1-1-1984

Math 752 Algebraic Topology II - Winter '84

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Math 752

Algebraic Topology II

Winter '84, WSU

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References: Spanier
MacLane, Homology
Milnor + Stasheff, Char. Classes (Princeton U. Press)
G. W. Whitehead, Homotopy Theory
Hu, Homotopy Theory

Topics: Cohomology, cup products
Homotopy Theory
Leray-Serre spectral sequence

Hom, cochain complexes, cohomology

Let R be a comm. ring with unit $1 \neq 0$. If A, B are R-modules, write $\mathrm{Hom}_R(A, B)$ = set of all R-homomorphisms from A into B. $\mathrm{Hom}_R(A,B)$ has the structure of an R-module with operations as follows: if $f, g \in \mathrm{Hom}_R(A,B)$ and $r \in R$, define $f+g$ and $r \cdot f$ by $(f+g)(a) = f(a) + g(a)$, $(r \cdot f)(a) = r \cdot f(a)$.

Example: If R is a field, $\mathrm{Hom}_R(A,R) = A^* =$ dual space of A.

Suppose $f: A' \to A$, $g: B \to B'$ are R-homomorphisms. Define $\mathrm{Hom}(f,g): \mathrm{Hom}_R(A,B) \to \mathrm{Hom}_R(A', B')$ by

$$\mathrm{Hom}(f,g)(\alpha) = g \alpha f.$$

Note: For fixed A, the assignment
$$B \longmapsto \mathrm{Hom}_R(A,B)$$
$$B \xrightarrow{g} B' \longmapsto \mathrm{Hom}(1_A, g)$$
is a covariant functor from cat. of R-modules to itself.
For fixed B ?

Def: $\mathcal{C}$, $\mathcal{D}$ categories. A contravariant functor $T: \mathcal{C} \to \mathcal{D}$ assigns to each object X in $\mathcal{C}$ an object $T(X)$ in $\mathcal{D}$, and to each morphism $f: X \to Y$ in $\mathcal{C}$, a morphism $T(f): T(Y) \to T(X)$ in $\mathcal{D}$ satisfying
1) $T(1_X) = 1_{T(X)}$
2) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms in $\mathcal{C}$, then $T(gf) = T(f)T(g)$.

Example: For fixed B, the assignment

$$A \longmapsto \mathrm{Hom}_R(A, B)$$

$$A' \xrightarrow{f} A \longmapsto \mathrm{Hom}(f, 1_B)$$

is a contravariant functor from cat. of R-modules to itself.

Note: $\mathrm{Hom}_R(R, A) \underset{\text{nat. iso.}}{\cong} A \qquad \forall$ R-modules A.

$$f \longmapsto f(1)$$

Def: A cochain complex of R modules C consists of a seq. of R-modules $C^0, C^1, C^2, \dots$ and R-homomorphisms $\delta^n: C^n \to C^{n+1}$, $n \geq 0$, $\ni$: $\delta^{n+1}\delta^n = 0 \quad \forall n$.

A cochain complex C is coaugmented over R if there is given an R-hom. $\eta: R \to C^0 \ni$. $\delta^0\eta = 0$.

If C and D are cochain complexes, a cochain map $f: C \to D$ consists of a seq of R-homomorphisms $f^n: C^n \to D^n \ni$

$$\begin{array}{ccc} C^n & \xrightarrow{\delta^n} & C^{n+1} \\ {\scriptstyle f^n}\downarrow & & \downarrow{\scriptstyle f^{n+1}} \\ D^n & \xrightarrow[\delta^n]{} & D^{n+1} \end{array}$$

commutes $\forall n$.

A cochain map f is coaugmented if

$$\begin{array}{ccc} & \overset{\eta}{\nearrow} & C^0 \\ R & & \downarrow{\scriptstyle f^0} \\ & \underset{\eta}{\searrow} & D^0 \end{array}$$

commutes.

Can form cat. of cochain complexes of R-modules + cat. of coaugmented cochain complexes of R-modules.

Example: Let C be a chain complex of R-modules, A an R-module. Define a cochain complex

$\mathrm{Hom}_R(C, A)$ as follows: $\mathrm{Hom}_R(C, A)^n = \mathrm{Hom}_R(C_n, A)$,

and $\delta^n = (-1)^n \mathrm{Hom}(\partial_{n+1}, 1_A): \mathrm{Hom}_R(C, A)^n \to \mathrm{Hom}_R(C, A)^{n+1}$

(Thus if $\alpha \in \mathrm{Hom}_R(C,A)^n$, $x \in C_{n+1}$,
$(\delta\alpha)(x) = (-1)^n \alpha(\partial x)$.).

Note: Some treatments omit the sign, others use the negative of this. If functions are consistently written on the left and their arguments on the right, our convention seems to be the most natural one.)

If $f: C \to D$ is a chain map, the homomorphisms $\mathrm{Hom}(f_n, 1_A): \mathrm{Hom}_R(D,A)^n \longrightarrow \mathrm{Hom}_R(C,A)^n$ constitute a cochain map $\mathrm{Hom}(f, 1_A): \mathrm{Hom}_R(D,A) \to \mathrm{Hom}_R(C,A)$.

For fixed A, the assignment $C \longmapsto \mathrm{Hom}_R(C,A)$,
$f \longmapsto \mathrm{Hom}(f, 1_A)$
is a contravariant functor from cat. of chain complexes of R-modules to cat. of cochain complexes of R-modules.

In case $A = R$, write $C^* = \mathrm{Hom}_R(C,R)$ and $f^* = \mathrm{Hom}(f, 1_R)$.

If C is R-augmented with aug. $\varepsilon: C_0 \to R$, the composition $R \cong \mathrm{Hom}_R(R,R) \xrightarrow[\mathrm{Hom}(\varepsilon, 1_A)]{} \mathrm{Hom}_R(C_0,R) = C^{*0}$

is an R-coaugmentation of C^*. The assignment $C \mapsto C^*$, $f \mapsto f^*$ is a contravariant functor from cat. of R-augmented chain complexes of R-modules to cat. of R-coaugmented cochain complexes of R-modules.

Def: Let C be a cochain complex of R-modules. Define
$Z^n(C) = \ker \delta^n: C^n \to C^{n+1}$ = module of <u>n-cocycles</u> of C,
$B^n(C) = \operatorname{im} \delta^{n-1}: C^{n-1} \to C^n$ = " " <u>n-coboundaries</u> of C.
Then $B^n(C) \subset Z^n(C)$. Define $H^n(C) = Z^n(C)/B^n(C)$ = n^{th} <u>cohomology group of C</u>.

If $f: C \to D$ is a cochain map, f maps cocycles to cocycles, coboundaries to coboundaries, and hence induces a map $H^n(f): H^n(C) \to H^n(D)$. For each n, H^n is a covariant functor from cat. of cochain complexes of R-modules to cat. of R-modules.

Note: For any cochain complex C, $B^0(C) = 0$ and so $H^0(C) = Z^0(C)$. If C is R-coaugmented with coaug. $\eta : R \to C^0$, then $\operatorname{im} \eta \subset Z^0(C)$ and so we get a coaug. $\eta : R \to H^0(C)$.

If $f : C \to D$ is a coaug. coch. map, then

$$\begin{tikzcd} & H^0(C) \arrow[d, "H^0(f)"] \\ R \arrow[ur, "\eta"] \arrow[r, "\eta"'] & H^0(D) \end{tikzcd}$$

commutes.

Def. (X,A) a top. pair, G an abel. group. Let $Q^*(X,A;G) = \operatorname{Hom}_{\mathbb{Z}}(Q(X,A), G)$ = cubical singular cochain complex of (X,A) with coefficients in G.

Define $H^n(X,A;G) = H^n(Q^*(X,A;G))$ = n^{th} singular cohomology group of (X,A) with coeffs. in G.

If $f : (X,A) \to (Y,B)$ is a map of top. pairs, write $f^* : H^n(Y,B;G) \to H^n(X,A;G)$ for

$H^n(\operatorname{Hom}(Q(f), 1_G))$.

$H^n(\ ;G)$ is a contravariant functor from cat. of top. pairs to cat. of abel. groups.

Note: If A an abel. group, B an R-module, then $\operatorname{Hom}_{\mathbb{Z}}(A,B)$ has the structure of an R-module via $(rf)(a) = r f(a)$. Moreover, if $f : A' \to A$ is a $\mathbb{Z}$-homomorphism, $g : B \to B'$ an R-homomorphism, then $\operatorname{Hom}(f,g)$ is an R-hom.

Prop: A an abel. group, B an R-module. Then $\exists$ nat. R-isom.

$$\operatorname{Hom}_{\mathbb{Z}}(A,B) \cong \operatorname{Hom}_R(A \otimes_{\mathbb{Z}} R, B).$$

Proof: If $f : A \to B$ is a $\mathbb{Z}$-hom., define $\bar{f} : A \otimes_{\mathbb{Z}} R \to B$ by $\bar{f}(a \otimes r) = r f(a)$. Can check $\bar{f}$ is well-defined, and $f \mapsto \bar{f}$ is a natural R-isom.

Thus if C is a chain complex of abel. groups, G an R-module, $\mathrm{Hom}_{\mathbb{Z}}(C, G)$ is a cochain complex of R-modules, nat. isom. to $\mathrm{Hom}_R(C \otimes_{\mathbb{Z}} {}^{o}R, G)$.

If $\varepsilon : C_0 \to \mathbb{Z}$ is a $\mathbb{Z}$-augmentation, then $\varepsilon \otimes 1_R : C_0 \otimes_{\mathbb{Z}} R \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} R \cong R$ is an R-aug. of $C \otimes_{\mathbb{Z}} {}^{o}R$, and thus $(C \otimes_{\mathbb{Z}} {}^{o}R)^* = \mathrm{Hom}_R(C \otimes_{\mathbb{Z}} {}^{o}R, R)$ is R-coaugmented.

In particular, for any R and top. space X, we have an R-coaug. $\eta : R \longrightarrow H^0(X, R)$. If $f : X \to Y$ is cont.,

$$\begin{tikzcd}
 & H^0(Y,R) \arrow[d, "f^*"] \\
R \arrow[ur, "\eta"] \arrow[dr, "\eta"'] & \\
 & H^0(X,R)
\end{tikzcd}$$

commutes.

<u>Def</u>: $f, g : C \to D$ R-cochain maps. An <u>R-cochain homotopy</u> $T : C \to D$ from f to g consists of a sequence of R-homomorphisms $T^n : C^n \longrightarrow D^{n-1}$ $\ni$: $g^n - f^n = \delta T^n + T^{n+1}\delta$. Write $f \underset{T}{\simeq} g$ in this case.

Just as in the homology case, cochain-homotopic cochain maps induce the same homomorphisms in cohomology.

<u>Example</u>: Suppose $f, g : C \to D$ are R-chain maps, S a chain homotopy from f to g, A an R-module. Define $T^n : \mathrm{Hom}_R(D, A)^n \longrightarrow \mathrm{Hom}_R(C, A)^{n-1}$ to be $(-1)^{n-1}\mathrm{Hom}(S_{n-1}, 1_A)$. Straightforward to check:

$$\mathrm{Hom}(g^n, 1_A) - \mathrm{Hom}(f^n, 1_A) = \delta T^n + T^{n+1}\delta.$$

If $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$ is a s.e.s. of cochain maps, just as in the homology case we get a natural ~~exact~~ seq.

$$\cdots \to H^n(C) \xrightarrow{f^*} H^n(D) \xrightarrow{g^*} H^n(E) \xrightarrow{\delta} H^{n+1}(C) \xrightarrow{f^*} H^{n+1}(D) \to \cdots$$

<u>Prop</u>: R arbitrary, $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ an exact seq. of R-modules. Then for any R-module D,

$$0 \to \mathrm{Hom}_R(C,D) \xrightarrow[\mathrm{Hom}(g,1_D)]{} \mathrm{Hom}_R(B,D) \xrightarrow[\mathrm{Hom}(f,1_D)]{} \mathrm{Hom}_R(A,D)$$

is exact.

If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a split s.e.s. of R-modules, then

$$0 \to \mathrm{Hom}_R(C,D) \xrightarrow[\mathrm{Hom}(g,1_D)]{} \mathrm{Hom}_R(B,D) \xrightarrow[\mathrm{Hom}(f,1_D)]{} \mathrm{Hom}_R(A,D) \to 0$$

is a split s.e.s.

<u>Proof</u>: Exercise.

Now let R be a PID, A an R-module, and

$$0 \to C_1(A) \xrightarrow{\partial_A} C_0(A) \xrightarrow{\varepsilon_A} A \to 0$$ the canonical s.f.r. of A.

For any R-module B, can form the cochain complex $\mathrm{Hom}_R(C(A),B)$.

<u>Note</u>: $H^n(\mathrm{Hom}_R(C(A),B)) = 0$ for $n \neq 0, 1$. We have

$$H^0(\mathrm{Hom}_R(C(A),B)) = Z^0(\mathrm{Hom}_R(C(A),B)) = \ker \mathrm{Hom}(\partial_A, 1_B).$$

By exactness of

$$0 \to \mathrm{Hom}_R(A,B) \xrightarrow{\mathrm{Hom}(\varepsilon_A,1_B)} \mathrm{Hom}_R(C_0(A),B) \xrightarrow{\mathrm{Hom}(\partial_A,1_B)} \mathrm{Hom}_R(C_1(A),B),$$

we have $H^0(\mathrm{Hom}_R(C(A),B)) \underset{\text{nat.}}{\cong} \mathrm{Hom}_R(A,B)$.

<u>Def</u>: $\mathrm{Ext}_R(A,B) = H^1(\mathrm{Hom}_R(C(A),B))$.

Since $Z^1(\mathrm{Hom}_R(C(A),B)) = \mathrm{Hom}_R(C_1(A),B)$,

we have $\mathrm{Ext}_R(A,B) = \dfrac{\mathrm{Hom}_R(C_1(A),B)}{\mathrm{im}\,\mathrm{Hom}(\partial_A,1_B)} = \mathrm{coker}\,\mathrm{Hom}(\partial_A,1_B),$

and so we obtain a natural exact sequence

$$0 \to \mathrm{Hom}_R(A,B) \xrightarrow{\mathrm{Hom}(\varepsilon_A, 1_B)} \mathrm{Hom}_R(C_0(A), B) \xrightarrow{\mathrm{Hom}(\partial_A, 1_B)} \mathrm{Hom}_R(C_1(A), B) \to \mathrm{Ext}_R(A,B) \to 0$$

If $f: A' \to A$, $g: B \to B'$ are R-homomorphisms, we obtain a chain map $C(f): C(A') \to C(A)$, and the $\mathrm{Hom}_R(C_n(f), g)$ constitute a cochain map $\mathrm{Hom}_R(C(f), g)$: $\mathrm{Hom}_R(C(A), B) \to \mathrm{Hom}_R(C(A'), B')$. Define

$$\mathrm{Ext}(f,g) = H^1(\mathrm{Hom}_R(C(f), g)) : \mathrm{Ext}_R(A,B) \to \mathrm{Ext}_R(A', B').$$

For fixed B, $A \longmapsto \mathrm{Ext}_R(A,B)$, $f \longmapsto \mathrm{Ext}(f, 1_B)$ is a contravariant functor from cat. of R-modules to itself, and for fixed A,

$B \longmapsto \mathrm{Ext}_R(A,B)$, $g \longmapsto \mathrm{Ext}(1_A, g)$ is a covariant functor from cat. of R-modules to itself.

<u>Recall</u>: If C is any s.f.r. of A, there is an A-augmented chain equivalence $f: C \to C(A)$, unique up to chain homotopy. Thus, for any R-module B,
$\mathrm{Hom}(f, 1_B): \mathrm{Hom}_R(C(A), B) \to \mathrm{Hom}_R(C, B)$ is a cochain equivalence, and hence we have an isom.
$f^*: \mathrm{Ext}_R(A,B) \to H^1(\mathrm{Hom}_R(C,B))$ which is indep. of the choice of A-augmented chain equiv. f.

Just as in the case of Tor, it follows that if C is a s.f.r. of A, C' a s.f.r. of A', $f: A' \to A$, $g: B \to B'$ R-homomorphisms, $h: C' \to C$ a chain map $\ni$:

$$\begin{array}{ccc} C_0' & \xrightarrow{\varepsilon'} & A' \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle f} \\ C_0 & \xrightarrow[\varepsilon]{} & A \end{array}$$

commutes, then

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}_R(A,B) & \to & \mathrm{Hom}_R(C_0,B) & \to & \mathrm{Hom}_R(C_1,B) & \to & \mathrm{Ext}_R(A,B) & \to 0 \\ & \downarrow{\scriptstyle \mathrm{Hom}(f,g)} & & \downarrow{\scriptstyle \mathrm{Hom}(h_0,g)} & & \downarrow{\scriptstyle \mathrm{Hom}(h_1,g)} & & \downarrow{\scriptstyle \mathrm{Ext}(f,g)} & \\ 0 \to & \mathrm{Hom}_R(A',B') & \to & \mathrm{Hom}_R(C_0',B') & \to & \mathrm{Hom}_R(C_1',B') & \to & \mathrm{Ext}_R(A',B') & \to 0 \end{array}$$

commutes.

<u>Prop:</u> For any R, $\mathrm{Hom}_R(A, \prod_\alpha B_\alpha) \underset{\text{nat. iso}}{\cong} \prod_\alpha \mathrm{Hom}_R(A, B_\alpha)$,

$$\mathrm{Hom}_R\left(\bigoplus_\alpha A_\alpha, B\right) \underset{\text{nat. iso.}}{\cong} \prod_\alpha \mathrm{Hom}_R(A_\alpha, B).$$

<u>Proof:</u> Exercise.

<u>Cor:</u> If R is a PID, then

1) $\mathrm{Ext}_R(A, \prod_\alpha B_\alpha) \underset{\text{nat. iso.}}{\cong} \prod_\alpha \mathrm{Ext}_R(A, B_\alpha)$,

2) $\mathrm{Ext}_R(\bigoplus_\alpha A_\alpha, B) \underset{\text{nat. iso.}}{\cong} \prod_\alpha \mathrm{Ext}_R(A_\alpha, B)$.

<u>Proof:</u> 1): For each α,

$$0 \to \mathrm{Hom}_R(A, B_\alpha) \to \mathrm{Hom}_R(C_0(A), B_\alpha) \to \mathrm{Hom}_R(C_1(A), B_\alpha) \to \mathrm{Ext}_R(A, B_\alpha) \to 0$$

is exact. Since a direct prod. of exact sequences is exact, we obtain the commutative diagram with exact rows

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \prod_\alpha \mathrm{Hom}_R(A, B_\alpha) \arrow[r] \arrow[d, "\cong"] & \prod_\alpha \mathrm{Hom}_R(C_0(A), B_\alpha) \arrow[r] \arrow[d, "\cong"] & \prod_\alpha \mathrm{Hom}_R(C_1(A), B_\alpha) \arrow[r] \arrow[d, "\cong"] & \prod_\alpha \mathrm{Ext}_R(A, B_\alpha) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(A, \prod_\alpha B_\alpha) \arrow[r] & \mathrm{Hom}_R(C_0(A), \prod_\alpha B_\alpha) \arrow[r] & \mathrm{Hom}_R(C_1(A), \prod_\alpha B_\alpha) \arrow[r] & \mathrm{Ext}_R(A, \prod_\alpha B_\alpha) \arrow[r] & 0
\end{tikzcd}$$

yielding the nat. isom. 1).

2): $\bigoplus_\alpha C(A_\alpha)$ is a s. f. r. of $\bigoplus_\alpha A_\alpha$. Thus, obtain comm. diagram with exact rows

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathrm{Hom}_R(\bigoplus_\alpha A_\alpha, B) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(\bigoplus_\alpha C_0(A_\alpha), B) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(\bigoplus_\alpha C_1(A_\alpha), B) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}_R(\bigoplus_\alpha A_\alpha, B) \arrow[r] & 0 \\
0 \arrow[r] & \prod_\alpha \mathrm{Hom}_R(A_\alpha, B) \arrow[r] & \prod_\alpha \mathrm{Hom}_R(C_0(A_\alpha), B) \arrow[r] & \prod_\alpha \mathrm{Hom}_R(C_1(A_\alpha), B) \arrow[r] & \prod_\alpha \mathrm{Ext}_R(A_\alpha, B) \arrow[r] & 0
\end{tikzcd}$$

yielding the nat. isom 2).

Prop: R a PID, A a free R-module. Then $\mathrm{Ext}_R(A,B)=0 \ \forall B$.

Proof: $0 \to 0 \to A \xrightarrow{1_A} A \to 0$ is a s.f.r. of A.

In general, if B is free it does not follow that $\mathrm{Ext}_R(A,B)=0$. (Hom-ing into a free module does not always preserve exactness, in contrast to tensoring with a free module).

Prop: For $n \geq 1$, $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/n, A) \cong A/nA \cong \mathbb{Z}/n \otimes_{\mathbb{Z}} A$ for all abel. groups A.

Proof: $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\rho_n} \mathbb{Z}/n \to 0$ is a s.f.r. of $\mathbb{Z}/n$, so

$$\begin{array}{ccccccc} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},A) & \xrightarrow{\mathrm{Hom}(n,1_A)} & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},A) & \longrightarrow & \mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/n,A) & \longrightarrow & 0 \text{ exact} \\ \wr\| & & \wr\| & & & & \\ A & \xrightarrow{\quad n \quad} & A & & & & \end{array}$$

and square commutes,

so $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/n, A) \cong A/nA$.

Now $A \cong \mathbb{Z} \otimes_{\mathbb{Z}} A \xrightarrow{\rho_n \otimes 1_A} \mathbb{Z}/n \otimes_{\mathbb{Z}} A$ is onto

with kernel nA, so $A/nA \cong \mathbb{Z}/n \otimes_{\mathbb{Z}} A$.

Let R be arbitrary, C a chain complex of R-modules, A an R-module. Let $\alpha \in \mathrm{Hom}_R(C_n, A) = \mathrm{Hom}_R(C,A)^n$ be a cocycle. Write $[\alpha]$ = cohomology class of α.

Claim: For any cycle $z \in C_n$, $\alpha(z) \in A$ depends only on $[\alpha]$ and on $[z]$. For $(\alpha + \delta x)(z + \partial y) = \alpha(z) + \alpha(\partial y) + (\delta x)(z) + (\delta x)(\partial y)$ $= \alpha(z) \pm (\delta\alpha)(y) \pm x(\partial z) \pm x(\partial\partial y) = \alpha(z)$ since $\delta\alpha = 0$ and $\partial z = 0$.

Thus, each cohomology class $[\alpha] \in H^n(\mathrm{Hom}_R(C,A))$ yields a function $H_n(C) \to A$, $[z] \longmapsto \alpha(z)$, which is easily seen to be an R-homomorphism.

Thus, we obtain a function $\nu: H^n(\mathrm{Hom}_R(C,A)) \to \mathrm{Hom}_R(H_n(C), A)$ given by $\nu[\alpha]([z]) = \alpha(z)$, which is easily seen to be a natural R-homomorphism.

Thm (Univ. Coeff. Thm. for cohomology): R a PID, C a chain complex of free R-modules, A an R-module. Then $\exists$ nat. exact seq. of R-homomorphisms

$$0 \to \mathrm{Ext}_R(H_{n-1}(C), A) \xrightarrow{\beta} H^n(\mathrm{Hom}_R(C, A)) \xrightarrow{\nu} \mathrm{Hom}_R(H_n(C), A) \to 0$$

which splits (splitting not natural).

Proof: We have a s.e.s. of chain complexes $0 \to Z \xrightarrow{i} C \xrightarrow{\bar\partial} \bar B \to 0$ where $Z_n = Z_n(C)$, $\partial_Z \equiv 0$, $\bar B_n = B_{n-1}(C) = B_{n-1}$, $\partial_{\bar B} \equiv 0$, and $\bar\partial_n = \partial|^{C_n}_{B_{n-1}}$. Since each $\bar B_n$ is free, above s.e.s. splits in each dim., and so we get a s.e.s. of cochain complexes

$$(*) \quad 0 \to \mathrm{Hom}_R(\bar B, A) \xrightarrow{\mathrm{Hom}(\bar\partial, 1_A)} \mathrm{Hom}_R(C, A) \xrightarrow{\mathrm{Hom}(i, 1_A)} \mathrm{Hom}_R(Z, A) \to 0,$$

Since $\delta_{\bar B} \equiv 0$ and $\delta_Z \equiv 0$ we have

$$H^n(\mathrm{Hom}_R(Z, A)) = \mathrm{Hom}_R(Z_n, A), \text{ and } H^{n+1}(\mathrm{Hom}_R(\bar B, A)) = \mathrm{Hom}_R(B_n, A).$$

Let $j_n : B_n \to Z_n$ denote the inclusion.

Claim 1: The connecting homomorphism $\delta : H^n(\mathrm{Hom}_R(Z, A)) \to H^{n+1}(\mathrm{Hom}_R(\bar B, A))$ arising from $(*)$ is $(-1)^n \mathrm{Hom}(j_n, 1_A) : \mathrm{Hom}_R(Z_n, A) \to \mathrm{Hom}_R(B_n, A)$.

$$\begin{array}{ccccc} & & \overset{g}{\mathrm{Hom}_R(C_n, A)} & \xrightarrow{\mathrm{Hom}(i_n, 1_A)} & \overset{f}{\mathrm{Hom}_R(Z_n, A)} \\ & & \downarrow{\scriptstyle \delta = (-1)^n \mathrm{Hom}(\partial, 1_A)} & & \\ \mathrm{Hom}_R(B_n, A) & \xrightarrow[\mathrm{Hom}(\bar\partial, 1_A)]{} & \mathrm{Hom}(C_{n+1}, A) & & \end{array}$$

Let $f \in \mathrm{Hom}_R(Z_n, A)$ and $g \in \mathrm{Hom}_R(C_n, A)$ $\ni$; $f = \mathrm{Hom}(i_n, 1_A)(g)$, i.e. $f = g i_n$. We have $\delta g = (-1)^n g\partial$. On the other hand, $\mathrm{Hom}(\bar\partial, 1_A)((-1)^n \mathrm{Hom}(j_n, 1_A)(f)) = (-1)^n f j_n \bar\partial = (-1)^n g i_n j_n \bar\partial = (-1)^n g\partial$, proving claim 1.

Let $p_n : Z_n \to H_n(C)$ denote the projection.

Claim 2: The diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(H_n(C), A) & \xrightarrow{\mathrm{Hom}(p_n, 1_A)} & \mathrm{Hom}_R(Z_n, A) \\ \nu \uparrow & & \| \\ H^n(\mathrm{Hom}_R(C, A)) & \xrightarrow[H^n(\mathrm{Hom}(i, 1_A))]{} & H^n(\mathrm{Hom}_R(Z, A)) \end{array}$$

commutes.

For if $f \in Z^n(\mathrm{Hom}_R(C,A))$, $z \in Z_n$,

$$\Big(\mathrm{Hom}(p_n, 1_A)\, \nu[f]\Big)(z) = \nu[f](p_n(z)) = \nu[f]([z]) = f(z) = f i_n(z),$$

while $\Big(H^n(\mathrm{Hom}(i, 1_A))[f]\Big)(z) = [f i_n](z) = f i_n(z)$.

Since $0 \to B_{n-1} \xrightarrow{(-1)^{n-1} j_{n-1}} Z_{n-1} \xrightarrow{p_{n-1}} H_{n-1}(C) \to 0$ is a f.r. of $H_{n-1}(C)$, we obtain the commutative diagram with exact rows

$$\begin{tikzcd}[column sep=small]
 & & 0 \arrow[r] & \mathrm{Hom}_R(H_n(C),A) \arrow[r, "\mathrm{Hom}(p_n,1_A)"] & \mathrm{Hom}_R(Z_n,A) \arrow[r, "(-1)^n \mathrm{Hom}(j_n,1_A)"] \arrow[d, equal] & \mathrm{Hom}_R(B_n,A) \arrow[d, equal] \\
H^{n-1}(\mathrm{Hom}_R(Z,A)) \arrow[r, "\delta"] \arrow[d, equal] & H^n(\mathrm{Hom}_R(\bar B,A)) \arrow[r] \arrow[d, equal] & H^n(\mathrm{Hom}_R(C,A)) \arrow[rr, "H^n(\mathrm{Hom}(i,1_A))"'] \arrow[u, "\nu"] & & H^n(\mathrm{Hom}_R(Z,A)) \arrow[r, "\delta"] & H^{n+1}(\mathrm{Hom}_R(\bar B,A)) \\
\mathrm{Hom}_R(Z_{n-1},A) \arrow[r, "(-1)^{n-1}\mathrm{Hom}(j_{n-1},1_A)"'] & \mathrm{Hom}_R(B_{n-1},A) \arrow[r] & \mathrm{Ext}_R(H_{n-1}(C),A) \arrow[r] & 0
\end{tikzcd}$$

The asserted natural exact sequence now follows by diagram chasing.

<u>Splitting</u>: For each n, the exact sequence $0 \to Z_n \xrightarrow{i_n} C_n \xrightarrow{\partial} B_{n-1} \to 0$ split. Hence $\exists$ R-hom. $r_n : C_n \to Z_n$ $\ni$; $r_n i_n = 1_{Z_n}$.
Write $p_n : Z_n \to H_n(C)$, $z \mapsto [z]$ for the projection.

<u>Claim</u>: im $\mathrm{Hom}(p_n r_n, 1_A) : \mathrm{Hom}_R(H_n(C), A) \to \mathrm{Hom}_R(C_n, A)$ is contained in $Z^n(\mathrm{Hom}_R(C,A))$. For if $f \in \mathrm{Hom}_R(H_n(C),A)$, $c \in C_{n+1}$

$$\delta\{\mathrm{Hom}(p_n r_n, 1_A)(f)\}(c) = \pm f p_n r_n(\partial c) = \pm f p_n r_n(i_n \partial c) \quad (\text{since } \partial c \in Z_n)$$
$$= \pm f p_n(\partial c) = \pm f[\partial c] = \pm f(0) = 0.$$

Define $\gamma : \mathrm{Hom}_R(H_n(C),A) \to H^n(\mathrm{Hom}_R(C,A))$ to be

$$p^n \, \mathrm{Hom}(p_n r_n, 1_A)\Big|^{\mathrm{Hom}_R(H_n(C),A)}_{Z^n(\mathrm{Hom}_R(C,A))} \qquad \text{where } p^n : Z^n(\mathrm{Hom}_R(C,A)) \to H^n(\mathrm{Hom}_R(C,A)),\ f \mapsto [f]$$

is the projection. Thus $\gamma(f) = [f p_n r_n]$. For any $f \in \mathrm{Hom}_R(H_n(C),A)$, $z \in Z_n(C)$, we have $\nu\gamma(f)[z] = \nu[f p_n r_n]([z])$
$= f p_n r_n(z) = f p_n r_n(i_n z) = f p_n(z) = f[z]$, and so $\nu\gamma = 1$.
Thus γ provides a splitting of the universal coefficient sequence.

Cor: (X, A) top. pair, G abel. grp. $\exists$ nat. exact seq.

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(H_{n-1}(X,A), G) \xrightarrow{\beta} H^n(X,A;G) \xrightarrow{\nu} \mathrm{Hom}_{\mathbb{Z}}(H_n(X,A),G) \to 0$$

which splits non-naturally.

If R a PID and G an R-module, $\exists$ nat. exact seq.

$$0 \to \mathrm{Ext}_R(H_{n-1}(X,A;R), G) \to H^n(X,A;G) \xrightarrow{\nu} \mathrm{Hom}_R(H_n(X,A;R), G) \to 0$$

which splits non-naturally.

For second part, use the nat. isom. of cochain complexes $\mathrm{Hom}_{\mathbb{Z}}(Q(X,A), G) \cong \mathrm{Hom}_R(Q(X,A) \otimes_{\mathbb{Z}} R, G)$.

Cor: If F is a field, $\nu : H^n(X,A;F) \to \mathrm{Hom}_F(H_n(X,A;F), F)$ is an isom.

Thm: Sing. coh. with coeffs in G satisfies the "Eilenberg–Steenrod axioms" for cohomology, i.e. for each $n \geq 0$, $H^n(\ ;G)$ is a contr. functor from top. pairs to abel. grps. (R-modules if G is an R-module) sat.

1) (Dimension axiom) If P a point, $H^i(P;G) \cong \begin{cases} G & \text{if } i=0 \\ 0 & \text{otherwise} \end{cases}$

2) (Exactness) If (X,A) a top. pair, $\exists$ nat. exact seq.

$$\to H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\delta} H^{n+1}(X,A;G) \to \cdots,$$

$i : A \hookrightarrow X$, $j : (X, \emptyset) \hookrightarrow (X,A)$ inclusions

3) (Homotopy) If $f \simeq g : (X,A) \to (Y,B)$, then

$$f^* = g^* : H^n(Y,B;G) \to H^n(X,A;G)$$

4) (Excision) If $\bar{U} \subset \mathrm{int}\, A$, (X,A) a top. pair, $i : (X-U, A-U) \to (X,A)$ inclusion, then $i^* : H^n(X,A;G) \to H^n(X-U, A-U;G)$ is an isom.

Proof similar to proving the $H_n(\ ;G)$ satisfies the E-S axioms. Use E.S. axioms for $H_n(\ ;\mathbb{Z})$, together with U.C.T.

Cohomology Cross Products

R a PID. Write $[H^*(X,A;R) \otimes_R H^*(Y,B;R)]^n =$

$\bigoplus_{p+q=n} H^p(X,A;R) \otimes_R H^q(Y,B;R)$.

Def: A top. pair (X,A) is admissible if A is open in X.

The cohomology cross product

$$\bar{\mu} : [H^*(X,A;R) \otimes_R H^*(Y,B;R)]^n \to H^n((X,A)\times(Y,B);R)$$

is a natural R-hom. which will be defined for admissible pairs (X,A), (Y,B). It will be defined as the composition of 3 natural maps, 2 of which are purely algebraic. 3rd involves $\{-\}$ map.

1st map: C, D cochain complexes of R-modules. Their tensor prod. $C \otimes_R D$ is defined just as for chain complexes, and as in the chain complex case we get a natural R-hom.

$$\mu : [H^*(C) \otimes_R H^*(D)]^n \longrightarrow H^n(C \otimes_R D)$$
$$[x] \otimes [y] \longmapsto [x \otimes y]$$

$\left([H^*(C) \otimes_R H^*(D)]^n = \bigoplus_{p+q=n} H^p(C) \otimes_R H^q(D) \right)$.

Thus for any top. pairs we get natural map

$$\mu : [H^*(X,A;R) \otimes_R H^*(Y,B;R)]^n \longrightarrow H^n(Q^*(X,A;R) \otimes_R Q^*(Y,B;R)).$$

2nd map: C, D chain complexes of abel. groups. $\exists$ nat. map of cochain complexes of R-modules

$$\eta : \mathrm{Hom}_{\mathbb{Z}}(C,R) \otimes_R \mathrm{Hom}_{\mathbb{Z}}(D,R) \to \mathrm{Hom}_{\mathbb{Z}}(C \otimes_{\mathbb{Z}} D, R)$$

given as follows: if $f \in \mathrm{Hom}_{\mathbb{Z}}(C_p,R)$, $g \in \mathrm{Hom}_{\mathbb{Z}}(D_q,R)$, $x \in C_r$, $y \in D_s$, then

$$\eta(f \otimes g)(x \otimes y) = \begin{cases} (-1)^{pq} f(x)\, g(y) & \text{if } p = r \text{ and } q = s \\ 0 & \text{otherwise} \end{cases}$$

Straightforward to check: η is a natural map of cochain complexes of R-modules. Thus, passing to cohomology, η induces a nat. R-hom

$$\eta^*: H^n\left(\mathrm{Hom}_{\mathbb{Z}}(C,R) \underset{R}{\otimes} \mathrm{Hom}_{\mathbb{Z}}(D,R)\right) \to H^n\left(\mathrm{Hom}_{\mathbb{Z}}(C \underset{\mathbb{Z}}{\otimes} D, R)\right).$$

Thus for any top. pairs (X,A), (Y,B), get nat. R-hom.

$$\eta^*: H^n\left(Q^*(X,A;R) \underset{R}{\otimes} Q^*(Y,B;R)\right) \to H^n(\mathrm{Hom}_{\mathbb{Z}}(Q(X,A) \underset{\mathbb{Z}}{\otimes} Q(Y,B)), R)\ \cancel{H^n(Q(X,A)\underset{\mathbb{Z}}{\otimes} Q(Y,B);R)}.$$

3rd map: Recall: For admissible pairs (X,A), (Y,B) we have a natural Eilenberg-Zilber chain map

$\theta: Q(X,A) \underset{\mathbb{Z}}{\otimes} Q(Y,B) \to Q((X,A)\times(Y,B))$ which induces $\cong$ in homology. (In the absolute case, we have proved θ is a chain equivalence. In the relative case we have not proved θ is a chain equivalence since in the proof of excision we did not prove that $Q(X,\mathcal{U}) \hookrightarrow Q(X)$ is a chain equivalence, but only that it induces $\cong$ in homology).

Since, for admissible pairs (X,A) and (Y,B), θ induces nat. $\cong$ in homology for all n, $\mathrm{Hom}(\theta, 1_R)$ induces a nat. isom.

$\theta^*: H^n((X,A)\times(Y,B);R) \to H^n(Q(X,A) \underset{\mathbb{Z}}{\otimes} Q(Y,B);R)$

(universal coeff. thm. + 5-lemma).

The 3rd map is $(\theta^*)^{-1}$.

$\bar{\mu}$ is defined to be the composition

$$\bar{\mu} = (\theta^*)^{-1}\eta^*\mu: \left[H^*(X,A;R) \underset{R}{\otimes} H^*(Y,B;R)\right]^n \to H^n((X,A)\times(Y,B);R)$$

If $a \in H^p(X,A;R)$, $b \in H^q(Y,B;R)$, write

$a \times b = \bar{\mu}(a \otimes b) \in H^{p+q}((X,A)\times(Y,B);R)$.

$a \times b$ is called the <u>cross-product</u> of a and b.

<u>Prop</u>: If $f:(X',A') \to (X,A)$, $g:(Y',B') \to (Y,B)$ are cont. maps of admissible pairs, $a \in H^p(X,A;R)$, $b \in H^q(Y,B;R)$, then $(f\times g)^*(a\times b) = f^*a \times g^*b$.

<u>Proof</u>: This is a restatement of naturality of $\bar{\mu}$. For

$(f\times g)^*(a\times b) = (f\times g)^*\bar{\mu}(a\otimes b) \underset{\text{nat. of } \bar{\mu}}{=} \bar{\mu}(f^*\otimes g^*)(a\otimes b)$

$= \bar{\mu}(f^*a \otimes g^*b) = f^*a \times g^*b$.

Prop: In the absolute case, cross prod. is assoc., i.e. if $a \in H^p(X;R)$, $b \in H^q(Y;R)$, $c \in H^r(Z;R)$, then $a\times(b\times c) = (a\times b)\times c \in H^{p+q+r}(X\times Y\times Z;R)$.

Proof: Suffices to show following commutes up to cochain homotopy:

$$\begin{tikzcd}
Q^*(X;R)\otimes_R Q^*(Y;R)\otimes_R Q^*(Z;R) \arrow[r, "1\otimes\eta"] \arrow[d, "\eta\otimes 1"'] \arrow[dr, phantom, "1"] & Q^*(X;R)\otimes_R \mathrm{Hom}_{\mathbb{Z}}(Q(Y)\otimes_{\mathbb{Z}} Q(Z),R) \arrow[d, "\eta"] \arrow[dr, phantom, "2"] & Q^*(X;R)\otimes_R Q^*(Y\times Z;R) \arrow[l, "1\otimes\theta^*"'] \arrow[d, "\eta"] \\
\mathrm{Hom}_{\mathbb{Z}}(Q(X)\otimes_{\mathbb{Z}} Q(Y),R)\otimes_R Q^*(Z;R) \arrow[r, "\eta"] & \mathrm{Hom}_{\mathbb{Z}}(Q(X)\otimes_{\mathbb{Z}} Q(Y)\otimes_{\mathbb{Z}} Q(Z),R) & \mathrm{Hom}_{\mathbb{Z}}(Q(X)\otimes_{\mathbb{Z}} Q(Y\times Z),R) \arrow[l, "(1\otimes\theta)^*"'] \\
Q^*(X\times Y;R)\otimes_R Q^*(Z;R) \arrow[u, "\theta^*\otimes 1"] \arrow[r, "\eta"'] \arrow[ur, phantom, "3"] & \mathrm{Hom}_{\mathbb{Z}}(Q(X\times Y)\otimes_{\mathbb{Z}} Q(Z),R) \arrow[u, "(\theta\otimes 1)^*"'] \arrow[ur, phantom, "4"] & Q^*(X\times Y\times Z;R) \arrow[l, "\theta^*"] \arrow[u, "\theta^*"']
\end{tikzcd}$$

1 commutes by direct check of def.

2 + 3 commute by nat. of η.

4 does not commute. However, only need 4 to commute up to cochain homotopy.

④ is obtained by applying $\mathrm{Hom}_{\mathbb{Z}}(\ \ , 1_R)$ to the diagram of chain maps

$$\begin{tikzcd}
Q(X)\otimes_{\mathbb{Z}} Q(Y)\otimes_{\mathbb{Z}} Q(Z) \arrow[r, "1\otimes\theta"] \arrow[d, "\theta\otimes 1"'] & Q(X)\otimes_{\mathbb{Z}} Q(Y\times Z) \arrow[d, "\theta"] \\
Q(X\times Y)\otimes_{\mathbb{Z}} Q(Z) \arrow[r, "\theta"'] & Q(X\times Y\times Z)
\end{tikzcd},$$

and so it suffices to show this last diagram commutes up to chain homotopy.

Recall: $\exists$ natural chain equivalences $F:\Delta\to Q$, $G: Q\to\Delta$ and natural chain homotopies $GF\simeq 1_\Delta$, $FG\simeq 1_Q$. Acyclic models yielded a natural augmented chain equivalence

$$\Delta(X)\otimes_{\mathbb{Z}}\Delta(Y)\xrightarrow{\ \varphi\ }\Delta(X\times Y)$$

, unique up to chain homotopy, and θ was taken to be the composition

$$Q(X)\otimes_{\mathbb{Z}} Q(Y)\xrightarrow{G\otimes G}\Delta(X)\otimes_{\mathbb{Z}}\Delta(Y)\xrightarrow{\ \varphi\ }\Delta(X\times Y)\xrightarrow{\ F\ } Q(X\times Y).$$

Consider diagram

$$\begin{tikzcd}
Q(X)\otimes_{\mathbb{Z}} Q(Y)\otimes_{\mathbb{Z}} Q(Z) \arrow[rrr, "1\otimes\theta"] \arrow[dr, "G\otimes G\otimes G"] \arrow[ddd, "\theta\otimes 1"'] & {} \arrow[r, phantom, "2"] & {} & Q(X)\otimes_{\mathbb{Z}} Q(Y\times Z) \arrow[ddd, "\theta"] \\
{} \arrow[d, phantom, "1"] & \Delta(X)\otimes\Delta(Y)\otimes\Delta(Z) \arrow[r, "1\otimes\varphi"] \arrow[d, "\varphi\otimes 1"'] \arrow[dr, phantom, "5"] & \Delta(X)\otimes\Delta(Y\times Z) \arrow[ur, "F\otimes F"'] \arrow[d, "\varphi"] & {} \arrow[d, phantom, "4"] \\
{} & \Delta(X\times Y)\otimes\Delta(Z) \arrow[r, "\varphi"'] \arrow[dl, "F\otimes F"'] & \Delta(X\times Y\times Z) \arrow[dr, "F"] & {} \\
Q(X\times Y)\otimes_{\mathbb{Z}} Q(Z) \arrow[rrr, "\theta"'] & {} \arrow[r, phantom, "3"] & {} & Q(X\times Y\times Z)
\end{tikzcd}$$

① and ② commute up to chain homotopy by def. of Θ and fact that $FG \simeq 1_Q$.

③ + ④ commute up to chain homotopy since

$$\Theta \circ (F \otimes F) = F \circ \varphi \circ (G \otimes G)(F \otimes F) \simeq F\varphi \quad \text{since } GF \simeq 1_\Delta.$$

⑤ commutes up to chain homotopy by acyclic models: Take cat. $\mathcal{C}$ of ordered triples of top. spaces. $\mathcal{M} = \{(\Delta^p, \Delta^q, \Delta^r)\}$, all proper (no degenerate maps). The functors $S, T: \mathcal{C} \to$ Cat of aug. ch. complexes given by $S(X; Y; Z) = \Delta(X) \otimes_{\mathbb{Z}} \Delta(Y) \otimes_{\mathbb{Z}} \Delta(Z)$, $T(X, Y, Z) = \Delta(X \times Y \times Z)$ are both $\mathcal{M}/\mathcal{B}$-free and acyclic on models.
$\varphi(1 \otimes \varphi)$ and $\varphi(\varphi \otimes 1): S(X, Y, Z) \to T(X, Y, Z)$ are natural augmented chain maps, and hence, by the acyclic models theorem, chain homotopic.

<u>Prop</u>: X, Y top. spaces, $T: Y \times X \to X \times Y$ the twist map $T(y, x) = (x, y)$. Then for $a \in H^p(X; R)$, $b \in H^q(Y; R)$, $T^*(a \times b) = (-1)^{pq} b \times a$.

<u>Proof</u>: For augmented chain complexes C, D, let $\tau: C \otimes D \to D \otimes C$ denote the natural augmented chain map $\tau(c \otimes d) = (-1)^{|c||d|} d \otimes c$, and similarly $\bar{\tau}$ for cochain complexes. Suff. to show following commutes up to cochain homotopy:

$$\begin{tikzcd}
Q^*(X; R) \otimes_R Q^*(Y; R) \arrow[r, "\eta"] \arrow[d, "\bar{\tau}"] \arrow[dr, phantom, "1"] & \mathrm{Hom}_{\mathbb{Z}}(Q(X) \otimes_{\mathbb{Z}} Q(Y); R) \arrow[d, "\tau^* = \mathrm{Hom}(\tau, 1_R)"] \arrow[dr, phantom, "2"] & Q^*(X \times Y; R) \arrow[l, "\Theta^*"'] \arrow[d, "Q(T)^* = \mathrm{Hom}(Q(T), 1_R)"] \\
Q^*(Y; R) \otimes_R Q^*(X; R) \arrow[r, "\eta"'] & \mathrm{Hom}_{\mathbb{Z}}(Q(Y) \otimes_{\mathbb{Z}} Q(X); R) & Q^*(Y \times X; R) \arrow[l, "\Theta^*"]
\end{tikzcd}$$

① commutes by direct check. ② is the dual of

$$\begin{tikzcd}
Q(X) \otimes_{\mathbb{Z}} Q(Y) \arrow[r, "\Theta"] & Q(X \times Y) \\
Q(Y) \otimes_{\mathbb{Z}} Q(X) \arrow[u, "\tau"] \arrow[r, "\Theta"'] & Q(Y \times X) \arrow[u, "Q(T)"']
\end{tikzcd},$$

so suffices to check this last diagram commutes up to chain homotopy.

By an argument similar to that of previous prop., suffices to show same diagram with Q replaced by Δ commutes up to chain homotopy. This follows by acyclic models.

Recall: For any top. space X, $Q^*(X;R)$ has an coaug. $\eta: R \to Q^0(X;R)$, which yields a natural coaug. $\eta: R \to H^0(X;R)$. (it is dual to $\varepsilon: H_0(X;R) \to R$ which is onto if $X \neq \emptyset$).
For $X \neq \emptyset$, let $\eta(1) \in H^0(X;R)$ be denoted by 1.
By nat. of η, if $f: X \to Y$ cont., $X, Y \neq \emptyset$, then $f^*(1) = 1$.

Prop: X, Y non-empty top spaces, $\pi_1 : X \times Y \to X$ proj. on 1^{st} factor. Then for each $a \in H^p(X;R)$, $\pi_1^*(a) = a \times 1$.

Proof: Let P be a point, $p: Y \to P$ the constant map. Suppose we know $\pi_1'^*(a) = a \times 1$ where $\pi_1' : X \times P \to X$ is proj. on 1^{st} factor. Then by commutativity of

$$\begin{tikzcd}
X \times Y \arrow[dr, "\pi_1"] \arrow[d, "1\times p"'] & \\
X \times P \arrow[r, "\pi_1'"'] & X
\end{tikzcd}$$

and naturality of cross prods.,
$$\pi_1^*(a) = (1\times p)^* \pi_1'^*(a) = (1 \times p)^*(a \times 1) = 1^*(a) \times p^*(1) = a \times 1.$$

Hence, suffices to prove prop. for $Y = P$. Suff. to show following commutes up to cochain homotopy:

$$\begin{tikzcd}
 & Q^*(X;R) \otimes_R Q^*(P;R) \arrow[r, "\eta\ (\text{not coaug})"] & \operatorname{Hom}_R(Q(X) \otimes_{\mathbb{Z}} Q(P), R) \\
Q^*(X;R) \arrow[ur, "\iota"] \arrow[urr, "\operatorname{Hom}(\kappa, 1_R)"'] \arrow[drr, "\operatorname{Hom}(Q(\pi_1), 1_R)"'] & \arrow[u, phantom, "1"] \arrow[r, phantom, "2"] & \\
 & & Q^*(X \times P; R) \arrow[uu, "\theta^*"']
\end{tikzcd}$$

where $\iota(x) = x \otimes \underset{\text{coaug.}}{\eta(1)}$

and κ is the composition

$$[Q(X) \otimes_{\mathbb{Z}} Q(P)]_n = Q_n(X) \otimes_{\mathbb{Z}} Q_0(P) \xrightarrow{\cong} Q_n(X)$$
$$u \otimes 1_0 \longmapsto u$$

(Recall: $Q_n(P) = 0$ for $n \neq 0$, $Q_0(P) \cong \mathbb{Z}$, gen. 1_0).

Easily checked: ① commutes, and K is a natural aug. chain map. ② is dual of

$$
\begin{tikzcd}
Q(X) & Q(X)\otimes_{\mathbb{Z}} Q(P) \arrow[l, "K"'] \arrow[d, "\theta"] \\
 & Q(X\times P) \arrow[ul, "Q(\pi_1)"]
\end{tikzcd}
$$

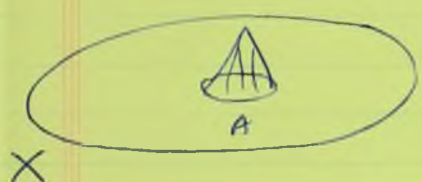
Proof is completed in same way as previous props: Can replace Q by Δ, and use acyclic models.

Similarly, or as a consequence of commutativity,

Prop: X, Y top. spaces $\neq \emptyset$, $b \in H^q(Y;R)$. Then $\pi_2^* b = 1\times b$.

<u>Extension of properties of cross prod. to relative case</u>

(X,A) a top. pair. Consider $(X\cup CA, CA)$, CA = cone on A.



By excision and def. retraction, the inclusion $i: (X,A) \to (X\cup CA, CA)$ induces $\cong$ in cohomology. <u>Note</u>: (X,A) admiss. $\Rightarrow$ $(X\cup CA, CA)$ admiss.

<u>Lemma</u>: (X,A), (Y,B) admissible pairs. Then the map induced by inclusion

$$H^p((X\cup CA, CA)\times(Y,B)) \longrightarrow H^p((X\cup CA)\times(Y,B)) \quad \text{is } 1-1$$

(arbitrary coeffs).

<u>Proof</u>: The s.e.s. of chain complexes

$$0 \to Q(CA)\otimes_{\mathbb{Z}} Q(Y,B) \to Q(X\cup CA)\otimes_{\mathbb{Z}} Q(Y,B) \to Q(X\cup CA, CA)\otimes Q(Y,B) \to 0,$$

together with E.Z. map θ, yields the exact seq

$$H^{p-1}((X\cup CA)\times(Y,B)) \xrightarrow{\text{incl}^*} H^{p-1}(CA\times(Y,B)) \to H^p((X\cup CA, CA)\times(Y,B)) \xrightarrow{\text{incl}^*} H^p((X\cup CA)\times(Y,B))$$

Hence, suff. to show $H^{p-1}((X\cup CA)\times(Y,B)) \to H^{p-1}(CA\times(Y,B))$ is onto. Write v = vertex of CA. Since CA is contractible, the inclusion $\{v\}\times(Y,B) \subset CA\times(Y,B)$ is a homotopy equivalence, and hence induces $\cong$ in cohomology. Hence, suff. to show the map induced by inclusion $H^{p-1}(X\cup CA \times (Y,B)) \to H^{p-1}(\{v\}\times(Y,B))$ is onto. But $\{v\}\times(Y,B)$ is a retract of $X\cup CA\times(Y,B)$, so coh. map induced by incl. is onto.

Prop: (X,A) admiss., $Y\neq\emptyset$, R a PID, $\pi_1 : (X,A)\times Y \to (X,A)$ proj. on first factor. Then for any $a\in H^p(X,A;R)$, $\pi_1^*(a) = a\times 1$.

Proof: Have comm. diagram

$$\begin{tikzcd}
a & H^p(X,A) \arrow[r, "\pi_1^*"] & H^p((X,A)\times Y) & \\
a' \arrow[u, mapsto] \arrow[d, mapsto] & H^p(X\cup CA, CA) \arrow[u, "\cong\ i^*"'] \arrow[r, "\pi_1^*"] \arrow[d, "j^*"'] & H^p((X\cup CA, CA)\times Y) \arrow[u, "(i\times 1)^*"'] \arrow[d, "(j\times 1)^*"] & \text{1-1 by lemma} \\
& H^p(X\cup CA) \arrow[r, "\pi_1^*"] & H^p((X\cup CA)\times Y) & \\
a'' \arrow[rr, mapsto] & & a''\times 1 &
\end{tikzcd}$$

We know $(j\times 1)^*(a'\times 1) = j^*a'\times 1^*(1) = a''\times 1$ by nat. of cross-prod. Since $(j\times 1)^*$ is 1-1 (by lemma) and bottom square commutes, must have $\pi_1^*(a') = a'\times 1$. Then

$$\pi_1^* a = \pi_1^* i^*(a') = (i\times 1)^*\pi_1^*(a') = (i\times 1)^*(a'\times 1) = i^*(a')\times 1^*(1) = a\times 1.$$

Prop. (X,A), (Y,B) admiss. pairs, R a PID. Suppose $a\in H^p(X,A;R)$, $b\in H^q(Y,B;R)$. Then $T^*(a\times b) = (-1)^{pq}(b\times a)$ where $T: (Y,B)\times(X,A)\to (X,A)\times(Y,B)$ interchanges factors.

Proof: Similar to proof of preceding using diagram

$a\otimes b \longmapsto a\times b \longmapsto T^*(a\times b)$

$$\begin{tikzcd}[column sep=small]
H^p(X,A)\otimes H^q(Y,B) \arrow[r, "\bar{u}"] & H^{p+q}((X,A)\times(Y,B)) \arrow[r, "T^*"] & H^{p+q}((Y,B)\times(X,A)) \\
H^p(X\cup cA, cA)\otimes H^q(Y\cup cB, cB) \arrow[u, "\cong"] \arrow[r, "\bar{u}"] \arrow[d] & H^{p+q}((X\cup cA, cA)\times(Y\cup cB, cB)) \arrow[u] \arrow[r, "T^*"] \arrow[d] & H^{p+q}((Y\cup cB, cB)\times(X\cup cA, cA)) \arrow[u] \arrow[d, "1\text{-}1\ (\text{lemma})"] \\
H^p(X\cup cA)\otimes H^q(Y\cup cB, cB) \arrow[r, "\bar{u}"] \arrow[d] & H^{p+q}((X\cup cA)\times(Y\cup cB, cB)) \arrow[r, "T^*"] \arrow[d] & H^{p+q}((Y\cup cB, cB)\times(X\cup cA)) \arrow[d, "1\text{-}1\ (\text{lemma})"] \\
H^p(X\cup cA)\otimes H^q(Y\cup cB) \arrow[r, "\bar{u}"] & H^{p+q}((X\cup cA)\times(Y\cup cB)) \arrow[r, "T^*"] & H^{p+q}((Y\cup cB)\times(X\cup cA))
\end{tikzcd}$$

$a'\otimes b' \longmapsto T^*(a'\times b')$

$a''\otimes b'' \longmapsto T^*(a''\times b'')$

$a'''\otimes b''' \longmapsto T^*(a'''\times b''') = (-1)^{pq}\, b'''\times a'''$ by abs. case

Prop: (X,A), (Y,B), (Z,C) admiss pairs, R a PID, $a\in H^p(X,A;R)$, $b\in H^q(Y,B;R)$, $c\in H^r(Z,C;R)$. Then $a\times(b\times c) = (a\times b)\times c$.

Proof: Similar to above, using diagram

$a\otimes b\otimes c \longmapsto a\times(b\times c) - (a\times b)\times c$

$$\begin{tikzcd}[column sep=small]
H^p(X,A)\otimes H^q(Y,B)\otimes H^r(Z,C) \arrow[r] & H^{p+q+r}((X,A)\times(Y,B)\times(Z,C)) \\
H^p(X\cup cA, cA)\otimes H^q(Y\cup cB, cB)\otimes H^r(Z\cup cC, cC) \arrow[u, "\cong"] \arrow[r] \arrow[d] & H^{p+q+r}((X\cup cA, cA)\times(Y\cup cB, cB)\times(Z\cup cC, cC)) \arrow[u] \arrow[d, "1\text{-}1\ (\text{lemma})"] \\
H^p(X\cup cA)\otimes H^q(Y\cup cB, cB)\otimes H^r(Z\cup cC, cC) \arrow[r] \arrow[d] & H^{p+q+r}((X\cup cA)\times(Y\cup cB, cB)\times(Z\cup cC, cC)) \arrow[d, "1\text{-}1\ (\text{lemma})"] \\
H^p(X\cup cA)\otimes H^q(Y\cup cB)\otimes H^r(Z\cup cC, cC) \arrow[r] \arrow[d] & H^{p+q+r}((X\cup cA)\times(Y\cup cB)\times(Z\cup cC, cC)) \arrow[d, "1\text{-}1\ (\text{lemma})"] \\
H^p(X\cup cA)\otimes H^q(Y\cup cB)\otimes H^r(Z\cup cC) \arrow[r] & H^{p+q+r}((X\cup cA)\times(Y\cup cB)\times(Z\cup cC)).
\end{tikzcd}$$

Cup Products

Def: (Absolute case): X a top. space, R a PID. Let $d: X \to X \times X$ denote the diagonal map $d(x) = (x,x)$. Let $a \in H^p(X;R)$, $b \in H^q(X;R)$. The cup product $a \smile b \in H^{p+q}(X;R)$ is defined by $a \smile b = d^*(a \times b)$.

Note: A cross-prod exists in homology, but a prod. analogous to cup prod. doesn't arise in homology since homology is covariant. Get

$$\left[H(X;R) \otimes_R H(X;R)\right]_n \xrightarrow{\text{cross-prod.}} H_n(X \times X;R) \xleftarrow{d_*} H_n(X;R).$$

If $(X;A,B)$ a triad and $d: X \to X \times X$ the diag map, $d^{-1}(X \times B \cup A \times X) = A \cup B$. Thus, get map of top. pairs $d: (X, A\cup B) \to (X,A) \times (Y,B)$

Def (Rel. cup prod): $(X;A,B)$ an admiss. triad (i.e. A, B open in X), R a PID. Let $a \in H^p(X,A;R)$, $b \in H^q(X,B;R)$. The cup prod of a and b is $a \smile b = d^*(a \times b) \in H^{p+q}(X, A\cup B; R)$.

Def: A map of admiss. triads $f: (X';A',B') \to (X;A,B)$ consists of a cont. map $X' \to X$ which carries A' into A, B' into B.

By restriction, such an f yields maps of admiss pairs $f_{A\cup B}: (X', A'\cup B') \to (X, A\cup B)$, $f_A: (X',A') \to (X,A)$, $f_B: (X',B) \to (X,B)$

Prop: If $f: (X';A',B') \to (X;A,B)$ is a map of admiss. triads, $a \in H^p(X,A;R)$, $b \in H^q(X,B;R)$, then
$$f^*_{A\cup B}(a \smile b) = f^*_A(a) \smile f^*_B(b)$$

Proof:
$$\begin{tikzcd}
(X', A'\cup B') \arrow[r, "d'"] \arrow[d, "f_{A\cup B}"'] & (X',A') \times (X',B') \arrow[d, "f_A \times f_B"] \\
(X, A\cup B) \arrow[r, "d"] & (X,A) \times (X,B)
\end{tikzcd}$$
commutes.

Thus $f^*_{A\cup B}(a \smile b) = f^*_{A\cup B}\, d^*(a \times b) = (d \circ f_{A\cup B})^*(a \times b) = \left[(f_A \times f_B)\circ d'\right]^*(a \times b)$
$= (d')^*(f_A \times f_B)^*(a \times b) = (d')^*\left[(f_A^* a) \times f_B^*(b)\right] = f_A^*(a) \smile f_B^*(b)$.

Prop: a) Cup prod. is assoc., i.e. if A, B, C open in X, R a PID, $a \in H^p(X, A; R)$, $b \in H^q(X, B; R)$, $c \in H^r(X, C; R)$, then
$a \cup (b \cup c) = (a \cup b) \cup c$.

b) Cup prod. is comm. in graded sense, i.e. if $a \in H^p(X, A; R)$, $b \in H^q(X, B; R)$, then $a \cup b = (-1)^{pq} b \cup a$.

c) If $X \neq \emptyset$, $a \in H^p(X, A; R)$, then $1 \cup a = a \cup 1 = a$.

d) $a \cup (b + c) = a \cup b + a \cup c$, $(ra) \cup b = a \cup (rb) = r(a \cup b)$, $r \in R$.

Proof: a)
$$\begin{tikzcd}
(X, A \cup B \cup C) \arrow[r, "d"] \arrow[d, "d"'] & (X, A) \times (X, B \cup C) \arrow[d, "1 \times d"] \\
(X, A \cup B) \times (X, C) \arrow[r, "d \times 1"'] & (X, A) \times (X, B) \times (X, C)
\end{tikzcd}$$
commutes.

Hence $a \cup (b \cup c) = d^*(a \times d^*(b \times c)) = d^*(1 \times d)^*(a \times (b \times c))$
$= d^*(d \times 1)^*((a \times b) \times c) = (a \cup b) \cup c$.

b)
$$\begin{tikzcd}
 & (X, A) \times (X, B) \\
(X, A \cup B) \arrow[ur, "d"] \arrow[dr, "d"'] & \\
 & (X, B) \times (X, A) \arrow[uu, "T"']
\end{tikzcd}$$
commutes.

Hence $a \cup b = d^*(a \times b) = d^* T^*(a \times b) = d^*\left[(-1)^{pq}(b \times a)\right] = (-1)^{pq} b \cup a$.

c)
$$\begin{tikzcd}
 & (X, A) \times X \arrow[dd, "\pi_1"] \\
(X, A) \arrow[ur, "d"] \arrow[dr, "1"'] & \\
 & (X, A)
\end{tikzcd}$$
commutes.

Hence $a = 1^* a = d^* \pi_1^* a = d^*(a \times 1) = a \cup 1$.
Similarly, $1 \cup a = a$.

d) Cross prod. is R-bilinear

Prop: Let $(X; A, A')$, $(Y; B, B')$ be admiss. triads. Let $a \in H^p(X, A; R)$, $b \in H^q(Y, B; R)$, $a' \in H^r(X, A'; R)$, $b' \in H^s(Y, B'; R)$. Then $(a \times b) \cup (a' \times b') = (-1)^{qr} (a \cup a') \times (b \cup b')$.

Proof:

$$\begin{tikzcd}
(X, A \cup A') \times (Y, B \cup B') \arrow[rr, "d"] \arrow[dr, "d_X \times d_Y"'] & & (X, A) \times (Y, B) \times (X, A') \times (Y, B') \\
& (X, A) \times (X, A') \times (Y, B) \times (Y, B') \arrow[ur, "1 \times T \times 1"'] &
\end{tikzcd}$$

commutes.

Hence $(a \times b) \cup (a' \times b') = d^* ((a \times b) \times (a' \times b'))$

$= (d_X \times d_Y)^* (1 \times T \times 1)^* (a \times (b \times a') \times b')$

$= (d_X \times d_Y)^* (1^* a \times T^*(b \times a') \times 1^* b')$

$= (-1)^{qr} (d_X \times d_Y)^* (a \times a' \times b \times b')$

$= (-1)^{qr} d_X^* (a \times a') \times d_Y^* (b \times b') = (-1)^{qr} (a \cup a') \times (b \cup b')$.

Cor: (X, A), (Y, B) admiss. pairs, $X \neq \emptyset \neq Y$, $a \in H^p(X, A; R)$, $b \in H^q(Y, B; R)$. Then $a \times b = \pi_1^*(a) \cup \pi_2^*(b)$ where $\pi_1 : (X, A) \times Y \to (X, A)$, $\pi_2 : X \times (Y, B) \to (Y, B)$ the projections.

Proof: $\pi_1^* a \cup \pi_2^* b = (a \times 1) \cup (1 \times b) = (a \cup 1) \times (1 \cup b) = a \times b$

Thm: Suppose $X = A \cup B$, A, B open in X, A, B both contractible. Let R be a PID, $a \in H^p(X; R)$, $b \in H^q(X; R)$, $p > 0$, $q > 0$. Then $a \cup b = 0$.

Proof: Have comm. diagram

$$\begin{tikzcd}
H^p(X) \otimes H^q(X) \arrow[r, "\cup"] & H^{p+q}(X) \\
H^p(X, A) \otimes H^q(X, B) \arrow[r, "\cup"] \arrow[u, "i_1^* \otimes i_2^*"] & H^{p+q}(X, A \cup B) \arrow[u, "i^*"']
\end{tikzcd}$$

Since A, B both contractible and $p, q > 0$, i_1^* and i_2^* are onto.

Since $A \cup B = X$, $H^{p+q}(X, A\cup B) = 0$. Hence, $H^p(X) \otimes H^q(X) \xrightarrow{\cup} H^{p+q}(X)$ is 0.

This proof generalizes to yield the following:

Thm: If X = union of n contractible open subspaces (or more generally, n open subspaces each of which is contractible in X), then all n-fold cup prods of pos. dim. elements in $H^*(X;R)$ vanish.

Cor: If X is the suspension of some space, then all cup products of pos dim. elements in $H^*(X;R)$ vanish.

A cohomology Künneth Theorem

R a PID. If C, D are cochain complexes of R-modules with each C^n free, then just as in the homology case we get a natural exact sequence

$$0 \to \left[H^*(C) \underset{R}{\otimes} H^*(D)\right]^n \xrightarrow{\mu} H^n(C\otimes D) \longrightarrow \left[\mathrm{Tor}^R(H^*(C), H^*(D))\right]^{n+1} \to 0$$

Def: A chain complex C of R-modules is of finite-type if each C_n is finitely-generated.

Recall: If C a free chain complex and each $H_n(C)$ is free, then C is chain equiv. to a chain complex C' with $\partial_{C'} = 0$. Thus if each $H_n(C)$ is finitely-gen., then C' will be of finite type.

Prop: C a free chain complex of abel. groups of finite type, D any chain complex of abel. groups. Then

$$\eta: \mathrm{Hom}_{\mathbb{Z}}(C,R) \underset{R}{\otimes} \mathrm{Hom}_{\mathbb{Z}}(D,R) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C \underset{\mathbb{Z}}{\otimes} D, R)$$

is an isom. of cochain complexes.

Proof: Immediate from def. of η and following fact, left as an exercise: If A, B abel. groups with A finitely-gen. and free, then $\eta: \mathrm{Hom}_{\mathbb{Z}}(A,R) \underset{R}{\otimes} \mathrm{Hom}_{\mathbb{Z}}(B,R) \to \mathrm{Hom}_{\mathbb{Z}}(A \underset{\mathbb{Z}}{\otimes} B, R)$ given by $\eta(f\otimes g)(a\otimes b) = f(a)g(b)$ is an isom.

Thm: (X,A), (Y,B) admiss. pairs; R a P.I.D. Suppose $H_n(X,A;\mathbb{Z})$ is finitely gen. + free for all n. Then the cross prod. map

$$\bar{\mu} : \left[H^*(X,A;R)\otimes_R H^*(Y,B;R)\right]^n \longrightarrow H^n((X,A)\times(Y,B);R)$$

is an isom.

Proof: $\bar{\mu} = (\theta^*)^{-1}\eta^*\mu$. Since $(\theta^*)^{-1}$ is an isom., will be done if we show $\eta^*\mu$ is an isom.

Since $H_n(X,A)$ is free and fin. gen. for all n, $\exists$ free chain complex of finite type C, and a chain equiv. $f: C \to Q(X,A)$. Then $\mathrm{Hom}_{\mathbb{Z}}(C,R)$ is a coch. complex of free R-modules since each C_n is finitely gen. free abel. (Not true that $Q^*(X,A;R)$ is free). f induces isoms. $f^*: H^n(X,A;R) \to H^n(\mathrm{Hom}(C,R))$. Since the $H_n(X,A)$ are free abel. and finitely-gen., by the U.C.T., all the $H^n(C;R)$ are free R-modules. Then $\mu : \left[H^*(C;R)\otimes_R H^*(Y,B;R)\right]^n \to H^n(\mathrm{Hom}_{\mathbb{Z}}(C,R)\otimes_R Q^*(Y,B;R))$

is an isom. (since all Tor terms are 0).

Following commutes:

$$\begin{tikzcd}
H^*(X,A;R)\otimes_R H^*(Y,B;R) \arrow[r, "\mu"] \arrow[d, "(f^*\otimes 1)^n"] & H^n(Q^*(X,A;R)\otimes_R Q^*(Y,B;R)) \arrow[r, "\eta^*"] \arrow[d, "{[\mathrm{Hom}(f,1)\otimes 1]^*}"] & H^n(\mathrm{Hom}_{\mathbb{Z}}(Q(X,A)\otimes_{\mathbb{Z}} Q(Y,B)), R) \arrow[d, "(f\otimes 1)^*"] \\
H^*(\mathrm{Hom}_{\mathbb{Z}}(C,R))\otimes_R H^*(Y,B;R) \arrow[r, "\cong", "\mu"'] & H^n(\mathrm{Hom}(C,R)\otimes_R Q^*(Y,B;R)) \arrow[r, "\cong", "\eta^*"'] & H^n(\mathrm{Hom}_{\mathbb{Z}}(C\otimes_{\mathbb{Z}} Q(Y,B)), R)
\end{tikzcd}$$

Vert. maps are $\cong$ since f is a chain equiv. Bottom maps are $\cong$, so top $\eta^*\mu$ is $\cong$.

Example: $X = S^2\times S^3$, $Y = \Sigma(S^1\vee S^2\vee S^4)$.

X and Y have isom. homology groups.
($H_0\cong\mathbb{Z}$, $H_1\cong 0$, $H_2\cong\mathbb{Z}$, $H_3\cong\mathbb{Z}$, $H_4\cong 0$, $H_5\cong\mathbb{Z}$).
Since Y is a suspension, cup prods. of pos. dim. elements are 0 in $H^*(Y)$.

Let a, b denote gens. of $H^2(S^2;\mathbb{Z})$, $H^3(S^3;\mathbb{Z})$, resp. By above Kunneth thm., $a\times b = \bar{\mu}(a\otimes b)\neq 0$. Hence $0 \neq a\times b = \pi_1^*a \cup \pi_2^*b$, so $H^*(X;\mathbb{Z})$ is not isom. to $H^*(Y;\mathbb{Z})$ as a graded ring. Hence X and Y cannot have the same homotopy type.

Compatibility of cup + cross prods. with connecting homomorphisms

Lemma: Let $0 \to C \to D \to E \to 0$ be a s.e.s. of cochain complexes of R-modules which splits in each dimension, and F another cochain complex of R-modules. (Thus $0 \to C \otimes_R F \longrightarrow D \otimes_R F \longrightarrow E \otimes_R F \to 0$ is also exact).

Then
$$
\begin{tikzcd}
H^p(E) \otimes_R H^q(F) \arrow[r, "\delta \otimes 1"] \arrow[d, "\mu"'] & H^{p+1}(C) \otimes H^q(F) \arrow[d, "\mu"] \\
H^{p+q}(E \otimes_R F) \arrow[r, "\bar{\delta}"'] & H^{p+q+1}(C \otimes_R F)
\end{tikzcd}
$$

commutes, where δ and $\bar{\delta}$ are the appropriate connecting homomorphisms.

Proof: Let $a = [z] \in H^p(E)$, $b = [w] \in H^q(F)$.
Recall: $\mu(a \otimes b) = [z \otimes w]$.
To obtain δa:

$$
\begin{tikzcd}
 & x \arrow[r, mapsto] \arrow[dddd, bend left, mapsto] & z \\
 & D^p \arrow[r] \arrow[d, "\delta"'] & E^p \\
 & & \\
C^{p+1} \arrow[r] & D^{p+1} & \\
y \arrow[r, mapsto] & \delta x &
\end{tikzcd}
\qquad \delta a = [y].
$$

Then $\mu((\delta a) \otimes b) = [y \otimes w]$.

To obtain $\bar{\delta}\,\mu(a \otimes b)$.

$$
\begin{tikzcd}
 & x \otimes w \arrow[r, mapsto] \arrow[ddd, bend left, mapsto] & z \otimes w \\
 & (D \otimes F)^{p+q} \arrow[r] \arrow[d, "\bar{\delta}"'] & (E \otimes F)^{p+q} \\
(C \otimes F)^{p+q+1} \arrow[r] & (D \otimes F)^{p+q+1} & \\
 & (\delta x) \otimes w \pm x \otimes \delta w & \\
 & \shortparallel & \\
y \otimes w \arrow[r, mapsto] & (\delta x) \otimes w & \text{since } \delta w = 0
\end{tikzcd}
$$

(same y as above). Hence $\bar{\delta}\,\mu(a \otimes b) = [y \otimes w] = \mu(\delta \otimes 1)(a \otimes b)$.

Thm: (X,A) admiss., Y a top. space, R a PID. Then

$$\begin{tikzcd}
H^p(A;R) \otimes_R H^q(Y;R) \arrow[r, "\delta\otimes 1"] \arrow[d, "\bar{u}"'] & H^{p+1}(X,A;R) \otimes_R H^q(Y;R) \arrow[d, "\bar{u}"] \\
H^{p+q}(A\times Y;R) \arrow[r, "\bar{\delta}"'] & H^{p+q+1}((X,A)\times Y;R)
\end{tikzcd}$$

commutes where δ is the conn. hom. in coh. seq. of (X,A), $\bar{\delta}$ conn. hom. in coh. seq. of $(X\times Y, A\times Y)$.

Proof: Have diagram (all coeffs in R, ~~$\otimes_R$~~)

$$\begin{tikzcd}
H^p(A)\otimes_R H^q(Y) \arrow[rrr, "\delta\otimes 1"] \arrow[dr, "u"] \arrow[ddd, "\bar{u}"'] & & & H^{p+1}(X,A)\otimes_R H^q(Y) \arrow[dl, "u"'] \arrow[ddd, "\bar{u}"] \\
& H^{p+q}(Q^*(A;R)\otimes_R Q^*(Y;R)) \arrow[r, "\delta'"] \arrow[d, "\eta^*"'] \arrow[r, phantom, "3", shift left=6ex] & H^{p+q+1}(Q^*(X,A;R)\otimes_R Q^*(Y;R)) \arrow[d, "\eta^*"] & \\
& H^{p+q}(\mathrm{Hom}_{\mathbb{Z}}(Q(A)\otimes_{\mathbb{Z}} Q(Y),R)) \arrow[r, "\delta''"] \arrow[r, phantom, "4", shift left=6ex] & H^{p+q+1}(\mathrm{Hom}_{\mathbb{Z}}(Q(X,A)\otimes_{\mathbb{Z}} Q(Y),R)) & \\
H^{p+q}(A\times Y) \arrow[rrr, "\bar{\delta}"'] \arrow[ur, "\theta^*", "\cong"'] & & & H^{p+q+1}((X,A)\times Y) \arrow[ul, "\theta^*"', "\cong"]
\end{tikzcd}$$

Here δ is conn. hom. arising from s.e.s.

(I) $0 \to Q^*(X,A;R) \to Q^*(X;R) \to Q^*(A;R) \to 0$,

δ' arises from

(II) ~~$\otimes$~~ (I) $\otimes\, Q^*(Y;R)$

δ'' arises from

(III) $0 \to \mathrm{Hom}(Q(X,A)\otimes Q(Y),R) \to \mathrm{Hom}(Q(X)\otimes Q(Y),R) \to \mathrm{Hom}(Q(A)\otimes Q(Y),R) \to 0$

$\bar{\delta}$ arises from

(IV) $0 \to Q^*((X,A)\times Y;R) \to Q^*(X\times Y;R) \to Q^*(A\times Y;R) \to 0$

1 + 2 commute by def. of $\bar{u}$. 3 commutes by lemma.

4 commutes since η is a morphism from (II) to (III).

5 commutes since $\mathrm{Hom}(\theta,1)$ is a morphism from (IV) to (III).

Note: In terms of elements, $(\delta a)\times b = \bar{\delta}(a\times b)$.

Rel. Mayer-Vietoris Sequence

Suppose G_1, G_2 are subgroups of an abel. group G, H_i a subgrp. of G_i, $i=1,2$. Consider homomorphism

$$\frac{G_1}{H_1}\oplus\frac{G_2}{H_2}\xrightarrow{\ p\ }\frac{G_1+G_2}{H_1+H_2}$$

given by $\quad p(g_1+H_1,\ g_2+H_2) = g_1+g_2+(H_1+H_2)$.

p is clearly onto.
Easily checked: $\ker p = \{(g+H_1, -g+H_2) \mid g\in G_1\cap G_2\}$.
Let $\bar{i}: G_1\cap G_2 \longrightarrow \frac{G_1}{H_1}\oplus\frac{G_2}{H_2}$ be given by

$\bar{i}(g) = (g+H_1, -g+H_2)$. Then $\operatorname{im}\bar{i} = \ker p$.
Easily checked: $\ker\bar{i} = H_1\cap H_2$. Thus, passing to quotients, $\bar{i}$ induces a monomorphism

$$i: \frac{G_1\cap G_2}{H_1\cap H_2}\longrightarrow\frac{G_1}{H_1}\oplus\frac{G_2}{H_2}$$ and we have an

exact seq.

$$0\longrightarrow\frac{G_1\cap G_2}{H_1\cap H_2}\xrightarrow{\ i\ }\frac{G_1}{H_1}\oplus\frac{G_2}{H_2}\xrightarrow{\ p\ }\frac{G_1+G_2}{H_1+H_2}\longrightarrow 0$$

where $i[g] = ([g],[-g])$, $\quad p([g_1],[g_2]) = [g_1+g_2]$.

Now suppose $X = X_1\cup X_2$, $A = A_1\cup A_2$, $A_i\subset X_i$, X_i, A_i all open in X. Above yields a s.e.s. of chain complexes

$$0\to\frac{Q(X_1\cap X_2)}{Q(A_1\cap A_2)}\longrightarrow\frac{Q(X_1)}{Q(A_1)}\oplus\frac{Q(X_2)}{Q(A_2)}\longrightarrow\frac{Q(X_1)+Q(X_2)}{Q(A_1)+Q(A_2)}\longrightarrow 0.$$

10 Note: Complex on right is free in each dim. since $Q(A_1)+Q(A_2)$ has a basis consisting of a portion of a basis of $Q(X_1)+Q(X_2)$. Hence, above s.e.s. splits in each dimension.

By the main lemma in proving the excision property for homology (small cube theorem), the inclusions

$Q(X_1)+Q(X_2)\to Q(X)$, $Q(A_1)+Q(A_2)\to Q(A)$ induce isomorphisms in homology. Hence, passage to quotients & 5 lemma yields

$$\frac{Q(X_1)+Q(X_2)}{Q(A_1)+Q(A_2)} \longrightarrow \frac{Q(X)}{Q(A)} = Q(X,A)$$

induces isoms in homology, and hence in cohomology by U.C.T. Thus, for arb. coeffs., obtain a natural exact seq.

$$\to H^n(X,A) \xrightarrow{(j_1^*,j_2^*)} H^n(X_1,A_1)\oplus H^n(X_2,A_2) \xrightarrow{i_1^*-i_2^*} H^n(X_1\cap X_2, A_1\cap A_2) \xrightarrow{\Delta} H^{n+1}(X,A)\to$$

called the relative Mayer-Vietoris sequence of the excisive triad $((X,A);(X_1,A_1),(X_2,A_2))$.

Exercise: The M.V. seq. of $((X,A);(X,\emptyset),(A,A))$ is the coh. seq. of the pair (X,A).

If $X\supset A\supset B$, the M.V. seq of $((X,A);(X,B),(A,A))$ is the coh. seq. of the triple (X,A,B).

Lemma: $(X;X_1,X_2)$ an excisive triad. Then the connecting hom. $\Delta: H^n(X_1\cap X_2)\to H^{n+1}(X)$ in the M.V seq of above triad is the comp.

$$H^n(X_1\cap X_2)\xrightarrow{\delta} H^{n+1}(X_1,X_1\cap X_2)\xrightarrow{j^{*-1}} H^{n+1}(X,X_2)\xrightarrow{i^*} H^{n+1}(X)$$

where i,j are the inclusions (j^* is an isom. by excision) and δ is the conn. hom. in the coh. seq. of the pair $(X_1,X_1\cap X_2)$. (arb. coeffs.).

Proof: Write $X_{12}=X_1\cap X_2$. The inclusions of triads

$$\begin{gathered}(X;X_1,X_2)\\ \cap\\ ((X,X_2);(X_1,\emptyset),(X_2,X_2))\\ \cup\\ ((X_1,X_{12});(X_1,\emptyset),(X_{12},X_{12}))\end{gathered}$$

and nat. of M.V. seq. yields the comm. diag.

$$\begin{tikzcd}
H^n(X_1\cap X_2) \arrow[r, "\Delta"] & H^{n+1}(X) \\
H^n(X_1\cap X_2) \arrow[u, "1"] \arrow[r, "\Delta"] \arrow[d, "="'] & H^{n+1}(X, X_2) \arrow[u, "i^*"'] \arrow[d, "j^*"] \\
H^n(X_1\cap X_2) \arrow[r, "\delta"'] & H^{n+1}(X_1, X_{12}).
\end{tikzcd}$$

Thm: $(X; X_1, X_2)$ an admiss. triad, (Y,B) an admiss. pair. Then for any PID R,

$$\begin{tikzcd}
H^p(X_{12}; R)\otimes_R H^q(Y,B; R) \arrow[r, "\Delta\otimes 1"] \arrow[d, "\bar{\mu}"'] & H^{p+1}(X; R)\otimes_R H^q(Y,B;R) \arrow[d, "\bar{\mu}"] \\
H^{p+q}(X_{12}\times(Y,B); R) \arrow[r, "\bar{\Delta}"'] & H^{p+q+1}(X\times(Y,B); R)
\end{tikzcd}$$

commutes, where Δ is the conn. hom. in the M.V. seq. of given triad, and $\bar{\Delta}$ for that of $(X\times(Y,B); X_1\times(Y,B), X_2\times(Y,B))$.

Proof: Case 1: $B=\phi$. Consider diagram

$$\begin{tikzcd}
H^p(X_{12})\otimes_R H^q(Y) \arrow[rrr, "\Delta\otimes 1"] \arrow[dr, "\delta\otimes 1"] \arrow[ddd, "\bar{\mu}"'] & & & H^{p+1}(X)\otimes_R H^q(Y) \arrow[ddd, "\bar{\mu}"] \\
& H^{p+1}(X_1, X_{12})\otimes_R H^q(Y) \arrow[r, "j^{*-1}\otimes 1"] \arrow[d, "\bar{\mu}"] \arrow[ur, phantom, "1"] & H^{p+1}(X, X_2)\otimes_R H^q(Y) \arrow[ur, "i^*\otimes 1"] \arrow[d, "\bar{\mu}"] & \\
\arrow[r, phantom, "3"] & H^{p+q+1}((X_1, X_{12})\times Y) \arrow[r, "\bar{J}^{*-1} = (j\times 1)^{*-1}"'] \arrow[ur, phantom, "4"] & H^{p+q+1}((X, X_2)\times Y) \arrow[dr, "(i\times 1)^* = \bar{i}^*"] \arrow[r, phantom, "5"] & \\
H^{p+q}(X_{12}\times Y) \arrow[ur, "\bar{\delta}"] \arrow[rrr, "\bar{\Delta}"] \arrow[rrr, phantom, bend left=20, "2"] & & & H^{p+q+1}(X\times Y)
\end{tikzcd}$$

1 and 2 commute by lemma. 3 commutes by compat. of $\bar{\mu}$ with conn. hom. for coh. seq. of pairs. 4 + 5 commute by nat. of $\bar{\mu}$.

Case 2: General case. Let $i : (Y, B) \to (Y \cup cB, cB)$, $j : (Y \cup cB, \emptyset) \to (Y \cup cB, cB)$ denote the inclusions.
Consider diagram

$$\begin{tikzcd}[column sep=small]
H^p(X_{12}) \otimes H^q(Y,B) \arrow[rrrrr, "\Delta\otimes 1"] \arrow[ddddd, "\bar{u}"'] & & & & & H^{p+1}(X) \otimes H^q(Y,B) \arrow[ddddd, "\bar{u}"] \\
& H^p(X_{12}) \otimes H^q(Y\cup cB, cB) \arrow[ul, "1\otimes i^*", "\cong"'] \arrow[rrr, "\Delta\otimes 1"] \arrow[dr, "1\otimes j^*"] \arrow[ddd, "\bar{u}"'] & & & H^{p+1}(X) \otimes H^q(Y\cup cB, cB) \arrow[ur, "1\otimes i^*"', "\cong"] \arrow[dl, "1\otimes j^*"'] \arrow[ddd, "\bar{u}"] & \\
& & H^p(X_{12}) \otimes H^q(Y\cup cB) \arrow[r, "\Delta\otimes 1"] \arrow[d, "\bar{u}"] & H^{p+1}(X) \otimes H^q(Y\cup cB) \arrow[d, "\bar{u}"] & & \\
& & H^{p+q}(X_{12} \times (Y\cup cB)) \arrow[r, "\bar{\Delta}"] & H^{p+q+1}(X \times (Y\cup cB)) & & \\
& H^{p+q}(X_{12} \times (Y\cup cB, cB)) \arrow[ur, "(1\times j)^*"] \arrow[rrr, "\bar{\Delta}"] \arrow[dl, "(1\times i)^*", "\cong"'] & & & H^{p+q+1}(X \times (Y\cup cB, cB)) \arrow[ul, "(1\times j)^*\ (-1)\ (\text{earlier lemma})"'] \arrow[dr, "(1\times i)^*"', "\cong\ \text{excis + def'n}"] & \\
H^{p+q}(X_{12} \times (Y,B)) \arrow[rrrrr, "\bar{\Delta}"] & & & & & H^{p+q+1}(X \times (Y,B))
\end{tikzcd}$$

1, 2, 3, 4 commute by nat. of $\bar{u}$.

5, 6 commute by direct check.

7, 8 commute by nat. of $\bar{\Delta}$ in M. V. seq.

9 commutes by case 1.

Cor: $(X; X_1, X_2)$ an admiss. triad, A open in X. Write $X_{12} = X_1 \cap X_2$, $A_i = X_i \cap A$, $i = 1, 2$, $A_{12} = A_1 \cap A_2 = X_{12} \cap A$.
R a PID, $a \in H^q(X, A; R)$. Then following commutes (coeffs in R)

$$\begin{tikzcd}
\cdots \arrow[r] & H^p(X_{12}) \arrow[r, "\Delta"] \arrow[d, "\cup\, i_{12}^* a"'] \arrow[dr, phantom, "3"] & H^{p+1}(X) \arrow[r] \arrow[d, "\cup\, a"] \arrow[dr, phantom, "2"] & H^{p+1}(X_1) \oplus H^{p+1}(X_2) \arrow[r] \arrow[d, "\cup\, i_1^* a \,\oplus\, \cup\, i_2^* a"] \arrow[dr, phantom, "1"] & H^{p+1}(X_{12}) \arrow[r] \arrow[d, "\cup\, i_{12}^* a"] & \cdots \\
\cdots \arrow[r] & H^{p+q}(X_{12}, A_{12}) \arrow[r, "\Delta'"'] & H^{p+q+1}(X, A) \arrow[r] & H^{p+q+1}(X_1, A_1) \oplus H^{p+q+1}(X_2, A_2) \arrow[r] & H^{p+q+1}(X_{12}, A_{12}) \arrow[r] & \cdots
\end{tikzcd}$$

where $i_j : (X_j, A_j) \to (X, A)$, $i_{12} : (X_{12}, A_{12}) \to (X, A)$ are the inclusions, and the rows are the approp. M.V. sequences.

<u>Proof</u>: ① , ② follow from nat. of cup prod.
Comm. of ③:

$$\begin{tikzcd}
 & H^p(X_{12}) \arrow[dl, "\times i_{12}^* a"'] \arrow[d, "\times a"] \arrow[rr, "\Delta"] \arrow[drr, phantom, "④"] & & H^{p+1}(X) \arrow[d, "\times a"] & (X; X_1, X_2) \\
H^{p+q}(X_{12}\times(X_{12},A_{12})) \arrow[dr, "d^*"'] & H^{p+q}(X_{12}\times(X,A)) \arrow[l, "(1\times i_{12})^*"'] \arrow[d, "d^*"] \arrow[rr, "\bar\Delta"] & & H^{p+q+1}(X\times(X,A)) \arrow[d, "d^*"] & \text{above} \times (X,A) \\
 & H^{p+q}(X_{12},A_{12}) \arrow[rr, "\Delta'"] & & H^{p+q+1}(X,A) & ((X,A);(X_1,A_1),(X_2,A_2)) \arrow[u, "d"']
\end{tikzcd}$$

b, Δb, $(\Delta b)\times a$, $(\Delta b)\smile a$, $b \times i_{12}^* a$, $b \smile i_{12}^* a$, $\Delta'(b \smile i_{12}^* a)$

④ commutes by previous thm. Other regions commute by nat.

<u>Vector Bundles</u>, <u>Orientations</u>, <u>Thom Isomorphism</u>

If V is a real n-dim. vector space, write $V^0 = V - \{0\}$. Then for any PID R, $H^n(V, V^0; R) \cong R$.

<u>Def</u>: An <u>R-orientation</u> of V is a choice of R-module generator u of $H^n(V, V^0; R)$.

Thus every V has exactly 2 $\mathbb{Z}$-orientations, 1 $\mathbb{Z}/2$ orientation (in general, many R-orientations).

Now consider a real n-plane bundle $p : E \to B$. For each $x \in B$, write $E_x = p^{-1}(x) =$ fibre over x. <u>Recall</u>: E_x has the structure of a real n-dim. vector space.

Let $E^0 = \bigcup_{x\in B} E_x^0 \subset E$. Let $i_x : (E_x, E_x^0) \subset (E, E^0)$ denote the inclusion.

<u>Def</u>: An <u>R-orientation</u> of the vector bundle $E \xrightarrow{p} B$ is an element $U \in H^n(E, E^0; R)$ $\ni$ $i_x^*(U)$ is an R-orient. of E_x for all $x \in B$.

<u>Note</u>: R-orientations do not always exist.

If P is a point, and $E \to P$ the essentially unique n-plane bundle, then $i_P : (E_P, E_P^0) \to (E, E^0)$ is the identity map, and so $E \to P$ is R-orientable.

Def. $E \xrightarrow{p} B$, $E' \xrightarrow{p'} B'$ n-plane bdles. A morphism f from 1st to 2nd consists of a pair of cont. maps $f_E : E \to E'$, $f_B : B \to B'$ $\ni$:

$$1)\quad \begin{tikzcd} E \arrow[r, "f_E"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f_B"'] & B' \end{tikzcd} \quad \text{commutes}$$

2) For each $x \in B$, f_E maps E_x isomorphically onto $E'_{f_B(x)}$.

Can form cat. of real n-plane bdles with morphisms as above. (Note: More general morphisms of vector bundles are sometimes useful, e.g. in differential geometry).

If f is a morphism from $E \xrightarrow{p} B$ to $E' \xrightarrow{p'} B'$, then $f_E(E^0) \subset (E')^0$ and so we get a map of top. pairs $\bar{f} : (E, E^0) \to (E', E'^0)$. Moreover, for each $x \in B$, f yields a homeo of pairs $\bar{f}_x : (E_x, E_x^0) \to (E'_{f_B(x)}, E'^0_{f_B(x)})$ $\ni$:

$$\begin{tikzcd} (E_x, E_x^0) \arrow[r, "\bar{f}_x"] \arrow[d, "i_x"'] & (E'_{f_B(x)}, E'^0_{f_B(x)}) \arrow[d, "i_{f_B(x)}"] \\ (E, E^0) \arrow[r, "\bar{f}"'] & (E', E'^0) \end{tikzcd}$$

commutes.

Prop: Let f be a morphism of n-plane bdles from $E \xrightarrow{p} B$ to $E' \xrightarrow{p'} B'$. Suppose U is an R-orientation of $E' \xrightarrow{p'} B'$. Then $\bar{f}^*(U)$ is an R-orient. of $E \xrightarrow{p} B$.

Proof: For each $x \in B$, $i_x^* \bar{f}^*(U) = \bar{f}_x^* i_{f_B(x)}^*(U)$ by commutativity of above. $i_{f_B(x)}^*(U)$ is an R-orient of $E'_{f_B(x)}$ since U is an R-orient. of $E' \xrightarrow{p'} B'$. Hence, since $\bar{f}_x$ is a homeo. of pairs,

$\bar{f}_x^{\,*}\, i^*_{f_B(x)}(U)$ is an R-orient. of E_x.

Cor: Any trivial n-plane bundle admits an R-orient. for any R.

Proof: If $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ trivial, $\exists$ morphism of n-pl. bundles

$$\begin{tikzcd} E \arrow[d, "p"] \arrow[r, "f"] & B\times\mathbb{R}^n \arrow[d, "\pi_1"] \\ B & B \end{tikzcd} \quad \text{with } f_B = 1_B.$$

We have a morphism of n-pl. bundles

$$\begin{tikzcd} B\times\mathbb{R}^n \arrow[d, "\pi_1"] \arrow[r, "g"] & \mathbb{R}^n \arrow[d] \\ B & P = pt. \end{tikzcd}$$

namely $g_B = \text{const.}$, $g_E = \pi_2$. Result now follows since $\begin{array}{c} \mathbb{R}^n \\ \downarrow \\ P \end{array}$ admits an R-orient.

Note: If $\begin{array}{c} E \\ p\downarrow \\ B \end{array}$ an n-pl. bundle and $X \subset B$, then the restriction $\begin{array}{c} E|X = p^{-1}(X) \\ \downarrow \\ X \end{array}$ is an n-plane bundle, and the inclusions

$$\begin{tikzcd} p^{-1}(X) \arrow[d] \arrow[r, phantom, "\subset"] & E \arrow[d, "p"] \\ X \arrow[r, phantom, "\subset"] & B \end{tikzcd}$$

constitute a morphism of n-plane bundles. In particular, if U is an R-orientation of $\begin{array}{c} E \\ p\downarrow \\ B \end{array}$ and $i: X \hookrightarrow B$ the inclusion, $i^*(U)$ is an R-orient. of $\begin{array}{c} p^{-1}(X) = E|X \\ \downarrow \\ X \end{array}$, the R-orientation <u>induced</u> by U.

Def: Let k be a pos. integer. A vector bdle $\begin{matrix} E \\ \downarrow p \\ B \end{matrix}$ is

k-presentable if it admits an atlas with $\le k$ members.
finitely-presentable if k-presentable for some finite k.

Thus if B compact, every vector bdle over B is finitely-presentable.

Thm (Thom isom. thm): $E \xrightarrow{p} B$ a finitely-presentable real n-plane bdle, R a PID. U an R-orient. of E. Then $\forall i$, $\Psi : H^i(E; R) \longrightarrow H^{i+n}(E, E^0; R)$ given by $\Psi(x) = x \smile U$ is an isom.

Proof: Induction on the presentability of E. Suppose E is 1-presentable. Then can identify
$(E, E^0) = B \times (V, V^0)$ for some n-dim. vector space V

Case 1': B path-conn. All coh. with R-coeffs. By the coh. Künneth thm. proved earlier,

$$\mu : H^0(B) \underset{R}{\otimes} H^n(V, V^0) \longrightarrow H^n(E, E^0) \text{ is an isom.}$$

Since $H^0(B) \cong R$ gen by 1 (since B is path-conn.), Can write $U = 1 \times \bar{U}$ for a unique [redacted] $\bar{U}$ [redacted] $H^n(V, V^0)$. Note that $\bar{U}$ is an R-module gen. of $H^n(V, V^0)$, for given any $x \in B$, the composition

$$(E_x, E_x^0) \xrightarrow{i_x} (E, E^0) \xrightarrow{\pi_2} (V, V^0) \text{ is a homeo.}$$

of pairs, and $(\pi_2 i_x)^*(\bar{U}) = i_x^* \pi_2^*(\bar{U}) = i_x^*(1 \times \bar{U}) = i_x^*(U) =$ orient. of E_x.

By the Künneth thm., for every i,

$$\mu : H^i(B) \underset{R}{\otimes} H^n(V, V^0) \to H^{n+i}(E, E^0) \text{ is an isom.} \quad \text{Thus}$$

$\bar{\Psi} : H^i(B) \longrightarrow H^{n+i}(E, E^0)$ given by $\bar{\Psi}(a) = a \times \bar{U}$ is an isom.
The diagram

$$\begin{tikzcd} H^i(B) \arrow[d, "\pi_1^*"'] \arrow[dr, "\overline{\Psi}", "\cong"'] & \\ H^i(E) \arrow[r, "\Psi"'] & H^{n+i}(E, E^0) \end{tikzcd}$$

commutes, for $\Psi\pi_1^*(a) = \Psi(a\times 1) = (a\times 1)\cup U$

$$= (a\times 1)\cup(1\times \overline{U}) = a\times \overline{U} = \overline{\Psi}(a).$$

Since π_1 is a homotopy equivalence, π_1^* is an isom., so Ψ is an isom., proving case 1'.

12 General 1-presentable case:

Write $B = \bigcup_\alpha B_\alpha$, B_α the path-components of B.

Let $i_\alpha : B_\alpha\times V \to B\times V$, $j_\alpha : B_\alpha\times(V,V^0) \to B\times(V,V^0)$ denote the inclusions. Write $U_\alpha = j_\alpha^*(U)$. Then U_α is an R-orient. of $B_\alpha\times V \to B_\alpha$, so by case 1',

$\Psi_\alpha : H^i(B_\alpha\times V) \to H^{i+n}(B_\alpha\times(V,V^0))$ is an isom

$x \longmapsto x\cup U_\alpha$

Following commutes

$$\begin{tikzcd} H^i(B\times V) \arrow[r, "\Psi"] \arrow[d, "(i_\alpha^*)"'] & H^{n+i}(B\times(V,V^0)) \arrow[d, "(j_\alpha^*)"] \\ \prod_\alpha H^i(B_\alpha\times V) \arrow[r, "\prod_\alpha \Psi_\alpha"'] & \prod_\alpha H^{n+i}(B_\alpha\times(V,V^0)), \end{tikzcd}$$

for $j_\alpha^*\Psi(x) = j_\alpha^*(x\cup U)$

$= i_\alpha^* x \cup j_\alpha^* U$ (map of admiss. triads $(B_\alpha\times V; \phi, B_\alpha\times V^0) \to (B\times V; \phi, B\times V^0)$)

$= i_\alpha^* x\cup U_\alpha = \Psi_\alpha i_\alpha^*(x)$.

Bottom map is $\cong$ since each Ψ_α is. Vertical maps are isoms since $\bigoplus_\alpha Q(B_\alpha\times V) \xrightarrow{\Sigma Q(i_\alpha)} Q(B\times V)$ and

$\bigoplus_\alpha Q(B_\alpha\times(V,V^0)) \xrightarrow{\Sigma Q(j_\alpha)} Q(B\times(V,V^0))$ are isomorphisms of ch. comp.

Now suppose $k > 1$ and that theorem holds for $k-1$-presentable n-plane bundles. Can write $B = B_1 \cup B_2$, B_i open in B $\ni$ $E \mid B_i$ is $k-1$-presentable, $i = 1, 2$. Follows that

$E \mid B_1 \cap B_2$ is also $k-1$-presentable. Write $B_{12} = B_1 \cap B_2$ and let U_1, U_2, U_{12} denote the R-orient. of $E|B_1$, $E|B_2$, $E|B_{12}$, resp. induced by U. Write $(E_\alpha, E_\alpha^\circ) = (E|B_\alpha, (E|B_\alpha)^\circ)$, $\alpha = 1, 2, 12$. Have comm. diag. with exact rows

$$\begin{tikzcd}[column sep=small]
\arrow[r] & H^{i-1}(E_1) \oplus H^{i-1}(E_2) \arrow[r] \arrow[d, "\smile U_1 \oplus \smile U_2"'] & H^{i-1}(E_{12}) \arrow[r] \arrow[d, "\smile U_{12}"'] & H^i(E) \arrow[r] \arrow[d, "\smile U"] & H^i(E_1) \oplus H^i(E_2) \arrow[r] \arrow[d, "\smile U_1 \oplus \smile U_2"] & H^i(E_{12}) \arrow[d, "\smile U_{12}"] \\
\arrow[r] & H^{n+i-1}(E_1, E_1^\circ) \oplus H^{n+i-1}(E_2, E_2^\circ) \arrow[r] & H^{n+i-1}(E_{12}, E_{12}^\circ) \arrow[r] & H^{n+i}(E, E^\circ) \arrow[r] & H^{n+i}(E_1, E_1^\circ) \oplus H^{n+i}(E_2, E_2^\circ) \arrow[r] & H^{n+i}(E_{12}, E_{12}^\circ)
\end{tikzcd}$$

whose rows are M.V. sequences. $\smile U_\alpha$ is an isom. for $\alpha = 1, 2, 12$ by ind. hyp. Hence by 5-lemma, $\smile U$ is an isom., i.e. Ψ is an isom.

<u>Def</u>: $\begin{matrix} E \\ p \downarrow \\ B \end{matrix}$ a real n-plane bdle, $\{B_\alpha\}$ an open cover of B,

U_α an R-orient. of $E \mid B_\alpha$ for each α. The U_α are said to be <u>compatible</u> if whenever $x \in U_\alpha \cap U_\beta$, if

$$\begin{tikzcd}
 & (E|B_\alpha, (E|B_\alpha)^\circ) \\
(E_x, E_x^\circ) \arrow[ur, "i_{x,\alpha}"] \arrow[dr, "i_{x,\beta}"'] & \\
 & (E|B_\beta, (E|B_\beta)^\circ)
\end{tikzcd}$$

are the inclusions, then $i_{x,\alpha}^* U_\alpha = i_{x,\beta}^* U_\beta$.

<u>Lemma</u>: E a fin. pres. R-orientable n-pl. bdle over B. Suppose $a \in H^n(E, E^\circ, R)$ is such that $i_x^*(a) = 0$ for one x in each path-component of B. Then $a = 0$.

<u>Proof</u>: Case 1: B path-conn. By Thom isom. thm, $H^n(E, E^\circ; R) \cong R$, gen. by U, so $a = rU$ for some $r \in R$. Hence $0 = i_x^* a = r i_x^* U$. Since $i_x^* U$ is a free gen. of $H^n(E_x, E_x^\circ, R)$ as

an R-module, must have $r=0$, so $a=0$.

General case: Let $\{B_\alpha\}$ be the path-components of B, $E_\alpha = E|B_\alpha$, $j_\alpha : (E_\alpha, E_\alpha^\circ) \to (E, E^\circ)$ inclusion. By hyp., $\exists\, x_\alpha \in B_\alpha$ for each α $\ni$ $i_{x_\alpha}^* a = 0$. Have comm. diag.

$$\begin{tikzcd} & (E, E^\circ) \\ (E_{x_\alpha}, E_{x_\alpha}^\circ) \arrow[ur, "i_{x_\alpha}"] \arrow[dr, "i'_{x_\alpha}"'] & \\ & (E_\alpha, E_\alpha^\circ) \arrow[uu, "j_\alpha"'] \end{tikzcd}$$

and so $i'^*_{x_\alpha} j_\alpha^* a = 0$. By case 1, $j_\alpha^* a = 0$ for each α.

Since $(j_\alpha^*) : H^n(E, E^\circ; R) \to \prod_\alpha H^n(E_\alpha, E_\alpha^\circ; R)$ is an isom.,

$\alpha = 0$.

Thm: E an n-pl. bdle over B, fin. pres., $\{B_i\}$ a finite open cover of B, R a PID, $\{U_i\}$ a compatible set of R-orient. of $\{E|B_i\}$. Then $\exists$ unique R-orient. U of E $\ni$ U_i is the R-orient. induced by U for each i.

Proof. Uniqueness: If U, V are 2 such R-orient., would have $i_x^* U = i_x^* V$ $\forall\, x \in B$, so $U = V$ by lemma.

Existence: Ind. on no. k of elts. in cover. Trivial if $k=1$. Assume $k>1$ and thm holds for covers with $<k$ elts. Write $B = A_1 \cup A_2$ where $A_1 = B_1$, $A_2 = B_2 \cup \cdots \cup B_k$. By ind. hyp., $\exists$ unique R-orient. U_2' of $E|A_2$ $\ni$ U_2' induces U_i for $2 \le i \le k$. U_1 and U_2' are compatible, for if $x \in A_1 \cap A_2$, say $x \in B_1 \cap B_i$.

$$\begin{tikzcd} & (E|B_i, (E|B_i)^\circ) \arrow[d, "j"] \\ (E_x, E_x^\circ) \arrow[ur, "i_{B_i}"] \arrow[r, "i'"'] & (E|A_2, (E|A_2)^\circ) \end{tikzcd}$$ commutes,

so $i'^* U_2' = i_{B_i}^* j^* U_2' = i_{B_i}^* U_i = i_{B_1}^* U_1$ since the U_i are compatible.

Write $E_i = E|A_i$, $i = 1, 2$, $E_{12} = E|A_1 \cap A_2$. Have M.V. seq. with coeffs. in R

$$H^n(E,E^\circ) \xrightarrow[(j_1^*,\ j_2^*)]{} H^n(E_1,E_1^\circ) \oplus H^n(E_2,E_2^\circ) \xrightarrow[i_1^* - i_2^*]{} H^n(E_{12},E_{12}^\circ).$$

$i_1^* U_1$, $i_2^* U_2'$ are R-orient. of $E|A_1 \cap A_2$ restricting to same orient. on each fibre since U_1, U_2' are compatible. Hence by lemma, $i_1^* U_1 - i_2^* U_2' = 0$. By exactness, $\exists\ U \in H^n(E,E^\circ) \ni$: $j_1^* U = U_1$, $j_2^* U = U_2'$. Easily checked: U is an R-orient. indexing U_1, U_2', and hence indexing $U_1, \dots, U_m$ (since U_2' indexes $U_2, \dots, U_n$).

Cor: E a fin. pres. n-pl. bdle. Then E has a unique $\mathbb{Z}/2$ orient.

Proof: $H^n(E_x, E_x^\circ; \mathbb{Z}/2)$ has only one generator, so any collection of $\mathbb{Z}/2$ orient. rel. to any open cover of base space is compatible. If $\{B_\alpha\}$ is a finite open cover of base space corresp. to an atlas, then $E|B_\alpha$ has a $\mathbb{Z}/2$ orient. for each α. Hence, by thm. E has a $\mathbb{Z}/2$ orient. Uniqueness: immediate from lemma.

13 Prop: $\begin{matrix} E \\ p\downarrow \\ B \end{matrix}$ the underlying real $2n$-plane bdle of a fin. pres. complex n-plane bdle. Then $\begin{matrix} E \\ p\downarrow \\ B \end{matrix}$ is R-orientable for every R.

Proof. Let $\{h_i : p^{-1}(B_i) = E_i \to B_i \times \mathbb{C}^n\}$ be a finite atlas for above complex n-pl. bdle. Let $u \in H^{2n}(\mathbb{C}^n, (\mathbb{C}^n)^\circ; R)$ be any R-module generator, and let $U_i = \bar{h}_i^*(1 \times u) \in H^{2n}(E_i, E_i^\circ; R)$ $(\bar{h}_i : (E_i, E_i^\circ) \to B_i \times (\mathbb{C}^n, (\mathbb{C}^n)^\circ))$. For each $x \in B_i$, let $g_{i,x}$ denote the composition

$$E_x \xrightarrow{i_{i,x}} E_i \xrightarrow{h_{i,x}} B_i \times \mathbb{C}^n \xrightarrow{\pi_2} \mathbb{C}^n.$$

Then $g_{i,x}$ is an isom. of complex vector spaces, and we have

$\bar{i}_{i,x}^*(U_i) = \bar{i}_{i,x}^* \bar{h}_{i,x}^*(1 \times u) = \bar{i}_{i,x}^* \bar{h}_{i,x}^* \pi_2^*(u) = \bar{g}_{i,x}^*(u) =$ orient. of E_x since $\bar{g}_{i,x}$ is a homeo. of pairs. Hence U_i is an R-orient. of $E|B_i$. Suff. to show: $\{U_i\}$ is compatible.

Suppose $x \in B_i \cap B_j$. Suff. to show composition $(\mathbb{C}^n, (\mathbb{C}^n)^\circ) \xrightarrow{\bar{g}_{i,x}^{-1}} E_x \xrightarrow{\bar{g}_{j,x}} (\mathbb{C}^n, (\mathbb{C}^n)^\circ)$ is $\simeq 1$, for then

$$\bar{\iota}_{i,x}^*(U_i) = \bar{g}_{i,x}^*(u) = \bar{g}_{i,x}^*\left(\bar{g}_{j,x}\bar{g}_{i,x}^{-1}\right)^*(u) = \left(\bar{g}_{j,x}\bar{g}_{i,x}^{-1}\bar{g}_{i,x}\right)^*(u)$$

$$= \bar{g}_{j,x}^*(u) = \bar{\iota}_{j,x}(U_j).$$

Now $g_{j,x}g_{i,x}^{-1} \in GL(n,\mathbb{C})$. Suff to show $GL(n,\mathbb{C})$ is path-conn., for then if $t \longmapsto \alpha(t)$ is a path from I_n ($n\times n$ identity matrix) to $g_{j,x}g_{i,x}^{-1}$, then

$$(\mathbb{C}^n, \mathbb{C}^n - \{0\}) \times I \longrightarrow (\mathbb{C}^n, \mathbb{C}^n - \{0\})$$
$$(y,t) \longmapsto \alpha(t)(y)$$

would be a homotopy from 1 to $\bar{g}_{j,x}\bar{g}_{i,x}^{-1}$.

Let $A \in GL(n,\mathbb{C})$. $\exists\ T \in GL(n,\mathbb{C}) \ni: TAT^{-1}$ is upper triang. (since $\mathbb{C}$ is alg. closed). Suff to show $\exists$ path $\alpha: [0,1] \to GL(n,\mathbb{C}) \ni: \alpha(0) = I_n,\ \alpha(1) = TAT^{-1}$ (for then $t \longmapsto T^{-1}\alpha(t)T$ would be a path from I_n to A in $GL(n,\mathbb{C})$.) Can write $TAT^{-1} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) + B$ where B is upper triang. with 0's on diag., and each $\lambda_i \in \mathbb{C} - \{0\}$. Since $\mathbb{C} - \{0\}$ is path-conn., $\exists$ path $\beta_i: [0,1] \to \mathbb{C} - \{0\} \ni: \beta_i(0) = 1,\ \beta_i(1) = \lambda_i$. Define $\alpha(t) = \mathrm{diag}(\beta_1(t), \ldots, \beta_n(t)) + tB$.

Projective spaces

Let $F = \mathbb{R}$ or $\mathbb{C}$, and let V be a finite-dim. vector space over F. The <u>F-projective space of V</u>, $P_F(V)$, is the quotient space obtained from V^0 by identifying $v \sim tv$ whenever $v \in V^0$, $t \in F - \{0\}$. Write $[v] \in P_F(V)$ for the image of $v \in V^0$ under the quot. map $V^0 \longrightarrow P_F(V)$.

Can identify $[v]$ with the 1-dim. F-subspace of V spanned by v, and will write $w \in [v] \iff w = tv$ for some $t \in F$.

Thus $P_{\mathbb{R}}(\mathbb{R}^{n+1}) = \mathbb{R}P^n$, $P_{\mathbb{C}}(\mathbb{C}^{n+1}) = \mathbb{C}P^n$.

If $f: V \to W$ is an isom. of F vector spaces, f induces a homeo. $P_F(f): P_F(V) \longrightarrow P_F(W)$

$$[v] \longmapsto [f(v)]$$

Thus, if $\dim_{\mathbb{R}} V = n+1$, by HW prob 3,

$$H^i(P_{\mathbb{R}}(V); \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & \text{if } 0 \le i \le n \\ 0 & \text{otherwise.} \end{cases}$$

Recall (last semester): $H_i(\mathbb{C}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } 0 \le i \le 2n, \ i \text{ even} \\ 0 & \text{otherwise.} \end{cases}$

By U.C.T., follows that

$$H^i(\mathbb{C}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } 0 \le i \le 2n \\ 0 & \text{otherwise,} \end{cases}$$

and same will be true if $\mathbb{C}P^n$ is replaced by $P_{\mathbb{C}}(V)$ when $\dim_{\mathbb{C}} V = n+1$.

We have an F-line bundle $p: L_F(V) \to P_F(V)$ as follows:

$$L_F(V) = \{([v], w) \in P_F(V) \times V \mid w \in [v]\},$$

$p([v], w) = [v]$. Vector space structure on $L_F(V)_{[v]}$ is given by

$([v], w_1) + ([v], w_2) = ([v], w_1 + w_2)$, $t([v], w) = ([v], tw)$.
Clearly, $\{([v], v)\}$ is an F-basis for $L_F(V)_{[v]}$.

Local triviality: For each $v \in V^\circ$, let $B_v = \{[w] \in P_F(V) \mid \langle v, w\rangle \neq 0\}$.
Then $\{B_v\}_{v \in V^\circ}$ is an open cover of $P_F(V)$.
Define $h_v : p^{-1}(B_v) \to B_v \times F$ by

$h_v([w], x) = ([w], \langle v, x\rangle)$. Then $\{h_v\}_{v \in V^\circ}$ constitute an atlas for above as an F-line bundle. Above is called the <u>canonical F-line bundle</u> over $P_F(V)$.

<u>Note</u>: The map $\begin{array}{ccc} L_F(V)^\circ & \longrightarrow & V^\circ \\ ([v], w) & \longmapsto & w \end{array}$ is a homeo.

Thus, if $\dim_F V = n+1$, $L_F(V)^\circ$ has the homotopy type of

$$\begin{cases} S^n & \text{if } F = \mathbb{R} \\ S^{2n+1} & \text{if } F = \mathbb{C} \end{cases}$$

14 Assume $n \geq 1$.

Thus for any coefficients, $j^*: H^i(L_F(V), L_F(V)^\circ) \to H^i(L_F(V))$ is

$$\begin{cases} \text{isom. for } i \leq n-1, \text{ 1-1 if } i = n & \text{if } F = \mathbb{R} \\ \text{isom. for } i \leq 2n & \text{if } F = \mathbb{C}. \end{cases}$$

Thm: Let $n \geq 1$, V a real $n+1$-dim. vector space. Let $a \in H^1(P_{\mathbb{R}}(V); \mathbb{Z}/2) \cong \mathbb{Z}/2$ be the generator. Then for $1 \leq i \leq n$, $\underbrace{a \cup a \cup \cdots \cup a}_{i}$ is the generator of $H^i(P_{\mathbb{R}}(V), \mathbb{Z}/2) \cong \mathbb{Z}/2$.
(i.e. $H^*(P_{\mathbb{R}}(V), \mathbb{Z}/2) \cong \mathbb{Z}/2[a]/(a^{n+1})$.)

Proof: Suff. to show: if $i < n$ and b generates $H^i(P_{\mathbb{R}}(V); \mathbb{Z}/2)$, then $b \cup a$ generates $H^{i+1}(P_{\mathbb{R}}(V), \mathbb{Z}/2)$.

Let $U \in H^1(L_{\mathbb{R}}(V), L_{\mathbb{R}}(V)^\circ; \mathbb{Z}/2)$ denote the unique $\mathbb{Z}/2$ orientation of $L_{\mathbb{R}}(V) \xrightarrow{p} P_{\mathbb{R}}(V)$. Since $p: L_{\mathbb{R}}(V) \to P_{\mathbb{R}}(V)$ is a homotopy equivalence and $j^*(U) \neq 0$ by above, we must have $j^*(U) = p^*(a)$. By the Thom isom, $p^*(b) \cup U \neq 0$, and so $j^*(p^*(b) \cup U) \neq 0$ by above. But
$j^*(p^*(b) \cup U) = p^*(b) \cup j^*(U) = p^*(b) \cup p^*(a) = p^*(b \cup a)$, and ~~so~~ $b \cup a \neq 0$. Thus $b \cup a$ generates $H^{i+1}(P_{\mathbb{R}}(V); \mathbb{Z}/2)$.

Thm: Let $n \geq 1$, V a complex $n+1$-dim. vector space. Let $a \in H^2(P_{\mathbb{C}}(V), \mathbb{Z}) \cong \mathbb{Z}$ ~~be~~ a generator. Then for $1 \leq i \leq n$,

$\underbrace{a \cup \cdots \cup a}_{i}$ is a generator of $H^{2i}(P_{\mathbb{C}}(V); \mathbb{Z}) \cong \mathbb{Z}$

(i.e. $H^*(P_{\mathbb{C}}(V); \mathbb{Z}) \cong \mathbb{Z}[a]/(a^{n+1})$).

Proof. Suff. to show if $i < n$ and b generates $H^{2i}(P_{\mathbb{C}}(V))$, then $b \cup a$ generates $H^{2i+2}(P_{\mathbb{C}}(V))$.

The underlying real 2-plane bdle of the complex line bdle $L_{\mathbb{C}}(V) \xrightarrow{p} P_{\mathbb{C}}(V)$ admits a $\mathbb{Z}$-orient. $U \in H^2(L_{\mathbb{C}}(V), L_{\mathbb{C}}(V)^\circ)$

As above, j^*U must be a gen. of $H^2(L_{\mathbb{C}}(V))$ and so must have $j^*U = \varepsilon p^*(a)$ where $\varepsilon = \pm 1$. By Thom isom., $(p^*b) \cup U$ generates $H^{2i+2}(L_{\mathbb{C}}(V), L_{\mathbb{C}}(V)^\circ)$, and so, by above, $j^*(p^*b \cup U)$ generates $H^{2i+2}(L_{\mathbb{C}}(V))$. As above, $j^*(p^*(b) \cup U) = \varepsilon p^*(b \cup a)$ and so $b \cup a$ generates $H^{2i+2}(P_{\mathbb{C}}(V))$.

Cor: For $1 \le m < n$, $\mathbb{R}P^m$ is not a retract of $\mathbb{R}P^n$, and $\mathbb{C}P^m$ is not a retract of $\mathbb{C}P^n$.

Proof: Let $r: \mathbb{R}P^n \to \mathbb{R}P^m$ be a retraction. Then

$$\begin{tikzcd} & \mathbb{R}P^n \arrow[dr, "r"] & \\ \mathbb{R}P^m \arrow[ur, "i"] \arrow[rr, "1"'] & & \mathbb{R}P^m \end{tikzcd}$$ commutes.

Let $a \in H^1(\mathbb{R}P^m, \mathbb{Z}/2)$, $b \in H^1(\mathbb{R}P^n, \mathbb{Z}/2)$ be the generators. Then $a = 1^*(a) = i^* r^*(a)$ and ~~so~~ must have $r^*(a) = b$. Since $a^{m+1} = 0$, we have $0 = r^*(a^{m+1}) = (r^*(a))^{m+1} = b^{m+1} \neq 0$, contradiction.

Proof that $\mathbb{C}P^m$ is not a retract of $\mathbb{C}P^n$ is similar.

Homotopy Theory

Cat. of pointed top. spaces

Objects: Pairs (X, x_0), X a top space, $x_0 \in X$. x_0 is the *base point* of (X, x_0) (or more briefly, the base pt. of X).

Morphisms: Cont. maps sending base pt. to base pt.

Def. $f, g : X \to Y$ pointed maps. $x_0 =$ base pt. of X, $y_0 =$ base pt. of Y. A *pointed homotopy* from f to g is a homotopy $h : X \times I \to Y$ from f to g $\ni$ $h(x_0, t) = y_0$ for all $t \in I$.

Write $f \underset{h}{\simeq_0} g$ if h is a pointed homotopy from f to g.

$\simeq_0$ is an equiv. rel. Moreover, if $f_1 \simeq_0 f_2 : X \to Y$, $g_1 \simeq_0 g_2 : Y \to Z$, then $g_1 f_1 \simeq_0 g_2 f_2$.

Write $[X, Y] =$ set of pointed homotopy classes of maps from X to Y.

Basic problem: Determine $[X, Y]$, i.e. find its cardinality and a complete set of representative maps.

A number of classification problems can be reduced to homotopy classification problems.

Examples 1): π an abel. group. $\exists$ space $K(\pi, n)$, $n \geq 0$, $\ni$ for any ~~[illegible]~~ CW space X, $H^n(X; \pi)$ is in natural 1-1 corresp. with $[X^+, K(\pi, n)]$ ($X^+ = X \cup$ disjoint base pt.).

2) $\exists$ space $BO(n)$ $\ni$ set of isom. classes of real n-pl. bdles over a pointed CW space X is in 1-1 corresp. with $[X, BO(n)]$.

3) Obstruction theory.

$$\begin{tikzcd} & X \arrow[d] \\ Z \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad\qquad \begin{tikzcd} X \arrow[d] \arrow[dr] & \\ Y \arrow[r, dashed] & Z \end{tikzcd}$$

lifting problem $\qquad\qquad$ extension problem

Homotopy groups (i.e. $[S^n, X]$) play an important role.

Problem of computing $[X, Y]$ in general is difficult. Has been solved in special cases.

Best chance for success: When X or Y (or both) possess some additional structure which gives rise to some sort of algebraic structure on $[X,Y]$ (e.g. group).

15 If (X,x_0), (Y,y_0) are pointed spaces, regard their wedge

$X\vee Y$ as $\{(x,y)\in X\times Y \mid \text{either } x=x_0 \text{ or } y=y_0\} \subset X\times Y$.
(x_0,y_0) will serve as base point for both $X\vee Y$ and $X\times Y$.

If (Y,y_0) is a pointed space, we have a cont. map $\varphi: Y\vee Y \to Y$ given by $\varphi(y_0,y)=\varphi(y,y_0)=y$. φ is called the <u>folding map</u>.

<u>Def</u>: An <u>H^*-space</u> consists of a pointed space (Y,y_0) together with pointed maps $\mu: Y\times Y\to Y$, $\eta: Y\to Y$ satisfying

i)
$$\begin{tikzcd} Y\vee Y \arrow[d, "i"'] \arrow[dr, "\varphi"] & \\ Y\times Y \arrow[r, "\mu"'] & Y \end{tikzcd}$$
commutes up to pointed homotopy.

ii)
$$\begin{tikzcd} Y\times Y\times Y \arrow[r, "\mu\times 1"] \arrow[d, "1\times\mu"'] & Y\times Y \arrow[d, "\mu"] \\ Y\times Y \arrow[r, "\mu"'] & Y \end{tikzcd}$$
commutes up to pointed homotopy

iii)
$$\begin{tikzcd} & Y\times Y \arrow[r, "1\times\eta"] & Y\times Y \arrow[dr, "\mu"] & \\ Y \arrow[ur, "d"] \arrow[rrr, "*"] \arrow[dr, "d"'] & & & Y \\ & Y\times Y \arrow[r, "\eta\times 1"'] & Y\times Y \arrow[ur, "\mu"'] & \end{tikzcd}$$

commutes up to pointed homotopy, where d = diag. map, $*$ = const. map.
μ is called the <u>mult. map</u>, η the <u>inversion map</u>.

<u>Example</u>: If Y is a top. group with mult. μ, and η is given by $\eta(y)=y^{-1}$, then Y is an H^*-space (y_0 = unit elt. of Y). In fact, i), ii), iii) actually commute in this case. i) expresses the fact that y_0 is a 2-sided unit element. ii) expresses the

assoc. of mult. iii) expresses $yy^{-1} = y^{-1}y = y_0$.

The terminology H^*-space is not standard. A point space Y together with a pointed map $\mu: Y \times Y \to Y$ sat. just i) is called an H-space (terminology H-space is standard).

Important example: (X, x_0) a pointed space. Let

$\Omega X = \{ \alpha : I \to X \mid \alpha(0) = \alpha(1) = x_0 \}$, topologized as a subspace of $\mathcal{C}(I, X)$ with the compact open topology. ΩX is called the loop space of X.

Define $\mu : \Omega X \times \Omega X \to \Omega X$, $\eta : \Omega X \to \Omega X$ by

$$\mu(\alpha, \beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}, \qquad \eta(\alpha)(t) = \alpha(1-t).$$

Not hard to show, using properties of C-O top., that μ, η are cont.

Let $* \in \Omega X$ denote the constant path at x_0.
$(\Omega X, *)$ is a pointed space
Claim: ΩX is an H^*-space with mult. μ, inversion η.
Proof:
i) If $\alpha \in \Omega X$,

$$\mu i(\alpha, *)(t) = \mu(\alpha, *)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ x_0 & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

$$\mu i(*, \alpha)(t) = \mu(*, \alpha)(t) = \begin{cases} x_0 & \text{if } 0 \le t \le \frac{1}{2} \\ \alpha(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Define $h : (\Omega X \vee \Omega X) \times I \to \Omega X$ by

$$h((\alpha, *), s)(t) = \begin{cases} \alpha\left(\frac{2t}{s+1}\right) & \text{if } 0 \le t \le \frac{1}{2}(s+1) \\ x_0 & \text{if } \frac{1}{2}(s+1) \le t \le 1 \end{cases},$$

$$h((*, \alpha), s)(t) = \begin{cases} x_0 & \text{if } 0 \le t \le \frac{1}{2}(1-s) \\ \alpha\left(\frac{2t+s-1}{s+1}\right) & \text{if } \frac{1}{2}(1-s) \le t \le 1 \end{cases}$$

h is well-defined and is a pointed homotopy from μi to ∇.

ii) If $\alpha, \beta, \gamma \in \Omega X$ it can be checked that

$$\mu(\mu\times 1)(\alpha,\beta,\gamma)(t) = \begin{cases} \alpha(4t) & \text{if } 0 \le t \le \frac{1}{4} \\ \beta(4t-1) & \text{if } \frac{1}{4} \le t \le \frac{1}{2} \\ \gamma(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases},$$

$$\mu(1\times\mu)(\alpha,\beta,\gamma)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \beta(4t-2) & \text{if } \frac{1}{2} \le t \le \frac{3}{4} \\ \gamma(4t-3) & \text{if } \frac{3}{4} \le t \le 1 \end{cases}$$

Define

$$h : (\Omega X\times\Omega X\times\Omega X)\times I \longrightarrow \Omega X \text{ by}$$

$$h(\alpha,\beta,\gamma,s)(t) = \begin{cases} \alpha\left(\frac{4t}{1+s}\right) & \text{if } 0 \le t \le \frac{1}{4}(1+s) \\ \beta(4t-1-s) & \text{if } \frac{1}{4}(1+s) \le t \le \frac{1}{4}(2+s) \\ \gamma\left(\frac{4t-s-2}{2-s}\right) & \text{if } \frac{1}{4}(2+s) \le t \le 1 \end{cases}$$

h is well-defined, and is a pointed homotopy from $\mu(\mu\times 1)$ to $\mu(1\times\mu)$.

iii) If $\alpha \in \Omega X$,

$$\mu(1\times\eta)d(\alpha)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \alpha(2-2t) & \text{if } \frac{1}{2} \le t \le 1 \end{cases},$$

Define $h : (\Omega X)\times I \longrightarrow \Omega X$ by

$$h(\alpha,s)(t) = \begin{cases} \alpha(2t(1-s)) & \text{if } 0 \le t \le \frac{1}{2} \\ \alpha((2-2t)(1-s)) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

h is a pointed homotopy from $\mu(1\times\eta)d$ to $*$.
Similarly $\mu(\eta\times 1)d \simeq_0 *$.

If Y an H^*-space with mult. μ, and $f, g : X \to Y$ are pointed maps, define $f\circ g$ to be the composition

$$X \xrightarrow{d} X\times X \xrightarrow{f\times g} Y\times Y \xrightarrow{\mu} Y$$

Note: The pointed homotopy class of $f \circ g$, $[f \circ g]$, depends only on $[f]$ and $[g]$, i.e. $[f\circ g]$ is indep. of the choice of representatives. Hence, can define a product in $[X,Y]$ by $[f][g] = [f\circ g]$.

Prop: $[X,Y]$ is a group under above prod. Unit elt is $[*]$, $[f]^{-1} = [\eta f]$.

Proof: Unit elt. $[*][f] = [*\circ f]$. $*\circ f$ is the composition

$$X \xrightarrow{d} X\times X \xrightarrow{*\times f} Y\times Y \xrightarrow{\mu} Y.$$

We have the $\simeq_0$ commutative diagram

$$\begin{tikzcd} X \arrow[r, "d"] \arrow[drr, "g"'] & X\times X \arrow[r, "*\times f"] & Y\times Y \arrow[r, "\mu"] & Y \\ & & Y\vee Y \arrow[u, "i"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

where $g(x) = (*, f(x))$. (LH side commutes, RH side $\simeq_0$ commutes). Moreover, $\varphi g = f$. Hence $*\circ f \simeq_0 f$ and so $[*][f] = [f]$. Similarly $[f][*] = [f]$.

Inverses: $[f][\eta f] = [f\circ \eta f]$. $f\circ\eta f$ is the composition

$$X \xrightarrow{d} X\times X \xrightarrow{f\times\eta f} Y\times Y \xrightarrow{\mu} Y.$$

We have the comm. diagram

$$\begin{tikzcd} X \arrow[r, "d"] \arrow[dr, "f"'] & X\times X \arrow[r, "f\times\eta f"] & Y\times Y \arrow[r, "\mu"] & Y \\ & Y \arrow[r, "d"'] & Y\times Y \arrow[u, "1\times\eta"'] & \end{tikzcd}$$

Since $\mu(1\times\eta)d \simeq_0 *$, it follows that $f\circ\eta f \simeq_0 *$, so $[f][\eta f] = [*]$. Similarly $[\eta f][f] = [*]$.

Associativity:

$$
\begin{tikzcd}
 & X\times X \arrow[r, "f\times(g\circ h)"] \arrow[d, "1\times d"] & Y\times Y \arrow[dr, "\mu"] & \\
X \arrow[ur, "d"] \arrow[dr, "d"'] \arrow[rrr, bend left=60, "f\circ(g\circ h)"] \arrow[rrr, bend right=60, "(f\circ g)\circ h"'] & X\times X\times X \arrow[r, "f\times g\times h"'] & Y\times Y\times Y \arrow[u, "1\times\mu"'] \arrow[d, "\mu\times 1"] \arrow[r, phantom, "\simeq_0"] & Y \\
 & X\times X \arrow[u, "d\times 1"] \arrow[r, "(f\circ g)\times h"'] & Y\times Y \arrow[ur, "\mu"'] &
\end{tikzcd}
$$

Note: $[X,Y]$ is a functor in each argument from pointed spaces to pointed sets; contravariant in 1st variable, covariant in 2nd variable, via composition of maps.

If $f: X' \to X$ a pointed map, $f^*: [X,Y] \to [X',Y]$ is given by $f^*[g] = [gf]$.

If $g: Y \to Y'$ a pointed map, $g_*: [X,Y] \to [X,Y']$ is given by $g_*[f] = [gf]$.

Prop: Y an H^*-space. If $f: X' \to X$ a pointed map, then $f^*: [X,Y] \to [X',Y]$ is a morphism of groups.

Proof: Have comm. diag.

$$
\begin{tikzcd}
X' \arrow[r, "f"] \arrow[dr, "d"'] \arrow[rrrr, bend right=50, "gf\circ hf"'] & X \arrow[r, "d"] \arrow[rrr, bend left=30, "g\circ h"] & X\times X \arrow[r, "g\times h"] & Y\times Y \arrow[r, "\mu"] & Y \\
 & X'\times X' \arrow[urr, "gf\times hf"'] & & &
\end{tikzcd}
$$

and so $f^*([g][h]) \overset{\text{def}}{=} [(g\circ h)f] = [gf\circ hf]$

$= [gf][hf] \overset{\text{def}}{=} f^*[g]\, f^*[h]$.

Thus if Y is an H^*-space, $[\ ,Y]$ is a contr. functor from pointed spaces to groups.

HW exercise: If Y is a pointed space such that $[X,Y]$ has a natural group structure for each pointed space X, then this natural group structure comes from an H^*-structure on Y.

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Def: X, Y pointed spaces. The smash product of X and Y, denoted $X \wedge Y$, is $X \times Y / X \vee Y$.

$X \wedge Y$ is again pointed with base pt. the point to which $X \vee Y$ is identified. Write $x \wedge y$ = image of (x,y) under the quot. map $X \times Y \to X \wedge Y$. Thus $* \wedge y = x \wedge * = *$.

If $f: X \to Y$, $f': X' \to Y'$ are pointed maps, then $f \times f'(X \vee X') \subset (Y \vee Y')$. Passing to quotients, we obtain a pointed map

$$f \wedge f' : X \wedge X' \longrightarrow Y \wedge Y'$$
$$x \wedge x' \longmapsto f(x) \wedge f(x')$$

Following properties are easily checked: $(f_1 \wedge f_2)(g_1 \wedge g_2) = f_1 g_1 \wedge f_2 g_2$, $f \wedge * = * = * \wedge g$, $f \simeq_0 f'$, $g \simeq_0 g' \Rightarrow f \wedge g \simeq_0 f' \wedge g'$.

We have natural homeos

$$(X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z),$$
$$(x \wedge y) \wedge z \longmapsto x \wedge (y \wedge z)$$

$$X \wedge Y \to Y \wedge X,$$
$$x \wedge y \longmapsto y \wedge x$$

$$X \wedge (Y \vee Z) \longrightarrow (X \wedge Y) \vee (X \wedge Z)$$
$$x \wedge (y, *) \longmapsto (x \wedge y, *)$$
$$x \wedge (*, z) \longmapsto (*, x \wedge z).$$

Note. $\wedge$ and $\vee$ in the pointed category are in many respects analogous to $\times$ and $\amalg$ in the unpointed cat.

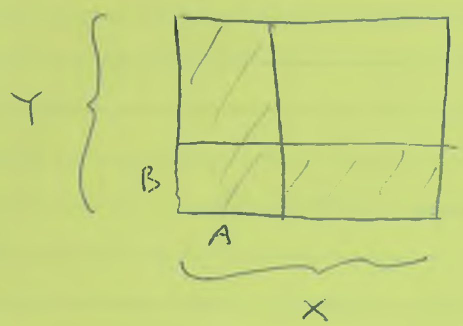
If (X, A) a top. pair (not pointed), then X/A is a pointed space with base pt. the point to which A is identified.
Convention: $X/\emptyset$ = disjoint union of X with a new point $*$, taken to be the base point of $X/\emptyset$.

Suppose (X,A), (Y,B) top. pairs. Let $p_A : X \to X/A$, $p_B : Y \to Y/B$ denote the quotient maps. The composition

$$X \times Y \xrightarrow{p_A \times p_B} (X/A) \times (Y/B) \xrightarrow{\text{quot.}} (X/A) \wedge (Y/B)$$

is cont. and sends $(X \times B) \cup (A \times Y)$ to $*$. Thus, passing to quotients, get a cont. map

$$f : \frac{X \times Y}{(X \times B) \cup (A \times Y)} \longrightarrow (X/A) \wedge (Y/B),$$ easily seen to be a bijection.



Easily seen: If $p_A \times p_B$ is a quot. map, then f is a homeo. A sufficient cond. for this is A, B both compact.

<u>Examples</u>: 1) Regard S^n as $I^n/\partial I^n$, $n \geq 0$. Then

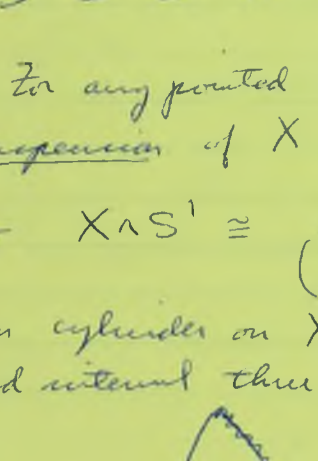
$$S^m \wedge S^n = (I^m/\partial I^m) \wedge (I^n/\partial I^n) \cong \frac{I^m \times I^n}{(I^m \times \partial I^n) \cup (\partial I^m \times I^n)}$$

$$= \frac{I^{m+n}}{\partial I^{m+n}} = S^{m+n}.$$ In particular, if $m \geq 1$,

$$S^m \cong S^{m-1} \wedge S^1$$

2) For any pointed X, the space $X \wedge S^1$ is called the <u>reduced suspension</u> of X. <u>Note</u>: $X \cong X/*$, $S^1 = I/\{0,1\}$,

so $$X \wedge S^1 \cong \frac{X \times I}{(X \times \{0,1\}) \cup (* \times I)}$$ = quot. space obtained from cylinder on X by collapsing top and bottom bases, and interval thru base pt, to a point.



3) $X \wedge S^0$ is canonically homeo. to X for any pointed X.

Def: A co-H^* space consists of a pointed space (X, x_0) together with pointed maps $\mu' : X \to X \vee X$ (the comultiplication) and $\eta' : X \to X$ (the coinversion) s.t. following commute up to pointed homotopy:

i)

$$\begin{tikzcd}
X \times X & \\
X \vee X \arrow[u, "i"] & X \arrow[l, "\mu'"] \arrow[ul, "d"']
\end{tikzcd}$$

ii)

$$\begin{tikzcd}
X \arrow[r, "\mu'"] \arrow[d, "\mu'"'] & X \vee X \arrow[d, "\mu' \vee 1"] \\
X \vee X \arrow[r, "1 \vee \mu'"'] & X \vee X \vee X
\end{tikzcd}$$

(Notation: if $f : X \to Y$, $g : Z \to W$ pointed maps, then $f \vee g : X \vee Z \to Y \vee W$ is the pointed map given by
$(f \vee g)(x, z_0) = (f(x), w_0)$, $(f \vee g)(x_0, z) = (y_0, g(x))$.

iii)

$$\begin{tikzcd}
 & X \vee X \arrow[dl, "\varphi"'] & X \vee X \arrow[l, "1 \vee \eta'"'] & \\
X & & & X \arrow[lll, "*"'] \arrow[ul, "\mu'"'] \arrow[dl, "\mu'"] \\
 & X \vee X \arrow[ul, "\varphi"] & X \vee X \arrow[l, "\eta' \vee 1"] &
\end{tikzcd}$$

Example: Let X be any pointed space. Claim: $X \wedge S^1$ is a co-H^* space with comult. $\mu' : X \wedge S^1 \to (X \wedge S^1) \vee (X \wedge S^1)$ and coinversion $\eta' : X \wedge S^1 \to X \wedge S^1$ given as follows:
Regarding S^1 as $I/\{0,1\}$, write $\langle t \rangle \in S^1$ for image of $t \in I$ under the quot. map $I \to S^1$. Define

$$\mu'(x \wedge \langle t \rangle) = \begin{cases} (x \wedge \langle 2t \rangle, *) & \text{if } 0 \le t \le \frac{1}{2} \\ (*, x \wedge \langle 2t-1 \rangle) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

 (μ' collapses equator to a point),

$\eta'(x \wedge \langle t \rangle) = x \wedge \langle 1-t \rangle$ (reflection in equator).

__Prop__: $X \wedge S^1$ is a co-H^* space with above μ' as comult., above η' as conversion.

__Proof__ i) Define $h : (X \wedge S^1) \times I \to (X \wedge S^1) \times (X \wedge S^1)$ by

$$h(x \wedge \langle t \rangle, s) = \begin{cases} (x \wedge \langle (1+s)t \rangle, \langle x \wedge (1-s)t \rangle) & \text{if } 0 \le t \le \frac{1}{2} \\ (x \wedge \langle s + (1-s)t \rangle, x \wedge \langle -s + (1+s)t \rangle) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

h is a pointed homotopy from d to $i\mu'$.

ii) Define $h : (X \wedge S^1) \times I \to (X \wedge S^1) \vee (X \wedge S^1) \vee (X \wedge S^1)$ by

$$h(x \wedge \langle t \rangle, s) = \begin{cases} (x \wedge \langle \frac{4t}{1+s} \rangle, *, *) & \text{if } 0 \le t \le \frac{1}{4}(1+s) \\ (*, x \wedge \langle 4t-1-s \rangle, *) & \text{if } \frac{1}{4}(1+s) \le t \le \frac{1}{4}(2+s) \\ (*, *, x \wedge \langle \frac{4t-s-2}{2-s} \rangle) & \text{if } \frac{1}{4}(2+s) \le t \le 1. \end{cases}$$

h is a pointed homotopy from $(\mu' \vee 1)\mu'$ to $(1 \vee \mu')\mu'$.

iii) Define $h : (X \wedge S^1) \times I \to X \wedge S^1$ by

$$h(x \wedge \langle t \rangle, s) = \begin{cases} x \wedge \langle 2t(1-s) \rangle & \text{if } 0 \le t \le \frac{1}{2} \\ x \wedge \langle (2-2t)(1-s) \rangle & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

h is a pointed homotopy from $\varphi(1 \vee \eta')\mu'$ to $*$. Similarly, $\varphi(\eta' \vee 1)\mu' \simeq_0 *$.

Let X be a co-H^* space with comult. μ', conv. η', and Y a pointed space. If $f, g : X \to Y$ are pointed maps, define $f \square g$ to be the composition

$$X \xrightarrow{\mu'} X \vee X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\varphi} Y.$$

Easily checked: $[f \vee g]$ depends only on $[f]$ and $[g]$. Thus, $[f \,\square\, g]$ depends only on $[f]$ and $[g]$. Hence, can define a prod. in $[X, Y]$ by $[f][g] = [f \,\square\, g]$.

<u>Prop.</u> X a co-H^* space with co-mult. μ', co-inv. η'. Then for any pointed Y, $[X, Y]$ is a group under the above prod. Unit elt. is $[*]$. $[f]^{-1} = [f\eta']$. Moreover, $[X, \ ]$ is a covariant functor from cat. of pointed spaces to cat. of groups.

<u>Proof</u>: HW exercise (essentially dual to proof of corresp. prop. for H^* spaces).

<u>Exercise</u>: X pointed $\ni$: $[X, \ ]$ factors through the cat. of groups. Then X is a co-H^*-space with the group structure on $[X, Y]$ coming from the co-H^* structure on X via the $\square$ construction.

<u>Notation</u>: Let (Y, y_0) be pointed. For $n \geq 0$, $[S^n, Y]$ is denoted $\pi_n(Y, y_0)$ (or more briefly $\pi_n(Y)$ if there is no danger of confusion). For $n \geq 1$, S^n is a reduced suspension, hence a co-H^* space, and $\pi_n(Y, y_0)$ is a group, the <u>n^{th} homotopy group</u> of (Y, y_0). $\pi_1(Y, y_0)$ is usually called the <u>fundamental group</u> of Y. $\pi_0(Y, y_0)$ is just a pointed set in general (in 1-1 corresp. with the path-components of Y). If Y is an H^*-space, $\pi_0(Y, y_0)$ is a group.

17 <u>Prop.</u> Suppose X is a co-H^* space, Y an H^*-space. Then the group structure on $[X, Y]$ coming from the co-H^* structure on X coincides with the group structure coming from the H^* structure on Y. Moreover, in this case, the group $[X, Y]$ is abelian.

<u>Proof</u>: Let $f, g : X \to Y$ be pointed maps. To show the 2 group structures coincide, must show $f \circ g \simeq_0 f \,\square\, g$.

$$\begin{tikzcd}
 & X \times X \arrow[r, "f \times g"] & Y \times Y \arrow[dr, "\mu"] & \\
X \arrow[ur, "d"] \arrow[dr, "\mu'"'] & & & Y \\
 & X \vee X \arrow[uu, "i"] \arrow[r, "f \vee g"'] & Y \vee Y \arrow[uu, "i"'] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Rectangle commutes. Side triangles $\simeq_0$ commute by props c) of H^* and co-H^* spaces, resp. Thus

$$f \circ g = \mu(f \times g)d \simeq_0 \varphi(f \vee g)\mu' = f \,\square\, g.$$

To show $[X, Y]$ abel, suff. to show $f \circ g \simeq_0 g \,\square\, f$.

$$\begin{tikzcd}
 & X \times X \arrow[r, "f \times g"] & Y \times Y \arrow[dr, "\mu"] & \\
X \arrow[ur, "d"] \arrow[r, "d"] \arrow[dr, "\mu'"'] & X \times X \arrow[u, "\tau"'] \arrow[r, phantom, "(1)"] & Y \vee Y \arrow[u, "i"'] \arrow[r, "\varphi"] & Y \\
 & X \vee X \arrow[u, "i"'] \arrow[r, "g \vee f"'] & Y \vee Y \arrow[u, "\tau"'] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Here τ interchanges factors. (1), (2), (3) commute.
(4), (5) $\simeq_0$ commute by (i) for coH^* and H^* spaces, resp.

Cor: If Y an H^*-space, $\pi_n(Y, y_0)$ is abel for $n \geq 1$.
(In particular, $\pi_1(Y, y_0)$ is abel.)

Recall: Facts about the C-O topology: If X, Y are top. spaces (unpointed), write $\mathcal{C}(X, Y)$ = space of cont. maps from X to Y with the C-O top.

If X is loc. compact and Hausdorff, then for any Y, Z,
$\psi : \mathcal{C}(Z \times X, Y) \to \mathcal{C}(Z, \mathcal{C}(X, Y))$ given by $\psi(f)(z)(x) = f(z, x)$ is a natural homeo.

Corresp. property for pointed spaces: If X, Y are pointed, write $\mathcal{C}_0(X, Y)$ = space of pointed cont. maps from X to Y with the C-O topology. If X is loc. compact Hausd., then for every pointed Z, Y,

$$\psi_0 : \mathcal{C}_0(Z \wedge X, Y) \to \mathcal{C}_0(Z, \mathcal{C}_0(X, Y))$$

given by $\psi_0(f)(z)(x) = f(z \wedge x)$ is a natural homeo.

Easily checked: $f \simeq_0 g \iff \psi_0(f) \simeq_0 \psi_0(g)$. Thus ψ_0 induces a natural bijection

$$\psi_0 : [Z \wedge X, Y] \to [Z, \mathcal{C}_0(X, Y)]$$ whenever X is loc. compact and Hausdorff.

Note: $\mathcal{C}_0(S^1, Y)$ is naturally homeo. with ΩY. Thus taking $X = S^1$ we obtain a natural bijection
$$\Psi_0 : [Z \wedge S^1, Y] \to [Z, \Omega Y].$$

Prop. Ψ_0 is a natural group isomorphism.

Proof: Remains only to show Ψ_0 is a group homomorphism. Straightforward to check: $\Psi_0(f \,\square\, g) = (\Psi_0 f) \circ (\Psi_0 g)$ using the explicit defs. of the comult. in $Z \wedge S^1$ and the mult. in ΩY.

Cor: For all $n \geq 1$ and all pointed (Y, y_0) we have a natural group isom. $\Psi_0 : \pi_n(Y, y_0) \to \pi_{n-1}(\Omega Y, *)$.

Cor: For $n \geq 2$, $\pi_n(Y, y_0)$ is abelian for any pointed Y.

Proof: $\pi_n(Y, y_0) \cong \pi_{n-1}(\Omega Y, *) = [S^{n-1}, \Omega Y]$ which is abel. since S^{n-1} is a co-H^* space (since $n-1 \geq 1$) and ΩY is an H^*-space.

Regard I as a pointed space with base pt. 0.

Def: If X is pointed, the reduced cone on X is $X \wedge I$.

Thus $X \wedge I = \dfrac{X \times I}{(X \times 0) \cup (* \times I)}$ = quot. space obtained from the unreduced cone $\dfrac{X \times I}{X \times \{0\}}$ by collapsing $* \times I$ to a point.

Note: $X \wedge I$ is contractible in the pointed sense, i.e. $1_{X \wedge I} \simeq_0 *$. For $1_{X \wedge I} = 1_X \wedge 1_I$. Since $1_I \simeq_0 *$, $1_{X \wedge I} \simeq_0 1_X \wedge * = *$.

Note: We have a canonical embedding $X \subset X \wedge I$ given by $x \mapsto x \wedge 1$. Regard X as a subspace of $X \wedge I$ in this way. Then

$$\frac{X \wedge I}{X} \underset{\text{homeo}}{\cong} X \wedge S^1.$$

Relative Homotopy Sets

A *pointed top. pair* consists of a top. pair (X, A) with a base point $* \in A$. If (X,A), (Y,B) are pointed top. pairs, a *map of pointed pairs* $f : (X,A) \to (Y,B)$ consists of a pointed map $f : X \to Y$ $\ni$: $f(A) \subset B$.

If $f, g : (X,A) \to (Y,B)$ are maps of pointed pairs, a *pointed homotopy* $h : (X,A) \times I \to (Y,B)$ from f to g is a homotopy from f to g $\ni$: $h(*,t) = *$ $\forall$ $t \in I$. Write $f \underset{h}{\simeq_0} g$ in this case. $\simeq_0$ is an equiv. rel.

Write $[(X,A),(Y,B)]$ = set of pointed homotopy classes of pointed maps from (X,A) to (Y,B). $[\ ,\ ]$ is a functor of 2 variables from pointed pairs to pointed sets, contravariant in 1st, covariant in 2nd.

Def: If (Y,B) a pointed pair and $n \geq 1$, define
$\pi_n(Y,B) = [(S^{n-1} \wedge I, S^{n-1}), (Y,B)]$ = nth *homotopy set* of (Y,B).

18 *Def*: (Y,B) a pointed pair. Define $P(Y;*,B) = \{\lambda \in \mathcal{C}_0(I,Y) \mid \lambda(1) \in B\}$.

(Thus $P(Y;*,*) = \Omega Y$.)

Let $\mathcal{C}_0((X,A),(Y,B))$ = subsp. of $\mathcal{C}_0(X,Y)$ consisting of those maps f $\ni$: $f(A) \subset B$.

Recall: Homeo. $\Psi_0 : \mathcal{C}_0(X \wedge I, Y) \to \mathcal{C}_0(X, \mathcal{C}_0(I,Y))$.
Can check: Ψ_0 carries $\mathcal{C}_0((X \wedge I, X),(Y,B))$ onto $\mathcal{C}_0(X, P(Y;*,B))$.
Moreover, $f \simeq_0 g \iff \Psi_0(f) \simeq_0 \Psi_0(g)$, so get natural bijection $[(X \wedge I, X),(Y,B)] \cong [X, P(Y;*,B)]$. In particular, have natural bijection

$$\pi_n(Y,B) \cong \pi_{n-1}(P(Y;*,B),*).$$

Thus, $\pi_n(Y,B)$ has a natural group structure for $n \geq 2$, and is abel. for $n \geq 3$.

When $B = *$, above reduces to the natural isom.
$\pi_n(Y,*) \cong \pi_{n-1}(\Omega Y, *)$ for $n \geq 1$.

Note: $\pi_0(Y,B)$ is undefined, except for case $B = *$.

If $n \geq 1$ and $f : (S^{n-1} \wedge I, S^{n-1}) \to (Y, B)$ a pointed map, then $f | S^{n-1} : S^{n-1} \to B$ is a pointed map. Moreover, $[f | S^{n-1}]$ depends only on $[f]$. Thus, get a natural morphism of pointed sets
$\partial : \pi_n(Y, B) \to \pi_{n-1}(B, *)$.

Let $p : P(Y; *, B) \longrightarrow B$ be given by $p(\lambda) = \lambda(1)$. p is continuous (by property of c-o top.) and pointed.

Prop. (Y, B) a pointed pair, $n \geq 1$. Then

$$\begin{tikzcd} \pi_n(Y, B) \arrow[rr, "\partial"] \arrow[dr, "\Psi_0"', "\cong"] & & \pi_{n-1}(B, *) \\ & \pi_{n-1}(P(Y; *, B), *) \arrow[ur, "p_*"'] & \end{tikzcd}$$

commutes.

Proof: If $f : (S^{n-1} \wedge I, S^{n-1}) \to (Y, B)$ pointed map,
$p\, \Psi_0(f)(x) = \Psi_0(f)(x)(1) = f(x \wedge 1) = f(x)$ (recall: X is identified with subspace of $X \wedge I$ via $x \longleftrightarrow x \wedge 1$).

Cor. For $n \geq 2$, $\partial : \pi_n(Y, B) \to \pi_{n-1}(B, *)$ is a group homomorphism.

Note: If $f : (X, A) \to (Y, B)$ a map of pointed pairs, get cont.
$Pf : P(X; *, A) \to P(Y; *, B)$ given by $Pf(\lambda) = f\lambda$.
P is a covariant functor from pointed pairs to pointed spaces.
In special case A, B are both $*$, write $\Omega f = Pf : \Omega X \to \Omega Y$.
Naturality of Ψ_0:

$$\begin{tikzcd} \pi_n(X, A) \arrow[r, "f_*"] \arrow[d, "\Psi_0"', "\cong"] & \pi_n(Y, B) \arrow[d, "\Psi_0", "\cong"'] \\ \pi_{n-1}(P(X; *, A), *) \arrow[r, "(Pf)_*"'] & \pi_{n-1}(P(Y; *, B), *) \end{tikzcd}$$

commutes. (Easy check). In particular f_* is a group homomorphism for $n \geq 2$.

Given a pointed pair (Y, B), let $i : (B, y_0) \to (Y, y_0)$ and $j : (Y, y_0) \to (Y, B)$ denote the inclusions. We obtain a sequence of morphisms of pointed sets (mostly group homomorphisms)

$$\to \pi_n(B,y_0) \xrightarrow{i_*} \pi_n(Y,y_0) \xrightarrow{j_*} \pi_n(Y,B) \xrightarrow{\partial} \pi_{n-1}(B,y_0) \xrightarrow{i_*} \pi_{n-1}(Y,y_0) \to \cdots$$

$$\cdots \to \pi_1(Y,B) \xrightarrow{\partial} \pi_0(B,y_0) \xrightarrow{i_*} \pi_0(Y,y_0).$$

Above sequence is called the <u>homotopy sequence of the pointed pair</u> (Y,B).

<u>Def</u>. A 3 term ~~sequence~~ of maps of pointed sets $A \xrightarrow{f} B \xrightarrow{g} C$ is <u>exact</u> if $f(A) = g^{-1}(*)$. An arbitrary sequence of maps of pointed sets is exact if each 3 term sequent of it is exact.

<u>Thm</u>: The homotopy sequence of a pointed pair (Y,B) is exact.

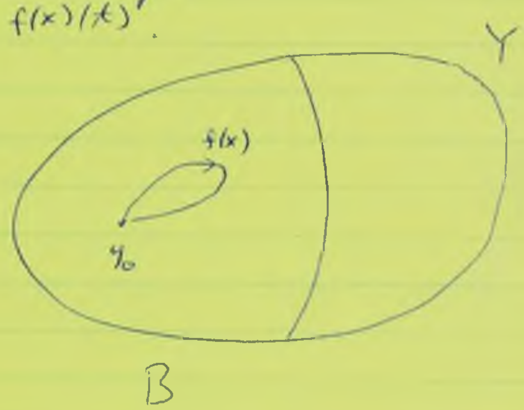
<u>Proof</u>: Exactness at $\pi_n(Y,y_0)$, $n \geq 1$.
Have commutative diagram

$$\begin{tikzcd}
\pi_n(B,y_0) \arrow[r, "i_*"] \arrow[d, "\Psi_0"', "\cong"] & \pi_n(Y,y_0) \arrow[r, "j_*"] \arrow[d, "\Psi_0"', "\cong"] & \pi_n(Y,B) \arrow[d, "\Psi_0"', "\cong"] \\
\pi_{n-1}(\Omega B,*) \arrow[r, "(Pi)_*"'] & \pi_{n-1}(\Omega Y,*) \arrow[r, "(Pj)_*"'] & \pi_{n-1}(P(Y,*,B),*)
\end{tikzcd}$$

so suff. to check bottom row exact.

Let $f: S^{n-1} \to \Omega B$ be a pointed map.
Then $(Pj)(Pi)f(x)(t) = j\,i\,f(x)(t) = f(x)(t)$.

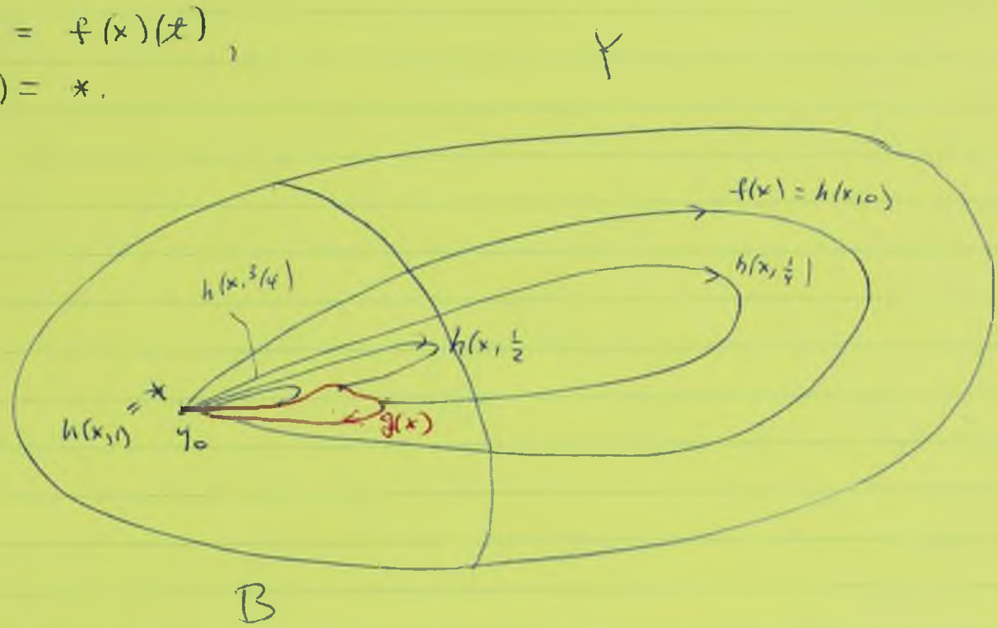
Define
$$h: S^{n-1} \times I \to P(Y,*,B)$$
by
$$h(x,s)(t) = f(x)(st).$$
h is a pointed homotopy from $*$ to $(Pj)(Pi)f$
and so
$$\operatorname{im}(Pi)_* \subset (Pj)_*^{-1}(*).$$



Now suppose $f : S^{n-1} \to \Omega Y$ is a pointed map $\ni$: $(Pj)_* [f] = *$. $\exists$ pointed homotopy

$$h : S^{n-1} \times I \to P(Y, *, B) \quad \ni:$$

$$h(x,0)(t) = f(x)(t),$$
$$h(x,1)(t) = *.$$



Define $g : S^{n-1} \to \Omega B$ by

$$g(x)(t) = h(x, 1-t)(1).$$

Suff. to show $(Pi)g \simeq_0 f$.

Define
$$k : S^{n-1} \times I \to \Omega Y \quad \text{by}$$

$$k(x,s)(t) = h(x, \min\{s, 1-t\})\Big(\underbrace{\min\{1, \tfrac{t}{1-s}\}}_{\text{interpret as 1 if } s=1.}\Big)$$

Continuity easily checked, using $h(x,1)(t) = *$.

Then $f \underset{k}{\simeq_0} (Pi)g$.

(Alternately, let $g(x)(t) = \begin{cases} y_0 & \text{if } 0 \le t \le \frac{1}{2} \\ h(x, 2-2t)(1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$

$g : S^{n-1} \to \Omega B$. Define $k : S^{n-1} \times I \to \Omega Y$ by

$$k(x,s)(t) = \begin{cases} h(x,s)\left(\frac{2t}{2-s}\right) & \text{if } 0 \le t \le 1 - \frac{s}{2} \\ h(x, 2-2t)(1) & \text{if } 1 - \frac{s}{2} \le t \le 1. \end{cases}$$ Then $f \underset{k}{\simeq_0} (Pi)(g)$)

19 Exactness at $\pi_n(Y,B)$, $n \geq 1$:
Have commutative diagram

$$\begin{tikzcd}
\pi_n(Y,y_0) \arrow[r, "j_*"] \arrow[d, "\Psi_0"', "\cong"] & \pi_n(Y,B) \arrow[r, "\partial"] \arrow[d, "\Psi_0"', "\cong"] & \pi_{n-1}(B,y_0) \arrow[d, "1", "\cong"'] \\
\pi_{n-1}(\Omega Y, *) \arrow[r, "(Pj)_*"'] & \pi_{n-1}(P(Y;*,B),*) \arrow[r, "p_*"'] & \pi_{n-1}(B,y_0)
\end{tikzcd}$$

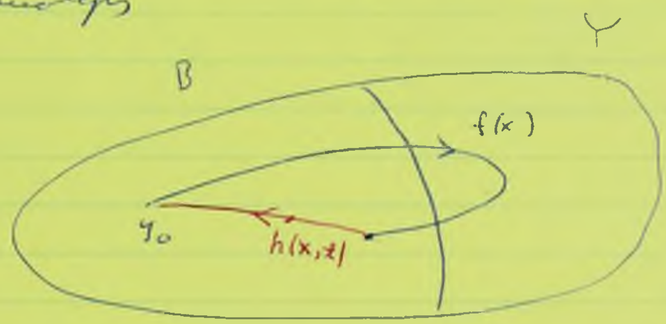
so suff. to show bottom row is exact.

$$p(Pj) = * \quad , \text{ so } \quad \mathrm{im}(Pj)_* \subset p_*^{-1}(*).$$

Suppose $f: S^{n-1} \to P(Y,*,B)$ a pointed map $\ni$ $pf \simeq_0 *$. $\exists$ pointed homotopy

$h: S^{n-1} \times I \to B$ $\ni$:

$h(x,0) = f(x)(1)$,
$h(x,1) = y_0$



Define $g: S^{n-1} \to \Omega Y$ by

$$g(x)(t) = \begin{cases} f(x)(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ h(x, 2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Suff. to show: $f \simeq_0 (Pj)g$. Define

$k: S^{n-1} \times I \to P(Y;*,B)$ by

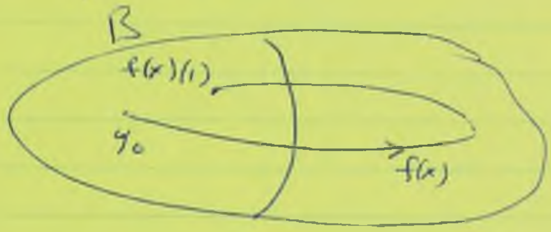
$$k(x,s)(t) = \begin{cases} f(x)\left(\frac{2t}{2-s}\right) & \text{if } 0 \leq t \leq 1 - \frac{s}{2} \\ h(x, 2t+s-2) & \text{if } 1 - \frac{s}{2} \leq t \leq 1. \end{cases}$$

Then $f \underset{k}{\simeq}_0 (Pj)g$.

Exactness at $\pi_n(B, y_0)$, $n \geq 0$:
Have commutative diagram

$$\begin{tikzcd}
\pi_{n+1}(Y,B) \arrow[r, "\partial"] \arrow[d, "\Psi_0"', "\cong"] & \pi_n(B,y_0) \arrow[r, "i_*"] \arrow[d, "1"', "\cong"] & \pi_n(Y,y_0) \arrow[d, "1"', "\cong"] \\
\pi_n(P(Y;*,B),*) \arrow[r, "p_*"'] & \pi_n(B,y_0) \arrow[r, "i_*"'] & \pi_n(Y,y_0)
\end{tikzcd}$$

so suff. to show bottom row exact.
Let $f : S^n \to P(Y;*,B)$ be a pointed map.
Then $i p f(x) = f(x)(1)$.
Define
$h : S^n \times I \to Y$ by



$h(x,t) = f(x)(t)$. Then
$* \underset{h}{\simeq_0} ipf$ and so

$\operatorname{im} p_* \subset i_*^{-1}(*)$.

Now suppose $f : S^n \to B$ a pointed map $\ni$: $if \simeq_0 *$. Then $\exists$ pointed homotopy $h : S^n \times I \to Y$ $\ni$: $h(x,0) = *$, $h(x,1) = f(x)$. Define $g : S^n \to P(Y;*,B)$ by $g(x)(t) = h(x,t)$. Then $f = pg$ and so $i_*^{-1}(*) \subset \operatorname{im} p_*$.

Fibrations

Def: A cont. map $p : E \to B$ (not necessarily pointed) has the homotopy lifting property for X if

$$\begin{tikzcd}
X \times 0 \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B
\end{tikzcd}$$

given continuous f, h $\ni$ rectangle commutes, $\exists$ continuous $\tilde{h}$ $\ni$ both triangles commute.

p is a fibration if it has the HLP for all X (sometimes called a Hurewicz fibration.

Examples: 1) $\begin{array}{c} B\times F \\ \downarrow \pi_1 \\ B \end{array}$ is a fibration. For given

$$\begin{tikzcd} X\times 0 \arrow[r, "f"] \arrow[d, hook] & B\times F \arrow[d, "\pi_1"] \\ X\times I \arrow[r, "h"'] & B \end{tikzcd}$$

, define $\tilde{h} : X\times I \longrightarrow B\times F$ by

$$\tilde{h}(x,t) = (h(x,t),\ \pi_2 f(x,0))$$

2) $p : E \to B$ is a fibre bundle projection with fibre F if for each $x\in B$, $\exists$ open nbhd U of x in B and a homeo.

$U\times F \xrightarrow{\ \theta\ } p^{-1}(U)$ s.t.

$$\begin{tikzcd} U\times F \arrow[rr, "\theta"] \arrow[dr, "\pi_1"'] & & p^{-1}(U) \arrow[dl, "p\,|\,p^{-1}(U)"] \\ & U & \end{tikzcd}$$

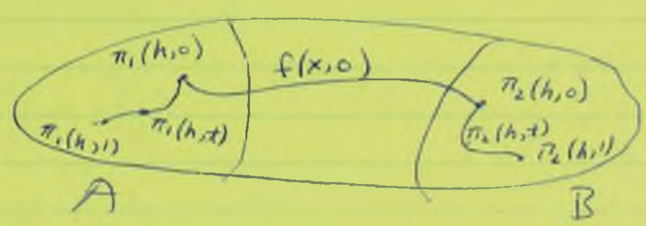
commutes.

Can be proved: If $p : E\to B$ is a fibre bundle projection with B paracompact, then p is a fibration.
References for proof: Steenrod, Top. of Fibre Bundles, p. 49-54
Husemoller, p. 48-51
Spanier, p. 92-96.

3) Let $(Y; A, B)$ be a top. triad. Write
$P(Y;A,B) = \{\lambda \in \mathcal{C}(I,Y) \mid \lambda(0)\in A,\ \lambda(1)\in B\}$.

Define $p : P(Y;A,B) \longrightarrow A\times B$ by $p(\lambda) = (\lambda(0),\lambda(1))$.
Then p is a fibration. For given

$$\begin{tikzcd} X\times 0 \arrow[r, "f"] \arrow[d, hook] & P(Y;A,B) \arrow[d, "p"] \\ X\times I \arrow[r, "h"'] & A\times B \end{tikzcd}$$



define $\tilde{h} : X\times I \longrightarrow P(Y;A,B)$ by

$$\tilde{h}(x,s)(t) = \begin{cases} \pi_1 h(x, s-3t) & \text{if } 0 \le t \le \frac{1}{3}s \\ f(x,0)\left(\dfrac{3t-s}{3-2s}\right) & \text{if } \frac{1}{3}s \le t \le 1-\frac{1}{3}s \\ \pi_2 h(x, 3t+s-3) & \text{if } 1-\frac{1}{3}s \le t \le 1 \end{cases}$$

4) If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are fibrations, easily proved $gf: X \to Z$ is a fibration.

Thus, combining examples 3) and 1),

$$P(Y; A, B) \to A, \quad \lambda \mapsto \lambda(0) \qquad , \qquad P(Y; A, B) \to B, \quad \lambda \mapsto \lambda(1) \qquad \text{are fibrations.}$$

In particular, $P(Y, *, B) \to B$, $\lambda \mapsto \lambda(1)$ is a fibration.

20 Thm (Cellular homotopy lifting extension theorem. CHLET): Let $p: E \to B$ be a fibration and (K, L) a CW pair. Suppose given continuous maps h, f such that following rectangle commutes.

$$\begin{tikzcd}
|K| \times 0 \cup |L| \times I \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\
|K| \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B
\end{tikzcd}$$

Then $\exists$ continuous $\tilde{h}: |K| \times I \to E$ s.t. both triangles commute.

Idea of proof: Extend one cell at a time.

Lemma: For $n \geq 1$, $\exists$ homeo. of top. pairs

$$(D^n \times I, D^n \times 0) \to (D^n \times I, D^n \times 0 \cup S^{n-1} \times I).$$

Proof:

$$(x,t) \longmapsto \begin{cases} \left(\left(\frac{2-t}{1+t}\right)x,\ t\right) & \text{if } 0 \le \|x\| \le \frac{1+t}{2} \\[2ex] \left(\left(1-\frac{t}{2}\right)\frac{x}{\|x\|},\ 2\|x\|-1\right) & \text{if } \frac{1+t}{2} \le \|x\| \le 1 \end{cases}$$

is the desired homeo.

Cor: Let $p: E \to B$ be a fibration. Suppose given continuous h, f $\ni$: rectangle

$$\begin{tikzcd} D^n\times 0 \cup S^{n-1}\times I \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ D^n \times I \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}$$

commutes.

Then $\exists$ continuous $\tilde{h}$ $\ni$: both triangles commute.

Proof of CHLET: Write $K_n^* = (|K|\times 0) \cup (K^n \cup |L|)\times I$.
Proof proceeds by inductively constructing a sequence of maps $\tilde{h}_n : K_n^* \to E$ satisfying

0) $\tilde{h}_{-1} = f$

1) $\tilde{h}_n | K_{n-1}^* = \tilde{h}_{n-1}$

2) $p\tilde{h}_n = h | K_n^*$

We would then have a well-defined function $\tilde{h}: |K|\times I \to E$ making the required triangles commute $\ni$: $\tilde{h}|K_n^* = \tilde{h}_n$.
$\tilde{h}$ would be continuous since each $\tilde{h}_n$ is and $K_n^* \supset K^n$.

<u>Construction of $\tilde{h}_0$</u>: For each 0-cell E^0_α of K not in L,

$$\begin{tikzcd}
E^0_\alpha \times 0 \arrow[r, "f|E^0_\alpha \times 0"] \arrow[d, hook] & E \arrow[d, "p"] \\
E^0_\alpha \times I \arrow[r, "h|E^0_\alpha \times I"'] \arrow[ur, dashed, "h^0_\alpha"] & B
\end{tikzcd}$$

$\exists$ continuous h^0_α $\ni$ both triangles commute

Define $\tilde{h}_0$ by $\tilde{h}_0 | (|K| \times 0) \cup (|L| \times I) = f$, $\tilde{h}_0 | E^0_\alpha \times I = h^0_\alpha$.

Suppose inductively $n > 0$ and $\tilde{h}_{n-1}$ constructed. Let E^n_α be an n-cell of K not in L, and $\eta^n_\alpha : D^n_\alpha \longrightarrow \bar{E}^n_\alpha$ the char. map for E^n_α. Note that

$\bar{E}^n_\alpha \cap K^*_{n-1} = \dot{E}^n_\alpha = \bar{E}^n_\alpha - E^n_\alpha$. Consider

$$\begin{tikzcd}
(D^n_\alpha \times 0) \cup (S^{n-1}_\alpha \times I) \arrow[r, "(\eta^n_\alpha \times 1)|"] \arrow[d, hook] & (\bar{E}^n_\alpha \times 0) \cup (\dot{E}^n_\alpha \times I) \arrow[r, "\tilde{h}_{n-1}|"] & E \arrow[d, "p"] \\
D^n_\alpha \times I \arrow[r, "\eta^n_\alpha \times 1"'] \arrow[urr, dashed, "k^n_\alpha"] & \bar{E}^n_\alpha \times I \arrow[r, "h|"'] & B
\end{tikzcd}$$

By cor. to lemma, $\exists$ cont. k^n_α making both triangles commute. Passing to quotients, k^n_α yields a continuous h^n_α making

$$\begin{tikzcd}
(\bar{E}_\alpha \times 0) \cup (\dot{E}^n_\alpha \times I) \arrow[r, "\tilde{h}_{n-1}|"] \arrow[d, hook] & E \arrow[d, "p"] \\
\bar{E}^n_\alpha \times I \arrow[r, "h|"'] & B
\end{tikzcd}$$

Define $\tilde{h}_n$ by $\tilde{h}_n | K^*_{n-1} = \tilde{h}_{n-1}$, $\tilde{h}_n | \bar{E}^n_\alpha \times I = h^n_\alpha$ for each n-cell E^n_α of K not in L.

Thm: Let $p : E \to B$ be a pointed fibration (i.e. a pointed map and a fibration). Write $F = p^{-1}(*)$. Then, writing $p : (E,F) \to (B,*)$ for the map of top. pairs which p yields, $p_* : \pi_n(E,F) \to \pi_n(B,*)$ is a bijection for all $n \geq 1$.

Proof: p_* is onto: Suppose $f : (S^{n-1}\wedge I, S^{n-1}) \to (B,*)$ is a pointed map. Consider

$$\begin{tikzcd} (S^{n-1}\times 0)\cup(*\times I) \arrow[rr, "*"] \arrow[d, hook] & & E \arrow[d, "p"] \\ S^{n-1}\times I \arrow[r, "\text{quot}"'] \arrow[urr, dashed, "\tilde{k}"] & S^{n-1}\wedge I \arrow[r, "f"'] & B \end{tikzcd}$$

$$(x,t) \longmapsto x\wedge t$$

Diagram commutes. Since $(S^{n-1},*)$ is the underlying space pair of a CW pair, by CHLET, $\exists$ continuous $\tilde{k}$ $\ni$ both triangles commute. Since $\tilde{k}(S^{n-1}\times 0 \cup *\times I) = *$, passage to quotients yields a pointed map $\tilde{h} : S^{n-1}\wedge I \to E$, $x\wedge t \longmapsto \tilde{k}(x,t)$

$\ni$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ S^{n-1}\wedge I \arrow[ur, "\tilde{h}"] \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

Since $\tilde{f}(x\wedge 1) = *$, we have $\tilde{h}(S^{n-1}) = \tilde{h}(S^{n-1}\wedge 1) \subset p^{-1}(*) = F$, i.e. $\tilde{h}$ yields a pointed map of pairs $\tilde{h} : (S^{n-1}\wedge I, S^{n-1}) \to (E,F)$ $\ni$ $p\tilde{h} = f$. Thus,

$[f] = p_*[\tilde{h}]$ and so p_* is onto.

p_* is 1-1: Suppose $p_*[f] = p_*[g]$ where $f, g : (S^{n-1}\wedge I, S^{n-1}) \to (E,F)$ are pointed. $\exists$ pointed homotopy

$h : (S^{n-1}\wedge I, S^{n-1})\times I \to (B,*)$ from pf to pg.

Consider

$$\begin{tikzcd} (S^{n-1}\times I)\times 0 \cup (S^{n-1}\times 0 \cup S^{n-1}\times 1 \cup *\times I)\times I \arrow[rr, "r"] \arrow[d, hook] & & E \arrow[d, "p"] \\ S^{n-1}\times I\times I \arrow[r, "q"'] \arrow[urr, dashed, "\tilde{k}"] & (S^{n-1}\wedge I)\times I \arrow[r, "h"'] & B \end{tikzcd}$$

where $q(x,s,t) = (x\wedge t, s)$,

$$\begin{aligned}
r(x,s,0) &= *\\
r(x,0,t) &= f(x\wedge t)\\
r(x,1,t) &= g(x\wedge t)\\
r(*,s,t) &= * \quad .
\end{aligned}$$

Straightforward to check r is continuous and diagram commutes. $(S^{n-1}\times I,\ S^{n-1}\times 0 \cup S^{n-1}\times 1 \cup * \times I)$ is the underlying space pair of a CW pair. Hence by CHLET, $\exists$ continuous $\tilde{k}$ making both triangles commute. Since $\tilde{k}(x,s,0) = r(x,s,0) = *$ and $\tilde{k}(*,s,t) = r(*,s,t) = *$, passage to quotients yields a continuous $\tilde{h}: (S^{n-1}\wedge I)\times I \longrightarrow E$ $\ni$

$$\begin{tikzcd}
(S^{n-1}\times I)\times 0 \cup (S^{n-1}\times 0 \cup S^{n-1}\times 1 \cup * \times I)\times I \arrow[rr, "r"] \arrow[d, hook] & & E \arrow[d, "p"] \\
S^{n-1}\times I \times I \arrow[r, "q"] & (S^{n-1}\wedge I)\times I \arrow[r, "h"] \arrow[ur, "\tilde{h}"] & B
\end{tikzcd}$$

commutes.

Since $h(S^{n-1}\times I) = *$, we have $\tilde{h}(S^{n-1}\times I) \subset p^{-1}(*) \doteq F$. We have $\tilde{h}(*,t) = \tilde{h}q(*,t,0) = r(*,t,0) = *$. Thus $\tilde{h}: (S^{n-1}\wedge I, S^{n-1})\times I \to (E,F)$ is a pointed homotopy. We have $\tilde{h}(x\wedge t, 0) = \tilde{h}q(x,0,t) = r(x,0,t) = f(x\wedge t)$, $\tilde{h}(x\wedge t, 1) = \tilde{h}q(x,1,t) = r(x,1,t) = g(x\wedge t)$. Thus p_* is 1-1.

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21 Thus for a pointed fibration $E \xrightarrow{p} B$ with fibre $F = p^{-1}(*)$, we have

exact

$$\begin{tikzcd}
\to \pi_n(F) \arrow[r, "i_*"] & \pi_n(E) \arrow[r, "j_*"] \arrow[dr, "p_*"'] & \pi_n(E,F) \arrow[r, "\partial"] \arrow[d, "p_*", "\cong"'] & \pi_{n-1}(F) \to \cdots \to \pi_1(E,F) \arrow[r, "\partial"] \arrow[d, "p_*"', "\cong"] & \pi_0(F) \arrow[r, "i_*"] & \pi_0(E) \\
& & \pi_n(B) & \pi_1(B) & &
\end{tikzcd}$$

and so we obtain the exact sequence

$$\to \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \to \cdots \to \pi_1(B) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(E)$$

called the <u>homotopy</u> <u>sequence</u> <u>of</u> <u>the</u> <u>pointed</u> <u>fibration</u> (E,p,B,F).

This exact sequence is natural w.r.t. to morphisms of pointed fibrations, i.e. if

$$\begin{tikzcd}
E_1 \arrow[r, "f_E"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\
B_1 \arrow[r, "f_B"'] & B_2
\end{tikzcd}$$

commutes where p_1, p_2 are pointed fibrations, f_E, f_B pointed maps, and $F_i = p_i^{-1}(*)$, then

$$\begin{tikzcd}
\arrow[r] & \pi_n(F_1) \arrow[r, "i_*"] \arrow[d, "f_{F*}"] \arrow[dr, phantom, "1"] & \pi_n(E_1) \arrow[r, "p_{1*}"] \arrow[d, "f_{E*}"] \arrow[dr, phantom, "2"] & \pi_n(B_1) \arrow[r, "\partial"] \arrow[d, "f_{B*}"] \arrow[dr, phantom, "3"] & \pi_{n-1}(F_1) \arrow[r] \arrow[d, "f_{F*}"] & \cdots \\
\arrow[r] & \pi_n(F_2) \arrow[r, "i_*"'] & \pi_n(E_2) \arrow[r, "p_{2*}"'] & \pi_n(B_2) \arrow[r, "\partial"'] & \pi_{n-1}(F_2) \arrow[r] & \cdots
\end{tikzcd}$$

commutes where f_F = restriction of f_E.

1, 2 commute since π_n is a functor.

Commutativity of 3:

$$\begin{tikzcd}
\pi_n(B_1) \arrow[rr, "\partial"] \arrow[dddd, "f_{B*}"'] & & \pi_{n-1}(F_1) \arrow[dddd, "f_{F*}"] \\
& \text{def. of } \partial \text{ on top} & \\
& \pi_n(E_1,F_1) \arrow[uul, "p_{1*}", "\cong"'] \arrow[uur, "\partial"'] \arrow[d, "f_*"] & \\
\pi_n \text{ is functor} & \pi_n(E_2,F_2) \arrow[dl, "p_{2*}"', "\cong"] \arrow[dr, "\partial"] & \text{nat. of } \partial \text{ in homotopy seq. of pair} \\
\pi_n(B_2) \arrow[rr, "\partial"'] & \text{def. of } \partial \text{ below} & \pi_{n-1}(F_2)
\end{tikzcd}$$

<u>Example</u>: $\begin{array}{c} \mathbb{R} \\ \downarrow p \\ S^1 = \mathbb{R}/\mathbb{Z} \end{array}$ is a fibre bundle with fibre $\mathbb{Z}$.

$\mathbb{R}$ is contractible, so $\pi_n(\mathbb{R}) = 0$ for all n. Thus for all $n \geq 2$, $\pi_n(S^1) \xrightarrow[\cong]{\partial} \pi_{n-1}(\mathbb{Z})$. $\mathbb{Z}$ is discrete, so $\pi_{n-1}(\mathbb{Z}) = 0$ for $n \geq 2$. Thus $\pi_n(S^1) = 0$ for all $n \geq 2$.

Example: Regard S^1 as $\{z \in \mathbb{C} \mid |z| = 1\}$, and S^{2n+1} as $\{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum_i |z_i|^2 = 1\}$. Then

$$\begin{array}{ccc} S^{2n+1} & & (z_1, \dots, z_{n+1}) \\ \downarrow p & & \downarrow \\ \mathbb{C}P^n & & [z_1, \dots, z_{n+1}] \end{array}$$

is a fibre bundle projection with fibre S^1.

Thus $p_* : \pi_i(S^{2n+1}) \to \pi_i(\mathbb{C}P^n)$ is an isom. for all $i \geq 3$.

Special case $n=1$: $\mathbb{C}P^1 \cong S^2$. Get fibre bundle $S^3 \to S^2$ with fibre S^1 (one of the Hopf fibrations). Thus

$\pi_i(S^3) \xrightarrow{\cong} \pi_i(S^2)$ for $i \geq 3$.

In particular $\pi_3(S^2) \cong \pi_3(S^3)$. Since $1 : S^3 \to S^3$ is not $\simeq *$ (from homology), $\pi_3(S^3) \neq 0$. Thus $\pi_3(S^2) \neq 0$.

Homotopy sequence of a pointed triple

Let (Y, A, B) be a pointed triple, i.e. $* \in B \subset A \subset Y$. Let $i : (A, B) \subset (Y, B)$, $j : (Y, B) \subset (Y, A)$, $k : (A, *) \subset (A, B)$ denote the inclusions. Define $\tilde{\partial} : \pi_n(Y, A) \to \pi_{n-1}(A, B)$, $n \geq 2$, to be the composition $\pi_n(Y, A) \xrightarrow{\partial} \pi_{n-1}(A, *) \xrightarrow{k_*} \pi_{n-1}(A, B)$ where ∂ is from the homotopy seq. of (Y, A). Then, we get a sequence

$$\to \pi_n(A, B) \xrightarrow{i_*} \pi_n(Y, B) \xrightarrow{j_*} \pi_n(Y, A) \xrightarrow{\tilde{\partial}} \pi_{n-1}(A, B) \to \cdots \to \pi_1(A, B) \to \pi_1(Y, B) \to \pi_1(Y, A)$$

called the homotopy sequence of the pointed triple. (If $B = *$, above is homotopy seq. of pointed pair (Y, A)).

HW Exercise: Homotopy seq. of pointed triple is exact and natural.

Note: If $p : E \to B$ a fibration, $B_0 \subset B$, then $p^{-1}(B_0) \xrightarrow[p|]{} B_0$ is a fibration.

Prop: $p : E \to B$ a pointed fibration with fibre F. Let $* \in B_0 \subset B$, and $E_0 = p^{-1}(B_0)$. Then $p_* : \pi_n(E, E_0) \to \pi_n(B, B_0)$ is an isom. for $n \geq 2$.

Proof: $p : (E, E_0, F) \to (B, B_0, *)$ is a map of pointed triples. Hence, get comm. diagram with exact rows

$$\begin{tikzcd} \pi_n(E_0,F) \arrow[r] \arrow[d, "p_*", "\cong"'] & \pi_n(E,F) \arrow[r] \arrow[d, "p_*", "\cong"'] & \pi_n(E,E_0) \arrow[r, "\tilde\partial"] \arrow[d, "p_*"] & \pi_{n-1}(E_0,F) \arrow[r] \arrow[d, "p_*", "\cong"'] & \pi_{n-1}(E,F) \arrow[d, "p_*", "\cong"'] \\ \pi_n(B_0,*) \arrow[r] & \pi_n(B,*) \arrow[r] & \pi_n(B,B_0) \arrow[r, "\tilde\partial", "\text{group hom.}"'] & \pi_{n-1}(B_0,*) \arrow[r] & \pi_{n-1}(B,*) \end{tikzcd}$$

Indicated bijections follow from earlier theorem. Hence middle p_* is $\cong$ by 5-lemma.

Note: Separate argument shows p_* is a bijection when $n=1$.

Hurewicz Map

For $n \geq 1$, let $p_n : (S^{n-1} \wedge I, S^{n-1}) \to (S^{n-1} \wedge S^1, *) = (S^n, *)$ be the map collapsing S^{n-1} to $*$. p_n induces $\cong$ in homology (good rel. homeo.).

Since $S^{n-1} \wedge I$ is contractible, $\partial : H_n(S^{n-1} \wedge I, S^{n-1}) \to H_{n-1}(S^{n-1}, *)$ in homology seq. of triple $(S^{n-1} \wedge I, S^{n-1}, *)$ is an isom., and $H_{n-1}(S^{n-1}, *) \cong \mathbb{Z}$ for all $n \geq 1$.

Regarding S^1 as $I/\partial I$, the quot. map $I \to I/\partial I$ is a singular 1-cube whose image in $Q_1(S^1, *)$ is a cycle representing a generator κ_1 of $H_1(S^1, *)$. Let $\iota_1 \in H_1(S^0 \wedge I, S^0)$ be the unique generator $\ni$ $p_{1*}(\iota_1) = \kappa_1$, and let $\kappa_0 = \partial \iota_1 \in H_0(S^0, *)$ (∂ as above).
For $n \geq 2$, inductively define generators $\iota_n \in H_n(S^{n-1} \wedge I, S^{n-1})$ and $\kappa_n \in H_n(S^n, *)$ by ι_n = unique generator $\ni$ $\partial \iota_n = \kappa_{n-1}$ and $\kappa_n = p_{n*}(\iota_n)$.

Def: Let (Y, B) be a pointed pair. For $n \geq 1$, define $h_n : \pi_n(Y, B) \to H_n(Y, B)$ as follows: If $f : (S^{n-1} \wedge I, S^{n-1}) \to (Y, B)$ a pointed map, $h_n[f] = f_*(\iota_n)$.

(Note: h_n is well-defined by the homotopy property for homology).

h_n is called the Hurewicz map.

Note: If $B=*$, the identification

$$[S^n, Y] \cong [(S^{n-1}\wedge I, S^{n-1}), (Y,*)]$$

is $[g] \longmapsto [g p_n]$ for every pointed $g:(S^n,*)\to(Y,*)$.
Thus $h_n[g] = (gp_n)_*(\iota_n) = g_* p_{n*}(\iota_n) = g_*(\kappa_n)$.

When $B=*$, also define $h_0 : \pi_0(Y,*) \to H_0(Y,*)$ by $h_0[f] = f_*(\kappa_0)$, $f:(S^0,*)\to(Y,*)$.

22 Prop: h_n is natural for all n. Moreover, if (Y,B) is a pointed pair,

$$\begin{array}{ccc} \pi_n(Y,B) & \xrightarrow{\partial} & \pi_{n-1}(B,*) \\ {\scriptstyle h_n}\downarrow & & \downarrow{\scriptstyle h_{n-1}} \\ H_n(Y,B) & \xrightarrow[\partial]{} & H_{n-1}(B,*) \end{array}$$

commutes for all $n\geq 1$,

where top ∂ is from homotopy seq. of (Y,B), bottom ∂ from homology seq. of triple $(Y,B,*)$.

Proof: Suppose $n\geq 1$ and $g:(X,A)\to(Y,B)$ a pointed map of top. pairs. If $f:(S^{n-1}\wedge I, S^{n-1})\to(X,A)$ is pointed, we have $g_*[f]=[gf]$ and so $g_* h_n[f] = g_* f_*(\iota_n) = (gf)_*(\iota_n)$
$= h_n[gf] = h_n g_*[f]$, so h_n is nat. Similarly h_0 is natural.

Let $n\geq 1$ and $f:(S^{n-1}\wedge I, S^{n-1})\to(Y,B)$ a pointed map. Then $\partial[f]$ is repr. by $f|S^{n-1}:(S^{n-1},*)\to(B,*)$. f yields a map of triples $(S^{n-1}\wedge I, S^{n-1},*)\to(Y,B,*)$, and by nat. of homology seq. of a triple, $\partial f_*\iota_n = (f|S^{n-1})_*\partial\iota_n$.
Hence, $\partial h_n[f] = \partial f_*(\iota_n) = (f|S^{n-1})_*(\partial\iota_n) = (f|S^{n-1})_*(\kappa_{n-1})$
$= h_{n-1}[f|S^{n-1}] = h_{n-1}\partial[f]$.

Next: Show when π_n is group-valued, h_n is a group hom.

Lemma: Let $n\geq 1$ and $\mu':(S^n,*)\to(S^n\vee S^n,*)$ the comult. Let $i_1,i_2:(S^n,*)\to(S^n\vee S^n,*)$ denote the inclusions on the 1st + 2nd leaves, resp. Then $\mu'_*(\kappa_n) = i_{1*}\kappa_n + i_{2*}\kappa_n$.

Proof. $H_n(S^n \vee S^n, *)$ is free abel. with basis $i_{1*}\kappa_n$, $i_{2*}\kappa_n$.
Hence $\mu'_*(\kappa_n) = a\, i_{1*}\kappa_n + b\, i_{2*}\kappa_n$ for unique $a, b \in \mathbb{Z}$.
Let $p_i : (S^n \vee S^n, *) \to (S^n, *)$ denote proj. on i^{th} leaf, $i = 1, 2$, and $q_i : (S^n \times S^n, *) \to (S^n, *)$ projection on i^{th} factor, $i = 1, 2$.
Following commutes up to homotopy:

$$\begin{tikzcd}
 & (S^n \vee S^n, *) \arrow[d, "i"] \arrow[dr, "p_i"] & \\
(S^n, *) \arrow[ur, "\mu'"] \arrow[r, "d"'] & (S^n \times S^n, *) \arrow[r, "q_i"'] & (S^n, *)
\end{tikzcd}$$

Moreover $p_1 i_1 = p_2 i_2 = 1_{S^n}$, $p_1 i_2 = p_2 i_1 = *$, $q_1 d = q_2 d = 1_{S^n}$. Hence $p_{1*} i_{1*} = 1$, $p_{1*} i_{2*} = 0$, $q_{1*} d_* = 1$ and so $\kappa_n = 1\,\kappa_n = q_{1*} d_* \kappa_n = p_{1*} \mu'_* \kappa_n$

$$= p_{1*}(a\, i_{1*}\kappa_n + b\, i_{2*}\kappa_n) = a\, p_{1*} i_{1*}\kappa_n + b\, p_{1*} i_{2*}\kappa_n$$
$$= a\,\kappa_n,$$ and so $a = 1$. Similarly, $b = 1$.

Prop: Let $n \geq 1$. Then $h_n : \pi_n(Y, *) \to H_n(Y, *)$ is a group hom.

Proof: Let $f, g : (S^n, *) \to (Y, *)$. Then $[f] + [g] = [f \square g]$ where $f \square g$ is the composition

$$(S^n, *) \xrightarrow{\mu'} (S^n \vee S^n, *) \xrightarrow{f \vee g} (Y \vee Y, *) \xrightarrow{\varphi} (Y, *).$$

Following commutes:

$$\begin{tikzcd}
 & (S^n, *) \arrow[r, "f"] \arrow[d, "i_1"] & (Y, *) \arrow[d, "i_1"] \arrow[dr, "1"] & \\
(S^n, *) \arrow[r, "\mu'"] & (S^n \vee S^n, *) \arrow[r, "f \vee g"] & (Y \vee Y, *) \arrow[r, "\varphi"] & (Y, *) \\
 & (S^n, *) \arrow[r, "g"'] \arrow[u, "i_2"] & (Y, *) \arrow[u, "i_2"] \arrow[ur, "1"'] &
\end{tikzcd}$$

Hence $h_n([f] + [g]) = h_n([f \square g]) = (f \square g)_*(\kappa_n) = \varphi_*(f \vee g)_* \mu'_*(\kappa_n)$
$= \varphi_*(f \vee g)_*(i_{1*}\kappa_n + i_{2*}\kappa_n) = f_*\kappa_n + g_*\kappa_n$

$= h_n[f] + h_n[g]$.

Note: For any pointed Y, $P(Y;*,Y)$ is contractible in the pointed sense. In fact $h: P(Y;*,Y)\times I \to P(Y;*,Y)$ given by $h(\lambda,s)(t)=\lambda(st)$ is a pointed homotopy from $*$ to 1.

Prop: For $n\geq 2$, $h_n: \pi_n(Y,B)\to H_n(Y,B)$ is a group homomorphism.

Proof: Have pointed fibration $p: P(Y;*,Y)\to Y$ and $p^{-1}(B)=P(Y;*,B)$. Have commutative diagram

$$\begin{array}{ccc}
\pi_n(P(Y;*,Y),P(Y;*,B)) & \xrightarrow{h_n} & H_n(P(Y;*,Y),P(Y;*,B)) \\
{\scriptstyle(\text{since } p \text{ a fibration})}\ \cong\big\downarrow p_* & & \big\downarrow p_* \\
\pi_n(Y,B) & \xrightarrow[h_n]{} & H_n(Y,B)
\end{array}$$

Hence, since vertical maps are group homs., suff. to show top h_n is a group hom.

Have comm. diagram

$$\begin{array}{ccc}
\pi_n(P(Y;*,Y),P(Y;*,B)) & \xrightarrow[\cong]{\partial} & \pi_{n-1}(P(Y;*,B),*) \\
\big\downarrow h_n & & \big\downarrow h_{n-1} \\
H_n(P(Y;*,Y),P(Y;*,B)) & \xrightarrow[\cong]{\partial} & H_{n-1}(P(Y;*,B),*)
\end{array}$$

with both ∂'s isomorphisms since $P(Y;*,Y)$ is contractible. h_{n-1} is a group hom. by absolute case, and so h_n is also.

The little Hurewicz Theorem

Thm: Suppose X is a path-connected pointed space. Then $h_1: \pi_1(X,*)\to H_1(X,*)$ is onto, and the kernel of h_1 is the commutator subgroup of $\pi_1(X,*)$.

Path composition: Suppose $\sigma, \tau : I \to X$ are singular 1-cubes $\ni$: $\sigma(1) = \tau(0)$. We can form a singular 1-cube $\sigma * \tau : I \to X$ given by

$$(\sigma * \tau)(t) = \begin{cases} \sigma(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \tau(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Notation: If $c_1, c_2 \in Q_n^u(X)$, write $c_1 \equiv c_2$ if $\exists$ $e \in Q_{n+1}^u(X) \ni$: $c_1 - c_2 - \partial e \in D_n(X)$.

Thus if $p_n : Q_n^u(X) \to Q_n(X,*)$ is the projection and if $p_n(c_i)$ are cycles, $i = 1,2$, $c_1 \equiv c_2 \Rightarrow p_n(c_1) \sim p_n(c_2)$ and hence $\{p_n(c_1)\} = \{p_n(c_2)\}$. (Write $\{\ \}$ for homology class).

Note: For $n \ge 1$, $Q_n(X,*) = Q_n(X)$ since $Q_n(*) = 0$ for $n \ge 1$. Thus $Q_n(X,*) = Q_n^u(X)/D_n(X)$ for $n \ge 1$. Easily checked: $\equiv$ is an equivalence rel., and $c_1 \equiv c_2, d_1 \equiv d_2 \Rightarrow m c_1 + n d_1 \equiv m c_2 + n d_2$ $\forall m,n \in \mathbb{Z}$.

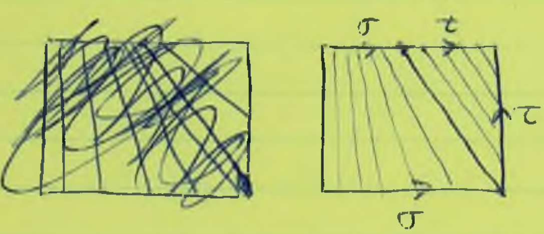
Homology properties of path-composition

23 **Lemma**: 1) Let $\sigma, \tau : I \to X$ be sing. 1-cubes $\ni$: $\sigma(1) = \tau(0)$. Then $\sigma * \tau \equiv \sigma + \tau$.

2) Given $\sigma : I \to X$ a sing. 1-cube, let $\bar\sigma : I \to X$ be the sing. 1-cube given by $\bar\sigma(t) = \sigma(1-t)$. Then $\bar\sigma \equiv -\sigma$.

Proof: 1)

Level curves

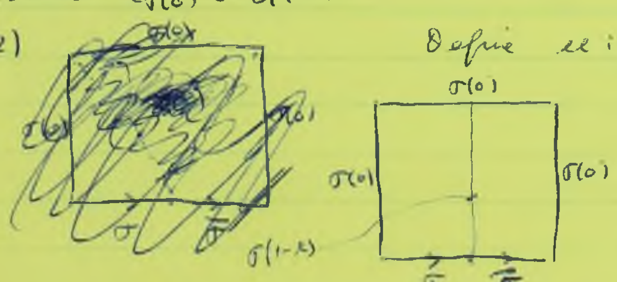


Define $u : I^2 \to X$ by

$$u(s,t) = \begin{cases} \sigma\left(\frac{2s}{2-t}\right) & \text{if } 0 \le s \le 1 - \frac{t}{2} \\ \tau(t + 2s - 2) & \text{if } 1 - \frac{t}{2} \le s \le 1. \end{cases}$$

Then $\partial u = \sigma + \tau - \sigma * \tau - c_{\sigma(0)}$ where $c_{\sigma(0)}$ = const. map with image $\sigma(0)$. Done since $c_{\sigma(0)} \in D_1(X)$.

2)



Define $u : I^2 \to X$ by

$$u(s,t) = \begin{cases} \sigma(2s(1-t)) & \text{if } 0 \le s \le \frac{1}{2} \\ \bar\sigma(2s - 1 + t(2-2s)) & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

Then $\partial u = \sigma * \bar\sigma - c_{\sigma(0)}$ and so $\sigma * \bar\sigma \equiv 0$. Thus by 1), $\sigma + \bar\sigma \equiv 0$ yielding $\bar\sigma \equiv -\sigma$.

<u>Note</u>: Let $g : I \to I/\partial I = S^1$ denote the projection. Then $\kappa_1 = \{p_1(g)\}$, $p_1 : Q_1^u(S^1) \to Q_1(S^1, *)$. Thus if $f : S^1 \to X$ is a pointed map, $h_1[f] = f_*(\kappa_1) = f_*\{p_1(g)\} = \{p_1(fg)\}$.

<u>Note</u>: If $\sigma : I \to X$ a singular 1-cube $\ni$ $\sigma(0) = \sigma(1) = *$, then $p_1(\sigma)$ is a cycle in $Q_1(X, *)$ and $\{p_1(\sigma)\} \in$ im h_1 since $\exists$ factorization

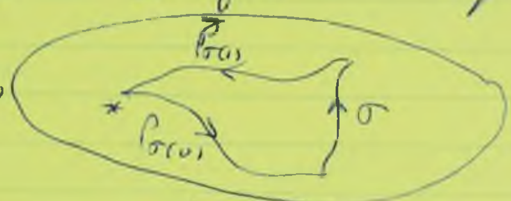
$$\begin{tikzcd} I \arrow[r, "g"] \arrow[dr, "\sigma"'] & S^1 \arrow[d, "f"] \\ & X \end{tikzcd}$$

, f pointed,

and so $h_1[f] = \{p_1(fg)\} = \{p_1(\sigma)\}$.

<u>Proof that h_1 is onto</u>: For each $x \in X$, choose a path $\rho_x : I \to X$ $\ni$ $\rho_x(0) = *$, $\rho_x(1) = x$ (possible since X is path-connected). For $x \in X$, let $c_x : I^0 \to X$ denote the sing. 0-cube with image
Let $\theta : Q_0(X) \to Q_1^u(X)$ denote the unique homomorphism $\ni$ $\theta(c_x) = \rho_x$.

For each sing. 1-cube $\sigma : I \to X$, let $\tilde{\sigma} = (\rho_{\sigma(0)} * \sigma) * \bar{\rho}_{\sigma(1)}$.



Then $\tilde{\sigma} \equiv \rho_{\sigma(0)} + \sigma + \bar{\rho}_{\sigma(1)}$, and $\{p_1(\tilde{\sigma})\} \in$ im h_1.

Let $a \in H_1(X, *)$. Then $a = \{p_1(\sum_i n_i \sigma_i)\}$, $n_i \in \mathbb{Z}$, $\sigma_i : I \to X$ a finite collection of sing. 1-cubes. Since

$$\begin{tikzcd} Q_1^u(X) \arrow[r, "p_1"] \arrow[d, "\partial"] & Q_1(X) = Q_1(X,*) \arrow[d] \\ Q_0^u(X) = Q_0(X) \arrow[r] & Q_0(X,*) \end{tikzcd}$$

$p_1(\sum_i a_i \sigma_i)$ is a cycle in $Q_1(X, *)$ we have $\partial(\sum_i n_i \sigma_i) \in Q_0(*)$. Since $\varepsilon\partial = 0$ and $\varepsilon : Q_0(*) \to \mathbb{Z}$ is an isom., we must have $\partial(\sum_i n_i \sigma_i) = 0$. Thus $0 = \sum_i n_i (c_{\sigma_i(1)} - c_{\sigma_i(0)})$, and so

$$0 = \theta(0) = \sum_i n_i (\rho_{\sigma_i(1)} - \rho_{\sigma_i(0)}).$$

Thus $\sum_i n_i \sigma_i = \sum_i n_i (\sigma_i - \rho_{\sigma_i(1)} + \rho_{\sigma_i(0)}) \equiv \sum_i n_i (\sigma_i + \bar{\rho}_{\sigma_i(1)} + \rho_{\sigma_i(0)})$

$\equiv \sum_i n_i \tilde{\sigma}_i$, and so $a = \{p_1(\sum_i n_i \tilde{\sigma}_i)\} = \sum_i n_i \{p_1(\tilde{\sigma}_i)\} \in$ im h_1

since each $\{p_1(\tilde{\sigma}_i)\} \in$ im h_1 and h_1 is a group hom.

Write Γ = commutator subgroup of $\pi_1(X,*)$. Since $H_1(X,*)$ is abelian, $\Gamma \subset \ker h_1$. Let $\pi : \pi_1(X,*) \to \pi_1(X,*)/\Gamma$ be the projection. To complete the proof of the little Hurewicz theorem, suffices to show $\exists$ group homomorphism
$$\alpha : H_1(X,*) \to \pi_1(X,*)/\Gamma \ni.$$

$$\begin{tikzcd}
\pi_1(X,*) \arrow[rr, "\pi"] \arrow[dr, "h_1"'] & & \pi_1(X,*)/\Gamma \\
& H_1(X,*) \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

commutes

(for since $\ker \pi = \Gamma$, it would then follow that $\ker h_1 \subset \Gamma$).

Path-homotopy rel endpoints.

Def: If $\sigma, \tau : I \to X$ are sing. 1-cubes $\ni$: $\sigma(0) = \tau(0)$, $\sigma(1) = \tau(1)$, we say σ is homotopic to τ rel. endpoints (written $\sigma \ {}_0{\simeq}_0\ \tau$) if $\exists$ homotopy $h : I \times I \to X \ni$. $h(t,0) = \sigma(t)$, $h(t,1) = \tau(t)$, $h(0,s) = \sigma(0) = \tau(0)$, $h(1,s) = \sigma(1) = \tau(1)$ for all s,t. Write $\sigma \ {}_0\underset{h}{\simeq}{}_0\ \tau$ in this case.

Same proof that $\simeq$ is an equivalence relation yields ${}_0{\simeq}_0$ is an equivalence relation.

Homotopy properties of path composition

Lemma: 1) If $\sigma_1 \ {}_0{\simeq}_0\ \sigma_2$, $\tau_1 \ {}_0{\simeq}_0\ \tau_2$, and $\sigma_i(1) = \tau_i(0)$, then $\sigma_1 * \tau_1 \ {}_0{\simeq}_0\ \sigma_2 * \tau_2$.

2) If $\sigma, \tau, u : I \to X$ satisfy $\sigma(1) = \tau(0)$, $\tau(1) = u(0)$, then $(\sigma * \tau) * u \ {}_0{\simeq}_0\ \sigma * (\tau * u)$.

3) $\sigma * \bar{\sigma} \ {}_0{\simeq}_0\ c_{\sigma(0)}$

4) $c_{\sigma(0)} * \sigma \ {}_0{\simeq}_0\ \sigma \ {}_0{\simeq}_0\ \sigma * c_{\sigma(1)}$.

Proof. 1) Say $\sigma_1 \ {}_0\underset{h}{\simeq}{}_0\ \sigma_2$, $\tau_1 \ {}_0\underset{k}{\simeq}{}_0\ \tau_2$. Define

$$\ell : I \times I \to X \text{ by } \ell(s,t) = \begin{cases} h(2t, s) & \text{if } 0 \le t \le \frac{1}{2} \\ k(2t-1, s) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then $\sigma_1 * \tau_1 \ {}_0\underset{\ell}{\simeq}{}_0\ \sigma_2 * \tau_2$.

2) Define $h : I \times I \to X$ by

$$h(t,s) = \begin{cases} \sigma\left(\frac{4t}{1+s}\right) & \text{if } 0 \le t \le \frac{1}{4}(1+s) \\ \tau(4t-1-s) & \text{if } \frac{1}{4}(1+s) \le t \le \frac{1}{4}(2+s) \\ u\left(\frac{4t-s-2}{2-s}\right) & \text{if } \frac{1}{4}(2+s) \le t \le 1. \end{cases}$$

Then $(\sigma * \tau) * u \underset{h}{\circ\!\simeq\!\circ} \sigma * (\tau * u)$.

3) Define $h : I \times I \to X$ by

$$h(t,s) = \begin{cases} \sigma(2t(1-s)) & \text{if } 0 \le t \le \frac{1}{2} \\ \bar{\sigma}(2t-1+s(2-2t)) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then $\sigma * \bar{\sigma} \underset{h}{\circ\!\simeq\!\circ} c_{\sigma(0)}$

4) Define $h : I \times I \to X$ by

$$h(t,s) = \begin{cases} \sigma(0) & \text{if } 0 \le t \le \frac{s}{2} \\ \sigma\left(\frac{2t-s}{2-s}\right) & \text{if } \frac{s}{2} \le t \le 1. \end{cases}$$

Then $\sigma \underset{h}{\circ\!\simeq\!\circ} c_{\sigma(0)} * \sigma$. Similarly $\sigma \,\circ\!\simeq\!\circ\, \sigma * c_{\sigma(1)}$.

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24 If $\sigma : I \to X$ a path, write $\langle \sigma \rangle$ = path homotopy class rel. end points of σ.

Suppose $\sigma_1, \dots, \sigma_n : I \to X$ are paths $\ni$: $\sigma_i(1) = \sigma_{i+1}(0)$, $1 \le i \le n-1$. Then because of assoc. of $*$ up to $\circ\!\simeq\!\circ$,

$\langle \sigma_1 * \cdots * \sigma_n \rangle$ is unambiguous, indep. of parenthesization.

If $\sigma : I \to X$ satisfies $\sigma(0) = \sigma(1) = *$, then σ factors as

$$\begin{tikzcd} I \arrow[r, "g"] \arrow[dr, "\sigma"'] & I/\partial I = S^1 \arrow[d, "\underline{\sigma}"] \\ & X \end{tikzcd}$$

where $\underline{\sigma}$ is a pointed map. Note that $[\underline{\sigma}] \in \pi_1(X,*)$

depends only on $\langle\sigma\rangle$, and if $\sigma_1, \dots, \sigma_n : I \to X$ satisfy $\sigma_1(0) = *$, $\sigma_i(1) = \sigma_{i+1}(0)$, $1 \le i \le n-1$, and $\sigma_n(1) = *$, then $[\underline{\sigma_1 * \cdots * \sigma_n}] \in \pi_1(X,*)$ is unambiguously defined (indep. of parenth.) If $\sigma, \tau : I \to X$ both send $0, 1$ to $*$, it is immediate that $\underline{\sigma * \tau} = \underline{\sigma} \,\square\, \underline{\tau}$ and so

$[\underline{\sigma * \tau}] = [\underline{\sigma}][\underline{\tau}]$. Note also $\underline{\bar\sigma} = \underline{\sigma}\,\eta'$ and so

$[\underline{\bar\sigma}] = [\underline{\sigma}]^{-1}$. Also $[\underline{c_*}] =$ unit elt of $\pi_1(X,*)$.

Note: $\overline{\sigma * \tau} = \bar\tau * \bar\sigma$, $\bar{\bar\sigma} = \sigma$.

In this notation, $h_1[\underline{\sigma}] = \{\rho_1(\sigma)\}$, since $\sigma = \underline{\sigma}\,\eta$.

Construction of $\alpha : H_1(X,*) \to \pi_1(X,*)/\Gamma$:

As before, for each $x \in X$, choose a path $\rho_x : I \to X$ from $*$ to x. Define a homomorphism $\beta : Q_1{}^u(X) \to \pi_1(X,*)/\Gamma$ as follows: For each sing. 1-cube $\sigma : I \to X$, define

$\beta(\sigma) = [\underline{\rho_{\sigma(0)} * \sigma * \bar\rho_{\sigma(1)}}]\,\Gamma$. (OK since $\pi_1(X,*)/\Gamma$ is abel. and the sing. 1-cubes form a free abel. basis of $Q_1{}^u(X)$).

If σ is degenerate, then $\sigma = c_x$ for some $x \in X$, and

$\langle \rho_x * c_x * \bar\rho_x \rangle = \langle \rho_x * \bar\rho_x \rangle = \langle c_* \rangle$ and hence

$\beta(c_x) = [\underline{c_*}]\,\Gamma = 1 \in \pi_1(X,*)/\Gamma$. Thus, passing to quotients, β induces a group hom.

$$\gamma : Q_1(X) = Q_1(X,*) \longrightarrow \pi_1(X,*)/\Gamma.$$

Let

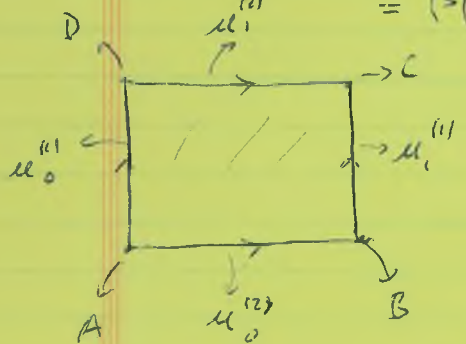
$$\delta = \gamma \,|\, Z_1(Q(X,*)) : Z_1(Q(X,*)) \longrightarrow \pi_1(X,*)/\Gamma.$$

Claim: $\delta(B_1(Q(X,*))) = 1 \in \pi_1(X,*)/\Gamma$.

For suppose $u : I^2 \to X$ is a singular 2-cube. Suffices to show $\beta(\partial u) = 1$.

We have $\beta(\partial u) = \beta(u_1^{(1)} - u_0^{(1)} - u_1^{(2)} + u_0^{(2)})$

$$= \beta(u_0^{(2)})\beta(u_1^{(1)})\beta(u_1^{(2)})^{-1}\beta(u_0^{(1)})^{-1}.$$



Write $u(0,0) = A$, $u(1,0) = B$, $u(1,1) = C$, $u(0,1) = D$. Then

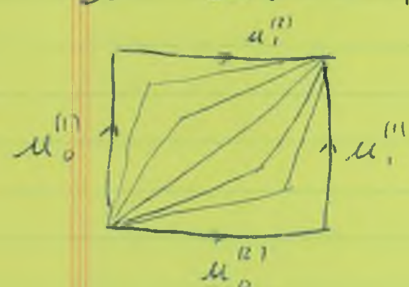
$$\beta(\partial u) = [\rho_A * u_0^{(2)} * \bar{\rho}_B][\rho_B * u_1^{(1)} * \bar{\rho}_C][\rho_D * u_1^{(2)} * \bar{\rho}_C]^{-1}[\rho_A * u_0^{(1)} * \bar{\rho}_D]^{-1}\Gamma$$

$$= [\rho_A * u_0^{(2)} * \bar{\rho}_B * \rho_B * u_1^{(1)} * \bar{\rho}_C * \rho_C * \bar{u}_1^{(2)} * \bar{\rho}_D * \rho_D * \bar{u}_0^{(1)} * \bar{\rho}_A]\Gamma$$

$$= [\rho_A * u_0^{(2)} * u_1^{(1)} * \bar{u}_1^{(2)} * \bar{u}_0^{(1)} * \bar{\rho}_A]\Gamma$$

$$= [\rho_A * (u_0^{(2)} * u_1^{(1)}) * (\overline{u_0^{(1)} * u_1^{(2)}}) * \bar{\rho}_A]\Gamma$$

But $u_0^{(2)} * u_1^{(1)} \simeq_0 u_0^{(1)} * u_1^{(2)}$.



In fact $u_0^{(2)} * u_1^{(1)} \simeq_h u_0^{(1)} * u_1^{(2)}$ where $h : I \times I \to X$ is given by

$$h(s,t) = \begin{cases} u(2s(1-t), 2st) & \text{if } 0 \le s \le \frac{1}{2} \\ u(1 + (2s-2)t, t + (2s-1)(1-t)) & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

Thus, writing $\sigma = u_0^{(1)} * u_1^{(2)}$,

$$\beta(\partial u) = [\rho_A * \sigma * \bar{\sigma} * \bar{\rho}_A]\Gamma = [\underline{c}_*]\Gamma = 1.$$

Thus, passing to quotients, β yields a group homomorphism

$\alpha : H_1(X,*) \to \pi_1(X,*)/\Gamma$ given by $\alpha\{\rho_1(c)\} = \beta(c)$.

We have $\alpha h_1[\sigma] = \alpha\{\rho_1(\sigma)\} = \beta(\sigma)$

$= [\rho_* * \sigma * \bar{\rho}_*]\Gamma = [\sigma][\rho_*][\bar{\rho}_*]\Gamma$ (since $\pi_1(X,*)/\Gamma$ is abel.)

$= [\sigma][\underline{c}_*][\underline{c}_*]^{-1}\Gamma = [\sigma]\Gamma = \pi[\sigma]$, completing the proof.

Def: Y a pointed space, $n \geq 0$. Y is n-connected if $\pi_i(Y,*) = 0$ for $0 \leq i \leq n$.

Thus 0-connected = path-connected.
1-connected is also called simply-connected.

Def: (Y, B) a pointed pair, $n \geq 1$. (Y, B) is n-connected if Y is 0-connected and $\pi_i(Y, B) = 0$ for $1 \leq i \leq n$.

The Hurewicz Theorem

Thm (Hurewicz): (Y, B) a pointed pair with both Y and B simply-connected. Suppose, for some $n \geq 1$, (Y, B) is n-connected. Then $h_i : \pi_i(Y, B) \to H_i(Y, B)$ is a bijection for $1 \leq i \leq n+1$. (i.e. $H_i(Y,B) = 0$ for $1 \leq i \leq n$ and $h_{n+1} : \pi_{n+1}(Y, B) \to H_{n+1}(Y, B)$ is an isom.).

Cor: If Y is simply-connected and $H_i(Y,*) = 0$ for $i \leq n$, then $\pi_i(Y,*) = 0$ for $i \leq n$ and $h_{n+1} : \pi_{n+1}(Y,*) \to H_{n+1}(Y,*)$ is an isom.

Example: S^n, $n \geq 2$, is simply-connected by Van Kampen's thm. $H_i(S^n,*) = 0$ for $i < n-1$. Hence $\pi_i(S^n,*) = 0$ for $i < n$ and $h_n : \pi_n(S^n,*) \to H_n(S^n,*) \cong \mathbb{Z}$ is an isom.

Start of proof of Hurewicz thm: Induction on n. Case $n=1$: By little Hurewicz thm, $h_1 : \pi_1(Y,*) \to H_1(Y,*)$ and $h_1 : \pi_1(B,*) \to H_1(B,*)$ are both onto, so $H_1(Y,*) = 0 = H_1(B,*)$. From homology seq of triple $(Y, B, *)$, follows easily that $H_i(Y,B) = 0$ for $i \leq 1$. Have comm. diagram

$$\begin{array}{ccccc}
\pi_2(Y,B) & \xleftarrow[\cong]{p_*} & \pi_2(P(Y;*,Y), P(Y;*,B)) & \xrightarrow[\cong]{\partial} & \pi_1(P(Y;*,B),*) \\
\downarrow h_2 & & \downarrow h_2 & & \downarrow h_1 \\
H_2(Y,B) & \xleftarrow[p_*]{} & H_2(P(Y;*,Y), P(Y;*,B)) & \xrightarrow[\cong]{\partial} & H_1(P(Y;*,B),*)
\end{array}$$

From exactness of $\pi_2(Y,*) \to \pi_2(Y,B) \to \underset{\overset{\parallel}{0}}{\pi_1(B,*)}$ and fact that $\pi_2(Y,*)$ is abelian, follows that $\pi_2(Y,B)$ is abel.

Hence $\pi_1(P(Y;*,B),*)$ is abelian. From the fibration

$$\Omega Y \longrightarrow P(Y;*,B) \xrightarrow{\;p\;} B,$$

$$\begin{array}{ccccc} \pi_0(\Omega Y,*) & \longrightarrow & \pi_0(P(Y;*,B),*) & \longrightarrow & \pi_0(B,*) \\ \cong & & & & \| \\ \pi_1(Y,*) & & & & 0 \\ \| & & & & \\ 0 & & & & \end{array}$$

exact

it follows that $P(Y;*,B)$ is path-connected.

Hence, by little Hurewicz theorem, indicated h_1 is an isom. Hence, proof for $n=1$ will be completed by

25 Lemma (proof deferred): If (Y,B), Y, B are 1-connected, then $p_*: H_2(P(Y;*,Y), P(Y;*,B)) \to H_2(Y,B)$ is an isom.

Now suppose $n>1$ and the Hurewicz thm. is true in lower dimensions. Then $h_i: \pi_i(Y,B) \to H_i(Y,B)$ is an isom. for $1 \le i \le n$ (so $H_i(Y,B)=0$ for $1 \le i \le n$). Have comm. diagram

$$\begin{array}{ccccc} \pi_{n+1}(Y,B) & \xleftarrow[\cong]{p_*} & \pi_{n+1}(P(Y;*,Y),P(Y;*,B)) & \xrightarrow[\cong]{\partial} & \pi_n(P(Y;*,B),*) \\ h_{n+1}\downarrow & & \downarrow h_{n+1} & & \downarrow h_n \\ H_{n+1}(Y,B) & \xleftarrow[p_*]{} & H_{n+1}(P(Y;*,Y),P(Y;*,B)) & \xrightarrow[\cong]{\partial} & H_n(P(Y;*,B),*) \end{array}$$

Since $\pi_i(P(Y;*,B),*) \cong \pi_{i+1}(Y,B)$ for $i \ge 1$ and (as before) $P(Y;*,B)$ is path-connected, it follows that $P(Y;*,B)$ is $n-1$-connected (hence at least 1-connected). Hence, by ind.-hyp., indicated h_n is an isom. Hence, proof will be completed by

Lemma (proof deferred): Suppose Y, B are 1-connected and for some $n \ge 1$, $H_i(Y,B)=0$ for $i \le n$. Then $p_*: H_{n+1}(P(Y;*,Y), P(Y;*,B)) \to H_{n+1}(Y,B)$ is an isom.

Note: 2nd deferred lemma includes 1st.

<u>Thm</u> (J. H. C. Whitehead): X, Y simply-connected, $f:(X,x_0)\to(Y,y_0)$ a pointed map. Let $n \geq 1$. Then following are equivalent.

(i) $f_*: \pi_k(X,x_0) \to \pi_k(Y,y_0)$ is an iso. for $k \leq n$ and onto for $k = n+1$.

(ii) $f_*: H_k(X) \to H_k(Y)$ is an iso. for $k \leq n$ and onto for $k = n+1$.

<u>Proof</u>: Let $Cyl(f)$ = reduced mapping cylinder of f = quot. space obtained from $(X\times I) \amalg Y$ by identifying $(x,0) \in X \times I$ with $f(x) \in Y$, and $(x_0,t) \in X\times I$ with y_0 $\forall t \in I$. Let $\pi: (X\times I)\amalg Y \to Cyl(f)$ be the quotient map. $Cyl(f)$ is pointed with basepoint $\pi(y_0)$ (which $= \pi(x_0,t)$ $\forall t \in I$).

We have pointed inclusions

$$i : X \to Cyl(f), \quad x \longmapsto \pi(x,1)$$

and $$j: Y \longrightarrow Cyl(f), \quad y \longmapsto \pi(y)$$ and a pointed map

$r: Cyl(f) \to Y$ given by $r(\pi(y)) = y$, $r(\pi(x,t)) = f(x)$.
j and r are pointed homotopy equivalences, inverse to one another. In fact, $rj = 1_Y$, $jr \underset{h}{\simeq} 1_{Cyl(f)}$ where

$h: Cyl(f)\times I \to Cyl(f)$ is given by $h(\pi(y),t) = \pi(y)$, $h(\pi(x,s),t) = \pi(x,st)$. Moreover,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & Cyl(f) \arrow[ur, "r"'] & \end{tikzcd}$$

commutes.

Identify X as a subspace of $Cyl(f)$ via Y. (Thus, base pt. of $Cyl(f)$ is identified with x_0. Thus, since r induces $\cong$ in homology and homotopy, (i) is equiv. to

(i)' $i_*: \pi_k(X,x_0) \to \pi_k(Cyl(f),x_0)$ is an iso. for $k \leq n$ and onto for $k = n+1$, and (ii) is equivalent to

(ii)' $i_*: H_k(X) \to H_k(Cyl(f))$ is an isom. for $k \le n$ and onto for $k = n+1$.

From homotopy seq. of $(Cyl(f), X)$, (i)' is equiv. to

(i)'': $\pi_k(Cyl(f), X) = 0$ for $k \le n+1$.

From homology seq. of $(Cyl(f), X)$, (ii)' is equiv. to

(ii)'': $H_k(Cyl(f), X) = 0$ for $k \le n+1$.

By Hurewicz theorem, (i)'' $\Longleftrightarrow$ (ii)''.

<u>Motivation</u> <u>for</u> <u>spectral</u> <u>sequences</u>

Consider first the case of a product bundle $T = B \times F \to B$. By the Künneth theorem and the U.C.T.,

$$H_n(T) \cong \bigoplus_{p+q=n} H_p(B; H_q(F)) = \bigoplus_{p+q=n} E_{p,q}.$$

In the case of a general fibration
$$\begin{array}{ccc} F & \to & T \\ & & \downarrow \\ & & B \end{array}$$

it is not true that $H_n(T) \cong \bigoplus_{p+q=n} E_{p,q}$. However, $\bigoplus_{p+q=n} E_{p,q}$ is a <u>first approximation</u> to $H_n(T)$. It turns out that $E = \bigoplus_{p+q=n} E_{p,q}$ has a <u>differential</u> $d: E \to E$ (which lowers degree by 1), and the homology $\frac{\ker d}{\operatorname{im} d} = E' = \bigoplus E'_{p,q}$ is a better approx. to $H_*(T)$. ($d = 0$ in case of a prod. bundle) E', in turn, has a differential d', and $E'' = \frac{\ker d'}{\operatorname{im} d'}$ is an even better approx. to $H_*(T)$, etc.

Spectral Sequences

R a comm. ring with $1 \neq 0$.

Def: A bigraded R-module E is a family $\{E_{p,q}\}$ of R-modules, where $p, q \in \mathbb{Z}$.

If $F = \{F_{p,q}\}$ is another bigraded R-module, a morphism of bigraded R-modules $f: E \to F$ consists of a family $f_{p,q}: E_{p,q} \to F_{p,q}$ of R-homomorphisms.

Can form the cat. of bigraded R-modules.

If $E = \{E_{p,q}\}$ a bigraded R-module, can associate with E the singly-graded R-module $\{E_n\}$, where $E_n = \bigoplus_{p+q=n} E_{p,q}$.

Elements of $E_{p,q}$ are said to have bidegree (p,q) and total degree $p+q$.

Def: If $E = \{E_{p,q}\}$, a differential of bidegree $(-r, r-1)$ is a family of homomorphisms $d: E_{p,q} \to E_{p-r, q+r-1}$ such that $d^2 = 0$, i.e. the compositions

$$E_{p,q} \xrightarrow{d} E_{p-r, q+r-1} \xrightarrow{d} E_{p-2r, q+2r-2} \quad \text{are all } 0.$$

Note: Such a d lowers total degree by 1. (E,d) will be called a $(-r, r-1)$-differential bigraded module.

Def: If (E,d), (F,d) are $(-r, r-1)$-diff. bigraded modules, a morphism of $(-r, r-1)$ diff. bigraded modules

$f: (E,d) \to (F,d)$ consists of a morphism of bigraded R-modules s.t.

$$\begin{tikzcd} E_{p,q} \arrow[r, "f_{p,q}"] \arrow[d, "d"'] & F_{p,q} \arrow[d, "d"] \\ E_{p-r, q+r-1} \arrow[r, "f_{p-r, q+r-1}"'] & F_{p-r, q+r-1} \end{tikzcd}$$

commutes for all p, q.

Can form cat of $(-r,r-1)$-diff. bigraded R-modules.

26 If (E,d) an $(-r,r-1)$-diff bigraded R-module, define $H(E,d)$, the <u>homology</u> of (E,d), to be the bigraded R-module $\{H_{p,q}(E)\}$ where $H_{p,q}(E) = \dfrac{\ker d: E_{p,q} \to E_{p-r,q+r-1}}{\operatorname{im} d: E_{p+r,q-r+1} \to E_{p,q}}$.

If $f: (E,d) \longrightarrow (F,d)$ a morphism of $(-r,r-1)$-diff. bigraded modules, then f induces
$H(f): H(E,d) \longrightarrow H(F,d)$, a morphism of bigraded R-modules. H is a covariant functor from cat. of $(-r,r-1)$-diff. bigraded R-modules to cat. of bigraded R-modules.

<u>Def</u>: A <u>homology spectral sequence</u> $E = \{(E^r,d^r)\}$ consists of a sequence (E^2,d^2), (E^3,d^3), (E^4,d^4), ... such that

1) (E^r,d^r) is a $(-r,r-1)$-diff. bigraded R-module, $r=2,3,\ldots$

2) For each $r \geq 2$, there is given an isomorphism $\theta^r: H(E^r,d^r) \cong E^{r+1}$ of bigraded R-modules.

Thus (E^r,d^r) determines the <u>bigraded R-module</u> E^{r+1}, but does not determine d^{r+1}!

<u>Def</u>. If $E = \{E^r,d^r)$ and $\{\bar{E}^r,\bar{d}^r\}$ are homology spectral sequences, a <u>morphism of spectral sequences</u> $f: E \to \bar{E}$ consists of a sequence $f^r: (E^r,d^r) \to (\bar{E}^r,\bar{d}^r)$ of $(-r,r-1)$-diff. bigraded R-module homomorphisms, $r = 2,3,\ldots$ such that

$$\begin{array}{ccc} H(E^r,d^r) & \xrightarrow{H(f^r)} & H(\bar{E}^r,\bar{d}^r) \\ \theta^r \downarrow \cong & & \downarrow \bar{\theta}^r \\ E^{r+1} & \xrightarrow{f^{r+1}} & \bar{E}^{r+1} \end{array}$$

commutes.

Can form cat of homology spectral sequences.

<u>Def</u>. A spectral sequence E is called a <u>first quadrant spectral sequence</u> if $E^r_{p,q} = 0$ if $p<0$ or $q<0$.

Note: If $E^2_{p,q} = 0$ for $p < 0$ or $q < 0$, then E is a 1^{st} quadrant spectral sequence. More generally, if $E^r_{p,q} = 0$ for some p, q, r, then $E^s_{p,q} = 0$ for $s \geq r$.

It is useful to picture a bigraded module E by associating with each lattice point (p,q) in the plane the module $E_{p,q}$. Then if E is a spectral sequence, the differentials are pictured as follows.

E^2 $\qquad\qquad$ E^3

Let E be a 1^{st} quad. homology spectral sequence. Note that if $r > p$, then $E^r_{p-r,\, q+r-1} = 0$ since $p - r < 0$,

and so $\ker\left[d^r : E^r_{p,q} \longrightarrow E^r_{p+r,\, q+r-1}\right] = E^r_{p,q}$.

Also if $r > q+1$, then $E^r_{p+r,\, q-r+1} = 0$ since $q - r + 1 < 0$

and so $\operatorname{im}\left[d^r : E^r_{p+r,\, q-r+1} \longrightarrow E^r_{p,q}\right] = 0$.

Hence $H_{p,q}(E^r, d^r) = E^r_{p,q}/0 = E^r_{p,q}$ if $r > p, q+1$.

Hence $E^r_{p,q} \xrightarrow[\cong]{\theta^r} E^{r+1}_{p,q} \xrightarrow[\cong]{\theta^{r+1}} E^{r+2}_{p,q} \xrightarrow[\cong]{\theta^{r+2}} \cdots$ if $r > p, q+1$.

Define $E^\infty_{p,q} = E^r_{p,q}$, $r > \max\{p, q+1\}$, and

$E^\infty = \{E^\infty_{p,q}\}$. E^∞ is a bigraded R-module.

If $f : E \to \bar{E}$ is a morphism of 1^{st} quad homology spectral sequences, f induces $f^\infty : E^\infty \to \bar{E}^\infty$, a morphism of bigraded modules, defined by

$$f^{\infty}_{p,q} = f^{r}_{p,q} : E^{r}_{p,q} \longrightarrow \bar{E}^{r}_{p,q} \quad , r > \max\{p, q+1\}.$$

It does not matter which $r > \max\{p,q+1\}$ we choose since

$$\begin{tikzcd} E^{r}_{p,q} \arrow[r, "f^{r}"] \arrow[d, "\theta^{r}"', "\cong"] & \bar{E}^{r}_{p,q} \arrow[d, "\bar{\theta}^{r}", "\cong"'] \\ E^{r+1}_{p,q} \arrow[r, "f^{r+1}"'] & \bar{E}^{r+1}_{p,q} \end{tikzcd}$$

commutes.

$E \longmapsto E^{\infty}$ is a covariant functor from cat. of 1^{st} quad. homology spectral sequences to cat. of bigraded modules.

Thm (Leray-Serre)(1^{st} version): Let $F \xrightarrow{i} T \xrightarrow{p} B$ be a pointed fibration with 1-connected base B. Let B' be any subspace of B, and $T' = p^{-1}(B')$. (B' need not contain the base point of B. In particular, allow $B' = \emptyset$). Let R be any commutative ring with $1 \neq 0$. Then $\exists$ 1^{st} quad homology spectral sequence E of R-modules, and for each $n \geq 0$ a nested family of R-modules

$$0 = J_{-1,n+1} \subset J_{0,n} \subset J_{1,n-1} \subset \cdots \subset J_{n-1,1} \subset J_{n,0} = H_n(T, T'; R)$$

such that

$$E^{2}_{p,q} \cong H_p(B, B'; H_q(F;R)), \quad E^{\infty}_{p,q} \cong J_{p,q}/J_{p-1,q+1}.$$

Proof will be given later.

Application: For Y a pointed space, write $PY = P(Y;*,Y)$. We have a pointed fibration $\Omega Y \longrightarrow PY \longrightarrow Y$ with PY contractible.

Let $n \geq 2$ and consider $\Omega S^n \longrightarrow PS^n \longrightarrow S^n$.

S^n is 1-connected, so $\exists$ 1^{st} quad. homology spectral seq. E $\ni$: $E^{2}_{p,q} = H_p(S^n; H_q(\Omega S^n))$, $\quad E^{\infty}_{p,q} = J_{p,q}/J_{p-1,q+1}$ where

$$0 = J_{-1,m+1} \subset J_{0,m} \subset \quad \subset J_{m,0} = H_m(PS^n).$$

If $m > 0$, $H_m(PS^n) = 0$ since PS^n is contractible. Hence $J_{p,q} = 0$ if $p+q > 0$, and so $E^{\infty}_{p,q} = 0$ if $(p,q) \neq (0,0)$.

By the universal coeff. thm.,

$$H_p(S^n; H_q(\Omega S^n)) \cong \begin{cases} H_q(\Omega S^n) & \text{if } p = 0 \text{ or } n \\ 0 & \text{if } p \neq 0, n. \end{cases}$$

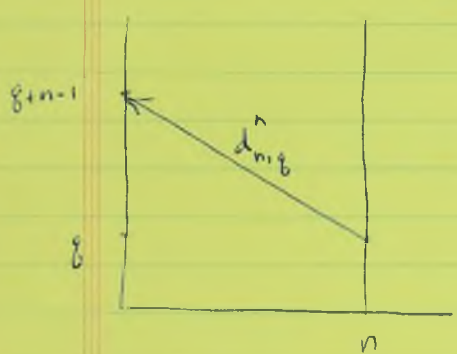
Hence $E^2_{p,q} = 0$ unless $p = 0$ or $p = n$, and

$E^2_{0,q} \cong H_q(\Omega S^n)$, $E^2_{n,q} \cong H_q(\Omega S^n)$. Thus $E^r_{p,q} = 0$ unless $p = 0$ or n.

Note that if $r \neq n$, every $d^r_{p,q}$ must be 0 since one of $E^r_{p,q}$, $E^r_{p-r,q+r-1}$ is 0. Hence $E^2_{p,q} \cong E^3_{p,q} \cong \cdots \cong E^n_{p,q}$,

and $E^{n+1}_{p,q} \cong E^{n+2}_{p,q} \cong \cdots \cong E^\infty_{p,q}$ for all p, q.

Thus for $(p,q) \neq (0,0)$, $E^{n+1}_{p,q} = 0$.



For each $q \in \mathbb{Z}$,

$E^{n+1}_{0,q+n-1} \cong E^{n}_{0,q+n-1} / \operatorname{im} d^n_{n,q}$,

$E^{n+1}_{n,q} \cong \ker d^n_{n,q}$. Thus if

$q > 1-n$, both $E^{n+1}_{0,q+n-1}$ and $E^{n+1}_{n,q}$ are 0

and so $d^n_{n,q} : E^n_{n,q} \longrightarrow E^n_{0,q+n-1}$ is an isom.

Since $E^n_{n,q} \cong E^2_{n,q} \cong H_q(\Omega S^n)$ and $E^n_{0,q+n-1} \cong E^2_{0,q+n-1} \cong H_{q+n-1}(\Omega S^n)$

we have $H_{q+n-1}(\Omega S^n) \cong H_q(\Omega S^n)$ whenever $q > 1-n$.
Since $\pi_0(\Omega S^n) \cong \pi_1(S^n) = 0$, ΩS^n is path-connected and so $H_0(\Omega S^n) \cong \mathbb{Z}$. Follows that

$$H_k(\Omega S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = m(n-1), \ m \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

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Base edge monomorphism

Let E be a 1st quadrant homology spectral sequence. For $p \geqslant 0$, $\operatorname{im}[d^r : E^r_{p+r,-r+1} \to E^r_{p,0}] = 0$ since $-r+1 < 0$. Hence

$$H_{p,0}(E^r) = \ker[d^r : E^r_{p,0} \to E^r_{p-r,r-1}] \subset E^r_{p,0}$$

and so we have monomorphisms $E^{r+1}_{p,0} \xrightarrow[\cong]{\sigma^{-1}} H_{p,0}(E^r) \subset E^r_{p,0}$. Thus we have a sequence of monomorphisms

$$E^\infty_{p,0} = E^{p+1}_{p,0} \hookrightarrow E^p_{p,0} \hookrightarrow E^{p-1}_{p,0} \hookrightarrow \cdots \hookrightarrow E^3_{p,0} \hookrightarrow E^2_{p,0}.$$

Denote this composite monomorphism by $i_B : E^\infty_{p,0} \hookrightarrow E^2_{p,0}$. i_B will be called the <u>base edge</u> monomorphism.

Easily checked: i_B is natural, i.e. if $f : E \to \overline{E}$ is a morphism of 1st quadrant spectral sequences, then

$$\begin{array}{ccc} E^\infty_{p,0} & \xrightarrow{i_B} & E^2_{p,0} \\ {\scriptstyle f^\infty_{p,0}}\downarrow & & \downarrow{\scriptstyle f^2_{p,0}} \\ \overline{E}^\infty_{p,0} & \xrightarrow[i_B]{} & \overline{E}^2_{p,0} \end{array}$$

commutes.

<u>Base-Edge Theorem</u>: Let $F \xrightarrow{i} T \xrightarrow{p} B$ be a pointed fibration with B 1-connected, F 0-connected. Let $B' \subset B$, $T' = p^{-1}(B')$. Let E be the Leray-Serre spectral sequence (coeffs. in R) of the above. Then following commutes:

$$\begin{array}{ccc} H_p(T,T';R) & \xrightarrow{p_*} & H_p(B,B';R) \\ {\scriptstyle \text{onto}}\ \downarrow{\scriptstyle P_B} & & \cong\uparrow{\scriptstyle \varepsilon_*} \\ E^\infty_{p,0} & \xrightarrow{i_B} & E^2_{p,0} \end{array}$$

where P_B is the composition

$$H_p(T,T';R) = J_{p,0} \xrightarrow{\text{onto}} J_{p,0}/J_{p-1,1} = E^\infty_{p,0},$$

$\varepsilon : H_0(F;R) \xrightarrow{\cong} R$ is the augmentation, and

$\varepsilon_* : E^2_{p,0} = H_p(B,B';H_0(F;R)) \longrightarrow H_p(B,B';R)$ is the isomorphism induced by ε.

Proof deferred till later.

Assuming the base-edge theorem + the Leray-Serre thm., we can now complete the proof of the Hurewicz theorem, using

<u>Lemma</u>: Let (Y,B) be a pointed pair with Y 1-connected. Suppose, for some $n \geq 1$, $H_i(Y,B)=0$ for $i \leq n$. Then $p_* : H_{n+1}(P(Y;*,Y),P(Y;*,B)) \to H_{n+1}(Y,B)$ is an isomorphism.

<u>Proof</u>: Since Y is 1-connected, ΩY is 0-connected. We have a fibration $\Omega Y \to P(Y;*,Y) \xrightarrow{p} Y$ with $p^{-1}(B) = P(Y;*,B)$. Let E be the Leray-Serre spectral sequence of the above. From commutativity of

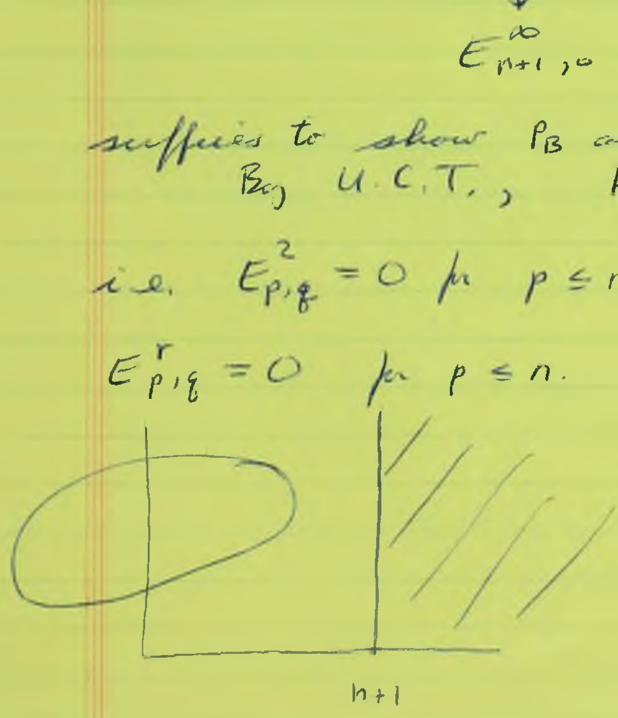
$$\begin{tikzcd}
H_{n+1}(P(Y;*,Y),P(Y;*,B)) \arrow[r, "p_*"] \arrow[d, "p_B"] & H_{n+1}(Y,B) \\
E^\infty_{n+1,0} \arrow[r, "i_B"'] & E^2_{n+1,0} \arrow[u, "\cong"', "\varepsilon_*"]
\end{tikzcd},$$

suffices to show p_B and i_B are isomorphisms.

By U.C.T., $H_p(Y,B;H_q(\Omega Y)) = 0$ for $p \leq n$,

i.e. $E^2_{p,q}=0$ for $p \leq n$. Hence for all $2 \leq r \leq \infty$,

$E^r_{p,q} = 0$ for $p \leq n$.



Thus $d^r_{n+1,0}$ is 0 for all $r \geq 2$ and so the inclusion $E^{r+1}_{n+1,0} \hookrightarrow E^r_{n+1,0}$ is an isomorphism for all $r \geq 2$, i.e. i_B is an isomorphism. Also, since $E^\infty_{p,q} = 0$ for $p \leq n$,

it follows that $0 = J_{-1,n+2} = J_{0,n+1} = J_{1,n} = \cdots = J_{n,1}$.
Thus $\rho_B : J_{n+1,0} \to J_{n+1,0}/J_{n,1}$ is an isomorphism.

Freudenthal Suspension Theorem

Let X be any pointed space. Write $SX = X \wedge S^1 =$ reduced suspension of X. If $f : X \to Y$ is pointed, write $Sf = f \wedge 1_{S^1} : SX \to SY$. Note that $[Sf]$ depends only on $[f]$. In particular, if $g : S^n \to X$ is pointed, $Sg : S^n \wedge S^1 = S^{n+1} \to SX$ and we obtain a well-defined function $S : \pi_n(X,*) \to \pi_{n+1}(SX,*)$, called the suspension map. S is natural: for if $f : X \to Y$ is pointed, consider

$$\begin{array}{ccc} \pi_n(X,*) & \xrightarrow{S} & \pi_{n+1}(SX,*) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (Sf)_*} \\ \pi_n(Y,*) & \xrightarrow[S]{} & \pi_{n+1}(SY,*) \end{array} .$$

If $g : S^n \to X$ is pointed, then

$$(Sf)_* S[g] = (Sf)_*[g \wedge 1_{S^1}] = [(f \wedge 1_{S^1})(g \wedge 1_{S^1})] = [fg \wedge 1_{S^1}]$$

$$= S[fg] = S f_*[g] \quad \text{, so above rectangle commutes.}$$

~~Lemma: For $n \geq 1$, $S : \pi_n(X,*) \to \pi_{n+1}(SX,*)$ is a group homomorphism.~~

~~Proof.~~

If $f : SX \to Y$ is pointed, let $\tilde{f} : X \to \Omega Y$ be given by $\tilde{f}(x)(t) = f(x \wedge \langle t \rangle)$.
Recall: We have a natural group isomorphism

$$\psi_0 : \pi_{n+1}(Y,*) \to \pi_n(\Omega Y,*) \quad \text{for all } n \geq 0.$$
$$[f] \longmapsto [\tilde{f}]$$

Let $i = \widetilde{1_{SX}} : X \to \Omega S X$.

i is a pointed inclusion map.

Lemma: For any pointed X,

$$\begin{tikzcd} \pi_n(X,*) \arrow[rr, "S"] \arrow[dr, "i_*"'] & & \pi_{n+1}(SX,*) \arrow[dl, "\psi_0", "\cong"'] \\ & \pi_n(\Omega S X,*) & \end{tikzcd}$$

commutes for all n.

Proof: $\psi_0 S[f] = \psi_0[f\wedge 1_{S^1}] = [\widetilde{f\wedge 1_{S^1}}]$, $i_*[f] = [f\circ \widetilde{1_{SX}}]$, and it is easily verified that $\widetilde{f\wedge 1_{S^1}} = f\circ \widetilde{1_{SX}}$.

Cor: $S : \pi_n(X,*) \to \pi_{n+1}(SX,*)$ is a group homomorphism for $n \geq 1$.

Example: Let $n \geq 1$. By the Hurewicz theorem,

$h_n : \pi_n(S^n,*) \to H_n(S^n,*)$ is an isom. (little Hurewicz thm in case $n=1$, using fact that S^1 is an H^*-space). Note that $h_n[1_{S^n}] = (1_{S^n})_*(\kappa_n) = \kappa_n$ = gen. of $H_n(S^n,*) \cong \mathbb{Z}$, and so $\pi_n(S^n,*) \cong \mathbb{Z}$ with generator $[1_{S^n}]$. Since $S(1_{S^n}) = 1_{S^{n+1}}$, $S : \pi_n(S^n,*) \to \pi_{n+1}(S^{n+1},*)$ is an isomorphism. Thus

$$i_* : \pi_n(S^n,*) \to \pi_n(\Omega S^{n+1},*) \text{ is an isomorphism.}$$

<u>Thm</u> (Freudenthal Suspension Theorem): Let $n \geq 2$. Then $S : \pi_k(S^n, *) \to \pi_{k+1}(S^{n+1}, *)$ is an isom. if $k \leq 2n-2$, onto if $k = 2n-1$.

<u>Cor</u>: For any $k \geq 0$, the isomorphism class of $\pi_{n+k}(S^n, *)$ is indep. of n for $n \geq k+2$.

28 <u>Proof</u>: By lemma, must show $i_* : \pi_k(S^n, *) \to \pi_k(\Omega S^{n+1}, *)$ is an isom. if $k \leq 2n-2$ and onto if $k = 2n-1$. Since $n \geq 2$, S^n and ΩS^{n+1} are both 1-connected. Hence, by the Whitehead theorem, suffices to show
$i_* : H_k(S^n, *) \to H_k(\Omega S^{n+1}, *)$ is an isom. if $k \leq 2n-2$ and onto if $k = 2n-1$.
For $k \leq 2n-1$ we have

$$H_k(S^n, *) \cong H_k(\Omega S^{n+1}, *) \cong \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k \neq 0 \end{cases}$$

Thus, remains only to show $i_* : H_n(S^n, *) \to H_n(\Omega S^{n+1}, *)$ is an isom. We have the commutative diagram

$$\begin{array}{ccc} \pi_n(S^n, *) & \xrightarrow[\cong]{i_*} & \pi_n(\Omega S^{n+1}, *) \\ \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle h_n} \\ H_n(S^n, *) & \xrightarrow[i_*]{} & H_n(\Omega S^{n+1}, *) \end{array}$$

S^n and ΩS^{n+1} are both $n-1$-connected, and so by the Hurewicz theorem, both h_n's are isomorphisms. Already noted top i_* is an isom. Hence, bottom i_* is an isom.

<u>Fibre edge epimorphism</u>

Let E be a 1st quad. homology spectral sequence. For each $q \geq 0$ we have the composite epimorphism

$$E^r_{0,q} \longrightarrow E^r_{0,q} / \mathrm{im}\left[d^r : E^r_{r,q-r+1} \to E^r_{0,q}\right] = H_{0,q}(E^r) \xrightarrow[\cong]{\theta^r} E^{r+1}_{0,q}.$$

Let $p_F : E^2_{0,q} \longrightarrow E^\infty_{0,q}$ denote the composition

$$E^2_{0,q} \xrightarrow{\text{onto}} E^3_{0,q} \xrightarrow{\text{onto}} \cdots \xrightarrow{\text{onto}} E^{q+2}_{0,q} = E^\infty_{0,q}.$$

p_F is called the fibre edge epimorphism.
Easily checked: p_F is natural, i.e. if $f: E \to \bar{E}$ is a morphism of 1st quadrant spectral sequences, then

$$\begin{tikzcd} E^2_{0,q} \arrow[r, "p_F"] \arrow[d, "f^2_{0,q}"'] & E^\infty_{0,q} \arrow[d, "f^\infty_{0,q}"] \\ \bar{E}^2_{0,q} \arrow[r, "p_F"'] & \bar{E}^\infty_{0,q} \end{tikzcd}$$

commutes.

Fibre-Edge Theorem: Let $F \xrightarrow{i} T \xrightarrow{p} B$ be a pointed fibration with B 1-connected. Let E be the Leray-Serre spectral sequence of this fibration (R-coeffs.). Then the following commutes for all q:

$$\begin{tikzcd} H_q(F;R) \arrow[r, "i_*"] \arrow[d, "\varphi", "\cong"'] & H_q(T;R) \\ E^2_{0,q} \arrow[r, "p_F"'] & E^\infty_{0,q} \arrow[u, "i_F"'] \end{tikzcd}$$

where i_F is the monomorphism $E^\infty_{0,q} = J_{0,q} \subset J_{q,0} = H_q(T;R)$, and φ is the composite isomorphism

$$\begin{tikzcd} H_q(F;R) \arrow[r, "\cong"] & \mathbb{Z} \otimes_{\mathbb{Z}} H_q(F;R) \arrow[r, "\varepsilon^{-1}\otimes 1"] & H_0(B;\mathbb{Z}) \otimes_{\mathbb{Z}} H_q(F;R) \arrow[d, "u", "\cong\ (\text{U.C.T.})"'] \\ & & H_0(B; H_q(F;R)) \\[-2ex] & & \overset{\shortparallel}{E^2_{0,q}} \end{tikzcd}$$

(Proof deferred).

Spectral Sequence of a Filtered Chain Complex

Def: A filtered chain complex (C,F) of R-modules consists of a chain complex C of R-modules, and a nested family of subchain complexes

$$0 = F_{-1}C \subset F_0C \subset F_1C \subset \cdots \subset C$$

Example: Let X be a filtered top. space, i.e. we are given a nest of subspaces $\phi = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X$. Let $Q(X)$ be the singular complex of X, $F_p Q(X) = Q(X_p)$. Then $(Q(X), F)$ is a filtered chain complex.

(The filtered chain complex yielding the Leray-Serre spectral sequence will be different).

Def: A morphism of filtered chain complexes $f:(C,F) \to (\bar{C}, \bar{F})$ consists of a chain map $f: C \to \bar{C}$ $\ni$: $f(F_pC) \subset \bar{F}_p\bar{C}$ for all p.

Can form cat. of filtered chain complexes of R-modules.

Convention: $F_pC = 0$ for $p < 0$.

We proceed to define a covariant functor from the cat. of filtered chain complexes of R-modules to the cat. of spectral sequences of R-modules.

Def: Let (C,F) be a filtered chain complex. For $r \geq 1$, define

$$E^r_{p,q}(C,F) = \operatorname{im}\left[H_{p+q}(F_pC/F_{p-r}C) \xrightarrow{i_*} H_{p+q}(F_{p+r-1}C/F_{p-1}C)\right]$$

Note: Sep of r between numerator + denominator. 1st quotient involves filtrations $\leq p$, 2nd quotient factors out filtrations $< p$.

The inclusion of triples

$$\begin{matrix} (F_pC, & F_{p-r}C, & F_{p-2r}C) \\ & \cap & \\ (F_{p+r-1}C, & F_{p-1}C, & F_{p-r-1}C) \end{matrix}$$

yields the commutative diagram

$$\begin{tikzcd}
H_{p+q}(F_pC/F_{p-r}C) \arrow[r, "i_*"] \arrow[d, "\partial"'] & H_{p+q}(F_{p+r-1}C/F_{p-1}C) \supset E^r_{p,q}(C,F) = \text{im}\, i_* \arrow[d, "\partial"] \\
H_{p+q-1}(F_{p-r}C/F_{p-2r}C) \arrow[r, "i_*"] & H_{p+q-1}(F_{p-1}C/F_{p-r-1}C) \supset E^r_{p-r,q+r-1}(C,F) = \text{im}\, i_*
\end{tikzcd}$$

where the vertical maps are the connecting homomorphisms from the homology sequences of the above triples. Thus

$\partial(E^r_{p,q}(C,F)) \subset E^r_{p-r,q+r-1}(C,F)$. Define

$d^r : E^r_{p,q}(C,F) \longrightarrow E^r_{p-r,q+r-1}(C,F)$ to be the restriction of ∂.

Thus

$$\begin{tikzcd}
H_{p+q}(F_{p+r-1}C/F_{p-1}C) \arrow[d, "\partial"'] & \supset & E^r_{p,q}(C,F) \arrow[d, "d^r"] \\
H_{p+q-1}(F_{p-1}C/F_{p-r-1}C) & \supset & E^r_{p-r,q+r-1}(C,F)
\end{tikzcd}$$ commutes.

<u>Thm</u>: $\{E^r_{p,q}(C,F), d^r\}$ is a spectral sequence of R-modules $\ni$:

1) $E^1_{p,q}(C,F) = H_{p+q}(F_pC/F_{p-1}C)$, and

$d^1 : E^1_{p,q}(C,F) \to E^1_{p-1,q}(C,F)$ is the connecting homomorphism

$\partial : H_{p+q}(F_pC/F_{p-1}C) \to H_{p+q-1}(F_{p-1}C/F_{p-2}C)$ in the homology seq. of the triple $(F_pC, F_{p-1}C, F_{p-2}C)$.

2) The isomorphisms $\theta^r : H(E^r, d^r) \to E^{r+1}$ are such that

$$\begin{tikzcd}
 & H_{p,q}(E^r(C,F), d^r) \arrow[r, "\theta^r", "\cong"'] & E^{r+1}_{p,q}(C,F) \arrow[d, phantom, "\cap"] \\
\operatorname{Ker}\left[d^r : E^r_{p,q}(C,F) \to E^r_{p-r,q+r-1}(C,F)\right] \arrow[ur, "\text{onto}"] \subset H_{p+q}(F_{p+r-1}C/F_{p-1}C) \arrow[rr, "j_*"] & & H_{p+q}(F_{p+r}C/F_{p-1}C)
\end{tikzcd}$$

commutes, where $j : (F_{p+r-1}C, F_{p-1}C) \to (F_{p+r}C, F_{p-1}C)$ is the inclusion.

3) $(C,F) \longmapsto \{E^r_{p,q}(C,F), d^r\}$ is a covariant functor from the cat. of filtered chain complexes of R-modules to the category of spectral sequences.

---

29 Proof: Write $F_p = F_p C$, $E^r_{p,q} = E^r_{p,q}(C,F)$.
Proof that $d^r d^r = 0$: Following commutes:

$$\begin{tikzcd}
 & H_{p+q}(F_{p+r-1}/F_{p-1}) \arrow[dl, "\partial"'] \arrow[dd, "\partial"] & \supset E^r_{p,q} \arrow[dd, "d^r"] \\
H_{p+q-1}(F_{p-1}) \arrow[dr, "k_*"] & & \\
 & H_{p+q-1}(F_{p-1}/F_{p-r-1}) \arrow[dl, "\partial"'] \arrow[dd, "\partial"] & \supset E^r_{p-r,q+r-1} \arrow[dd, "d^r"] \\
H_{p+q-2}(F_{p-r-1}) \arrow[dr, "k_*"] & & \\
 & H_{p+q-2}(F_{p-r-1}/F_{p-2r-1}) & \supset E^2_{p-2r,q+2r-2}
\end{tikzcd}$$

and enclosed composition is 0 from homology seq. of pair (F_{p-1}, F_{p-r-1}).
Thus $d^r d^r = 0$.

Write $Z^r_{p,q} = \ker\left[E^r_{p,q} \xrightarrow{d^r} E^r_{p-r,q+r-1}\right]$,

$B^r_{p,q} = \operatorname{im}\left[E^r_{p+r,q-r+1} \xrightarrow{d^r} E^r_{p,q}\right]$.

Then $H_{p,q}(E^r, d^r) = Z^r_{p,q}/B^r_{p,q}$. To prove $\{E^r, d^r\}$ is a spectral sequence and to establish (2), it suffices to show

a) $j_*(Z^r_{p,q}) = E^{r+1}_{p,q}$

b) $\ker(j_* | Z^r_{p,q}) = B^r_{p,q}$

where j is as in (2).

Proof of a): Have commutative diagram

$$
\begin{tikzcd}
 & H_{p+q}(F_p/F_{p-r-1}) \arrow[rr, "i_*"] \arrow[dd] \arrow[dr, "\text{onto}"'] & & H_{p+q}(F_{p+r}/F_{p-1}) \\
 & & E^{r+1}_{p,q} \arrow[ur, hook] & \\
 & H_{p+q}(F_p/F_{p-r}) \arrow[rr, "i^*"] \arrow[dd, "\partial"'] \arrow[dr, "\text{onto}"'] & & H_{p+q}(F_{p+r-1}/F_{p-1}) \arrow[uu, "j_*"'] \arrow[dd, "\partial"] \\
 & & E^r_{p,q} \arrow[ur, hook] & \\
\text{exact} \quad H_{p+q}(F_{p-1}/F_{p-r}) \arrow[uur, "k_*"] \arrow[r, "\partial"'] & H_{p+q-1}(F_{p-r}/F_{p-r-1}) \arrow[rr] & & H_{p+q-1}(F_{p-1}/F_{p-r-1})
\end{tikzcd}
$$

exact

Indicated exactness is from homology sequences of appropriate triples. $Z^r_{p,q} = \operatorname{im} i^* \cap \ker \partial$

Note $i_* k_* = 0$ since it factors through $H_{p+q}(F_{p-1}/F_{p-1}) = 0$.
a) now follows by diagram chasing in above diagram.

Proof of b): Will prove the stronger result:

$\ker (j_* | E^r_{p,q}) = B^r_{p,q}$. This follows from diagram chasing in following diagram, with indicated exactness from homology sequences of appropriate triples:

$$
\begin{tikzcd}
\text{exact} & H_{p+q+1}(F_{p+r}/F_p) \arrow[rrr] \arrow[ddd, "\partial"'] \arrow[dr, "\text{onto}"] & & & H_{p+q+1}(F_{p+r}/F_{p+r-1}) \arrow[r, "\partial"] \arrow[ddd, "\partial"] & H_{p+q}(F_{p+r-1}/F_p) \\
& & E^r_{p+r,q-r+1} \arrow[d, "d^r"] \arrow[r, "\text{onto}"] & B^r_{p,q} \arrow[dl, hook] & & \\
& & E^r_{p,q} \arrow[drr, hook] & & & \\
& H_{p+q}(F_p/F_{p-r}) \arrow[rrr, "i_*"'] \arrow[ur, "\text{onto}"] \arrow[dr] & & & H_{p+q}(F_{p+r-1}/F_{p-1}) \arrow[uuur] \arrow[d, "j_*"] & \\
& & H_{p+q}(F_p/F_{p-1}) \arrow[urr] & & H_{p+q}(F_{p+r}/F_{p-1}) & \\
& \text{exact} & & & \text{exact} &
\end{tikzcd}
$$

1) is immediate.
3) is straightforward.

<u>Note</u>: $E^r_{p,q}(C,F) = 0$ for $p < 0$. For $H_{p+q}(F_pC/F_{p-r}C) = 0$ since $F_pC = 0$ for $p < 0$.

<u>Def</u>: A filtered chain complex (C,F) is <u>canonically bounded</u> if, for each p, $(F_pC)_i = C_i$ for all $i \le p$.

(Filtration of $Q(X)$ arising from a filtered top. space X is not usually canonically bounded).

<u>Note</u>: If (C,F) is canonically bounded, then $E(C,F)$ is a 1^{st} quad. spectral sequence, for if $q < 0$,
$(F_pC)_{p+q} = C_{p+q} = (F_{p-1}C)_{p+q}$ since $p+q \le p-1$, and so

$$E^1_{p,q}(C,F) = H_{p+q}(F_pC/F_{p-1}C) = 0.$$

For any filtered chain complex (C,F), define

$$J_{p,q}(C,F) = \operatorname{im}\left[H_{p+q}(F_pC) \xrightarrow{i_*} H_{p+q}(C)\right].$$

It is immediate that $J_{p,q}$ is natural w/r. to morphisms of filtered chain complexes. From commutativity of

$$\begin{tikzcd}
H_{p+q}(F_{p-1}C) \arrow[d] \arrow[dr, "i_*"] & \\
H_{p+q}(F_pC) \arrow[r, "i_*"'] & H_{p+q}(C)
\end{tikzcd}$$

it is immediate that $J_{p-1,q+1}(C,F) \subset J_{p,q}(C,F)$.
Since $F_{-1}C = 0$, $J_{-1,n+1}(C,F) = 0$ for all n. Thus we have a natural filtration

$$0 = J_{-1,n+1}(C,F) \subset J_{0,n}(C,F) \subset J_{1,n-1}(C,F) \subset \cdots \subset H_n(C).$$

Thm: Suppose (C,F) is canonically bounded. Then $J_{n,0}(C,F) = H_n(C)$, and there is a natural isomorphism

$$\varphi : J_{p,q}(C,F)/J_{p-1,q+1}(C,F) \longrightarrow E^{\infty}_{p,q}(C,F)$$

such that for all $r > \max\{p, q+1\}$, the diagram

$$\begin{tikzcd}
\operatorname{im}\left[H_{p+q}(F_pC) \xrightarrow{i_*} H_{p+q}(F_{p+r-1}C)\right] \arrow[r, "\cong"', "{\substack{(\text{can. idd.})\\ p+r-1>p+q\\ \text{for } r>q+1}}"] \arrow[d] & \operatorname{im}\left[H_{p+q}(F_pC) \to H_{p+q}(C)\right] \arrow[d, equal] \\
\operatorname{im}\left[H_{p+q}(F_pC) \xrightarrow{i_*} H_{p+q}(F_{p+r-1}C/F_{p-1}C)\right] \arrow[d, equal, "(\text{since } F_{p-r}C=0 \text{ for } r>p)"] & J_{p,q}(C,F) \arrow[d, "\text{onto}"] \\
\operatorname{im}\left[H_{p+q}(F_pC/F_{p-r}C) \xrightarrow{i_*} H_{p+q}(F_{p+r-1}C/F_{p-1}C)\right] = E^r_{p,q}(C,F) = E^{\infty}_{p,q}(C,F) & J_{p,q}(C,F)/J_{p-1,q+1}(C,F) \arrow[l, "\cong", "\varphi"']
\end{tikzcd}$$

commutes.

30 Proof: By canonical boundedness, $(F_nC)_n = C_n$ and so $H_n(F_nC) \xrightarrow{i_*} H_n(C)$ is onto. Hence $J_{n,0}(C,F) = H_n(C)$.

Let $r > \max\{p, q+1\}$, and let $f^r_{p,q} : J_{p,q}(C,F) \to E^r_{p,q}(C,F)$ denote the composition of the maps in the diagram moving counterclockwise. $f^r_{p,q}$ is natural. Note that $f^r_{p,q}$ is onto since

$$\begin{tikzcd}
H_{p+q}(F_pC) \arrow[r, "i_*"] \arrow[dr, "i_*"'] & H_{p+q}(F_{p+r-1}C) \arrow[d] \\
& H_{p+q}(F_{p+r-1}C/F_{p-1}C)
\end{tikzcd}$$

commutes.

Must first check

1) $\ker f^r_{p,q} = J_{p-1,q+1}(C,F)$. ($f^r_{p,q}$ would then induce a natural isom. $\varphi^r : J_{p,q}(C,F)/J_{p-1,q+1}(C,F) \to E^r_{p,q}(C,F)$.

Must then check

2) For $r > \max\{p, q+1\}$,

$$\begin{tikzcd}
 & E^r_{p,q}(C,F) \arrow[d, equal] \\
J_{p,q}(C,F)/J_{p-1,q+1}(C,F) \arrow[ur, "\varphi^r"] \arrow[r, "\varphi^{r+1}"'] & E^{r+1}_{p,q}(C,F)
\end{tikzcd}$$

commutes.

Proof of 1): Write $F_p = F_pC$, $J_{p,q} = J_{p,q}(C,F)$. We have the commutative diagram, the left-hand column exact:

$$\begin{tikzcd}
H_{p+q}(F_{p-1}) \arrow[rr] \arrow[dr, "\text{onto}"'] \arrow[dd] & & H_{p+q}(F_p) \arrow[dr, "\text{onto}"] \arrow[dd] & \\
 & J_{p-1,q+1} \arrow[dr] & & J_{p,q} \arrow[dl] \\
H_{p+q}(F_{p+r-1}) \arrow[rr, "i^r_*", "\cong \text{ for } r>q+1 \ (\text{conn. cond.})"'] \arrow[d, "j^r_*"] & & H_{p+q}(C) & \\
H_{p+q}(F_{p+r-1}/F_{p-1}) & & &
\end{tikzcd}$$

exact

Note that $\ker \varphi^r_{p,q} = \ker\left(j^r_*(i^r_*)^{-1}\big|J_{p,q}\right)$. 1) now follows easily from above diagram.

Proof of 2): Write $E^r_{p,q} = E^r_{p,q}(C,F)$ and let $Z^r_{p,q}$, $B^r_{p,q}$ denote $\ker d^r$ and $\operatorname{im} d^r$, resp. in $E^r_{p,q}$.

Recall: For any r, the isom. $\theta^r : H_{p,q}(E^r, d^r) \xrightarrow{\cong} E^{r+1}_{p,q}$ is given as follows: For $x \in Z^r_{p,q}$, write $[x] = x + B^r_{p,q}$. Then $\theta^r[x] = j_*(x)$ where $j_* : H_{p+q}(F_{p+r-1}/F_{p-1}) \to H_{p+q}(F_{p+r}/F_{p-1})$. In particular, for $r > \max\{p, q+1\}$, the identification of $E^r_{p,q}$ with $E^{r+1}_{p,q}$ is such that

$$\begin{tikzcd}
E^r_{p,q} \arrow[r, equal] \arrow[d, hook] & E^{r+1}_{p,q} \arrow[d, hook] \\
H_{p+q}(F_{p+r-1}/F_{p-1}) \arrow[r, "j_*"'] & H_{p+q}(F_{p+r}/F_{p-1})
\end{tikzcd}$$

commutes.

Let $y \in J_{p,q}$ and write $[y] = y + J_{p-1,q+1}$. Then by def. $\varphi^r[y] = j^r_*(i^r_*)^{-1}(y)$. We have the commutative diagram

$$\begin{tikzcd}
 & H_{p+q}(C) & \\
H_{p+q}(F_{p+r-1}) \arrow[ur, "i_*^r", "\cong"'] \arrow[rr] \arrow[d, "j_*^r"'] & & H_{p+q}(F_{p+r}) \arrow[ul, "i_*^{r+1}"', "\cong"] \arrow[d, "j_*^{r+1}"] \\
H_{p+q}(F_{p+r-1}/F_{p-1}) \arrow[rr, "j_*"'] & & H_{p+q}(F_{p+r}/F_{p-1})
\end{tikzcd}$$

The result is now immediate.

Math 752 Notes

The category of map pairs

Def: A map pair $\pi : (T, T_0) \to (B, B_0)$ is a map of topological pairs such that $T_0 = \pi^{-1}(B_0)$.

π is not necessarily pointed. In particular, allow $T_0 = B_0 = \emptyset$.

Def: A morphism of map pairs f from $\pi : (T, T_0) \to (B, B_0)$ to $\bar{\pi} : (\bar{T}, \bar{T}_0) \to (\bar{B}, \bar{B}_0)$ consists of a pair of maps of topological pairs $f_T : (T, T_0) \to (\bar{T}, \bar{T}_0)$, $f_B : (B, B_0) \to (\bar{B}, \bar{B}_0)$ such that

$$\begin{array}{ccc} (T, T_0) & \xrightarrow{f_T} & (\bar{T}, \bar{T}_0) \\ \pi \downarrow & & \downarrow \bar{\pi} \\ (B, B_0) & \xrightarrow[f_B]{} & (\bar{B}, \bar{B}_0) \end{array}$$

commutes.

Form the category of map pairs.

The Serre filtration

Let $\pi : T \to B$ be a map (not necessarily a fibration), and $\sigma : I^n \to T$ a non-degenerate singular n-cube. The *weight* of σ (with respect to π), $\omega(\sigma)$, is the smallest non-negative $p \le n$ such that $\pi\sigma(t_1, \dots, t_p, t_{p+1}, \dots, t_n)$ is independent of $t_{p+1}, \dots, t_n$.

Thus $\omega(\sigma) = 0$ if and only if $\pi\sigma$ is constant. $\omega(\sigma) = n$ if and only if $\pi\sigma$ is non-degenerate.

Prop: Let $\pi : T \to B$ be a map. Let $\sigma : I^n \to T$ be a non-degenerate singular n-cube, $n \ge 1$. Suppose $\omega(\sigma) = p$. Then for each $1 \le i \le n$ and $\varepsilon = 0, 1$, either $\sigma_\varepsilon^{(i)}$ is degenerate, or $\omega(\sigma_\varepsilon^{(i)})$ is $\begin{cases} p & \text{if } p < i \le n \\ < p & \text{if } 1 \le i \le p. \end{cases}$

The proof is straightforward.

Thus for fixed p and any commutative ring with unit R, the non-degenerate singular cubes of weight $\leq p$ span a subcomplex $F_p^{\pi} Q(T;R)$ of $Q(T;R)$.

<u>Convention</u>: $F_p^{\pi} Q(T;R) = 0$ for $p < 0$.

<u>Note</u>: $\left[F_p^{\pi} Q(T;R) \right]_i = Q_i(T;R)$ if $i \leq p$. Thus $(Q(T;R), F^{\pi})$ is a canonically-bounded filtered chain complex.

If $\pi : (T, T_0) \to (B, B_0)$ is a map pair, set

$$F_p^{\pi} Q(T, T_0; R) = \frac{F_p^{\pi} Q(T;R) + Q(T_0;R)}{Q(T_0;R)} \subset \frac{Q(T;R)}{Q(T_0;R)} = Q(T, T_0; R).$$

This yields a canonically-bounded filtered chain complex $(Q(T, T_0; R), F^{\pi})$.

It is clear that if f is a morphism of map pairs, for each non-degenerate $\sigma : I^n \to T$, either $f_T \sigma$ is degenerate or $w(f_T \sigma) \leq w(\sigma)$. Hence f induces a morphism of filtered chain complexes, and it is straightforward to check that $\pi \longmapsto (Q(T, T_0; R), F^{\pi})$ is a covariant functor from the category of map pairs to the category of canonically-bounded filtered chain complexes of R-modules.

<u>The Leray-Serre Spectral Sequence of a map pair</u>

Composing the above functor from map pairs to canonically-bounded filtered chain complexes with the earlier algebraic functor from canonically-bounded filtered chain complexes to 1^{st} quadrant spectral sequences, we obtain a covariant functor $\pi \longmapsto E(\pi)$ from the category of map pairs to the category of 1^{st} quadrant spectral sequences of R-modules. The following is immediate from the earlier algebra on spectral sequences of filtered chain complexes:

Prop: Let $\pi : (T, T_0) \to (B, B_0)$ be a map pair. Then for each $n \geq 0$ there exists a natural filtration

$$0 = J_{-1,n+1}(\pi) \subset J_{0,n}(\pi) \subset \cdots \subset J_{n,0}(\pi) = H_n(T, T_0; R)$$

and natural isomorphisms $E^{\infty}_{p,q}(\pi) \cong J_{p,q}(\pi) / J_{p-1,q+1}(\pi)$ for all (p,q).

To prove the Leray-Serre theorem, it remains to prove: If $\pi : (T, T_0) \to (B, B_0)$ is a map pair for which $F \to T \xrightarrow{\pi} B$ is a pointed fibration with B 1-connected, then there are natural isomorphisms

$$E^2_{p,q}(\pi) \cong H_p(B, B_0; H_q(F; R)).$$

To show this, we investigate $E^1(\pi)$ and show that the chain complex $(E^1_{*,q}(\pi), d^1)$ is naturally chain-isomorphic to $Q(B, B_0) \underset{\mathbb{Z}}{\otimes} H_q(F; R)$.

31 Special Case 1: $1 : (B, B_0) \to (B, B_0)$. For any $\sigma : I^n \to B$, $w(\sigma) < n$ if and only if σ is degenerate. Thus

$$F^1_p Q_n(B, B_0; R) = \begin{cases} 0 & \text{if } p < n \\ Q_n(B, B_0; R) & \text{if } p \geq n, \end{cases}$$

and so

$$\left(\frac{F^1_p Q(B, B_0; R)}{F^1_{p-1} Q(B, B_0; R)} \right)_i = \begin{cases} 0 & \text{if } i \neq p \\ Q_p(B, B_0; R) & \text{if } i = p. \end{cases}$$

Thus

$$E^1_{p,q}(1) = \begin{cases} 0 & \text{if } q \neq 0 \\ Q_p(B, B_0; R) & \text{if } q = 0. \end{cases}$$

From the diagram

$$\begin{tikzcd}
 & (F_p^1/F_{p-2}^1)_p \arrow[rr] \arrow[d, "\partial"] \arrow[drr, "="] & & (F_p^1/F_{p-1}^1)_p \arrow[d, "="] \\
 & & & Q_p(B, B_0; R) \arrow[d, "\partial"] \\
(F_{p-1}^1/F_{p-2}^1)_{p-1} \arrow[r] \arrow[rrr, bend right=10, "="'] & (F_p^1/F_{p-2}^1)_{p-1} \arrow[rr, "="] & & Q_{p-1}(B, B_0; R)
\end{tikzcd}$$

it follows that the connecting homomorphism

$H_p(F_p^1/F_{p-1}^1) \longrightarrow H_{p-1}(F_{p-1}^1/F_{p-2}^1)$ is $\partial : Q_p(B, B_0; R) \to Q_{p-1}(B, B_0; R)$. Thus the chain complex $(E^1_{*,0}(1), d^1)$ is $Q(B, B_0; R)$. It follows that $E^2_{p,q}(1)$ is naturally isomorphic to

$$\begin{cases} 0 & \text{if } q \neq 0 \\ H_p(B, B_0; R) & \text{if } q = 0. \end{cases}$$

In this case the spectral sequence is concentrated on the base edge, and $E^r_{p,q}(1) \cong E^{r+1}_{p,q}(1)$ for all $r \geq 2$.

Claim:

$$\begin{tikzcd}
H_p(B, B_0; R) \arrow[d, "p_B"'] \arrow[dr, "\theta'", "\cong"'] & \\
E^\infty_{p,0}(1) \arrow[r, "i_B"'] & E^2_{p,0}(1)
\end{tikzcd}$$

commutes.

Proof of claim: Write $E^r_{p,q} = E^r_{p,q}(1)$, $F_p = F_p^1 Q(B, B_0; R)$, $Q = Q(B, B_0; R)$

Let $z \in Z_p Q$. Since

$$\begin{tikzcd}
 & H_{p,0}(E^1, d^1) \arrow[r, "\cong", "\theta'"'] & E^2_{p,0} \arrow[dd, hook] \\
Z_p Q \arrow[d, hook] \arrow[r, phantom, "="] & Z^1_{p,0} \arrow[u, "\text{onto}"'] \arrow[d, hook] & \\
Q_p \arrow[r, phantom, "="] & H_p(F_p/F_{p-1}) \arrow[r, "j_*"] & H_p(F_{p+1}/F_{p-1})
\end{tikzcd}$$

commutes, $\theta'(z + B_p Q) = (z + F_{p-1}) + B_p(F_{p+1}/F_{p-1}) \in E^2_{p,0} \subset H_p(F_{p+1}/F_{p-1})$.

Let $r > \max\{p, 1\}$. From commutativity of

$$\begin{tikzcd}
H_p(Q) \arrow[dd, "p_B"'] \arrow[r, phantom, "="] & J_{p,0} \arrow[d, equal] \arrow[r, phantom, "="] & \operatorname{im}\left[H_p(F_p) \longrightarrow H_p(Q)\right] \\
& J_{p,0}/J_{p-1,1} \arrow[dl, "\varphi"', "\cong"] \arrow[d, "\varphi^r", "\cong"'] & H_p(F_{p+r-1}) \arrow[u, "i^r_*"', "\cong"] \arrow[d, "j^r_*"] \\
E^\infty_{p,0} \arrow[r, equal] & E^r_{p,0} \arrow[r, hook] & H_p(F_{p+r-1}/F_{p-1})
\end{tikzcd}$$

and the fact that $(F_{p+r-1})_i = Q_i$ for $i \le p+r-1$, it follows that $z \in Z_p(F_{p+r-1})$. Since $i^r_*(z + B_p(F_{p+r-1})) = z + B_p Q$, it follows that

$$p_B(z + B_p Q) = j^r(z) + B_p(F_{p+r-1}/F_{p-1}) = (z + F_{p-1}) + B_p(F_{p+r-1}/F_{p-1})$$

$$\rotatebox{90}{$\in$}$$

$$E^\infty_{p,0} = E^r_{p,0} \subset H_p(F_{p+r-1}/F_{p-1}) \,.$$

From commutativity of

$$\begin{tikzcd}
E^\infty_{p,0} \arrow[rr, "i_B"] \arrow[d, equal] & & E^2_{p,0} \arrow[dd, hook] \\
E^r_{p,0} \arrow[d, hook] & & \\
H_p(F_{p+r-1}/F_{p-1}) & & H_p(F_{p+1}/F_{p-1}) \arrow[ll, "\cong", "j_*"']
\end{tikzcd}$$

for $r > \max\{p, 1\}$ (which follows from definition of the isomorphisms $\theta^r : H_{p,0}(E^r, d^r) \longrightarrow E^{r+1}_{p,0}$), and the fact that $(F_{p+1})_i = Q_i$ for $i \le p+1$, $z + F_{p-1} \in Z_p(F_{p+1}/F_{p-1})$. Thus since $j(z + F_{p-1}) = z + F_{p-1} \in (F_{p+r-1}/F_{p-1})_p$,

$$i_B(z + F_{p-1} + B_p(F_{p+r-1}/F_{p-1})) = (z + F_{p-1}) + B_p(F_{p+1}/F_{p-1}).$$

Thus

$$i_B\, p_B(z + B_p Q) = (z + F_{p-1}) + B_p(F_{p+1}/F_{p-1}) = \theta^1(z + B_p Q),$$

proving the claim.

Special Case 2: $\pi : X \to P$, P a point. Then for each non-degenerate $\sigma : I^n \to X$, $\pi\sigma$ is constant and so $\omega(\sigma) = 0$ for all σ. Thus $F_0^\pi Q(X;R) = Q(X;R)$, and so

$$\frac{F_p^\pi Q(X;R)}{F_{p-1}^\pi Q(X;R)} = \begin{cases} 0 & \text{if } p \neq 0 \\ Q(X;R) & \text{if } p = 0. \end{cases}$$

Thus $$E^1_{p,q}(\pi) = \begin{cases} 0 & \text{if } p \neq 0 \\ H_q(X;R) & \text{if } p = 0. \end{cases}$$

In this case the spectral sequence is concentrated on the fibre edge. All the differentials (including d^1) are thus 0.

In fact, for $r \geq 1$,

$$E^r_{0,q}(\pi) = \operatorname{im}\left[H_q(F_0^\pi) \longrightarrow H_q(F^\pi_{r-1})\right] = H_q(X;R)$$

and from commutativity of

$$\begin{tikzcd}
H_{0,q}(E^r,d^r) \arrow[r, "\theta^r", "\cong"'] & E^{r+1}_{0,q}(\pi) \arrow[dd, equal] \\
Z^r_{0,q} \arrow[u, "="'] \arrow[d, equal] & \\
H_q(F_{r-1}) \arrow[r, "j_*"] \arrow[d, equal] & H_q(F_r) \arrow[d, equal] \\
H_q(X;R) \arrow[r, "1"'] & H_q(X;R)
\end{tikzcd}$$

it follows that each θ^r is the identity map on $H_q(X;R)$. It follows easily that both $p_F : E^2_{0,q}(\pi) \longrightarrow E^\infty_{0,q}(\pi)$ and $i_F : E^\infty_{0,q}(\pi) \to H_q(X;R)$ (the composition $E^\infty_{0,q} \xrightarrow[\cong]{\varphi^{-1}} J_{0,q} \hookrightarrow H_q(X;R)$) are the identity map on $H_q(X;R)$.

The $Q(B,B_0)$-decomposition of $E^1(\pi)$

Let $\pi : (T,T_0) \to (B,B_0)$ be a map pair. Recall: $E^1_{p,q}(\pi) = H_{p+q}(F_p^\pi Q(T,T_0;R) / F_{p-1}^\pi Q(T,T_0;R))$.

Write $\bar{F}_p^\pi = F_p^\pi Q(T;R) + Q(T_0;R) \subset Q(T;R)$.
By definition,
$$F_p^\pi Q(T,T_0;R) = \bar{F}_p^\pi / Q(T_0;R).$$
Thus
$$\frac{F_p^\pi Q(T,T_0;R)}{F_{p-1}^\pi Q(T,T_0;R)} = \frac{\bar{F}_p^\pi / Q(T_0;R)}{\bar{F}_{p-1}^\pi / Q(T_0;R)} \cong \bar{F}_p^\pi / \bar{F}_{p-1}^\pi$$

and so we have a natural isomorphism $E^1_{p,q}(\pi) \cong H_{p+q}(\bar{F}_p^\pi / \bar{F}_{p-1}^\pi)$.

Moreover, from the map of triples

$$(\bar{F}_p^\pi, \bar{F}_{p-1}^\pi, \bar{F}_{p-2}^\pi) \longrightarrow (\bar{F}_p^\pi/Q(T_0;R), \bar{F}_{p-1}^\pi/Q(T_0;R), \bar{F}_{p-2}^\pi/Q(T_0;R))$$

it follows that $d^1 : E^1_{p,q}(\pi) \to E^1_{p-1,q}(\pi)$ is identified with the connecting homomorphism $H_{p+q}(\bar{F}_p^\pi/\bar{F}_{p-1}^\pi) \to H_{p+q-1}(\bar{F}_{p-1}^\pi/\bar{F}_{p-2}^\pi)$

in the homology sequence of the triple $(\bar{F}_p^\pi, \bar{F}_{p-1}^\pi, \bar{F}_{p-2}^\pi)$.

$(\bar{F}_p^\pi/\bar{F}_{p-1}^\pi)_n$ is the free R-module on the non-degenerate singular n-cubes σ of T such that $w(\sigma) = p$ and $\sigma(I^n) \not\subset T_0$.

If $\sigma : I^n \to T$ is non-degenerate and $w(\sigma) \le p$, write $\langle\sigma\rangle \in (\bar{F}_p^\pi/\bar{F}_{p-1}^\pi)_n$ for the image of σ.

Thus $\langle\sigma\rangle = 0$ if either $w(\sigma) \le p-1$ or $\sigma(I^n) \subset T_0$.

<u>Note</u>: $(\bar{F}_p^\pi/\bar{F}_{p-1}^\pi)_n = 0$ for $n < p$.

Let $\sigma : I^n \to T$ be non-degenerate. Say $w(\sigma) = p$.
Define $\sigma_B : I^p \to B$ by $\sigma_B(t_1, \dots, t_p) = \pi\sigma(t_1, \dots, t_p, *, \dots, *)$

(the last $n-p$ coordinates in this last expression can be arbitrary since $w(\sigma) = p$). Note that σ_B is a non-degenerate singular p-cube in B (if it were degenerate, $\pi\sigma$ would be independent of the last $n-p+1$ coordinates, and so we would have $\varpi(\sigma) \le p-1$, contradiction.)

It is easily checked that for $p < i \le n$ and $\varepsilon = 0, 1$, $(\sigma_\varepsilon^{(i)})_B = \sigma_B$ if $\sigma_\varepsilon^{(i)}$ is non-degenerate.

Recall: $\varpi(\sigma_\varepsilon^{(i)}) = \begin{cases} p & \text{if } p < i \le n \\ < p & \text{if } 1 \le i \le p \end{cases}$ when $\sigma_\varepsilon^{(i)}$ is non-degenerate. Thus $\langle \sigma_\varepsilon^{(i)} \rangle = 0$ for $1 \le i \le p$. Thus, since $\partial \sigma = \sum_{i=1}^{n} (-1)^i \{ \sigma_0^{(i)} - \sigma_1^{(i)} \}$, it follows that

$$\partial \langle \sigma \rangle = \sum_{i=p+1}^{n} (-1)^i \{ \langle \sigma_0^{(i)} \rangle - \langle \sigma_1^{(i)} \rangle \}.$$

(<u>Convention</u>: $\langle \sigma_\varepsilon^{(i)} \rangle = 0$ if $\sigma_\varepsilon^{(i)}$ is degenerate).

Thus, for a fixed non-degenerate $\tau : I^p \to B$, $\{ \langle \sigma \rangle \mid \sigma_B = \tau \}$ span a subcomplex $C^\tau(\pi)$ of $\bar{F}_p^\pi / \bar{F}_{p-1}^\pi$.

If $\tau : I^p \to B$ is degenerate or has image in B_0, define $C^\tau(\pi) = 0$.

$C^\tau(\pi)$ is natural with respect to morphisms of map pairs, i.e. if $f : \pi \to \pi'$ is a morphism of map pairs and $\tau : I^p \to B$, then the chain map induced by f,

$$f_\# : \bar{F}_p^\pi / \bar{F}_{p-1}^\pi \longrightarrow \bar{F}_p^{\pi'} / \bar{F}_{p-1}^{\pi'} \quad \text{carries } C^\tau(\pi) \text{ into } C^{f_B \tau}(\pi').$$

(If $f_B \tau$ is degenerate, then $\varpi(f_T \sigma) < p$ for any σ such that $\sigma_B = \tau$, and so $f_\# \langle \sigma \rangle = \langle f_T \sigma \rangle = 0$.)

For a map pair $\pi : (T, T_0) \to (B, B_0)$, let $N_p(\pi) =$ set of all non-degenerate $\tau : I^p \to B$ such that $\tau(I^p) \not\subset B_0$. For each non-degenerate $\sigma : I^n \to T$ such that $\varpi(\sigma) = p$ and $\sigma(I^n) \not\subset T_0$, there exists a unique $\tau \in N_p(\pi)$ such that $\sigma_B = \tau$ (since $T_0 = \pi^{-1}(B_0)$). Thus

$$\bar{F}_p^\pi / \bar{F}_{p-1}^\pi = \bigoplus_{\tau \in N_p(\pi)} C^\tau(\pi), \text{ and so}$$

$$E^1_{p,q}(\pi) = \bigoplus_{\tau \in N_p(\pi)} H_{p+q}(C^\tau(\pi)).$$

We must now describe d^1 in terms of this decomposition.

Def: Let C, $\bar{C}$ be chain complexes. A chain map of degree r $f: C \to \bar{C}$ consists of a sequence of homomorphisms $f_n : C_n \to \bar{C}_{n+r}$ such that for all n,

$$\begin{array}{ccc} C_n & \xrightarrow{f_n} & \bar{C}_{n+r} \\ \partial \downarrow & & \downarrow (-1)^r \bar{\partial} \\ C_{n-1} & \xrightarrow[f_{n-1}]{} & \bar{C}_{n+r-1} \end{array}$$

commutes.

Just as for ordinary chain maps (i.e chain maps of degree 0), such an f sends cycles to cycles, boundaries to boundaries, and hence induces $f_* : H_n(C) \to H_{n+r}(\bar{C})$. Moreover if $f: C \to \bar{C}$, $g: \bar{C} \to \bar{\bar{C}}$ are chain maps of degrees r and s, resp., then gf is a chain map of degree $r+s$ and $(gf)_* = g_* f_*$.

Terminology. If $\sigma : I^n \to T$, $\tau : I^p \to B$ are such that $\sigma_B = \tau$, we will say σ covers τ.

Note: If σ covers $\tau : I^p \to B$, it is easily checked that for $1 \le i \le p$, $\varepsilon = 0, 1$, $\sigma_\varepsilon^{(i)}$ covers $\tau_\varepsilon^{(i)}$ (provided $\sigma_\varepsilon^{(i)}$ is non-degenerate).

For each $\tau \in N_p(\pi)$, and $1 \le i \le p$, $\varepsilon = 0, 1$, define $\varphi_\varepsilon^{(i)} : C^\tau(\pi) \to C^{\tau_\varepsilon^{(i)}}(\pi)$ by

$$\varphi_\varepsilon^{(i)} \langle \sigma \rangle = \langle \sigma_\varepsilon^{(i)} \rangle .$$

Prop: The $\varphi_\varepsilon^{(i)}$, $1 \le i \le p$, $\varepsilon = 0, 1$, are chain maps of degree -1.

Proof: Recall: If $i < j$, then $\left(\sigma_\varepsilon^{(j)}\right)_\eta^{(i)} = \left(\sigma_\eta^{(i)}\right)_\varepsilon^{(j-1)}$, $\varepsilon = 0, 1$, $\eta = 0, 1$. Thus if $\sigma : I^{p+q} \to T$ is non-degenerate and covers τ,

$$\partial \varphi_\varepsilon^{(i)} \langle \sigma \rangle = \partial \langle \sigma_\varepsilon^{(i)} \rangle = \sum_{j=p}^{p+q-1} (-1)^j \left\{ \langle (\sigma_\varepsilon^{(i)})_0^{(j)} \rangle - \langle (\sigma_\varepsilon^{(i)})_1^{(j)} \rangle \right\}$$

$$= \sum_{j=p+1}^{p+q} (-1)^{j-1} \left\{ \langle (\sigma_\varepsilon^{(i)})_0^{(j-1)} \rangle - \langle (\sigma_\varepsilon^{(i)})_1^{(j-1)} \rangle \right\}$$

$$= - \sum_{j=p+1}^{p+q} (-1)^{j} \left\{ \langle (\sigma_0^{(j)})_\varepsilon^{(i)} \rangle - \langle (\sigma_1^{(j)})_\varepsilon^{(i)} \rangle \right\}$$

$$= - \varphi_\varepsilon^{(i)} \sum_{j=p+1}^{p+q} (-1)^{j} \left\{ \langle \sigma_0^{(j)} \rangle - \langle \sigma_1^{(j)} \rangle \right\}$$

$$= - \varphi_\varepsilon^{(i)} \partial \langle \sigma \rangle .$$

The $\varphi_\varepsilon^{(i)}$ are natural: If $f : \pi \to \pi'$ is a morphism of map pairs, it is easily verified that

$$\begin{tikzcd} C^{\tau}(\pi) \arrow[r, "\varphi_\varepsilon^{(i)}"] \arrow[d, "\hat{f}_\#"'] & C^{\tau_\varepsilon^{(i)}}(\pi) \arrow[d, "f_\#"] \\ C^{f_B\tau}(\pi') \arrow[r, "\varphi_\varepsilon^{(i)}"'] & C^{f_B \tau_\varepsilon^{(i)}}(\pi') \end{tikzcd}$$

commutes.

<u>Theorem</u>: The differential

$$d' : \bigoplus_{\tau \in N_p(\pi)} H_{p+q}(C^{\tau}(\pi)) \longrightarrow \bigoplus_{\tau \in N_{p-1}(\pi)} H_{p+q-1}(C^{\tau}(\pi))$$

is $\displaystyle\sum_{i=1}^{p} (-1)^i \left(\varphi_{0\,*}^{(i)} - \varphi_{1\,*}^{(i)} \right)$

<u>Proof</u>: Let $a \in H_{p+q}(C^{\tau}(\pi))$. Say

$\displaystyle\sum_\sigma r_\sigma \langle \sigma \rangle \in C^{\tau}(\pi) \subset \bar{\bar{F}}_p^{\pi} / \bar{\bar{F}}_{p-1}^{\pi}$ is a cycle representing a, where the $r_\sigma \in R$ and the σ are non-degenerate of weight p.

Then $\displaystyle\sum_\sigma r_\sigma \sigma$ is a chain in $\bar{\bar{F}}_p^{\pi}$ such that $\partial \left(\sum_\sigma r_\sigma \sigma \right) \in \bar{\bar{F}}_{p-1}^{\pi}$. By definition of the connecting homomorphism in the homology sequence of a triple, $d'a$ is the homology class of the image of

$\partial\left(\sum_{\sigma} r_\sigma \sigma\right)$ in $\overline{F}^{\pi}_{p-1} / \overline{F}^{\pi}_{p-2}$.

We have $\partial\left(\sum_{\sigma} r_\sigma \sigma\right) = \sum_{\sigma} r_\sigma \sum_{i=1}^{p+q} (-1)^i \{\sigma_0^{(i)} - \sigma_1^{(i)}\}$

$$= \sum_{\sigma} r_\sigma \left(\sum_{i=1}^{p} (-1)^i \{\sigma_0^{(i)} - \sigma_1^{(i)}\} + \sum_{i=p+1}^{p+q} (-1)^i \{\sigma_0^{(i)} - \sigma_1^{(i)}\} \right).$$

Now for $i > p$ each $\sigma_\varepsilon^{(i)}$ has weight p, while for $i \leq p$ each $\sigma_\varepsilon^{(i)}$ has weight $< p$. Since singular cubes of different weights are independent and since

$\partial\left(\sum_{\sigma} r_\sigma \sigma\right) \in \overline{F}^{\pi}_{p-1}$, we must have

$$\sum_{\sigma} r_\sigma \sum_{i=p+1}^{p+q} (-1)^i \{\sigma_0^{(i)} - \sigma_1^{(i)}\} \in Q(T_0; R).$$

Thus $\partial\left(\sum_{\sigma} r_\sigma \sigma\right) = \sum_{\sigma} r_\sigma \sum_{i=1}^{p} (-1)^i \{\sigma_0^{(i)} - \sigma_1^{(i)}\} \mod Q(T_0; R)$

and hence $d'a$ = homology class of

$$\sum_{\sigma} r_\sigma \sum_{i=1}^{p} (-1)^i \{\langle \sigma_0^{(i)} \rangle - \langle \sigma_1^{(i)} \rangle\}$$

$$= \sum_{\sigma} r_\sigma \sum_{i=1}^{p} (-1)^i \{\varphi_0^{(i)} - \varphi_1^{(i)}\} \langle \sigma \rangle$$

$$= \sum_{i=1}^{p} (-1)^i \{\varphi_0^{(i)} - \varphi_1^{(i)}\} \left(\sum_{\sigma} r_\sigma \langle \sigma \rangle \right)$$

and so $d'a = \sum_{i=1}^{p} (-1)^i \{\varphi_{0*}^{(i)} - \varphi_{1*}^{(i)}\}(a)$.

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Special Case 1: $1 : (B, B_0) \to (B, B_0)$.
Each $\tau \in N_p(1)$ has a unique non-degenerate lift, namely τ.
Thus
$$C^\tau(1)_n = \begin{cases} 0 & \text{if } n \neq p \\ \text{free } R\text{-module on } \langle\tau\rangle & \text{if } n = p. \end{cases}$$

Thus $C^\tau(1)$ has trivial boundary operator, and $H_p(C^\tau(1)) = C^\tau(1)_p =$ free R-module on $\langle\tau\rangle$. Since $\varphi^{(i)}_{\varepsilon *}\langle\tau\rangle = \langle\tau^{(i)}_\varepsilon\rangle$ we have $d'\langle\tau\rangle = \sum_{i=1}^{p}(-1)^i\{\langle\tau^{(i)}_0\rangle - \langle\tau^{(i)}_1\rangle\}$.

Special Case 2: $\pi : X \to P$, P a point.

Let $\tau_0 : I^0 \to P$ denote the unique map. τ_0 is the only non-degenerate singular cube in P, and every non-degenerate singular cube in X covers τ_0. Thus $C^{\tau_0}(\pi) = Q(X; R)$, and $E^1_{0,q}(\pi) = H_q(C^{\tau_0}(\pi)) = H_q(X; R)$. τ_0 has no faces, and as noted earlier, $d' = 0$.

The function space decomposition of $E^1(\pi)$

Let $\pi : (T, T_0) \to (B, B_0)$ be a map pair, and $\tau \in N_p(\pi)$. Let $X^\tau(\pi) = \{u : I^p \to T \mid u \text{ covers } \tau\}$ with the compact-open topology.

If $\sigma : I^{p+q} \to T$ is non-degenerate and covers τ, define $ad^\tau\sigma : I^q \to X^\tau(\pi)$ by
$$(ad^\tau\sigma)(t_1, \dots, t_q)(u_1, \dots, u_p) = \sigma(u_1, \dots, u_p, t_1, \dots, t_q).$$
Note that $(ad^\tau\sigma)(t_1, \dots, t_q) \in X^\tau(\pi)$, for $\pi\sigma(u_1, \dots, u_p, t_1, \dots, t_q) = \tau(u_1, \dots, u_p)$ since $\sigma_B = \tau$. Also, since $\sigma(u_1, \dots, u_p, t_1, \dots, t_q)$ is not independent of t_q (since σ is non-degenerate), $ad^\tau\sigma$ is non-degenerate.

Conversely suppose $u : I^q \to X^\tau(\pi)$ is non-degenerate. Define $ad_\tau u : I^{p+q} \to T$ by
$$(ad_\tau u)(u_1, \dots, u_p, t_1, \dots, t_q) = u(t_1, \dots, t_q)(u_1, \dots, u_p).$$
Clearly, $ad_\tau u$ is non-degenerate. Since $u(t_1, \dots, t_q)$ covers τ, $\pi\, ad_\tau u(u_1, \dots, u_p, t_1, \dots, t_q) = \pi u(t_1, \dots, t_q)(u_1, \dots, u_p) = \tau(u_1, \dots, u_p)$ and so $ad_\tau u$ covers τ.

It is easily checked that $ad_\tau\, ad^\tau(\sigma) = \sigma$, $ad^\tau ad_\tau(u) = u$, and so ad_τ is a bijection from the set of non-degenerate singular q-cubes of $X^\tau(\pi)$ to the set of non-degenerate singular $p+q$ - cubes of T which cover τ. We thus obtain an R-isomorphism which raises degree by p

$$\alpha_\tau : Q(X^\tau(\pi); R) \longrightarrow C^\tau(\pi)$$

given by $\alpha_\tau(u) = \langle \mathrm{ad}_\tau u \rangle$, $u : I^q \to X^\tau(\pi)$ non-degenerate.
We wish to show α_τ is a chain map of degree p.

Lemma: Let $\tau \in N_p(\pi)$, and suppose $u : I^q \to X^\tau(\pi)$ is non-degenerate. Then for $1 \le i \le q$ and $\varepsilon = 0, 1$,

$$\mathrm{ad}_\tau(u_\varepsilon^{(i)}) = (\mathrm{ad}_\tau u)_\varepsilon^{(p+i)}.$$

(It is understood that $\mathrm{ad}_\tau(u_\varepsilon^{(i)}) = 0$ if $u_\varepsilon^{(i)}$ is degenerate).

Proof: $\mathrm{ad}_\tau(u_\varepsilon^{(i)})(u_1, \ldots, u_p, t_1, \ldots, t_{q-1}) =$

$$u_\varepsilon^{(i)}(t_1, \ldots, t_{q-1})(u_1, \ldots, u_p) = u(t_1, \ldots, \underset{i^{\text{th}}}{\varepsilon}, \ldots, t_{q-1})(u_1, \ldots, u_p),$$

$$(\mathrm{ad}_\tau u)_\varepsilon^{(p+i)}(u_1, \ldots, u_p, t_1, \ldots, t_{q-1}) = (\mathrm{ad}_\tau u)(\overbrace{u_1, \ldots, u_p, \underbrace{t_1, \ldots,}_{i}}^{p+i} \varepsilon, \ldots, t_{q-1})$$

$$= u(t_1, \ldots, \underset{i^{\text{th}}}{\varepsilon}, \ldots, t_{q-1})(u_1, \ldots, u_p).$$

Theorem: α_τ is a chain isomorphism of degree p for each $\tau \in N_p(\pi)$.

Proof: It remains only to show $\partial \alpha_\tau = (-1)^p \alpha_\tau \partial$.
Let $u : I^q \to X^\tau(\pi)$ be non-degenerate. Then

$$\alpha_\tau \partial(u) = \alpha_\tau \sum_{i=1}^{q} (-1)^i \{ u_0^{(i)} - u_1^{(i)} \}$$

$$= \sum_{i=1}^{q} (-1)^i \{ \langle \mathrm{ad}_\tau(u_0^{(i)}) \rangle - \langle \mathrm{ad}_\tau(u_1^{(i)}) \rangle \},$$

$$\partial \alpha_\tau(u) = \partial \langle \mathrm{ad}_\tau u \rangle = \sum_{i=p+1}^{p+q} (-1)^i \{ \langle (\mathrm{ad}_\tau u)_0^{(i)} \rangle - \langle (\mathrm{ad}_\tau u)_1^{(i)} \rangle \}$$

$$= \sum_{i=1}^{q} (-1)^{p+i} \{ \langle (\mathrm{ad}_\tau u)_0^{(p+i)} \rangle - \langle (\mathrm{ad}_\tau u)_1^{(p+i)} \rangle \}$$

$$= (-1)^p \sum_{i=1}^{q} (-1)^i \left\{ \langle ad_\tau(u_0^{(i)}) \rangle - \langle ad_\tau(u_1^{(i)}) \rangle \right\}$$
$$= (-1)^p \alpha_\tau \partial(u).$$

We next deal with naturality of α_τ. If $\tau : I^p \to B$ is either degenerate or has image in B_0, define $X^\tau(\pi) = \emptyset$.

If $f : \pi \to \pi'$ is a morphism of map pairs, and $\tau \in N_p(\pi)$ is such that $f_B\tau \in N_p(\pi')$, f induces a continuous map
$$X(f) : X^\tau(\pi) \longrightarrow X^{f_B\tau}(\pi')$$
given by $X(f)(u) = f_T u$. Define

$X(f)_\# : Q(X^\tau(\pi), R) \to Q(X^{f_B\tau}(\pi'); R)$ to be

$$\begin{cases} Q(X(f)) & \text{if } f_B\tau \in N_p(\pi') \\ 0 & \text{if } f_B\tau \text{ is either degenerate or has image in } B_0'. \end{cases}$$

In either case, it is easily verified that

$$\begin{tikzcd} Q(X^\tau(\pi); R) \arrow[r, "\alpha_\tau"] \arrow[d, "X(f)_\#"'] & C^\tau(\pi) \arrow[d, "f_\#"] \\ Q(X^{f_B\tau}(\pi'); R) \arrow[r, "\alpha_{f_B\tau}"'] & C^{f_B\tau}(\pi') \end{tikzcd}$$

commutes. (Recall: $C^{f_B\tau}(\pi') = 0$ if $f_B\tau \notin N_p(\pi')$).

Thus the isomorphisms $(\alpha_\tau)_*^{-1} : H_{p+q}(C^\tau(\pi)) \to H_q(X^\tau(\pi); R)$ yield a natural isomorphism

$$E^1_{p,q}(\pi) \cong \bigoplus_{\tau \in N_p(\pi)} H_q(X^\tau(\pi); R).$$

We must now describe d^1 in terms of this decomposition.
<u>Recall</u>: If $\tau \in N_p(\pi)$ and $\sigma \in X^\tau(\pi)$, then for $1 \le i \le p$, $\varepsilon = 0, 1$, $\sigma_\varepsilon^{(i)}$ covers $\tau_\varepsilon^{(i)}$ (provided $\tau_\varepsilon^{(i)}$ is non-degenerate). Thus we obtain a continuous map $\rho_\varepsilon^{(i)} : X^\tau(\pi) \to X^{\tau_\varepsilon^{(i)}}(\pi)$ given by $\rho_\varepsilon^{(i)}(\sigma) = \sigma_\varepsilon^{(i)}$ when $\tau_\varepsilon^{(i)} \in N_{p-1}(\pi)$. In any case, define

$\rho^{(i)}_{\varepsilon\,\#} : Q(X^{\tau}(\pi)) \to Q(X^{\tau^{(i)}_{\varepsilon}}(\pi))$ to be

$$\begin{cases} Q(\rho^{(i)}_{\varepsilon}) & \text{if } \tau^{(i)}_{\varepsilon} \in N_{p-1}(\pi) \\ 0 & \text{if either } \tau^{(i)}_{\varepsilon} \text{ is degenerate or } \tau^{(i)}_{\varepsilon}(I^{p-1}) \subset B_0. \end{cases}$$

<u>Lemma</u>: Let $\tau \in N_p(\pi)$. Then for $1 \le i \le p$ and $\varepsilon = 0, 1$,

$$\begin{tikzcd} Q(X^{\tau}(\pi); R) \arrow[r, "\rho^{(i)}_{\varepsilon\,\#}"] \arrow[d, "\alpha_{\tau}"'] & Q(X^{\tau^{(i)}_{\varepsilon}}(\pi); R) \arrow[d, "\alpha_{\tau^{(i)}_{\varepsilon}}"] \\ C^{\tau}(\pi) \arrow[r, "\varphi^{(i)}_{\varepsilon}"'] & C^{\tau^{(i)}_{\varepsilon}}(\pi) \end{tikzcd}$$

commutes.

<u>Proof</u>: The result is trivial if $\tau^{(i)}_{\varepsilon} \notin N_{p-1}(\pi)$. Assume $\tau^{(i)}_{\varepsilon} \in N_{p-1}(\pi)$. Let $\mu : I^{q} \to X^{\tau}(\pi)$ be non-degenerate. Then

$$\alpha_{\tau^{(i)}_{\varepsilon}} \rho^{(i)}_{\varepsilon\,\#}(\mu) = \alpha_{\tau^{(i)}_{\varepsilon}}(\rho^{(i)}_{\varepsilon} \circ \mu) = \langle ad_{\tau^{(i)}_{\varepsilon}}(\rho^{(i)}_{\varepsilon} \circ \mu) \rangle,$$

$$\varphi^{(i)}_{\varepsilon} \alpha_{\tau}(\mu) = \varphi^{(i)}_{\varepsilon} \langle ad_{\tau}(\mu) \rangle = \langle (ad_{\tau}(\mu))^{(i)}_{\varepsilon} \rangle.$$

Thus we must check $\quad ad_{\tau^{(i)}_{\varepsilon}}(\rho^{(i)}_{\varepsilon} \circ \mu) = (ad_{\tau}(\mu))^{(i)}_{\varepsilon}$.

We have $ad_{\tau^{(i)}_{\varepsilon}}(\rho^{(i)}_{\varepsilon} \circ \mu)(u_1, \dots, u_{p-1}, t_1, \dots, t_q)$

$$= (\rho^{(i)}_{\varepsilon} \circ \mu)(t_1, \dots, t_q)(u_1, \dots, u_{p-1}) = (\mu(t_1, \dots, t_q))^{(i)}_{\varepsilon}(u_1, \dots, u_{p-1})$$
$$= \mu(t_1, \dots, t_q)(u_1, \dots, \underset{i^{th}}{\varepsilon}, \dots, u_{p-1}),$$

$$(ad_{\tau}(\mu))^{(i)}_{\varepsilon}(u_1, \dots, u_{p-1}, t_1, \dots, t_q) = ad_{\tau}(\mu)(u_1, \dots, \underset{i^{th}}{\varepsilon}, \dots, u_{p-1}, t_1, \dots, t_q)$$

$$= \mu(t_1, \dots, t_q)(u_1, \dots, \underset{i^{th}}{\varepsilon}, \dots, u_{p-1}).$$

<u>Theorem</u>: $\quad d' : \bigoplus_{\tau \in N_p(\pi)} H_q(X^{\tau}(\pi); R) \longrightarrow \bigoplus_{\tau \in N_{p-1}(\pi)} H_q(X^{\tau}(\pi); R)$

is given by
$$d' = \sum_{i=1}^{p} (-1)^i \{ \rho^{(i)}_{0\,*} - \rho^{(i)}_{1\,*} \}.$$

Proof: By the lemma,

$$\begin{array}{ccc}
\bigoplus_{\tau\in N_p(\pi)} H_{p+q}(C^\tau(\pi)) & \xrightarrow{d' = \sum_{i=1}^{p}(-1)^i\{\varphi_{0*}^{(i)} - \varphi_{1*}^{(i)}\}} & \bigoplus_{\tau\in N_{p-1}(\pi)} H_{p+q-1}(C^\tau(\pi)) \\
\oplus\alpha_{\tau*}\uparrow\cong & & \cong\uparrow\oplus\alpha_{\tau*} \\
\bigoplus_{\tau\in N_p(\pi)} H_q(X^\tau(\pi);R) & \xrightarrow[\sum_{i=1}^{p}(-1)^i\{\rho_{0*}^{(i)} - \rho_{1*}^{(i)}\}]{} & \bigoplus_{\tau\in N_{p-1}(\pi)} H_q(X^\tau(\pi);R)
\end{array}$$

commutes.

Special Case 1: $1:(B,B_0)\to(B,B_0)$. Since each $\tau\in N_p(1)$ has τ as its unique lift, the space $X^\tau(1)$ consists of the single element τ. Let $u_\tau: I^0\to X^\tau(1)$ denote the unique map. We have $ad_\tau(u_\tau)(u_1,\dots,u_p) = u_\tau(\,)(u_1,\dots,u_p) = \tau(u_1,\dots,u_p)$ and so $ad_\tau(u_\tau)=\tau$. Thus $\alpha_\tau: Q(X^\tau(1);R)\to C^\tau(1)$ is the chain isomorphism of degree p given by $\alpha_\tau(u_\tau) = \langle\tau\rangle$. Write $\{u_\tau\}\in H_0(X^\tau(1);R)$ for the homology class of u_τ. We have

$$E'_{p,0}(1) \cong \bigoplus_{\tau\in N_p(\pi)} H_0(X^\tau(1);R).$$

For each $1\le i\le p$, $\varepsilon=0,1$ for which $\tau_\varepsilon^{(i)}\in N_{p-1}(1)$,

$\rho_\varepsilon^{(i)}: X^\tau(1)\to X^{\tau_\varepsilon^{(i)}}(1)$ is the unique map of 1-point spaces. Since $\{u_\tau\}$ and $\{u_{\tau_\varepsilon^{(i)}}\}$ both have augmentation 1, it follows that $\rho_{\varepsilon*}^{(i)}\{u_\tau\} = \{u_{\tau_\varepsilon^{(i)}}\}$. Thus

$$d'\{u_\tau\} = \sum_{i=1}^{p}(-1)^i\left(\{u_{\tau_0^{(i)}}\} - \{u_{\tau_1^{(i)}}\}\right).$$

Special Case 2: $\pi: X\to P$, P a point. Let $\tau_0: I^0\to P$ denote the unique non-degenerate singular cube in P. For each $x\in X$, let $\sigma_x: I^0\to X$ denote the singular 0-cube with image x. Then $X^{\tau_0}(\pi) = \{\sigma_x \mid x\in X\}$, and the function $h: X^{\tau_0}(\pi)\to X$ given by $h(\sigma_x) = x = \sigma_x(\,)$ is a homeomorphism. For each non-degenerate $u: I^q\to X^{\tau_0}(\pi)$, $ad_{\tau_0}(u)(t_1,\dots,t_q) = u(t_1,\dots,t_q)(\,) = hu(t_1,\dots,t_q)$ and so

$ad_{\tau_0}(u) = hu$. Thus $\alpha_{\tau_0} : Q(X^{\tau_0}(\pi); R) \to C^{\tau_0}(\pi) = Q(X; R)$ is the chain map $Q(h)$, and $(\alpha_{\tau_0 *})^{-1} = (h^{-1})_* : H_q(X; R) \to H_q(X^{\tau_0}(\pi); R)$

The fibration lemma

Let $\pi : (T, T_0) \to (B, B_0)$ be a map pair. For each $x \in B$, write $F_x = \pi^{-1}(x)$. Let $\tau \in N_p(\pi)$. For each $u \in I^p$, define

$$e_u^\tau : X^\tau(\pi) \longrightarrow F_{\tau(u)} \quad \text{by} \quad e_u^\tau(\sigma) = \sigma(u).$$

Each e_u^τ is continuous. The object of this section is to prove the following:

<u>Fibration Lemma</u>: Suppose $\pi : T \to B$ is a fibration. Then for each $\tau \in N_p(\pi)$ and $u \in I^p$, $e_u^\tau : X^\tau(\pi) \to F_{\tau(u)}$ is a homotopy equivalence.

33 The proof involves some basics about fibrations, which we proceed to describe.

Suppose $\pi : T \to B$ is a fibration, and $f : X \to B$ a continuous map. Define $f^*(T) = \{(x,y) \in X \times T \mid f(x) = \pi(y)\}$, and $\rho : f^*(T) \to X$, $\tilde{f} : f^*(T) \to T$ by $\rho(x,y) = x$, $\tilde{f}(x,y) = y$. Then

$$\begin{tikzcd} f^*(T) \arrow[r, "\tilde{f}"] \arrow[d, "\rho"'] & T \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

<u>Claim</u>: $\rho : f^*(T) \to X$ is a fibration.

For suppose given a commutative diagram of continuous maps

$$\begin{tikzcd} Y \times 0 \arrow[r, "g"] \arrow[d, hook] & f^*(T) \arrow[d, "\rho"] \\ Y \times I \arrow[r, "G"'] & X \end{tikzcd}$$

Then the outer rectangle

$$\begin{tikzcd} Y\times 0 \arrow[r, "\tilde{f} g"] \arrow[d, hook] & T \arrow[d, "\pi"] \\ Y\times I \arrow[r, "fG"'] \arrow[ur, dashed, "F"] & B \end{tikzcd}$$

commutes, and so since π is a fibration, a continuous F exists so that both triangles commute. Define $H: Y\times I \to f^*(T)$ by $H(z) = (G(z), F(z))$. Then H is continuous, and both triangles in

$$\begin{tikzcd} Y\times 0 \arrow[r, "g"] \arrow[d, hook] & f^*T \arrow[d, "\rho"] \\ Y\times I \arrow[r, "G"'] \arrow[ur, "H"] & X \end{tikzcd}$$

commute.

$\rho: f^*T \to X$ is called the <u>fibration induced from π by f</u>. Note that for each $x\in X$, $\tilde{f}$ carries $\rho^{-1}(x)$ homeomorphically onto $\pi^{-1}(f(x))$.

Let τ, u be as in the statement of the fibration lemma. Form the induced fibration $\rho: \tau^*(T) \to I^P$. Write $F_u = \rho^{-1}(u)$. Then $\tilde{\tau}$ maps F_u homeomorphically onto $F_{\tau(u)}$.

Let $\Gamma(\rho)$ = space of sections of ρ

$= \{\sigma: I^P \to \tau^*(T) \mid \rho\sigma = 1_{I^P}\}$ with the compact-open topology. Define $f: \Gamma(\rho) \to X^\tau(\pi)$, $g: X^\tau(\pi) \to \Gamma(\rho)$ by $f(\sigma)(v) = \tilde{\tau}\sigma(v)$, $g(\mu)(v) = (v, \mu(v))$. f and g are continuous, and it is easily checked that they are homeomorphisms, inverse to one another.

For each $u \in I^P$ define $e_u: \Gamma(\rho) \to F_u$ by $e_u(\sigma) = \sigma(u)$. Then the diagram

$$\begin{tikzcd} F_u \arrow[r, "\tilde{\tau}|F_u"] & F_{\tau(u)} \\ \Gamma(\rho) \arrow[u, "e_u"] \arrow[r, "f"'] & X^\tau(\pi) \arrow[u, "e_u^\tau"'] \end{tikzcd}$$

commutes.

Since f and $\tilde{\tau}|F_u$ are homeomorphisms, the proof of the Fibration Lemma will be completed by showing e_u is a homotopy equivalence.

<u>Def</u>: Two fibrations $\pi_1 : T_1 \to B$, $\pi_2 : T_2 \to B$ are said to be <u>fibre-homotopy equivalent</u> if there exist fibre-preserving maps $\alpha : T_1 \to T_2$, $\beta : T_2 \to T_1$

(i.e.
$$\begin{tikzcd} T_1 \arrow[rr, "\alpha"] \arrow[dr, "\pi_1"'] & & T_2 \arrow[dl, "\pi_2"] \\ & B & \end{tikzcd} \quad , \quad \begin{tikzcd} T_2 \arrow[rr, "\beta"] \arrow[dr, "\pi_2"'] & & T_1 \arrow[dl, "\pi_1"] \\ & B & \end{tikzcd}$$
both commute)

such that $\alpha\beta$ is <u>fibrewise</u> homotopic to 1_{T_2} and $\beta\alpha$ is <u>fibrewise</u> homotopic to 1_{T_1} (i.e. there exist homotopies $h_i : T_i \times I \to T_i$, $i = 1, 2$ such that

$$\beta\alpha \underset{h_1}{\simeq} 1_{T_1}, \quad \alpha\beta \underset{h_2}{\simeq} 1_{T_2}, \quad \text{and} \quad \pi_i\, h_i(x,t) = \pi_i(x)$$

for all $(x,t) \in T_i \times I$, $i = 1, 2$.)

<u>Prop</u>: Let $\pi : T \to B$ be a fibration with B contractible. Then this fibration is fibre-homotopy equivalent to the product bundle $\pi_1 : B \times F_u \to B$ for any $u \in B$.

The proof of this proposition can be found in Spanier, pp. 100-102 (it is corollary 15 on p. 102).

The proof of the Fibration Lemma is now completed as follows: Since I^p is contractible, there exist fibre-homotopy equivalences

$$\tau^*(T) \xrightarrow{\alpha} I^p \times F_u \xrightarrow{\beta} \tau^*(T)$$

and a fibrewise homotopy $h : \tau^*(T) \times I \to \tau^*(T)$ from $\beta\alpha$ to $1_{\tau^*(T)}$. Define $g : F_u \to \Gamma(\rho)$ by

$$g(y)(x) = \beta(x, \pi_2\alpha(y)) \quad \text{where} \quad \pi_2 : I^p \times F_u \to F_u \quad \text{is}$$

projection on the 2nd factor. $g(y) \in \Gamma(\rho)$ for each $y \in F_u$,

$$\begin{aligned} \text{for} \quad \rho\, g(y)(x) &= \rho\,\beta(x, \pi_2\alpha(y)) \\ &= \pi_1(x, \pi_2\alpha(y)) \quad (\text{since } \beta \text{ is fibre-preserving}) \\ &= x. \end{aligned}$$

We proceed to show $e_u g \simeq 1_{F_u}$ and $g\, e_u \simeq 1_{\Gamma(\rho)}$.

For all $y \in F_u$, $\pi_1\alpha(y) = u$ since α is fibre-preserving.

Thus $e_u g(y) = g(y)(u) = \beta(u, \pi_2 \alpha(y)) = \beta(\pi_1 \alpha(y), \pi_2 \alpha(y))$ $= \beta\alpha(y)$ and so $e_u g = \beta\alpha | F_u : F_u \to F_u$. Since h is a fibrewise homotopy, $h | F_u \times I$ is a homotopy from $\beta\alpha | F_u$ to 1_{F_u}. Thus $e_u g \simeq 1_{F_u}$.

For each $\sigma \in \Gamma(\rho)$, $g e_u(\sigma)(x) = g(\sigma(u))(x) = \beta(x, \pi_2 \alpha\, \sigma(u))$. Let $k : I^p \times I \to I^p$ be a homotopy from the constant map with value u to 1_{I^p} (k exists since I^p is contractible). For each $t \in I$, the map $\omega_t^\sigma : I^p \to \tau^*(T)$ given by $\omega_t^\sigma(x) =$ $\beta(x, \pi_2 \alpha\, \sigma(k(x,t)))$ is a section, for $\rho\beta(x, \pi_2 \alpha\, \sigma(k(x,t))) =$ $\pi_1(x, \text{———}) = x$. We have $\omega_0^\sigma(x) = \beta(x, \pi_2 \alpha\, \sigma(k(x,0)))$ $= \beta(x, \pi_2 \alpha\, \sigma(u)) = g e_u(\sigma)(x)$, $\omega_1^\sigma(x) = \beta(x, \pi_2 \alpha\, \sigma(k(x,1)))$ $= \beta(x, \pi_2 \alpha\, \sigma(x)) = \beta(\pi_1 \alpha\, \sigma(x), \pi_2 \alpha\, \sigma(x))$ (since $\pi_1 \alpha = \rho$) $= \beta\alpha\,\sigma(x)$. Thus the map $\Gamma(\rho) \times I \to \Gamma(\rho)$ given by $(\sigma, t) \mapsto \omega_t^\sigma$ is a homotopy from $g e_u$ to the map ℓ given by $\ell(\sigma) = \beta\alpha\sigma$. Finally, using the fibrewise homotopy h from $\beta\alpha$ to $1_{\tau^*(T)}$, the map $L : \Gamma(\rho) \times I \to \Gamma(\rho)$ given by $L(\sigma, t)(x) = h(\sigma(x), t)$ is a homotopy from ℓ to $1_{\Gamma(\rho)}$, completing the proof of the Fibration Lemma.

<u>Special Case 1:</u> $1 : (B, B_0) \to (B, B_0)$. As noted earlier, $X^\tau(1)$ is a 1-point space for each $\tau \in N_p(1)$. Each fibre is also a 1-point space, and so each e_u^τ is a homeomorphism of 1-point spaces.

<u>Special Case 2:</u> $\pi : X \to P$, P a point. There is only one fibre, namely $F_P = X$, and $e_P^{\tau_0} : X^{\tau_0}(\pi) \to X$ is the homeomorphism h described earlier.

<u>The twisted $Q(B, B_0) \otimes H(F)$ description of E^1 for a fibration pair</u>

Let $\pi : (E, E_0) \to (B, B_0)$ be a fibration pair, i.e. a map pair with $\pi : T \to B$ a fibration.

Write $0 = (0, \dots, 0) \in I^p$. If $\tau \in N_p(\pi)$, write $F_\tau = F_{\tau(0)}$. The isomorphisms $(e_0^\tau)_* : H_q(X^\tau(\pi); R) \to H_q(F_\tau; R)$, together with the function space description of E^1, yield a natural isomorphism $E^1_{p,q}(\pi) \cong \bigoplus_{\tau \in N_p(\pi)} H_q(F_\tau; R)$.

<u>Note</u>: This last direct sum is indexed by the $\tau \in N_p(\pi)$. If it happens that $\tau \neq \tau'$ but $\tau(0) = \tau'(0)$, then $F_\tau = F_{\tau'}$, but $H_q(F_\tau; R)$ and $H_q(F_{\tau'}; R)$ occur as <u>separate summands</u>. In order to keep track of summands, write $\tau \times H_q(F_\tau; R)$ as an isomorphic copy of $H_q(F_\tau; R)$. Thus if $\tau \neq \tau'$ but $F_\tau = F_{\tau'} = F$, $\tau \times H_q(F; R)$ is notationally different from $\tau' \times H_q(F; R)$.

We will write $\overline{e^\tau_0} : H_q(X^\tau(\pi); R) \to \tau \times H_q(F_\tau; R)$ for the map
$$\overline{e^\tau_0}(a) = (\tau, (e^\tau_0)_*(a)).$$
Thus the $\overline{e^\tau_0}$ yield a natural isomorphism
$$E^1_{p,q}(\pi) \cong \bigoplus_{\tau \in N_p(\pi)} \tau \times H_q(F_\tau; R).$$

It is convenient to define $\tau \times H_q(F_\tau; R) = 0$ if τ is either degenerate or has image in B_0.

<u>Note</u>: If it were possible to naturally identify the various $H_q(F_\tau; R)$ with $H_q(F; R)$ for a single F, then the map
$$Q_p(B, B_0) \underset{\mathbb{Z}}{\otimes} H_q(F; R) \longrightarrow \bigoplus_{\tau \in N_p(\pi)} \tau \times H_q(F; R)$$
sending $\tau \otimes a \mapsto (\tau, a)$ would be an isomorphism. An additional hypothesis (<u>orientability</u>) will later be required to achieve this.

We must now identify d^1 in terms of this last description of $E^1(\pi)$.

Let $\pi : (T, T_0) \to (B, B_0)$ be a fibration pair. Write $\pi_a : T \to B$ for its underlying absolute pair.

Let $\lambda : I \to B$ be a path from x to y. If λ is not constant, then $\lambda \in N_1(\pi_a)$ and by the fibration lemma we have homotopy equivalences

$$F_x \xleftarrow{e_0^\lambda} X^\lambda(\pi_a) \xrightarrow{e_1^\lambda} F_y .$$

Thus we have an isomorphism

$$h_\lambda \underset{\text{def}}{=\!=} (e_0^\lambda)_*(e_1^\lambda)_*^{-1} : H_q(F_y; R) \to H_q(F_x; R).$$

If λ is the constant map at x, define $h_\lambda = 1_{H_q(F_q;R)}$.

Let $\tau \in N_p(\pi)$, $p \geq 1$. For $1 \leq i \leq p$, $\tau_0^{(i)}(0) = \tau(0)$, and so $F_\tau = F_{\tau_0^{(i)}}$.

We have $\tau_1^{(i)}(0) = \tau(0, \ldots, \underset{i^{th}}{1}, 0, \ldots, 0)$.

Let $\lambda^{(i)} : I \to I^p$ be given by $\lambda^{(i)}(t) = (0, \ldots, \underset{i^{th}}{1-t}, \ldots, 0)$.

Then $\tau\lambda^{(i)}$ is a path in B from $\tau_1^{(i)}(0)$ to $\tau(0)$.

Lemma: Let $\tau \in N_p(\pi)$, $p \geq 1$. Then for $1 \leq i \leq p$,

$$\begin{tikzcd} X^\tau(\pi) \arrow[r, "e_0^\tau"] \arrow[d, "\rho_0^{(i)}"'] & F_\tau \arrow[d, "1"] \\ X^{\tau_0^{(i)}}(\pi) \arrow[r, "e_0^{\tau_0^{(i)}}"'] & F_{\tau_0^{(i)}} \end{tikzcd}$$

commutes (provided $\tau_0^{(i)} \in N_{p-1}(\pi)$), and

$$\begin{tikzcd} H_q(X^\tau(\pi); R) \arrow[r, "(e_0^\tau)_*"] \arrow[d, "\rho_{1*}^{(i)}"'] & H_q(F_\tau; R) \arrow[d, "h_{\tau\lambda^{(i)}}"] \\ H_q(X^{\tau_1^{(i)}}(\pi); R) \arrow[r, "(e_0^{\tau_1^{(i)}})_*"'] & H_q(F_{\tau_1^{(i)}}; R) \end{tikzcd}$$

commutes (provided $\tau_1^{(i)} \in N_{p-1}(\pi)$.)

Proof: Commutativity of the first diagram is immediate.

Define $\alpha : X^\tau(\pi) \to X^{\tau\lambda^{(i)}}(\pi_a)$ by $\alpha(\sigma)(t) = \sigma(0, \ldots, \underset{i^{th}}{1-t}, \ldots, 0)$. The latter does lie in $X^{\tau\lambda^{(i)}}(\pi_a)$,

for $\pi\sigma(0,\ldots,1-t,\ldots,0) = \tau(0,\ldots,1-t,\ldots,0) = \tau\lambda^{(i)}(t)$.
It is easily checked that the following commutes:

$$\begin{tikzcd}
X^{\tau}(\pi) \arrow[rr, "e_0^{\tau}"] \arrow[dr, "\alpha"] \arrow[dd, "\rho_1^{(i)}"'] & & F_{\tau} \\
& X^{\tau\lambda^{(i)}}(\pi_a) \arrow[ur, "e_1^{\tau\lambda^{(i)}}"'] \arrow[dr, "e_0^{\tau\lambda^{(i)}}"] & \\
X^{\tau_1^{(i)}}(\pi) \arrow[rr, "e_0^{\tau_1^{(i)}}"'] & & F_{\tau_1^{(i)}}
\end{tikzcd}$$

The commutativity of the 2nd diagram now follows.

<u>Theorem</u>: Let π be a fibration pair. Then for $p \geq 1$,

$$d^1 : \bigoplus_{\tau \in N_p(\pi)} \tau \times H_q(F_\tau; R) \longrightarrow \bigoplus_{\tau \in N_{p-1}(\pi)} \tau \times H_q(F_\tau; R)$$

is given as follows: For $\tau \in N_p(\pi)$, $a \in H_q(F_\tau; R)$,

$$d^1(\tau, a) = \sum_{i=1}^{p} (-1)^i \left\{ (\tau_0^{(i)}, a) - (\tau_1^{(i)}, h_{\tau\lambda^{(i)}}(a)) \right\}.$$

<u>Proof</u>: Write $\bar{d}^1 : \bigoplus_{\tau \in N_p(\pi)} H_q(X^\tau(\pi); R) \longrightarrow \bigoplus_{\tau \in N_{p-1}(\pi)} H_q(X^\tau(\pi); R)$

for d^1 using the function space description of E^1. We have the commutative diagram

$$\begin{tikzcd}
\bigoplus_{\tau \in N_p(\pi)} H_q(X^\tau(\pi); R) \arrow[r, "\bar{d}^1"] \arrow[d, "\bigoplus_{\tau \in N_p(\pi)} \overline{e_0^\tau}"', "\cong"] & \bigoplus_{\tau \in N_{p-1}(\pi)} H_q(X^\tau(\pi); R) \arrow[d, "\cong"', "\bigoplus_{\tau \in N_{p-1}(\pi)} \overline{e_0^\tau}"] \\
\bigoplus_{\tau \in N_p(\pi)} \tau \times H_q(F_\tau; R) \arrow[r, "d^1"] & \bigoplus_{\tau \in N_{p-1}(\pi)} \tau \times H_q(F_\tau; R)
\end{tikzcd}$$
.

Given $\sigma \in N_p(\pi)$, $a \in H_q(F_\sigma; R)$, we can write $(\sigma, a) = \overline{e_0^\sigma}(b)$ for some $b \in H_q(X^\sigma(\pi); R)$. Thus $a = (e_0^\sigma)_*(b)$. Then

$$d^1(\sigma, a) = d^1(\overline{e_0^\sigma}(b)) = \Big(\bigoplus_{\tau \in N_{p-1}(\pi)} \overline{e_0^\tau}\Big)(\bar{d}^1(b))$$

$$= \Big(\bigoplus_{\tau \in N_{p-1}(\pi)} \overline{e_0^\tau}\Big) \sum_{i=1}^{p} (-1)^i \left\{ \rho_{0*}^{(i)}(b) - \rho_{1*}^{(i)}(b) \right\}$$

$$= \sum_{i=1}^{p} (-1)^i \left\{ \overline{e}_0^{\sigma_0^{(i)}} \rho_{0*}^{(i)}(b) - \overline{e}_{0\,*}^{\sigma_1^{(i)}} \rho_{1*}^{(i)}(b) \right\}$$

$$= \sum_{i=1}^{p} (-1)^i \left\{ (\sigma_0^{(i)}, e_{0*}^{\sigma_0^{(i)}} \rho_{0*}^{(i)}(b)) - (\sigma_1^{(i)}, e_{0*}^{\sigma_1^{(i)}} \rho_{1*}^{(i)}(b)) \right\}$$

$$= \sum_{i=1}^{p} (-1)^i \left\{ (\sigma_0^{(i)}, e_{0*}^{\sigma}(b)) - (\sigma_1^{(i)}, h_{\sigma\lambda^{(i)}}\, e_{0*}^{\sigma}(b)) \right\} \quad \text{(by lemma)}$$

$$= \sum_{i=1}^{p} (-1)^i \left\{ (\sigma_0^{(i)}, a) - (\sigma_1^{(i)}, h_{\sigma\lambda^{(i)}}(a)) \right\}.$$

34 Special Case 1: $1:(B,B_0)\to(B,B_0)$. Each $e_0^\tau : X^\tau(1)\to F_\tau$ is a homeomorphism of 1-point spaces. For each $x\in B$, let $a_x\in H_0(F_x;R)$ be the element of augmentation 1. Let $\lambda: I\to B$ be a path in B from x to y. Since e^λ_{0*} and e^λ_{1*} are augmentation-preserving, so is $h_\lambda: H_0(F_y;R)\to H_0(F_x;R)$ and so $h_\lambda(a_y)=a_x$. If we write $\bar a_\tau = (\tau, a_{\tau(0)})\in \tau\times H_0(F_\tau;R)$,

$$\bigoplus_{\tau\in N_p(1)} \tau\times H_0(F_\tau;R)$$ is the free R-module with basis $\{\bar a_\tau \mid \tau\in N_p(1)\}$.

If $p>0$ and $1\le i\le p$ we have $h_{\tau\lambda^{(i)}}(a_\tau)=a_{\tau_1^{(i)}}$ and we obtain

$$d^1(\bar a_\tau)=\sum_{i=1}^{p}(-1)^i\left(\bar a_{\tau_0^{(i)}}-\bar a_{\tau_1^{(i)}}\right).$$

Special Case 2: $\pi: X\to P$, P a point. If $\tau_0: I^p\to P$ is the unique 0-cube in P, $F_{\tau_0}=X$ and

$\bar e_0^{\tau_0}: H_q(X^{\tau_0}(\pi);R)\to \tau_0\times H_q(X;R)$ is given by

$\bar e_0^\tau(a)=(\tau_0, h_*(a))$ where $h: X^{\tau_0}(\pi)\to X$ is the homeomorphism described earlier.

Properties of the isomorphisms h_λ

Lemma 1: Let $\pi:(T,T_0)\to(B,B_0)$ be a fibration pair. Let $\lambda,\mu: I\to B$ be paths such that $\lambda(1)=\mu(0)$. Then $h_{\lambda*\mu}=h_\lambda h_\mu$.

Proof: We have continuous maps $\alpha: X^{\lambda*\mu}(\pi)\to X^\lambda(\pi)$,

$\beta: X^{\lambda*\mu}(\pi)\to X^\mu(\pi)$ given by

$$\alpha(\sigma)(t)=\sigma\left(\tfrac{t}{2}\right),\quad \beta(\sigma)(t)=\sigma\left(\tfrac{1+t}{2}\right).$$

Write $x=\lambda(0)$, $y=\lambda(1)=\mu(0)$, $z=\mu(1)$. The following diagram commutes:

$$\begin{tikzcd}
& F_x & & F_z & \\
X^\lambda(\pi) \arrow[ur, "e_0^\lambda"] \arrow[drr, "e_1^\lambda"'] & & X^{\lambda*\mu}(\pi) \arrow[ll, "\alpha"] \arrow[rr, "\beta"'] \arrow[ul, "e_0^{\lambda*\mu}"'] \arrow[ur, "e_1^{\lambda*\mu}"] \arrow[d, "e_{1/2}^{\lambda*\mu}"] & & X^\mu(\pi) \arrow[ul, "e_1^\mu"'] \arrow[dll, "e_0^\mu"] \\
& & F_y & &
\end{tikzcd}$$

Thus $h_\lambda = (e_0^\lambda)_*(e_1^\lambda)_*^{-1} = (e_0^{\lambda*\mu})_*(e_{1/2}^{\lambda*\mu})_*^{-1}$,

$h_\mu = (e_0^\mu)_*(e_1^\mu)_*^{-1} = (e_{1/2}^{\lambda*\mu})_*(e_1^{\lambda*\mu})_*^{-1}$

and so $h_\lambda h_\mu = (e_0^{\lambda*\mu})_*(e_{\frac{1}{2}}^{\lambda*\mu})_*^{-1}(e_{\frac{1}{2}}^{\lambda*\mu})_*(e_1^{\lambda*\mu})_*^{-1}$

$$= (e_0^{\lambda*\mu})_*(e_1^{\lambda*\mu})_*^{-1} = h_{\lambda*\mu}.$$

Lemma 2: Let $f: \pi \to \pi'$ be a morphism of fibration pairs. Let $\lambda: I \to B$ be a path from x to y. Then

$$\begin{tikzcd}
H_q(F_y;R) \arrow[r, "h_\lambda"] \arrow[d, "(f_y)_*"'] & H_q(F_x;R) \arrow[d, "(f_x)_*"] \\
H_q(F_{f(y)};R) \arrow[r, "h_{f_B\lambda}"'] & H_q(F_{f(x)};R)
\end{tikzcd}$$

commutes where $f_x: F_x \to F_{f(x)}$, $f_y: F_y \to F_{f(y)}$ are the restrictions of f_T to the fibres.

Proof: We have the commutative diagram

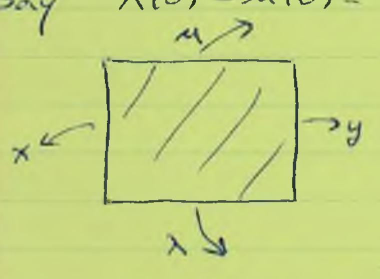
$$\begin{tikzcd}
F_x \arrow[r, "f_x"] & F_{f(x)} \\
X^{\lambda}(\pi) \arrow[u, "e_0^{\lambda}"] \arrow[r, "X(f)"] \arrow[d, "e_1^{\lambda}"'] & X^{f_B\lambda}(\pi') \arrow[u, "e_0^{f_B\lambda}"'] \arrow[d, "e_1^{f_B\lambda}"] \\
F_y \arrow[r, "f_y"'] & F_{f(y)}
\end{tikzcd}$$

and hence $(e_0^{f_B\lambda})_*(e_1^{f_B\lambda})_*^{-1}(f_y)_* = (f_x)_*(e_0^{\lambda})_*(e_1^{\lambda})_*^{-1}$,

i.e. $h_{f_B\lambda}(f_y)_* = (f_x)_* h_\lambda$.

Lemma 3: Let $\pi : (T, T_0) \to (B, B_0)$ be a fibration pair. Suppose $\lambda, \mu : I \to B$ are paths which are homotopic rel endpoints. Then $h_\lambda = h_\mu$.

Proof: Let $H : I^2 \to B$ be a homotopy rel endpoints from λ to μ. Say $\lambda(0) = \mu(0) = x$, $\lambda(1) = \mu(1) = y$.



Note that $\lambda = H_0^{(2)}$, $\mu = H_1^{(2)}$ and we have the maps

$\rho_0^{(2)} : X^H(\pi) \to X^{\lambda}(\pi)$, $\rho_1^{(2)} : X^H(\pi) \to X^{\mu}(\pi)$.

The following diagram commutes:

$$\begin{tikzcd}
& F_x & & F_x & \\
X^{\lambda}(\pi) \arrow[ur, "e_0^{\lambda}"] \arrow[dr, "e_1^{\lambda}"'] & & X^H(\pi) \arrow[ll, "\rho_0^{(2)}"'] \arrow[ul, "e^H_{(0,0)}"'] \arrow[ur, "e^H_{(0,1)}"] \arrow[dl, "e^H_{(1,0)}"] \arrow[dr, "e^H_{(1,1)}"'] \arrow[rr, "\rho_1^{(2)}"] & & X^{\mu}(\pi) \arrow[ul, "e_0^{\mu}"'] \arrow[dl, "e_1^{\mu}"] \\
& F_y & & F_y &
\end{tikzcd}$$

Since $H(0,t) = x$ for all $t \in I$, we obtain a homotopy $G : X^H(\pi) \times I \longrightarrow F_x$ from $e^H_{(0,0)}$ to $e^H_{(0,1)}$ given by

$G(\sigma, t) = e^H_{(0,t)}(\sigma)$. Hence $e^H_{(0,0)*} = e^H_{(0,1)*}$. Similarly $e^H_{(1,0)*} = e^H_{(1,1)*}$. Hence $h_\lambda = (e^\lambda_0)_*(e^\lambda_1)_*^{-1} =$
$(e^H_{(0,0)})_*(e^H_{(1,0)})_*^{-1} = (e^H_{(0,1)})_*(e^H_{(1,1)})_*^{-1} = (e^\mu_0)_*(e^\mu_1)_*^{-1} = h_\mu$.

Orientability

<u>Def</u>: Let $\pi : T \to B$ be a fibration with B path-connected. π is said to be <u>R-orientable</u> if whenever $x, y \in B$ and $\lambda, \mu : I \to B$ are paths from x to y, then
$$h_\lambda = h_\mu : H_q(F_y; R) \to H_q(F_x; R) \quad \text{for all } q.$$

<u>Proposition</u>: Let $\pi : T \to B$ be a fibration with B simply-connected. Then π is R-orientable for every R.

<u>Proof</u>: This is immediate from Lemma 3 of the preceding section since the simple-connectivity of B implies that any two paths from x to y are homotopic rel end points.

<u>Proposition</u>: Let $\pi : T \to B$ be an R-orientable fibration. Then for any continuous map $f : X \to B$ with X path-connected, the induced fibration $\rho : f^*T \to X$ is R-orientable.

<u>Proof</u>: The maps $\tilde{f} : f^*T \to T$, $f : X \to B$ constitute a morphism of fibrations $\rho \to \pi$, and $\tilde{f}_x : F_x \to F_{f(x)}$, $\tilde{f}_y : F_y \to F_{f(y)}$ are homeomorphisms. Hence $(\tilde{f}_x)_*$ and $(\tilde{f}_y)_*$ are isomorphisms for any $x, y \in X$.

Let $\lambda, \mu : I \to X$ be paths from x to y. By Lemma 2 of the previous section, $h_\lambda = (\tilde{f}_x)_*^{-1} h_{f\lambda} (\tilde{f}_y)_*$,

$h_\mu = (\tilde{f}_x)_*^{-1} h_{f\mu} (\tilde{f}_y)_*$. But since π is R-orientable

and $f\lambda$, $f\mu$ are paths in B from $f(x)$ to $f(y)$, we have $h_{f\lambda} = h_{f\mu}$. Hence $h_\lambda = h_\mu$.

<u>Note</u>: If $\pi : T \to B$ is a fibration such that $F_x = \pi^{-1}(x)$ is path-connected for every $x \in B$, and if $\lambda : I \to B$ is a path

from x to y, then from commutativity of

$$\begin{tikzcd} H_0(F_y;R) \arrow[rr, "h_\lambda"] \arrow[dr, "\varepsilon"', "\cong"] & & H_0(F_x;R) \arrow[dl, "\varepsilon", "\cong"'] \\ & R & \end{tikzcd}$$

it follows that h_λ is independent of λ in dimension 0. Thus if B is path-connected and if for some $x \in B$, F_x is path-connected and $H_i(F_x;R) = 0$ for $i > 0$, π will be R-orientable.

In particular if B is path-connected, $1 : B \to B$ is R-orientable for every R.

Notation: If $\pi : T \to B$ is an R-orientable fibration and $x, y \in B$, write $h_{x,y} : H_q(F_y;R) \to H_q(F_x;R)$ for h_λ where λ is any path in B from x to y.

Proposition: Let $\pi : T \to B$ be an R-orientable fibration. Then whenever $x, y, z \in B$, $h_{x,y}\, h_{y,z} = h_{x,z}$.

Proof: This follows immediately from Lemma 1 of the previous section.

Proposition: Let $f : \pi \to \pi'$ be a morphism of R-orientable fibrations. Then for any $x, y \in B$, the diagram

$$\begin{tikzcd} H_q(F_y;R) \arrow[r, "h_{x,y}"] \arrow[d, "(f_y)_*"'] & H_q(F_x;R) \arrow[d, "(f_x)_*"] \\ H_q(F_{f(y)};R) \arrow[r, "h_{f(x),f(y)}"'] & H_q(F_{f(x)};R) \end{tikzcd}$$

commutes.

Proof: This follows immediately from Lemma 2 of the previous section.

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The Leray-Serre Theorem

Theorem: Let R be a commutative ring with unit $1 \neq 0$. Let $\mathcal{F}_R$ denote the category of map pairs $\pi : (T, T_0) \to (B, B_0)$ such that B is path-connected and $F \to T \xrightarrow{\pi} B$ is an R-orientable pointed fibration. Then there is a covariant functor $\pi \longmapsto E(\pi)$ from $\mathcal{F}_R$ to the category of 1^{st} quadrant spectral sequences of R-modules satisfying

1) there are natural R-isomorphisms

$$\Psi : H_p(B, B_0; H_q(F; R)) \xrightarrow{\cong} E^2_{p,q}(\pi) \quad ,$$

2) for each n there is a natural filtration
$0 = J_{-1,n+1}(\pi) \subset J_{0,n}(\pi) \subset \cdots \subset J_{n,0}(\pi) = H_n(T, T_0; R)$
and natural R-isomorphisms

$$\varphi : J_{p,q}(\pi) / J_{p-1,q+1}(\pi) \xrightarrow{\cong} E^\infty_{p,q}(\pi).$$

Proof: Write $F = \pi^{-1}(x_0)$, x_0 the base-point of B. All that remains in the proof is to produce natural isomorphisms

$$\gamma : \bigoplus_{\tau \in N_p(\pi)} \tau \times H_q(F_\tau; R) \longrightarrow Q_p(B, B_0) \underset{\mathbb{Z}}{\otimes} H_q(F; R)$$

which commute with the boundary maps. The boundary map on the left is d', and that on the right is $\partial \otimes 1$.

Define γ by $\gamma(\tau, a) = \tau \otimes h_{x_0, \tau(0)}(a)$ whenever $\tau \in N_p(\pi)$, $a \in H_q(F_\tau; R)$. Since the $h_{x,y}$ are isomorphisms and $Q_p(B, B_0)$ is free abelian on $N_p(\pi)$, it is immediate that γ is an isomorphism. Naturality of γ follows easily from the naturality property of the $h_{x,y}$. It remains only to show
$\gamma d'(\tau, a) = (\partial \otimes 1)\gamma(\tau, a)$ whenever $\tau \in N_p(\pi)$, $a \in H_q(F_\tau; R)$. We have

$$\gamma d'(\tau, a) = \gamma\left(\sum_{i=1}^{p} (-1)^i \left\{ (\tau^{(i)}_0, a) - (\tau^{(i)}_1, h_{\tau\lambda^{(i)}}(a)) \right\} \right)$$

$$= \gamma\left(\sum_{i=1}^{p} (-1)^i \left\{ (\tau^{(i)}_0, a) - (\tau^{(i)}_1, h_{\tau^{(i)}_1(0), \tau(0)}(a)) \right\} \right)$$

$$= \sum_{i=1}^{p} (-1)^i \left\{ \tau^{(i)}_0 \otimes h_{x_0, \tau^{(i)}_0(0)}(a) - \tau^{(i)}_1 \otimes h_{x_0, \tau^{(i)}_0(0)} h_{\tau^{(i)}_1(0), \tau(0)}(a) \right\}$$

$$= \sum_{i=1}^{p} (-1)^i \left\{ \tau_0^{(i)} \otimes h_{x_0,\tau(0)}(a) - \tau_1^{(i)} \otimes h_{x_0,\tau(0)}(a) \right\}$$

(since $\tau_0^{(i)}(0) = \tau(0)$, and $h_{x,y}\, h_{y,z} = h_{x,z}$)

$$= \left(\sum_{i=1}^{p} (-1)^i \left\{ \tau_0^{(i)} - \tau_1^{(i)} \right\} \right) \otimes h_{x_0,\tau(0)}(a)$$

$$= (\partial\tau) \otimes h_{x_0,\tau(0)}(a) = (\partial \otimes 1)(\tau \otimes h_{x_0,\tau(0)}(a))$$

$= (\partial \otimes 1)\, \gamma(\tau, a)$, completing the proof.

<u>Note</u>: Naturality of Ψ means the following: By definition, if $\pi: (T, T_0) \to (B, B_0)$ is an object in $\mathcal{F}_R$, then B is pointed. For any morphism f in $\mathcal{F}_R$, f_B is a pointed map.

Let $f: \pi \to \pi'$ be a morphism in $\mathcal{F}_R$. Write $F = \pi^{-1}(x_0)$, $F' = (\pi')^{-1}(x_0')$ where x_0, x_0' are the base points in B and B', respectively. By restriction of f_T we get a map $f_F: F \to F'$, and thus a chain map

$$Q(f_B) \otimes H(f_F) : Q(B, B_0) \underset{\mathbb{Z}}{\otimes} H_q(F; R) \to Q(B', B_0') \underset{\mathbb{Z}}{\otimes} H_q(F'; R).$$

Naturality of Ψ means that

$$\begin{array}{ccc}
H_p(B, B_0; H_q(F;R)) & \xrightarrow{[Q(f_B) \otimes H(f_F)]_*} & H_p(B', B_0'; H_q(F';R)) \\
\Psi(\pi) \downarrow \cong & & \cong \downarrow \Psi(\pi') \\
E^2_{p,q}(\pi) & \xrightarrow{E^2_{p,q}(f)} & E^2_{p,q}(\pi')
\end{array}$$

commutes.

<u>Note</u>: $Q(f_B) \otimes H(f_F) = (Q(f_B) \otimes 1)(1 \otimes H(f_F)) = (1 \otimes H(f_F))(Q(f_B) \otimes 1)$.

Thus we obtain the commutative diagram

$$\begin{tikzcd}
 & H_p(B', B'_0; H_q(F;R)) \arrow[dr, "H(f_F)_*"] & \\
H_p(B, B_0; H_q(F;R)) \arrow[ur, "(f_B)_*"] \arrow[rr, "{[Q(f_B) \otimes H(f_F)]_*}"] \arrow[dr, "H(f_F)_*"'] & & H_p(B', B'_0; H_q(F';R)) \\
 & H_p(B, B_0; H_q(F';R)) \arrow[ur, "(f_B)_*"'] &
\end{tikzcd}$$

.

The Base - Edge Theorem

Special Case 1: $1 : (B, B_0) \to (B, B_0)$, B path-connected. We investigate the chain isomorphism γ in the proof of the Leray-Serre Theorem in this case. Choose a base-point $x_0 \in B$, and write $a_0 = a_{x_0} \in H_0(F_{x_0}; R) = H_0(\{x_0\}; R)$, using the earlier notation (p. 25). Recall that for any $x \in B$, $h_{x_0, x}(a_x) = a_0$. Thus for any $\tau \in N_p(1)$, $\gamma(\bar{a}_\tau) = \gamma(\tau, a_{\tau(0)})$ $= \tau \otimes h_{x_0, \tau(0)}(a_{\tau(0)}) = \tau \otimes a_0$. We now look at the composition of all the natural chain isomorphisms in the preceding from the original $E^1_{*,0}(1)$ to $Q(B, B_0) \underset{\mathbb{Z}}{\otimes} H_0(\{x_0\}; R)$:

$$\begin{array}{lcc}
\text{Original} & E^1_{p,0}(1) = Q_p(B, B_0; R) = Q_p(B, B_0) \underset{\mathbb{Z}}{\otimes} R & \tau \otimes 1 \\
 & \downarrow & \downarrow \\
Q(B, B_0) \text{ decomposition} & \bigoplus_{\tau \in N_p(\pi)} H_p(C^\tau(1); R) = \bigoplus_{\tau \in N_p(\pi)} C^\tau(1)_p & \langle \tau \rangle \\
 & \downarrow \oplus \alpha_\tau^{-1} & \downarrow \\
\text{Function space decomposition} & \bigoplus_{\tau \in N_p(\pi)} H_0(X^\tau(1); R) & \mu_\tau \\
 & \downarrow & \downarrow \ (\text{since } \varepsilon(\mu_\tau) = 1) \\
\text{Twisted } Q(B, B_0) \otimes H(F) \text{ description} & \bigoplus_{\tau \in N_p(\pi)} \tau \times H_0(F_\tau; R) & \bar{a}_\tau \\
 & \downarrow \gamma & \downarrow \\
 & Q_p(B, B_0) \underset{\mathbb{Z}}{\otimes} H_0(\{x_0\}; R) & \tau \otimes a_0
\end{array}$$

Thus, since all the vertical maps are isomorphisms and $\varepsilon(a_0)=1$, the composition of the inverses of the above is the chain map

$$1\otimes\varepsilon : Q(B,B_0)\underset{\mathbb{Z}}{\otimes} H_0(\{x_0\};R) \longrightarrow Q(B,B_0)\underset{\mathbb{Z}}{\otimes} R = E^1_{*,0}(1).$$

Thus the natural isomorphism

$\Psi : H_p(B,B_0;H_0(\{x_0\};R)) \longrightarrow E^2_{p,0}(1)$ is the composition

$$H_p(B,B_0;H_0(\{x_0\};R)) \xrightarrow{\varepsilon_*} H_{p,0}(E^1(1),d^1) \underset{\cong}{\xrightarrow{\Theta^1}} E^2_{p,0}(1).$$

Recall (p. 4): Θ^1 is the composition

$$H_p(B,B_0;R) \xrightarrow{\rho_B} E^\infty_{p,0}(1) \xrightarrow{i_B} E^2_{p,0}(1) \quad, \quad \text{i.e.}$$

$$\begin{tikzcd}
H_p(B,B_0;H_0(\{x_0\};R)) \arrow[rr, "\Psi", "\cong"'] \arrow[d, "\varepsilon_*", "\cong"'] & & E^2_{p,0}(1) \\
H_p(B,B_0;R) \arrow[rr, "\rho_B"'] & & E^\infty_{p,0}(1) \arrow[u, "i_B"']
\end{tikzcd}$$

commutes.

Recall: For a general $\pi:(T,T_0)\to(B,B_0)$ in $\mathcal{F}_R$, $\rho_B : H_p(T,T_0;R)\to E^\infty_{p,0}(\pi)$ is the composition

$$H_p(T,T_0;R) = J_{p,0}(\pi) \longrightarrow J_{p,0}(\pi)/J_{p-1,1}(\pi) \underset{\cong}{\xrightarrow{\varphi}} E^\infty_{p,0}(\pi).$$

Since φ and J are natural with respect to morphisms in $\mathcal{F}_R$, so is ρ_B.

Recall: $i_B : E^\infty_{p,0} \to E^2_{p,0}$ is defined for any 1^{st} quadrant spectral sequence, and is natural with respect to morphisms of 1^{st} quadrant spectral sequences. Hence $i_B : E^\infty_{p,0}(\pi) \longrightarrow E^2_{p,0}(\pi)$ is natural with respect to morphisms in $\mathcal{F}_R$.

36 <u>Theorem</u>: Let $\pi : (T, T_0) \to (B, B_0)$ be an object in $\mathcal{F}_R$. Let x_0 be the base-point of B, and assume $F = \pi^{-1}(x_0)$ is path-connected. Then

$$\begin{tikzcd}
H_p(T, T_0; R) \arrow[rr, "\pi_*"] \arrow[d, "p_B"'] & & H_p(B, B_0; R) \\
E^\infty_{p,0}(\pi) \arrow[r, "i_B"'] & E^2_{p,0}(\pi) \arrow[r, "\cong", "\psi^{-1}"'] & H_p(B, B_0; H_0(F; R)) \arrow[u, "\cong", "\varepsilon_*"']
\end{tikzcd}$$

commutes.

<u>Proof</u>: Since B and F are path-connected, so is T. (For choose a base point $y_0 \in \pi^{-1}(x_0)$. If $y \in T$, there exists a path $\lambda : I \to B$ from $\pi(y)$ to x_0. By the homotopy lifting property there exists a lift $\tilde{\lambda} : I \to T$ of λ such that $\tilde{\lambda}(0) = y$. Then $\tilde{\lambda}(1) \in F = \pi^{-1}(x_0)$ and since F is path-connected, there exists a path u in F from $\tilde{\lambda}(1)$ to y_0. Then $\tilde{\lambda} * u$ is a path in T from y to y_0). Thus $1 : (T, T_0) \to (T, T_0)$ is an object in $\mathcal{F}_R$.

We have a morphism $f : 1 \to \pi$ in $\mathcal{F}_R$ as follows:

$$\begin{tikzcd}
(T, T_0) \arrow[r, "1 = f_T"] \arrow[d, "1"'] & (T, T_0) \arrow[d, "\pi"] \\
(T, T_0) \arrow[r, "\pi = f_B"'] & (B, B_0)
\end{tikzcd}$$

By naturality of p_B, i_B and ψ and commutativity of

$$\begin{tikzcd}
H_0(\{y_0\}; R) \arrow[rr, "H(f_F)"] \arrow[dr, "\varepsilon"'] & & H_0(F; R) \arrow[dl, "\varepsilon"] \\
& R &
\end{tikzcd}$$

we have the commutative diagram

$$\begin{tikzcd}
H_p(T,T_0;R) \arrow[rr, "1"] \arrow[d, "p_B(1)"'] & & H_p(T,T_0;R) \arrow[d, "p_B(\pi)"] \\
E^\infty_{p,0}(1) \arrow[rr, "E^\infty_{p,0}(f)"] \arrow[d, "i_B(1)"'] & & E^\infty_{p,0}(\pi) \arrow[d, "i_B(\pi)"] \\
E^2_{p,0}(1) \arrow[rr, "E^2_{p,0}(f)"] \arrow[d, "\Psi^{-1}(1)"'] & & E^2_{p,0}(\pi) \arrow[d, "\Psi^{-1}(\pi)"] \\
H_p(T,T_0;H_0(\{y_0\};R)) \arrow[r, "\pi_*"] \arrow[d, "\varepsilon_*"'] & H_p(B,B_0;H_0(\{y_0\};R)) \arrow[r, "H(f_F)_*"] \arrow[d, "\varepsilon_*"'] & H_p(B,B_0;H_0(F;R)) \arrow[dl, "\varepsilon_*"] \\
H_p(T,T_0;R) \arrow[r, "\pi_*"'] & H_p(B,B_0;R) &
\end{tikzcd}$$

By Special Case i (p. 33) the composition of the vertical maps on the left is the identity map on $H_p(T,T_0;R)$. The Theorem now follows.

The Fibre Edge Theorem

Special Case 2: $\pi : X \to P$, P a point, P is simply-connected, so π is an object in $\mathcal{F}_R$. Write $\tau_0 : I^0 \to P$ for the unique map. Since $h_{P,P} = 1$, it follows that

$\gamma : \tau_0 \times H_q(X;R) \longrightarrow Q_0(P) \otimes_{\mathbb{Z}} H_q(X;R)$ is given by

$\gamma(\tau_0, a) = \tau_0 \otimes a$. We now look at the composition of all the natural chain isomorphisms in the preceding from the original $E^1_{0,q}(\pi)$ to $Q_0(P) \otimes_{\mathbb{Z}} H_q(X;R)$ in special case 2:

$$\begin{array}{lrcc}
\text{Original } E^1_{0,q}(\pi) & = & H_q(X;R) & a \\
 & & \downarrow 1 & \downarrow \\
Q(B,B_0) \text{ decomposition} & H_q(C^{\tau_0}(\pi)) = & H_q(X;R) & a \\
h : X^{\tau_0}(\pi) \xrightarrow[\text{homeo.}]{} X & & \downarrow (h^{-1})_* & \downarrow \\
\text{Function space decomposition} & & H_q(X^{\tau_0}(\pi);R) & (h^{-1})_*(a) \\
 & & \downarrow \bar{e}^{\tau_0}_0 & \downarrow \\
\text{Twisted } Q(B,B_0)\otimes H(F) \text{ description} & & \tau_0 \times H_q(X;R) & (\tau_0, h_*(h^{-1})_*(a)) \\
 & & & \parallel \\
 & & & (\tau_0, a) \\
 & & \downarrow \gamma & \downarrow \\
 & & Q_0(P) \otimes_{\mathbb{Z}} H_q(X;R) & \tau_0 \otimes a
\end{array}$$

Thus the composition of the inverses of the above isomorphisms is the composition

$$Q_0(P) \otimes_{\mathbb{Z}} H_q(X;R) \xrightarrow{\varepsilon \otimes 1} \mathbb{Z} \otimes_{\mathbb{Z}} H_q(X;R) \xrightarrow[\text{nat. isom.}]{\cong} H_q(X;R).$$

Call this last composition α. α is a map of chain complexes with trivial boundary.

Recall that $d^r = 0$ for $r \geq 1$ in $E(\pi)$ and so we canonically identify $E^r_{0,q}(\pi) = E^1_{0,q}(\pi) = H_q(X;R)$ for all $r \geq 1$. Each $\Theta^r : H_{0,q}(E^r(\pi), d^r) = H_q(X;R) \to E^{r+1}_{0,q}(\pi) = H_q(X;R)$ is the identity map. Thus the natural isomorphism $\Psi : H_0(P; H_q(X;R)) \to E^2_{0,q}(\pi) = H_q(X;R)$ is α_*.

It is easily checked that α_* (and hence Ψ) is the composition

$$H_0(P; H_q(X;R)) \xrightarrow{\mu^{-1}} H_0(P) \underset{\mathbb{Z}}{\otimes} H_q(X;R) \xrightarrow{\varepsilon \otimes 1} \mathbb{Z} \underset{\mathbb{Z}}{\otimes} H_q(X;R) \xrightarrow[\text{nat. isom.}]{\cong} H_q(X;R)$$

where $\mu : H_0(P) \underset{\mathbb{Z}}{\otimes} H_q(X;R) \to H_0(P; H_q(X;R))$ is the map in the universal coefficient Theorem ($\mu(\{c\} \otimes a) = \{c \otimes a\}$).

<u>Recall</u>: (p. 6) In Special Case 2, $p_F : E^2_{0,q}(\pi) \to E^\infty_{0,q}(\pi)$ and $i_F : E^\infty_{0,q}(\pi) \to H_q(X;R)$ are both the identity map on $H_q(X;R)$.

<u>Theorem</u>: Let $F \xrightarrow{i} T \xrightarrow{\pi} B$ be an R-orientable fibration. Then the diagram

$$\begin{array}{ccccc}
H_q(F;R) & & \xrightarrow{\quad i_* \quad} & & H_q(T;R) \\
\zeta \downarrow \cong & & & & \uparrow i_F \\
H_0(B; H_q(F;R)) & \xrightarrow[\Psi]{\cong} & E^2_{0,q}(\pi) & \xrightarrow[p_F]{} & E^\infty_{0,q}(\pi)
\end{array}$$

commutes, where ζ is the composition

$$H_q(F;R) \xrightarrow[\text{nat. isom.}]{\cong} \mathbb{Z} \underset{\mathbb{Z}}{\otimes} H_q(F;R) \xrightarrow[\cong]{\varepsilon^{-1} \otimes 1} H_0(B) \underset{\mathbb{Z}}{\otimes} H_q(F;R) \xrightarrow[\cong]{\mu} H_0(B; H_q(F;R)).$$

<u>Recall</u>: Implicitly, B is path-connected if π is in $\mathcal{F}_R$.

<u>Proof</u>: Let x_0 be the base-point of B (thus $F = \pi^{-1}(x_0)$). We have the Special Case 2 fibration $\rho : F \to \{x_0\}$ and a morphism $f : \rho \to \pi$ in $\mathcal{F}_R$ as follows:

$$\begin{tikzcd}
F \arrow[r, "i = f_T"] \arrow[d, "\rho"'] & T \arrow[d, "\pi"] \\
\{x_0\} \arrow[r, "i_0 = f_B"'] & B
\end{tikzcd}$$

By naturality of i_F, p_F and Ψ, the diagram

$$\begin{tikzcd}
H_q(F;R) \arrow[r, "i_*"] & H_q(T;R) \\
E^\infty_{0,q}(\rho) \arrow[u, "i_F(\rho)"] \arrow[r, "E^\infty_{0,q}(f)"] & E^\infty_{0,q}(\pi) \arrow[u, "i_F(\pi)"'] \\
E^2_{0,q}(\rho) \arrow[u, "p_F(\rho)"] \arrow[r, "E^2_{0,q}(f)"] & E^2_{0,q}(\pi) \arrow[u, "p_F(\pi)"'] \\
H_0(\{x_0\}; H_q(F;R)) \arrow[u, "\Psi(\rho)", "\cong"'] \arrow[r, "i_{0*}"'] & H_0(B; H_q(F;R)) \arrow[u, "\cong", "\Psi(\pi)"']
\end{tikzcd}$$

commutes.

Since $i_F(\rho)$ and $p_F(\rho)$ are both the identity on $H_q(F;R)$, the result will follow from the above diagram if we show $i_{0*}\Psi(\rho)^{-1} = \zeta$. This is immediate from commutativity of

$$\begin{tikzcd}
H_q(F;R) \arrow[rrr, "1"] \arrow[dr, "\text{nat. isom.}", "\cong"'] \arrow[ddd, "\Psi(\rho)^{-1}"'] & & & H_q(F;R) \arrow[dl, "\text{nat. isom.}"', "\cong"] \arrow[ddd, "\zeta"] \\
\arrow[d, phantom, "\text{Special Case 2}"] & \mathbb{Z} \otimes_{\mathbb{Z}} H_q(F;R) \arrow[r, "1"] \arrow[d, "\varepsilon^{-1}\otimes 1"'] \arrow[dr, phantom, "\text{nat. of } \varepsilon"] & \mathbb{Z} \otimes_{\mathbb{Z}} H_q(F;R) \arrow[d, "\varepsilon^{-1}\otimes 1"] & \arrow[d, phantom, "\text{Def. of } \zeta"] \\
{} & H_0(\{x_0\}) \otimes_{\mathbb{Z}} H_q(F;R) \arrow[r, "i_{0*}\otimes 1"'] \arrow[dl, "\mu"] \arrow[dr, phantom, "\text{nat. of } \mu"] & H_0(B) \otimes_{\mathbb{Z}} H_q(F;R) \arrow[dr, "\mu"'] & {} \\
H_0(\{x_0\}; H_q(F;R)) \arrow[rrr, "i_{0*}"'] & & & H_0(B; H_q(F;R))
\end{tikzcd}$$

Converting a map to a fibration

Prop: $f : X \to Y$ a pointed map. $\exists$ pointed homotopy equiv. $i_f : X \to E_f$ and a fibration $p_f : E_f \to Y$ $\ni$

$$\begin{tikzcd} X \arrow[r, "i_f"] \arrow[dr, "f"'] & E_f \arrow[d, "p_f"] \\ & Y \end{tikzcd}$$

commutes.

Proof: Let $E_f = \{(x,\lambda) \in X \times \mathcal{C}(I,Y) \mid f(x) = \lambda(0)\}$,

$$i_f : X \longrightarrow E_f$$
$$x \longmapsto (x, c_{f(x)}) = \text{const. path at } f(x),$$

$$p_f : E_f \longrightarrow Y$$
$$(x,\lambda) \longmapsto \lambda(1)$$

Then above diagram commutes.

Proof that i_f is a pointed homotopy equivalence:

Let $\pi_1 : E_f \to X$ be given by $\pi_1(x,\lambda) = x$. Then $\pi_1 i_f = 1_X$.
For $\lambda \in \mathcal{C}(I,Y)$, $t \in I$, define $\lambda_t \in \mathcal{C}(I,Y)$ by $\lambda_t(s) = \lambda(ts)$.
Then $\lambda_0 = c_{\lambda(0)}$, $\lambda_1 = \lambda$, and $\lambda_t(0) = \lambda(0)$ $\forall\, t \in I$.
Define $h : E_f \times I \to E_f$ by $h((x,\lambda),t) = (x, \lambda_t)$
h is a homotopy from $i_f \pi_1$ to 1_{E_f}
Note: $h((x, c_{f(x)}), t) = (x, c_{f(x)})$ for all $t \in I$ and $x \in X$.
Thus h is a pointed homotopy for any choice of base point in X.

Proof that $p_f : E_f \to Y$ is a fibration:
Suppose given

$$\begin{tikzcd} Z \times 0 \arrow[r, "g"] \arrow[d, hook] & E_f \arrow[d, "p_f"] \\ Z \times I \arrow[r, "G"'] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

Write $g(z,0) = (k(z), \ell(z))$ where $k : Z \to X$, $\ell : Z \to \mathcal{C}(I,Y)$.
Thus $\ell(z)(0) = f k(z)$ and $\ell(z)(1) = p_f g(z,0) = G(z,0)$.

Define H by
$$H(z,t)(s) = \begin{cases} \ell(z)\left(\frac{2s}{2-t}\right) & \text{if } 0 \le s \le 1-\frac{t}{2} \\ G(z, 2s+t-2) & \text{if } 1-\frac{t}{2} \le s \le 1. \end{cases}$$

Then H is continuous and both triangles commute.

__Thm:__ Let X and Y be 0-connected and $f: X \to Y$ a pointed map such that $f_*: \pi_n(X) \to \pi_n(Y)$ is an isom. for all n. Then $f_*: H_n(X) \to H_n(Y)$ is an isom. for all n.

__Note:__ Have previously seen this if X, Y are both 1-connected (Whitehead Theorem).

__Proof:__ Write F = fibre of $E_f \xrightarrow{p_f} Y$. From commutativity of

$$\begin{array}{ccc} X & \xrightarrow{i_f} & E_f \\ & f\searrow \quad \swarrow p_f & \\ & Y & \end{array}$$

and fact that i_f is a homotopy equivalence, follows that suffices to show $(p_f)_*: H_n(E_f) \to H_n(Y)$ is an isom. for all n.

Since $i_{f*}: \pi_n(X) \to \pi_n(E_f)$ is an isom. $\forall n$ (since i_f is a pointed homotopy equivalence), it follows that $p_{f*}: \pi_n(E_f) \to \pi_n(Y)$ is an isom. $\forall n$. Hence, from the homotopy sequence of the fibration $F \to E_f \xrightarrow[p_f]{} Y$ it follows that $\pi_n(F) = 0$ $\forall n$. Hence, by the Hurewicz theorem, $H_n(F) = 0$ for all $n > 0$. F is path-connected (since $\pi_0(F) = 0$), so by earlier remark, the fibration p_f is $\mathbb{Z}$-orientable. $E^2_{p,q}(p_f) = 0$ for $q \neq 0$,

from which it follows easily that p_B and i_B are isomorphisms. Hence by the base-edge theorem, each $(p_f)_*: H_n(E_f) \to H_n(Y)$ is an isomorphism.

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Cohomology Spectral Sequences

Def: A cohomology spectral sequence of R-modules E consists of a sequence of bigraded diff. $(r,-r+1)$-modules (E_r, d_r) (i.e. $d_r : E_r^{p,q} \longrightarrow E_r^{p+r, q-r+1}$) together with isomorphisms $E_{r+1}^{p,q} \underset{\cong}{\xleftarrow{\theta_r}} H^{p,q}(E_r, d_r) = \dfrac{\ker\left[E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}\right]}{\operatorname{im}\left[E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q}\right]}$

Def: A cohomology spectral sequence of R-modules E is multiplicative if there are given R-homomorphisms

$$\mu_r : E_r^{p,q} \underset{R}{\otimes} E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'}$$
$$a \otimes b \longmapsto a \cdot b$$

satisfying

1) $d_r(a\cdot b) = (d_r a)\cdot b + (-1)^{p+q} a \cdot (d_r b)$ where $a \in E_r^{p,q}$.

(Note that as a consequence of 1), if $a \in E_r^{p,q}$, $b \in E_r^{p',q'}$ are in the kernel of d_r, so is $a\cdot b$, and $\{a\cdot b\} \in H^{p+p',q+q'}(E_r, d_r)$ depends only on $\{a\} \in H^{p,q}(E_r, d_r)$ and $\{b\} \in H^{p',q'}(E_r, d_r)$. We thus obtain an R-homomorphism

$$H(\mu_r) : H^{p,q}(E_r, d_r) \underset{R}{\otimes} H^{p',q'}(E_r, d_r) \to H^{p+p',q+q'}(E_r, d_r)$$
$$\{a\} \otimes \{b\} \longmapsto \{a\cdot b\}.$$

2) For each r and all p, q, p', q', the diagram

$$\begin{tikzcd}
H^{p,q}(E_r,d_r) \underset{R}{\otimes} H^{p',q'}(E_r,d_r) \arrow[r, "H(\mu_r)"] \arrow[d, "\theta_r \otimes \theta_r"', "\cong"] & H^{p+p',q+q'}(E_r,d_r) \arrow[d, "\theta_r", "\cong"'] \\
E_{r+1}^{p,q} \underset{R}{\otimes} E_{r+1}^{p',q'} \arrow[r, "\mu_{r+1}"'] & E_{r+1}^{p+p',q+q'}
\end{tikzcd}$$

commutes.

Note: If E is a 1st quadrant cohomology spectral sequence (i.e. $E_r^{p,q} = 0$ for $p < 0$ or $q < 0$), then E_∞ is defined just as in the homology case. Moreover E_∞ has prod. maps $\mu_\infty : E_\infty^{p,q} \underset{R}{\otimes} E_\infty^{p',q'} \longrightarrow E_\infty^{p+p',q+q'}$, namely μ_r for r suff. large w/r. to p, q, p', q'.

Form cat. of 1st quad. multiplicative cohomology spectral sequences of R-modules.

Note: If μ_r is associative for some r, so is $H(\mu_r)$ and consequently, so is $\mu_{r'}$ for all $r' > r$.

Similarly for graded-commutativity ($a \cdot b = (-1)^{|a||b|} b \cdot a$ where $|a| = p+q$ if $a \in E_r^{p,q}$).

Cup Products with Coefficients

Let R be a PID. Suppose A, B, C are R-modules and we are given an R-homomorphism
$$\alpha : A \otimes_R B \to C.$$

Then for any top. space there is given an R-hom.

$$\mu_\alpha : H^p(X;A) \otimes_R H^q(X;B) \longrightarrow H^{p+q}(X;C)$$
$$a \otimes b \longmapsto a \cup b$$

satisfying

1) μ_α is natural with respect to continuous maps, i.e. if $f : X \to Y$ is continuous,

$$\begin{tikzcd} H^p(X;A) \otimes_R H^q(X;B) \arrow[r, "\mu_\alpha"] & H^{p+q}(X;C) \\ H^p(Y;A) \otimes_R H^q(Y,B) \arrow[u, "f^* \otimes f^*"] \arrow[r, "\mu_\alpha"'] & H^{p+q}(Y;C) \arrow[u, "f^*"'] \end{tikzcd}$$ commutes.

2) μ_α is natural with respect to α, i.e. if we are given another R-homomorphism $\alpha' : A' \otimes_R B' \to C'$

and R-homomorphisms $f : A \to A'$, $g : B \to B'$, $h : C \to C'$ such that

$$\begin{tikzcd} A \otimes B \arrow[r, "\alpha"] \arrow[d, "f \otimes g"'] & C \arrow[d, "h"] \\ A' \otimes B' \arrow[r, "\alpha'"] & C' \end{tikzcd}$$ commutes,

then for each each X,

$$\begin{array}{ccc} H^p(X;A)\otimes_R H^q(X;B) & \xrightarrow{\mu_\alpha} & H^{p+q}(X;C) \\ \downarrow{\scriptstyle f_*\otimes g_*} & & \downarrow{\scriptstyle h_*} \\ H^p(X;A')\otimes_R H^q(X;B') & \xrightarrow[\mu_{\alpha'}]{} & H^{p+q}(X;C') \end{array}$$ commutes.

3) If $\alpha : R\otimes_R R \longrightarrow R$, $a\otimes b \longmapsto ab$ is the canonical isomorphism,

then $\mu_\alpha : H^p(X;R)\otimes_R H^q(X;R) \to H^{p+q}(X;R)$ is the cup product map previously studied.

μ_α is the composition

$$H^p(X;A)\otimes_R H^q(X;B) \xrightarrow{\mu} H^{p+q}\left(Q^*(X;A)\otimes_R Q^*(X;B)\right)$$

$$\{u\}\otimes\{v\} \longmapsto \{u\otimes v\}$$

$$\downarrow \eta_\alpha^* \qquad \eta_\alpha(f\otimes g)(x\otimes y) = (-1)^{|x||y|}\alpha\left(f(x)\otimes g(y)\right)$$

$$H^{p+q}\left(\mathrm{Hom}_{\mathbb{Z}}\left(Q(X)\otimes_{\mathbb{Z}} Q(X), C\right)\right)$$

$$\downarrow (\text{Eilenberg - Zilber})^*$$

$$H^{p+q}(X\times X;C) \xrightarrow{d^*} H^{p+q}(X;C)$$

<u>Leray - Serre Theorem</u> (Mult. Coh. version): R a PID, $F \xrightarrow{i} T \xrightarrow{\pi} B$ an R-orient. fibration. Then there is a natural 1st quad. mult. coh. spectral sequence of R-modules $E(\pi)$ satisfying

1) there are natural R-isomorphisms
$$\Psi : E_2^{p,q}(\pi) \to H^p(B;H^q(F;R))$$

such that the diagram

$$\begin{tikzcd}
E_2^{p,q}(\pi) \otimes_R E_2^{p',q'}(\pi) \arrow[r, "\mu_2"] \arrow[d, "\Psi\otimes\Psi"'] & E_2^{p+p',q+q'}(\pi) \arrow[d, "\Psi"] \\
H^p(B; H^q(F;R)) \otimes_R H^{p'}(B; H^{q'}(F;R)) \arrow[r, "\mu_{\alpha_F}"'] & H^{p+p'}(B; H^{q+q'}(F;R))
\end{tikzcd}$$

where $\alpha_F : H^q(F;R) \otimes_R H^{q'}(F;R) \to H^{q+q'}(F;R)$ is the cup product pairing.

2) There is a natural filtration

$$H^n(T;R) = J^{0,n}(\pi) \supset J^{1,n-1}(\pi) \supset \cdots \supset J^{n,0}(\pi) \supset J^{n+1,-1}(\pi) = 0$$

and natural isomorphisms $\varphi : E_\infty^{p,q}(\pi) \to J^{p,q}(\pi)/J^{p+1,q-1}(\pi)$.

Moreover, $u \in J^{p,q}(\pi)$, $v \in J^{p',q'}(\pi) \Rightarrow u \cup v \in J^{p+p',q+q'}(\pi)$ and the diagram

$$(u + J^{p+1,q-1}) \otimes (v + J^{p'+1,q'-1}) \longmapsto (u \cup v + J^{p+p'+1,q+q'-1})$$

$$\begin{tikzcd}
J^{p,q}(\pi)/J^{p+1,q-1}(\pi) \otimes_R J^{p',q'}(\pi)/J^{p'+1,q'-1}(\pi) \arrow[r] & J^{p+p',q+q'}(\pi)/J^{p+p'+1,q+q'-1}(\pi) \\
E_\infty^{p,q}(\pi) \otimes_R E_\infty^{p',q'}(\pi) \arrow[r, "\mu_\infty"'] \arrow[u, "\varphi\otimes\varphi"] & E_\infty^{p+p',q+q'}(\pi) \arrow[u, "\varphi"']
\end{tikzcd}$$

commutes.

38 Let $F \longrightarrow T \xrightarrow{\pi} B$ be an R-orientable fibration and suppose F is 0-connected. We have the commutative diagram

$$\begin{tikzcd}
H^0(F;R) \underset{R}{\otimes} H^0(F;R) \arrow[r, "\smile"] & H^0(F;R) \\
R \underset{R}{\otimes} R \arrow[u, "\eta\otimes\eta", "\cong"'] \arrow[r, "\text{mult.}"'] & R \arrow[u, "\cong", "\eta"']
\end{tikzcd}$$

where η is the coaugmentation. Thus by naturality of μ_α with respect to α, the diagram

$$\begin{tikzcd}
H^p(B;H^0(F;R)) \underset{R}{\otimes} H^{p'}(B;H^0(F;R)) \arrow[r, "\mu_{\alpha_F}"] & H^{p+p'}(B;H^0(F;R)) \\
H^p(B;R) \underset{R}{\otimes} H^{p'}(B;R) \arrow[u, "\eta_*\otimes\eta_*", "\cong"'] \arrow[r, "\smile"'] & H^{p+p'}(B;R) \arrow[u, "\cong", "\eta_*"']
\end{tikzcd}$$

commutes. Thus $E_2^{*,0}(\pi) \underset{\text{nat.}}{\cong} H^*(B;R)$ as a graded ring.

For any R-module A we have a coaugmentation

$\eta_A : A \to H^0(X;A)$, which is an isomorphism if X is 0-connected. η_A is induced by

$$A \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},A) \xrightarrow{\mathrm{Hom}(\varepsilon,1_A)} \mathrm{Hom}_{\mathbb{Z}}(Q_0(X);A) = Q^0(X;A).$$

If $\alpha : A \underset{R}{\otimes} B \to C$ is an R-homomorphism,

$$\begin{tikzcd}
H^0(X;A) \underset{R}{\otimes} H^0(X;B) \arrow[r, "\mu_\alpha"] & H^0(X;C) \\
A \underset{R}{\otimes} B \arrow[u, "\eta_A\otimes\eta_B"] \arrow[r, "\alpha"'] & C \arrow[u, "\eta_C"']
\end{tikzcd}$$

commutes.

Thus for $F \longrightarrow T \xrightarrow{\pi} B$ as above,

$$\begin{tikzcd}
H^0(B; H^q(F;R)) \otimes_R H^0(B; H^{q'}(F;R)) \arrow[r, "\mathcal{U}_{\alpha_F}"] & H^0(B; H^{q+q'}(F;R)) \\
H^q(F;R) \otimes_R H^{q'}(F;R) \arrow[u, "\eta\otimes\eta"', "\cong"] \arrow[r, "\cup"'] & H^{q+q'}(F;R) \arrow[u, "\cong"', "\eta"]
\end{tikzcd}$$

commutes, and so $E_2^{0,*}(\pi) \underset{\text{nat.}}{\cong} H^*(F;R)$ as a graded ring.

Suppose $\alpha_i : A \otimes_R B_i \to C_i$ are R-homomorphisms, $1 \le i \le n$. Write

$$\alpha = \bigoplus_i \alpha_i : A \otimes_R \bigoplus_i B_i \longrightarrow \bigoplus_i C_i .$$

For any X,

$$\begin{tikzcd}
H^p(X;A) \otimes_R H^{p'}(X; \bigoplus_i B_i) \arrow[r, "\mathcal{U}_\alpha"] \arrow[d, "\cong"'] & H^{p+p'}(X; \bigoplus_i C_i) \arrow[d, "\cong"] \\
\bigoplus_i H^p(X;A) \otimes_R H^{p'}(X;B_i) \arrow[r, "\bigoplus_i \mathcal{U}_{\alpha_i}"'] & \bigoplus_i H^{p+p'}(X;C_i)
\end{tikzcd}$$

commutes. In particular, if for some p, p' each $\mathcal{U}_{\alpha_i}$ is an isomorphism, then $\mathcal{U}_\alpha$ is an isom.

In particular, suppose X is 0-connected. Then

$H^p(X;R) \otimes_R H^0(X;R) \xrightarrow{\ \cup\ } H^p(X;R)$ is an isom.

Thus if A is a finitely-generated free R-module and $\alpha : R \otimes_R A \to A$ the canonical isomorphism,

$H^p(X;R) \otimes_R H^0(X;A) \xrightarrow{\ \mathcal{U}_\alpha\ } H^p(X;A)$ is an isom.

Thus if $F \to T \xrightarrow{\pi} B$ is an R-orientable fibration with F 0-connected, and if each $H^q(F;R)$ is free and finitely generated as an R-module, then commutativity of

$$\begin{tikzcd} H^0(F;R) \otimes_R H^q(F;R) \arrow[r, "\cup"] & H^q(F;R) \\ R \otimes_R H^q(F;R) \arrow[u, "\eta\otimes 1", "\cong"'] \arrow[r, "\text{can. iso.}"'] & H^q(F;R) \arrow[u, "1"'] \end{tikzcd}$$

implies

$$\begin{tikzcd} H^p(B; H^0(F;R)) \otimes_R H^0(B; H^q(F;R)) \arrow[r, "\mu_{\alpha_F}"] & H^p(B; H^q(F;R)) \\ H^p(B;R) \otimes_R H^0(B; H^q(F;R)) \arrow[u, "\eta_*\otimes 1", "\cong"'] \arrow[r, "\cong"', "\mu_{\text{can. iso.}}"'] & H^p(B; H^q(F;R)) \arrow[u, "1"'] \end{tikzcd}$$

commutes, and so top μ_{α_F} is an isom. Thus, if each $H^q(F;R)$ is a free and finitely generated R-module, and F is 0-connected,

$$E_2^{p,0}(\pi) \otimes_R E_2^{0,q}(\pi) \xrightarrow{\mu_2} E_2^{p,q}(\pi)$$

is an isomorphism.

<u>Example:</u> $S^\infty = \bigcup_n S^n$ with the weak topology,

$\mathbb{C}P^\infty = \bigcup_n \mathbb{C}P^n$ with the weak topology. We have fibre bundles

$$\begin{array}{cc} S^{2n+1} & (z_1, \dots, z_{n+1}) \\ \downarrow & \mapsto \\ \mathbb{C}P^n & [z_1, \dots, z_{n+1}] \end{array}$$

with fibre S^1 such that

$$\begin{tikzcd} S^{2n+1} \arrow[r, hook] \arrow[d] & S^{2n+3} \arrow[d] \\ \mathbb{C}P^n \arrow[r, hook] & \mathbb{C}P^{n+1} \end{tikzcd}$$

commutes.

This yields a fibre bundle $S^\infty \to \mathbb{C}P^\infty$ with fibre S^1.

<u>Claim:</u> $\pi_n(S^\infty) = 0$ for all n. For if $f: S^n \to S^\infty$ is a pointed continuous map, then since S^n is compact and S^∞ has the weak topology, $\exists\, N > n$ and a factorization

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[dr, "g"'] & S^\infty \\ & S^N \arrow[u, "i"'] \end{tikzcd}$$ and hence $[f] = i_*[g]$. But

$[g] = 0$ since $\pi_n(S^N) = 0$ for $n < N$.

Thus, by the Hurewicz theorem, $H_n(S^\infty) = 0$ for $n \neq 0$.

It follows from the homotopy sequence of the above fibration that

$$\pi_n(\mathbb{C}P^\infty) \cong \begin{cases} 0 & \text{if } n \neq 2 \\ \mathbb{Z} & \text{if } n = 2. \end{cases}$$

In particular $\mathbb{C}P^\infty$ is 1-connected, so above fibration is $\mathbb{Z}$-orientable.

Let E denote the cohomology spectral sequence of this fibration with $\mathbb{Z}$ coefficients. Since $H^q(S^1) = 0$ for $q \neq 0, 1$ we have $E_2^{p,q} = 0$ for $q \neq 0, 1$. Thus $E_\infty^{p,q} = E_3^{p,q}$. Since $H^n(S^\infty) = 0$ for $n \neq 0$, $E_\infty^{p,q} = 0$ for $(p,q) \neq (0,0)$. Thus $E_3^{p,q} = 0$ for $(p,q) \neq (0,0)$. From exactness of

$$0 \to \underset{\substack{\parallel \\ 0}}{E_3^{p,1}} \longrightarrow E_2^{p,1} \xrightarrow{d_2} E_2^{p+2,0} \longrightarrow \underset{\substack{\parallel \\ 0 \text{ for } p \neq -2}}{E_3^{p+2,0}} \to 0$$

it follows that $d_2 : E_2^{p,1} \xrightarrow{\cong} E_2^{p+2,0}$ for $p \neq -2$.

Write x = a gen. of $E_2^{0,1} = H^0(\mathbb{C}P^\infty; H^1(S^1)) \cong \mathbb{Z}$.

Since

$$0 = E_2^{-1,1} \underset{\cong}{\xrightarrow{d_2}} E_2^{1,0} \cong H^1(\mathbb{C}P^\infty; H^0(S^1))$$

we have $H^1(\mathbb{C}P^\infty) = 0$. Since $E_2^{1,0} \otimes E_2^{0,1} \xrightarrow{\cong} E_2^{1,1}$ is an isom., $E_2^{1,1} = 0$. Thus since $E_2^{1,1} \xrightarrow{d_2} E_2^{3,0} \cong H^3(\mathbb{C}P^\infty)$, it follows that $H^3(\mathbb{C}P^\infty) = 0$.

By induction, $H^{odd}(\mathbb{C}P^\infty) = 0$.

$$E_2^{0,1} \underset{\cong}{\xrightarrow{d_2}} E_2^{2,0} = H^2(\mathbb{C}P^\infty)$$ Write $y = d_2 x$.

Thus $H^2(\mathbb{C}P^\infty) \cong \mathbb{Z}$, gen. y. We have $d_2 y = 0$.

Since $E_2^{2,0} \otimes E_2^{0,1} \xrightarrow{\mu_2} E_2^{2,1}$ is an isom.,

$E_2^{2,1} \cong \mathbb{Z}$, gen. $y \cdot x$. We have $d_2 : E_2^{2,1} \xrightarrow{\cong} E_2^{4,0}$,
gen. $d_2(y \cdot x)$. But

$$d_2(y \cdot x) = (d_2 y) \cdot x + y \cdot d_2 x = 0 \cdot x + y \cdot y = y^2.$$

Thus $H^4(\mathbb{C}P^\infty) \cong \mathbb{Z}$, gen. y^2.

Assume inductively $H^{2n}(\mathbb{C}P^\infty) \cong \mathbb{Z}$, gen. y^n.
Since $E_2^{2n,0} \otimes E_2^{0,1} \xrightarrow{\mu_2} E_2^{2n,1}$ is an isom.,

$E_2^{2n,1} \cong \mathbb{Z}$, gen. $y^n \cdot x$. We have $d_2(y^n) = 0$.

Since $d_2 : E_2^{2n,1} \longrightarrow E_2^{2n+2,0}$ is an isom.,

$E_2^{2n+2,0} \cong \mathbb{Z}$, gen. $d_2(y^n \cdot x)$. But $d_2(y^n \cdot x) = d_2(y^n) \cdot x + y^n \cdot d_2 x$

$$= 0 \cdot x + y^n \cdot y = y^{n+1}.$$

Thus, we obtain

$$H^*(\mathbb{C}P^\infty) = \mathbb{Z}[y], \quad y \in H^2(\mathbb{C}P^\infty).$$

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Serre's $\mathcal{C}$ - theory

Def: A class $\mathcal{C}$ of abelian groups is called a Serre class if all 0 groups are in $\mathcal{C}$, and whenever $0 \to A \to B \to C \to 0$ is exact, $B \in \mathcal{C} \iff A \in \mathcal{C}$ and $C \in \mathcal{C}$.

Examples:
1) all 0 groups
2) all abelian groups
3) all finitely-generated abel groups
4) all finite abel. groups
5) all finite abel p-groups, p a fixed prime
6) Let S be a set of primes. All finite abel. groups of order a prod. of primes in S. (this generalizes 5).

Prop. $\mathcal{C}$ a Serre class. Then
i) If $A \subset B$, then $B \in \mathcal{C} \iff A \in \mathcal{C}$ and $B/A \in \mathcal{C}$.
ii) If $A \cong B$, then $B \in \mathcal{C} \iff A \in \mathcal{C}$
iii) If $A \to B \to C$ exact and $A \in \mathcal{C}$, $C \in \mathcal{C}$, then $B \in \mathcal{C}$.
iv) $A \oplus B \in \mathcal{C} \iff A \in \mathcal{C}$ and $B \in \mathcal{C}$.

Proofs are all trivial.

Def: Let $\mathcal{C}$ be a Serre class, $f: A \to B$ a homomorphism of abelian groups. (A, B not nec. in $\mathcal{C}$). f is called a $\mathcal{C}$ - monomorphism if $\ker f \in \mathcal{C}$, a $\mathcal{C}$ - epimorphism if $\operatorname{coker} f = B/\operatorname{im} f \in \mathcal{C}$, a $\mathcal{C}$ - isomorphism if it is both a $\mathcal{C}$ - monomorphism and a $\mathcal{C}$ - epimorphism.

Prop: $\mathcal{C}$ a Serre class, $f: A \to B$ a $\mathcal{C}$ - isomorphism. Then $A \in \mathcal{C} \iff B \in \mathcal{C}$.

Proof is trivial.

Prop: $\mathcal{C}$ a Serre class. Suppose $f: A \to B$, $g: B \to C$ are $\mathcal{C}$ - isomorphisms. Then $gf: A \to C$ is a $\mathcal{C}$ - isomorphism.

Proof left as exercise.

Main goal is to obtain the following:

Thm (Mod $\mathcal{C}$ Hurewicz thm): Let $\mathcal{C}$ be a strongly homologically complete Serre class. Let X be a 1-connected pointed space. Suppose for some $n \geq 1$, $\pi_i(X,*) \in \mathcal{C}$ for $i \leq n$. Then $H_i(X,*) \in \mathcal{C}$ for $i \leq n$ and $h_{n+1} : \pi_{n+1}(X,*) \to H_{n+1}(X,*)$ is a $\mathcal{C}$-isomorphism.

(Will give def. of strongly homologically complete later. Turns out: all 6 examples above are strongly homologically complete).

Cor: If $\mathcal{C}$ is strongly homologically complete and X is a 1-connected pointed space, then $\pi_i(X,*) \in \mathcal{C}$ for all i $\iff$ $H_i(X,*) \in \mathcal{C}$ for all i.

In particular, if X is 1-connected, $\pi_i(X,*)$ is finitely-generated for all i $\iff$ $H_i(X,*)$ is finitely-generated for all i.

Lemma: E a homology spectral sequence of abel. groups, $\mathcal{C}$ a Serre class. Suppose for some r, p, q, $E^r_{p,q} \in \mathcal{C}$. Then $E^s_{p,q} \in \mathcal{C}$ for all $s \geq r$.

Proof: $E^{r+1}_{p,q} \cong$ quotient of a subgroup of $E^r_{p,q}$, and hence is in $\mathcal{C}$. Result now follows by induction.

Def: A Serre class $\mathcal{C}$ is homologically complete if $A \in \mathcal{C}$, $B \in \mathcal{C} \Rightarrow A \underset{\mathbb{Z}}{\otimes} B \in \mathcal{C}$ and $\mathrm{Tor}^{\mathbb{Z}}(A,B) \in \mathcal{C}$.

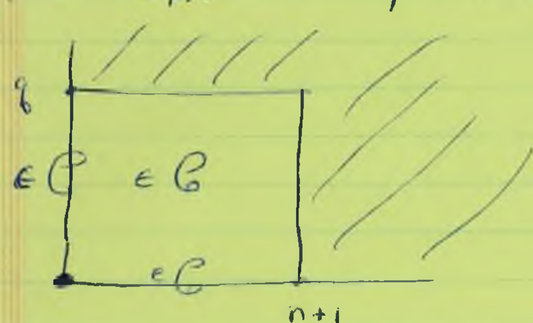
Examples: It is immediate that examples 1-6 above are homologically complete.

Lemma: Let $\mathcal{C}$ be a homologically complete Serre class. Suppose X is a 1-connected pointed space such that for some $n \geq 2$, $H_i(X) \in \mathcal{C}$ for $1 \leq i \leq n$. Then $H_i(\Omega X) \in \mathcal{C}$ for $1 \leq i \leq n-1$.

Proof: The fibration $\Omega X \to PX \to X$ is $\mathbb{Z}$-orientable since X is 1-connected. Let E be the Leray-Serre spectral sequence of this fibration with $\mathbb{Z}$ coefficients. Since PX is contractible, $E^\infty_{p,q} = 0$ for $(p,q) \neq (0,0)$. In particular $E^\infty_{0,1} = E^\infty_{2,0} = 0$, and so $d^2 : E^2_{2,0} \to E^2_{0,1}$ must be an isomorphism. Now $E^2_{2,0} = H_2(X; H_0(\Omega X)) \cong H_2(X) \in \mathcal{C}$, so $E^2_{0,1} \in \mathcal{C}$. But $E^2_{0,1} = H_0(X; H_1(\Omega X)) \cong H_1(\Omega X)$, so $H_1(\Omega X) \in \mathcal{C}$.

Suppose $1 < q \le n-1$ and that, inductively, $H_i(\Omega X) \in \mathcal{C}$ for $1 \le i \le q-1$. Remains only to show: $H_q(\Omega X) \in \mathcal{C}$. Since $E^2_{0,q} = H_0(X; H_q(\Omega X)) \cong H_q(\Omega X)$, remains only to show $E^2_{0,q} \in \mathcal{C}$.

By the U.C.T. and homological completeness of $\mathcal{C}$, $E^2_{p,i} = H_p(X; H_i(\Omega X)) \in \mathcal{C}$ for $0 \le p \le n$ and $1 \le i < q$. Thus $E^r_{p,i} \in \mathcal{C}$ for $0 \le p \le n$, $1 \le i < q$, and $2 \le r \le \infty$. Also for $1 \le p \le n$, $E^2_{p,0} = H_p(X; H_0(\Omega X)) \cong H_p(X) \in \mathcal{C}$, so $E^r_{p,0} \in \mathcal{C}$ for $1 \le p \le n$ and $2 \le r \le \infty$.



The proof will be completed by proving, by decreasing induction on r, that $E^r_{0,q} \in \mathcal{C}$ for all r. We have $0 = E^\infty_{0,q} = E^{q+2}_{0,q}$, so certainly $E^\infty_{0,q} \in \mathcal{C}$. Suppose that for some r in the range $2 < r \le q+2$, $E^r_{0,q} \in \mathcal{C}$. We have the exact sequence

$$E^{r-1}_{r-1,q-r+2} \xrightarrow{d^{r-1}} E^{r-1}_{0,q} \xrightarrow{\text{onto}} E^r_{0,q} \quad \text{exact.}$$

Note that $1 < r-1 \le q+1 \le n$, $q-r+2 < q$. Hence by 1st inductive assumption, $E^{r-1}_{r-1,q-r+2} \in \mathcal{C}$. By 2nd inductive assumption, $E^r_{0,q} \in \mathcal{C}$. Hence, $E^{r-1}_{0,q} \in \mathcal{C}$, completing the inductive proof.

Killing Homotopy groups.

Thm: Let X be a 0-connected pointed space which admits a universal covering space (e.g. if X is 1-connected). Then for each $n \ge 1$, there exists a space $X(0,\dots,n)$ and an inclusion $i : X \to X(0,\dots,n)$ such that $i_* : \pi_q(X) \to \pi_q(X(0,\dots,n))$ is an isomorphism for $q \le n$, and $\pi_q(X(0,\dots,n)) = 0$ for $q > n$. Moreover, $X(0,\dots,n)$ can be obtained by first attaching a family of $n+2$-cells to X, then a family of $n+3$-cells to this new space, then a family of $n+4$-cells to this new space, etc.

Idea of construction: For each $\alpha \in \pi_{n+1}(X)$, attach an $n+2$-cell to X with attaching map representing α. Call this new space K_{n+2}. Can prove: $X \hookrightarrow K_{n+2}$ induces $\cong$ in π_i for $i \le n$, and $\pi_{n+1}(K_{n+2}) = 0$. For each $\beta \in \pi_{n+2}(K_{n+2})$, attach an $n+3$-cell to K_{n+2} with attaching map representing β. Call this new space K_{n+3}. Can prove: $K_{n+2} \hookrightarrow K_{n+3}$ induces $\cong$ in π_i for $i \le n+1$, and $\pi_{n+2}(K_{n+3}) = 0$, etc. Take $X(0,\dots,n) = \bigcup_m K_m$ with the weak topology.

40 Def: A pointed space X is said to be of <u>type (π, n)</u>, n a given integer ≥ 1, π a group (abelian if $n > 1$) if

$$\pi_i(X) \cong \begin{cases} \pi & \text{if } i = n \\ 0 & \text{if } i \ne n. \end{cases}$$

Examples: 1) S^1 is of type $(\mathbb{Z}, 1)$.
2) $\mathbb{CP}^\infty$ is of type $(\mathbb{Z}, 2)$
3) $\mathbb{RP}^\infty$ is of type $(\mathbb{Z}/2, 1)$ (have fibration $S^0 \to S^\infty \to \mathbb{RP}^\infty$)
4) If X is of type (π, n), $n > 1$, then ΩX is of type $(\pi, n-1)$.

Spaces of type (π, n) for some π, n are called <u>Eilenberg-MacLane spaces</u>.

Thm: Let $n \ge 1$, π a group (abel. if $n \ge 2$). Then $\exists$ a CW complex K with $|K|$ of type (π, n). Moreover if Y is any other space of type (π, n), $\exists$ pointed map $f : |K| \to Y$ s.t. $f_* : \pi_n(|K|) \to \pi_n(Y)$ is an isomorphism.

Cor: Any two spaces of type (π, n) have isomorphic homology groups, and isomorphic cohomology rings with arbitrary coefficients.

Idea of construction of K: Write $\pi = F/R$ where F is free abelian (free if $n = 1$). Let K^n be a wedge of S^n's in

1-1 corresp. with a set of free abelian generators (free generators if $n=1$) of F. Then K^n is $n-1$-connected, and $\pi_n(K^n) = F$. Let K^{n+1} be obtained from K^n by attaching, for each $r \in R$, one $n+1$-cell with attaching map representing r. Can prove: K^{n+1} is $n-1$-connected, and $\pi_n(K^{n+1}) \cong \pi$. Take $|K| = K^{n+1}(0,\dots,n)$.

Given a Y of type (π, n), define $f_n : K^n \to Y$ as follows. Let $p: F \to F/R = \pi_n(Y)$ be the projection. On the wedge summand S^n_α in K^n corresponding to $\alpha \in F$, define $f_n | S^n_\alpha$ = a representative of $p(\alpha) \in \pi_n(Y)$. Each $f_n \circ$ (attaching map for any $n+1$-cell) is homotopically trivial, so f_n extends to a map $f_{n+1} : K^{n+1} \to Y$. Since $\pi_i(Y) = 0$ for $i > n$, can successively extend over all skeleta.

Def: A Serre class $\mathcal{C}$ is <u>strongly homologically complete</u> if it is homologically complete, and if whenever $\pi \in \mathcal{C}$, and X is a space of type $(\pi, 1)$, then $H_i(X) \in \mathcal{C}$ for all $i > 0$.

Thm: Let $\mathcal{C}$ be a homologically complete Serre class consisting of finitely-generated abelian groups. Then $\mathcal{C}$ is strongly homologically complete.

<u>Sketch of proof:</u> For each $n > 1$, $\mathbb{Z}/n$ is a subgroup of S^1. S^1 acts freely on S^{2m+1} $\forall m$, and hence on S^∞. $L(n) = S^\infty/\mathbb{Z}/n$ is a space of type $(\mathbb{Z}/n, 1)$. $H_*(L(n))$ can be calculated from a cell decomposition (generalization of the cell decomposition for real projective spaces) and in fact

$$H_i(L(n)) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/n & \text{if } i > 0,\ i \text{ odd} \\ 0 & \text{if } i > 0,\ i \text{ even.} \end{cases}$$

If $\pi = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r} \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_q$, then

$\underbrace{S^1 \times \cdots \times S^1}_{r} \times L(n_1) \times \cdots \times L(n_q)$ is a space of type $(\pi, 1)$.

It follows from the Künneth theorem that the homology groups of this space lie in any homologically complete Serre class which contains π.

Thm: Suppose X is 0-connected and admits a universal covering space. Then for each $n \geq 1$ there exists a space $X(n+1,\dots,\infty)$ and a map $p: X(n+1,\dots,\infty) \to X$ such that $X(n+1,\dots,\infty)$ is n-connected and $p_*: \pi_q(X(n+1,\dots,\infty)) \to \pi_q(X)$ is an isomorphism for all $q \geq n+1$.

Proof: Convert $X \xrightarrow{i} X(0,\dots,n)$ to a fibration.

$$\begin{tikzcd}
 & X \arrow[dr, "i"] & \\
F \arrow[r, "j"'] & X' \arrow[u, "h"', "\text{homotopy equivalence}"] \arrow[r, "\text{fibration}"'] & X(0,\dots,n)
\end{tikzcd}$$

Take $X(n+1,\dots,\infty) = F$ and $p = hj$.

Note: $X(n+1,\dots,\infty)$ is sometimes called the <u>n-connective covering</u> of X. Without loss of generality, we can suppose p is a fibration. The universal covering space $\tilde{X}$ of X can be taken as $X(2,\dots,\infty)$.

<u>Proof of the Mod $\mathcal{C}$ Hurewicz Thm</u>: The case $n=1$ is immediate from the standard Hurewicz Theorem (in fact h_2 is an isomorphism).
Case $n=2$: It must be proved that $H_2(X) \in \mathcal{C}$ and $h_3: \pi_3(X) \to H_3(X)$ is a $\mathcal{C}$-isomorphism.

Write $p: Y \to X$ for the 2-connective covering of X, regarded as a fibration. Write F for the fibre. From the homotopy sequence of this fibration, it follows that F is a space of type $(\pi_2(X), 1)$. Thus, since $\pi_2(X) \in \mathcal{C}$ and $\mathcal{C}$ is strongly homologically complete, $H_i(F) \in \mathcal{C}$ for all $i > 0$. Thus $E^2_{0,q} = H_0(X; H_q(F)) \cong H_q(F) \in \mathcal{C}$ for $q \neq 0$. Since X is 1-connected, $E^2_{1,q} = H_1(X; H_q(F)) = 0$ for all q, and so $E^\infty_{1,q} = 0$ for all q.

Since Y is 2-connected, $H_2(Y) = 0$ and so $E^\infty_{p,q} = 0$ for $p+q=2$. In particular, $E^\infty_{2,0} = 0 = E^\infty_{0,1}$ and so

41 Lemma: Let $\mathcal{C}$ be a homologically complete Serre class. Let $\pi : (T, T_0) \to (B, B_0)$ be a fibration pair with $F \to T \to B$ $\mathbb{Z}$-orientable. Suppose $H_1(B, B_0) = 0$ and, for some $n \geq 1$, $H_i(B, B_0) \in \mathcal{C}$ for $0 < i \leq n$, and

$$H_j(F) \in \mathcal{C} \text{ for } 0 < j \leq \begin{cases} n-1 & \text{if } B_0 \neq \emptyset \\ n+1 & \text{if } B_0 = \emptyset \end{cases}.$$

Suppose F is 0-connected.

Then $\pi_* : H_{n+1}(T, T_0) \to H_{n+1}(B, B_0)$ is a $\mathcal{C}$-isomorphism.

Proof: Let E be the Leray-Serre spectral sequence of the above fibration pair with $\mathbb{Z}$-coefficients. By the base-edge theorem it suffices to show $p_B : H_{n+1}(T, T_0) \to E^\infty_{n+1,0}$ and $i_B : E^\infty_{n+1,0} \to E^2_{n+1,0}$ are $\mathcal{C}$-isomorphisms.

Proof that p_B is a $\mathcal{C}$-isomorphism: p_B is the composition

$$H_{n+1}(T, T_0) = J_{n+1,0} \longrightarrow J_{n+1,0}/J_{n,1} \cong E^\infty_{n+1,0}.$$

Thus, must show $J_{n,1} \in \mathcal{C}$. We prove by induction on i that $J_{i,n+1-i} \in \mathcal{C}$ for $0 \leq i \leq n$.

$$J_{0,n+1} = E^\infty_{0,n+1}, \text{ and } E^2_{0,n+1} \cong H_0(B, B_0; H_{n+1}(F))$$

$$\cong \begin{cases} 0 & \text{if } B_0 \neq \emptyset \\ H_{n+1}(F) & \text{if } B_0 = \emptyset \end{cases} \in \mathcal{C}.$$

Thus $J_{0,n+1} \in \mathcal{C}$.

We have the exact sequence $J_{0,n+1} \to J_{1,n} \to E^\infty_{1,n}$. Now $E^2_{1,n} \cong H_1(B, B_0; H_n(F)) = 0$, and so $E^\infty_{1,n} = 0 \in \mathcal{C}$. Hence $J_{1,n} \in \mathcal{C}$.

Suppose for some $1 \leq i < n$, $J_{i,n+1-i} \in \mathcal{C}$. We have the exact sequence

$$\underset{\in \mathcal{C}}{J_{i,n+1-i}} \longrightarrow J_{i+1,n-i} \to E^\infty_{i+1,n-i}$$

Now $E^2_{i+1,n-i} \cong H_{i+1}(B, B_0; H_{n-i}(F))$. By hypothesis,

$H_{n-i}(F) \in \mathcal{C}$ for i in above range, and $H_{i+1}(B, B_0)$, $H_i(B, B_0)$ are either $\mathbb{Z}$ or in $\mathcal{C}$. Thus, by U.C.T. and homological completeness of $\mathcal{C}$, $E^2_{i+1,n-i} \in \mathcal{C}$ and hence $E^\infty_{i+1,n-i} \in \mathcal{C}$. Thus $J_{i+1,n-i} \in \mathcal{C}$, completing proof that p_B is a $\mathcal{C}$-isom-

Proof that i_B is a $\mathcal{C}$-isom: $i_B : E^\infty_{n+1,0} \to E^2_{n+1,0}$ is the composition
$$E^\infty_{n+1,0} = E^{n+2}_{n+1,0} \hookrightarrow E^{n+1}_{n+1,0} \hookrightarrow \cdots \hookrightarrow E^2_{n+1,0},$$
so it suffices to show the inclusion $E^{r+1}_{n+1,0} \hookrightarrow E^r_{n+1,0}$ is a $\mathcal{C}$-isomorphism for $2 \le r \le n+1$.
$$E^{r+1}_{n+1,0} = \ker\left[E^r_{n+1,0} \xrightarrow{d^r} E^r_{n+1-r,r-1}\right], \text{ so}$$
$$E^r_{n+1,0}/E^{r+1}_{n+1,0} \cong \operatorname{im}\left[E^r_{n+1,0} \xrightarrow{d^r} E^r_{n+1-r,r-1}\right].$$
Thus, suffices to show $E^r_{n+1-r,r-1} \in \mathcal{C}$. We have
$$E^2_{n+1-r,r-1} \cong H_{n+1-r}(B, B_0; H_{r-1}(F)).$$

For $2 \le r \le n-1$, $H_{r-1}(F)$, $H_{n+1-r}(B,B_0)$, $H_{n-r}(B,B_0) \in \mathcal{C}$, and so $E^2_{n+1-r,r-1} \in \mathcal{C}$ by U.C.T. and homological completeness of $\mathcal{C}$.

When $r = n$: $H_1(B,B_0; H_{n-1}(F)) = 0$ by U.C.T. and hypothesis.

When $r = n+1$:
$$H_0(B,B_0; H_n(F)) \cong \begin{cases} 0 & \text{if } B_0 \neq \emptyset \\ H_n(F) & \text{if } B_0 = \emptyset \end{cases} \in \mathcal{C}$$

Thus, $E^2_{n+1-r,r-1} \in \mathcal{C}$ for $2 \le r \le n+1$, and so $E^r_{n+1-r,r-1} \in \mathcal{C}$ for $2 \le r \le n+1$, completing proof of lemma.

<u>Proof of the Mod $\mathcal{C}$ Hurewicz Theorem</u>: The case $n=1$ is immediate from the standard Hurewicz theorem (in fact h_2 is an isomorphism).

Suppose $n \geq 2$ and theorem holds for $n-1$. By the inductive hypothesis, $H_i(X,*) \in \mathcal{C}$ for $0 < i \le n$ and it remains only to prove $h_{n+1} : \pi_{n+1}(X,*) \to H_{n+1}(X,*)$ is a $\mathcal{C}$-isomorphism.

Write $p : Y \to X$ for the 2-connective covering of X, regarded as a fibration. Write F for the fibre. From the homotopy sequence of this fibration it follows that F is a space of type $(\pi_2(X), 1)$. Thus since $\pi_2(X) \in \mathcal{C}$ and $\mathcal{C}$ is strongly homologically complete, $H_i(F) \in \mathcal{C}$ for all $i > 0$.

We have the commutative diagram

$$\begin{tikzcd} \pi_{n+1}(Y,*) \arrow[r, "p_*", "\cong"'] \arrow[d, "h_{n+1}"'] & \pi_{n+1}(X,*) \arrow[d, "h_{n+1}"] \\ H_{n+1}(Y,*) \arrow[r, "p_*"'] & H_{n+1}(X,*) \end{tikzcd}$$

By lemma above, $p_* : H_{n+1}(Y) \longrightarrow H_{n+1}(X)$ is a $\mathcal{C}$-isomorphism. Thus bottom p_* in above diagram is a $\mathcal{C}$-isomorphism, and it remains only to show $h_{n+1} : \pi_{n+1}(Y,*) \longrightarrow H_{n+1}(Y,*)$ is a $\mathcal{C}$-isomorphism.

Consider the fibration $\Omega Y \longrightarrow PY \xrightarrow{\pi} Y$. We have the commutative diagram

$$\begin{tikzcd} \pi_{n+1}(Y,*) \arrow[d, "h_{n+1}"'] & \pi_{n+1}(PY,\Omega Y) \arrow[l, "\pi_*"', "\cong"] \arrow[r, "\partial", "\cong"'] \arrow[d, "h_{n+1}"] & \pi_n(\Omega Y,*) \arrow[d, "h_n"] \\ H_{n+1}(Y,*) & H_{n+1}(PY,\Omega Y) \arrow[l, "\pi_*"] \arrow[r, "\cong", "\partial"'] & H_n(\Omega Y,*) \end{tikzcd}$$

with indicated isomorphisms from fact that π is a fibration and PY contractible.

Since $\pi_i(Y) \cong \pi_i(X)$ for all $i \geq 3$, it follows that $\pi_i(Y) \in \mathcal{C}$ for $i \leq n$. Hence by the inductive hypothesis, $H_i(Y,*) \in \mathcal{C}$ for $i \leq n$. Thus by an earlier lemma, $H_i(\Omega Y,*) \in \mathcal{C}$ for $i \leq n-1$. Thus, by above lemma $\pi_* : H_{n+1}(PY,\Omega Y) \to H_{n+1}(Y,*)$ is a $\mathcal{C}$-isomorphism. Also ΩY is 1-connected since Y is 2-connected and so by the induction hypothesis, $h_n : \pi_n(\Omega Y,*) \to H_n(\Omega Y,*)$ is a $\mathcal{C}$-isomorphism. Thus $h_{n+1} : \pi_{n+1}(Y,*) \to H_{n+1}(Y,*)$ is a $\mathcal{C}$-isomorphism, completing the proof.

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Calculation of $\pi_4(S^3)$

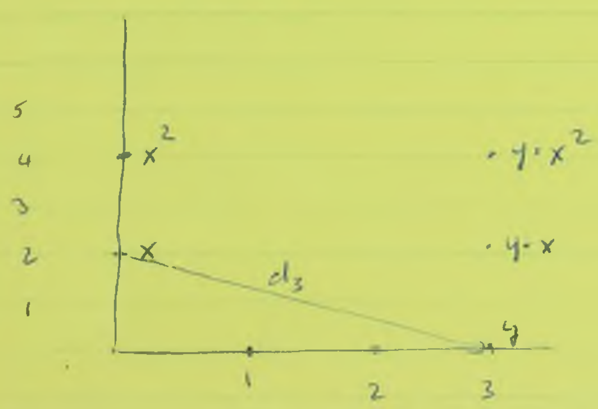
Let $p: Y \to S^3$ denote the 3-connective covering of S^3, regarded as a fibration. Then $p_*: \pi_4(Y) \to \pi_4(S^3)$ is an isomorphism. Since Y is 3-connected, $h_4: \pi_4(Y) \to H_4(Y)$ is an isomorphism, and so $\pi_4(S^3) \cong H_4(Y)$.

From the homotopy sequence of the above fibration it follows that the fibre F is a space of type $(\mathbb{Z}, 2)$. Thus $H^*(F) \cong H^*(\mathbb{C}P^\infty)$ as a graded ring, and so $H^*(F) = \mathbb{Z}[x]$, $x \in H^2(F)$.

Let $\mathcal{C}$ denote the class of all finitely-generated abelian groups. By the Mod $\mathcal{C}$ Hurewicz theorem, $\pi_n(S^3) \in \mathcal{C}$ for all n. Thus $\pi_n(Y) \in \mathcal{C}$ for all n, so again, by the mod $\mathcal{C}$ Hurewicz theorem, $H_n(Y) \in \mathcal{C}$ for all $n > 0$. Thus, can write $H_4(Y) = F \oplus T$ where F is free-abelian on a finite no. of generators and T is a finite abelian group. By the universal coefficient theorem, $F \cong H^4(Y)$, $T \cong$ torsion part of $H^5(Y)$.

Let E denote the cohomology Leray-Serre spectral sequence with $\mathbb{Z}$-coefficients for $F \to Y \to S^3$. We can identify $E_2^{0,*}$ with $H^*(F) = \mathbb{Z}[x]$, and $E_2^{*,0} = H^*(S^3)$. Since each $H^n(F)$ is free and finitely-generated, the product map $E_2^{p,0} \otimes_{\mathbb{Z}} E_2^{0,q} \to E_2^{p,q}$ is an isomorphism for all (p,q). Since $d_2 \equiv 0$, $E_3 = H^*(E_2, d_2) = E_2$ as bigraded rings. Note that since $d_r = 0$ for $r > 3$, $E_4 = E_\infty$.

Since Y is 3-connected, $E_\infty^{0,2} = 0 = E_\infty^{3,0}$. Hence $d_3: E_3^{0,2} \to E_3^{3,0}$ is an isomorphism. Write $y = d_3(x)$. Then y generates $E_3^{3,0} \cong \mathbb{Z}$



$$d_3(x^2) = d_3(x)\cdot x + (-1)^2 x\cdot d_3(x) = y\cdot x + x\cdot y$$

$$= y\cdot x + (-1)^{2\cdot 3} y\cdot x = 2y\cdot x$$

Thus $d_3 | E_3^{0,4}$ is 1-1 and so $E_\infty^{0,4} = E_4^{0,4} = 0$. Thus $E_\infty^{p,q} = 0$ whenever $p+q=4$ and so $H^4(Y)=0$.

(Thus, free abelian part of $H_4(Y) = 0$).

We have $E_4^{3,2} = E_3^{3,2} / \operatorname{im}\left[E_3^{0,4} \xrightarrow{d_3} E_3^{3,2} \right]$

$$= \frac{\text{free on } y\cdot x}{\text{subgp. gen. by } 2y\cdot x} \cong \mathbb{Z}/2 .$$ Thus $E_\infty^{3,2} \cong \mathbb{Z}/2$.

Note that $E_2^{p,q} = 0$ whenever $p+q=5$ and $(p,q) \neq (3,2)$.

Thus

$$0 = J^{6,-1} = J^{5,0} = J^{4,1},$$

$$\mathbb{Z}/2 = E_\infty^{3,2} = J^{3,2} = J^{2,3} = J^{1,4} = J^{0,5} = H^5(Y).$$

Thus, torsion part of $H_4(Y) \cong \mathbb{Z}/2$.

<u>Conclusion</u>: $\pi_4(S^3) \cong \mathbb{Z}/2$

Further Topics and References

1. Cohomology operations
   - Steenrod + Epstein, Cohomology Operations
   - Spanier
   - Milnor, "The Steenrod algebra and its dual", Ann. of Math. 1958.

2. Characteristic classes
   - Milnor + Stasheff, Char. Classes, Princeton U. Press
   - Husemoller, Fibre Bundles

3. Poincaré duality
   - Milnor + Stasheff (appendix)
   - Spanier
   - Dold, Algebraic Topology

4. Generalized homology + cohomology, spectra
   - G. W. Whitehead, Generalized homology theories, Trans. A.M.S. 1962
   - E. H. Brown, Cohomology theories, Ann. of Math 1962
   - J. F. Adams, Stable Homotopy + Generalized Homology, U. of Chicago Lecture Notes, 1971
   - M. F. Atiyah, K-Theory
   - R. Stong, Notes on Cobordism Theory (Princeton U. Press)

5. General:
   - J. F. Adams, Algebraic Topology: A Student's Guide. (Cambridge U. Press).

Math 752 Problems Due Jan. 23, 1984

1. Let F be a field, and $F[x]$ the polynomial ring over F on one indeterminate x. It is an easy algebraic fact that $F[x]$ is a principal ideal domain.

F is an $F[x]$-module with module action given by

$(a_0 + a_1 x + \cdots + a_n x^n)\cdot b = a_0 b$, $a_i, b \in F$.

a) Find $Ext_{F[x]}(F, F)$.

b) Find $Ext_{F[x]}(F, F[x])$.

2. Let X be a path-connected topological space, and G an abelian group.

<u>Prove</u>: For every 0-cocycle $f \in Q^0(X;G) = \mathrm{Hom}_{\mathbb{Z}}(Q_0(X), G)$, $f(\sigma) = f(\tau)$ for all singular 0-cubes $\sigma, \tau : I^0 \to X$.

3. Find $H^i(\mathbb{R}P^n; G)$ for all n and all i in each of the following cases:

a) $G = \mathbb{Z}/2$

b) $G = \mathbb{Z}/p$ where p is an odd prime

c) $G = \mathbb{Q}$, the rationals

d) $G = \mathbb{Z}$.

Math 752 Problems Due Feb. 20, 1994

Let R be a PID. An <u>R-oriented real n-plane bundle</u> (E, p, B, U) consists of a real n-plane bundle $E \xrightarrow{p} B$ together with an R-orientation $U \in H^n(E, E^0; R)$.

A <u>morphism of R-oriented n-plane bundles</u> $f : (E_1, p_1, B_1, U_1) \longrightarrow (E_2, p_2, B_2, U_2)$ consists of a morphism of n-plane bundles

$$\begin{tikzcd} E_1 \arrow[r, "f_E"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f_B"'] & B_2 \end{tikzcd}$$

such that $\bar{f}^*(U_2) = U_1$ where $\bar{f} : (E_1, E_1^0) \to (E_2, E_2^0)$ is the relative map coming from f_E.

If (E, p, B, U) is an R-oriented real n-plane bundle, the <u>Euler class</u> of (E, p, B, U), written $\chi(E, U) \in H^n(B; R)$ is the unique class satisfying $p^* \chi(E, U) = j^* U$ where $j : (E, \phi) \to (E, E^0)$ is the inclusion. (<u>Note</u>: p^* is an isomorphism since p is a homotopy equivalence).

4. Let $f : (E_1, p_1, B_1, U_1) \to (E_2, p_2, B_2, U_2)$ be a morphism of R-oriented n-plane bundles.

<u>Prove</u>: $f_B^* \chi(E_2, U_2) = \chi(E_1, U_1)$.

5. a) Let V be an $n+1$-dimensional real vector space, $n \geq 1$. Let $\mathcal{U}$ be the unique $\mathbb{Z}/2$ orientation of $L_{\mathbb{R}}(V) \to P_{\mathbb{R}}(V)$.
<u>Prove</u>: $\chi(L_{\mathbb{R}}(V), \mathcal{U})$ is the generator of $H^1(P_{\mathbb{R}}(V); \mathbb{Z}/2)$.

b) Let V be an $n+1$-dimensional complex vector space, $n \geq 1$. Let $\mathcal{U}$ be a $\mathbb{Z}$-orientation of the underlying real 2-plane bundle of $L_{\mathbb{C}}(V) \to P_{\mathbb{C}}(V)$.
<u>Prove</u>: $\chi(L_{\mathbb{C}}(V), \mathcal{U})$ is a generator of $H^2(P_{\mathbb{C}}(V); \mathbb{Z})$.

6. A continuous map $f: S^m \to S^n$ will be called <u>equivariant</u> if $f(-x) = -f(x)$ for all $x \in S^m$.

a) Suppose $f: S^m \to S^n$ is equivariant. Define $F_E : L_{\mathbb{R}}(\mathbb{R}^{m+1}) \to L_{\mathbb{R}}(\mathbb{R}^{n+1})$ and $F_B : \mathbb{R}P^m \to \mathbb{R}P^n$ by

$$F_E([v], w) = \left(\left[f\left(\frac{v}{\|v\|}\right) \right], \frac{\langle v, w \rangle}{\|v\|^2} f\left(\frac{v}{\|v\|}\right) \right),$$

$$F_B([v]) = \left[f\left(\frac{v}{\|v\|}\right) \right].$$

Verify that F_E, F_B are well-defined and constitute a map of real 1-plane bundles

$$\begin{array}{ccc} L_{\mathbb{R}}(\mathbb{R}P^{m+1}) & \xrightarrow{F} & L_{\mathbb{R}}(\mathbb{R}P^{n+1}) \\ \downarrow & & \downarrow \\ \mathbb{R}P^m & & \mathbb{R}P^n \end{array}$$

b) <u>Prove</u>: If $m > n$, a continuous equivariant map $S^m \to S^n$ does not exist.

c) (Borsuk-Ulam Theorem) <u>Prove</u>: If $f: S^n \to \mathbb{R}^n$ is continuous, there exists an $x \in S^n$ such that $f(x) = f(-x)$.

Math 752 Problems Due Mar. 5, 1984

7. Let (E, p, B, F) be a finitely-presentable fibre bundle, i.e. $p : E \to B$ is continuous, and there is a finite open cover $\{U_1, \dots, U_n\}$ of B and homeomorphisms $h_i : U_i \times F \to p^{-1}(U_i)$ such that

$$\begin{tikzcd} U_i \times F \arrow[rr, "h_i"] \arrow[dr, "\pi_1"'] & & p^{-1}(U_i) \arrow[dl, "p|p^{-1}(U_i)"] \\ & U_i & \end{tikzcd}$$

commutes.

For $x \in B$, write $E_x = p^{-1}(x)$, and $i_x : E_x \to E$ for the inclusion.

Let R be a PID. Suppose there exists a finite set $\{e_i \mid e_i \in H^{n_i}(E;R),\ 1 \le i \le k\}$ of cohomology classes such that for each $x \in B$, $\bigoplus_{n \ge 0} H^n(E_x;R)$ is a free R-module with basis $\{i_x^*(e_i) \mid 1 \le i \le k\}$.

<u>Prove</u>: For each $x \in H^n(E;R)$, there exist unique elements $b_i \in H^{n-n_i}(B;R)$ such that $x = \sum_{i=1}^{k} p^*(b_i) \cup e_i$.

(<u>Hint</u>: Let $\Psi_n : \bigoplus_{i=1}^{k} H^{n-n_i}(B;R) \to H^n(E;R)$ be the map given by $\Psi_n(b_1, \dots, b_k) = \sum_{i=1}^{k} p^*(b_i) \cup e_i$. It must be shown that Ψ_n is an isomorphism. Imitate the proof of the Thom Isomorphism Theorem).

8. Let $p : E \to B$ be a real n-plane bundle. Let $P_{\mathbb{R}}(E)$ = quotient space obtained from E^0 by identifying $v \sim tv$ whenever $t \in \mathbb{R} - \{0\}$. (Thus, as a set, $P_{\mathbb{R}}(E) = \coprod_{x \in B} P_{\mathbb{R}}(E_x)$). Let $L_{\mathbb{R}}(E) = \{([v], w) \in P_{\mathbb{R}}(E) \times E \mid w \in [v]\}$. (As in the lectures we are regarding $[v] \in P_{\mathbb{R}}(E)$ as the 1-dimensional subspace of $E_{p(v)}$ spanned by v).

Let $q : L_{\mathbb{R}}(E) \to P_{\mathbb{R}}(E)$ be given by $q([v], w) = [v]$, and $r : P_{\mathbb{R}}(E) \to B$ be given by $r[v] = p(v)$.

It can be shown that

1) r is a fibre bundle projection with fibres homeomorphic to $P_{\mathbb{R}}(\mathbb{R}^n)$, and this fibre bundle is finitely-presentable if $p: E \to B$ is.

2) $\xi: L_{\mathbb{R}}(E) \to P_{\mathbb{R}}(E)$ is a real line bundle, and is finitely-presentable if $p: E \to B$ is.

(Try to prove 1) and 2), but you do not have to turn in the proofs).

Suppose $p: E \to B$ is finitely-presentable, and let $U \in H^1(L_{\mathbb{R}}(E), L_{\mathbb{R}}(E)^\circ; \mathbb{Z}/2)$ denote the unique $\mathbb{Z}/2$-orientation of $L_{\mathbb{R}}(E) \to P_{\mathbb{R}}(E)$.

Prove: For each $x \in H^m(P_{\mathbb{R}}(E); \mathbb{Z}/2)$, there are unique elements $b_i \in H^{m-i}(B; \mathbb{Z}/2)$, $0 \le i \le n-1$, such that

$$x = \sum_{i=0}^{n-1} r^*(b_i) \cup \chi(L_{\mathbb{R}}(E), U)^i.$$

Note: In particular, there are unique elements $w_i(E) \in H^i(B; \mathbb{Z}/2)$, $1 \le i \le n$, such that

$$\chi(L_{\mathbb{R}}(E), U)^n = \sum_{i=1}^{n} r^* w_i(E) \cup \chi(L_{\mathbb{R}}(E), U)^{n-i}.$$

The $w_i(E)$ are called the Stiefel-Whitney classes of the real n-plane bundle $p: E \to B$. We also define $w_0(E) = 1$ and $w_i(E) = 0$ if $i > n$.

Note: A similar treatment can be given for complex vector bundles, yielding Chern classes.

9. Let Y be a pointed space such that $[X,Y]$ has a natural group structure for pointed spaces X (i.e. the functor $[\ ,Y]$ factors through the category of groups).

<u>Prove</u>: This natural group structure comes from an H^*-structure on Y.

10. <u>Prove</u>: If X is a co-H^* space with comultiplication μ' and coinversion η', then for any pointed Y, $[X,Y]$ is a group with operation given by $[f][g]=[f \,\square\, g]$ where $f \,\square\, g$ is defined as in the lectures. The unit element is $[*]$, and $[f]^{-1}=[f\eta']$. Moreover, $[X,\ ]$ is a covariant functor from the category of pointed spaces to the category of groups.

Math 752 Problems Due Mar. 26, 1984

11. <u>Prove</u>: The homotopy sequence of a pointed triple is exact.

12. Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ be real m and n-plane bundles, resp. Write $E_1 \oplus E_2 = \{(u,v) \in E_1 \times E_2 \mid p_1(u) = p_2(v)\}$ and let $p : E_1 \oplus E_2 \to B$ be given by $p(u,v) = p_1(u) = p_2(v)$. Note that for each $x \in B$, $p^{-1}(x) = (E_1)_x \oplus (E_2)_x$ and hence has the structure of a real $m+n$-dimensional vector space.
<u>Prove</u>: $p : E_1 \oplus E_2 \to B$ is a real $(m+n)$-plane bundle, and is finitely-presentable if the $E_i \to B$ are, $i = 1, 2$.
($E_1 \oplus E_2 \to B$ is called the <u>Whitney</u> <u>sum</u> of the vector bundles $E_i \to B$).

13. Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ be finitely-presentable m and n-plane bundles, respectively. Write $P(E) = P_{\mathbb{R}}(E)$, $L(E) = L_{\mathbb{R}}(E)$.
We have inclusions $i_k : P(E_k) \to P(E_1 \oplus E_2)$,
$j_k : L(E_k) \to L(E_1 \oplus E_2)$, $k = 1, 2$, given by
$i_1[u] = [u, 0]$, $i_2[v] = [0, v]$, $j_1([u], w) = ([u, 0], (w, 0))$,
$j_2([v], x) = ([0, v], (0, x))$.

a) <u>Prove</u>: i_k, j_k constitute a map of real line bundles

$$\begin{array}{ccc} L(E_k) & & L(E_1 \oplus E_2) \\ \downarrow & \longrightarrow & \downarrow \\ P(E_k) & & P(E_1 \oplus E_2) \end{array}, \quad k = 1, 2.$$

b) Identify $P(E_1)$, $P(E_2)$ with subspaces of $P(E_1 \oplus E_2)$ via i_1, i_2, respectively. Let $U = P(E_1 \oplus E_2) - P(E_2)$,
$V = P(E_1 \oplus E_2) - P(E_1)$.
<u>Prove</u>: $P(E_1)$ is a deformation retract of U, and $P(E_2)$ is a deformation retract of V.

14. Suppose $X = U \cup V$ where U, V are open in X. Let $i_1 : U \to X$, $i_2 : V \to X$ denote the inclusions. Let R be a PID and suppose $a \in H^p(X; R)$, $b \in H^q(X; R)$ are such that $i_1^*(a) = 0$, $i_2^*(b) = 0$. Prove: $a \cup b = 0$.

15. (Whitney Product Formula) Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ be finitely-presentable real m and n-plane bundles, resp. Prove: $$w_k(E_1 \oplus E_2) = \sum_{i=0}^{k} w_i(E_1) \cup w_{k-i}(E_2).$$

(Hint: Write $\chi_{E_1 \oplus E_2} = \chi(E_1 \oplus E_2, U)$, U the unique $\mathbb{Z}/2$-orientation. Note that
$$\sum_{k=0}^{m+n} r^*\Big(\sum_{i=0}^{k} w_i(E_1) \cup w_{k-i}(E_2)\Big) \cup \chi_{E_1 \oplus E_2}^{m+n-k}$$
$$= \Big(\sum_{i=0}^{m} r^* w_i(E_1) \cup \chi_{E_1 \oplus E_2}^{m-i}\Big) \cup \Big(\sum_{j=0}^{n} r^* w_j(E_2) \cup \chi_{E_1 \oplus E_2}^{n-j}\Big)$$
where $r : P(E_1 \oplus E_2) \to B$ is the projection.)

Math 752 Problems Due April 6, 1984

16. Compute $H_n(\Omega(S^2 \vee S^3))$ for all n.

17. Let $F \xrightarrow{i} T \xrightarrow{p} S^n$ be a pointed fibration, $n \geq 2$.
Let R be any commutative ring with unit.
Prove: there exists an exact sequence

$$\cdots \xrightarrow{\alpha} H_{i-n+1}(F;R) \xrightarrow{\beta} H_i(F;R) \xrightarrow{i_*} H_i(T;R) \xrightarrow{\alpha} H_{i-n}(F;R) \xrightarrow{\beta} \cdots$$

18. Let $S^n \xrightarrow{i} T \xrightarrow{p} B$ be a pointed fibration with B simply-connected and $n \geq 1$. Let $B_0 \subset B$ and $T_0 = p^{-1}(B_0)$.
Let R be any commutative ring with unit.
Prove: There exists an exact sequence

$$\xrightarrow{\gamma} H_{i-n}(B, B_0; R) \xrightarrow{\delta} H_i(T, T_0; R) \xrightarrow{p_*} H_i(B, B_0; R) \xrightarrow{\gamma} H_{i-n-1}(B, B_0; R) \xrightarrow{\delta} \cdots$$

19. Let $F \xrightarrow{i} T \xrightarrow{p} B$ be a pointed fibration with B 1-connected. Assume $H_i(F;\mathbb{Q})$ and $H_i(B;\mathbb{Q})$ are finite-dimensional over $\mathbb{Q}$ for all i, and non-zero for only finitely many i.
Prove: $H_i(T;\mathbb{Q})$ is finite-dimensional over $\mathbb{Q}$ for all i, non-zero for only finitely many i, and $\chi(T) = \chi(B)\chi(F)$.
(Recall: The Euler characteristic $\chi(X) = \sum_i (-1)^i \dim_{\mathbb{Q}} H_i(X;\mathbb{Q})$.)

20. The tangent bundle $\tau(S^n)$ has the following description:
$\tau(S^n) = \{(x,y) \in S^n \times \mathbb{R}^{n+1} \mid \langle x,y \rangle = 0\}$ where $\langle\ ,\ \rangle$ is the standard Euclidean inner product. The projection $\tau(S^n) \to S^n$ is projection on the first factor.

a) Let ε^k denote the product k-plane bundle over S^n.
Prove: $\tau(S^n) \oplus \varepsilon^1 \cong \varepsilon^{n+1}$.

b) Find $w_i(\tau(S^n))$ for all i.

Math 752 Solutions

1. $0 \to F[x] \xrightarrow{\partial} F[x] \xrightarrow{\varepsilon} F \to 0$ is a short free resolution of F, where $\varepsilon(f(x)) = f(0)$, $\partial(f(x)) = x f(x)$.

a) Using the above short free resolution we have the exact sequence

$$\mathrm{Hom}_{F[x]}(F[x], F) \xrightarrow{\mathrm{Hom}(\partial, 1_F)} \mathrm{Hom}_{F[x]}(F[x], F) \to \mathrm{Ext}_{F[x]}(F, F) \to 0.$$

Note that $\mathrm{Hom}(\partial, 1_F)$ is the 0-homomorphism, for if $\alpha : F[x] \to F$ is an $F[x]$-homomorphism and $f(x) \in F[x]$,

$$\begin{aligned}\mathrm{Hom}(\partial, 1_F)(\alpha)(f(x)) &= \alpha\partial(f(x)) = \alpha(x f(x)) \\ &= x \cdot \alpha(f(x)) \quad (\text{since } \alpha \text{ is an } F[x]\text{-homomorphism}) \\ &= 0 \quad (\text{by def. of action of } F[x] \text{ on } F, \text{ since constant term of } x \text{ is } 0).\end{aligned}$$

Hence $\mathrm{Ext}_{F[x]}(F, F) \cong \mathrm{Hom}_{F[x]}(F[x], F) \cong F$

(since $\mathrm{Hom}_R(R, A) \cong A$ in general).

b) Using the same short free resolution above, we have the exact sequence

$$\mathrm{Hom}_{F[x]}(F[x], F[x]) \xrightarrow{\mathrm{Hom}(\partial, 1_F)} \mathrm{Hom}_{F[x]}(F[x], F[x]) \to \mathrm{Ext}_{F[x]}(F, F[x]) \to 0.$$

We have the natural $F[x]$-isomorphism $\theta : \mathrm{Hom}_{F[x]}(F[x], F[x]) \to F[x]$ given by $\theta(\alpha) = \alpha(1)$, and it is easily checked that

$$\begin{tikzcd} \mathrm{Hom}_{F[x]}(F[x], F[x]) \arrow[r, "\mathrm{Hom}(\partial, 1_{F[x]})"] \arrow[d, "\theta"', "\cong"] & \mathrm{Hom}_{F[x]}(F[x], F[x]) \arrow[d, "\theta"', "\cong"] \\ F[x] \arrow[r, "\partial"] & F[x] \end{tikzcd}$$

commutes. Thus, $\mathrm{Ext}_{F[x]}(F, F[x]) \cong \dfrac{F[x]}{\mathrm{im}\,\partial} = \dfrac{F[x]}{xF[x]} \cong F$.

2. Since X is path-connected, for any 0-cubes $\sigma, \tau : I^0 \to X$, $\exists$ continuous $\rho : I^1 \to X$ such that $\rho(0) = \sigma(\,)$, $\rho(1) = \tau(\,)$. Thus $\partial\rho = \tau - \sigma$. Hence, for any 0-cocycle $f \in Q^0(X; G)$,
$0 = (\delta f)(\rho) = f(\partial\rho) = f(\tau - \sigma) = f(\tau) - f(\sigma)$.

3. For any field F, $H^i(X;F) \cong \mathrm{Hom}_F(H_i(X;F),F)$. Thus for parts a), b), c), using the results of HW problem 27 of Math 751,

a) $H^i(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathrm{Hom}_{\mathbb{Z}/2}(H_i(\mathbb{R}P^n;\mathbb{Z}/2),\mathbb{Z}/2)$

$$\cong \begin{cases} \mathrm{Hom}_{\mathbb{Z}/2}(\mathbb{Z}/2,\mathbb{Z}/2) & \text{if } 0 \le i \le n \\ \mathrm{Hom}_{\mathbb{Z}/2}(0,\mathbb{Z}/2) & \text{otherwise} \end{cases} \cong \begin{cases} \mathbb{Z}/2 & \text{if } 0 \le i \le n \\ 0 & \text{otherwise.} \end{cases}$$

b) $H^i(\mathbb{R}P^n; \mathbb{Z}/p) \cong \mathrm{Hom}_{\mathbb{Z}/p}(H_i(\mathbb{R}P^n;\mathbb{Z}/p),\mathbb{Z}/p)$

$$\cong \begin{cases} \mathrm{Hom}_{\mathbb{Z}/p}(\mathbb{Z}/p,\mathbb{Z}/p) & \text{if } i=0 \text{ or } i=n \text{ with } n \text{ odd} \\ \mathrm{Hom}_{\mathbb{Z}/p}(0,\mathbb{Z}/p) & \text{otherwise} \end{cases} \cong \begin{cases} \mathbb{Z}/p & \text{if } i=0 \text{ or } i=n \text{ with } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

c) $H^i(\mathbb{R}P^n; \mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Q}}(H_i(\mathbb{R}P^n;\mathbb{Q}),\mathbb{Q})$

$$\cong \begin{cases} \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q},\mathbb{Q}) & \text{if } i=0 \text{ or } i=n \text{ with } n \text{ odd} \\ \mathrm{Hom}_{\mathbb{Q}}(0,\mathbb{Q}) & \text{otherwise} \end{cases} \cong \begin{cases} \mathbb{Q} & \text{if } i=0 \text{ or } i=n \text{ with } n \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

d) By the universal coefficient theorem,

$$H^i(\mathbb{R}P^n;\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_i(\mathbb{R}P^n),\mathbb{Z}) \oplus \mathrm{Ext}_{\mathbb{Z}}(H_{i-1}(\mathbb{R}P^n),\mathbb{Z}).$$

Using the fact that

$$H_i(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & \text{if } i=0 \text{ or } i=n \text{ with } n \text{ odd} \\ \mathbb{Z}/2 & \text{if } 0<i<n \text{ with } i \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

and the facts that

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}) = 0$, $\mathrm{Hom}_{\mathbb{Z}}(0,\mathbb{Z}) = 0$,

$\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0$, $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}) \cong \mathbb{Z}/2$, $\mathrm{Ext}_{\mathbb{Z}}(0,\mathbb{Z}) = 0$,

we obtain

$$H^i(\mathbb{R}P^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i=0 \text{ or } i=n \text{ with } n \text{ odd} \\ \mathbb{Z}/2 & \text{if } 0<i\le n \text{ with } i \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

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4. We have the commutative diagram

$$\begin{tikzcd}
(E_1, E_1^\circ) \arrow[r, "\bar{f}"] & (E_2, E_2^\circ) \\
(E_1, \emptyset) \arrow[u, "j_1"] \arrow[r, "f_E"'] \arrow[d, "p_1"'] & (E_2, \emptyset) \arrow[u, "j_2"'] \arrow[d, "p_2"] \\
(B_1, \emptyset) \arrow[r, "f_B"'] & (B_2, \emptyset)
\end{tikzcd}$$

It suffices to show $p_1^* f_B^* \chi(E_2, U_2) = j_1^* U_1$.
We have

$p_1^* f_B^* \chi(E_2, U_2) = f_E^* p_2^* \chi(E_2, U_2)$ (commutativity of bottom square)

$= f_E^* j_2^* (U_2)$ (def. of $\chi(E_2, U_2)$)

$= j_1^* \bar{f}^* (U_2)$ (commutativity of top square)

$= j_1^* U_1$ (sine f is a morphism of R-oriented n-plane bundles).

5. Write $i : L_F(V)^\circ \to L_F(V)$ and $j : (L_F(V), \emptyset) \to (L_F(V), L_F(V)^\circ)$ for the inclusions and $p : L_F(V) \to P_F(V)$ for the projections, $F = \mathbb{R}$ or $\mathbb{C}$. In the lectures it was shown that $L_{\mathbb{R}}(V)^\circ$ has the homotopy type of S^n, and $L_{\mathbb{C}}(V)^\circ$ has the homotopy type of S^{2n+1}.

a) $H^0(L_{\mathbb{R}}(V); \mathbb{Z}/2) \xrightarrow{i^*} H^0(L_{\mathbb{R}}(V)^\circ; \mathbb{Z}/2) \xrightarrow{\delta} H^1(L_{\mathbb{R}}(V), L_{\mathbb{R}}(V)^\circ; \mathbb{Z}/2) \xrightarrow{j^*} H^1(L_{\mathbb{R}}(V), \mathbb{Z}/2)$

is exact. i^* is an isomorphism since $L_{\mathbb{R}}(V)$ (which has the homotopy type of $\mathbb{R}P^n$) and $L_{\mathbb{R}}(V)^\circ$ (which has the homotopy type of S^n with $n \geq 1$) are both path-connected. Hence j^* is a monomorphism, and so $j^*(U) \neq 0$. Thus $p^* \chi(L_{\mathbb{R}}(V), U) \neq 0$, and so $\chi(L_{\mathbb{R}}(V), U) \neq 0$. Thus since $H^1(P_{\mathbb{R}}(V); \mathbb{Z}/2) \cong \mathbb{Z}/2$, $\chi(L_{\mathbb{R}}(V), U)$ must be the generator.

b) $H^1(L_{\mathbb{C}}(V)^\circ; \mathbb{Z}) \xrightarrow{\delta} H^2(L_{\mathbb{C}}(V), L_{\mathbb{C}}(V)^\circ; \mathbb{Z}) \xrightarrow{j^*} H^2(L_{\mathbb{C}}(V); \mathbb{Z}) \longrightarrow H^2(L_{\mathbb{C}}(V)^\circ; \mathbb{Z})$

is exact. Since $L_{\mathbb{C}}(V)^\circ$ has the homotopy type of S^{2n+1} with $n \geq 1$, both $H^1(L_{\mathbb{C}}(V)^\circ; \mathbb{Z})$ and $H^2(L_{\mathbb{C}}(V)^\circ; \mathbb{Z})$ are 0. Hence, above j^* is an isomorphism. By the Thom isomorphism,

$\cup U: H^0(L_{\mathbb{C}}(V); \mathbb{Z}) \to H^2(L_{\mathbb{C}}(V), L_{\mathbb{C}}(V)^\circ; \mathbb{Z})$ is an isomorphism and since $H^0(L_{\mathbb{C}}(V); \mathbb{Z}) \cong \mathbb{Z}$ with generator 1, $H^2(L_{\mathbb{C}}(V), L_{\mathbb{C}}(V)^\circ; \mathbb{Z}) \cong \mathbb{Z}$ with generator U. Thus $H^2(L_{\mathbb{C}}(V); \mathbb{Z}) \cong \mathbb{Z}$ with generator j^*U. Thus, since $p^*: H^2(P_{\mathbb{C}}(V); \mathbb{Z}) \to H^2(L_{\mathbb{C}}(V); \mathbb{Z})$ is an isomorphism and $p^*\chi(L_{\mathbb{C}}(V), U) = j^*U$, $\chi(L_{\mathbb{C}}(V), U)$ must be a generator of $H^2(P_{\mathbb{C}}(V); \mathbb{Z})$.

6.a) Define $g: \mathbb{R}^{n+1} - \{0\} \longrightarrow \mathbb{R}^{n+1} - \{0\}$ by $g(v) = f\left(\frac{v}{\|v\|}\right)$.

g is continuous. For any $t \in \mathbb{R} - \{0\}$ we have

$$g(tv) = f\left(\frac{tv}{\|tv\|}\right) = f\left(\frac{t}{|t|}\frac{v}{\|v\|}\right) = \frac{t}{|t|} f\left(\frac{v}{\|v\|}\right)$$ by equivariance

of f, and $\left[\frac{t}{|t|} f\left(\frac{v}{\|v\|}\right)\right] = \left[f\left(\frac{v}{\|v\|}\right)\right]$ in $\mathbb{R}P^n$.

Thus F_B is well-defined, and is continuous since it arises from g by passage to quotients.

Define $h: (\mathbb{R}^{m+1} - \{0\}) \times \mathbb{R}^{m+1} \longrightarrow (\mathbb{R}^{n+1} - \{0\}) \times \mathbb{R}^{n+1}$ by $h(v, w) = \left(f\left(\frac{v}{\|v\|}\right), \frac{\langle v, w\rangle}{\|v\|} f\left(\frac{v}{\|v\|}\right)\right)$. h is continuous, and the second component of this last expression is always a scalar multiple of the first.

For any $t \in \mathbb{R} - \{0\}$ we have, using the equivariance of f,

$$\begin{aligned} h(tv, w) &= \left(f\left(\frac{tv}{\|tv\|}\right), \frac{\langle tv, w\rangle}{\|tv\|} f\left(\frac{tv}{\|tv\|}\right)\right) \\ &= \left(\frac{t}{|t|} f\left(\frac{v}{\|v\|}\right), \frac{t}{|t|}\frac{\langle v, w\rangle}{\|v\|}\frac{t}{|t|} f\left(\frac{v}{\|v\|}\right)\right) \\ &= \left(\frac{t}{|t|} f\left(\frac{v}{\|v\|}\right), \frac{\langle v, w\rangle}{\|v\|} f\left(\frac{v}{\|v\|}\right)\right) \end{aligned}$$

Since $\left[\frac{t}{|t|} f\left(\frac{v}{\|v\|}\right)\right] = \left[f\left(\frac{v}{\|v\|}\right)\right]$ in $\mathbb{R}P^n$, F_E is well-defined

and is continuous since it arises, by passage to quotients, from the restriction of h to $\{(v, w) \mid w = \text{a scalar multiple of } v\}$.

Commutativity of
$$\begin{array}{ccc} L_{\mathbb{R}}(\mathbb{R}^{m+1}) & \xrightarrow{F_E} & L_{\mathbb{R}}(\mathbb{R}^{n+1}) \\ \downarrow & & \downarrow \\ P_{\mathbb{R}}(\mathbb{R}^{m+1}) & \xrightarrow[F_B]{} & P_{\mathbb{R}}(\mathbb{R}^{n+1}) \end{array}$$
is immediate. Finally to check that F_E maps fibres isomorphically onto fibres it suffices to check (since the fibres are 1-dimensional) that $F_E([v], rw) = r F_E([v], w)$ for $r \in \mathbb{R}$, $([v], w) \in L_{\mathbb{R}}(\mathbb{R}^{m+1})$, and that $F_E([v], w) \neq 0$ for $w \neq 0$.

We have
$$F_E([v], rw) = \left(\left[f\left(\tfrac{v}{\|v\|}\right) \right], \frac{\langle v, rw \rangle}{\|v\|} f\left(\tfrac{v}{\|v\|}\right) \right)$$
$$= \left(\left[f\left(\tfrac{v}{\|v\|}\right) \right], r \frac{\langle v, w \rangle}{\|v\|} f\left(\tfrac{v}{\|v\|}\right) \right)$$
$$= r \left(\left[f\left(\tfrac{v}{\|v\|}\right) \right], \frac{\langle v, w \rangle}{\|v\|} f\left(\tfrac{v}{\|v\|}\right) \right) = r F_E([v], w).$$

If $w \neq 0$, then $w = tv$ for some $t \neq 0$, and 2nd component of $F_E([v], w) = \dfrac{\langle v, tv \rangle}{\|v\|} f\left(\tfrac{v}{\|v\|}\right) = t \|v\| f\left(\tfrac{v}{\|v\|}\right) \neq 0$, and so $F_E([v], w) \neq 0$.

b) If an equivariant $S^m \to S^n$ existed, by part a) there would exist a map of real 1-plane bundles
$$\begin{array}{ccc} L_{\mathbb{R}}(\mathbb{R}^{m+1}) & \xrightarrow{F_E} & L_{\mathbb{R}}(\mathbb{R}^{n+1}) \\ \downarrow & & \downarrow \\ \mathbb{R}P^m & \xrightarrow{F_B} & \mathbb{R}P^n \end{array}$$
Write $u =$ generator of $H^1(\mathbb{R}P^m; \mathbb{Z}/2)$, $v =$ generator of $H^1(\mathbb{R}P^n; \mathbb{Z}/2)$. By problem 5, $u = \chi(L_{\mathbb{R}}(\mathbb{R}^{m+1}), \mathbb{Z}/2\text{-orient.})$, $v = \chi(L_{\mathbb{R}}(\mathbb{R}^{n+1}), \mathbb{Z}/2\text{-orient.})$, and by problem 4, $F_B^*(v) = u$. Thus, since F_B^* is a ring homomorphism,

$F_B^*(v^m) = u^m$. If $m > n$, $v^m = 0$ since $H^m(\mathbb{R}P^n; \mathbb{Z}/2) = 0$, and so we would have $u^m = 0$. But it was proved in lecture that $u^m \neq 0$, contradiction.

c) Suppose $f: S^n \to \mathbb{R}^n$ is continuous, and $f(x) \neq f(-x)$ for all $x \in S^n$. Then $g: S^n \to S^{n-1}$ given by

$$g(x) = \frac{f(x) - f(-x)}{\| f(x) - f(-x) \|}$$ would be a continuous

equivariant map, which is impossible by part b).

<u>Note</u>: When $n=2$, the Borsuk-Ulam Theorem has the following meteorological consequence: If temperature and atmospheric pressure at any given moment are assumed to be continuous functions of position on the surface of the earth, then at every moment there exists a pair of antipodal points on the surface of the earth at which the temperature and atmospheric pressure are the same.

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7. Proceed by induction on the presentability (no. of charts required) of the bundle.

The 1-presentable case: Case 1: B path-connected. We can suppose $E = B\times F$ and $p: B\times F \to B$ is projection on the first factor. Each E_x is then identified with F, $x\in B$, and $i_x : F \to B\times F$ is given by $i_x(y) = (x,y)$. By hypothesis, each $H^i(F;R)$ is a free R-module on a finite no. of generators. Thus the cross-product map $\bar{\mu}: [H^*(B;R)\otimes_R H^*(F;R)]^n \longrightarrow H^n(B\times F;R)$ is an isomorphism. (This would be immediate from the cohomology Künneth theorem proved in the lectures if we assumed each $H_i(F;R)$ is a free and finitely-generated R-module. The general case needs more technical argument. Actually, I would have been satisfied with a correct proof assuming each $H_i(F,R)$ is a free and finitely-generated R-module. Sorry about that.)

Choose a base-point $x_0\in B$ and write $\tilde{e}_i = i_{x_0}^*(e_i)$, $f_i = \pi_2^*(\tilde{e}_i) = 1\times\tilde{e}_i$. Let

$$\tilde{\Psi}_n : \bigoplus_{i=1}^{k} H^{n-n_i}(B;R) \to H^n(E;R)$$

be given by

$$\tilde{\Psi}_n(b_1,\dots,b_k) = \sum_{i=1}^{k} p^*(b_i)\cup f_i = \sum_{i=1}^{k}(b_i\times 1)\cup(1\times\tilde{e}_i) = \sum_{i=1}^{k} b_i\times\tilde{e}_i .$$

Thus, writing

$$\alpha : \bigoplus_{i=1}^{k} H^{n-n_i}(B;R)\to [H^*(B;R)\otimes_R H^*(F;R)]^n$$

for the map $\alpha(b_1,\dots,b_k) = \sum_{i=1}^{k} b_i\otimes\tilde{e}_i$, α is an isomorphism and the diagram

$$\begin{tikzcd}
\bigoplus_{i=1}^{k} H^{n-n_i}(B;R) \arrow[rr, "\alpha", "\cong"'] \arrow[dr, "\tilde{\Psi}_n"'] & & {[H^*(B;R)\otimes_R H^*(F;R)]^n} \arrow[dl, "\cong"', "\bar{\mu}"] \\
& H^n(B\times F;R) &
\end{tikzcd}$$

commutes. Thus $\tilde{\Psi}_n$ is an isomorphism. In particular, for each i there exist unique $c_{ij}\in H^*(B;R)$ such that

$$e_i = \sum_{j=1}^{k} p^*(c_{ij})\cup f_j = \sum_{j=1}^{k} c_{ij}\times\tilde{e}_j . \quad \text{Then}$$

$$\tilde{e}_i = i_{x_0}^*(e_i) = \sum_{j=1}^{k} i_{x_0}^* p^*(c_{ij})\cup i_{x_0}^* f_j = \sum_{j=1}^{k}(p\, i_{x_0})^*(c_{ij})\cup i_{x_0}^*\pi_2^*(\tilde{e}_j).$$

Now pi_{x_0} is a constant map, and so $(pi_{x_0})^*$ is 0 in positive dimensions. Since $\pi_2 i_{x_0} = 1_F$, $i_{x_0}^* \pi_2^*(\tilde{e}_j) = \tilde{e}_j$. Thus

$\tilde{e}_i = \sum_{\substack{j \ni: \\ n_j = n_i}} (pi_{x_0})^*(c_{ij}) \cup \tilde{e}_j$. Since each $c_{ij} \in H^0(B;R)$

whenever $n_j = n_i$, we have $c_{ij} = m_{ij} 1$ for some $m_{ij} \in R$ (since B is path-connected), and so

$\tilde{e}_i = \sum_{\substack{j \ni: \\ n_j = n_i}} m_{ij} \tilde{e}_j$. Since $H^{n_i}(F;R)$ is a free R-module

with basis $\{\tilde{e}_j \mid n_j = n_i\}$, it follows that $m_{ii} = 1$ and $m_{ij} = 0$ if $j \neq i$. Thus

$$e_i = 1 \times \tilde{e}_i + \sum_{\substack{j \ni: \\ n_j < n_i}} c_{ij} \times \tilde{e}_j ,$$

and it follows easily that

$$(*) \quad \psi_n(b_1, \dots, b_k) = \sum_{i=1}^{k} b_i \times \tilde{e}_i + \sum_{i=1}^{k} \sum_{\substack{j \ni: \\ n_j < n_i}} (b_i \cup c_{ij}) \times \tilde{e}_j$$

$$= \bar{\mu}\left(\sum_{i=1}^{k} b_i \otimes \tilde{e}_i + \sum_{i=1}^{k} \sum_{\substack{j \ni: \\ n_j < n_i}} (b_i \cup c_{ij}) \otimes \tilde{e}_j \right).$$

Proof that ψ_n is 1-1: If $\psi_n(b_1, \dots, b_k) = 0$, then since $\bar{\mu}$ is an isomorphism, it follows from $(*)$ that

$$\sum_{i=1}^{k} b_i \otimes \tilde{e}_i + \sum_{i=1}^{k} \sum_{\substack{j \ni: \\ n_j < n_i}} (b_i \cup c_{ij}) \otimes \tilde{e}_j = 0 .$$

Suppose $(b_1, \dots, b_k) \neq 0$. Let $m = \max\{n_i \mid b_i \neq 0\}$. Then for each $(b_i \cup c_{ij}) \otimes \tilde{e}_j$ in the right-hand summation for which $n_j = m$, we have $n_i > m$ and so $b_i = 0$. Thus

$\sum_{\substack{i \ni: \\ n_i = m}} b_i \otimes \tilde{e}_i = 0$. Thus, since α is an isomorphism, each b_i such that $n_i = m$ must be 0, contradiction. Thus $(b_1, \dots, b_k) = 0$.

Proof that Ψ_n is onto: Since $\tilde{\Psi}_n$ is onto, suffices to show each $b \times \tilde{e}_i$ lies in the image of Ψ_n. We proceed by induction on n_i. If $n_i = 0$, $(*)$ yields

$\Psi_n(0,\dots,0,\underset{i}{b},0,\dots,0) = b \times \tilde{e}_i$. Suppose $n_i > 0$ and, inductively, $c \times \tilde{e}_j \in \operatorname{im} \Psi_n$ whenever $n_j < n_i$. $(*)$ yields

$$\Psi_n(0,\dots,0,\underset{i}{b},0,\dots,0) = b \times \tilde{e}_i + \sum_{\substack{j \ni \\ n_j < n_i}} (b \cup c_{ij}) \times \tilde{e}_j .$$ Since

each $(b \cup c_{ij}) \times \tilde{e}_j \in \operatorname{im} \Psi_n$ by the induction hypothesis, it follows that $b \times \tilde{e}_i \in \operatorname{im} \Psi_n$.

The general 1-presentable case: Let $\{B_\alpha\}_{\alpha \in I}$ be the path-components of B, $j_\alpha : B_\alpha \to B$ the inclusion maps, and $e_i^\alpha = (i_\alpha \times 1_F)^*(e_i) \in H^{n_i}(B_\alpha \times F; R)$. Then each $\Psi_n^\alpha : \bigoplus_{i=1}^{k} H^{n-n_i}(B_\alpha; R) \to H^n(B_\alpha \times F; R)$, given by $\Psi_n^\alpha(b_1,\dots,b_k) = \sum_{i=1}^{k} \pi_1^*(b_i) \cup e_i^\alpha$, is an isomorphism by case 1. Moreover, it is easily checked that following commutes:

$$\begin{tikzcd}
\prod_\alpha \bigoplus_{i=1}^{k} H^{n-n_i}(B_\alpha; R) \arrow[r, "\prod_\alpha \Psi_n^\alpha", "\cong"'] & \prod_\alpha H^n(B_\alpha \times F; R) \\
\bigoplus_{i=1}^{k} H^{n-n_i}(B, R) \arrow[u, "(i_\alpha^*)"', "\cong"] \arrow[r, "\Psi_n"'] & H^n(B \times F; R) \arrow[u, "\cong"', "((i_\alpha \times 1_F)^*)"]
\end{tikzcd}$$

Thus Ψ_n is an isomorphism.

Now suppose the given bundle is m-presentable with $m > 1$. Assume result true for $m-1$-presentable bundles. We can write $B = B_1 \cup B_2$ where B_1, B_2 are open in B and the given bundle restricted to B_j is $m-1$-presentable, $j = 1, 2$. Write $B_{12} = B_1 \cap B_2$. Then the given bundle restricted to B_{12} is also $m-1$-presentable. Write $E_j = p^{-1}(B_j)$, $j = 1, 2, 12$, $p_j : E_j \to B_j$ for the projections, and $k_j : E_j \to E$ for the inclusions. Let $e_i^{(j)} = k_j^*(e_i)$ and

$$\Psi_n^j : \bigoplus_{i=1}^{k} H^{n-n_i}(B_j ; R) \to H^n(E_j ; R) \text{ be given by}$$

$$\Psi_n^j(b_1, \ldots, b_n) = \sum_{i=1}^{k} p_j^*(b_i) \cup e_i^{(j)}.$$ By the inductive hypothesis, the Ψ_n^j are isomorphisms for $j = 1, 2, 12$ and all n. By naturality of cup products and the behovior of cup products with respect to connecting homomorphisms in Mayer-Vietoris sequences, the following commutes (R coefficients):

$$\begin{array}{ccccccccc}
\bigoplus_{i=1}^{k}\left[H^{n-1-n_i}(B_1) \oplus H^{n-1-n_i}(B_2)\right] & \to & \bigoplus_{i=1}^{k} H^{n-1-n_i}(B_{12}) & \to & \bigoplus_{i=1}^{k} H^{n-n_i}(B) & \to & \bigoplus_{i=1}^{k}\left[H^{n-n_i}(B_1) \oplus H^{n-n_i}(B_2)\right] & \to & \bigoplus_{i=1}^{k} H^{n-n_i}(B_{12}) \\
\cong \downarrow \Psi_{n-1}^1 \oplus \Psi_{n-1}^2 & & \cong \downarrow \Psi_{n-1}^{12} & & \downarrow \Psi_n & & \cong \downarrow \Psi_n^1 \oplus \Psi_n^2 & & \cong \downarrow \Psi_n^{12} \\
H^{n-1}(E_1) \oplus H^{n-1}(E_2) & \to & H^{n-1}(E_{12}) & \to & H^n(E) & \to & H^n(E_1) \oplus H^n(E_2) & \to & H^n(E_{12})
\end{array}$$

where top row is the direct sum of Mayer-Vietoris sequences for $(B; B_1, B_2)$ and bottom row is the Mayer-Vietoris sequence for $(E; E_1, E_2)$. The result now follows by the 5-lemma.

Note: This result is known as the Leray-Hirsch Theorem.

8. For each $x \in B$, the inclusions $L_{\mathbb{R}}(E_x) \hookrightarrow L_{\mathbb{R}}(E)$, $P_{\mathbb{R}}(E_x) \xhookrightarrow{i_x} P_{\mathbb{R}}(E)$ constitute a map of real line bundles

$$\begin{array}{ccc}
L_{\mathbb{R}}(E_x) & & L_{\mathbb{R}}(E) \\
\downarrow & \to & \downarrow \\
P_{\mathbb{R}}(E_x) & & P_{\mathbb{R}}(E)
\end{array}.$$

Write $\chi_x = i_x^* \chi(L_{\mathbb{R}}(E), u)$. Then by naturality of the Euler class (Problem 4), $\chi_x = \chi(L_{\mathbb{R}}(E_x), i_x^* u)$ which, by Problem 5, is the generator of $H^1(P_{\mathbb{R}}(E_x); \mathbb{Z}/2)$. From the lectures, the elements $1, \chi_x, \chi_x^2, \ldots, \chi_x^{n-1}$ form a $\mathbb{Z}/2$-basis for $H^*(P_{\mathbb{R}}(E_x); \mathbb{Z}/2)$. The conclusion now follows from Problem 7.

9. If P is a point, then $[P, Y]$ has exactly one element, and so must be the identity element of the group $[P, Y]$. If X is any pointed space and $c_X : X \to Y$ the constant pointed map, then c_X factors as

$$\begin{tikzcd} & P \arrow[dr, "c_P"] & \\ X \arrow[ur, "c'"] \arrow[rr, "c_X"'] & & Y \end{tikzcd}$$

and so $[c_X] = (c')^*[c_P] = (c')^*(\text{identity element})$
$= \text{identity element of } [X, Y]$ since $(c')^*$ is a group homomorphism.

Let $\pi_i : Y \times Y \to Y$ denote the projections on the 1^{st} and 2^{nd} factors, resp. Let $\mu : Y \times Y \to Y$ be any representative of $[\pi_1] * [\pi_2]$ where $*$ denotes the group operation in $[Y \times Y, Y]$, and $\eta : Y \to Y$ any representative of $[1_Y]^{-1}$, where the inverse is with respect to the group operation in $[Y, Y]$. We first check that the following commute up to pointed homotopy:

i)
$$\begin{tikzcd} Y \vee Y \arrow[d, "i"'] \arrow[dr, "\varphi"] & \\ Y \times Y \arrow[r, "\mu"'] & Y \end{tikzcd}$$

ii)
$$\begin{tikzcd} Y \times Y \times Y \arrow[r, "\mu \times 1_Y"] \arrow[d, "1_Y \times \mu"'] & Y \times Y \arrow[d, "\mu"] \\ Y \times Y \arrow[r, "\mu"'] & Y \end{tikzcd}$$

iii)
$$\begin{tikzcd} & Y \times Y \arrow[r, "1_Y \times \eta"] & Y \times Y \arrow[dr, "\mu"] & \\ Y \arrow[ur, "d"] \arrow[rrr, "c_Y"] \arrow[dr, "d"'] & & & Y \\ & Y \times Y \arrow[r, "\eta \times 1_Y"'] & Y \times Y \arrow[ur, "\mu"'] & \end{tikzcd}$$

Proof of i): Note that i) is equivalent to $\simeq_0$-commutativity of the 2 diagrams

$$\begin{tikzcd} Y \arrow[d, "i_j"'] \arrow[dr, "1_Y"] & \\ Y\times Y \arrow[r, "\mu"'] & Y \end{tikzcd} \quad , j=1,2$$

where $i_1(y)=(y,y_0)$, $i_2(y)=(y_0,y)$, y_0 the base-point of Y.
We have $[\mu i_1] = i_1^*[\mu] = i_1^*([\pi_1]*[\pi_2])$

$= i_1^*[\pi_1] * i_1^*[\pi_2]$ (since i_1^* is a group hom.)
$= [\pi_1 i_1] * [\pi_2 i_1] = [1_Y]*[c_Y]$
$= [1_Y]$ since $[c_Y]$ = identity element of $[Y,Y]$.
Thus $\mu i_1 \simeq_0 1_Y$. Similarly, $\mu i_2 \simeq_0 1_Y$.

Proof of ii): Write $\rho_i : Y\times Y\times Y \to Y$ for projection on the i^{th} factor, $i=1,2,3$, and $\rho_{12}: Y\times Y\times Y \to Y\times Y$ for the map $\rho_{12}(y_1,y_2,y_3) = (y_1,y_2)$. We have
$[\mu(\mu\times 1_Y)] = (\mu\times 1_Y)^*[\mu] = (\mu\times 1_Y)^*([\pi_1]*[\pi_2])$

$= (\mu\times 1_Y)^*[\pi_1] * (\mu\times 1_Y)^*[\pi_2]$ (since $(\mu\times 1_Y)^*$ is a group hom.)
$= [\pi_1(\mu\times 1_Y)] * [\pi_2(\mu\times 1_Y)] = [\mu\rho_{12}] * [\rho_3]$
$= \rho_{12}^*[\mu] * [\rho_3] = \rho_{12}^*([\pi_1]*[\pi_2]) * [\rho_3]$
$= \rho_{12}^*[\pi_1] * \rho_{12}^*[\pi_2] * [\rho_3]$ (since ρ_{12}^* is a group hom).
$= [\pi_1\rho_{12}] * [\pi_2\rho_{12}] * [\rho_3] = [\rho_1]*[\rho_2]*[\rho_3]$.
Similarly, $[\mu(1_Y\times\mu)] = [\rho_1]*[\rho_2]*[\rho_3]$, proving ii).

Proof of iii): $[\mu(1_Y\times\eta)d] = ((1_Y\times\eta)d)^*[\mu] = ((1_Y\times\eta)d)^*([\pi_1]*[\pi_2])$
$= [\pi_1(1_Y\times\eta)d] * [\pi_2(1_Y\times\eta)d]$ (since $((1_Y\times\eta)d)^*$ is a group hom.)
$= [1_Y]*[\eta] = [1_Y]*[1_Y]^{-1}$ (def. of η)
$= [c_Y]$ (since c_Y = id. elt. of group $[Y,Y]$).
Similarly, $[\mu(\eta\times 1_Y)d] = [c_Y]$.

Thus Y is an H^*-space with above μ as multiplication, η as inversion.

If $f, g : X \to Y$ are pointed maps, let $f \circ g$ denote the composition
$$X \xrightarrow{d} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow{\mu} Y.$$
Then
$$[f \circ g] = [\mu (f\times g) d] = ((f\times g)d)^* [\mu] = ((f\times g)d)^* ([\pi_1] * [\pi_2])$$
$$= [\pi_1 (f\times g) d] * [\pi_2 (f \times g) d] \quad (\text{since } ((f\times g)d)^* \text{ is a group hom.})$$
$= [f] * [g]$, proving the group structure on $[X, Y]$ arises from the above H^*-structure on Y via the $\circ$-construction.

10. Proof that $[X, Y]$ is a group under the given operation:

i) Let $f : X \to Y$ be pointed. Consider

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "d"'] \arrow[rrr, bend left=40, "f \square *"] \arrow[rrr, bend right=70, "f"'] & X \vee X \arrow[r, "f \vee *"'] \arrow[d, "i"] & Y \vee Y \arrow[r, "\varphi"] & Y \\
& X \times X \arrow[r, "\pi_1"'] & X \arrow[ur, "f"'] &
\end{tikzcd}$$

(1), (2), (3) commute. (4) $\simeq_0$-commutes by 1^{st} property of co-H^*-spaces. Thus $f \square * \simeq_0 f$ and so $[f][*] = [f]$. Similarly, $[*][f] = [f]$. Thus $[*]$ is a unit element for the given operation in $[X, Y]$.

ii) Let $f, g, h : X \to Y$ be pointed. Consider

$$\begin{tikzcd}
& X \vee X \arrow[dr, "1 \vee \mu'"] \arrow[rrr, "f \vee (g \square h)"] & & & Y \vee Y \arrow[dr, "\varphi"] & \\
X \arrow[ur, "\mu'"] \arrow[dr, "\mu'"'] \arrow[rrrrr, bend left=50, "f \square (g \square h)"] \arrow[rrrrr, bend right=50, "(f \square g) \square h"'] & & X \vee X \vee X \arrow[r, "f \vee g \vee h"] & Y \vee Y \vee Y \arrow[ur, "1 \vee \varphi"] \arrow[dr, "\varphi \vee 1"'] & & Y \\
& X \vee X \arrow[ur, "\mu' \vee 1"] \arrow[rrr, "(f \square g) \vee h"'] & & & Y \vee Y \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

①, ②, ③, ④, ⑤ commute. ⑥ $\simeq_0$ commutes by 2^{nd} property for co-H^*-spaces. Thus $f\square(g\square h)\simeq_0(f\square g)\square h$ and so $([f][g])[h]=[(f\square g)\square h]=[f\square(g\square h)]=[f]([g][h])$, and so the given operation in $[X,Y]$ is associative.

(iii) For any pointed $f : X\to Y$, consider

$$\begin{tikzcd}
X \arrow[r, "\mu'"] \arrow[rrrr, bend left, "f\square(f\eta')"] \arrow[ddrr, bend right, "*"'] & X\vee X \arrow[rr, "f\vee f\eta'"] \arrow[d, "1\vee\eta'"] & & Y\vee Y \arrow[r, "\varphi"] & Y \\
& X\vee X \arrow[dr, "\varphi"] & & & \\
& & X \arrow[uurr, "f"'] & &
\end{tikzcd}$$

① and ② commute. ③ $\simeq_0$-commutes by 3^{rd} property of co-H^*-spaces. Thus $f\square(f\eta')\simeq_0 *$. Similarly $(f\eta')\square f\simeq_0 *$. Thus $[f\eta']$ is an inverse to $[f]$ with respect to the given operation in $[X,Y]$.

Thus, $[X,Y]$ is a group with respect to the given operation.

For any pointed X, $[X,\ ]$ is a covariant functor from the category of pointed spaces to the category of pointed sets. To complete the proof, it must be checked that if $f : Y\to Z$ is a pointed map, then $f_* : [X,Y]\to[X,Z]$ is a group homomorphism.

Let $g,h : X\to Y$ be pointed. It is easily checked that

$$f(g\square h)=(fg)\square(fh).\quad \text{Thus}$$

$$f_*([g][h])=f_*[g\square h]=[f(g\square h)]=[(fg)\square(fh)]$$

$$=[fg][fh]=f_*[g]\,f_*[h]\text{, completing the proof.}$$

11. Let (X, A, B) be a pointed triple.
(I) Exactness of $\pi_n(A; B) \xrightarrow{i_*} \pi_n(X, B) \xrightarrow{j_*} \pi_n(X, A)$, $n \geq 1$:
Note that ji factors through (A, A), and $\pi_n(A, A) = 0$. Hence $j_* i_* = 0$, and so $\mathrm{im}\, i_* \subset \ker j_*$.
Proof that $\ker j_* \subset \mathrm{im}\, i_*$:
Case 1: $n \geq 2$. We have the commutative diagram

$$\begin{array}{ccccc}
\pi_n(A) & \xrightarrow{q_*} & \pi_n(X) & \xrightarrow{r_*} & \pi_n(X, A) \\
p_* \downarrow & & \downarrow l_* & & \downarrow 1 \\
\pi_n(A, B) & \xrightarrow{i_*} & \pi_n(X, B) & \xrightarrow{j_*} & \pi_n(X, A) \\
\partial_1 \downarrow & & \downarrow \partial_2 & & \downarrow \partial_3 \\
\pi_{n-1}(B) & \xrightarrow{1} & \pi_{n-1}(B) & \xrightarrow{k_*} & \pi_{n-1}(A) \\
k_* \downarrow & & & & \\
\pi_{n-1}(A) & & & &
\end{array}$$

All maps are group homomorphisms since $n \geq 2$. The left and middle columns, and the top row are exact, being portions of homotopy sequences of appropriate pairs. Let $a \in \ker j_*$. Then from commutativity of the lower right rectangle, $\partial_2 a \in \ker k_*$. Thus by exactness of the left column, $\partial_2 a = \partial_1 b$ for some $b \in \pi_n(A, B)$. Then $a - i_*(b) \in \ker \partial_2$ and so $a - i_*(b) = l_*(c)$ for some $c \in \pi_n(X)$ (exactness of middle column). Since we already know $j_* i_* = 0$ we have $j_* i_*(b) = 0$, and hence $j_*(a - i_*(b)) = 0$. It follows that $r_*(c) = 0$ and so, by exactness of the top row, there exists a $d \in \pi_n(A)$ such that $q_*(d) = c$. Then $i_*(p_*(d) + b) = a$ and so $\ker j_* \subset \mathrm{im}\, i_*$ if $n \geq 2$.
Case 2: $n = 1$. $(S^0 \wedge I, S^0) \cong (I, \{0, 1\})$ and so we can regard $\pi_1(X, B)$ as $[(I, \{0,1\}), (X, B)]$. Take 0 as base point and suppose $f: (I, \{0,1\}) \to (X, B)$ is a pointed map such that $j_*[f] = *$. Then there exists a pointed homotopy $h: I \times I \to X$ such that $h(s, 0) = f(s)$, $h(s, 1) = *$, and $h(1, t) \in A$ for all $t \in I$. Define $H: I \times I \to X$ by

$$H(s, t) = \begin{cases} h\left(\frac{2s}{2-t}, t\right) & \text{if } 0 \leq s \leq 1 - \frac{t}{2} \\ h(1, 2-2s) & \text{if } 1 - \frac{t}{2} \leq s \leq 1. \end{cases}$$

H is continuous. Note that $H(0,t) = h(0,t) = *$ for all t since h is a pointed homotopy,

$H(1,t) = h(1,0) = f(1) \in B$ for all t,

$H(s,0) = h(s,0) = f(s)$,

$$H(s,1) = \begin{cases} h(2s,1) & \text{if } 0 \le s \le \frac{1}{2} \\ h(1,2-2s) & \text{if } \frac{1}{2} \le s \le 1 \end{cases} = \begin{cases} * & \text{if } 0 \le s \le \frac{1}{2} \\ \in A & \text{if } \frac{1}{2} \le s \le 1 \end{cases}.$$

Define $g : (I, \{0,1\}) \to (A, B)$ by $g(s) = H(s,1)$. Then

$H : (I, \{0,1\}) \times I \to (X, B)$ is a pointed homotopy from f to ig, and so $[f] \in \operatorname{im} i_*$.

(II) Exactness of $\pi_{n+1}(X,A) \xrightarrow{\bar\partial} \pi_n(A,B) \xrightarrow{i_*} \pi_n(X,B)$, $n \ge 1$:

By definition, $\bar\partial$ is the composition

$$\pi_{n+1}(X,A) \xrightarrow{\partial} \pi_n(A) \xrightarrow{k_*} \pi_n(A,B)$$

where ∂ is from the homotopy sequence of the pair (X,A).

We have the commutative diagram

$$\begin{tikzcd} \pi_{n+1}(X,A) \arrow[r, "\partial"] \arrow[dr, "\bar\partial"'] & \pi_n(A) \arrow[r] \arrow[d, "k_*"] & \pi_n(X) \arrow[d] \\ & \pi_n(A,B) \arrow[r, "i_*"'] & \pi_n(X,B) \end{tikzcd}$$

with the top row exact. Hence $i_* \bar\partial = 0$ and so $\operatorname{im} \bar\partial \subset \ker i_*$.

Proof that $\ker i_* \subset \operatorname{im} \bar\partial$:

Case 1: $n \ge 2$. We have the commutative diagram

$$\begin{tikzcd} & \pi_n(B) \arrow[r, "1"] \arrow[d] & \pi_n(B) \arrow[d] \\ \pi_{n+1}(X,A) \arrow[r, "\partial"] \arrow[dr, "\bar\partial"'] & \pi_n(A) \arrow[r] \arrow[d, "k_*"] & \pi_n(X) \arrow[d] \\ & \pi_n(A,B) \arrow[r, "i_*"] \arrow[d, "\partial"] & \pi_n(X,B) \arrow[d] \\ & \pi_{n-1}(B) \arrow[r, "1"'] & \pi_{n-1}(B) \end{tikzcd}$$

with both columns and the 2^{nd} row exact. Since $n \geq 2$, all the maps are group homomorphisms. The result now follows from chasing in the above diagram, similar to that of part (I).

Case 2: $n = 1$. The only difficulty with the argument in Case 1 is that $k_* : \pi_1(A) \to \pi_1(A,B)$ is not a group homomorphism (since $\pi_1(A,B)$ is not a group). However $\pi_1(A)$ is a group. The argument given in Case 1 can be carried out if it is shown that whenever $a, b \in \pi_1(A)$ with $b \in \operatorname{im}\left[\pi_1(B) \xrightarrow{\ell_*} \pi_1(A)\right]$, then $k_*(a+b) = k_*(a)$.

Let $f : (I, \{0,1\}) \to (A, *)$, $g : (I, \{0,1\}) \to (B, *)$ be continuous. Then $[f] + \ell_*[g]$ is represented by $(f * g) : (I, \{0,1\}) \to (A, *)$ where

$$(f*g)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Define $h : I \times I \to A$ by

$$h(s,t) = \begin{cases} f\left(\frac{2s}{2-t}\right) & \text{if } 0 \leq s \leq 1 - \frac{t}{2} \\ g(2s + t - 2) & \text{if } 1 - \frac{t}{2} \leq s \leq 1 \end{cases}$$

Note that $h(0,t) = *$, $h(1,t) \in B$ for all t, and

$$kf \underset{h}{\simeq_0} k(f*g).$$

(III) Exactness of $\pi_{n+1}(X,B) \xrightarrow{j_*} \pi_{n+1}(X,A) \xrightarrow{\bar{\partial}} \pi_n(A,B)$, $n \geq 1$:
we have the commutative diagram

$$\begin{array}{ccccc}
\pi_{n+1}(X,B) & \xrightarrow{j_*} & \pi_{n+1}(X,A) & & \\
\downarrow \partial & & \downarrow \partial & \searrow \bar{\partial} & \\
\pi_n(B) & \longrightarrow & \pi_n(A) & \xrightarrow{k_*} & \pi_n(A,B)
\end{array}$$

with the bottom row exact. Hence $\bar{\partial} j_* = 0$, and so $\operatorname{im} j_* \subset \ker \bar{\partial}$.

Proof that $\ker \bar\partial \subset \operatorname{im} j_*$: For all $n \geq 1$ we have the commutative diagram

$$\begin{tikzcd}
\pi_{n+1}(X) \arrow[r, "1"] \arrow[d] & \pi_{n+1}(X) \arrow[d] & \\
\pi_{n+1}(X,B) \arrow[r, "j_*"] \arrow[d, "\partial"'] & \pi_{n+1}(X,A) \arrow[d, "\partial"'] \arrow[dr, "\bar\partial"] & \\
\pi_n(B) \arrow[r] \arrow[d] & \pi_n(A) \arrow[r, "k_*"'] \arrow[d] & \pi_n(A,B) \\
\pi_n(X) \arrow[r, "1"] & \pi_n(X) &
\end{tikzcd}$$

with both columns and the 3rd row exact. The result follows from chasing in the above diagram (even when $n = 1$).

12. Let $\{ h_i : U_i \times \mathbb{R}^m \longrightarrow p_1^{-1}(U_i) \}_{i \in I}$ and

$\{ k_j : V_j \times \mathbb{R}^n \longrightarrow p_2^{-1}(V_j) \}_{j \in J}$ be vector bundle atlases for p_1 and p_2, respectively. For $(i,j) \in I \times J$, define

$l_{ij} : (U_i \cap V_j) \times \mathbb{R}^{m+n} \longrightarrow p^{-1}(U_i \cap V_j)$ by

$l_{ij}(x,(y,z)) = (h_i(x,y), k_j(x,z))$ where $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$. Then $\{ l_{ij} \}_{(i,j) \in I \times J}$ is a vector bundle atlas for p, and is finite if I and J are finite.

13. a) Write $q_1 : L(E_1) \to P(E_1)$, $q : L(E_1 \oplus E_2) \to P(E_1 \oplus E_2)$ for the projections. Then $q j_1([u], w) = q([u,0],(w,0)) = [u,0]$, $i_1 q_1([u],w) = i_1([u]) = [u,0]$ and so $q j_1 = i_1 q_1$.

For all $t \in \mathbb{R}$, $u \in (E_1)_0$ we have $j_1(t([u],u))$ $= j_1([u],tu) = ([u,0],(tu,0)) = ([u,0], t(u,0))$ $= t([u,0],(u,0)) = t j_1([u],u)$. Thus, since the fibre $L(E_1)_{[u]}$ is 1-dimensional with basis $([u],u)$, it follows that $j_1 | L(E_1)_{[u]}$ is a linear isomorphism onto $L(E_1 \oplus E_2)_{[u,0]}$. Thus i_1, j_1 constitute a map of real line bundles. Similarly for i_2, j_2.

b) Define $r : U \to P(E_1)$ by $r[u,v] = [u,0]$. (Note that $u \neq 0$ if $[u,v] \in U$). Then $r|P(E_1) = 1_{P(E_1)}$. Define $h : U \times I \to U$ by $h([u,v],t) = [u,tv]$. h is a homotopy from $i_1 r$ to 1_U, and so r is a deformation retraction of U onto $P(E_1)$.

Similarly, $P(E_2)$ is a deformation retract of V.

14. Let $(X,U) \xleftarrow{j_1} (X,\emptyset) \xrightarrow{j_2} (X,V)$ be the inclusions. From the cohomology sequences of (X,U) and (X,V) respectively, there exist $\bar{a} \in H^p(X,U;R)$, $\bar{b} \in H^q(X,V;R)$ such that $j_1^*(\bar{a}) = a$, $j_2^*(\bar{b}) = b$. By naturality of the cup product, the diagram

$$\begin{array}{ccc} H^p(X,U;R) \otimes_R H^q(X,V;R) & \xrightarrow{\smile} & H^{p+q}(X,U\cup V;R) \\ \downarrow{\scriptstyle j_1^* \otimes j_2^*} & & \downarrow{\scriptstyle j^*} \\ H^p(X;R) \otimes_R H^q(X;R) & \xrightarrow{\smile} & H^{p+q}(X;R) \end{array}$$

commutes, where j is the inclusion. Hence $a \smile b = j^*(\bar{a} \smile \bar{b})$. But since $U \cup V = X$, $H^{p+q}(X, U\cup V; R) = 0$ and so $j^*(\bar{a}\smile\bar{b}) = 0$.

15. We have the commutative diagram

$$\begin{tikzcd} P(E_1) \arrow[r, hook, "i_1"] \arrow[dr, "r_2"'] & P(E_1 \oplus E_2) \arrow[d, "r"] & P(E_2) \arrow[l, hook', "i_2"'] \arrow[dl, "r_1"] \\ & B & \end{tikzcd}$$

Write χ_{E_k} for the Euler class of $L(E_k)$ with respect to the unique $\mathbb{Z}/2$ orientation, $k = 1,2$. From problem 13c) and naturality of the Euler class, $i_k^* \chi_{E_1 \oplus E_2} = \chi_{E_k}$, $k = 1,2$. By definition of the w_i we have

$$\chi_{E_1}^m = \sum_{i=1}^{m} r_1^* w_i(E_1) \smile \chi_{E_1}^{m-i} \text{ and so } \sum_{i=0}^{m} r_1^* w_i(E_1) \smile \chi_{E_1}^{m-i} = 0.$$

Similarly $\sum_{j=0}^{n} r_2^* w_i(E_2) \smile \chi_{E_2}^{n-i} = 0$. Write

$$a = \sum_{i=0}^{m} r^* w_i(E_1) \smile \chi_{E_1\oplus E_2}^{m-i} \in H^m(P(E_1 \oplus E_2); \mathbb{Z}/2),$$

$$b = \sum_{j=0}^{n} r^* w_j(E_2) \cup \chi_{E_1 \oplus E_2}^{n-j} \in H^n(P(E_1 \oplus E_2); \mathbb{Z}/2)$$

Then $$i_1^*(a) = \sum_{i=0}^{m} i_1^* r^* w_i(E_1) \cup i_1^* \chi_{E_1 \oplus E_2}^{m-i}$$
$$= \sum_{i=0}^{m} r_1^* w_i(E_1) \cup \chi_{E_1}^{m-i} = 0$$ by naturality of the cup product. Similarly $i_2^*(b) = 0$. Let U, V be as in problem (3b), and

$$U \xrightarrow{j_1} P(E_1 \oplus E_2) \xleftarrow{j_2} V$$ the inclusions.

From commutativity of

$$\begin{tikzcd} P(E_1) \arrow[r, hook] \arrow[dr, "i_1"'] & U \arrow[d, "j_1"] \\ & P(E_1 \oplus E_2) \end{tikzcd}$$

and the fact that $P(E_1)$ is a deformation retract of U, it follows that $j_1^*(a) = 0$. Similarly $j_2^*(b) = 0$. Since $U \cup V = P(E_1 \oplus E_2)$, it follows from problem 14 that $a \cup b = 0$, i.e.

$$0 = \left(\sum_{i=0}^{m} r^* w_i(E_1) \cup \chi_{E_1 \oplus E_2}^{m-i}\right) \cup \left(\sum_{j=0}^{n} r^* w_j(E_2) \cup \chi_{E_1 \oplus E_2}^{n-j}\right)$$

$$= \sum_{k=0}^{m+n} r^* \left(\sum_{i=0}^{k} w_i(E_1) \cup w_{k-i}(E_2)\right) \cup \chi_{E_1 \oplus E_2}^{m+n-k}$$

Thus

$$\chi_{E_1 \oplus E_2}^{m+n} = \sum_{k=1}^{m+n} r^* \left(\sum_{i=0}^{k} w_i(E_1) \cup w_{k-i}(E_2)\right) \cup \chi_{E_1 \oplus E_2}^{m+n-k}$$

and so by definition, $w_k(E_1 \oplus E_2) = \sum_{i=0}^{k} w_i(E_1) \cup w_{k-i}(E_2)$.

16. Let E be the Leray-Serre spectral sequence with $\mathbb{Z}$-coefficients of the fibration $\Omega(S^2 \vee S^3) \to P(S^2 \vee S^3) \to S^2 \vee S^3$. ($S^2 \vee S^3$ is 1-connected, so above fibration is $\mathbb{Z}$-orientable). Since $P(S^2 \vee S^3)$ is contractible, $E^\infty_{p,q} = 0$ for $(p,q) \neq (0,0)$. Since

$$H_p(S^2 \vee S^3) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0,2,3 \\ 0 & \text{otherwise} \end{cases}$$

it follows, from the universal coefficient theorem, that

$$E^2_{p,q} \cong H_p(S^2 \vee S^3; H_q(\Omega(S^2 \vee S^3))) \cong \begin{cases} H_q(\Omega(S^2 \vee S^3)) & \text{if } p = 0,2,3 \\ 0 & \text{otherwise} \end{cases}.$$

Thus the only possible non-zero differentials are the $d^2: E^2_{2,q} \to E^2_{0,q+1}$ and $d^3: E^3_{3,q} \to E^3_{0,q+2}$. Thus $E^3_{2,q} = E^\infty_{2,q} = 0$ for all q, and so $E^2_{2,q} \xrightarrow{d^2} E^2_{0,q+1}$ is injective. Since $E^2_{0,q+1}/\operatorname{im} d^2 \cong E^3_{0,q+1}$, we have an exact sequence

$$(*) \qquad 0 \to E^2_{2,q} \xrightarrow{d^2} E^2_{0,q+1} \longrightarrow E^3_{0,q+1} \to 0$$

for all q.

For all (p,q) we have $E^\infty_{p,q} = E^4_{p,q}$. Thus since $E^\infty_{3,q-1}$ and $E^\infty_{0,q+1}$ are 0 for $q \geq 0$, we must have $E^3_{3,q-1} \xrightarrow{d^3} E^3_{0,q+1}$ is an isomorphism for all $q \geq 0$. Since $E^2_{3,q-1} \xrightarrow{d^2} E^2_{1,q}$ is 0, we have $E^2_{3,q-1} = E^3_{3,q-1}$ and hence $E^3_{0,q+1} \cong E^2_{3,q-1}$ for $q \geq 0$. Thus $(*)$ yields an exact sequence

$$0 \to E^2_{2,q} \longrightarrow E^2_{0,q+1} \longrightarrow E^2_{3,q-1} \to 0 \qquad \text{for all } q \geq 0,$$

i.e. an exact sequence

$$(**) \quad 0 \to H_q(X) \to H_{q+1}(X) \longrightarrow H_{q-1}(X) \to 0 \qquad \text{for all } q \geq 0$$

where $X = \Omega(S^2 \vee S^3)$. Since X is 0-connected (since $S^2 \vee S^3$ is 1-connected), $H_0(X) \cong \mathbb{Z}$. It follows from $(**)$, with $q = 0$, that $H_1(X) \cong \mathbb{Z}$.

Let $q \geq 1$ and assume inductively $H_i(X)$ is free abelian on f_i generators where f_i is the i^{th} Fibonacci number (i.e. $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$). Then $H_{q-1}(X)$ is free abelian and so $(**)$ splits, and hence

$$\begin{aligned} H_{q+1}(X) \cong H_q(X) \oplus H_{q-1}(X) &= \text{free abelian on } f_q + f_{q-1} \text{ generators} \\ &= \text{free abelian on } f_{q+1} \text{ generators.} \end{aligned}$$

Thus, it follows by induction that $H_n(X)$ is free abelian on f_n generators where $f_n = n^{\text{th}}$ Fibonacci number.

17. Let E be the Leray-Serre spectral sequence of $F \xrightarrow{i} T \xrightarrow{p} S^n$ with coefficients in R. We have

$$E^2_{p,q} \cong H_p(S^n; H_q(F;R)) \cong \begin{cases} H_q(F;R) & \text{if } p = 0 \text{ or } n \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by the Fibre Edge Theorem, it suffices to show the existence of a commutative diagram with exact row

$$\begin{tikzcd} E^2_{n,i-n+1} \arrow[r, "\bar{\beta}"] & E^2_{0,i} \arrow[rr] \arrow[dr, "p_F"'] & & H_i(T;R) \arrow[r, "\bar{\alpha}"] & E^2_{n,i-n} \arrow[r, "\bar{\beta}"] & E^2_{0,i-1} \\ & & E^\infty_{0,i} \arrow[ur, "i_F"'] & & & \end{tikzcd}$$

The only possible non-zero differentials are the $d^n : E^n_{n,q} \to E^n_{0,q+n-1}$. Thus $E^n_{p,q} = E^2_{p,q}$ and $E^{n+1}_{p,q} = E^\infty_{p,q}$ for all p, q.

Since $E^\infty_{n+j,i-n-j} = 0$ for $j > 0$ we have

$$J_{n,i-n} = J_{n+1,i-n-1} = \cdots = J_{i,0} = H_i(T;R).$$

Since $E^\infty_{j,i-j} = 0$ for $0 < j < n$ we have

$$E^\infty_{0,i} = J_{0,i} = J_{1,i-1} = \cdots = J_{n-1,i-n+1}.$$

Define $\bar{\alpha} : H_i(T;R) \to E^2_{n,i-n}$ to be the composition

$$H_i(T;R) = J_{i,0} = J_{n,i-n} \longrightarrow J_{n,i-n}/J_{n-1,i-n+1} = E^\infty_{n,i-n} \hookrightarrow E^n_{n,i-n} = E^2_{n,i-n}.$$

Define $\bar\beta : E^2_{n,i-n} \longrightarrow E^2_{0,i-1}$ to be the composition

$$E^2_{n,i-n} = E^n_{n,i-n} \xrightarrow{d^n} E^n_{0,i-1} = E^2_{0,i-1}.$$

Since p_F is onto and i_F is 1-1 it suffices to check exactness of

(I) $$E^2_{n,i-n+1} \xrightarrow{\bar\beta} E^2_{0,i} \xrightarrow{p_F} E^\infty_{0,i},$$

(II) $$E^\infty_{0,i} \xrightarrow{i_F} H_i(T;R) \xrightarrow{\bar\alpha} E^2_{n,i-n},$$

(III) $$H_i(T;R) \xrightarrow{\bar\alpha} E^2_{n,i-n} \xrightarrow{\bar\beta} E^2_{0,i-1}.$$

Exactness of (I) is immediate from exactness of

$$E^n_{n,i-n} \xrightarrow{d^n} E^n_{0,i-1} \longrightarrow E^\infty_{0,i-1}.$$

Exactness of (II) is immediate from exactness of

$$J_{n-1,i-n+1} \longrightarrow J_{n,i-n} \longrightarrow E^\infty_{n,i-n}.$$

Exactness of (III) is immediate from exactness of

$$E^\infty_{n,i-n} \hookrightarrow E^n_{n,i-n} \xrightarrow{d^n} E^n_{0,i-1}.$$

<u>Note</u>: The exact sequence of this problem is called the <u>Wang</u> sequence.

18. Let E be the Leray-Serre spectral sequence of $p:(T,T_0)\to(B,B_0)$. Then

$$E^2_{p,q} \cong H_p(B,B_0;H_q(S^n;R)) \cong \begin{cases} H_p(B,B_0;R) & \text{if } q=0,n \\ 0 & \text{otherwise.} \end{cases}$$

By the base-edge theorem it suffices to show the existence of a commutative diagram with exact row

$$\begin{tikzcd}
E^2_{i-n,n} \arrow[r, "\bar\delta"] & H_i(T,T_0;R) \arrow[rr] \arrow[dr, "p_B"'] & & E^2_{i,0} \arrow[r, "\bar\gamma"] & E^2_{i-n-1,n} \arrow[r, "\bar\delta"] & H_{i-1}(T,T_0;R) \\
& & E^\infty_{i,0} \arrow[ur, "i_B"'] & & &
\end{tikzcd}$$

The only possible non-zero differentials are the $d^{n+1}: E^{n+1}_{q,0} \to E^{n+1}_{q-n-1,n}$.
Thus $E^2_{p,q} = E^{n+1}_{p,q}$ and $E^\infty_{p,q} = E^{n+2}_{p,q}$ for all p, q.

Since $E^\infty_{i-j,j} = 0$ for $0 < j < n$ we have

$J_{i-1,1} = J_{i-2,2} = \cdots = J_{i-n,n}$. Since $E^\infty_{i-j,j} = 0$ for $j > n$ we have $J_{i-n-1,n+1} = J_{i-n-2,n+2} = \cdots = 0$, and $E^\infty_{i-n,n} = J_{i-n,n}$.

Define $\bar\gamma : E^2_{i,0} \to E^2_{i-n-1,n}$ to be the composition

$$E^2_{i,0} = E^{n+1}_{i,0} \xrightarrow{d^{n+1}} E^{n+1}_{i-n-1,n} = E^2_{i-n-1,n}.$$

Define $\bar\delta : E^2_{i-n,n} \to H_i(T;T_0;R)$ to be the composition

$$E^2_{i-n,n} = E^{n+1}_{i-n,n} \xrightarrow{\text{onto}} E^\infty_{i-n,n} = J_{i-n,n} = J_{i-1,1} \hookrightarrow J_{i,0} = H_i(T,T_0;R).$$

Since p_B is onto and i_B is 1-1 it suffices to check exactness of

(I) $\quad E^2_{i-n,n} \xrightarrow{\bar\delta} H_i(T,T_0;R) \xrightarrow{p_B} E^\infty_{i,0}$,

(II) $\quad E^\infty_{i,0} \xrightarrow{i_B} E^2_{i,0} \xrightarrow{\bar\gamma} E^2_{i-n-1,n}$,

(III) $\quad E^2_{i,0} \xrightarrow{\bar\gamma} E^2_{i-n-1,n} \xrightarrow{\bar\delta} H_{i-1}(T,T_0;R)$.

Exactness of (I) is immediate from exactness of

$$J_{i-1,1} \longrightarrow J_{i,0} \longrightarrow E^\infty_{i,0}.$$

Exactness of (II) is immediate from exactness of

$$E^\infty_{i,0} \to E^{n+1}_{i,0} \xrightarrow{d^{n+1}} E^{n+1}_{i-n-1,n}.$$

Exactness of (III) is immediate from exactness of

$$E^{n+1}_{i,0} \xrightarrow{d^{n+1}} E^{n+1}_{i-n-1,n} \longrightarrow E^\infty_{i-n-1,n}.$$

Note: The exact sequence of this problem is called the Gysin sequence.

19. Let E be the Leray - Serre spectral sequence of $F \to T \to B$ with $\mathbb{Q}$-coefficients. We have $E^2_{p,q} \cong H_p(B; H_q(F;\mathbb{Q})) \cong H_p(B;\mathbb{Q}) \underset{\mathbb{Q}}{\otimes} H_q(F;\mathbb{Q})$. Thus, since

the $H_p(B;\mathbb{Q})$ and $H_q(F;\mathbb{Q})$ are finite - dimensional over $\mathbb{Q}$ and non-zero for only finitely many (p,q), each $E^2_{p,q}$ is finite-dimensional over $\mathbb{Q}$ and non-zero for only finitely many (p,q). Thus since $E^{r+1}_{p,q}$ is a quotient of a sub-$\mathbb{Q}$-module of $E^r_{p,q}$, each $E^r_{p,q}$ is finite-dimensional over $\mathbb{Q}$ and non-zero for only finitely many (p,q) for $2 \le r \le \infty$. Thus, for all but a finite number of n, the $E^\infty_{p,n-p}$ are all 0. If n is such that the $E^\infty_{p,n-p}$ are all 0, it follows that

$$0 = J_{-1,n+1} = J_{0,n} = J_{1,n-1} = \quad = J_{n,0} = H_n(T;\mathbb{Q}),$$ and so the $H_n(T;\mathbb{Q})$ are 0 for all but finitely many n.

For a general n, exactness of $J_{p,n-p} \to J_{p+1,n-p-1} \to E^\infty_{p+1,n-p-1}$, and the facts that $E^\infty_{p+1,n-p-1}$ and $E^\infty_{0,n} = J_{0,n}$ are finite-dimensional over $\mathbb{Q}$, yields, by induction on p, that all the $J_{p,n-p}$ are finite-dimensional over $\mathbb{Q}$. In particular each $J_{n,0} = H_n(T;\mathbb{Q})$ is finite-dimensional over $\mathbb{Q}$.

Exactness of $0 \to J_{p-1,n-p+1} \to J_{p,n-p} \to E^\infty_{p,n-p} \to 0$

yields $\dim_{\mathbb{Q}} E^\infty_{p,n-p} = \dim_{\mathbb{Q}} J_{p,n-p} - \dim_{\mathbb{Q}} J_{p-1,n-p+1}$ for $0 \le p \le n$.

Thus $$\sum_{p=0}^{n} \dim_{\mathbb{Q}} E^\infty_{p,n-p} = \sum_{p=0}^{n} \left[\dim_{\mathbb{Q}} J_{p,n-p} - \dim_{\mathbb{Q}} J_{p-1,n-p+1} \right]$$

$$= \dim_{\mathbb{Q}} J_{n,0} - \dim_{\mathbb{Q}} J_{-1,n+1} = \dim_{\mathbb{Q}} H_n(T;\mathbb{Q}).$$

Thus
$$\chi(T) = \sum_n (-1)^n \dim_{\mathbb{Q}} H_n(T;\mathbb{Q}) = \sum_{p,q} (-1)^{p+q} \dim_{\mathbb{Q}} E^\infty_{p,q}.$$

Since $E^2_{p,q} = 0$ for all but finitely many (p,q), there exists a finite r, independent of (p,q), such that $E^\infty_{p,q} = E^r_{p,q}$ for all (p,q).

Since $E^2_{p,q} \cong H_p(B;\mathbb{Q}) \underset{\mathbb{Q}}{\otimes} H_q(F;\mathbb{Q})$,

$\dim_{\mathbb{Q}} E^2_{p,q} = \dim_{\mathbb{Q}} H_p(B;\mathbb{Q}) \cdot \dim_{\mathbb{Q}} H_q(F;\mathbb{Q})$. Thus

$$\sum_{p,q} (-1)^{p+q} \dim_{\mathbb{Q}} E^2_{p,q} = \sum_{p,q} (-1)^{p+q} \dim_{\mathbb{Q}} H_p(B;\mathbb{Q}) \cdot \dim_{\mathbb{Q}} H_q(F;\mathbb{Q})$$

$$= \left[\sum_p (-1)^p \dim_{\mathbb{Q}} H_p(B;\mathbb{Q})\right]\left[\sum_q (-1)^q \dim_{\mathbb{Q}} H_q(F;\mathbb{Q})\right] = \chi(B)\cdot\chi(F).$$

Thus we will be done if we prove: For each $r \geq 2$,

$$\sum_{p,q} (-1)^{p+q} \dim_{\mathbb{Q}} E^r_{p,q} = \sum_{p,q} (-1)^{p+q} \dim_{\mathbb{Q}} E^{r+1}_{p,q}.$$

For $2 \leq r < \infty$ we have a chain complex (E^r, d^r) with $E^r_n = \bigoplus_{p+q=n} E^r_{p,q}$. It follows, by the Hopf trace formula (using the identity map on E^r), that

$$\sum_n (-1)^n \dim_{\mathbb{Q}} E^r_n = \sum_n (-1)^n \dim_{\mathbb{Q}} H_n(E^r, d^r).$$

Since $H_n(E^r, d^r) \cong E^{r+1}_n$, this yields

$$\sum_{p,q} (-1)^{p+q} \dim_{\mathbb{Q}} E^r_{p,q} = \sum_{p,q} (-1)^{p+q} \dim_{\mathbb{Q}} E^{r+1}_{p,q},$$ completing the proof.

20 a). The map $E(\tau(S^n) \oplus \varepsilon^1) \longrightarrow E(\varepsilon^{n+1})$ sending $((x,y),(x,t)) \longmapsto (x, y+tx)$ is the desired vector bundle isomorphism.

b) If

$$f: \begin{array}{c} E_1 \\ \downarrow \\ B_1 \end{array} \longrightarrow \begin{array}{c} E_2 \\ \downarrow \\ B_2 \end{array}$$

is a morphism of finitely-presentable n-plane bundles, f induces maps $f_P : P_{\mathbb{R}}(E_1) \to P_{\mathbb{R}}(E_2)$, $f_L : L_{\mathbb{R}}(E_1) \to L_{\mathbb{R}}(E_2)$ which constitute a map of real line bundles such that

$$\begin{tikzcd} P_{\mathbb{R}}(E_1) \arrow[r, "f_P"] \arrow[d, "r_1"'] & P_{\mathbb{R}}(E_2) \arrow[d, "r_2"] \\ B_1 \arrow[r, "f_B"'] & B_2 \end{tikzcd}$$

commutes.

It follows, by naturality of the Euler class, that $f_P^* \chi_{L(E_2)} = \chi_{L(E_1)}$. Thus, since

$$\chi^n_{L(E_2)} - \sum_{i=1}^n r_2^* w_i(E_2) \smile \chi^{n-i}_{L(E_2)} = 0,$$ we obtain

$$0 = f_P^*\left[\chi^n_{L(E_2)} - \sum_{i=1}^n r_2^* w_i(E_2) \cup \chi^{n-i}_{L(E_2)} \right]$$

$$= \chi^n_{L(E_1)} - \sum_{i=1}^n f_P^* r_2^* w_i(E_2) \cup \chi^{n-i}_{L(E_1)}$$

$$= \chi^n_{L(E_1)} - \sum_{i=1}^n r_1^* f_B^* w_i(E_2) \cup \chi^{n-i}_{L(E_1)}, \quad \text{i.e. the } f_B^* w_i(E_2)$$

satisfy the defining equation of the $w_i(E_1)$. Thus $f_B^* w_i(E_2) = w_i(E_1)$ for all i.

For any B, we have a map of n-plane bundles

$$\begin{tikzcd} B \times \mathbb{R}^n \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathbb{R}^n \arrow[d] \\ B \arrow[r, "c"'] & pt \end{tikzcd}$$

Since $H^i(pt; \mathbb{Z}/2) = 0$ for $i \neq 0$, it follows that $w_i(B \times \mathbb{R}^n) = 0$ for $i \neq 0$.

Thus $w_i(\varepsilon^k) = \begin{cases} 0 & \text{if } i \neq 0 \\ 1 & \text{if } i = 0 \end{cases}$ for all k.

Thus, by the Whitney Product Formula, for $i > 0$,

$$0 = w_i(\varepsilon^{n+1}) = w_i(\tau(S^n) \oplus \varepsilon) = \sum_j w_{i-j}(\tau(S^n)) \cup w_j(\varepsilon^1)$$

$$= w_i(\tau(S^n)) \cup w_0(\varepsilon^1) = w_i(\tau(S^n)).$$

Thus $w_i(\tau(S^n)) = \begin{cases} 0 & \text{if } i \neq 0 \\ 1 & \text{if } i = 0. \end{cases}$