C$ is strongly homologically complete.

Sketch of proof: For each $n > 1$, $Z/n$ is a subgroup of $S^1$. $S^1$ acts freely on $S^{n+1}/V$, and hence on $Z/n$. $L(n) = S^n/V \cong Z/n$ is a space of type $(Z/n, 1)$. $H_i(L(n))$ can be calculated from a cell decomposition (generalization of the cell decomposition for real projective spaces) and in fact:

$$H_i(L(n)) = \begin{cases} Z & \text{if } i = 0 \\ Z/n & \text{if } i > 0, \text{ odd} \\ 0 & \text{if } i > 0, \text{ even} \end{cases}$$

So $T = \mathbb{Z} \oplus \cdots \oplus Z \oplus Z/n \oplus \cdots Z/n$, then

$$S^1 \times \cdots \times S^1 \times L(n) \times \cdots L(n)$$

is a space of type $(\Pi, 1)$. 

It follows from the Kneser theorem, that the homology groups of this space are in any homologically complete class which contains $\Pi_1$.

This: Suppose $X$ is $0$-connected and admits a universal covering space. Then for each $n > 1$ there exists a space $X(n+1, \ldots, \infty)$ and a map $p: X(n+1, \ldots, \infty) \to X$ such that $X(n+1, \ldots, \infty)$ is $n$-connected and $p: \Pi_1(X(n+1, \ldots, \infty)) \to \Pi_1(X)$ is an isomorphism for all $\ell \geq n+1$.

Proof: Connet $X \overset{i}{\to} X(0, \ldots, n)$ to a fibration.

\[
\begin{array}{ccc}
X & \xrightarrow{\text{homotopy}} & F \\
& \text{equiv.} & \\
\downarrow & & \downarrow
\end{array}
\]  

\[
F \xrightarrow{f} X' \xrightarrow{\text{retract}} X(0, \ldots, n)
\]

Take $X(n+1, \ldots, \infty) = F$ and $p = h_j$.

Note: $X(n+1, \ldots, \infty)$ is sometimes called the $n$-connected covering of $X$. Without loss of generality, we can suppose $p$ is a fibration. The universal covering space $X$ of $X$ can be taken as $X(2, \ldots, \infty)$.

Proof of the Morel & Hurewicz Theorem: The case $n=1$ is immediate from the Standard Hurewicz Theorem (since $h_2$ is an isomorphism).

Case $n=2$: We must prove that $H_2(X) \in G$ and $h_3: \Pi_2(X) \to H_3(X)$ is a $G$-isomorphism.

Write $p: Y \to X$ for the 2-connected covering of $X$, regarded as a fibration. Write $F$ for the fibre. From the homotopy sequence of this fibration, it follows that $F$ is a space of type $(H_2(X), 1)$. Thus, since $H_2(X) \in G$ and $G$ is strongly homologically complete, $H_i(F) \in G$ for all $i > 0$. Thus $E_{2, \ell} = H_2(X, H_2(F)) \cong H_2(F) \in G$ for $\ell \neq 0$. Since $X$ is 1-connected, $E_{1, \ell} = H_1(X, H_2(F)) = 0$ for all $\ell > 0$, and so $E_{1, \ell} = 0$ for all $\ell$.

Since $Y$ is 2-connected, $H_2(Y) = 0$ and so $E_{2, \ell} = 0$ for $p+\ell = 2$. In particular, $E_{2, 0} = 0$ and $E_{2, \ell} = 0$ for $\ell > 0$. Therefore, $E_{2, \ell} = 0$ and $E_{1, \ell} = 0$ for all $\ell$.
Lemma: Let $E$ be a homologically complete fibre class. Let $\pi : (T, T_0) \to (B, B_0)$ be a fibration pair with $F \to T \to B$ $\mathbb{Z}$-oriented. Suppose $H_i(B, B_0) = 0$ and, for some $n \geq 1$, $H_n(B, B_0) \in \mathbb{C}$ for $1 \leq i \leq n$, and $H_j(F) \in \mathbb{C}$ for $0 < j \leq \begin{cases} n-1 & \text{if } B_0 \neq \emptyset \\ n+1 & \text{if } B_0 = \emptyset \end{cases}$. Suppose $F$ is $0$-ultrametric.

Then $\pi_* : H_{n+1}(F) \to H_{n+1}(B, B_0)$ is a $C$-isomorphism.

Proof: Let $E$ be the Leray-Serre spectral sequence of the above fibration pair with $\mathbb{Z}$-coefficients. By the base-change theorem, it suffices to show $P_B : H_{n-1}(F) \to E_{n-1,0}$ and $I_B : E_{n,0} \to E_{n,0}$ are $C$-isomorphisms.

Prove that $P_B$ is a $C$-isomorphism. $P_B$ is the composition $H_{n-1}(F) \xrightarrow{H_{n-1}(F) \to E_{n,0}} H_{n-1}(F) \otimes \mathbb{Z} \to E_{n,0}$. Thus, we must show $H_{n-1}(F) \in \mathbb{C}$. We prove by induction on $i$ that $H_{n-1-i}(F) \in \mathbb{C}$ for $0 \leq i \leq n$.

$H_{n}(F) = E_{n,i}$, and $E_{n,i} = H_0(B, B_0; H_{n+1-i}(F)) = \begin{cases} 0 & \text{if } B_0 \neq \emptyset \\ H_{n+1-i}(F) & \text{if } B_0 = \emptyset \end{cases} \in \mathbb{C}$. Hence $H_{n-i}(F) \in \mathbb{C}$.

We have the exact sequence $0 \to H_{n-i}(F) \to E_{n-i}$. Now, $E_{n-i} = H_0(B, B_0; H_{n-i}(F)) = 0$, and hence $E_{n-i} = 0 \in \mathbb{C}$.

Hence $H_{n-i}(F) \in \mathbb{C}$.

Suppose for some $1 \leq i < n$, $H_{n-i}(F) \in \mathbb{C}$.

We have the exact sequence $H_{n-i}(F) \to H_{n-i}(F) \to E_{n-i}$. Now, $E_{n-i} = H_0(B, B_0; H_{n-i}(F))$. By hypothesis, $H_{n-i}(F) \in \mathbb{C}$ for all fibre ranges, and $H_{n-i}(B, B_0), H_0(B, B_0)$ are either $\mathbb{Z}$ or $\mathbb{C}$. Thus, by U.C.T. and homological completeness of $C$, $E_{n-i} \in \mathbb{C}$ and hence $E_{n-i} \in \mathbb{C}$.

Thus, $H_{n-i}(F) \in \mathbb{C}$ is complete. Prove that $P_B$ is a $C$-isomorphism.
Proof that \( i_B \) is a \( G \)-morphism: \( i_B : E_{n+1}^2 \to E_{n+1}^2 \) is the composition 
\[ E_{n+1}^2 = E_{n+1}^2 \to E_{n+1}^2 \to \cdots \to E_{n+1}^2, \]
so it suffices to show that inclusion \( E_{n+1}^2 \to E_{n+1}^2 \) is a \( G \)-morphism \( \forall \ 2 \leq r \leq n+1 \).

\[ E_{n+1}^r / E_{n+1}^r \cong \ker \left[ E_{n+1}^r \to E_{n+1-r}^r \right], \]

Thus, it suffices to show \( E_{n+1-r}^r \in G \). We have

\[ E_{n+1-r}^r \equiv H_{n+1-r}(B, B_0; H_{-1}(F)). \]

For \( 2 \leq r \leq n-1 \), \( H_{-1}(F) \), \( H_{n+1-r}(B, B_0) \), \( H_{n-r}(B, B_0) \in G 
and \( E_{n+1-r,v}^r \in G \) by U.C.T. and homological completeness of \( G \).

When \( r = n \), \( H_{n}(B, B_0; H_{-1}(F)) = 0 \) by U.C.T. and hypothesis.
When \( r = n+1 \),

\[ H_{n}(B, B_0; H_{0}(F)) = \begin{cases} 0 & \text{if } B_0 \neq \emptyset \\ H_{n}(F) & \text{if } B_0 = \emptyset \end{cases}, \]

Thus, \( E_{n+1-r,v}^r \in G \) for \( 2 \leq r \leq n+1 \), and so \( E_{n+1-r,v}^r \in G \) for \( 2 \leq r \leq n+1 \), completing proof of lemma.

Proof of the Weak \( G \)-Homotopy Theorem: The case \( n=1 \) is immediate from the standard homotopy theorem (as \( d_2 \) is an isomorphism).

Suppose \( n \geq 2 \) and theorem holds for \( n-1 \). By the inductive hypothesis, \( H_n(X, *) \in G \) for \( 0 \leq i \leq n \) and it remains only to prove \( H_{n+1} : \Pi_{n+1}(X, *) \to H_{n+1}(X, *) \) is a \( G \)-morphism.

Write \( f : Y \to X \) for the \( 2 \)-connective covering of \( X \), regarded as a fibration. Write \( F \) for the fiber. From the homotopy sequence of this fibration it follows that \( F \) is a space of type \(( \Pi_2(X), 1) \). Thus \( \Pi_2(X) \in G \) and \( G \) is strongly homologically complete, \( H_2(F) \in G \) for all \( i > 0 \).

We have the commutative diagram.
\[
\begin{align*}
\Pi_{n+1}(Y, \star) & \xrightarrow{p_*} \Pi_{n+1}(X, \star) \\
\text{by lemma above, } p_* : H_{n+1}(Y) & \rightarrow H_{n+1}(X) \text{ is a } G- \\
\text{isomorphism. Thus, bottom } p_* \text{ is above diagram is a } \\
\text{G-isomorphism, and it remains only to show } \\
k_{n+1} : \Pi_{n+1}(Y, \star) & \rightarrow \Pi_{n+1}(X, \star) \text{ is a } G- \\
\text{isomorphism.} \\
\text{Consider the fibrations } \mathcal{L}X \rightarrow PY \rightarrow Y. \text{ We have } \\
\text{the commutative diagram,} \\
\Pi_{n+1}(Y, \star) & \xleftarrow{\pi_*} \Pi_{n+1}(PY, \mathcal{L}Y) \xrightarrow{\phi} \Pi_n(\mathcal{L}Y, \mathcal{L}Y) \\
k_{n+1} & \downarrow \\
H_{n+1}(Y, \star) & \xleftarrow{\pi_*} H_{n+1}(PY, \mathcal{L}Y) \xrightarrow{\phi} H_n(\mathcal{L}Y, \mathcal{L}Y), \\
\text{with indicated isomorphisms from fact that } \pi \text{ is a } \text{fibration} \text{ and } PY \text{ contractible.} \\
\text{Since } \Pi_n(Y) \cong \Pi_n(X) \text{ for all } n \geq 3 \text{, it follows that } \\
\Pi_n(Y) \in G \text{ for } n \leq n. \text{ Hence by the inductive hypothesis, } \\
H_n(Y, \star) \in G \text{ for } n \leq n. \text{ Thus by an earlier lemma, } \\
H_{n-1}(\mathcal{L}Y, \mathcal{L}Y) \in G \text{ for } n \leq n-1. \text{ Thus, by above lemma } \\
p_\pi : H_{n+1}(PY, \mathcal{L}Y) \rightarrow H_{n+1}(Y, \star) \text{ is a } G- \\
isomorphism. \text{ Also } \mathcal{L}Y \text{ is } 1- \text{connected since } Y \text{ is } 2- \text{connected and so by the } \\
induction hypothesis, k_{n+1} : \Pi_{n+1}(\mathcal{L}Y, \mathcal{L}Y) \rightarrow H_{n+1}(\mathcal{L}Y, \mathcal{L}Y) \text{ is a } G- \\
isomorphism. \text{ Thus } k_{n+1} : H_{n+1}(Y, \star) \rightarrow H_{n+1}(Y, \star) \text{ is a } G- \\
isomorphism, \text{ completing the proof.} 
\end{align*}
\]
\[ d_3(x^2) = d_3(x) \cdot x + (-1)^2 x \cdot d_3(x) = y \cdot x + x \cdot y \]
\[ = y \cdot x + (-1)^2 y \cdot x = 2y \cdot x \]
Thus \( d_3 \vert E_3 \otimes \mathbb{Z} \) is \(-1\) and \( E_6^0,4 = E_4^0,4 = 0 \). Thus \( E_4^p,q = 0 \) whenever \( p + q = 4 \) and so \( H_4(Y) = 0 \).
(Thus, the algebraic part of \( H_4(Y) = 0 \)).
We have
\[ E_4 \sim E_3^2 / \text{ker} \{ E_3^0 \otimes d_3 \rightarrow E_3^1 \} \]
\[ = \frac{\text{ker} \{ E_3^2 \}}{\text{ker} \{ E_3^0 \}} \approx \mathbb{Z}/2. \]
Thus \( E_4^3 = \mathbb{Z}/2. \)

Note that \( E_2^p,q = 0 \) whenever \( p + q = 5 \) and \( (p, q) \neq (3, 2) \).
Thus \( 0 = J_5^{0,1} = J_5^{1,0} = J_5^{0,1} \).
\[ \mathbb{Z}/2 = E_4^{3,2} = J_3^{3,2} = J_3^{2,3} = J_3^{1,4} = J_3^{0,5} = H^5(Y). \]
Thus, the torsion part of \( H_4(Y) \approx \mathbb{Z}/2 \).

Conclusion: \( \pi_4(S^3) \approx \mathbb{Z}/2 \)
Further Topics and References

1. Cohomology operations
   Spanier

2. Characteristic classes
   Husemoller, "Fibre Bundles"

3. Poincaré duality
   Milnor + Stasheff (appendix)
   Spanier
   Gold, Algebraic Topology

4. Generalized cohomology and cohomology, spectra
   M. F. Atiyah, K. Thom

5. General:
Math 752 Problems

Due Jan. 23, 1984

1. Let $F$ be a field, and $F[x]$ the polynomial ring over $F$ on one indeterminate $x$. It is an easy algebraic fact that $F[x]$ is a principal ideal domain.

Let $F$ be an $F[x]$-module with module action given by $(a_0 + a_1 x + \ldots + a_n x^n) \cdot b = a_0 b$, $a_i, b \in F$.

a) Find $\text{Ext}^1_{F[x]}(F, F)$.

b) Find $\text{Ext}^1_{F[x]}(F, F[x])$.

2. Let $X$ be a path-connected topological space, and $G$ an abelian group.

Prove: For every 0-cocycle $f \in \mathcal{C}^0(X; G) = \text{Hom}_\mathbb{Z}(\mathcal{C}_0(X), G)$, $f(\sigma) = f(\tau)$ for all singular 0-cubes $\sigma, \tau: I^0 \to X$.

3. Find $H^2(\mathbb{R}P^n; G)$ for all $n$ and all $i$ in each of the following cases:

a) $G = \mathbb{Z}/2$

b) $G = \mathbb{Z}/p$ where $p$ is an odd prime

c) $G = \mathbb{Q}$, the rationals

d) $G = \mathbb{R}$
Let $R$ be a PID. An \underline{R-oriented real $n$-plane bundle} $(E, p, B, U)$ consists of a real $n$-plane bundle $E \overset{p}{\rightarrow} B$ together with an \underline{$R$-orientation} $U \in H^n(E, E^0, R)$.

A \underline{morphism} of \underline{$R$-oriented $n$-plane bundles} $f: (E_1, p_1, B_1, U_1) \rightarrow (E_2, p_2, B_2, U_2)$ consists of a morphism of $n$-plane bundles $f_E: E_1 \overset{\cong}{\rightarrow} E_2$ and $f_B: B_1 \overset{\cong}{\rightarrow} B_2$ such that $\overline{f}^*(U_2) = U_1$, where $\overline{f}: (E_1, E_1^0) \rightarrow (E_2, E_2^0)$ is the relative map coming from $f_E$.

If $(E, p, B, U)$ is an \underline{$R$-oriented real $n$-plane bundle}, the \underline{Euler class} of $(E, p, B, U)$, written $\chi(E, U) \in H^n(B, R)$, is the unique class satisfying $p^*\chi(E, U) = j^*U$ where $j: (E, \emptyset) \rightarrow (E, E^0)$ is the inclusion. (Note: $p^*$ is an isomorphism since $p$ is a homotopy equivalence).

4. Let $f: (E_1, p_1, B_1, U_1) \rightarrow (E_2, p_2, B_2, U_2)$ be a \underline{morphism} of \underline{$R$-oriented $n$-plane bundles}.

\textbf{Prove:} $f_B^*\chi(E_2, U_2) = \chi(E_1, U_1)$. 

\textbf{Date Feb. 20, 1994}
5. a) Let $V$ be an $n+1$-dimensional real vector space, $n \geq 1$.
Let $U$ be the unique $\mathbb{Z}/2$ orientation of $L_{\mathbb{R}}(V) \to \mathbb{P}_{\mathbb{R}}(V)$.
Prove: $\chi(L_{\mathbb{R}}(V), U)$ is the generator of $H^1(\mathbb{P}_{\mathbb{R}}(V); \mathbb{Z}/2)$.

b) Let $V$ be an $n+1$-dimensional complex vector space, $n \geq 1$.
Let $U$ be a $\mathbb{Z}$-orientation of the underlying real 2-plane bundle of $L_{\mathbb{C}}(V) \to \mathbb{P}_{\mathbb{C}}(V)$.
Prove: $\chi(L_{\mathbb{C}}(V), U)$ is a generator of $H^2(\mathbb{P}_{\mathbb{C}}(V); \mathbb{Z})$.

6. A continuous map $f : S^m \to S^n$ will be called **equivariant** if $f(-x) = -f(x)$ for all $x \in S^m$.

a) Suppose $f : S^m \to S^n$ is equivariant. Define

$$F_E : L_{\mathbb{R}}(\mathbb{R}^{m+1}) \to L_{\mathbb{R}}(\mathbb{R}^{n+1})$$
and $F_B : \mathbb{R}P^m \to \mathbb{R}P^n$ by

$$F_E([v], w) = \left( [f(v)/\|v\|], \frac{\langle v, w \rangle}{\|v\|^2} f\left(\frac{v}{\|v\|}\right) \right)$$

$$F_B([v]) = \left[ f\left(\frac{v}{\|v\|}\right) \right].$$

Verify that $F_E, F_B$ are well-defined and constitute a map of real 1-plane bundles $L_{\mathbb{R}}(\mathbb{R}P^{m+1}) \xrightarrow{F} L_{\mathbb{R}}(\mathbb{R}P^{n+1})$

$b)$ **Prove**: If $m > n$, a continuous equivariant map $S^m \to S^n$ does not exist.

$c)$ (Borsuk-Ulam Theorem) **Prove**: If $f : S^n \to \mathbb{R}^n$ is continuous, there exists an $x \in S^n$ such that $f(x) = f(-x)$.
Math 752 Problems Due Mar. 5, 1984

7. Let \((E, p, B, F)\) be a finitely presentable fibre bundle, i.e. \(p : E \to B\) is continuous, and there is a finite open cover \(\{ U_1, \ldots, U_n \}\) of \(B\) and homeomorphisms \(h_i : U_i \times F \to p^{-1}(U_i)\) such that \(\pi \circ h_i = p\). For \(x \in B\), write \(E_x = p^{-1}(x)\), and \(j_x : E_x \to E\) for the inclusion.

Let \(R\) be a PID. Suppose there exists a finite set of cohomology classes \(\{ c_i \in H^n(E; R) \mid 1 \leq i \leq k \}\) such that for each \(x \in B\), \(\bigoplus_{n \geq 0} H^n(E_x; R)\) is a free \(R\)-module with basis \(\{ c_i^* \mid 1 \leq i \leq k \}\).

Prove: For each \(x \in H^n(E; R)\), there exist unique elements \(b_i^x \in H^{n-n_i}(B; R)\) such that \(x = \sum_{i=1}^{k} p^*(b_i^x) \cup c_i\).

(Hint: Let \(\psi_n : \bigoplus_{i=1}^{k} H^{n-n_i}(B; R) \to H^n(E; R)\) be the map given by \(\psi_n(b_1^x, \ldots, b_k^x) = \sum_{i=1}^{k} p^*(b_i^x) \cup c_i\). It must be shown that \(\psi_n\) is an isomorphism. Imitate the proof of the Thom Isomorphism Theorem.)

8. Let \(p : E \to B\) be a real \(n\)-plane bundle. Let \(P_{\mathbb{R}}(E)\) be quotient space obtained from \(E^\mathbb{R}\) by identifying \(x^\mathbb{R} \sim y^\mathbb{R}\) whenever \(x \in \mathbb{R} - \{0\}\). (Thus, as a set, \(P_{\mathbb{R}}(E) = \bigsqcup_{x \in B} P_{\mathbb{R}}(E_x)\).) Let \(L_{\mathbb{R}}(E) = \{(x^\mathbb{R}, \omega) \in P_{\mathbb{R}}(E) \times E \mid \omega \in [\omega]\}\). (As in the lectures we are regarding \([\omega]\) as \(P_{\mathbb{R}}(E)\) as the 1-dimensional subspace of \(E_{\mathbb{R}}\) spanned by \(\omega\).)

Let \(q : L_{\mathbb{R}}(E) \to P_{\mathbb{R}}(E)\) be given by \(q([x], \omega) = [\omega]\), and \(r : P_{\mathbb{R}}(E) \to B\) be given by \(r([x]) = p(x^\mathbb{R})\).
It can be shown that

1) \( \pi \) is a fibre bundle projection with fibres homeomorphic to \( \mathbb{RP}^n \), and this fibre bundle is finitely-presentable if \( p : E \rightarrow B \) is.

2) \( \beta : L_{\mathbb{R}}(E) \rightarrow \mathbb{RP}(E) \) is a real line bundle, and is finitely-presentable if \( p : E \rightarrow B \) is.

(Try to prove 1) and 2), but you do not have to turn in the proofs).

Suppose \( p : E \rightarrow B \) is finitely-presentable, and let \( U \in H^1(L_{\mathbb{R}}(E), L_{\mathbb{R}}(E)^0; \mathbb{Z}/2) \) denote the unique \( \mathbb{Z}/2 \)-orientation of \( L_{\mathbb{R}}(E) \rightarrow \mathbb{RP}(E) \).

Prove: For each \( x \in H^n(\mathbb{RP}(E), \mathbb{Z}/2) \), there are unique elements \( b_i \in H^{n-i}(B; \mathbb{Z}/2) \), \( 0 \leq i \leq n-1 \), such that

\[
x = \sum_{i=0}^{n-1} r^*(b_i) \cup \chi(L_{\mathbb{R}}(E), U)^i.
\]

Note: In particular, there are unique elements

\[
w_i(E) \in H^i(B; \mathbb{Z}/2), \quad 1 \leq i \leq n,
\]

such that

\[
\chi(L_{\mathbb{R}}(E), U)^n = \sum_{i=1}^{n} r^*w_i(E) \cup \chi(L_{\mathbb{R}}(E), U)^{n-i}.
\]

The \( w_i(E) \) are called the \textit{Stiefel-Whitney classes} of the real \( n \)-plane bundle \( p : E \rightarrow B \). We also define \( w_0(E) = 1 \) and \( w_i(E) = 0 \) if \( i > n \).

Note: A similar treatment can be given for complex vector bundles, yielding \textit{Chern classes}.
9. Let $Y$ be a pointed space such that $[X,Y]$ has a natural group structure for pointed spaces $X$ (i.e. the functor $[X,Y]$ factors through the category of groups).

Prove: This natural group structure comes from an $H^*$-structure on $Y$.

10. Prove: If $X$ is a co-$H^*$ space with comultiplication $\mu'$ and coinversion $\tau'$, then for any pointed $Y$, $[X,Y]$ is a group with operation given by $[f][g] = [f \circ g]$ where $f \circ g$ is defined as in the lectures. The unit element is $[\ast]$ and $[f]^{-1} = [\tau']$. Moreover, $[X, \_]$ is a covariant functor from the category of pointed spaces to the category of groups.
11. Prove: The homotopy sequence of a pointed triple is exact.

12. Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ be real $m$ and $n$-plane bundles, resp. Write $E_1 \oplus E_2 = \{(u,v) \in E_1 \times E_2 | p_1(u) = p_2(v)\}$ and let $p : E_1 \oplus E_2 \to B$ be given by $p(u,v) = p_1(u) = p_2(v)$. Note that for each $x \in B$, $p^{-1}(x) = (E_1)_x \oplus (E_2)_x$ and hence has the structure of a real $m+n$-dimensional vector space.

Prove: $p : E_1 \oplus E_2 \to B$ is a real $(m+n)$-plane bundle, and is finitely-presentable if the $E_i \to B$ are, $i = 1,2$. ($E_1 \oplus E_2 \to B$ is called the Whitney sum of the vector bundles $E_1 \to B$, $E_2 \to B$).

13. Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ be finitely-presentable $m$ and $n$-plane bundles, respectively. Write $P(E) = P_\mathbb{R}(E)$, $L(E) = L_{\mathbb{R}}(E)$.

We have inclusions $\imath_k : P(E_k) \to P(E_1 \oplus E_2)$, $\jmath_k : L(E_k) \to L(E_1 \oplus E_2)$, $k = 1,2$, given by

$$\imath_1([u]) = [u,0], \quad \imath_2([v]) = [0,v], \quad \jmath_1([u],w) = ([uw],(w,0)), \quad \jmath_2([v],x) = ([0,v],(0,x)).$$

(a) Prove: $\imath_k, \jmath_k$ constitute a map of real line bundles

$$
\begin{align*}
P(E_k) & \quad \to \quad P(E_1 \oplus E_2) \\
\downarrow & \quad \downarrow \\
L(E_k) & \quad \to \quad L(E_1 \oplus E_2)
\end{align*}
$$

(b) Identify $P(E_1), P(E_2)$ with subspaces of $P(E_1 \oplus E_2)$ via $\imath_1, \imath_2$, respectively. Let $U = P(E_1 \oplus E_2) - P(E_2)$, $V = P(E_1 \oplus E_2) - P(E_1)$.

Prove: $P(E_1)$ is a deformation retract of $U$, and $P(E_2)$ is a deformation retract of $V$. 
14. Suppose $X = U \cup V$ where $U, V$ are open in $X$. Let $i_1 : U \to X$, $i_2 : V \to X$ denote the inclusions. Let $R$ be a PID and suppose $a \in H^1(X; R)$, $b \in H^1(X; R)$ are such that $i_1^*(a) = 0$, $i_2^*(b) = 0$. Prove: $a \cup b = 0$.

15. (Whitney Product Formula) Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ be finitely-presentable real $m$- and $n$-plane bundles, resp. Prove: $w_k(E_1 \oplus E_2) = \sum_{i=0}^k w_i(E_1) \cup w_{k-i}(E_2)$.

(Hint: Write $\chi_{E_1 \oplus E_2} = \chi(E_1 \oplus E_2, U)$, $U$ the unique $\mathbb{Z}/2$-orientation. Note that
\[\sum_{k=0}^{m+n} r^*(\sum_{i=0}^k w_i(E_1) \cup w_{k-i}(E_2)) = \chi_{E_1 \oplus E_2} \]
\[= \left(\sum_{i=0}^m r^*w_i(E_1) \cup \chi_{E_1 \oplus E_2}^{m-i}\right) \cup \left(\sum_{j=0}^n r^*w_j(E_2) \cup \chi_{E_1 \oplus E_2}^{n-j}\right)\]
where $r : p(E_1 \oplus E_2) \to B$ is the projection.)
16. Compute $H_n(S^2 \vee S^3)$ for all $n$.

17. Let $F \to T \to S^n$ be a pointed fibration, $n \geq 2$.
Let $R$ be any commutative ring with unit.
Prove: There exists an exact sequence

\[ \cdots \to H_{i-1+n+1} (F; R) \xrightarrow{p^*} H_i (F; R) \xrightarrow{d^*} H_i (T; R) \to H_{i-n} (F; R) \xrightarrow{p^*} \to \cdots \]

18. Let $S^n \to T \to B$ be a pointed fibration with $B$ simply-connected and $n \geq 1$.
Let $B_0 \subset B$ and $T_0 = p^{-1}(B_0)$.
Let $R$ be any commutative ring with unit.
Prove: There exists an exact sequence

\[ \cdots \to H_{i-n} (B_0; R) \xrightarrow{\delta} H_i (T_0; R) \xrightarrow{p^*} H_i (B, B_0; R) \to H_{i-n-1} (B, B_0; R) \xrightarrow{\delta} \cdots \]

19. Let $F \to T \to B$ be a pointed fibration with $B$ 1-connected. Assume $H_i (F; \mathbb{Q})$ and $H_i (B; \mathbb{Q})$ are finite-dimensional over $\mathbb{Q}$ for all $i$, and non-zero for only finitely many $i$.
Prove: $H_i (T; \mathbb{Q})$ is finite-dimensional over $\mathbb{Q}$ for all $i$, non-zero for only finitely many $i$, and $\chi (T) = \chi (B) \chi (F)$.
(Recall: The Euler characteristic $\chi (X) = \sum (-1)^i \dim_{\mathbb{Q}} H_i (X; \mathbb{Q})$.)

20. The tangent bundle $\tau (S^n)$ has the following description:
$\tau (S^n) = \{(x, y) \in S^n \times \mathbb{R}^{n+1} | \langle x, y \rangle = 0\}$ where $\langle \ , \rangle$ is the standard Euclidean inner product. The projection $\tau (S^n) \to S^n$ is projection on the first factor.

a) Let $\mathbb{E}^k$ denote the product $k$-plane bundle over $S^n$.
Prove: $\tau (S^n) \otimes \mathbb{E}^1 \approx \mathbb{E}^{n+1}$.

b) Find $\omega_i (\tau (S^n))$ for all $i$. 

1. $0 \rightarrow F[x] \xrightarrow{\partial} F[x] \xrightarrow{\varepsilon} F \rightarrow 0$ is a short free resolution of $F$, where $\varepsilon(f(x)) = f(0)$, $\partial(f(x)) = x f(x)$.

a) Using the above short free resolution we have the exact sequence

$$\text{Hom}_{F[x]} (F[x], F) \xrightarrow{\text{Hom} (\partial, 1_F)} \text{Hom}_{F[x]} (F[x], F) \xrightarrow{\text{Ext}^1_{F[x]} (F, F)} 0.$$

Note that $\text{Hom} (\partial, 1_F)$ is the $O$-homomorphism, for if

$\alpha : F[x] \rightarrow F$ is an $F[x]$-homomorphism and $x(\alpha) \in F[x]$, then

$$\text{Hom} (\partial, 1_F) (\alpha) (x(f(x))) = x \partial (f(x)) = x (x f(x)) = 0 \quad \text{(since $\alpha$ is an $F[x]$-homomorphism)}.$$

Hence $\text{Ext}^1_{F[x]} (F, F) \cong \text{Hom}_{F[x]} (F[x], F) \cong F$

(since $\text{Hom}_R (R, A) \cong A$ in general).

b) Using the same short free resolution above, we have the exact sequence

$$\text{Hom}_{F[x]} (F[x], F[x]) \xrightarrow{\text{Hom} (\partial, 1_F)} \text{Hom}_{F[x]} (F[x], F[x]) \xrightarrow{\text{Ext}^1_{F[x]} (F, F[x])} 0.$$

We have the natural $F[x]$-isomorphism $\Theta : \text{Hom}_{F[x]} (F[x], F[x]) \rightarrow F[x]$ given by $\Theta (\alpha) = \alpha (1)$, and it is easily checked that

$$\text{Hom}_{F[x]} (F[x], F[x]) \xrightarrow{\text{Hom} (\partial, 1_{F[x]})} \text{Hom}_{F[x]} (F[x], F[x])$$

$\Theta \cong \text{Ext}^1_{F[x]} (F, F[x]) \cong F.$

2. Since $X$ is path-connected, for any $O$-cubes $\delta, \tau : I^0 \rightarrow X$, $\exists$ continuous $\varphi : I^1 \rightarrow X$ such that $\varphi(0) = \delta(1)$, $\varphi(1) = \tau(1)$. Thus $\partial \varphi = \tau - \delta$. Hence, for any $O$-cocycle $f \in \text{C}^0(X, G)$,

$0 = (\delta \varphi)(\partial) = f(\partial \varphi) = f(\tau - \delta) = f(\tau) - f(\delta), \varphi(1)$.
3. For any field $F$, $H^i(X,F) \cong \text{Hom}_F (H_i(X,F), F)$. Thus for parts a), b), c), using the results of HW problem 27 of Math 751,

a) $H^i(\mathbb{R}P^n, \mathbb{Z}/2) \cong \text{Hom}_{\mathbb{Z}/2} (H_i(\mathbb{R}P^n, \mathbb{Z}/2), \mathbb{Z}/2) \cong \begin{cases} \text{Hom}_{\mathbb{Z}/2} (\mathbb{Z}/2, \mathbb{Z}/2) & \text{if } 0 \leq i \leq n \\ \text{Hom}_{\mathbb{Z}/2} (\mathbb{Z}/2, \mathbb{Z}/2) & \text{otherwise} \end{cases} = \begin{cases} \mathbb{Z}/2 & \text{if } 0 \leq i \leq n \\ 0 & \text{otherwise} \end{cases}$

b) $H^i(\mathbb{R}P^n, \mathbb{Z}/p) \cong \text{Hom}_{\mathbb{Z}/p} (H_i(\mathbb{R}P^n, \mathbb{Z}/p), \mathbb{Z}/p) \cong \begin{cases} \text{Hom}_{\mathbb{Z}/p} (\mathbb{Z}/p, \mathbb{Z}/p) & \text{if } i = 0 \text{ or } i = n \\ \text{Hom}_{\mathbb{Z}/p} (\mathbb{Z}/p, \mathbb{Z}/p) & \text{otherwise} \end{cases} = \begin{cases} \mathbb{Z}/p & \text{if } i = 0 \text{ or } i = n \\ 0 & \text{otherwise} \end{cases}$

c) $H^i(\mathbb{R}P^n, \mathbb{Q}) \cong \text{Hom}_{\mathbb{Q}} (H_i(\mathbb{R}P^n, \mathbb{Q}), \mathbb{Q}) \cong \begin{cases} \text{Hom}_{\mathbb{Q}} (\mathbb{Q}, \mathbb{Q}) & \text{if } i = 0 \text{ or } i = n \\ \text{Hom}_{\mathbb{Q}} (\mathbb{Q}, \mathbb{Q}) & \text{otherwise} \end{cases} = \begin{cases} \mathbb{Q} & \text{if } i = 0 \text{ or } i = n \\ 0 & \text{otherwise} \end{cases}$

d) By the universal coefficient theorem,

$H^i(\mathbb{R}P^n, \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}} (H_i(\mathbb{R}P^n, \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}_{\mathbb{Z}}^{i-1} (H_{i-1}(\mathbb{R}P^n), \mathbb{Z})$.

Using the fact that $H_* (\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } i = n \text{ with } i \text{ odd} \\ \mathbb{Z}/2 & \text{if } 0 < i < n \text{ with } i \text{ even} \\ 0 & \text{otherwise} \end{cases}$

and the facts that $\text{Hom}_{\mathbb{Z}} (\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$, $\text{Hom}_{\mathbb{Z}} (\mathbb{Z}/2, \mathbb{Z}) = 0$, $\text{Hom}_{\mathbb{Z}} (0, \mathbb{Z}) = 0$, $\text{Ext}_{\mathbb{Z}} (\mathbb{Z}, \mathbb{Z}) = 0$, $\text{Ext}_{\mathbb{Z}} (\mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}/2$, $\text{Ext}_{\mathbb{Z}} (0, \mathbb{Z}) = 0$,

we obtain

$H^i(\mathbb{R}P^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } i = n \text{ with } i \text{ odd} \\ \mathbb{Z}/2 & \text{if } 0 < i < n \text{ with } i \text{ even} \\ 0 & \text{otherwise} \end{cases}$
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4. We have the commutative diagram

\[
\begin{array}{ccc}
(E_1, E_1) & \xrightarrow{f} & (E_2, E_2) \\
\uparrow \quad j_1 & & \uparrow \quad j_2 \\
(E_1, \emptyset) & \xrightarrow{f_E} & (E_2, \emptyset) \\
\downarrow \quad \phi & & \downarrow \quad \phi_2 \\
(B_1, \emptyset) & \xrightarrow{f_B} & (B_2, \emptyset)
\end{array}
\]

It suffices to show \( f_1^* f_B^* \chi(E_2, U_2) = j_1^* U_1 \).

We have

\[
f_1^* f_B^* \chi(E_2, U_2) = f_E^* j_2^* \chi(E_2, U_2) \quad \text{(commutativity of bottom square)}
\]

\[
= f_E^* j_2^* (U_2) \quad \text{(def. of} \ \chi(E_2, U_2))
\]

\[
= j_1^* f_2^* (U_2) \quad \text{(commutativity of top square)}
\]

\[
= j_1^* U_1 \quad \text{(since} \ f \ \text{is a morphism of} \ R \text{-oriented} \ n \text{-plane bundles}).
\]

5. Write \( i : L_F(V)^0 \to L_F(V) \) and \( j : (L_F(V), \emptyset) \to (L_F(V), L_F(V)^0) \) for the inclusions and \( p : L_F(V) \to \mathbb{P}^1(V) \) for the projections, \( F = \mathbb{R} \) or \( \mathbb{C} \). In the lectures it was shown that \( L_F(V)^0 \) has the homotopy type of \( S^n \), and \( L_\mathbb{C}(V)^0 \) has the homotopy type of \( S^{2n+1} \).

a) \( H^0(L_R(V), \mathbb{Z}/2) \to H^0(L_R(V)^0, \mathbb{Z}/2) \to H^1(L_R(V), L_R(V)^0, \mathbb{Z}/2) \to H^1(L_R(V), \mathbb{Z}/2) \)

is exact. \( i^* \) is an isomorphism since \( L_R(V) \) (which has the homotopy type of \( \mathbb{R}^n \)) and \( L_R(V)^0 \) (which has the homotopy type of \( S^n \)) are both path-connected. Hence \( j^* \) is a monomorphism, and so \( j^*(U) \neq 0 \), thus \( p^* \chi(L_R(V), U) \neq 0 \), and so \( \chi(L_R(V), U) \neq 0 \). Thus since \( H^1(p_R(V), \mathbb{Z}/2) \simeq \mathbb{Z}/2 \), \( \chi(L_R(V), U) \) must be the generator.

b) \( H^1(L_\mathbb{C}(V)^0, \mathbb{Z}) \to H^1(L_\mathbb{C}(V), L_\mathbb{C}(V)^0, \mathbb{Z}) \to H^2(L_\mathbb{C}(V), \mathbb{Z}) \to H^2(L_\mathbb{C}(V)^0, \mathbb{Z}) \)

is exact. Since \( L_\mathbb{C}(V)^0 \) has the homotopy type of \( S^{2n+1} \) with \( n \geq 1 \),
both \( H^1(L\epsilon(V)^n, \mathbb{Z}) \) and \( H^2(L\epsilon(V)^n, \mathbb{Z}) \) are 0. Hence, above \( j^* \) is an isomorphism. By the Thom isomorphism, 
\[ u: H^0(L\epsilon(V); \mathbb{Z}) \to H^2(L\epsilon(V), L\epsilon(V)^n, \mathbb{Z}) \] is an isomorphism and since \( H^0(L\epsilon(V); \mathbb{Z}) \cong \mathbb{Z} \) with generator 1, \( H^0(L\epsilon(V), L\epsilon(V)^n, \mathbb{Z}) \cong \mathbb{Z} \) with generator \( j^*U \). Thus \( H^2(L\epsilon(V); \mathbb{Z}) \cong \mathbb{Z} \) with generator \( j^*U \). Thus, since \( p^*: H^2(P_\epsilon(V), \mathbb{Z}) \to H^2(L\epsilon(V); \mathbb{Z}) \) is an isomorphism and \( p^*\mathcal{L}(L\epsilon(V), U) = j^*U \), \( \mathcal{L}(L\epsilon(V), U) \) must be a generator of \( H^2(P_{\epsilon}(V); \mathbb{Z}) \).

6. a) Define \( g: \mathbb{R}^{m+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\} \) by \( g(\nu) = f(\frac{\nu}{\|\nu\|}) \).

\( g \) is continuous. For any \( \nu \in \mathbb{R}^{R - \{0\}} \) we have 
\[ g(\nu) = f(\frac{\nu}{\|\nu\|}) = f(\frac{\nu}{\|\nu\|}) = \frac{\|\nu\|}{1\|\nu\|} f(\frac{\nu}{\|\nu\|}) \] by equivariance of \( f \), and 
\[ \left[ \frac{\|\nu\|}{1\|\nu\|} f(\frac{\nu}{\|\nu\|}) \right] = \left[ f(\frac{\nu}{\|\nu\|}) \right] \text{ in } \mathbb{R}^n. \]

Thus \( F_\mathcal{B} \) is well-defined, and is continuous since it arises from \( g \) by passage to quotients.

Define \( h: (\mathbb{R}^{m+1} - \{0\}) \times \mathbb{R}^{n+1} \to (\mathbb{R}^{n+1} - \{0\}) \times \mathbb{R}^{m+1} \)
by \( h(\nu, \omega) = (f(\frac{\nu}{\|\nu\|}), \frac{\langle \nu, \omega \rangle}{\|\nu\|} f(\frac{\nu}{\|\nu\|})) \). \( h \) is continuous, and the second component of this last expression is always a scalar multiple of the first.

For any \( \nu \in \mathbb{R}^{R - \{0\}} \) we have, using the equivariance of \( f \),
\[ h(\nu, \omega) = (f(\frac{\nu}{\|\nu\|}), \frac{\langle \nu, \omega \rangle}{\|\nu\|} f(\frac{\nu}{\|\nu\|})) \]
\[ = (\frac{\|\nu\|}{1\|\nu\|} f(\frac{\nu}{\|\nu\|}), \frac{\|\nu\|}{1\|\nu\|} \frac{\langle \nu, \omega \rangle}{\|\nu\|} f(\frac{\nu}{\|\nu\|})) \]
\[ = (\frac{\|\nu\|}{1\|\nu\|} f(\frac{\nu}{\|\nu\|}), \frac{\|\nu\|}{1\|\nu\|} \frac{\langle \nu, \omega \rangle}{\|\nu\|} f(\frac{\nu}{\|\nu\|})) \].

Since \( \left[ \frac{\|\nu\|}{1\|\nu\|} f(\frac{\nu}{\|\nu\|}) \right] = \left[ f(\frac{\nu}{\|\nu\|}) \right] \text{ in } \mathbb{R}^n \), \( F_\mathcal{B} \) is well-defined and is continuous since it arises, by passage to quotients, from the restriction of \( h \) to \( \{ (\nu, \omega) \mid \omega = \text{a scalar multiple of } \nu \} \).
Commutativity of \( L_R(R^{m+1}) \xrightarrow{F_E} L_R(R^{m+1}) \)
\[ \xrightarrow{P_R(R^{m+1})} \xrightarrow{F_B} P_R(R^{m+1}) \]

is immediate. Finally, to check that \( F_E \) maps fibres isomorphically onto fibres it suffices to check (since the fibres are 1-dimensional) that \( F_E([v], rw) = r F_E([v], w) \) for \( r \in R \), \((v, w) \in L_R(R^{m+1})\), and that \( F_E([v], w) \neq 0 \) for \( w \neq 0 \).

We have
\[
F_E([v], rw) = \left[ f\left( \frac{\alpha}{||v||} \right) \right], \quad \left< \frac{\alpha}{||v||}, rw \right> = f\left( \frac{\alpha}{||v||} \right)
\]
\[
= r \left( \left[ f\left( \frac{\alpha}{||v||} \right) \right], \quad \left< \frac{\alpha}{||v||}, w \right> = f\left( \frac{\alpha}{||v||} \right) = r F_E([v], w).
\]

If \( w \neq 0 \), then \( w = \pm \alpha \) for some \( \pm \alpha \neq 0 \), and 2nd component of \( F_E([v], w) = \left< \frac{\alpha}{||v||}, w \right> = t \frac{\alpha}{||v||} \neq 0 \), and so \( F_E([v], w) \neq 0 \).

b) If an equivariant \( S^m \to S^n \) existed, by part a) there would exist a map of real 1-plane bundles
\[ L_R(R^{m+1}) \xrightarrow{F_E} L_R(R^{n+1}) \]
\[ \xrightarrow{P_R(R^{m+1})} \xrightarrow{F_B} P_R(R^{n+1}) \]

Write \( u = \text{generator of } H^1(RP^m, \mathbb{Z}/2) \), \( \pi = \text{generator of } H^1(RP^n, \mathbb{Z}/2) \). By problem 5, \( u = \chi(L_R(R^{m+1}), \mathbb{Z}/2\text{-orient.}) \), \( \pi = \chi(L_R(R^{n+1}), \mathbb{Z}/2\text{-orient.}) \), and by problem 4, \( F_B^*(\pi) = u \). Thus, since \( F_B^* \) is a ring homomorphism, \( F_B^*(\pi^m) = u^m \). If \( m > n \), \( \pi^m = 0 \) since \( H^m(RP^n, \mathbb{Z}/2) = 0 \), and so we would have \( u^m = 0 \). But it was proved in lecture that \( u^m \neq 0 \); contradiction.
c) Suppose $f: S^n \to \mathbb{R}^n$ is continuous, and $f(x) \neq f(-x)$ for all $x \in S^n$. Then $g: S^n \to S^n$ given by

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

would be a continuous equivariant map, which is impossible by part b).

Note: When $n=2$, the Borsuk–Ulam Theorem has the following meteorological consequence: If temperature and atmospheric pressure at any given moment are assumed to be continuous functions of position on the surface of the earth, then at every moment there exists a pair of antipodal points on the surface of the earth at which the temperature and atmospheric pressure are the same.
7. Proceed by induction on the presentability (no. of charts required) of the bundle.

The 1-presentable case: Case 1: B path-connected. We can suppose $E = B \times F$ and $p: B \times F \to B$ is projection on the first factor. Each $E_x$ is then identified with $F$, $x \in B$, and $i_x: F \to B \times F$ is given by $i_x(y) = (x, y)$. By hypothesis, each $H^*(F; R)$ is a free $R$-module on a finite no. of generators. Thus the cross-product map $\bar{\alpha}: \left[ H^*(B; R) \otimes H^*(F; R) \right]^n \to H^n(B \times F; R)$ is an isomorphism. (This would be immediate from the cohomology Künneth theorem proved in the lectures if we assumed each $H^*_x(F; R)$ is a free and finitely-generated $R$-module. The general case needs more technical argument. Actually, I would have been satisfied with a correct proof assuming each $H^*_x(F; R)$ is a free and finitely-generated $R$-module. Sorry about that.)

Choose a base-point $x_0 \in B$ and write $\tilde{\gamma}_i = \pi^* (\gamma^i)$, $\tilde{\gamma}_0 = \pi^* (\gamma^0) = 1 \times \tilde{\gamma}_0$. Let $\mathbf{V}_n: \bigoplus_{i=1}^R H^{n-n_i}(B; R) \to H^n(E; R)$ be given by $\mathbf{V}_n(b_1, \ldots, b_R) = \sum_{i=1}^R p^*(b_i) \cdot \tilde{f}_i$.

Thus, writing $\alpha: \bigoplus_{i=1}^R H^{n-n_i}(B; R) \to \left[ H^*(B; R) \otimes H^*(F; R) \right]^n$ for the map $\alpha(b_1, \ldots, b_R) = \sum_{i=1}^R b_i \otimes \tilde{\gamma}_i$, $\alpha$ is an isomorphism and the diagram

$$
\begin{array}{ccc}
\bigoplus_{i=1}^R H^{n-n_i}(B; R) & \xrightarrow{\alpha} & \left[ H^*(B; R) \otimes H^*(F; R) \right]^n \\
\mathbf{V}_n & \xrightarrow{\cong} & H^n(B \times F; R)
\end{array}
$$

commutes. Thus $\mathbf{V}_n$ is an isomorphism. In particular, for each $i$ there exist unique $c_{i,j} \in H^*_x(B; R)$ such that

$\mathbf{V}_i = \sum_{j=1}^R p^*(c_{i,j}) \cdot \tilde{f}_j = \sum_{j=1}^R c_{i,j} \times \tilde{\gamma}_j$. Then

$\tilde{\gamma}_i = i_{x_0}^*(\gamma_i) = \sum_{j=1}^R \pi^* (c_{i,j}) \cdot \pi^* (\tilde{\gamma}_j) = \sum_{j=1}^R (\pi^* (c_{i,j})) \cdot \pi^* (\tilde{\gamma}_j) \cdot \pi^* (\tilde{f}_j) = \sum_{j=1}^R \pi^* (c_{i,j}) \cdot \tilde{\gamma}_j \cdot \pi^* (\tilde{f}_j)$. 

Now \( p|_{\mathcal{X}_0} \) is a constant map, and so \( (p|_{\mathcal{X}_0})^* \) is 0 in positive dimensions. Since \( \pi_2 \mathcal{X}_0 = 1 \), \( \pi^*_2 \mathcal{X}_0 \pi^*_2 (\tilde{e}_j) = \tilde{e}_j \). Thus
\[
\tilde{e}_j = \sum_{j \geq i} (p|_{\mathcal{X}_0})^* (c_{i,j}) \cdot \tilde{e}_j,
\]
Since each \( c_{i,j} \in H^0(B; R) \)

whenever \( n_j = n_i \), we have \( c_{i,j} = m_{i,j} \) for some \( m_{i,j} \in R \)
(since \( B \) is path-connected), and so
\[
\tilde{e}_j = \sum_{j \geq i} m_{i,j} \tilde{e}_j.
\]
Since \( H^n_{\mathsf{R}}(F; R) \) is a free \( R \)-module

with basis \( \{ \tilde{e}_j \mid n_j = n_i \} \), it follows that \( m_{i,i} = 1 \) and
\( m_{i,j} = 0 \) if \( j \neq i \). Thus
\[
\tilde{e}_i = \sum_{n_j < n_i} c_{i,j} \tilde{e}_j,
\]
and it follows easily that
\[
\Psi_n (b_1, \ldots, b_k) = \sum_{i=1}^k b_i \cdot \tilde{e}_i = \sum_{i=1}^k \sum_{n_j < n_i} (b_i \circ c_{i,j}) \cdot \tilde{e}_j.
\]

Proof that \( \Psi_n \) is 1-1: If \( \Psi_n (b_1, \ldots, b_k) = 0 \), then since \( \alpha \) is an isomorphism, it follows from (*) that
\[
\sum_{i=1}^k b_i \otimes \tilde{e}_i + \sum_{i=1}^k \sum_{n_j < n_i} (b_i \circ c_{i,j}) \otimes \tilde{e}_j = 0.
\]

Suppose \( (b_1, \ldots, b_k) \neq 0 \). Let \( m = \max \{ n_i \mid b_i \neq 0 \} \). Then for each \( b_i \circ c_{i,j} \otimes \tilde{e}_j \) in the right-hand summation for which \( n_j = m \), we have \( n_i > m \) and so \( b_i = 0 \). Thus
\[
\sum_{i=1}^k b_i \otimes \tilde{e}_i = 0.
\]
Thus, since \( \alpha \) is an isomorphism, each \( b_i \) such that \( n_i = m \) must be 0; contradiction. Thus \( (b_1, \ldots, b_k) = 0 \).
Proof that $\Psi_n$ is onto: Since $\tilde{\Psi}_n$ is onto, suffice to show each $b \times \tilde{e}_i$ lies in the image of $\Psi_n$. We proceed by induction on $n_i$. If $n_i = 0$, (1) yields

$$\psi_n(0, \ldots, 0, b, 0, \ldots, 0) = b \times \tilde{e}_i.$$  Suppose $n_i > 0$ and, inductively, $c \times \tilde{e}_i \in \text{im } \Psi_n$ whenever $n_j < n_i$. (2) yields

$$\psi_n(0, \ldots, 0, b, 0, \ldots, 0) = b \times \tilde{e}_i + \sum_{j > n_i} (b \cdot c_{ij}) \times \tilde{e}_j.$$

Each $(b \cdot c_{ij}) \times \tilde{e}_j \in \text{im } \Psi_n$ by the induction hypothesis, it follows that $b \times \tilde{e}_i \in \text{im } \Psi_n$.

The general $1$-presentable case: Let $\{B_x \times x \mid x \in X\}$ be the path-components of $B$, $i_x : B_x \to B$ the inclusion maps, and

$$E^x = (i_x \times 1_F)^*(E_x) \in H^{n_x}(B_x \times F, R).$$

Then each

$$\Psi_n^x : \bigoplus_{i=1}^k H^{n_{i-1}}(B_x, R) \to H^n(B_x \times F, R),$$

given by

$$\psi_n^x(b_1, \ldots, b_k) = \sum_{i=1}^k \pi_i^*(b_i) \cdot E_x^i,$$

is an isomorphism by case 1. Moreover, it is easily checked that following commutes:

$$\begin{array}{ccc}
\prod_{\alpha} \bigoplus_{i=1}^k H^{n_{i-1}}(B_x, R) & \xrightarrow{\prod_{\alpha} \psi_n^x} & \prod_{\alpha} H^n(B_x \times F, R) \\
(i_{\alpha}^*) \downarrow & & \downarrow ((i_{\alpha} \times 1_F)^*) \\
\bigoplus_{i=1}^k H^{n_{i-1}}(B, R) & \xrightarrow{\Psi_n} & H^n(B \times F, R)
\end{array}$$

Thus $\Psi_n$ is an isomorphism.

Now suppose the given bundle is $m$-presentable with $m > 1$. Assume result true for $m-1$-presentable bundles. We can write $B = B_1 \cup B_2$, where $B_1, B_2$ are open in $B$ and the given bundle restricted to $B_j$ is $m-1$-presentable, $j = 1, 2$. Write $B_{12} = B_1 \cap B_2$. Then the given bundle restricted to $B_{12}$ is
also m-1-presentable. Write $E_j = \mathbb{P}^{-1}(B_j)$, $j = 1, 2, 12$.
$p_j : E_j \to B_j$ for the projections, and $k_j : E_j \to E$ for the inclusions, let $e_j^{-1} = k_j^*(e_i)$ and

\[
\psi^{-1}_n : \bigoplus_{i=1}^k H^{n,-n} \left( B_j \right) \to H^n \left( E_j \right)
\]

be given by

\[
\psi^{-1}_n (b_1, \ldots, b_k) = \sum_{i=1}^k p_i^* (b_i) \cdot e_i^{-1}
\]

By the inductive hypothesis, the $\psi^{-1}_n$ are isomorphisms for $j = 1, 2, 12$ and all $n$. By naturality of cup products and the behavior of cup products with respect to connecting homeomorphisms in Mayer–Vietoris sequences, the following commutes (R coefficient):

\[
\begin{align*}
\bigoplus_{i=1}^k H^{n-1,-n} \left( B_i \right) & \to \bigoplus_{i=1}^k H^{n-1,-n} \left( B_{12} \right) \to \bigoplus_{i=1}^k H^{n,0} \left( B_i \right) \\
& \to \bigoplus_{i=1}^k H^{n,0} \left( B_{12} \right) \\
\end{align*}
\]

where top row is the direct sum of Mayer–Vietoris sequences for $(B_1, B_2, B_{12})$ and bottom row is the Mayer–Vietoris sequence for $(E_1, E_1, E_2)$. The result now follows by the S–lemma.

Note: This result is known as the Leray–Hirsch Theorem.

8. For each $x \in B$, the inclusions $L_R(E_x) \hookrightarrow L_R(E)$, $P_R(E_x) \hookrightarrow P_R(E)$ constitute a map of real line bundles

\[
\begin{align*}
L_R(E_x) & \to L_R(E) \\
P_R(E_x) & \to P_R(E)
\end{align*}
\]

Write $\xi_x = L_x^* \xi (L_R(E), U)$. Then by naturality of the Euler class (Problem 4), $\xi_x = \xi (L_R(E_x), L_x^* U)$ which, by Problem 5, is the generator of $H^1(P_R(E_x) ; \mathbb{Z}/2)$. From the lectures, the elements

\[
1, \xi_x, \xi_x^2, \ldots, \xi_x^{n-1}
\]

form a $\mathbb{Z}/2$-basis for $H^* (P_R(E_x) ; \mathbb{Z}/2)$. The conclusion now follows from Problem 7.
9. If \( P \) is a point, then \([P, Y]\) has exactly one element, and so must be the identity element of the group \([P, Y]\). If \( X \) is any pointed space and \( C_x : X \to Y \) the constant pointed map, then \( C_x \) factors as

\[
X \xrightarrow{c} \frac{P}{C_x} \xrightarrow{c_p} Y
\]

and so \([C_x] = (c')^*[c_p] = (c')^* (\text{identity element})\) = identity element of \([X, Y]\) since \((c')^*\) is a group homomorphism.

Let \( \pi_1 : Y \times Y \to Y \) denote the projections on the 1st and 2nd factors, resp. Let \( \mu : Y \times Y \to Y \) be any representative of \([\pi_1]^* [\pi_2]\) where \( \ast \) denotes the group operation in \([Y \times Y, Y]\), and \( \eta : Y \to Y \) any representative of \([1_Y]^{-1}\), where the inverse is with respect to the group operation in \([Y, Y]\). We first check that the following commute up to pointed homotopy:

\[c\) \]

\[
\begin{array}{c}
Y \times Y \\
\downarrow i \\
Y \times Y \\
\downarrow \mu
\end{array}
\]

\[\mu \times 1_Y \]

\[1_Y \times \mu \]

\[\mu \]

\[\mu \]

\[\mu \]

\[\mu \]

\[\mu \]

\[\mu \]

\[\mu \]

\[\mu \]
Proof of i): Note that i) is equivalent to \( \cong_0 \) commutativity of the 2 diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{1_X} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{Y} & Y
\end{array}
\]

where \( \lambda_1(y) = (y_1, y_0), \lambda_2(y) = (y_0, y) \), \( y_0 \) the base point of \( Y \).
We have \( [\mu \lambda_1] = \lambda_1^* [\mu] = \lambda_1^* ([\pi_1] * [\pi_2]) \)

\[
= \lambda_1^* [\pi_1] * \lambda_1^* [\pi_2] \quad \text{(since \( \lambda_1^* \) is a group hom.)}
\]

\[
= [\pi_1 \lambda_1^*] * [\pi_2 \lambda_1^*] = [1_Y] * [C_Y]
\]

\[
= [1_Y] \quad \text{since \([C_Y] = \text{identity element of } [Y, Y]\).}
\]

Thus \( \mu \lambda_1 \cong 1_Y \). Similarly \( \mu \lambda_2 \cong 1_Y \).

Proof of ii): Write \( \rho_i : Y \times Y \times Y \to Y \) for projection on the \( i \)-th factor, \( i = 1, 2, 3 \), and \( \rho_{12} : Y \times Y \times Y \to Y \times Y \) for the map \( \rho_{12} (y_1, y_2, y_3) = (y_1, y_2) \).
We have

\[
[\mu (\mu \times 1_Y)] = (\mu \times 1_Y)^* [\mu] = (\mu \times 1_Y)^* ([\pi_1] * [\pi_2])
\]

\[
= (\mu \times 1_Y)^* [\pi_1] * (\mu \times 1_Y)^* [\pi_2] \quad \text{(since \( (\mu \times 1_Y)^* \) is a group hom.)}
\]

\[
= [\pi_1 (\mu \times 1_Y)] * [\pi_2 (\mu \times 1_Y)] = [\mu \rho_{12}] * [\rho_3]
\]

\[
= \rho_{12}^* [\mu] \times [\rho_3] = \rho_{12}^* ([\pi_1] * [\pi_2]) \times [\rho_3]
\]

\[
= \rho_{12}^*[\pi_1] \times \rho_{12}^*[\pi_2] \times [\rho_3] = \rho_{12}^* [\rho_2] \times [\rho_3] \quad \text{(since \( \rho_{12}^* \) is a group hom.)}
\]

\[
= [\rho_1 \rho_{12}] \times [\rho_2] \times [\rho_3], \text{ proving ii).
\]

Proof of iii): \( [\mu (1_Y \times \eta)] = ((1_Y \times \eta) d)^* [\mu] = ((1_Y \times \eta) d)^* ([\pi_1] * [\pi_2]) \)

\[
= [\pi_1 (1_Y \times \eta) d] * [\pi_2 (1_Y \times \eta) d] \quad \text{(since \( (1_Y \times \eta) d \) is a group hom.)}
\]

\[
= [2 \gamma] * [?] = [2 \gamma] * [2 \gamma]^{-1} \quad \text{(def. of \( \gamma \))}
\]

\[
= [C_Y] \quad \text{(since \( C_y = \text{id. elt. of group } [Y, Y] \)).}
\]

Similarly, \( [\mu (\eta \times 1_Y)] = [C_Y]. \)

Thus \( Y \) is an \( H^* \)-space with above \( \mu \) as multiplication, \( \eta \) as inversion.
If \( f, g: X \to Y \) are pointed maps, let \( f \circ g \) denote the composition
\[
X \xrightarrow{d} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow{\mu} Y.
\]
Then
\[
[f \circ g] = [\mu (f \times g)_d] = ([f \times g]_d)^*[\mu_1] \ast [\mu_2] = [\pi_1 (f \times g)_d] \ast [\pi_2 (f \times g)_d]
\]
(since \((f \times g)_d)^*\) is a group hom.)
\[
= [f] \ast [g],
\]
proving the group structure on \([X, Y]\) arises from the above \(H^*\)-structure on \(Y\) via the \(O\)-construction.

10. Proof that \([X, Y]\) is a group under the given operation:

i) Let \( f: X \to Y \) be pointed. Consider
\[
\text{Diagram 1}
\]
(\(1), (2), (3) commute. (4) \( \circ \) - commutes by 1st property of co-\(H^*\)-spaces. Thus \( f \circ \ast \) \( \circ \) \( f \) and so \([f] \ast [\ast] = [f\ast] \).
Similarly, \([\ast] \ast [f] = [f\ast] \). Thus \([\ast] \) is a unit element for the given operation in \([X, Y]\).

ii) Let \( f, g, h: X \to Y \) be pointed. Consider
\[
\text{Diagram 2}
\]
For any pointed $f : X \to Y$, consider

\[
\begin{array}{ccc}
X & \xrightarrow{f \circ (g \circ h)} & Y \\
\downarrow{f \circ (g \circ h)} & & \downarrow{f \circ (g \circ h)} \\
X & \xrightarrow{f \circ g \circ h} & Y \\
\end{array}
\]

(1) and (2) commute. (3) commutes by 3rd property of $co - H^*$-spaces. Thus $f \circ (g \circ h) \## (f \circ g) \circ h$ and so

$$((f \circ g) \circ h)[h] = [(f \circ g) \circ h] = [f \circ (g \circ h)] = [f \circ (g \circ h)]$$

and so the given operation in $[X, Y]$ is associative.

Thus, $[X, Y]$ is a group with respect to the given operation.

For any pointed $X$, $[X, -]$ is a covariant functor from the category of pointed spaces to the category of pointed sets. To complete the proof, it must be checked that if $f : X \to Z$ is a pointed map, then $f_* : [X, Y] \to [X, Z]$ is a group homomorphism.

Let $g, h : X \to Y$ be pointed. It is easily checked that

$$f \circ (g \circ h) = (f \circ g) \circ (f \circ h).$$

Thus

$$f_* ([g][h]) = f_* [g \circ h] = f_* [f (g \circ h)] = [(f \circ g) \circ (f \circ h)]$$

$$= [f \circ g][f \circ h] = f_* [g] \circ [h],$$

completing the proof.
Let \((X, A, B)\) be a pointed triple.

**Exactness of** \(\pi_n(A, B) \xrightarrow{j_*} \pi_n(X, B) \xrightarrow{r_*} \pi_n(X, A)\), \(n \geq 1\):

Note that \(j_*\) factors through \((A, A)\), and \(\pi_n(A, A) = 0\). Hence \(j_*i_* = 0\), and so \(\ker j_* \subseteq \ker i_*\).

**Proof** that \(\ker j_* \subseteq \ker j_*\):

**Case 1**: \(n \geq 2\). We have the commutative diagram:

\[
\begin{array}{cccc}
\pi_n(A) & \xrightarrow{f_*} & \pi_n(X) & \xrightarrow{r_*} & \pi_n(X, A) \\
\downarrow{p_*} & & \downarrow{i_*} & & \downarrow{1} \\
\pi_n(A, B) & \xrightarrow{i_*} & \pi_n(X, B) & \xrightarrow{j_*} & \pi_n(X, A) \\
\downarrow{\partial_1} & & \downarrow{\partial_2} & & \downarrow{\partial_3} \\
\pi_{n-1}(B) & \xrightarrow{1} & \pi_{n-1}(B) & \xrightarrow{k_*} & \pi_{n-1}(A) \\
\downarrow{k_*} & & & & \\
\pi_{n-1}(A) \\
\end{array}
\]

All maps are group homomorphisms since \(n \geq 2\). The left and middle columns, and the top row are exact, being portions of homotopy sequences of appropriate pairs. Let \(a \in \ker j_*\). Then from commutativity of the lower right rectangle, \(\partial_2 a = \partial_1 b\) for some \(b \in \pi_n(A, B)\). Then \(a - i_* (b) \in \ker \partial_2\), and so \(a - i_* (b) = j_* (c)\) for some \(c \in \pi_n(X)\) (exactness of middle column). Since we already know \(j_* i_* = 0\), we have \(j_* i_* (b) = 0\), and hence \(j_* (a) = 0\).

It follows that \(r_* (c) = 0\) and so, by exactness of the top row, there exists a \(d \in \pi_n(A)\) such that \(j_* (d) = c\). Then \(a = i_* (j_* (d) + b)\), and so \(\ker j_* \subseteq \ker i_*\) if \(n \geq 2\).

**Case 2**: \(n = 1\). \((S^0 \times I, S^0) \cong (I, I \times I)\) and so we can regard \(\pi_1(X, B)\) as \([(I, I \times I), (X, B)]\). Take 0 as base point and suppose \(f: (I, I \times I) \to (X, B)\) is a pointed map such that \(j_* [f] = \ast\). Then there exists a pointed homotopy \(h: I \times I \to X\) such that \(h(s, 0) = f(s), h(s, 1) = \ast\), and \(h(1, x) \in A\) for all \(x \in I\). Define \(H: I \times I \to X\) by

\[
H(s, x) = \begin{cases} 
\frac{h(2s - x)}{2} & \text{if } 0 \leq s \leq 1 - \frac{x}{2} \\
\frac{h(1, 2 - 2s)}{2} & \text{if } 1 - \frac{x}{2} \leq s \leq 1.
\end{cases}
\]
H is continuous. Note that $H(0, x) = h(0, x) = *$ for all $x$.

Since $h$ is a pointed homotopy,

$H(1, x) = h(1, 0) = f(1) \in B$ for all $x$,

$H(s, 0) = h(s, 0) = f(s)$,

$H(s, 1) = \begin{cases} h(2s, 1) & \text{if } 0 \leq s \leq \frac{1}{2} \\ h(1, 2s - 2s) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases} = \begin{cases} * & \text{if } 0 \leq s \leq \frac{1}{2} \\ \in A & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$

Define $g: (I \setminus \{0, 1\}) \to (A, B)$ by $g(s) = H(s, 1)$. Then $H: (I \setminus \{0, 1\}) \times I \to (X, B)$ is a pointed homotopy from $f$ to $i y$, and so $[\gamma] \in \operatorname{im} i_*$.

\[(\forall n) \text{ Exactness of } \Pi_{n+1}(X, A) \overset{\partial}{\longrightarrow} \Pi_{n}(A, B) \overset{i_*}{\longrightarrow} \Pi_{n}(X, B), n \geq 2\]

By definition, $\partial$ is the composition

$\Pi_{n+1}(X, A) \overset{\partial}{\longrightarrow} \Pi_{n}(A, B) \overset{k_*}{\longrightarrow} \Pi_{n}(X, B)$

where $i$ is from the homotopy sequence of the pair $(X, A)$.

We have the commutative diagram

\[
\begin{array}{ccc}
\Pi_{n+1}(X, A) & \xrightarrow{\partial} & \Pi_{n}(A, B) \\
\downarrow & & \downarrow k_* \downarrow \uparrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\Pi_{n+1}(X, A) & \xrightarrow{\partial} & \Pi_{n}(A) \\
\downarrow & & \downarrow k_* \\
\Pi_{n}(A, B) & \xrightarrow{i_*} & \Pi_{n}(X, B) \\
\end{array}
\]

with the top row exact. Hence $i_* \partial = 0$ and so $\operatorname{im} \partial \subseteq \ker i_*$.

**Proof that** $\ker i_* \subseteq \operatorname{im} \partial$:

**Case 1:** $n \geq 2$. We have the commutative diagram

\[
\begin{array}{ccc}
\Pi_{n}(B) & \xrightarrow{1} & \Pi_{n}(B) \\
\downarrow & & \downarrow \\
\Pi_{n+1}(X, A) & \xrightarrow{\partial} & \Pi_{n}(A) \\
\downarrow & & \downarrow k_* \\
\Pi_{n}(A, B) & \xrightarrow{i_*} & \Pi_{n}(X, B) \\
\downarrow & & \downarrow \\
\Pi_{n-1}(B) & \xrightarrow{1} & \Pi_{n-1}(B) \\
\end{array}
\]
with both columns and the 2nd row exact. Since $n \geq 2$, all the maps are group homomorphisms. The result now follows from chasing in the above diagram, similar to that of part (I).

Case 2: $n = 1$. The only difficulty with the argument in Case 1 is that $k_\ast : \pi_n(A) \to \pi_n(A,B)$ is not a group homomorphism (since $\pi_n(A,B)$ is not a group). However, $\pi_1(A)$ is a group. The argument given in Case 1 can be carried out if it is shown that whenever $a, b \in \pi_1(A)$ with $a + b \in \lim \begin{array}{c} \pi_1(B) \overset{\partial}{\longrightarrow} \pi_1(A) \end{array}$, then $k_\ast(a + b) = k_\ast(a)$.

Let $f : (I, \{0, 1\}) \to (A, \ast)$, $g : (I, \{0, 1\}) \to (B, \ast)$ be continuous. Then $[f] + [g]$ is represented by $(f \ast g) : (I, \{0, 1\}) \to (A, \ast)$ where

$$(f \ast g)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Define $h : I \times I \to A$ by

$$h(s, t) = \begin{cases} f\left(\frac{2s}{2-t}\right) & \text{if } 0 \leq s \leq 1 - \frac{t}{2} \\ g\left(2s + t - 2\right) & \text{if } 1 - \frac{t}{2} \leq s \leq 1. \end{cases}$$

Note that $h(0, t) = \ast$, $h(1, t) \in B$ for all $t$, and

$$k \circ \frac{\partial h}{h} \circ k(f \ast g).$$

$$(\text{III}) \text{ Exactness of } \pi_{n+1}(X,B) \xrightarrow{J_*} \pi_{n+1}(X,A) \xrightarrow{\overline{\partial}} \pi_{n+1}(A,B), \quad n \geq 1:$$

we have the commutative diagram

$$\begin{array}{ccc}
\pi_{n+1}(X,B) & \xrightarrow{J_*} & \pi_{n+1}(X,A) \\
\downarrow & & \downarrow \\
\pi_n(B) & \xrightarrow{\partial} & \pi_n(A) \xrightarrow{k_*} \pi_n(A,B)
\end{array}$$

with the bottom row exact. Hence $\overline{\partial}j_* = 0$, and so $\im j_* \subseteq \ker \overline{\partial}$.
Proof that \( \ker \hat{T} \subset \imath_n \varphi \): For all \( n \geq 1 \) we have the commutative diagram

\[
\begin{array}{ccc}
\Pi_{n+1}(X) & \xrightarrow{1} & \Pi_n(X) \\
\downarrow & & \downarrow \\
\Pi_{n+1}(X, B) & \xrightarrow{\varphi} & \Pi_n(X, A) \\
\downarrow & & \downarrow \\
\Pi_n(B) & \xrightarrow{\varphi} & \Pi_n(A) \\
\downarrow & & \downarrow \\
\Pi_n(X) & \xrightarrow{1} & \Pi_n(X) \\
\end{array}
\]

with both columns and the 3rd row exact. The result follows from chasing in the above diagram (even when \( n = 1 \)).

12. Let \( \{ \phi_i : U_i \times \mathbb{R}^m \rightarrow \mathbb{R}^{n_1} \}, i \in I \) and \( \{ \phi_j : V_j \times \mathbb{R}^n \rightarrow \mathbb{R}^{n_2} \}, j \in J \) be vector bundle atlases for \( p_1 \) and \( p_2 \), respectively. For \( (i,j) \in I \times J \), define

\[
\xi_{ij} : (U_i \cap V_j) \times \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{n_1} \text{ by}
\]

\[
\xi_{ij}(x, (y, z)) = \left( \phi_i(x, y), \phi_j(x, z) \right)
\]

where \( y \in \mathbb{R}^m \), \( z \in \mathbb{R}^n \). Then \( \{ \xi_{ij} \}_{(i,j) \in I \times J} \) is a vector bundle atlas for \( p_1 \) and is finite if \( I \) and \( J \) are finite.

13. a) Write \( \varphi_1 : L(E_1) \rightarrow P(E_1) \), \( \varphi_2 : L(E_1 \oplus E_2) \rightarrow P(E_1 \oplus E_2) \) for the projections. Then \( \varphi_1([u], w) = \varphi_2([u], w, 0) = [u], \varphi_2([u], w) = \varphi_1([u]) = [u] \) and so \( \varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 \).

For all \( x \in \mathbb{R}^n \) \( u \in (E_1)_0 \) we have \( \hat{j}_1(x([u, u])) = \hat{j}_1([u], xu) = ([u], xu, 0) = ([u], x([u], x)) = \hat{j}_1([u], x) \). Thus, since the fibre \( L(E_1)[u] \) is 1-dimensional with basis \( ([u], u) \), it follows that \( \hat{j}_1|L(E_1)[u] \) is a linear isomorphism onto \( L(E_1 \oplus E_2)[u] \). Thus \( \xi_{ij} \) constitute
Inĭ, we have the commutative diagram:

\[
\begin{array}{ccc}
P(E) & \xrightarrow{f} & P(E') \\
\downarrow & & \downarrow \\
\mathbb{H}^p(X,\mathcal{F}) & \xrightarrow{\otimes} & \mathbb{H}^p(X',\mathcal{F'})
\end{array}
\]

By naturality of the cup product, the diagram commutes, where $j$ is the inclusion. Hence $\mathbb{H}^p(X,\mathcal{F}) \otimes \mathbb{H}^q(X',\mathcal{F'}) \cong \mathbb{H}^p \oplus \mathbb{H}^q \cap \mathbb{H}^{p+q}(X \times X')$.

Similarly, $P(E')$ is a deformation retract of $V'$ and so it is a deformation retract of $u \cap u'$ (cf. [1.2]).
\[
\beta = \sum_{j=0}^{n} r^* w_j^*(E_2) \cup \Upsilon_{E_1 \oplus E_2}^{n-j} \in H^n(\text{P}(E_1 \oplus E_2), \mathbb{Z}/2) \\
\]

Then \[i_1^* (a) = \sum_{i=0}^{m} i_1^* r^* w_i^*(E_1) \cup \Upsilon_{E_1 \oplus E_2}^{m-i} = \sum_{i=0}^{m} r_i^* w_i^*(E_1) \cup \Upsilon_{E_1} = 0 \quad \text{by naturality of the cup product.} \]

Similarly \[j_2^* (b) = 0. \]

Let \( U, V \) be as in problem (3b), and \[ U \xrightarrow{j_1} \text{P}(E_1 \oplus E_2) \xrightarrow{j_2} V \] the inclusions. From commutativity of

\[
\begin{array}{c}
\text{P}(E_1) \xrightarrow{i_1} \text{P}(E_1 \oplus E_2) \\
\downarrow \\
\text{P}(E_1 \oplus E_2) \xrightarrow{j_1} \end{array}
\]

and the fact that \( \text{P}(E_1) \) is a deformation retract of \( U \), it follows that \( j_1^*(a) = 0 \). Similarly, \( j_2^*(b) = 0 \). Since \( U \cup V = \text{P}(E_1 \oplus E_2) \), it follows from problem 14 that \( a \cup b = 0 \), i.e.

\[
0 = \left( \sum_{i=0}^{m} r^* w_i^*(E_1) \cup \Upsilon_{E_1 \oplus E_2}^{m-i} \right) \cup \left( \sum_{j=0}^{n} r^* w_j^*(E_2) \cup \Upsilon_{E_1 \oplus E_2}^{n-j} \right)
\]

\[
= \sum_{k=0}^{m+n} r^* \left( \sum_{i=0}^{k} w_i^*(E_1) \cup w_{k-i}^*(E_2) \right) \cup \Upsilon_{E_1 \oplus E_2}^{m+n-k}
\]

Thus

\[
\Upsilon_{E_1 \oplus E_2}^{m+n} = \sum_{k=1}^{m+n} r^* \left( \sum_{i=0}^{k} w_i^*(E_1) \cup w_{k-i}^*(E_2) \right) \cup \Upsilon_{E_1 \oplus E_2}^{m+n-k}
\]

and so by definition, \( \omega_k(E_1 \oplus E_2) = \sum_{i=0}^{k} w_i^*(E_1) \cup w_{k-i}^*(E_2) \).
16. Let $E$ be the Leray-Serre spectral sequence with $\mathbb{Z}$-coefficients of the fibration $\Sigma \lambda(S^2vS^3) \to \lambda(S^2vS^3) \to S^2vS^3$. ($S^2vS^3$ is 1-connected, so above fibration is $\mathbb{Z}$-orientable). Since $\lambda(S^2vS^3)$ is contractible, $E^\infty_{p,q} = 0$ for $(p,q) \neq (0,0)$. Since

$$H_p(S^2vS^3) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0,2,3 \\ 0 & \text{otherwise} \end{cases}$$

it follows, from the universal coefficient theorem, that

$$E^2_{p,q} \cong H_p(S^2vS^3) H_q(\lambda(S^2vS^3)) \cong \begin{cases} H^2_q(\lambda(S^2vS^3)) & \text{if } p = 0,2,3 \\ 0 & \text{otherwise} \end{cases}$$

Thus the only possible non-zero differentials are the $d^2: E^2_{2,\overline{q}} \to E^2_{0,\overline{q}+2}$ and $d^2: E^2_{3,\overline{q}} \to E^2_{0,\overline{q}+1}$. Thus

$$E^3_{2,\overline{q}} = E^\infty_{2,\overline{q}} = 0 \quad \text{for all } \overline{q},$$

and so $E^2_{2,\overline{q}} \to E^2_{0,\overline{q}+1}$ is injective. Since $E^\infty_{0,\overline{q}+1}/\text{im } d^2 \cong E^3_{0,\overline{q}+1}$, we have an exact sequence

$$(*) \quad 0 \to E^2_{2,\overline{q}} \xrightarrow{d^2} E^2_{0,\overline{q}+1} \xrightarrow{E^2_{2,\overline{q}}} E^2_{0,\overline{q}+1} \to 0$$

for all $\overline{q}$.

For all $(p,\overline{q})$ we have $E^\infty_{p,\overline{q}} = E^4_{p,\overline{q}}$. Thus since $E^\infty_{3,\overline{q}-1}$ and $E^\infty_{0,\overline{q}+1}$ are 0 for $\overline{q} > 0$, we must have

$$E^3_{3,\overline{q}-1} \xrightarrow{d^2} E^3_{0,\overline{q}+1}$$

is an isomorphism for all $\overline{q} > 0$. Since $E^2_{3,\overline{q}-1} \xrightarrow{d^2} E^2_{1,\overline{q}}$ is 0, we have $E^2_{3,\overline{q}-1} = E^3_{3,\overline{q}-1}$ and hence

$$E^3_{0,\overline{q}+1} \cong E^3_{3,\overline{q}-1} \quad \text{for } \overline{q} > 0.$$ Thus $(*)$ yields an exact sequence

$$0 \to E^2_{2,\overline{q}} \xrightarrow{d^2} E^2_{0,\overline{q}+1} \xrightarrow{d^2} E^2_{3,\overline{q}-1} \to 0 \quad \text{for all } \overline{q} > 0,$$

i.e., an exact sequence

$$(***) \quad 0 \to H^1_\varphi(X) \to H^1_{\overline{q}+1}(X) \to H^1_{\overline{q}-1}(X) \to 0 \quad \text{for all } \overline{q} > 0,$$

where $X = SL(S^2vS^3)$. Since $X$ is 0-connected
(since $S^2 \times S^2$ is 1-connected), $H_0(X) \cong \mathbb{Z}$. It follows from (*) that $H_1(X) \cong \mathbb{Z}$.

Let $i \geq 1$ and assume inductively $H_i(X)$ is free abelian on $f_i$ generators where $f_i$ is the $i$th Fibonacci number (i.e., $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$). Then $H_{i+1}(X)$ is free abelian and so (**) splits, and hence

$$H_{i+1}(X) \cong H_i(X) \oplus H_{i-1}(X)$$

is free abelian on $f_i + f_{i-1}$ generators.

Thus, it follows by induction that $H_n(X)$ is free abelian on $f_n$ generators where $f_n$ is the $n$th Fibonacci number.

17. Let $E$ be the Leray-Serre spectral sequence of $F \to T \to \mathbb{S}^n$ with coefficients in $R$. We have

$$E^{i,j}_{p,q} = H_p(S^n, H_q(F;R)) \cong \begin{cases} H_q(T;R) & \text{if } p = 0 \text{ or } n \\ 0 & \text{otherwise} \end{cases}$$

Thus, by the Fibre Edge Theorem, it suffices to show the existence of a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
E^{2}_{0,i-n+1} & \rightarrow & E^{2}_{0,i} & \rightarrow & H_2(T;R) & \rightarrow & E^{2}_{n,i-n} & \rightarrow & E^{2}_{0,i-1} \\
\downarrow P_F & & \downarrow \phi & & \downarrow \beta & & \downarrow \gamma & & \end{array}$$

The only possible non-zero differentials are the $d^n_i: E^{n}_{i,j} \rightarrow E^{n}_{i,j+n-1}$. Thus $E^{n}_{i,j} = E_{i,j}$ and $E^{n+1}_{i,j} = E^{\infty}_{i,j}$ for all $p,q$.

Since $E^{\infty}_{n+1,i-n-1} = 0$ for $j > 0$ we have

$$J_n,i-n = J_{n+1},i-n-1 = \ldots = J_{i,0} = H_2(T;R).$$

Since $E^{\infty}_{j,i-j} = 0$ for $0 < j < n$ we have

$$E^{\infty}_{0,i} = J_0,i = J_{1,i-1} = \ldots = J_{n-1,i-n+1}.$$ 

Define $\phi: H_2(T;R) \rightarrow E^{\infty}_{n,i-n}$ to be the composition
\[ H_i(T; R) = \bigoplus_i H_i \to \bigoplus_i H_{i-1} \to \bigoplus_i H_{i-2} \to \bigoplus_i H_{i-3} \to \cdots \to \bigoplus_i H_{i-n} \to 0. \]

Define \( \beta : E_{n-1,i-n}^2 \to E_{0,i-1}^2 \) to be the composition
\[ E_{n-1,i-n}^2 \to E_{n,i-n}^n \xrightarrow{d^n} E_{0,i-1}^n = E_{0,i-1}^2. \]

Since \( \beta_F \) is onto and \( \delta_F \) is 1-1, it suffices to check exactness of

- (I) \( E_{n-1,i-n+1}^2 \xrightarrow{\beta} E_{0,i}^2 \xrightarrow{\beta} E_{0,i}^\infty \),
- (II) \( E_{0,i}^\infty \xrightarrow{\delta_F} H_i(T; \mathbb{R}) \xrightarrow{\delta} E_{n-1,i-n}^2 \),
- (III) \( H_i(T; \mathbb{R}) \xrightarrow{\delta} E_{n,i-n}^2 \xrightarrow{\beta} E_{0,i-1}^2 \).

Exactness of (I) is immediate from exactness of
\[ E_{n,i-n}^n \xrightarrow{d^n} E_{0,i-1}^n \to E_{0,i-1}^\infty. \]

Exactness of (II) is immediate from exactness of
\[ J_{n-1,i-n+1} \to J_{n,i-n} \to E_{n,i-n}^\infty. \]

Exactness of (III) is immediate from exactness of
\[ E_{n,i-n}^\infty \xrightarrow{d^n} E_{n,i-n}^n \to E_{0,i-1}^n. \]

Note: The exact sequence of this problem is called the Wang sequence.

18. Let \( E \) be the Leray–Serre spectral sequence of \( p: (T, T_0) \to (B, B_0) \). Then
\[ E_1^{p,q} = H_p(B, B_0; H_q(S^0, R)) \]
\[ = \begin{cases} H_p(B, B_0; R) & \text{if } q = 0, n \\ 0 & \text{otherwise.} \end{cases} \]

By the base-edge theorem it suffices to show the existence of a commutative diagram with exact rows.
The only possible non-zero differentials are the $d^{n+1}: E^{n+1}_{i+1,j} \rightarrow E^{n+1}_{i,j}$.
Thus $E^{n+1}_{p,q} = E^{n+1}_{p+1,q}$ and $E^{n+2}_{p,q} = E^{n+2}_{p,q}$ for all $p,q$.

Since $E^{\infty}_{i,j} = 0$ for $0 < j < n$ we have

$J_i = \cdots = J_{i-n+1} = J_i = \cdots = 0$,
and
$E^{\infty}_{i-n,n} = J_i = \cdots = 0$.

Define $\delta: E^{2}_{i,2} \rightarrow E^{2}_{i-1,2}$ to be the composition

$E^{2}_{i,2} = E^{n+1}_{i,2} \xrightarrow{d^{n+1}} E^{n+1}_{i-1,2} = E^{2}_{i-1,2}$.

Define $\delta: E^{2}_{i-1,2} \rightarrow H_i(T, T_j R)$ to be the composition

$E^{2}_{i-1,2} \xrightarrow{\delta} E^{\infty}_{i-1,2} = J_i = J_{i-1} = \cdots = 0$.

Since $p_B$ is onto and $\delta_B$ is 1-1 it suffices to check exactness of

(I) $E^{2}_{i-n,n} \xrightarrow{\delta} H_i(T, T_j R) \xrightarrow{p_B} E^{\infty}_{i,0}$,

(II) $E^{2}_{i,0} \xrightarrow{\delta_B} E^{2}_{i,0} \xrightarrow{\delta} E^{2}_{i-1,0}$,

(III) $E^{2}_{i,0} \xrightarrow{\delta} E^{2}_{i-1,0} \xrightarrow{\delta} H_{i-1}(T, T_j R)$.

Exactness of (I) is immediate from exactness of

$J_{i-1} \rightarrow J_{i,0} \rightarrow E^{\infty}_{i,0}$.

Exactness of (II) is immediate from exactness of

$E^{\infty}_{i,0} \xrightarrow{d^{n+1}} E^{n+1}_{i,0} \xrightarrow{d^{n+1}} E^{n+1}_{i-1,0}$.

Exactness of (III) is immediate from exactness of

$E^{n+1}_{i,0} \xrightarrow{d^{n+1}} E^{n+1}_{i-1,0} \xrightarrow{d^{n+1}} E^{n+1}_{i-2,0}$.

Note: The exact sequence of this problem is called the Gysin sequence.
Let $E$ be the Leray–Serre spectral sequence of $F \to T \to B$ with $\mathcal{A}$-coefficients. We have $E^2_{p,q} \cong H_p(B; H_q(F; \mathcal{A})) \cong H_p(B; \mathcal{A}) \otimes H_q(F; \mathcal{A})$. Thus, since $H_p(B; \mathcal{A})$ and $H_q(F; \mathcal{A})$ are finite-dimensional over $\mathcal{A}$ and non-zero for only finitely many $(p,q)$, each $E^2_{p,q}$ is finite-dimensional over $\mathcal{A}$ and non-zero for only finitely many $(p,q)$. Thus, since $E^{\infty}_{p,q}$ is a quotient of a sub-$\mathcal{A}$-module of $E^2_{p,q}$, each $E^{\infty}_{p,q}$ is finite-dimensional over $\mathcal{A}$ and non-zero for only finitely many $(p,q)$ for $2 \leq t < \infty$. Thus, for all but a finite number of $n$, the $E^{\infty}_{p,n,p}$ are all 0. If $n$ is such that the $E^{\infty}_{p,n,p}$ are all 0, it follows that

$0 = J_{-1,n+1} = J_{0,n} = J_{1,n-1} = \cdots = J_{n,0} = H_n(T; \mathcal{A})$, and so the $H_n(T; \mathcal{A})$ are 0 for all but finitely many $n$.

For a general $n$, exactness of $J_{p,n} \to J_{p+1,n-1} \to E^{\infty}_{p,n}$ and the facts that $E^{\infty}_{p+n-1,n}$ and $E^{\infty}_{0,n} = J_{0,n}$ are finite-dimensional over $\mathcal{A}$, yields, by induction on $p$, that all the $J_{p,n}$ are finite-dimensional over $\mathcal{A}$. In particular, each $J_{n,0} = H_n(T; \mathcal{A})$ is finite-dimensional over $\mathcal{A}$.

Exactness of $0 \to J_{p-1,n-1} \to J_{p,n} \to E^{\infty}_{p,n-1} \to 0$

yields $\dim_{\mathcal{A}} E^{\infty}_{p,n} = \dim_{\mathcal{A}} J_{p,n} - \dim_{\mathcal{A}} J_{p-1,n-1}$ for $0 \leq p \leq n$.

Thus $\sum_{p=0}^{n} \dim_{\mathcal{A}} E^{\infty}_{p,n} = \sum_{p=0}^{n} \left[ \dim_{\mathcal{A}} J_{p,n} - \dim_{\mathcal{A}} J_{p-1,n-1} \right]$

$= \dim_{\mathcal{A}} J_{n,0} - \dim_{\mathcal{A}} J_{-1,n+1} = \dim_{\mathcal{A}} H_n(T; \mathcal{A})$.

Thus $\chi(T) = \sum_{n=1}^{\infty} (-1)^n \dim_{\mathcal{A}} H_n(T; \mathcal{A}) = \sum_{p,q} (-1)^{p+q} \dim_{\mathcal{A}} E^{\infty}_{p,q}$.

Since $E^{2}_{p,q} = 0$ for all but finitely many $(p,q)$, there exists a finite $r$, independent of $(p,q)$, such that $E^{\infty}_{p,q} = E^r_{p,q}$ for all $(p,q)$.

Since $E^{2}_{p,q} \cong H_p(B; \mathcal{A}) \otimes H_q(F; \mathcal{A})$, $\dim_{\mathcal{A}} E^{2}_{p,q} = \dim_{\mathcal{A}} H_p(B; \mathcal{A}) \cdot \dim_{\mathcal{A}} H_q(F; \mathcal{A})$. Thus $\sum_{p,q} (-1)^{p+q} \dim_{\mathcal{A}} E^{2}_{p,q} = \sum_{p,q} (-1)^{p+q} \dim_{\mathcal{A}} H_p(B; \mathcal{A}) \cdot \dim_{\mathcal{A}} H_q(F; \mathcal{A})$.
\[
= \left[ \sum_{p} (-1)^p \dim_{\mathbb{Q}} H_p(B, \mathcal{Q}) \right] \left[ \sum_{q} (-1)^q \dim_{\mathbb{Q}} \mathcal{H}_q(F; \mathcal{Q}) \right] = \chi(B). \chi(F).
\]

Thus we will be done if we prove: For each \( r \geq 2 \),
\[
\sum_{p \geq 1} (-1)^{p+r} \dim_{\mathbb{Q}} E_{p+r}^r = \sum_{q \geq 1} (-1)^{q+r} \dim_{\mathbb{Q}} E_{q+r}^{r+1}
\]
For \( 2 \leq r < \infty \) we have a chain complex \((E^r, d^r)\) with \( E_n^r = \bigoplus_{p+r} E_{p+r}^r \). It follows, by the Hopf trace formula (using the identity map on \( E^r \)), that
\[
\sum_{n \geq 1} (-1)^n \dim_{\mathbb{Q}} E_n^r = \sum_{n \geq 1} (-1)^n \dim_{\mathbb{Q}} H_n(E^r, d^r).
\]

Since \( H_n(E^r, d^r) \cong E^{r+1} \), this yields
\[
\sum_{p \geq 1} (-1)^{p+r} \dim_{\mathbb{Q}} E_{p+r}^r = \sum_{q \geq 1} (-1)^{q+r} \dim_{\mathbb{Q}} E_{q+r}^{r+1},
\]
completing the proof.

20 a). The map \( E(T(s^n) \oplus \mathbb{E}^1) \to E(\mathbb{E}^{n+1}) \) sending \((x, y, z) \mapsto (x, y+z+x)\) is the desired vector bundle isomorphism.

b) If \( f \) is a morphism of finitely-presentable \( n \)-plane bundles, \( f \) induces maps \( f_p : P_R(E_1) \to P_R(E_2) \) and \( f_\ell : L_R(E_1) \to L_R(E_2) \), which constitute a map of real line bundles such that
\[
P_R(E_1) \xrightarrow{f_p} P_R(E_2)
\]
\[
\xrightarrow{f_\ell} L_R(E_1) \xrightarrow{f_\ell} L_R(E_2)
\]
It follows, by naturality of the Euler class, that
\[
f_p^* \chi_L(E_1) = \chi_L(E_2).
\]
Thus, since
\[
\chi_L^n(E_1) = \sum_{i=1}^n \tau^*_i \omega^*_i(E_1) \cdot \chi_L^n(E_2) = 0,
\]
we obtain
\[ o = \tau_0^* \left[ \mathcal{K}_n^{(e_i)} - \sum_{i=1}^{n} r_i^* \omega_i (E_2) \cup \mathcal{K}_n^{n-i} \right] \]

\[ = \mathcal{K}_n^{(e_i)} - \sum_{i=1}^{n} \tau_{i-1}^* \omega_i (E_2) \cup \mathcal{K}_n^{n-i} \]

\[ = \mathcal{K}_n^{(e_i)} - \sum_{i=1}^{n} \tau_{i-1}^* \omega_i (E_2) \cup \mathcal{K}_n^{n-i}, \text{ i.e. the } \tau_{i-1}^* \omega_i (E_2) \]

satisfies the defining equation of the \( \omega_i (E_1) \). Thus \( \tau_{i-1}^* \omega_i (E_2) = \omega_i (E_1) \) for all \( i \).

For any \( B \), we have a map of \( n \)-plane bundles

\[
\begin{align*}
B \times \mathbb{R}^n & \xrightarrow{\pi_1} \mathbb{R}^n \\
\pi_1 & \downarrow \quad \downarrow \\
B & \xrightarrow{\rho} \mathbb{R}^n
\end{align*}
\]

Since \( H^k (pt, \mathbb{Z}) = 0 \) for \( k > 0 \), it follows that \( \omega_i (B \times \mathbb{R}^n) = 0 \) for \( i > 0 \).

Thus \( \omega_i (\mathbb{R}^n) = \begin{cases} 0 & \text{if } i > 0 \\ 1 & \text{if } i = 0 \end{cases} \) for all \( i \).

Thus, by the Whitney Product Formula, for \( i > 0 \),

\[ o = \omega_i (\mathbb{R}^{n+1}) = \omega_i (\tau (S^n) \Theta \mathbb{R}) = \sum_j \omega_{i-j} (\tau (S^n)) \cup \omega_j (\mathbb{R}) \]

\[ = \omega_i (\tau (S^n)) \cup \omega_0 (\mathbb{R}) = \omega_i (\tau (S^n)). \]

Thus \( \omega_i (\tau (S^n)) = \begin{cases} 0 & \text{if } i > 0 \\ 1 & \text{if } i = 0 \end{cases} \).