1-1-2006

Notes for MAT 7500 – Winter '93, revised Winter '06

David Handel
Wayne State University

Preface to MAT 7500 Notes by David Handel

These notes developed from a one semester course at Wayne State University which I taught several times during the last 20 years. The subject matter is analysis on manifolds, consisting of the theory of smooth manifolds, differential forms, integration of forms, the generalized Stokes' Theorem, de Rham cohomology, and some related topics. The course is intended for first or second year graduate students in Mathematics with a background in Advanced Calculus, General Topology, linear algebra (including quotient spaces), and a little elementary group theory (including some familiarity with the symmetric groups). Given the above background, the notes are self-contained. In particular, we develop from scratch the vector bundle theory and exterior algebra that we make use of. We include a fair number of examples, as well as exercises. The notes are divided into 19 sections, which we proceed to describe.

In Section 1 we prove that the norm topologies arising from any two norms on a finite-dimensional real vector space V are the same, and consequently we can define the *standard topology* on V to be that which arises from any norm on V.

In Section 2, topological manifolds are introduced and examples are given. Fiber bundles are defined, and we observe that if the fiber and base of a fiber bundle are topological manifolds, then so is the total space. We also treat connected sums of manifolds to generate further examples.

Section 3 is devoted to advanced calculus. We assume the multivariate Chain Rule, the Inverse Function Theorem, and Fundamental Theorem of Calculus. Two special cases of Taylor's Theorem, in a form suitable for later purposes, are deduced. We reformulate some of multivariate calculus in a coordinate-free form.

Smooth ($= C^\infty$) manifolds and smooth maps between such are introduced in Section 4. Smooth fiber bundles and smooth connected sums are treated, and examples are given.

Section 5 deals with the tangent space at a point of a smooth manifold and the tangent map (linear approximation or differential) of tangent spaces induced by a smooth map. We take the "directional derivative" approach to defining tangent vectors, entailing derivations of algebras of germs. Using advanced calculus, we prove that the tangent space at a point of an n-dimensional smooth manifold is n-dimensional.

Smooth submanifolds, immersions, regular values, and submersions are studied in Section 6. Among other things it is proved (with the help of the Inverse Function Theorem) that inverse images of regular values of smooth maps are smooth submanifolds. Some examples are worked out.

In the case of smooth manifolds of Euclidean space, a more intuitive notion of tangent vector in terms of velocity vectors to smooth curves is possible. We present this in Section 7 and construct a natural isomorphism between this notion of tangent space and the one given in Section 5.

Vector bundles and vector bundle homomorphisms (both topological and smooth, with the emphasis on smooth) are introduced in Section 8. The individual tangent spaces to a smooth manifold are assembled to construct the tangent bundle of that manifold, and the individual tangent maps for a smooth map are assembled to construct a vector bundle homomorphism. Smooth sections of a smooth vector bundle are introduced.

The language of categories and functors is introduced in Section 9, and various examples are given. In particular we reformulate some of the earlier material on smooth maps, tangent maps, and vector bundles in terms of categories and functors.

Section 10 gives a treatment of exterior powers of real vector spaces, and it is noted that the exterior power functors are smooth. This is used in Section 11 to construct exterior powers of smooth vector bundles.

Differential forms on a smooth manifold are introduced in Section 12 as smooth sections of exterior powers of the tangent bundle. We establish the functorial properties of differential forms, wedge products and their formal properties, and the explicit classical description of differential forms on open subsets of Euclidean space.

Exterior differentiation of differential forms is introduced in Section 13. The de Rham complex and the de Rham cohomology ring are introduced, and some formal properties are established.

Section 14 deals with integration of differential forms over smooth cubical chains. A generalized Stokes' Theorem is proved, which enables one to establish the non-triviality of certain de Rham cohomology groups. The classical Stokes' Theorem, Divergence Theorem, and Green's Theorem are deduced.

In Section 15, the concept of smooth homotopy between smooth maps is introduced, and it is proved that smoothly homotopic maps induce the same homomorphisms in de Rham cohomology. This enables the determination of further de Rham cohomology information. The latter, together with the Weierstrass Approximation Theorem (which enables us to replace an arbitrary continous map without fixed points by a smooth map without fixed points) yields a proof of the purely topological Brouwer Fixed Point Theorem.

Section 16 deals with technicalities concerning paracompactness, smooth partitions of unity, and the piecing together of local smooth sections of smooth vector bundles.

Orientations of smooth manifolds are studied in Section 17. Various examples of orientable and non-orientable smooth manifolds are given. For example, we show:

(i) Spheres are orientable.

(ii) For real projective spaces, those of odd dimension are orientable, while those of positive even dimension are not.

(iii) The total space of the tangent bundle of any smooth paracompact manifold is orientable.

(iv) Any paracompact complex analytic manifold is orientable.

The section ends with a statement (but no proof) of the Poincaré Duality Theorem.

Riemannian metrics on smooth manifolds are introduced in Section 18, and it is shown that the following three conditions on a smooth manifold M are equivalent:

(i) M admits a Riemannian metric.

(ii) M is paracompact.

(iii) M is metrizable.

Section 19 gives a brief introduction to smooth cubical homology and cohomology on smooth manifolds, and the de Rham map from de Rham cohomology to real smooth cubical cohomology. The de Rham Theorem, that the de Rham map is an isomorphism under appropriate conditions, is stated but not proved.

NOTES FOR MAT 750 – Winter '93, revised Winter '06

David Handel

1. The Standard Topology on a Finite-Dimensional Real Vector Space

Definition 1.1. Let V be a vector space over the real numbers $\mathbf{R}$. A *norm* N on V is a function $N : V \to \mathbf{R}$ satisfying
(i) For all $v \in V$, $N(v) \geq 0$. $N(v) = 0$ if and only if $v = 0$.
(ii) For all $v \in V$ and $r \in \mathbf{R}$, $N(rv) = |r|N(v)$.
(iii) For all $v, w \in V$, $N(v+w) \leq N(v) + N(w)$.

Example 1.2. Suppose V is finite-dimensional over $\mathbf{R}$ with basis B. The *Euclidean norm* on V relative to B is given by

$$N\left(\sum_{b\in B} r_b b\right) = \left(\sum_{b\in B} r_b^2\right)^{1/2}$$

where the r_b are real. The proof of property (iii) uses the Schwarz inequality.

Example 1.3. Let V and B be as in Example 1.2. The *box norm* on V relative to B is given by

$$N\left(\sum_{b\in B} r_b b\right) = \max_{b\in B} |r_b| .$$

Example 1.4. Let V and B be as in Example 1.2. The *diamond norm* on V relative to B is given by

$$N\left(\sum_{b\in B} r_b b\right) = \sum_{b\in B} |r_b| .$$

If N is a norm on the real vector space V, the function $d_N : V \times V \to \mathbf{R}$ given by $d_N(x, y) = N(x - y)$ is easily checked to be a metric on V. Our main goal in this section is to show that if V is finite-dimensional over $\mathbf{R}$, then the topology on V arising from the metric d_N is independent of the choice of the norm N on V. (This is false for infinite-dimensional vector spaces over $\mathbf{R}$.)

If N_1 and N_2 are norms on V, we say these norms are *equivalent* if there exist positive constants α and β such that for all $x \in V$, $N_1(x) \leq \alpha N_2(x)$ and $N_2(x) \leq \beta N_1(x)$. It is an easy exercise to check that the metric topologies arising from equivalent norms are the same. Thus, to achieve the above goal, it remains only to show that any two norms on a finite-dimensional real vector space are equivalent.

Theorem 1.5. *Let V be a real finite-dimensional vector space. Then any two norms on V are equivalent.*

Proof. Let $B = \{b_1, \ldots, b_n\}$ be any $\mathbf{R}$-basis for V and N_e the Euclidean norm on V with respect to B. Clearly, equivalence of norms is an equivalence relation on the set of all norms on V, and so it suffices to prove that each norm on V is equivalent to N_e. Let N be a norm on V. We will refer to the topology on V arising from the metric d_N as the *N-topology* on V.

A metric space (X, d) is said to have the *Heine-Borel property* if a subspace of X is compact if and only if it is closed in X, and bounded in the metric d. It is well-known that $\mathbf{R}^n$, with the standard Euclidean metric, has the Heine-Borel property. The map $V \to \mathbf{R}^n$ which sends $\sum_i r_i b_i$ to $(r_1, \ldots, r_n)$ is an isometry with respect to the metric d_{N_e} on V and the standard Euclidean metric on $\mathbf{R}^n$. It follows that (V, d_{N_e}) has the Heine-Borel property.

It is an easy exercise to check that $N : V \to \mathbf{R}$ is continuous with respect to the N-topology on V and the standard topology on $\mathbf{R}$. We proceed to show that N is also continuous with respect to the N_e-topology on V and the standard topology on $\mathbf{R}$.

Let $x = \sum_{b \in B} r_b b$, $y = \sum_{b \in B} s_b b$ where r_b, $s_b \in \mathbf{R}$ for each $b \in B$. Then $|N(x) - N(y)| \leq N(x - y) = N\left(\sum_{b \in B}(r_b - s_b)b\right) \leq \sum_{b \in B} |r_b - s_b| N(b)$. Note that $|r_b - s_b| \leq N_e(x - y)$ for all $b \in B$. Thus, if $C = \max_{b \in B} N(v)$, we have $|N(x) - N(y)| \leq nCN_e(x - y)$ for all x, $y \in V$. It now follows easily, from the continuity of N_e with respect to the N_e-topology on V and the standard topology on $\mathbf{R}$, that N is continuous with respect to the N_0-topology on V and the standard topology on $\mathbf{R}$.

Let $S = \{x \in V \mid N_e(x) = 1\}$, the unit sphere in V relative to N_e. By the Heine-Borel property, S is compact in the N_e-topology. Thus, by continuity of $N : V \to \mathbf{R}$ with respect to the N_e-topology on V and the standard topology on $\mathbf{R}$, $N(S)$ is compact. Since $N(x) > 0$ for all $x \in S$, it follows that there exist positive constants m and M such that $m \leq N(x) \leq M$ for all $x \in S$. For non-zero $v \in V$, $\dfrac{v}{N_e(v)} \in S$ and so $m \leq N\left(\dfrac{v}{N_e(v)}\right) \leq M$. Thus $mN_e(v) \leq N(v) \leq MN_e(v)$ for all $v \in V$. It follows easily that N and N_e are equivalent norms on V. □

Corollary 1.6. *Let V be a finite-dimensional real vector space and N any norm on V. Then (V, d_N) has the Heine-Borel property.*

Proof. Exercise. □

By virtue of Theorem 1.5 we can make the following definition:

Definition 1.7. Let V be a real finite-dimensional vector space. The *standard topology* on V is the N-topology on V for any norm N on V.

Note that what is usually called the standard topology on $\mathbf{R}$ (e.g. in the proof of Theorem 1.5) is the N-topology where $N(x) = |x|$. Thus, in the case of $\mathbf{R}$, Definition 1.7 agrees with the usual usage.

If N is a norm on the finite-dimensional real vector space V, $v \in V$, and $r > 0$, let $B_N(v, r) = \{x \in V \mid N(x - v) < r\}$, the *open ball of radius r centered at v with respect to N*. $\{B_N(v, r) \mid v \in V, r > 0\}$ is a basis for the standard topology on V.

Corollary 1.8. *Let m be a positive integer and $V_1, \ldots, V_m$ finite-dimensional real vector spaces. Give each V_i its standard topology. Then the product topology on the real vector space $V_1 \times \cdots \times V_m$ coincides with its standard topology.*

Proof. For $1 \le i \le m$ let N_i be any norm on V_i and define $N : V_1 \times \cdots \times V_m \to \mathbf{R}$ by $N(v_1, \ldots, v_m) = \max_i N_i(v_i)$. It is easily checked that N is a norm on $V_1 \times \cdots \times V_m$. If $v = (v_1, \ldots, v_m)$, $v_i \in V_i$, and $r > 0$, one checks that $B_N(v, r) = B_{N_1}(v_1, r) \times \cdots \times B_{N_m}(v_m, r)$. Since $\{B_N(v, r) \mid v \in V_1 \times \cdots \times V_m,\ r > 0\}$ is a basis for the standard topology while $\{B_{N_1}(v_1, r) \times \cdots \times B_{N_m}(v_m, r) \mid v_i \in V_i,\ r > 0\}$ is a basis for the product topology, the result now follows. ☐

Corollary 1.9. *Let V be a finite-dimensional real vector space and W an $\mathbf{R}$-linear subspace of V. Then the subspace topology on W induced by the standard topology on V coincides with the standard topology on W.*

Proof. Let N be any norm on V. Then N_W, the restriction of N to W, is a norm on W. For each $w \in W$ and $r > 0$, $B_{N_W}(w, r) = W \cap B_N(w, r)$. Since $\{B_{N_W}(w, r) \mid w \in W,\ r > 0\}$ is a basis for the standard topology on W while $\{W \cap B_N(w, r) \mid w \in W,\ r > 0\}$ is a basis for the subspace topology on W induced by the standard topology on V, the result now follows. ☐

Each real finite-dimensional vector space below will be assumed to be equipped with the standard topology, unless otherwise mentioned.

Lemma 1.10. *Let V, W be finite-dimensional real vector spaces, and $f : V \to W$ an $\mathbf{R}$-isomorphism. Then f is a homeomorphism.*

Proof. Let B be any $\mathbf{R}$-basis of V. Then $f(B) = \{f(b) \mid b \in B\}$ is an $\mathbf{R}$-basis of W. If N_1 and N_2 are the Euclidean norms on V and W, respectively, with respect to these bases, then f is an isometry with respect to the metrics d_{N_1} and d_{N_2}. Thus, in particular, f is a homeomorphism. ☐

Let V be a finite-dimensional real vector space and B a basis of V over $\mathbf{R}$. Let X be a topological space, $Y \subset V$, and $f : X \to Y$ a function. Then there are unique functions $f_b^B : X \to \mathbf{R}$, $b \in B$, satisfying $f(x) = \sum_{b \in B} f_b^B(x) b$. We call the f_b^B the *coordinate functions of f relative to B.*

Proposition 1.11. *Let V be a finite-dimensional real vector space, B a basis of V over $\mathbf{R}$, X a topological space, $Y \subset V$, and $f : X \to Y$ a function. Then f is continuous if and only if the coordinate functions f_b^B of f relative to B are all continuous.*

Proof. Let $i : Y \to V$ denote the inclusion map. Note that $f_b^B = (if)_b^B$ for all $b \in B$, and that by the characteristic property of the subspace topology on Y, f is continuous if and only if if is continuous. Thus, it suffices to treat the case $Y = V$.

Say $B = \{b_1, \ldots, b_n\}$. Define $h : V \to \mathbf{R}^n$ by $h\left(\sum_i r_i b_i\right) = (r_1, \ldots, r_n)$. Then h is an $\mathbf{R}$-isomorphism and so, by Lemma 1.10, h is a homeomorphism. Thus f will be continuous if and only if hf is continuous.

Let $\pi_i : \mathbf{R}^n \to \mathbf{R}$ be projection on the i^{th} factor, $1 \le i \le n$. By the characteristic property of the product topology on $\mathbf{R}^n$, hf is continuous if and only if $\pi_i \circ (hf)$ is continuous for $1 \le i \le n$. Since $\pi_i \circ (hf) = f_{bi}^B$, we are done. ☐

Let V be an n-dimensional real vector space, X a subset of V, Y a topological space, and $f : X \to Y$ a function. If $h : \mathbf{R}^n \to V$ is any $\mathbf{R}$-isomorphism, the map $f \circ h : h^{-1}(X) \to Y$ will be called a *coordinate representation of f*. For example, given an ordered basis $(b_1, \ldots, b_n)$ of V, we get a coordinate representation $f \circ h$ of f where $h : \mathbf{R}^n \to V$ is given by $h(r_1, \ldots, r_n) = \sum_i r_i b_i$. The following is immediate from Lemma 1.10:

Proposition 1.12. *Let V be a finite-dimensional real vector space, X a subset of V, Y a topological space, and $f : X \to Y$ a function. Then the following are equivalent:*

(1) *f is continuous.*
(2) *At least one coordinate representation of f is continuous.*
(3) *Every coordinate representation of f is continuous.* □

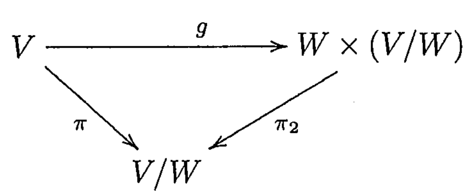
We will assume the continuity of the elementary real-valued functions of several real variables (where defined) studied in Calculus and/or Elementary Analysis.

Let V be a finite-dimensional real vector space, and W an $\mathbf{R}$-linear subspace of V. Then V/W has two natural topologies: (1) the standard topology, by virtue of V/W being a finite-dimensional real vector space, and (2) the quotient topology arising from the projection $\pi : V \to V/W$ given by $\pi(v) = v + W$ for all $v \in V$. We proceed to show that these two topologies are the same.

Proposition 1.13. *Suppose V is a finite-dimensional real vector space and W an $\mathbf{R}$-linear subspace of V. Then the standard topology on V/W coincides with the quotient topology arising from the projection $\pi : V \to V/W$.*

Proof. Give V and V/W their respective standard topologies. It suffices to show that with these topologies, $\pi : V \to V/W$ is a quotient map.

We can choose an $\mathbf{R}$-linear map $f : V \to W$ such that $f(w) = w$ for all $w \in W$. Then the map $g : V \to W \times (V/W)$ given by $g(v) = (f(v), v + W)$ is an $\mathbf{R}$-isomorphism. Thus, giving $W \times (V/W)$ its standard topology, g is a homeomorphism by Lemma 1.10. Since, by Corollary 1.8, the latter standard topology coincides with the product topology on $W \times (V/W)$, projection on the second factor $\pi_2 : W \times (V/W) \to V/W$ is a quotient map. The diagram



commutes, and so π is a quotient map, being the composition of the quotient maps π_2 and g. □

Exercises for §1

1. Establish the following claims made in this section. In each case, in order to avoid circular reasoning, only use material which precedes the claim in the text above.

(a) The metric topologies arising from equivalent norms are the same.

(b) If N is a norm on a real vector space V, then $N : V \to \mathbf{R}$ is continuous with respect to the N-topology on V and the standard topology on $\mathbf{R}$.

(c) If N_i is a norm on the real vector space V_i, $1 \le i \le m$, and $N : V_1 \times \cdots \times V_m \to \mathbf{R}$ is given by $N(v_1, \ldots, v_m) = \max\limits_i N_i(v_i)$, then N is a norm on $V_1 \times \cdots \times V_m$. If $v = (v_1, \ldots, v_m)$, $v_i \in V_i$, and $r > 0$, then $B_N(v, r) = B_{N_1}(v_1, r) \times \cdots \times B_{N_m}(v_m, r)$.

2. If V and W are real vector spaces, $\mathrm{Hom}_{\mathbf{R}}(V, W)$ denotes the set of all $\mathbf{R}$-homomorphisms from V to W. $\mathrm{Hom}_{\mathbf{R}}(V, W)$ is itself a real vector space with operations as follows: If f, $g \in \mathrm{Hom}_{\mathbf{R}}(V, W)$ and $r \in \mathbf{R}$, then for all $v \in V$, $(f+g)(v) = f(v)+g(v)$, $(rf)(v) = rf(v)$. Suppose V and W are finite-dimensional over $\mathbf{R}$ with norms N_1 and N_2, respectively. Define $N : \mathrm{Hom}_{\mathbf{R}}(V, W) \to \mathbf{R}$ by $N(f) = \sup\limits_{N_1(v)=1} N_2(f(v))$. Prove that N is a norm on $\mathrm{Hom}_{\mathbf{R}}(V, W)$.

3. Let V and W be finite-dimensional real vector spaces with bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$, respectively, over $\mathbf{R}$. Let N_1 and N_2 denote the box norms on V and W, respectively, relative to these bases. Let N be the norm on $\mathrm{Hom}_{\mathbf{R}}(V, W)$ which arises from N_1 and N_2 by the construction of Problem 2. If $f \in \mathrm{Hom}_{\mathbf{R}}(V, W)$ and (a_{ij}) is the matrix of f relative to the above bases, i.e. $f(v_i) = \sum_{j=1}^{n} a_{ji} w_j$ for $1 \le i \le m$, prove $N(f) \le m \max\limits_{i,j} |a_{ij}|$.

2. Topological Manifolds

Definition 2.1. Let n be a non-negative integer. A *topological manifold of dimension n* (or, more briefly, a *topological n-manifold*) is a Hausdorff space M which is a union of open subsets, each homeomorphic to an open subset of $\mathbf{R}^n$. If V is a real n-dimensional vector space, any homeomorphism from an open subset of M onto an open subset of V is called a *chart* for M. A set of charts for M whose domains cover M is called an *atlas* for M.

The adjective *topological* serves to distinguish the gadgets defined above from gadgets with more structure, called *differentiable manifolds* or *smooth manifolds,* which will be defined later. We will drop the adjective *topological* when there is no danger of confusion (e.g. now).

1-manifolds are sometimes called *curves.* 2-manifolds are sometimes called *surfaces.* A topological space is a 0-manifold if and only if it is discrete. (By convention, $\mathbf{R}^0 = \{0\}$, a one-point space.)

Example 2.2. If V is a real n-dimensional vector space, any open subset A of V is an n-manifold which admits an atlas with exactly one chart, namely 1_A, the identity map on A.

For example, let n be a positive integer and $M_n(\mathbf{R})$ the set of all $n \times n$ matrices with real entries. Then $M_n(\mathbf{R})$ is an n^2-dimensional real vector space with basis $\{E_{ij} \mid 1 \le i \le n,\ 1 \le j \le n\}$ where E_{ij} is the matrix with 1 in the ij position and 0's elsewhere. Define $GL_n(\mathbf{R}) = \{A \in M_n(\mathbf{R}) \mid A \text{ is invertible}\}$. Then $GL_n(\mathbf{R}) = \det^{-1}(\mathbf{R} - \{0\})$ where $\det : M_n(\mathbf{R}) \to \mathbf{R}$ is the map which assigns to each $A \in M_n(\mathbf{R})$ its determinant $\det(A)$. For any ordering of the above basis of $M_n(\mathbf{R})$, the resulting coordinate representation of det is a polynomial function over $\mathbf{R}$ in the n^2 coordinates, and hence continuous. Thus, by Proposition 1.12, $\det : M_n(\mathbf{R}) \to \mathbf{R}$ is continuous. Therefore, $GL_n(\mathbf{R})$ is an open subset of $M_n(\mathbf{R})$, and hence is an n^2-manifold. $GL_n(\mathbf{R})$ is called the n^{th} *general linear group over* $\mathbf{R}$.

Example 2.3. Let $n \ge 0$. For $x \in \mathbf{R}^{n+1}$ let $\|x\|$ denote the standard Euclidean norm of x. The *n-sphere* S^n is the unit sphere with center at the origin in $\mathbf{R}^{n+1}$ with respect to the standard Euclidean norm, i.e. $S^n = \left\{x \in \mathbf{R}^{n+1} \mid \|x\| = 1\right\}$. Thus S^0 is a discrete two-point space, S^1 is a circle, and S^2 is a spherical surface in $\mathbf{R}^3$. We proceed to show that S^n is an n-manifold by exhibiting an atlas for S^n.

For $x \in \mathbf{R}^{n+1}$ and $1 \le i \le n+1$, let x_i denote the i^{th} coordinate of x. For $1 \le i \le n+1$ let $U_i^+ = \{x \in S^n \mid x_i > 0\}$ and $U_i^- = \{x \in S^n \mid x_i < 0\}$. Then $\left\{U_1^+, \dots, U_{n+1}^+, U_1^-, \dots, U_{n+1}^-\right\}$ is easily checked to be an open cover of S^n. Let $E^n = \{x \in \mathbf{R}^n \mid \|x\| < 1\}$, the open unit disc in $\mathbf{R}^n$. Define maps $\varphi_i^+ : U_i^+ \to E^n$, $\psi_i^+ : E^n \to U_i^+$, $\varphi_i^- : U_i^- \to E^n$, $\psi_i^- : E^n \to U_i^-$ by

$$\varphi_i^{\pm}(x) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}),$$

$$\psi_i^+(y) = \left(y_1, \dots, y_{i-1}, \left(1 - \|y\|^2\right)^{1/2}, y_i, \dots, y_n\right),$$

$$\text{and} \quad \psi_i^-(y) = \left(y_1, \dots, y_{i-1}, -\left(1 - \|y\|^2\right)^{1/2}, y_i, \dots, y_n\right).$$

One checks easily that φ_i^+ and ψ_i^+ are homeomorphisms, inverse to one another, and similarly for φ_i^- and ψ_i^-. Thus $\{\varphi_1^+, \ldots, \varphi_{n+1}^+, \varphi_1^-, \ldots, \varphi_{n+1}^-\}$ is an atlas for S^n.

Example 2.4. An atlas for S^n with exactly two charts can be given as follows: Let $U = S^n - \{(0, \ldots, 0, 1)\}$, $V = S^n - \{(0, \ldots, 0, -1)\}$. Then $\{U, V\}$ is an open cover of S^n. The charts have U and V as their respective domains, and are given by stereographic projection from the north and south poles, respectively. Precisely, define maps $\varphi^+ : U \to \mathbf{R}^n$, $\psi^+ : \mathbf{R}^n \to U$, $\varphi^- : V \to \mathbf{R}^n$, $\psi^- : \mathbf{R}^n \to V$ by

$$\begin{aligned} \varphi^+(x) &= (1 - x_{n+1})^{-1}(x_1, \ldots, x_n), \\ \psi^+(y) &= \left(1 + \|y\|^2\right)^{-1} \left(2y_1, \ldots, 2y_n, \|y\|^2 - 1\right), \\ \varphi^-(x) &= (1 + x_{n+1})^{-1}(x_1, \ldots, x_n), \\ \text{and} \quad \psi^-(y) &= \left(1 + \|y\|^2\right)^{-1} \left(2y_1, \ldots, 2y_n, 1 - \|y\|^2\right). \end{aligned}$$

It is straightforward to check that φ^+ and ψ^+ are homeomorphisms, inverse to one another, and similarly for φ^- and ψ^-. Thus $\{\varphi^+, \varphi^-\}$ is an atlas for S^n.

Example 2.5. For $n \geq 0$, *real projective n-space* $P^n(\mathbf{R})$ is defined to be the quotient space obtained from S^n by identifying x with $-x$ for each $x \in S^n$. Write $[x]$ for the image of $x \in S^n$ under the quotient map $\pi : S^n \to P^n(\mathbf{R})$. Thus for $x, y \in S^n$, $[x] = [y]$ if and only if $y = \pm x$. We proceed to show that $P^n(\mathbf{R})$ is an n-manifold.

We first observe that the quotient map π is an open map. For let $\alpha : S^n \to S^n$ be the antipodal map, i.e. $\alpha(z) = -z$ for each $z \in S^n$. α is a homeomorphism and thus, in particular, an open map. For any subset W of S^n, note that $\pi^{-1}(\pi(W)) = W \cup \alpha(W)$. In particular, if W is open in S^n, so is $\pi^{-1}(\pi(W))$, and hence, since π is a quotient map, $\pi(W)$ is open in $P^n(\mathbf{R})$. Thus π is an open map.

We next check that $P^n(\mathbf{R})$ is Hausdorff. Suppose $x, y \in S^n$ are such that $[x] \neq [y]$. Then x, y, $-x$, and $-y$ are distinct points of S^n. Since S^n is Hausdorff, there exist mutually disjoint open neighborhoods A, B, C, and D of x, y, $-x$, and $-y$, respectively, in S^n. Let $U = \pi(A) \cap \pi(C)$ and $V = \pi(B) \cap \pi(D)$. Note that $[x] \in U$, $[y] \in V$. Since π is an open map, it follows easily that U and V are open in $P^n(\mathbf{R})$. We have $\pi^{-1}(U) = (A \cap \alpha(C)) \cup (C \cap \alpha(A))$ and $\pi^{-1}(V) = (B \cap \alpha(D)) \cup (D \cap \alpha(B))$. Since the latter two sets are disjoint, U and V are disjoint. Thus $P^n(\mathbf{R})$ is Hausdorff.

For $1 \leq i \leq n+1$, let $V_i = \pi(U_i^+)$ where U_i^+ is as in Example 2.3. Since π is an open map, and the V_i clearly cover $P^n(\mathbf{R})$, $\{V_1, \ldots, V_{n+1}\}$ is an open cover of $P^n(\mathbf{R})$. The restriction of π maps U_i^+ bijectively to V_i. Thus, since π is a continuous open map, V_i is homeomorphic to U_i^+. Since, from Example 2.3, U_i^+ is homeomorphic to an open set in $\mathbf{R}^n$, so is V_i. It now follows that $P^n(\mathbf{R})$ is an n-manifold and, in fact, admits an atlas with $n+1$ charts $\theta_1, \ldots, \theta_{n+1}$ where $\theta_i : V_i \to E^n$ is given by

$$\theta_i([x_1, \ldots, x_{n+1}]) = \frac{x_i}{|x_i|}(x_1, \ldots, \widehat{x}_i, \ldots, x_{n+1})$$

where the notation $\widehat{x}_i$ means that x_i is skipped.

Using algebraic topology, it can be proved that if n is even, every atlas for $P^n(\mathbf{R})$ must contain at least $n+1$ charts. If we insist that the domains of the charts be simply-connected (as is the case in Example 2.5), then at least $n+1$ charts are required even if n is odd. $P^3(\mathbf{R})$ admits an atlas with exactly 2 charts (see the exercises at the end of this section).

Proposition 2.6. *Suppose M and N are manifolds of dimensions m and n, respectively. Then $M \times N$ is an $(m+n)$-manifold. If $\mathcal{A}$ and $\mathcal{B}$ are atlases for M and N, respectively, then $\{\varphi \times \psi \mid \varphi \in \mathcal{A}, \psi \in \mathcal{B}\}$ is an atlas for $M \times N$.*

Proof. Since M and N are Hausdorff, so is $M \times N$. The rest is immediate. □

Lemma 2.7. (a) *Any open subspace of an n-manifold is an n-manifold.*

(b) *Suppose X is a Hausdorff space which is the union of some open subspaces, each of which is an n-manifold. Then X is an n-manifold.*

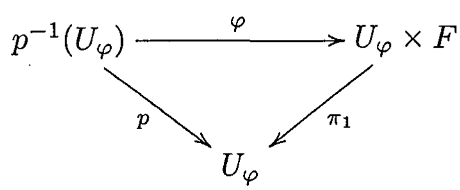
We leave the proof as an exercise.

Definition 2.8. A *fiber bundle* ξ is a quadruple (F, E, B, p) satisfying:

(i) F, E, and B are topological spaces called, respectively, the *fiber*, *total space*, and *base space of* ξ.

(ii) $p : E \to B$ is a continuous map called the *projection for* ξ; for each $x \in B$, $p^{-1}(x)$ is called the *fiber over* x.

A *chart* for ξ is a homeomorphism $\varphi : p^{-1}(U_\varphi) \to U_\varphi \times F$, where U_φ is open in B, such that the diagram



commutes, where π_1 denotes projection on the first factor, and we abuse notation by writing p for the indicated restriction of p.

(iii) There exists a collection $\mathcal{A}$ of charts for ξ such that $\{U_\varphi \mid \varphi \in \mathcal{A}\}$ covers B. Such an $\mathcal{A}$ is called an *atlas for* ξ.

We will frequently commit notational abuses as above, i.e. use the same notation for both a map and a restriction of that map (e.g. if $f : X \to Y$ is such that $f(A) \subset B$, we may write $f : A \to B$ for the restriction of f rather than a precise but cumbersome notation such as $f|_A^B : A \to B$) provided the domain and codomain of the restriction are displayed as above, or are clear from context. If situations arise where this abuse can cause trouble, we will use a non-ambiguous notation.

Note that if $\xi = (F, E, B, p)$ is a fiber bundle with atlas $\mathcal{A}$, then if $x \in B$ and $\varphi \in \mathcal{A}$ are such that $x \in U_\varphi$, it follows that $\varphi : p^{-1}(x) \to \{x\} \times F \cong F$ is a homeomorphism. However this homeomorphism is not canonical since it depends on the choice of chart φ for ξ for which $x \in U_\varphi$ and there is usually no preferred choice for the latter.

Example 2.9. Let F and B be arbitrary topological spaces. Then the quadruple $(F, B \times F, B, \pi_1)$ is a fiber bundle with atlas $\{1_{B\times F}\}$. This fiber bundle is called the *product bundle with base* B (or *over* B) *and fiber* F.

Example 2.10. For $n \geq 1$, let L_n denote the quotient space obtained from $S^n \times \mathbf{R}$ by identifying (x, r) with $(-x, -r)$ whenever $x \in S^n$, $r \in \mathbf{R}$. Let $q : S^n \times \mathbf{R} \to L_n$ denote the quotient map and write $[x, r] = q(x, r)$ for $x \in S^n$, $r \in \mathbf{R}$. Define $p_n : L_n \to P^n(\mathbf{R})$ by $p_n([x, r]) = [x]$. Let V_i and U_i^+, $1 \leq i \leq n+1$, be as in Examples 2.5 and 2.3. Let $\varphi_i : p_n^{-1}(V_i) \to V_i \times \mathbf{R}$ be given by $\varphi_i([x, r]) = ([x], x_i r)$. Note that φ_i is well-defined since for all $(x, r) \in U_i^+ \times \mathbf{R}$, $[-x] = [x]$ and $(-x_i)(-r) = x_i r$. Let $\mathcal{A}_n = \{\varphi_i \mid 1 \leq i \leq n+1\}$. We proceed to show that $(\mathbf{R}, L_n, P^n(\mathbf{R}), p_n)$ is a fiber bundle with atlas $\mathcal{A}_n$. We must show that p_n is continuous and that each φ_i is a homeomorphism; the remaining conditions of Definition 2.8 are easily checked.

Continuity of p_n is immediate from commutativity of the diagram

$$\begin{CD} S^n \times \mathbf{R} @>{\pi_1}>> S^n \\ @V{q}VV @VV{\pi}V \\ L_n @>>{p_n}> P^n(\mathbf{R}) \end{CD}$$

and the facts that π_1, π are both continuous, and that q is a quotient map.

Let $\widetilde{\varphi}_i : U_i^+ \times \mathbf{R} \to U_i^+ \times \mathbf{R}$ be given by $\widetilde{\varphi}_i(x, r) = (x, x_i r)$. Clearly, each component function of $\widetilde{\varphi}_i$ is continuous, and hence so is $\widetilde{\varphi}_i$. Continuity of φ_i is now immediate from commutativity of the diagram

$$\begin{CD} U_i^+ \times \mathbf{R} @>{\widetilde{\varphi}_i}>> U_i^+ \times \mathbf{R} \\ @V{q}VV @VV{\pi \times 1_{\mathbf{R}}}V \\ p_n^{-1}(V_i) @>>{\varphi_i}> V_i \times \mathbf{R}, \end{CD}$$

the continuity of $\widetilde{\varphi}_i$ and $\pi \times 1_{\mathbf{R}}$, and the fact that the indicated q (the restriction of the quotient map q above to q^{-1} of an open set) is a quotient map.

Define $\psi_i : V_i \times \mathbf{R} \to p_n^{-1}(V_i)$ by $\psi_i([x], r) = [x, r/x_i]$. ψ_i is well-defined since $x_i \neq 0$ for $[x] \in V_i$ and $[-x, r/(-x_i)] = [x, r/x_i]$. It is easily checked that φ_i and ψ_i are inverses of one another. Thus, to prove that φ_i is a homeomorphism, it remains only to check that ψ_i is continuous. Since $\pi : U_i^+ \to V_i$ is a quotient map and $\mathbf{R}$ is locally compact and Hausdorff, $\pi \times 1_{\mathbf{R}} : U_i^+ \times \mathbf{R} \to V_i \times \mathbf{R}$ is a quotient map. Define $\widetilde{\psi}_i : U_i^+ \times \mathbf{R} \to U_i^+ \times \mathbf{R}$ by $\widetilde{\psi}_i(x, r) = (x, r/x_i)$. $\widetilde{\psi}_i$ is clearly continuous. Continuity of ψ_i now follows from commutativity of

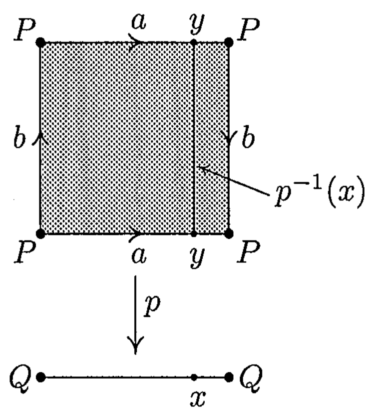
$$\begin{CD} U_i^+ \times \mathbf{R} @>{\widetilde{\psi}_i}>> U_i^+ \times \mathbf{R} \\ @V{\pi \times 1_{\mathbf{R}}}VV @VV{q}V \\ V_i \times \mathbf{R} @>>{\psi_i}> p_n^{-1}(V_i), \end{CD}$$

the continuity of $\widetilde{\psi}_i$ and q, and the fact that $\pi \times 1_{\mathbf{R}}$ is a quotient map.

This fiber bundle is called the *canonical line bundle over* $P^n(\mathbf{R})$.

Example 2.11. We construct a fiber bundle having the *Klein bottle* K as total space, with both base space and fiber homeomorphic to the circle S^1. Write $I = [0,1]$, the closed unit interval. K is the quotient space $I \times I/\sim$ where $\sim$ is the equivalence relation on $I \times I$ generated by $(s,0) \sim (s,1)$, $(0,t) \sim (1, 1-t)$ for all $s, t \in I$. Write $[s,t] \in K$ for the image of $(s,t) \in I \times I$ under the quotient map $I \times I \to K$.

Write $\partial I = \{0,1\}$. Note that $I/\partial I$, the quotient space obtained from I by identifying 0 and 1, is homeomorphic to S^1. In fact, writing $[\,t\,] \in I/\partial I$ for the image of $t \in I$ under the quotient map $I \to I/\partial I$, $h : I/\partial I \to S^1$ given by $h([\,t\,]) = (\cos(2\pi t), \sin(2\pi t))$ is well-defined and is a homeomorphism. (We leave the details as an exercise.) Define $p : K \to I/\partial I$ by $p([s,t]) = [\,s\,]$. It is easily checked that if $(s_1,t_1) \sim (s_2,t_2)$ in $I \times I$, then $[\,s_1\,] = [\,s_2\,]$, and so p is well-defined. p is illustrated in the figure below:



We proceed to prove that $\xi_K = (I/\partial I, K, I/\partial I, p)$ is a fiber bundle. We first check that p is continuous.

We have the commutative diagram

$$\begin{array}{ccc} I \times I & \xrightarrow{\pi_1} & I \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle q_1} \\ K & \xrightarrow[p]{} & I/\partial I \end{array}$$

where the q_i are the respective quotient maps and π_1 is projection on the first factor. Since π_1 and q_1 are both continuous, it follows that pq_2 is continuous. Thus, since q_2 is a quotient map, p is continuous.

We will show that there exists an atlas for ξ_K consisting of 2 charts. Suppose $c \in (0,1)$, the open unit interval. Write $U_c = I/\partial I - \{[c]\}$. Since $I/\partial I$ is Hausdorff, it follows that U_c is open in $I/\partial I$. If c_1, c_2 are distinct points in $(0,1)$, $U_{c_1} \cup U_{c_2} = I/\partial I$. Thus we will be done if we show that for each $c \in (0,1)$, there exists a chart $\varphi_c : p^{-1}(U_c) \to U_c \times I/\partial I$ for ξ_K.

For $c \in (0,1)$ define $\varphi_c : p^{-1}(U_c) \to U_c \times I/\partial I$ and $\psi_c : U_c \times I/\partial I \to p^{-1}(U_c)$ by

$$\varphi_c([s,t]) = \begin{cases} ([\,s\,],[\,t\,]) & \text{if } 0 \le s < c, \\ ([\,s\,],[\,1-t\,]) & \text{if } c < s \le 1, \end{cases}$$

$$\psi_c([s],[t]) = \begin{cases} [s,t] & \text{if } 0 \le s < c, \\ [s,1-t] & \text{if } c < s \le 1. \end{cases}$$

It is straightforward to check that both φ_c and ψ_c are well-defined, are inverses of one another, and that the diagram

$$\begin{array}{ccc} p^{-1}(U_c) & \xrightarrow{\varphi_c} & U_c \times I/\partial I \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & U_c & \end{array}$$

commutes. Thus it remains only to prove continuity of φ_c and ψ_c.

We have the commutative diagram

$$\begin{array}{ccccc} (I-\{c\})\times I & \xrightarrow{\theta} & (I-\{c\})\times I & \xrightarrow{\theta} & (I-\{c\})\times I \\ \downarrow{\scriptstyle q_2'} & & \downarrow{\scriptstyle q_1'\times q_1} & & \downarrow{\scriptstyle q_2'} \\ p^{-1}(U_c) & \xrightarrow[\varphi_c]{} & U_c \times I/\partial I & \xrightarrow[\psi_c]{} & p^{-1}(U_c) \end{array}$$

where

$$\theta(s,t) = \begin{cases} (s,t) & \text{if } 0 \le s < c, \\ (s,1-t) & \text{if } c < s \le 1, \end{cases}$$

and the q_i' are restrictions of the quotient maps q_i above. The restrictions of θ to $[0,c) \times I$ and $(c,1] \times I$ are clearly continuous, and so since the latter two sets form an open cover of $(I - \{c\}) \times I$, θ is continuous.

Since $(I-\{c\}) \times I$ is the complete inverse image of $p^{-1}(U_c)$ under q_2 and is open in K, it follows that q_2' is a quotient map. Since θ, q_1' and q_1 are all continuous, it follows from commutativity of the left-hand square in the above diagram that $\varphi_c q_2'$ is continuous. Thus, since q_2' is a quotient map, it follows that φ_c is continuous.

Since $I - \{c\}$ is the complete inverse image of U_c under q_1 and is open in I, it follows that q_1' is a quotient map. Since both $I-\{c\}$ and I are locally compact and Hausdorff, it follows that $q_1' \times q_1$ is a quotient map. Since θ and q_2' are continuous, it follows from commutativity of the right-hand square that $\psi_c(q_1' \times q_1)$ is continuous. Thus, since $q_1' \times q_1$ is a quotient map, ψ_c is continuous, completing the proof.

Example 2.12. Every covering map $p : \widetilde{X} \to X$ is a fiber bundle projection. The fibers are discrete spaces.

Lemma 2.13. *Suppose $\xi = (F,E,B,p)$ is a fiber bundle with F and B Hausdorff. Then E is Hausdorff.*

Proof. Let x, y be distinct points of E. We first consider the case $p(x) \neq p(y)$. Then since B is Hausdorff, there exist disjoint open neighborhoods X and Y of $p(x)$ and $p(y)$, respectively, in B. Then $p^{-1}(X)$ and $p^{-1}(Y)$ are disjoint open neighborhoods of x and y, respectively, in E.

Now suppose $p(x) = p(y)$. Choose a chart φ for ξ such that $p(x) \in U_\varphi$. Since φ is injective, there exist distinct points u, v in F such that $\varphi(x) = (p(x),u)$, $\varphi(y) = (p(y),v)$. Since F is Hausdorff, there exist disjoint open neighborhoods U and V of u and v, respectively, in F. Then $\varphi^{-1}(U_\varphi \times U)$ and $\varphi^{-1}(U_\varphi \times V)$ are disjoint open neighborhoods of x and y, respectively, in E. □

Proposition 2.14. *Suppose (F, E, B, p) is a fiber bundle with F an m-manifold and B an n-manifold. Then E is an $(m+n)$-manifold.*

Proof. By Lemma 2.13, E is Hausdorff. Let $\mathcal{A}$ be an atlas for (F, E, B, p). For each $\varphi \in \mathcal{A}$, U_φ is an n-manifold by Lemma 2.7(a). Thus, by Proposition 2.6, each $U_\varphi \times F$ is an $(m+n)$-manifold. Thus, since $U_\varphi \times F$ is homeomorphic to $p^{-1}(U_\varphi)$, each $p^{-1}(U_\varphi)$ is an $(m+n)$-manifold. The conclusion now follows from Lemma 2.7(b) since $\{p^{-1}(U_\varphi) \mid \varphi \in \mathcal{A}\}$ is an open cover of E. □

If $\{X_j \mid j \in J\}$ is a collection of topological spaces where J is an index set, their *disjoint union* $\coprod_{j\in J} X_j$ is $\bigcup_{j\in J} X_j \times \{j\}$. We topologize $\coprod_{j\in J} X_j$ by taking all $\bigcup_{j\in J} U_j \times \{j\}$, U_j open in X_j for each $j \in J$, as its open sets. In particular each $X_j \times \{j\}$ is open (and closed) in $\coprod_{j\in J} X_j$. The function $X_j \to X_j \times \{j\}$ sending x to (x, j) is a homeomorphism. In case $J = \{1, \ldots, n\}$, we sometimes write $\coprod_{j\in J} X_j = X_1 \amalg \cdots \amalg X_n$.

The index coordinate assures us that if $i \neq j$, then $X_i \times \{i\}$ and $X_j \times \{j\}$ are disjoint. In case the X_j's are already mutually disjoint, we drop the index coordinate and identify $\coprod_{j\in J} X_j$ with $\bigcup_{j\in J} X_j$, where the open sets of the latter are precisely all $\bigcup_{j\in J} U_j$, U_j open in X_j. In proving general results about disjoint unions we can often suppose, without loss of generality, that the X_j are mutually disjoint and drop the index coordinate. In specific examples where the X_j might not be disjoint (e.g. $\mathbf{R} \amalg \mathbf{R}$), the index coordinate is required to avoid ambiguity.

Example 2.15. Suppose $\{M_j \mid j \in J\}$ is a collection of n-manifolds. Then their disjoint union $\coprod_{j\in J} M_j$ is an n-manifold.

Our next goal is a construction called *connected sum* which manufactures new n-manifolds from old ones. The intuitive idea is to take two n-manifolds, punch a hole in each, and connect the punctured manifolds together by a pipe. For topological manifolds, the construction we give below is a little more complicated than necessary. However, when we come to smooth manifolds later, this more complicated construction will facilitate putting smooth structures on connected sums.

Suppose X and Y are topological spaces, $A \subset X$, $B \subset Y$, and $h : A \to B$ a homeomorphism. We write $X \cup_h Y$ for the quotient space obtained from the disjoint union $X \amalg Y$ by identifying $(a, 1) \sim (h(a), 2)$ for all $a \in A$ ($a \sim h(a)$ if X and Y are disjoint).

Lemma 2.16. *Let X and Y be topological spaces, A open in X, B open in Y, and $h : A \to B$ a homeomorphism. Let $q : X \amalg Y \to X \cup_h Y$ denote the quotient map. Then:*

(a) *q is an open map.*

(b) *The restrictions $q : X \times \{1\} \to q(X \times \{1\})$ and $q : Y \times \{2\} \to q(Y \times \{2\})$ are homeomorphisms.*

(c) *$q(X \times \{1\})$ and $q(Y \times \{2\})$ are both open in $X \cup_h Y$.*

(d) *$X \cup_h Y = q(X \times \{1\}) \cup q(Y \times \{2\})$.*

Proof. We can suppose X and Y as disjoint and drop index coordinates. Note that for any $U \subset X$ and $V \subset Y$, $q^{-1}q(U \cup V) = U \cup h(U \cap A) \cup V \cup h^{-1}(V \cap B)$. If U is open in X, then since h is a homeomorphism, $h(U \cap A)$ is open in B, and hence open in Y since B is open in Y. Similarly if V is open in Y, then $h^{-1}(V \cap B)$ is open in X. Thus if U and V are open in X and Y, respectively, $q^{-1}q(U \cup V)$ is open in $X \amalg Y$. Thus, since q is a quotient map, $q(U \cup V)$ is open in $X \cup_h Y$, establishing part (a).

Note that the restrictions $q : X \to q(X)$ and $q : Y \to q(Y)$ are both bijective. Thus, since q is an open continuous map and X, Y are both open in $X \amalg Y$, these restrictions are homeomorphisms, proving part (b).

Part (c) follows immediately from part (b) and the fact that both X and Y are open in $X \amalg Y$.

Part (d) follows immediately from the fact that $X \amalg Y = X \cup Y$. □

In particular, if M and N are n-manifolds and h is a homeomorphism from an open subset of M to an open subset of N, $M \cup_h N$ is a union of two open subspaces, each of which is an n-manifold. The only thing that prevents us from concluding that $M \cup_h N$ is an n-manifold is that it may fail to be Hausdorff.

Example 2.17. Consider $\mathbf{R} \cup_h \mathbf{R}$ where $h : (0, \infty) \to (0, \infty)$ is the identity map. We proceed to show that $\mathbf{R} \cup_h \mathbf{R}$ is not Hausdorff.

Let $q : \mathbf{R} \amalg \mathbf{R} \to \mathbf{R} \cup_h \mathbf{R}$ denote the quotient map. For each $x \in \mathbf{R}$, $S \subset \mathbf{R}$, and $j = 0, 1$, write x_j instead of $(x, j) \in \mathbf{R} \times \{j\} \subset \mathbf{R} \amalg \mathbf{R}$ and S_j instead of $S \times \{j\} \subset \mathbf{R} \amalg \mathbf{R}$ to make the notation less cumbersome. Then $q(0_1)$ and $q(0_2)$ are distinct points of $\mathbf{R} \cup_h \mathbf{R}$. We will see that these two points cannot be separated in $\mathbf{R} \cup_h \mathbf{R}$. Let U and V be open neighborhoods of $q(0_1)$ and $q(0_2)$, respectively, in $\mathbf{R} \cup_h \mathbf{R}$. Then $q^{-1}(U)$ and $q^{-1}(V)$ are open neighborhoods of 0_1 and 0_2, respectively, in $\mathbf{R} \amalg \mathbf{R}$. Thus there exists a $\delta > 0$ such that the open intervals $(0, \delta)_1$ and $(0, \delta)_2$ are contained, respectively, in $q^{-1}(U)$ and $q^{-1}(V)$. But then $q((0, \delta)_1) \subset U \cap V$ and so $U \cap V \neq \emptyset$.

Thus $\mathbf{R} \cup_h \mathbf{R}$ is a non-Hausdorff space which is the union of two open subspaces, each homeomorphic to $\mathbf{R}$. This shows that the Hausdorff condition in the definition of manifold is not superfluous.

We next give a sufficient condition to ensure that $X \cup_h Y$ is Hausdorff. If $h : A \to B$ is a homeomorphism where A and B are open in X and Y, respectively, we say X and Y are <u>unpinched</u> by h if there exists an open subspace U of X such that $\bar{A} - A \subset U$ and $\overline{h(U \cap A)} \subset B$.

Lemma 2.18. *Let X and Y be Hausdorff spaces, A open in X, B open in Y, and $h : A \to B$ a homeomorphism. Suppose X and Y are unpinched by h. Then $X \cup_h Y$ is Hausdorff.*

Proof. Let $q : X \amalg Y \to X \cup_h Y$ denote the quotient map. We may assume X is disjoint from Y and drop the index coordinate. Suppose v and w are distinct points of $X \cup_h Y$. We first consider the case where v and w both lie in $q(X)$. From Lemma 2.16(b), $q(X)$ is homeomorphic to X and hence Hausdorff. Thus there exist disjoint open neighborhoods V and W of v and w, respectively, in $q(X)$. Since, by Lemma 2.16(c), $q(X)$ is open in $X \cup_h Y$, V and W are both open in $X \cup_h Y$ and thus separate v and w in $X \cup_h Y$. The case where v, w both lie in $q(Y)$ is the same. Thus it remains only to treat the case $v \in q(X) - q(Y)$, $w \in q(Y) - q(X)$.

Since q is injective on X, $q(X - \bar{A})$ is disjoint from $q(A)$. Thus, since $q(X) \cap q(Y) = q(A)$, $q(X - \bar{A})$ is disjoint from $q(Y)$. Thus if $v \in q(X - \bar{A})$, $q(X - \bar{A})$ and $q(Y)$ are disjoint open neighborhoods of v and w, respectively, in $X \cup_h Y$.

Thus it remains only to treat the case $v \in q(X) - q(Y)$ but $v \notin q(X - \bar{A})$, and $w \in q(Y) - q(X)$. Since q is injective on X and $v \notin q(X - \bar{A})$, we must have $v \in q(\bar{A})$. But since $v \notin q(Y)$ and $q(X) \cap q(Y) = q(A)$, it follows that $v \in q(\bar{A} - A)$.

By hypothesis, there exists an open subspace U of X such that $\bar{A} - A \subset U$ and $\overline{h(U \cap A)} \subset B$. Thus $q(U)$ is an open neighborhood of v in $X \cup_h Y$, and $w \in q(Y) - q(X) = q(Y - B) \subset q(Y - \overline{h(U \cap A)})$. Thus $q(U)$ and $q(Y - \overline{h(U \cap A)})$ are open neighborhoods of v and w, respectively, in $X \cup_h Y$. It remains only to show that these neighborhoods are disjoint.

Proceed by contradiction. Suppose $x \in U$ and $y \in Y - \overline{h(U \cap A)}$ are such that $q(x) = q(y)$. Then $q(x) \in q(X) \cap q(Y) = q(A)$ and so, since q is injective on X, we must have $x \in A$. Thus $x \in U \cap A$. Since $q(x) = q(h(x))$, we have $q(y) = q(h(x))$. Since q is injective on Y, $y = h(x)$ and so $y \in h(U \cap A) \subset \overline{h(U \cap A)}$, a contradiction. □

Corollary 2.19. *Let M and N be n-manifolds. Suppose h is a homeomorphism from an open subset of M to an open subset of N such that M and N are unpinched by h. Then $M \cup_h N$ is an n-manifold.*

Example 2.20. Let $n \geq 1$. We now construct the connected sum $M \# N$ of two non-empty connected n-manifolds M and N. Recall that E^n denotes the open unit disk in $\mathbf{R}^n$, i.e. $\{x \in \mathbf{R}^n \mid \|x\| < 1\}$ where $\| \ \|$ denotes the standard Euclidean norm. Suppose $\varphi : \operatorname{dom} \varphi \to E^n$ and $\psi : \operatorname{dom} \psi \to E^n$ are homeomorphisms where $\operatorname{dom} \varphi$ and $\operatorname{dom} \psi$ are open sets in M and N, respectively. Say $\varphi(P) = \psi(Q) = 0$. For any subinterval J of $(0,1)$, write $E^n_J = \{ x \in E^n \mid \|x\| \in J \}$. For $r \in (0,1)$, write $E^n_r = \{ x \in E^n \mid \|x\| = r \}$.

Let $A = \varphi^{-1}\left(E^n_{(0,1/2)}\right) \subset M - \{P\}$, $B = \psi^{-1}\left(E^n_{(0,1/2)}\right) \subset N - \{Q\}$. A is open in $\operatorname{dom} \varphi$, and hence open in $M - \{P\}$ since $\operatorname{dom} \varphi$ is open in M. Similarly, B is open in $N - \{Q\}$. Write $\varphi_1 : A \to E^n_{(0,1/2)}$ and $\psi_1 : B \to E^n_{(0,1/2)}$ for the restrictions of φ and ψ, respectively. φ_1 and ψ_1 are homeomorphisms.

Note that for any $r \in (0,1)$, the map $\alpha_r : E^n_{(0,r)} \to E^n_{(0,r)}$ given by $\alpha_r(x) = (r - \|x\|) \dfrac{x}{\|x\|}$ is continuous and satisfies $\alpha_r \alpha_r = 1_{E^n_{(0,r)}}$. Thus each α_r is a homeomorphism. Descriptively, α_r reflects each open ray of length r emanating from the origin about its midpoint. Define $h : A \to B$ to be the composition

$$A \xrightarrow{\varphi_1} E^n_{(0,1/2)} \xrightarrow{\alpha_{1/2}} E^n_{(0,1/2)} \xrightarrow{\psi_1^{-1}} B \, .$$

Each map in this composition is a homeomorphism, and so h is a homeomorphism. We proceed to check that $M - \{P\}$ and $N - \{Q\}$ are unpinched by h.

We claim that if $r \in (0,1)$, then $\varphi_1^{-1}\left(E^n_{(0,r]}\right)$ is closed in $M - \{P\}$ and that $\psi^{-1}\left(E^n_{(0,r]}\right)$ is closed in $N - \{Q\}$. For writing $D^n_r = \{\, x \in \mathbf{R}^n \mid \|x\| \leq r \,\}$, D^n_r is compact and so, since $\varphi : \operatorname{dom} \varphi \to E^n$ is a homeomorphism, $\varphi^{-1}(D^n_r)$ is compact, and hence closed in M. Thus $\varphi^{-1}(D^n_r) \cap (M - \{P\})$ is closed in $M - \{P\}$. The claim for $\varphi_1^{-1}\left(E^n_{(0,r]}\right)$ now follows since $\varphi^{-1}(D^n_r) \cap (M - \{P\}) = \varphi_1^{-1}\left(E^n_{(0,r]}\right)$. The proof of the claim for $\psi^{-1}\left(E^n_{(0,r]}\right)$ is the same.

Thus, since $A = \varphi_1^{-1}\left(E^n_{(0,1/2)}\right) \subset \varphi_1^{-1}\left(E^n_{(0,1/2]}\right)$ and, by the above claim, this last set is closed in $M - \{P\}$, it follows that $\bar{A} \subset \varphi_1^{-1}\left(E^n_{(0,1/2]}\right)$ (where the bar indicates closure in $M - \{P\}$). Thus $\bar{A} - A \subset \varphi_1^{-1}\left(E^n_{1/2}\right)$. Take $U = \varphi^{-1}\left(E^n_{(1/4,3/4)}\right)$. Then U is open in $\operatorname{dom} \varphi$, and hence open in $M - \{P\}$ since $\operatorname{dom} \varphi$ is open in M. Note that $U \cap A = \varphi_1^{-1}\left(E^n_{(1/4,1/2)}\right)$, and $h(U \cap A) = \psi_1^{-1}\left(E^n_{(0,1/4)}\right)$. Thus $h(U \cap A) \subset \psi_1^{-1}\left(E^n_{(0,1/4]}\right)$ and, by the claim above, this latter set is closed in $N - \{Q\}$. Thus, $\overline{h(U \cap A)} \subset \psi_1^{-1}\left(E^n_{(0,1/4]}\right) \subset \psi_1^{-1}\left(E^n_{(0,1/2)}\right) = B$ where this last closure is in $N - \{Q\}$, completing the proof that $M - \{P\}$ and $N - \{Q\}$ are unpinched by h.

We thus obtain, by Corollary 2.19, an n-manifold $M \# N$ as indicated above.

The construction of $M \# N$ above depends on the choices of the charts φ and ψ. It can be shown that for compact connected 2-manifolds, the homeomorphism type of $M \# N$ is independent of these choices, but for higher dimensional manifolds the homeomorphism type of $M \# N$ may depend on the "orientation classes" of these charts, a concept which we will deal with later.

If M and N are compact connected n-manifolds, so is $M \# N$ (see the Exercises for §2). We inductively define two sequences of compact connected 2-manifolds $T_1, T_2, \ldots$ and $P_1, P_2, \ldots$ as follows: $T_1 = S^1 \times S^1$, the *2-dimensional torus*, and $T_n = T_1 \# T_{n-1}$ for $n > 1$. $P_1 = P^2(\mathbf{R})$, real projective 2-space (or the *real projective plane*), and $P_n = P_1 \# P_{n-1}$ for $n > 1$. It is known that every compact connected 2-manifold is homeomorphic to exactly one of $S^2, T_1, T_2, \ldots, P_1, P_2, \ldots$. References for the classification of compact connected 2-manifolds are the following:

(1) W. S. Massey, *Algebraic Topology: An Introduction*, Harcourt, Brace & World, Inc. 1967, Chapter 1.

(2) A. J. Sieradski, *An Introduction to Topology and Homotopy*, PWS-Kent Publishing Co. 1992, Chapter 13.

3) D. B. Fuks & V. A. Rokhlin, *Beginner's Course in Topology*, Springer-Verlag 1984, Chapter 3, §5.3.

It is elementary, though tricky, to show that every compact connected 1-manifold is homeomorphic to S^1. A proof is given in the third of the above references, pp. 139–140.

Exercises for §2

1.(a) Check the claim in Example 2.3 that φ_i^+ and ψ_i^+ are inverses of one another.
(b) Check the claim in Example 2.4 that φ^+ and ψ^+ are inverses of one another.

2. Prove Lemma 2.7.

3. Prove that every connected n-manifold is path-connected.

4. Prove that if M and N are connected n-manifolds, then $M \# N$ is connected. If, further, M and N are both compact, then $M \# N$ is compact.

5. Using informal cut and paste arguments, show
(a) $P^2(\mathbf{R}) \# P^2(\mathbf{R})$ is homeomorphic to the Klein bottle.
(b) $P^2(\mathbf{R}) \# (S^1 \times S^1)$ is homeomorphic to $P^2(\mathbf{R}) \# P^2(\mathbf{R}) \# P^2(\mathbf{R})$.

6. Let G be a Hausdorff topological group. Suppose G contains a non-empty open subset which is homeomorphic to an open subset of $\mathbf{R}^n$. Prove that G is an n-manifold.

7. The *orthogonal group* $O(n)$ consists of all $T \in GL_n(\mathbf{R})$ such that $T^{-1} = T^*$ where T^* denotes the transpose of T, topologized as a subspace of $GL_n(\mathbf{R})$.
(a) Prove that $O(n)$ is a Hausdorff topological group.
(b) Let $\text{Skew}(n) = \{A \in M_n(\mathbf{R}) \mid A^* = -A\}$. Show that if $A \in \text{Skew}(n)$, then $(I_n + A)(I_n - A)^{-1} \in O(n)$ where I_n denotes the $n \times n$ identity matrix. (Recall that the eigenvalues of a real skew-symmetric matrix are pure imaginaries, and so $I - A$ is invertible.)
(c) Show that if $A \in \text{Skew}(n)$, then -1 is not an eigenvalue of $(I + A)(I - A)^{-1}$.
(d) Prove that $O(n)$ is a $\frac{1}{2}n(n-1)$-manifold.

8. For $n \geq 1$, let $T(S^n) = \{(x, y) \in S^n \times \mathbf{R}^{n+1} \mid x \perp y\}$ where $\perp$ denotes "orthogonal with respect to the standard inner product on $\mathbf{R}^{n+1}$". Define $p : T(S^n) \to S^n$ by $p(x, y) = x$. Show that $(\mathbf{R}^n, T(S^n), S^n, p)$ is a fiber bundle.

9. The goal of this exercise is to show that $P^3(\mathbf{R})$ is the union of two open subsets, each homeomorphic to an open subset of $\mathbf{R}^3$.

Identify $\mathbf{R}^2$ with $\mathbf{C}$ in the standard way, i.e. (a, b) is identified with $a + bi$. Then for $n \geq 1$, S^{2n-1} can be identified with $\{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid \sum_i |z_i|^2 = 1\}$, and $P^{2n-1}(\mathbf{R})$ with the quotient space obtained from S^{2n-1} by identifying $(z_1, \ldots, z_n)$ with $(-z_1, \ldots, -z_n)$ for all $(z_1, \ldots, z_n) \in S^{2n-1}$. Write $[z_1, \ldots, z_n] \in P^{2n-1}(\mathbf{R})$ for the image of $(z_1, \ldots, z_n) \in S^{2n-1}$ under the quotient map.

(a) Let $h : P^1(\mathbf{R}) \to S^1$ be given by $h([z]) = z^2$. Show that h is well-defined and is a homeomorphism.
(b) Find an explicit homeomorphism from $S^1 \times \mathbf{R}$ onto an open subset of $\mathbf{R}^2$.
(c) Deduce that $P^1(\mathbf{R}) \times \mathbf{C}$ is homeomorphic to an open subset of $\mathbf{R}^3$.
(d) Let $U = \{[z, w] \in P^3(\mathbf{R}) \mid z \neq 0\}$ and $V = \{[z, w] \in P^3(\mathbf{R}) \mid w \neq 0\}$. Show that U and V are both open in $P^3(\mathbf{R})$ and that $P^3(\mathbf{R}) = U \cup V$.
(e) Let $f : P^1(\mathbf{R}) \times \mathbf{C} \to U$ be given by

$$f([z], w) = \left[z/\sqrt{1+|w|^2}, zw/\sqrt{1+|w|^2}\right].$$

Show that f is well-defined, and is a homeomorphism. Similarly, show that V is homeomorphic to $P^1(\mathbf{R}) \times \mathbf{C}$. Deduce that $P^3(\mathbf{R})$ is the union of two subspaces, each homeomorphic to an open subspace of $\mathbf{R}^3$.

(f) Generalize this to a result about $P^{2n-1}(\mathbf{R})$.

10. For $n \geq 0$ regard S^{2n+1} as the unit sphere in $\mathbf{C}^{n+1}$ as in Problem 9. In particular we identify S^1 with the set of complex numbers of absolute value 1. Define *complex projective n-space* $P^n(\mathbf{C})$ to be the quotient space obtained from S^{2n+1} by identifying $(z_1, \ldots, z_{n+1}) \sim (\lambda z_1, \ldots, \lambda z_{n+1})$ for all $(z_1, \ldots, z_{n+1}) \in S^{2n+1}$ and $\lambda \in S^1$.

(a) Prove that $P^n(\mathbf{C})$ is a $2n$-manifold.

(b) Write $[z, w] \in P^1(\mathbf{C})$ for the image of $(z, w) \in S^3$ under the quotient map. Let $\mathbf{C}^+$ denote the one-point compactification of $\mathbf{C}$. Let $f : P^1(\mathbf{C}) \to \mathbf{C}^+$ be given by

$$f([z,w]) = \begin{cases} \dfrac{z}{w} & \text{if } w \neq 0, \\ \infty & \text{if } w = 0. \end{cases}$$

Show that f is well-defined, and is a homeomorphism.

3. Differential Calculus

We assume a basic knowledge of Advanced Calculus. In particular, we assume the multivariate Chain Rule and the Inverse Function Theorem (stated below) as well as other standard analytic facts at that level. We include proofs of two cases of Taylor's Theorem in a form which will be needed later, but not found in some treatments of Advanced Calculus. We will reformulate some of Advanced Calculus in a coordinate-free form suitable for our purposes.

If f is a real-valued function defined on an open subset U of $\mathbf{R}^n$, we write $D_i f$ for the partial derivative of f with respect to the i^{th} variable. If $I = (i_1, \ldots, i_r)$ where $i_k \in \{1, \ldots, n\}$ for $1 \le k \le r$, we write $D_I f = D_{i_1} \ldots D_{i_r} f$ and call the latter an r^{th} *order partial derivative of* f.

Definition 3.1. Let U be open in $\mathbf{R}^n$ and $f : U \to \mathbf{R}$ a function. We say f is C^∞ if f and all its possible partial derivatives of all orders exist and are continuous on U.

If $g : U \to V$ where U is open in $\mathbf{R}^m$ and V open in $\mathbf{R}^n$, we say g is C^∞ if each of its coordinate functions $g_1, \ldots, g_n$ (relative to the standard basis of $\mathbf{R}^n$) is C^∞.

Thus if a real-valued function is C^∞, so are all its partial derivatives of all orders. In what follows, the results can usually be made more general by replacing C^∞ hypotheses by weaker differentiability hypotheses. Since we only need the C^∞ case, we sacrifice this generality in exchange for simplicity of statement.

Definition 3.2. If $f : U \to V$ is C^∞ where U is open in $\mathbf{R}^m$ and V open in $\mathbf{R}^n$, then for each $x \in U$, the *derivative of* f *at* x, denoted $Df(x)$, is the linear transformation $Df(x) : \mathbf{R}^m \to \mathbf{R}^n$ whose matrix with respect to the standard bases is the Jacobian matrix of f at x, i.e.

$$Df(x) = \begin{pmatrix} D_1 f_1(x) & D_2 f_1(x) & \ldots & D_m f_1(x) \\ D_1 f_2(x) & D_2 f_2(x) & \ldots & D_m f_2(x) \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f_n(x) & D_2 f_n(x) & \ldots & D_m f_n(x) \end{pmatrix}.$$

(Here, and possibly later, we identify a linear transformation from $\mathbf{R}^m$ to $\mathbf{R}^n$ with its matrix with respect to the standard bases.)

Theorem 3.3. **(Chain Rule)** *Suppose* $f : U \to V$ *and* $g : V \to W$ *are* C^∞ *maps where* U, V, *and* W *are open in* $\mathbf{R}^m$, $\mathbf{R}^n$, *and* $\mathbf{R}^q$, *respectively. Then the composition* gf *is* C^∞ *and for all* $x \in U$, $D(gf)(x) = Dg(f(x))Df(x)$. □

Theorem 3.4. **(Inverse Function Theorem)** *Suppose* $f : U \to V$ *is a* C^∞ *map where* U *and* V *are open in* $\mathbf{R}^n$. *Suppose* $x_0 \in U$ *is such that the linear transformation* $Df(x_0) : \mathbf{R}^n \to \mathbf{R}^n$ *is invertible. Then there exist open neighborhoods* W *of* x_0 *in* U, X *of* $f(x_0)$ *in* V, *such that* f *maps* W *bijectively onto* X *and* $f^{-1} : X \to W$ *is* C^∞. □

Theorem 3.5. **(First Taylor Theorem)** *Let U be a convex open set in $\mathbf{R}^n$ and f a C^∞ real-valued function defined on U. Let $a = (a_1, \ldots, a_n) \in U$. Then there exist C^∞ real-valued functions $g_1, \ldots, g_n$ on U satisfying:*

(a) $g_i(a) = D_i f(a)$ *for* $1 \le i \le n$.

(b) *For all* $x = (x_1, \ldots, x_n) \in U$,

$$f(x) = f(a) + \sum_{i=1}^{n} g_i(x)(x_i - a_i).$$

Proof. By the convexity of U, for all $t \in [0,1]$ and $x \in U$ we have $tx + (1-t)a \in U$. By the Chain Rule,

$$\frac{d}{dt}\left(f\left(tx + (1-t)a\right)\right) = \sum_{i=1}^{n} D_i f\left(tx + (1-t)a\right)(x_i - a_i).$$

Thus, by the Fundamental Theorem of Calculus,

$$f(x) - f(a) = \sum_{i=1}^{n} \left(\int_0^1 D_i f\left(tx + (1-t)a\right) dt\right)(x_i - a_i).$$

Define $g_i(x) = \int_0^1 D_i f\left(tx + (1-t)a\right) dt$. It follows from the Advanced Calculus theorem on differentiating under the integral sign that each g_i is C^∞. Since $g_i(a) = \int_0^1 D_i f(a)\, dt = D_i f(a)$, the proof is complete. □

Theorem 3.6. **(Second Taylor Theorem)** *Let U be a convex open set in $\mathbf{R}^n$ and f a C^∞ real-valued function on U. Let $a \in U$. Then there exist C^∞ real-valued functions h_{ij} on U, $1 \le i \le n$, $1 \le j \le n$, such that for all $x \in U$,*

$$f(x) = f(a) + \sum_{i=1}^{n} D_i f(a)(x_i - a_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} h_{ij}(x)(x_i - a_i)(x_j - a_j).$$

Proof. By the First Taylor Theorem there exist C^∞ functions g_i on U, $1 \le i \le n$, such that $g_i(a) = D_i f(a)$ and $f(x) = f(a) + \sum_{i=1}^{n} g_i(x)(x_i - a_i)$. Applying the First Taylor Theorem to each of the g_i, we deduce that there exist C^∞ functions h_{ij} on U, $1 \le j \le n$, such that for each $x \in U$, $g_i(x) = g_i(a) + \sum_{j=1}^{n} h_{ij}(x)(x_j - a_j)$. The result now follows by substituting this into the above and the fact that $g_i(a) = D_i f(a)$ for each i. □

Theorem 3.7. *Let $f : U \to V$ be a C^∞ map where U is open in $\mathbf{R}^m$ and V is open in $\mathbf{R}^n$. Let $\| \ \|_1$ and $\| \ \|_2$ be arbitrary norms on $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively. Then for each $a \in U$ there exists a neighborhood N of a in U and a positive constant C such that for all $x \in N$,*

$$\|f(x) - f(a) - Df(a)(x-a)\|_2 \le C\|x-a\|_1^2.$$

Proof. Clearly, if the result holds for one choice of norms $\| \ \|_1$ and $\| \ \|_2$, it holds for all choices by Theorem 1.5. Thus we may suppose $\| \ \|_1$ and $\| \ \|_2$ are the respective box norms relative to the standard bases (see Example 1.3). Given $a \in U$, choose $\varepsilon > 0$ such that the open ball $B(a, \varepsilon)$ (with respect to the standard Euclidean norm) is contained in U. Take $N = B(a, \varepsilon/2)$.

For each $x \in U$,

$$\|f(x) - f(a) - Df(a)(x-a)\|_2 = \max_i \left| f_i(x) - f_i(a) - \sum_{j=1}^{m} D_j f_i(a)(x_j - a_j) \right|.$$

Applying the Second Taylor Theorem to each f_i, there exist C^∞ real-valued functions h_{ijk} on $B(a, \varepsilon)$ such that for all $x \in B(a, \varepsilon)$,

$$f_i(x) - f_i(a) - \sum_{j=1}^{m} D_j f_i(a)(x_j - a_j) = \sum_{j=1}^{m} \sum_{k=1}^{m} h_{ijk}(x)(x_j - a_j)(x_k - a_k).$$

Since $\overline{N}$ is compact and the h_{ijk} are all continuous on $\overline{N}$, there exists a positive constant M such that $|h_{ijk}(x)| \le M$ for all $x \in \overline{N}$ and all i, j, and k. Thus for all $x \in N$,

$$\begin{aligned}\|f(x) - f(a) - Df(a)(x-a)\|_2 &\le \sum_{j=1}^{m} \sum_{k=1}^{m} M\,|x_j - a_j|\,|x_k - a_k| \\ &\le m^2 M \|x-a\|_1^2. \qquad \square\end{aligned}$$

The next result, together with Theorem 3.7, characterizes the derivative $Df(a)$.

Theorem 3.8. *Let $f : U \to V$ be a C^∞ map where U is open in $\mathbf{R}^m$ and V is open in $\mathbf{R}^n$. Let $a \in U$ and suppose $T : \mathbf{R}^m \to \mathbf{R}^n$ is an $\mathbf{R}$-linear transformation with the property that with respect to some choice of norms $\| \ \|_1$ and $\| \ \|_2$ on $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, there exists a neighborhood N of a in U and a positive constant C such that for all $x \in N$,*

$$\|f(x) - f(a) - T(x-a)\|_2 \le C\|x-a\|_1^2.$$

Then $T = Df(a)$.

Proof. By Theorem 3.7 we can suppose, by passing to smaller neighborhoods and larger constants, if necessary, that N is convex and that for all $x \in N$ we have, in addition to the postulated inequality,

$$\|f(x) - f(a) - Df(a)(x-a)\|_2 \le C\|x-a\|_1^2.$$

We then have, for all $x \in N$,

$$\begin{aligned}
&\left\|\Big(T - Df(a)\Big)(x-a)\right\|_2 \\
&\qquad = \left\|f(x) - f(a) - Df(a)(x-a) - \Big(f(x) - f(a) - T(x-a)\Big)\right\|_2 \\
&\qquad \le \|f(x) - f(a) - Df(a)(x-a)\|_2 + \|f(x) - f(a) - T(x-a)\|_2 \\
&\qquad \le 2C\|x-a\|_1^2.
\end{aligned}$$

If $T - Df(a) \neq 0$, we can choose $x_0 \in N$ such that $\Big(T - Df(a)\Big)(x_0 - a) \neq 0$ and hence $\left\|\Big(T - Df(a)\Big)(x_0 - a)\right\|_2 \neq 0$. For any positive $\delta \le 1$ we have

$$\begin{aligned}
\delta\left\|\Big(T - Df(a)\Big)(x_0 - a)\right\|_2 &= \left\|\Big(T - Df(a)\Big)\Big(\delta(x_0 - a)\Big)\right\|_2 \\
&= \left\|\Big(T - Df(a)\Big)(\delta x_0 + (1-\delta)a - a)\right\|_2.
\end{aligned}$$

By convexity of N, $\delta x_0 + (1-\delta)a \in N$ and so

$$\begin{aligned}
&\left\|\Big(T - Df(a)\Big)(\delta x_0 + (1-\delta)a - a)\right\|_2 \le 2C\,\|\delta x_0 + (1-\delta)a - a\|_1^2 \\
&\qquad = 2C\,\|\delta(x_0 - a)\|_1^2 = 2C\delta^2\,\|x_0 - a\|_1^2.
\end{aligned}$$

Thus $\left\|\Big(T - Df(a)\Big)(x_0 - a)\right\|_2 \le 2C\delta\,\|x_0 - a\|_1^2$ whenever $0 < \delta \le 1$. It follows that $\left\|\Big(T - Df(a)\Big)(x_0 - a)\right\|_2 = 0$, a contradiction. □

Example 3.9. Let $T : \mathbf{R}^m \to \mathbf{R}^n$ be an $\mathbf{R}$-linear transformation. T is C^∞ since its coordinate functions are linear functions. For any a and x in $\mathbf{R}^m$,

$$T(x) - T(a) - T(x-a) = 0.$$

It follows from Theorem 3.8 that $DT(a) = T$ for all $a \in \mathbf{R}^m$.

We now wish to talk about C^∞ maps between open subsets of general finite-dimensional real vector spaces, perhaps without any preferred bases. This is made possible by the next lemma.

Lemma 3.10. *Let X and Y be finite-dimensional real vector spaces of dimensions m and n, respectively. Let $f : U \to V$ be a function where U and V are open in X and Y, respectively. Suppose $\alpha_1, \alpha_2 : X \to \mathbf{R}^m$ and $\beta_1, \beta_2 : Y \to \mathbf{R}^n$ are $\mathbf{R}$-isomorphisms. Then $\beta_1 f \alpha_1^{-1} : \alpha_1(U) \to \beta_1(V)$ is C^∞ if and only if $\beta_2 f \alpha_2^{-1} : \alpha_2(U) \to \beta_2(V)$ is C^∞.*

Proof. We have $\beta_2 f \alpha_2^{-1} = \left(\beta_2\beta_1^{-1}\right)\left(\beta_1 f \alpha_1^{-1}\right)\left(\alpha_1\alpha_2^{-1}\right)$. The functions $\alpha_1\alpha_2^{-1} : \alpha_2(U) \to \alpha_1(U)$ and $\beta_2\beta_1^{-1} : \beta_1(V) \to \beta_2(V)$ are both C^∞ since they are restrictions to open sets of $\mathbf{R}$-linear transformations $\alpha_1\alpha_2^{-1} : \mathbf{R}^m \to \mathbf{R}^m$ and $\beta_2\beta_1^{-1} : \mathbf{R}^n \to \mathbf{R}^n$, respectively. Thus if $\beta_1 f \alpha_1^{-1}$ is C^∞, so is $\beta_2 f \alpha_2^{-1}$. □

Definition 3.11. Suppose X and Y are finite-dimensional real vector spaces of dimensions m and n, respectively. Suppose $f : U \to V$ is a function where U and V are open in X and Y, respectively. We say f is C^∞ if for some (and hence all) $\mathbf{R}$-isomorphisms $\alpha : X \to \mathbf{R}^m$ and $\beta : Y \to \mathbf{R}^n$, $\beta f \alpha^{-1} : \alpha(U) \to \beta(V)$ is C^∞.

Proposition 3.12. *Let X, Y, and Z be finite-dimensional real vector spaces. Suppose $f : U \to V$ and $g : V \to W$ are C^∞ maps where U, V, and W are open in X, Y, and Z, respectively. Then $gf : U \to W$ is C^∞.*

We leave the proof as an exercise.

Proposition 3.13. *Let X and Y be real finite-dimensional vector spaces, and $T : X \to Y$ an $\mathbf{R}$-linear transformation. Then T is C^∞.*

Proof. Let $\alpha : X \to \mathbf{R}^m$ and $\beta : Y \to \mathbf{R}^n$ be $\mathbf{R}$-isomorphisms. Then $\beta T \alpha^{-1} : \mathbf{R}^m \to \mathbf{R}^n$ is an $\mathbf{R}$-linear transformation, and hence C^∞. □

Proposition 3.14. *Let X and Y be finite-dimensional real vector spaces, and B an $\mathbf{R}$-basis for Y. Let $f : U \to V$ be a function where U and V are open in X and Y, respectively. Then f is C^∞ if and only if the coordinate functions of f relative to B are C^∞.*

Proof. Say $B = \{b_1, \ldots, b_n\}$. Recall from Section 1 (just preceding Proposition 1.11), that the coordinate functions $f^B_{b_i} : U \to \mathbf{R}$ of f relative to B are characterized by $f(x) = \sum_i f^B_{b_i}(x) b_i$ for all $x \in X$. Let $\alpha : Y \to \mathbf{R}^n$ be given by $\alpha\left(\sum_i r_i b_i\right) = (r_1, \ldots, r_n)$. Thus $\alpha f(x) = \left(f^B_{b_1}(x), \ldots, f^B_{b_n}(x)\right)$ for all $x \in U$.

Let $\beta : X \to \mathbf{R}^m$ be any $\mathbf{R}$-isomorphism. Then for all $u \in \mathbf{R}^n$, $\alpha f \beta^{-1}(u) = \left(f^B_{b_1}(\beta^{-1}(u)), \ldots, f^B_{b_n}(\beta^{-1}(u))\right)$. Thus the coordinate functions of $\alpha f \beta^{-1}$ relative to the standard basis of $\mathbf{R}^n$ are given by $(\alpha f \beta^{-1})_i = f^B_{b_i} \circ \beta^{-1}$, $1 \le i \le n$. We have the equivalences

$$
\begin{aligned}
f \text{ is } C^\infty &\Leftrightarrow \alpha f \beta^{-1} \text{ is } C^\infty \quad \text{(Definition 3.11)} \\
&\Leftrightarrow (\alpha f \beta^{-1})_i \text{ is } C^\infty \text{ for } 1 \le i \le n \quad \text{(Definition 3.1)} \\
&\Leftrightarrow f^B_{b_i} \circ \beta^{-1} \text{ is } C^\infty \text{ for } 1 \le i \le n \quad \text{(by the equality above)} \\
&\Leftrightarrow f^B_{b_i} \text{ is } C^\infty \text{ for } 1 \le i \le n \quad \text{(by Proposition 3.12 since,} \\
&\qquad\qquad \text{by Proposition 3.13, both } \beta \text{ and } \beta^{-1} \text{ are } C^\infty.) \quad \square
\end{aligned}
$$

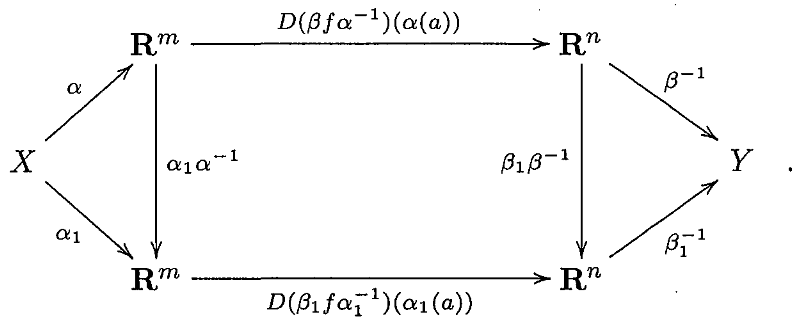
Our next task is to extend the notion of derivative to the class of C^∞ functions of Definition 3.11.

Lemma 3.15. *Let X and Y be finite-dimensional real vector spaces of dimensions m and n, respectively. Suppose $f : U \to V$ is a C^∞ map where U and V are open in X and Y, respectively. Let $\alpha : X \to \mathbf{R}^m$ and $\beta : Y \to \mathbf{R}^n$ be choices of $\mathbf{R}$-isomorphisms. Then for each $a \in U$ the composition*

$$X \xrightarrow{\alpha} \mathbf{R}^m \xrightarrow{D(\beta f \alpha^{-1})(\alpha(a))} \mathbf{R}^n \xrightarrow{\beta^{-1}} Y$$

is an $\mathbf{R}$-linear transformation which depends only on f and a, and not on the choices of α and β.

Proof. Let $\alpha_1 : X \to \mathbf{R}^m$ and $\beta_1 : Y \to \mathbf{R}^n$ be other choices of $\mathbf{R}$-isomorphisms. Consider the diagram



The two triangles trivially commute, and so it remains only to check commutativity of the rectangle.

By Example 3.9, $\alpha_1\alpha^{-1} = D(\alpha_1\alpha^{-1})(\alpha(a))$ and $\beta_1\beta^{-1} = D(\beta_1\beta^{-1})(\beta f(a))$. Thus

$$D(\beta_1 f \alpha_1^{-1})(\alpha_1(a))\alpha_1\alpha^{-1} = D(\beta_1 f \alpha_1^{-1})(\alpha_1(a))D(\alpha_1\alpha^{-1})(\alpha(a)) \qquad \text{and}$$
$$\beta_1\beta^{-1}D(\beta f \alpha^{-1})(\alpha(a)) = D(\beta_1\beta^{-1})(\beta f(a))D(\beta f \alpha^{-1})(\alpha(a)).$$

By the Chain Rule (Theorem 3.3),

$$\begin{aligned} D(\beta_1 f \alpha_1^{-1})(\alpha_1(a))D(\alpha_1\alpha^{-1})(\alpha(a)) &= D\Big((\beta_1 f \alpha_1^{-1}) \circ (\alpha_1\alpha^{-1})\Big)(\alpha(a)) \\ &= D(\beta_1 f \alpha^{-1})(\alpha(a)), \qquad \text{and} \\ D(\beta_1\beta^{-1})(\beta f(a))D(\beta f \alpha^{-1})(\alpha(a)) &= D\Big((\beta_1\beta^{-1}) \circ (\beta f \alpha^{-1})\Big)(\alpha(a)) \\ &= D(\beta_1 f \alpha^{-1})(\alpha(a)). \end{aligned}$$

By comparison, we are done. $\square$

Lemma 3.15 permits the following definition.

Definition 3.16. If X and Y are finite-dimensional real vector spaces of dimensions m and n, respectively, and $f : U \to V$ is a C^∞ map where U and V are open in X and Y, respectively, then for each $a \in U$ the *derivative of f at a* is the $\mathbf{R}$-linear transformation $Df(a) : X \to Y$ defined by

$$Df(a) = \beta^{-1}D(\beta f \alpha^{-1})\big(\alpha(a)\big)\alpha$$

where $\alpha : X \to \mathbf{R}^m$ and $\beta : Y \to \mathbf{R}^n$ are arbitrary $\mathbf{R}$-isomorphisms.

Theorem 3.17. *Let X and Y be finite-dimensional real vector spaces and $f : U \to V$ a C^∞ map where U and V are open in X and Y, respectively. Suppose $\| \ \|_X$ and $\| \ \|_Y$ are norms on X and Y, respectively. Then for each $a \in U$ there exists a neighborhood N of a in U and a positive constant C such that for all $x \in N$,*

$$\|f(x) - f(a) - Df(a)(x-a)\|_Y \le C\|x-a\|_X^2.$$

Proof. Let $\alpha : X \to \mathbf{R}^m$ and $\beta : Y \to \mathbf{R}^n$ be $\mathbf{R}$-isomorphisms. Let $\| \ \|_1$ and $\| \ \|_2$ be the norms on $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, such that α and β are isometries with respect to $\| \ \|_X$, $\| \ \|_1$ and $\| \ \|_Y$, $\| \ \|_2$, respectively. By Theorem 3.7 there exists a neighborhood N of a in U and a positive constant C such that for each $x \in N$,

$$\|(\beta f \alpha^{-1})(\alpha(x)) - (\beta f \alpha^{-1})(\alpha(a)) - D(\beta f \alpha^{-1})(\alpha(a))(\alpha(x) - \alpha(a)\|_2 \\ \leq C\|\alpha(x) - \alpha(a)\|_1^2$$

i.e.,

$$\|\beta(f(x) - f(a)) - \beta Df(a)(x-a)\|_2 \leq C\|\alpha(x-a)\|_1^2.$$

Since α and β are isometries, the left-hand side of this last inequality is equal to $\|f(x) - f(a) - Df(a)(x-a)\|_Y$ while the right-hand side equals $C\|x-a\|_X^2$. □

The next theorem generalizes Theorem 3.8. Its proof is the same as that of Theorem 3.8.

Theorem 3.18. *Suppose X and Y are finite-dimensional real vector spaces, and that $f : U \to V$ is a C^∞ map where U and V are open in X and Y, respectively. Let $a \in U$ and suppose $T : X \to Y$ is an $\mathbf{R}$-linear transformation with the property that with respect to some choice of norms $\| \ \|_X$ and $\| \ \|_Y$ on X and Y, respectively, there exists a neighborhood N of a in U and a positive constant C such that for all $x \in N$,*

$$\|f(x) - f(a) - T(x-a)\|_Y \leq C\|x-a\|_X^2.$$

Then $T = Df(a)$. □

Example 3.19. Suppose X and Y are finite-dimensional real vector spaces and $T : X \to Y$ is an $\mathbf{R}$-linear transformation. Using the same argument that was used to deduce Example 3.9 from Theorem 3.8, it follows from Theorem 3.18 that $DT(a) = T$ for all $a \in X$.

The next theorem generalizes Theorem 3.3.

Theorem 3.20. **(Chain Rule)** *Let X, Y, and Z be finite-dimensional real vector spaces. Suppose $f : U \to V$ and $g : V \to W$ are C^∞ where U, V, and W are open in X, Y, and Z, respectively. Then for each $a \in U$, $D(gf)(a) = Dg(f(a))Df(a)$.*

Proof. Let $\alpha : X \to \mathbf{R}^m$, $\beta : Y \to \mathbf{R}^n$, and $\gamma : Z \to \mathbf{R}^q$ be $\mathbf{R}$-isomorphisms. Then

$$\begin{aligned} D(gf)(a) &= \gamma^{-1}D(\gamma g f \alpha^{-1})(\alpha(a))\alpha \\ &\qquad\qquad \text{(by Definition 3.16)} \\ &= \gamma^{-1}D(\gamma g \beta^{-1}\beta f \alpha^{-1})(\alpha(a))\alpha \\ &= \gamma^{-1}D(\gamma g \beta^{-1})\Big(\beta(f(a))\Big)D(\beta f \alpha^{-1})(\alpha(a))\alpha \\ &\qquad\qquad \text{(by Theorem 3.3)} \\ &= \gamma^{-1}D(\gamma g \beta^{-1})\Big(\beta(f(a))\Big)\beta\beta^{-1}D(\beta f \alpha^{-1})(\alpha(a))\alpha \\ &= Dg(f(a))Df(a) \\ &\qquad\qquad \text{(by Definition 3.16).} \end{aligned}$$

□

Proposition 3.21. **(Local Property)** *Suppose X and Y are finite-dimensional real vector spaces and $f : U \to V$ is a function where U and V are open in X and Y, respectively.*

(a) *Suppose U_0 and V_0 are open in U and V, respectively, and that $f(U_0) \subset V_0$. If f is C^∞, then the restriction $f : U_0 \to V_0$ is C^∞.*

(b) *If U is a union of open subsets such that the restriction of f to each of these is C^∞, then f is C^∞.*

(c) *If f is C^∞, $a \in U$, and g is a C^∞ function from an open subset of X to an open subset of Y which agrees with f on some neighborhood of a, then $Df(a) = Dg(a)$.*

Proof. Part (c) is a formal consequence of Theorems 3.17 and 3.18. It also, of course, follows quickly from the definitions since the partial derivatives $D_i f(a)$ only depend on f in a neighborhood of a. In case $X = \mathbf{R}^m$ and $Y = \mathbf{R}^n$ for some m and n, parts (a) and (b) are immediate since existence and continuity of the $D_I f$ at a point only depends on f in a neighborhood of that point. Using Definition 3.11, the general case now follows easily. □

We conclude this section with a version of the Product Rule suitable for our later needs. If X is a real finite-dimensional vector space, U open in X, and $f, g : U \to \mathbf{R}$ are C^∞, let $f \cdot g : U \to \mathbf{R}$ be given by $(f \cdot g)(x) = f(x)g(x)$ for all $x \in U$.

Theorem 3.22. **(Product Rule)** *Let X be a real finite-dimensional vector space and suppose U is an open subset of X. Suppose $f, g : U \to \mathbf{R}$ are C^∞. Then $f \cdot g : U \to \mathbf{R}$ is C^∞, and for all $x \in U$,*

$$D(f \cdot g)(x) = \big(Df(x)\big)g(x) + f(x)Dg(x).$$

Proof. We first consider the case $X = \mathbf{R}^n$. Then $f \cdot g$ is C^∞ by the classical Product Rule. We have

$$D(f \cdot g)(x) = \begin{pmatrix} D_1(f \cdot g)(x) & D_2(f \cdot g)(x) & \dots & D_n(f \cdot g)(x) \end{pmatrix}.$$

By the classical Product Rule, the i^{th} component of the latter is $\big(D_i f(x)\big)g(x) + f(x)D_i g(x)$, and the result follows.

For the general case, choose an arbitrary $\mathbf{R}$-isomorphism $\alpha : X \to \mathbf{R}^n$. Note that $(f \cdot g)\alpha^{-1} = (f\alpha^{-1}) \cdot (g\alpha^{-1})$. It now follows from the case above and Definition 3.11 (using $\beta = 1_{\mathbf{R}}$) that $f \cdot g$ is C^∞. By Definition 3.16 (using $\beta = 1_{\mathbf{R}}$),

$$D(f \cdot g)(x) = D\big((f \cdot g)\alpha^{-1}\big)\big(\alpha(x)\big)\alpha.$$

By the case already done,

$$\begin{aligned} &D\big((f \cdot g)\alpha^{-1}\big)\big(\alpha(x)\big) \\ &\qquad = D(f\alpha^{-1})\big(\alpha(x)\big)(g\alpha^{-1})\big(\alpha(x)\big) + (f\alpha^{-1})\big(\alpha(x)\big)D(g\alpha^{-1})\big(\alpha(x)\big) \\ &\qquad = \Big(D(f\alpha^{-1})\big(\alpha(x)\big)\Big)g(x) + f(x)D(g\alpha^{-1})\big(\alpha(x)\big). \end{aligned}$$

The result now follows by composing with α on the right and invoking Definition 3.16. □

Exercises for §3

1. Prove Proposition 3.12.

2. Let n and q be positive integers. Write $M_{n,q}(\mathbf{R})$ for the vector space of all n-rowed, q-columned real matrices. Let $\mathrm{Sym}(n) = \{T \in M_n(\mathbf{R}) \mid T = T^*\}$ where T^* denotes the transpose of T. Define $f : M_{n,q}(\mathbf{R}) \to \mathrm{Sym}(n)$ by $f(T) = TT^*$ for $T \in M_{n,q}$. (Note: $(TT^*)^* = T^{**}T^* = TT^*$ and so $TT^* \in \mathrm{Sym}(n)$.)

(a) Show that f is a C^∞ map.

(b) Show that for each $A \in M_{n,q}(\mathbf{R})$, $Df(A) : M_{n.q}(\mathbf{R}) \to \mathrm{Sym}(n)$ is given by $Df(A)(T) = AT^* + TA^*$.

3. Let m, n, and q be positive integers. Define $f : M_{m,n}(\mathbf{R}) \times M_{n,q}(\mathbf{R}) \to M_{m,q}(\mathbf{R})$ by $f(S,T) = ST$ (matrix multiplication).

(a) Show that f is C^∞.

(b) Show that for each $(A, B) \in M_{m,n}(\mathbf{R}) \times M_{n,q}(\mathbf{R})$,

$$Df(A,B) : M_{m,n}(\mathbf{R}) \times M_{n,q}(\mathbf{R}) \to M_{m,q}(\mathbf{R})$$

is given by $Df(A,B)(S,T) = AT + SB$.

4. Suppose X_1, X_2, Y_1, and Y_2 are finite-dimensional real vector spaces. Suppose U_1, U_2, V_1, and V_2 are open, respectively, in X_1, X_2, Y_1, and Y_2. Suppose $f : U_1 \to V_1$ and $g : U_2 \to V_2$ are C^∞.

(a) Show that $f \times g : U_1 \times U_2 \to V_1 \times V_2$, given by $(f \times g)(a,b) = \big(f(a), g(b)\big)$ is C^∞.

(b) Show that for each $(a, b) \in U_1 \times U_2$, $D(f \times g)(a,b) : X_1 \times X_2 \to Y_1 \times Y_2$ is equal to $Df(a) \times Dg(b)$.

5. Let n be a positive integer. Recall, from Example 2.2, that $GL_n(\mathbf{R})$ is open in the real vector space $M_n(\mathbf{R})$. Let $f : GL_n(\mathbf{R}) \to GL_n(\mathbf{R})$ be given by $f(T) = T^{-1}$.

(a) Show that f is C^∞.

(b) Show that for each $A \in GL_n(\mathbf{R})$, $Df(A) : M_n(\mathbf{R}) \to M_n(\mathbf{R})$ is given by $Df(A)(T) = -A^{-1}TA^{-1}$.

4. Smooth Manifolds

If $f : X \to Y$ is a function, we will sometimes write $X = \operatorname{dom} f$, the *domain of* f, and $Y = \operatorname{codom} f$, the *codomain of* f. Recall (from Definition 2.1) that if M is a topological n-manifold, a *chart* φ for M is a homeomorphism where $\operatorname{dom}\varphi$ is an open subset of M and $\operatorname{codom}\varphi$ is an open subset of some real n-dimensional vector space.

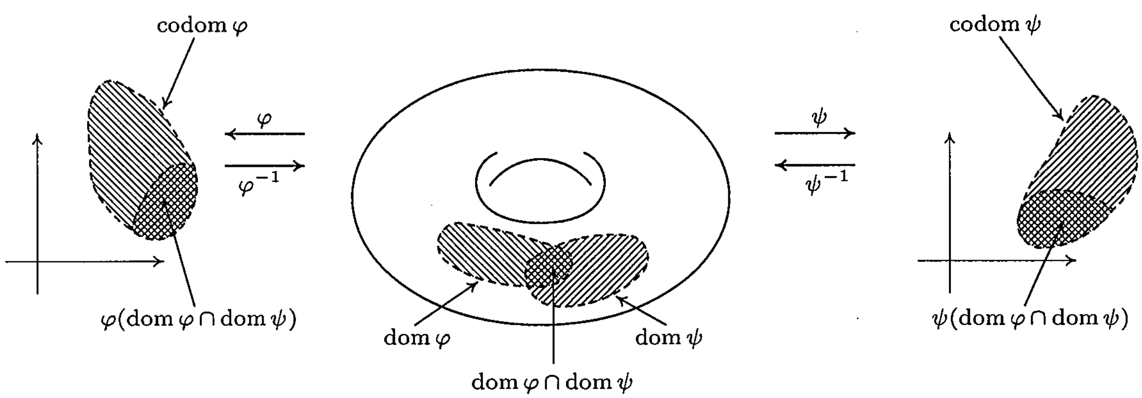
Definition 4.1. Let M be a topological n-manifold. Suppose φ and ψ are charts for M. We say φ and ψ are *smoothly related* if the compositions

$$\varphi(\operatorname{dom}\varphi \cap \operatorname{dom}\psi) \xrightarrow{\varphi^{-1}} \operatorname{dom}\varphi \cap \operatorname{dom}\psi \xrightarrow{\psi} \psi(\operatorname{dom}\varphi \cap \operatorname{dom}\psi)$$

and

$$\psi(\operatorname{dom}\varphi \cap \operatorname{dom}\psi) \xrightarrow{\psi^{-1}} \operatorname{dom}\varphi \cap \operatorname{dom}\psi \xrightarrow{\varphi} \varphi(\operatorname{dom}\varphi \cap \operatorname{dom}\psi)$$

(i.e. the *overlap maps for φ and ψ*) are both C^∞. (By convention, if $\operatorname{dom}\varphi$ and $\operatorname{dom}\psi$ are disjoint, we consider φ and ψ as being smoothly related.)



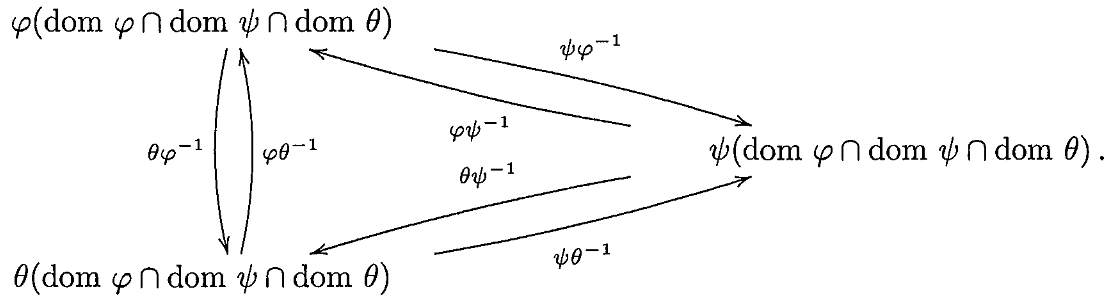
An atlas for M is a *smooth atlas* if all pairs of charts in that atlas are smoothly related.

If $\mathcal{A}$ and $\mathcal{B}$ are smooth atlases for M, $\mathcal{A}$ is *smoothly equivalent to* $\mathcal{B}$ if every chart in $\mathcal{A}$ is smoothly related to every chart in $\mathcal{B}$.

The relation "is smoothly related to" on charts is clearly reflexive and symmetric. Note, however, that it is *not* transitive. (A trivial example is given by two charts φ_1 and φ_3 which are not smoothly related, and a chart φ_2 such that $\operatorname{dom}\varphi_2$ is disjoint from both $\operatorname{dom}\varphi_1$ and $\operatorname{dom}\varphi_3$. Then φ_1 is smoothly related to φ_2 and φ_2 is smoothly related to φ_3, but φ_1 is not smoothly related to φ_3.) However we have the following weaker transitivity.

Lemma 4.2. *Let M be a topological manifold and suppose $\mathcal{A}$, $\mathcal{B}$ are smooth atlases for M such that $\mathcal{A}$ is smoothly equivalent to $\mathcal{B}$. Suppose φ is a manifold chart for M which is smoothly related to each member of $\mathcal{A}$. Then φ is smoothly related to each member of $\mathcal{B}$.*

Proof. Let $\psi \in \mathcal{B}$. $\{\, \varphi(\operatorname{dom} \varphi \cap \operatorname{dom} \psi \cap \operatorname{dom} \theta) \mid \theta \in \mathcal{A} \,\}$ is an open cover of $\varphi(\operatorname{dom} \varphi \cap \operatorname{dom} \psi)$ and $\{\, \psi(\operatorname{dom} \varphi \cap \operatorname{dom} \psi \cap \operatorname{dom} \theta) \mid \theta \in \mathcal{A} \,\}$ is an open cover of $\psi(\operatorname{dom} \varphi \cap \operatorname{dom} \psi)$. Thus, by Proposition 3.21(b) (Local Property), to prove that the overlap maps $\psi\varphi^{-1}$ and $\varphi\psi^{-1}$ are C^∞, it suffices to show that for each $\theta \in \mathcal{A}$, the restrictions of these overlap maps to $\varphi(\operatorname{dom} \varphi \cap \operatorname{dom} \psi \cap \operatorname{dom} \theta)$ and $\psi(\operatorname{dom} \varphi \cap \operatorname{dom} \psi \cap \operatorname{dom} \theta)$, respectively, are C^∞. We have the diagram



with both the outside triangle and inside triangle commuting. $\theta\varphi^{-1}$ and $\varphi\theta^{-1}$ are both C^∞ since φ is smoothly related to every member of $\mathcal{A}$. $\psi\theta^{-1}$ and $\theta\psi^{-1}$ are both C^∞ since $\mathcal{A}$ is smoothly equivalent to $\mathcal{B}$. Thus $\varphi\psi^{-1}$ and $\psi\varphi^{-1}$ are compositions of C^∞ maps and hence are C^∞. □

Proposition 4.3. *The relation "smooth equivalence" on the class of all smooth atlases for a given topological manifold is an equivalence relation.*

Proof. The reflexive and symmetric properties are immediate. Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are smooth atlases for M such that $\mathcal{A}$ is smoothly equivalent to $\mathcal{B}$ and $\mathcal{B}$ is smoothly equivalent to $\mathcal{C}$. Suppose $\varphi \in \mathcal{A}$. Then since $\mathcal{A}$ is smoothly equivalent to $\mathcal{B}$, φ is smoothly related to each member of $\mathcal{B}$. Since $\mathcal{B}$ is smoothly equivalent to $\mathcal{C}$, it follows from Lemma 4.2 that φ is smoothly related to each member of $\mathcal{C}$. □

Definition 4.4. A *smooth structure* $\mathcal{S}$ on a topological n-manifold M is a smooth equivalence class of smooth atlases for M. The pair $(M, \mathcal{S})$ is then called a *smooth n-manifold.* Any smooth atlas for M which represents $\mathcal{S}$ is called an *$\mathcal{S}$-admissible atlas.* Any manifold chart for M which is a member of an $\mathcal{S}$-admissible atlas is called an *$\mathcal{S}$-admissible chart.*

Not all topological manifolds admit smooth structures. The first example, due to M. Kervaire [*A manifold which does not admit any differentiable structure*, Comment. Math. Helv. **34** (1960), 257–270] was of a compact topological 10-manifold admitting no smooth structure. The non-existence of a smooth structure on the Kervaire example was proved using methods of algebraic topology.

Frequently, a particular smooth structure on a topological manifold M will be understood, and we will abuse notation and talk about the "smooth manifold M" rather than the "smooth manifold $(M, \mathcal{S})$". This is analogous to using the same notation to denote both a topological space and its underlying set.

Example 4.5. If U is an open subset of a finite-dimensional real vector space, the atlas consisting of the single chart 1_U is clearly smooth. The smooth structure on U represented by this atlas will be called the *standard smooth structure on U.* If a smooth structure on such a U is not specified, the standard smooth structure will be understood.

Example 4.6. For $n \geq 1$, the atlas for S^n given in Example 2.4 is smooth. In fact the overlap maps

$$\varphi^- \left(\varphi^+\right)^{-1}, \varphi^+ \left(\varphi^-\right)^{-1} : \mathbf{R}^n - 0 \to \mathbf{R}^n - 0$$

are given by $\varphi^- \left(\varphi^+\right)^{-1}(y) = \varphi^+ \left(\varphi^-\right)^{-1}(y) = \|y\|^{-2} y$, which is C^∞. The smooth structure on S^n represented by this smooth atlas is called the *standard smooth structure on* S^n.

It is not hard to show (see the exercises for §4) that the atlas for S^n given in Example 2.3 is smooth, and that it represents the standard smooth structure on S^n.

Example 4.7. For $n \geq 1$, it follows from Example 2.5 that $P^n(\mathbf{R})$ has an atlas $\{\theta_1, \ldots, \theta_{n+1}\}$ where $\theta_i : V_i \to E^n$ is given by

$$\theta_i([x_1, \ldots, x_{n+1}]) = \frac{x_i}{|x_i|}(x_1, \ldots, \widehat{x}_i, \ldots, x_{n+1}).$$

We proceed to check that the above atlas is smooth.

For $i \neq j$,

$$\theta_i(V_i \cap V_j) = \begin{cases} \{y \in E^n \mid y_j \neq 0\} & \text{if } j < i, \\ \{y \in E^n \mid y_{j-1} \neq 0\} & \text{if } j > i. \end{cases}$$

One checks that for $i \neq j$, $\theta_j \theta_i^{-1} : \theta_i(V_i \cap V_j) \to \theta_j(V_i \cap V_j)$ is given by

$$\theta_j \theta_i^{-1}(y) = \begin{cases} \dfrac{y_j}{|y_j|}(y_1, \ldots, \widehat{y}_j, \ldots, y_{i-1}, \sqrt{1 - \|y\|^2}, y_i, \ldots, y_n) & \text{if } j < i, \\ \dfrac{y_{j-1}}{|y_{j-1}|}(y_1, \ldots, y_{i-1}, \sqrt{1 - \|y\|^2}, y_i, \ldots, \widehat{y}_{j-1}, \ldots, y_n) & \text{if } j > i, \end{cases}$$

which is C^∞. The smooth structure on $P^n(\mathbf{R})$ represented by the above smooth atlas is called the *standard smooth structure on* $P^n(\mathbf{R})$.

Example 4.8. Let $h : \mathbf{R} \to \mathbf{R}$ be any homeomorphism. Then the one element set $\{h\}$ is a smooth atlas for $\mathbf{R}$. If h is not C^∞, e.g. if

$$h(x) = \begin{cases} x & \text{if } x \leq 0, \\ 2x & \text{if } x > 0, \end{cases}$$

then $\{h\}$ is not smoothly equivalent to $\{1_{\mathbf{R}}\}$ since the overlap map $h 1_{\mathbf{R}}^{-1} = h : \mathbf{R} \to \mathbf{R}$ is not C^∞.

Proposition 4.9. *Let $(M, \mathcal{S})$ be a smooth manifold. Then:*

(a) *Any two $\mathcal{S}$-admissible charts are smoothly related.*

(b) *If $\mathcal{A}$ and $\mathcal{B}$ are atlases for M such that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B}$ is $\mathcal{S}$-admissible, then $\mathcal{A}$ is $\mathcal{S}$-admissible.*

(c) *If $\mathcal{A}$ and $\mathcal{B}$ are smooth atlases for M such that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{A}$ is $\mathcal{S}$-admissible, then $\mathcal{B}$ is $\mathcal{S}$-admissible.*

(d) *If φ is an $\mathcal{S}$-admissible chart and U is open in* dom φ*, then the restriction $\varphi : U \to \varphi(U)$ is an $\mathcal{S}$-admissible chart.*

(e) *If φ is a manifold chart for M and $\mathcal{A}$ an $\mathcal{S}$-admissible atlas, then φ is an $\mathcal{S}$-admissible chart if and only if φ is smoothly related to each member of $\mathcal{A}$.*

(f) *If $\mathcal{A}$ is an $\mathcal{S}$-admissible atlas and $\mathcal{C}$ is any set of $\mathcal{S}$-admissible charts, then $\mathcal{A} \cup \mathcal{C}$ is an $\mathcal{S}$-admissible atlas.*

Proof of (a). Suppose φ and ψ are $\mathcal{S}$-admissible. Then there exist $\mathcal{S}$-admissible atlases $\mathcal{A}$ and $\mathcal{B}$ such that $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B}$ are both $\mathcal{S}$-admissible, $\mathcal{A}$ is smoothly equivalent to $\mathcal{B}$. Thus every chart in $\mathcal{A}$ is smoothly related to every chart in $\mathcal{B}$. In particular, φ is smoothly related to ψ.

Proofs of (b) *and* (c) Let $\mathcal{A}$ and $\mathcal{B}$ be as in either (b) or (c). In either case, since $\mathcal{B}$ is a smooth atlas, it follows trivially that $\mathcal{A}$ is a smooth atlas and that each chart in $\mathcal{A}$ is smoothly related to each chart in $\mathcal{B}$. Thus $\mathcal{A}$ is smoothly equivalent to $\mathcal{B}$. Thus if one of these atlases is $\mathcal{S}$-admissible, so is the other.

Proof of (d). Say φ belongs to the $\mathcal{S}$-admissible atlas $\mathcal{A}$. Let $\varphi' : U \to \varphi(U)$ denote the restriction of φ. Certainly, φ' is a chart for M, and so $\mathcal{A} \cup \{\varphi'\}$ is an atlas for M. For any $\psi \in \mathcal{A}$, smoothness of $\varphi\psi^{-1}$ and $\psi\varphi^{-1}$ and the Local Property (Proposition 3.21(a)) yield smoothness of the respective restrictions $\varphi'\psi^{-1}$ and $\psi\varphi'^{-1}$. It follows that $\mathcal{A} \cup \{\varphi'\}$ is a smooth atlas for M. Thus, since $\mathcal{A} \subset \mathcal{A} \cup \{\varphi'\}$, it follows from part (c) that $\mathcal{A} \cup \{\varphi'\}$ is $\mathcal{S}$-admissible, and hence each of its members (in particular φ') is an $\mathcal{S}$-admissible chart.

Proof of (e). Suppose φ is $\mathcal{S}$-admissible. Then there exists an $\mathcal{S}$-admissible atlas $\mathcal{B}$ for M such that $\varphi \in \mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B}$ are both $\mathcal{S}$-admissible, $\mathcal{B}$ is smoothly related to $\mathcal{A}$, and so every chart of $\mathcal{B}$ is smoothly related to every chart of $\mathcal{A}$. In particular, φ is smoothly related to each member of $\mathcal{A}$.

Conversely, suppose φ is smoothly related to each member of $\mathcal{A}$. Then $\mathcal{A} \cup \{\varphi\}$ is a smooth atlas for M which contains $\mathcal{A}$, and hence is $\mathcal{S}$-admissible by part (c). It follows that φ is $\mathcal{S}$-admissible.

Proof of (f). Since $\mathcal{A} \cup \mathcal{C}$ consists of $\mathcal{S}$-admissible charts, it follows from part (a) that any two charts in $\mathcal{A} \cup \mathcal{C}$ are smoothly related, and so $\mathcal{A} \cup \mathcal{C}$ is a smooth atlas for M. Since $\mathcal{A} \subset \mathcal{A} \cup \mathcal{C}$ and $\mathcal{A}$ is $\mathcal{S}$-admissible, it follows from part (c) that $\mathcal{A} \cup \mathcal{C}$ is an $\mathcal{S}$-admissible atlas. $\square$

It follows from Proposition 4.9(f) that if $(M, \mathcal{S})$ is a smooth n-manifold, the set of all $\mathcal{S}$-admissible charts having codomains contained in a fixed n-dimensional real vector space V is an $\mathcal{S}$-admissible atlas, the unique maximal $\mathcal{S}$-admissible atlas with charts having codomains contained in V.

We next consider products of smooth manifolds. Recall from Proposition 2.6 that if M is a topological m-manifold with atlas $\mathcal{A}$ and N is a topological n-manifold with atlas $\mathcal{B}$, then $M \times N$ is a topological $(m+n)$-manifold with atlas $\{\varphi \times \psi \mid \varphi \in \mathcal{A}, \psi \in \mathcal{B}\}$. We denote this atlas on $M \times N$ by $\mathcal{A} \times \mathcal{B}$.

Proposition 4.10. *Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds. Suppose $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{S}$- and $\mathcal{T}$-admissible atlases, respectively. Then $\mathcal{A} \times \mathcal{B}$ is a smooth atlas for $M \times N$, and the smooth structure on $M \times N$ determined by $\mathcal{A} \times \mathcal{B}$ depends only on the smooth structures $\mathcal{S}$ and $\mathcal{T}$.*

Proof. If $\varphi_1, \varphi_2 \in \mathcal{A}$ and $\psi_1, \psi_2 \in \mathcal{B}$, note that

$$(\varphi_1 \times \psi_1)\Big(\text{dom } (\varphi_1 \times \psi_1) \cap \text{dom } (\varphi_2 \times \psi_2)\Big) \\ = \varphi_1(\text{dom } \varphi_1 \cap \text{dom } \varphi_2) \times \psi_1(\text{dom } \psi_1 \cap \text{dom } \psi_2)$$

and that the overlap map

$$(\varphi_2 \times \psi_2)(\varphi_1 \times \psi_1)^{-1} : \varphi_1(\text{dom } \varphi_1 \cap \text{dom } \varphi_2) \times \psi_1(\text{dom } \psi_1 \cap \text{dom } \psi_2) \\ \to \varphi_2(\text{dom } \varphi_1 \cap \text{dom } \varphi_2) \times \psi_2(\text{dom } \psi_1 \cap \text{dom } \psi_2)$$

is equal to $(\varphi_2\varphi_1^{-1}) \times (\psi_2\psi_1^{-1})$. Since $\varphi_2\varphi_1^{-1}$ and $\psi_2\psi_1^{-1}$ are both C^∞, so is their product. It follows that the atlas $\mathcal{A} \times \mathcal{B}$ is smooth.

If $\mathcal{A}'$ and $\mathcal{B}'$ are other atlases which are $\mathcal{S}$- and $\mathcal{T}$-admissible, respectively, then whenever $\varphi \in \mathcal{A}$, $\varphi' \in \mathcal{A}'$, $\psi \in \mathcal{B}$, and $\psi' \in \mathcal{B}'$, the overlap maps $\varphi'\varphi^{-1}$ and $\psi'\psi^{-1}$ are both C^∞. Thus, since $(\varphi' \times \psi')(\varphi \times \psi)^{-1} = (\varphi'\varphi^{-1}) \times (\psi'\psi^{-1})$, it follows that the overlap map $(\varphi' \times \psi')(\varphi \times \psi)^{-1}$ is C^∞. It follows that $\mathcal{A} \times \mathcal{B}$ is smoothly equivalent to $\mathcal{A}' \times \mathcal{B}'$. ☐

We denote the smooth structure on $M \times N$ arising from Proposition 4.10 by $\mathcal{S} \times \mathcal{T}$.

Proposition 4.11. *Let $(M, \mathcal{S})$ be a smooth n-manifold. Suppose $\mathcal{A}$ is an $\mathcal{S}$-admissible atlas and U is an open subset of M. Let*

$$\mathcal{A}|U = \{\varphi : U \cap \text{dom } \varphi \to \varphi(U \cap \text{dom } \varphi) \mid \varphi \in \mathcal{A}\}.$$

Then $\mathcal{A}|U$ is a smooth n-manifold atlas on U, and the smooth structure on U represented by $\mathcal{A}|U$ depends only on $\mathcal{S}$.

Proof. Since each $\varphi \in \mathcal{A}$ is a homeomorphism from an open set in M onto an open set in some real n-dimensional vector space X, and U is open in M, each restriction of such a φ to $U \cap \text{dom } \varphi$ is a homeomorphism from an open set in U onto an open set in X. Since $\{\text{dom } \varphi \mid \varphi \in \mathcal{A}\}$ is an open cover of M, $\{U \cap \text{dom } \varphi \mid \varphi \in \mathcal{A}\}$ is an open cover of U. Thus $\mathcal{A}|U$ is a topological n-manifold atlas for U. If $\varphi, \psi \in \mathcal{A}$, the overlap map

$$\psi\varphi^{-1} : \varphi(U \cap \text{dom } \varphi \cap \text{dom } \psi) \to \psi(U \cap \text{dom } \varphi \cap \text{dom } \psi)$$

is the restriction of the overlap map

$$\psi\varphi^{-1} : \varphi(\text{dom } \varphi \cap \text{dom } \psi) \to \psi(\text{dom } \varphi \cap \text{dom } \psi).$$

Since the latter map is C^∞, it follows from the Local Property that the former map is C^∞. Thus $\mathcal{A}|U$ is a smooth atlas for U. If $\mathcal{A}'$ is another $\mathcal{S}$-admissible atlas, and if $\varphi \in \mathcal{A}$, $\varphi' \in \mathcal{A}'$, the overlap map

$$\varphi'\varphi^{-1} : \varphi(U \cap \text{dom } \varphi \cap \text{dom } \varphi') \to \varphi(U \cap \text{dom } \varphi \cap \text{dom } \varphi')$$

is the restriction of the overlap map

$$\varphi'\varphi^{-1} : \varphi(\text{dom } \varphi \cap \text{dom } \varphi') \to \varphi'(\text{dom } \varphi \cap \text{dom } \varphi').$$

The latter map is C^∞ since $\mathcal{A}$ and $\mathcal{A}'$ are smoothly equivalent. Thus, by the Local Property, the former map is C^∞. Similarly, the restriction of $\varphi\varphi'^{-1}$ to $\varphi'(U \cap \text{dom } \varphi \cap \text{dom } \varphi')$ is C^∞. Thus $\mathcal{A}|U$ is smoothly equivalent to $\mathcal{A}'|U$. □

We write $\mathcal{S}|U$ for the smooth structure on U arising from Proposition 4.11 and call it the *restriction of* $\mathcal{S}$ *to* U. We leave the proof of the following proposition as an exercise.

Proposition 4.12. *Let* $(M, \mathcal{S})$ *and* $(N, \mathcal{T})$ *be smooth manifolds. Suppose* U *and* V *are open in* M *with* $U \subset V$*, and suppose* W *is open in* N*. Then:*

(a) $(\mathcal{S}|V)|U = \mathcal{S}|U$.
(b) $(\mathcal{S} \times \mathcal{T})|(U \times W) = (\mathcal{S}|U) \times (\mathcal{T}|W)$.
(c) *Every* $\mathcal{S}|U$*-admissible chart is* $\mathcal{S}$*-admissible.* □

In many parts of mathematics, the objects of interest are sets with special structure and one studies special functions between these objects which have special properties with respect to these structures. For example, in group theory the objects are groups and the special functions are group homomorphisms; in general topology, the objects are topological spaces and the special functions are continuous functions. Our present objects of study are smooth manifolds, and we now proceed to describe the special functions between these which we will study.

Definition 4.13. Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds and $f : M \to N$ a function. We say f is *smooth with respect to* $\mathcal{S}$ *and* $\mathcal{T}$ if f is continuous and for each $\mathcal{S}$-admissible chart φ and $\mathcal{T}$-admissible chart ψ, the composition

$$\psi f \varphi^{-1} : \varphi\Big(\text{dom } \varphi \cap f^{-1}(\text{dom } \psi)\Big) \to \text{codom } \psi$$

is C^∞. If smooth structures on M and N are understood from context, we will simply say *f is smooth* if it is smooth with respect to the understood structures.

The assumption that f is continuous in Definition 4.13 is needed to ensure that $f^{-1}(\text{dom } \psi)$ is open in M, and hence that the domain of the above restriction of $\psi f \varphi^{-1}$ is an open subset of a finite-dimensional real vector space. (Recall, from Chapter 3, that the domain of a C^∞ function must be open in the containing vector space.)

If M and N are open subsets of finite-dimensional real vector spaces and $f : M \to N$ is a function, we have the notion of f being C^∞ in the sense of Definition 3.1, and the notion of f being smooth with respect to the standard smooth structures in the sense of Definition 4.13. We observe next that these two notions agree.

Proposition 4.14. *Let M and N be open subsets of finite-dimensional real vector spaces and $f : M \to N$ a function. Then f is smooth with respect to the standard smooth structures on M and N, respectively, if and only if f is C^∞.*

Proof. Suppose f is smooth with respect to the standard smooth structures. Then since 1_M and 1_N are admissible charts for these respective structures, the composition $(1_N)f(1_M)^{-1}$ is C^∞, i.e. f is C^∞.

Conversely, suppose f is C^∞. Then f is certainly continuous. Suppose φ and ψ are charts for M and N, respectively, which are smoothly related to 1_M and 1_N, respectively. Then $1_M\varphi^{-1}$ and $\psi 1_N^{-1}$ are both C^∞, i.e. φ^{-1} and ψ are both C^∞. By proposition 3.12, the composition $\psi f \varphi^{-1}$ is C^∞. Thus f is smooth. □

Thus the concept of *"smooth function"* between general smooth manifolds is a generalization of the concept of *"C^∞ function"* between open subsets of finite-dimensional real vector spaces. In practice, one of these two terms is often used for both situations. However, since the former is defined in terms of the latter, it is perhaps clearer to use different terminologies for the two situations (as we have done here).

The next result shows that to check smoothness of a map, it is only necessary to test it with "enough" pairs of admissible charts.

Proposition 4.15. *Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds and $f : M \to N$ a continuous function. Then the following are equivalent:*

(1) *f is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$.*

(2) *For some $\mathcal{S}$-admissible atlas $\mathcal{A}$ for M and some $\mathcal{T}$-admissible atlas $\mathcal{B}$ for N, the composition*

$$\psi f \varphi^{-1} : \varphi\Big(\text{dom } \varphi \cap f^{-1}(\text{dom } \psi)\Big) \to \text{codom } \psi$$

is C^∞ for all $\varphi \in \mathcal{A}$ and all $\psi \in \mathcal{B}$.

(3) *For each $x \in M$, there exist $\mathcal{S}$-admissible and $\mathcal{T}$-admissible charts φ_x for M and ψ_x for N, respectively, such that $x \in \text{dom } \varphi_x$, $f(\text{dom } \varphi_x) \subset \text{dom } \psi_x$, and the composition*

$$\psi_x f \varphi_x^{-1} : \text{codom } \varphi_x \to \text{codom } \psi_x$$

is C^∞.

Proof. The implication (1) $\implies$ (2) is trivial.

Suppose (2) and let $x \in M$. Since $\{\text{dom } \varphi \mid \varphi \in \mathcal{A}\}$ covers M, we can choose a chart $\varphi \in \mathcal{A}$ such that $x \in \text{dom } \varphi$. Similarly, we can choose a chart $\psi \in \mathcal{B}$ such that $f(x) \in \text{dom } \psi$. By continuity of f, $f^{-1}(\text{dom } \psi)$ is open in M. Writing $U = \text{dom } \varphi \cap f^{-1}(\text{dom } \psi)$, U is an open neighborhood of x in $\text{dom } \varphi$, and $f(U) \subset \text{dom } \psi$. By Proposition 4.9(d), the restriction of φ to U is an $\mathcal{S}$-admissible chart for M. Taking φ_x to be the above restriction and $\psi_x = \psi$, it follows from the fact that $\psi f \varphi^{-1}$ is C^∞ and the Local Property that $\psi_x f \varphi_x^{-1}$ is C^∞, completing the proof that (2) $\implies$ (3).

Assume (3), and suppose φ and ψ are arbitrary $\mathcal{S}$- and $\mathcal{T}$-admissible charts, respectively. To show that $\psi f \varphi^{-1} : \varphi\Big(\text{dom } \varphi \cap f^{-1}(\text{dom } \psi)\Big) \to \text{codom } \psi$ is C^∞, it suffices, by the Local Property, to check that each $y \in \varphi\Big(\text{dom } \varphi \cap f^{-1}(\text{dom } \psi)\Big)$ has an open neighborhood U_y in $\varphi\Big(\text{dom } \varphi \cap f^{-1}(\text{dom } \psi)\Big)$ such that the restriction of $\psi f \varphi^{-1}$ to U_y is C^∞. Given such a y, write $x = \varphi^{-1}(y)$. By (3) there exist $\mathcal{S}$- and $\mathcal{T}$-admissible charts φ_x and ψ_x, respectively, such that $x \in \text{dom } \varphi_x$, $f(\text{dom } \varphi_x) \subset \text{dom } \psi_x$, and the composition $\psi_x f \varphi_x^{-1} : \text{codom } \varphi_x \to \text{codom } \psi_x$ is C^∞. Take

$$U_y = \varphi\Big(\text{dom } \varphi \cap \text{dom } \varphi_x \cap f^{-1}(\text{dom } \psi)\Big).$$

Since $x \in \text{dom } \varphi \cap f^{-1}(\text{dom } \psi)$ and $x \in \text{dom } \varphi_x$, it follows that $y \in U_y$. By continuity of f, $f^{-1}(\text{dom } \psi)$ is open in M, and hence $\text{dom } \varphi \cap \text{dom } \varphi_x \cap f^{-1}(\text{dom } \psi)$ is open in M (and hence open in dom φ). Since $\varphi : \text{dom } \varphi \to \text{codom } \varphi$ is a homeomorphism, it follows that N_y is open in codom φ, and hence open in $\varphi\Big(\text{dom } \varphi \cap f^{-1}(\text{dom } \psi)\Big)$. Since $f(\text{dom } \varphi_x) \subset \text{dom } \psi_x$, it follows that

$$\begin{aligned}(\psi_x f \varphi_x^{-1})\Big(\phi_x\big(\text{dom } \varphi \cap \text{dom } \varphi_x \cap f^{-1}(\text{dom } \psi)\big)\Big) &\subset \psi_x f\Big(\text{dom } \varphi_x \cap f^{-1}(\text{dom } \psi)\Big) \\ &\subset \psi_x(\text{dom } \psi_x \cap \text{dom } \psi).\end{aligned}$$

The restriction of $\psi f \varphi^{-1}$ to U_y is the composition $(\psi\psi_x^{-1})(\psi_x f \varphi_x^{-1})(\varphi_x \varphi^{-1})$ where the maps in this last composition have domains and codomains restricted as follows:

$$\varphi_x\varphi^{-1} : \varphi\big(\text{dom } \varphi \cap \text{dom } \varphi_x \cap f^{-1}(\text{dom } \psi)\big) \to \varphi_x\big(\text{dom } \varphi \cap \text{dom } \varphi_x \cap f^{-1}(\text{dom } \psi)\big),$$

$$\psi_x f \varphi_x^{-1} : \varphi_x\big(\text{dom } \varphi \cap \text{dom } \varphi_x \cap f^{-1}(\text{dom } \psi)\big) \to \psi_x(\text{dom } \psi \cap \text{dom } \psi_x),$$

$$\psi\psi_x^{-1} : \psi_x(\text{dom } \psi \cap \text{dom } \psi_x) \to \text{codom } \psi.$$

The first of these is a restriction of an overlap map for φ and φ_x, which is C^∞ since φ and φ_x are both $\mathcal{S}$-admissible. The second of these is C^∞ by condition (3). The third of these is a restriction of an overlap map for the ψ_x and ψ, which is C^∞ since ψ_x and ψ are both $\mathcal{T}$-admissible. Thus, the composition of these maps is C^∞ by Proposition 3.12. □

Example 4.16. Suppose $(M, \mathcal{S})$ and $(N, \mathcal{T})$ are smooth manifolds. Let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ denote the respective projections. We claim π_1 is smooth with respect to $\mathcal{S} \times \mathcal{T}$ and $\mathcal{S}$, and π_2 is smooth with respect to $\mathcal{S} \times \mathcal{T}$ and $\mathcal{T}$. For if $(x, y) \in M \times N$, choose any $\mathcal{S}$- and $\mathcal{T}$-admissible charts φ_x and ψ_y, respectively, such that $x \in \text{dom } \varphi_x$ and $y \in \text{dom } \psi_y$. Then $\varphi_x \times \psi_y$ is an $\mathcal{S} \times \mathcal{T}$-admissible chart with $(x, y) \in \text{dom } (\varphi_x \times \psi_y)$ and $\pi_1\big(\text{dom } (\varphi_x \times \psi_y)\big) = \text{dom } \varphi_x$. The composition $\varphi_x \pi_1 (\varphi_x \times \psi_y)^{-1}$ is the projection on the first factor

$$(\text{codom } \varphi_x) \times (\text{codom } \psi_y) \to \text{codom } \varphi_x$$

which is C^∞. Thus π_1 satisfies condition (3) of Proposition 4.15 and hence, by the latter, is smooth. Similarly, π_2 is smooth.

We leave the proof of the following as an exercise.

Proposition 4.17. *Let $(M, \mathcal{S})$, $(N, \mathcal{T})$, and $(Q, \mathcal{U})$ be smooth manifolds, and $f : Q \to M \times N$ a function. Then f is smooth with respect to $\mathcal{U}$ and $\mathcal{S} \times \mathcal{T}$ if and only if $\pi_1 f : Q \to M$ is smooth with respect to $\mathcal{U}$ and $\mathcal{S}$, and $\pi_2 f : Q \to N$ is smooth with respect to $\mathcal{U}$ and $\mathcal{T}$.* □

Proposition 4.18. *Suppose $(M, \mathcal{S})$ is a smooth manifold. Then:*

(a) *$1_M : M \to M$ is smooth with respect to $\mathcal{S}$ and $\mathcal{S}$.*

(b) *If $(N, \mathcal{T})$ and $(Q, \mathcal{U})$ are smooth manifolds, $f : M \to N$ is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$, and $g : N \to Q$ is smooth with respect to $\mathcal{T}$ and $\mathcal{U}$, then $gf : M \to Q$ is smooth with respect to $\mathcal{S}$ and $\mathcal{U}$.*

Proof. 1_M is certainly continuous. If φ and ψ are $\mathcal{S}$-admissible charts, they are smoothly related and hence $\psi\varphi^{-1}$ is C^∞, i.e. $\psi 1_M \varphi^{-1}$ is C^∞, proving (a).

If f and g are as in (b), certainly gf is continuous. Let $x \in M$. By the continuity of f and g and Proposition 4.9(d), it is possible to choose charts φ_x, ψ_x, θ_x which are $\mathcal{S}$-, $\mathcal{T}$-, $\mathcal{U}$-admissible, respectively, such that $x \in \text{dom } \varphi_x$, $f(\text{dom } \varphi_x) \subset \text{dom } \psi_x$, and $g(\text{dom } \psi_x) \subset \text{dom } \theta_x$. From the smoothness of f and g, $\psi_x f \varphi_x^{-1}$, $\theta_x g \psi_x^{-1}$ are both C^∞. By Proposition 3.12, their composition $\theta_x g f \varphi_x^{-1}$ is C^∞. Thus gf satisfies condition (3) of Proposition 4.15, and so, by the latter, is smooth. □

We have the following extension of the first two parts of the Local Property (Proposition 3.21(a)(b)):

Proposition 4.19. *Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds and $f : M \to N$ a function.*

(a) *Suppose U and V are open in M and N, respectively, and that $f(U) \subset V$. If f is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$, then the restriction $f : U \to V$ is smooth with respect to $\mathcal{S}|U$ and $\mathcal{T}|V$.*

(b) *Suppose $\mathcal{O}$ is an open cover of M such that for each $U \in \mathcal{O}$, the restriction of f to U is smooth with respect to $\mathcal{S}|U$ and $\mathcal{T}$. Then f is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$.*

Proof. Suppose U and V satisfy the hypotheses of part (a), and that f is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$. Then if φ and ψ are $\mathcal{S}$- and $\mathcal{T}$-admissible charts, respectively, the composition

$$\psi f \varphi^{-1} : \varphi\Big(\text{dom } \varphi \cap f^{-1}(\text{dom } \psi)\Big) \to \text{codom } \psi$$

is C^∞. By Proposition 3.21(a), the composition

$$\psi f \varphi^{-1} : \varphi\Big(U \cap \text{dom } \varphi \cap f^{-1}(V \cap \text{dom } \psi)\Big) \to \psi(V \cap \text{dom } \psi)$$

is C^∞. Thus, since the restrictions to U and V, respectively, of all $\mathcal{S}$- and $\mathcal{T}$-admissible charts, respectively, constitute $\mathcal{S}|U$- and $\mathcal{T}|V$-admissible atlases, respectively, it follows that $f : U \to V$ satisfies condition (2) of Proposition 4.15, and hence, by the latter, is smooth with respect to $\mathcal{S}|U$ and $\mathcal{T}|V$, proving part (a).

Let $\mathcal{O}$ satisfy the hypotheses of part (b). Since the restriction of f to each member of $\mathcal{O}$ is continuous, f is continuous. Let $x \in M$ and choose a $U_x \in \mathcal{O}$ such that $x \in U_x$. By the smoothness of the restriction $f : U_x \to N$ with respect to $\mathcal{S}|U_x$

and $\mathcal{T}$, it follows, from the equivalence of conditions (1) and (3) of Proposition 4.15, that there exist $\mathcal{S}|U_x$- and $\mathcal{T}$-admissible charts φ_x and ψ_x, respectively, such that $x \in \operatorname{dom} \varphi_x$, $f(\operatorname{dom} \varphi_x) \subset \operatorname{dom} \psi_x$, and $\psi_x f \varphi_x^{-1}$: codom $\varphi_x \to$ codom ψ_x is C^∞. By Proposition 4.12(c), φ_x is $\mathcal{S}$-admissible. Thus f satisfies condition (3) of Proposition 4.15 and hence, by the latter, is smooth. □

Definition 4.20. Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds. A *diffeomorphism from* $(M, \mathcal{S})$ *to* $(N, \mathcal{T})$ is a homeomorphism $f : M \to N$ such that f is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$, and f^{-1} is smooth with respect to $\mathcal{T}$ and $\mathcal{S}$. $(M, \mathcal{S})$ is said to be *diffeomorphic to* $(N, \mathcal{T})$ if a diffeomorphism from $(M, \mathcal{S})$ to $(N, \mathcal{T})$ exists.

We will sometimes say "*$f : M \to N$ is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{T}$*" if f is a diffeomorphism from $(M, \mathcal{S})$ to $(N, \mathcal{T})$.

Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds, U open in M, V open in N, and $f : U \to V$ a function. We will sometimes abbreviate the statement "*f is smooth with respect to $\mathcal{S}|U$ and $\mathcal{T}|V$*" by "*f is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$*". Similarly, we will sometimes abbreviate the statement "*f is a diffeomorphism with respect to $\mathcal{S}|U$ and $\mathcal{T}|V$*" by "*f is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{T}$*". This should cause no confusion.

The following proposition is immediate:

Proposition 4.21. *Let $(M, \mathcal{S})$, $(N, \mathcal{T})$, and $(Q, \mathcal{U})$ be smooth manifolds. Then:*

(a) $1_M : M \to M$ *is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{S}$.*

(b) *If $f : M \to N$ is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{T}$, then $f^{-1} : N \to M$ is a diffeomorphism with respect to $\mathcal{T}$ and $\mathcal{S}$.*

(c) *If $f : M \to N$ is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{T}$, and $g : N \to Q$ is a diffeomorphism with respect to $\mathcal{T}$ and $\mathcal{U}$, then $gf : M \to Q$ is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{U}$.* □

It follows that the relation "is diffeomorphic to" is an equivalence relation on the class of smooth manifolds. This is the fundamental equivalence relation of differential topology. Diffeomorhism plays the role in differential topology that homeomorphism plays in general topology.

Example 4.22. Let $h : \mathbf{R} \to \mathbf{R}$ be any homeomorphism which is not C^∞. Let $\mathcal{S}_{\mathbf{R}}$ denote the standard smooth structure on $\mathbf{R}$, and $\mathcal{T}_h$ the smooth structure on $\mathbf{R}$ determined by the smooth atlas $\{h\}$ as in Example 4.8. There, we observed that $\mathcal{S}_{\mathbf{R}}$ and $\mathcal{T}_h$ were different smooth structures on $\mathbf{R}$. However, since $1_{\mathbf{R}} h h^{-1} = 1_{\mathbf{R}}$ which is C^∞, h satisfies condition (2) of Proposition 4.15 with $\mathcal{A} = \{h\}$ and $\mathcal{B} = \{1_{\mathbf{R}}\}$ and so, by Proposition 4.15, h is smooth with respect to $\mathcal{T}_h$ and $\mathcal{S}_{\mathbf{R}}$. Similarly, since $h h^{-1} 1_{\mathbf{R}}^{-1} = 1_{\mathbf{R}}$ which is C^∞, h^{-1} satisfies condition (2) of Proposition 4.15 with $\mathcal{A} = \{1_{\mathbf{R}}\}$ and $\mathcal{B} = \{h\}$ and so, by Proposition 4.15, h^{-1} is smooth with respect to $\mathcal{S}_{\mathbf{R}}$ and $\mathcal{T}_h$. It follows that h is a diffeomorphism with respect to $\mathcal{T}_h$ and $\mathcal{S}_{\mathbf{R}}$.

Similarly, given any smooth manifold $(M, \mathcal{S})$ and a homeomorphism $h : M \to M$ which is not smooth with respect to $\mathcal{S}$ and $\mathcal{S}$ one can construct a smooth structure $\mathcal{T}$ on M which is strictly different from $\mathcal{S}$, but such that h is a diffeomorphism from $(M, \mathcal{T})$ to $(M, \mathcal{S})$. Such different smooth structures are not particularly interesting. The real interest is in smooth structures $\mathcal{S}$ and $\mathcal{T}$ on a given topological manifold M such that $(M, \mathcal{S})$ and $(M, \mathcal{T})$ *are not diffeomorphic.* Prior to the 1950's, no example of a topological manifold admitting non-diffeomorphic smooth structures was known and, indeed, although it had not been proved that this phenomenon was impossible, it was widely guessed that it was. The first example of this phenomenon was due to J. Milnor (*On manifolds homeomorphic to the 7-sphere*, Ann. of Math. **64** (1956), 399–405)who proved that S^7 admits 28 distinct diffeomorphic classes of smooth structures. Later, work of J. Milnor and M. Kervaire (*Groups of homotopy spheres.* I., Ann. of Math. **77** (1963), 504–537) classified the diffeomorphism classes of smooth structures on all S^n for $n \geq 5$.

It had been known for some time before the 1980's that $\mathbf{R}^n$ admits a unique smooth structure up to diffeomorphism, with the possible exception of the case $n = 4$. In the early 1980's, the mathematical world was astounded by work of S. Donaldson (*Self-dual connections and the topology of 4-manifolds*, Bull. Amer. Math. Soc. **8** (1983), 81–83) which showed that $\mathbf{R}^4$ admits *uncountably many* diffeomorphically distinct smooth structures.

Proposition 4.23. *Let $(M, \mathcal{S})$ be a smooth manifold and φ a chart for M. Then φ is $\mathcal{S}$-admissible if and only if φ is a diffeomorphism with respect to $\mathcal{S}$ and the standard smooth structure on* codom φ.

Proof. Write $\mathcal{S}_\varphi$ for the standard smooth structure on codom φ. Then $\{ 1_{\text{codom } \varphi} \}$ is an $\mathcal{S}_\varphi$-admissible atlas for codom φ.

Suppose φ is $\mathcal{S}$-admissible. Then $\{\varphi\}$ is an $(\mathcal{S}|\text{dom } \varphi)$-admissible atlas for dom φ. Since $(1_{\text{codom } \varphi})\varphi\varphi^{-1} = 1_{\text{codom } \varphi}$, and $1_{\text{codom } \varphi}$ is C^∞, it follows that φ satisfies condition (2) of Proposition 4.15 with $\mathcal{A} = \{\varphi\}$ and $\mathcal{B} = \{1_{\text{codom } \varphi}\}$. Thus by Proposition 4.15, φ is smooth with respect to $\mathcal{S}$ and $\mathcal{S}_\varphi$. Similarly, since $\varphi\varphi^{-1}(1_{\text{codom } \varphi})^{-1} = 1_{\text{codom } \varphi}$, it follows that φ^{-1} is smooth with respect to $\mathcal{S}_\varphi$ and $\mathcal{S}$. Thus φ is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{S}_\varphi$.

Conversely, suppose φ is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{S}_\varphi$. Let ψ be an arbitrary $\mathcal{S}$-admissible chart. By the above, ψ is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{S}_\psi$, where $\mathcal{S}_\psi$ denotes the standard smooth structure on codom ψ. Thus, ψ is smooth with respect to $\mathcal{S}$ and $\mathcal{S}_\psi$, and ψ^{-1} is smooth with respect to $\mathcal{S}_\psi$ and $\mathcal{S}$. By Proposition 4.19(a), the restriction

$$\varphi^{-1} : \varphi(\text{dom } \varphi \cap \text{dom } \psi) \to \text{dom } \varphi \cap \text{dom } \psi$$

is smooth with respect to $\mathcal{S}_\varphi$ and $\mathcal{S}$, and the restriction

$$\psi : \text{dom } \varphi \cap \text{dom } \psi \to \psi(\text{dom } \varphi \cap \text{dom } \psi)$$

is smooth with respect to $\mathcal{S}$ and $\mathcal{S}_\psi$. By Proposition 4.18(b), it follows that the composition of these is smooth with respect to the standard smooth structures. Thus, by Proposition 4.14, $\psi\varphi^{-1} : \varphi(\text{dom } \varphi \cap \text{dom } \psi) \to \psi(\text{dom } \varphi \cap \text{dom } \psi)$ is C^∞. Similarly, $\varphi\psi^{-1} : \psi(\text{dom } \varphi \cap \text{dom } \psi) \to \varphi(\text{dom } \varphi \cap \text{dom } \psi)$ is C^∞. Thus φ is smoothly related to every $\mathcal{S}$-admissible chart. Thus, by Proposition 4.9(e), φ is $\mathcal{S}$-admissible. □

Proposition 4.24. *Let M be a topological n-manifold and $\mathcal{U}$ an open cover of M. Suppose $\mathcal{S}$ and $\mathcal{T}$ are smooth structures on M such that for all $U \in \mathcal{U}$, $\mathcal{S}|U = \mathcal{T}|U$. Then $\mathcal{S} = \mathcal{T}$.*

Proof. Let φ and ψ be $\mathcal{S}$-admissible and $\mathcal{T}$-admissible charts, respectively. We must show that φ and ψ are smoothly related.

For $U \in \mathcal{U}$, let φ_U and ψ_U denote the restrictions of φ and ψ, respectively, to dom $\varphi \cap U$ and dom $\psi \cap U$, respectively. Thus φ_U is $\mathcal{S}|U$-admissible and ψ_U is $\mathcal{T}|U$-admissible. Since $\mathcal{S}|U = \mathcal{T}|U$, it follows that φ_U and ψ_U are smoothly related. Thus the restriction of $\psi\varphi^{-1}$ to $\varphi(\text{dom}\,\varphi \cap \text{dom}\,\psi \cap U)$ is C^∞. Since $\{\varphi(\text{dom}\,\varphi \cap \text{dom}\,\psi \cap U) \mid U \in \mathcal{U}\}$ is an open cover of $\varphi(\text{dom}\,\varphi \cap \text{dom}\,\psi)$, it follows from the Local Property that $\psi\varphi^{-1}$ is C^∞. Similarly, $\varphi\psi^{-1}$ is C^∞, proving that φ and ψ are smoothly related. □

Definition 4.25. Let M be a topological n-manifold and $\mathcal{U}$ a collection of open subsets of M. Suppose, for each $U \in \mathcal{U}$, we are given a smooth structure $\mathcal{S}_U$ on U. We say the collection $\{\mathcal{S}_U \mid U \in \mathcal{U}\}$ is *compatible* if whenever $U, V \in \mathcal{U}$, we have $\mathcal{S}_U \,|\, U \cap V = \mathcal{S}_V \,|\, U \cap V$.

Theorem 4.26. *Let M be a topological n-manifold, and $\mathcal{U}$ an open cover of M. Suppose, for each $U \in \mathcal{U}$, we are given a smooth structure $\mathcal{S}_U$ on U. Suppose the collection $\{\mathcal{S}_U \mid U \in \mathcal{U}\}$ is compatible. Then there exists a unique smooth structure $\mathcal{S}$ on M such that for each $U \in \mathcal{U}$, $\mathcal{S} \,|\, U = \mathcal{S}_U$.*

Proof. The uniqueness is immediate from Theorem 4.24.

For each $U \in \mathcal{U}$, choose an $\mathcal{S}_U$-admissible atlas $\mathcal{A}_U$ for U, and let $\mathcal{A} = \bigcup_{U \in \mathcal{U}} \mathcal{A}_U$. Since each $U \in \mathcal{U}$ is open in M, the members of each $\mathcal{A}_U$ are charts for M. Since $\mathcal{U}$ is a cover of M, the domains of the charts in $\mathcal{A}$ cover M, and so $\mathcal{A}$ is an atlas for M. We proceed to show that $\mathcal{A}$ is a smooth atlas for M, and that the smooth structure $\mathcal{S}$ which it represents satisfies $\mathcal{S}|U = \mathcal{S}_U$ for all $U \in \mathcal{U}$.

Suppose $U, V \in \mathcal{U}$ and $\varphi \in \mathcal{A}_U$, $\psi \in \mathcal{A}_V$. Write φ' and ψ' for the restrictions of φ and ψ, respectively, to dom $\varphi \cap$ dom ψ. Since dom $\varphi \cap$ dom $\psi \subset U \cap V$, φ' is $\mathcal{S}_U|U \cap V$-admissible and ψ' is $\mathcal{S}_V|U \cap V$-admissible. Since, by the compatibility hypothesis, $\mathcal{S}_U|U \cap V = \mathcal{S}_V|U \cap V$, it follows that φ' and ψ' are both $\mathcal{S}_U|U \cap V$-admissible, and hence are smoothly related. It follows that φ and ψ are smoothly related, and so $\mathcal{A}$ is a smooth atlas for M.

Let $\mathcal{S}$ denote the smooth structure on M which is represented by $\mathcal{A}$. For each $U \in \mathcal{U}$, $\mathcal{S}|U$ is represented by $\mathcal{A}|U$ where $\mathcal{A}|U$ is as in Proposition 4.11, while $\mathcal{S}_U$ is represented by $\mathcal{A}_U$. From the construction of $\mathcal{A}$ and $\mathcal{A}|U$, $\mathcal{A}_U \subset \mathcal{A}|U$. By Proposition 4.9(c), $\mathcal{A}|U$ is $\mathcal{S}_U$-admissible. Thus $\mathcal{A}|U$ is both $\mathcal{S}|U$-admissible and $\mathcal{S}_U$-admissible. It follows that $\mathcal{S}|U = \mathcal{S}_U$, completing the proof. □

The proof of the following is routine, and left as an exercise.

Proposition 4.27. *Let $(M, \mathcal{S})$ be a smooth manifold, X a topological space, and $f : X \to M$ a homeomorphism. For each chart φ for M, let $f^*\varphi$ denote the composition*

$$f^{-1}(\text{dom}\,\varphi) \xrightarrow{f} \text{dom}\,\varphi \xrightarrow{\varphi} \text{codom}\,\varphi\,.$$

(a) *Let $\mathcal{A}$ be an $\mathcal{S}$-admissible atlas for M. Write $f^*\mathcal{A} = \{f^*\varphi \mid \varphi \in \mathcal{A}\}$. Then $f^*\mathcal{A}$ is a smooth atlas for X whose smooth equivalence class depends only on $\mathcal{S}$. Let $f^*\mathcal{S}$ denote the smooth equivalence class of $f^*\mathcal{A}$.*

(b) *f is a diffeomorphism with respect to $f^*\mathcal{S}$ and $\mathcal{S}$.* □

We next consider fiber bundles with smoothness conditions. Recall the basic definitions without smoothness given in Definition 2.8. Also, to make the terminology less cumbersome, if $(M, \mathcal{S})$ is a smooth manifold and U, V are open in M and $f : U \to V$ is a function, we will abbreviate "f is smooth with respect to $\mathcal{S}$ and $\mathcal{S}$" to "f is smooth with respect to $\mathcal{S}$", and we will abbreviate "f is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{S}$" to "f is a diffeomorphism with respect to $\mathcal{S}$".

Definition 4.28. Suppose $(F, \mathcal{S}_F)$ and $(B, \mathcal{S}_B)$ are smooth m- and n-manifolds, respectively, and let $\xi = (F, E, B, p)$ be a fiber bundle. Let $\varphi : p^{-1}(U_\varphi) \to U_\varphi \times F$ and $\psi : p^{-1}(U_\psi) \to U_\psi \times F$ be charts for ξ. We say φ and ψ are *$\mathcal{S}_B$–$\mathcal{S}_F$-related* if the compositions

$$(U_\varphi \cap U_\psi) \times F \xrightarrow{\varphi^{-1}} p^{-1}(U_\varphi \cap U_\psi) \xrightarrow{\psi} (U_\varphi \cap U_\psi) \times F \qquad \text{and}$$

$$(U_\varphi \cap U_\psi) \times F \xrightarrow{\psi^{-1}} p^{-1}(U_\varphi \cap U_\psi) \xrightarrow{\varphi} (U_\varphi \cap U_\psi) \times F$$

are smooth with respect to $\mathcal{S}_B \times \mathcal{S}_F$. (Equivalently, if $\varphi\psi^{-1}$ and $\psi\varphi^{-1}$ are diffeomorphisms with respect to $\mathcal{S}_B \times \mathcal{S}_F$.)

An atlas $\mathcal{A}$ for ξ is said to be *$\mathcal{S}_B$–$\mathcal{S}_F$–smooth* if whenever $\varphi, \psi \in \mathcal{A}$, then φ and ψ are $\mathcal{S}_B$–$\mathcal{S}_F$–related.

If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{S}_B$–$\mathcal{S}_F$–smooth atlases for ξ, we say $\mathcal{A}$ and $\mathcal{B}$ are *$\mathcal{S}_B$–$\mathcal{S}_F$–equivalent* if for all $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{B}$, φ and ψ are $\mathcal{S}_B$–$\mathcal{S}_F$–related.

Proposition 4.29. *Let $\xi = (F, E, B, p)$ be a fiber bundle. Suppose $\mathcal{S}_B$ and $\mathcal{S}_F$ are smooth structures on B and F, respectively. Then the relation $\mathcal{S}_B$–$\mathcal{S}_F$–equivalence on the set of all $\mathcal{S}_B$–$\mathcal{S}_F$–smooth atlases for ξ is an equivalence relation.*

Proof. For the purpose of this proof, write $\equiv$ for "is $\mathcal{S}_B$–$\mathcal{S}_F$–equivalent to". It is immediate that $\equiv$ is reflexive and symmetric.

Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are $\mathcal{S}_B$–$\mathcal{S}_F$–smooth atlases for ξ with $\mathcal{A} \equiv \mathcal{B}$ and $\mathcal{B} \equiv \mathcal{C}$. Let $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{C}$. The overlap maps $\psi\varphi^{-1}$ and $\varphi\psi^{-1}$ on $(U_\varphi \cap U_\psi) \times F$ are homeomorphisms (*any* chart for ξ is a homeomorphism). Thus, to prove that these overlap maps are diffeomorphisms with respect to $\mathcal{S}_B \times \mathcal{S}_F$, it remains only to prove that they are smooth with respect to $\mathcal{S}_B \times \mathcal{S}_F$.

$\{ (U_\theta \cap U_\varphi \cap U_\psi) \times F \mid \theta \in \mathcal{B} \}$ is an open cover of $(U_\varphi \cap U_\psi) \times F$. By the Local Property, it suffices to prove that the restrictions of the above overlap maps to each $(U_\theta \cap U_\varphi \cap U_\psi) \times F$, $\theta \in \mathcal{B}$, are smooth with respect to $\mathcal{S}_B \times \mathcal{S}_F$. Let $\theta \in \mathcal{B}$. Then the restriction of $\varphi\psi^{-1}$ to $(U_\theta \cap U_\varphi \cap U_\psi) \times F$ is the composition

$$(U_\theta \cap U_\varphi \cap U_\psi) \times F \xrightarrow{\theta\psi^{-1}} (U_\theta \cap U_\varphi \cap U_\psi) \times F \xrightarrow{\varphi\theta^{-1}} (U_\theta \cap U_\varphi \cap U_\psi) \times F\,.$$

The above $\theta\psi^{-1}$ is smooth with respect to $\mathcal{S}_B \times \mathcal{S}_F$ since $\mathcal{B} \equiv \mathcal{C}$, while the above $\varphi\theta^{-1}$ is smooth with respect to $\mathcal{S}_B \times \mathcal{S}_F$ since $\mathcal{A} \equiv \mathcal{B}$. Thus their composition, $\varphi\psi^{-1}$ restricted to $(U_\theta \cap U_\varphi \cap U_\psi) \times F$, is smooth with respect to $\mathcal{S}_B \times \mathcal{S}_F$, and similarly for $\psi\varphi^{-1}$. $\square$

Definition 4.30. Let $\xi = (F, E, B, p)$ be a fiber bundle and suppose $\mathcal{S}_B$ and $\mathcal{S}_F$ are smooth structures on F and B, respectively. An *$\mathcal{S}_B$–$\mathcal{S}_F$–smooth structure for* ξ is an $\mathcal{S}_B$–$\mathcal{S}_F$–equivalence class $\mathcal{S}$ of $\mathcal{S}_B$–$\mathcal{S}_F$–smooth atlases for ξ.

An $\mathcal{S}_B$–$\mathcal{S}_F$–smooth atlas for ξ which represents $\mathcal{S}$ will be said to be *$\mathcal{S}$-admissible.*

A chart for ξ which belongs to an $\mathcal{S}$-admissible $\mathcal{S}_B$–$\mathcal{S}_F$–smooth atlas for ξ will be said to be *$\mathcal{S}$-admissible.*

Definition 4.31. A *smooth fiber bundle* is a quintuple $\big((F, \mathcal{S}_F), E, (B, \mathcal{S}_B), p, \mathcal{S}\big)$ satisfying:

(i) (F, E, B, p) is a fiber bundle.

(ii) $(F, \mathcal{S}_F)$ and $(B, \mathcal{S}_B)$ are smooth manifolds.

(iii) $\mathcal{S}$ is an $\mathcal{S}_B$–$\mathcal{S}_F$–smooth structure for (F, E, B, p).

If $\xi = \big((F, \mathcal{S}_F), E, (B, \mathcal{S}_B), p, \mathcal{S}\big)$ is a smooth fiber bundle, write $\underline{\xi} = (F, E, B, p)$ for its underlying fiber bundle.

The following proposition is analogous to Proposition 4.9. The proof is left as an exercise.

Proposition 4.32. *Let $\xi = \big((F, \mathcal{S}_F), E, (B, \mathcal{S}_B), p, \mathcal{S}\big)$ be a smooth fiber bundle. Then:*

(a) *Any two $\mathcal{S}$-admissible charts are $\mathcal{S}_B$–$\mathcal{S}_F$–related.*

(b) *If $\mathcal{A}$ and $\mathcal{B}$ are atlases for $\underline{\xi}$ such that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B}$ is $\mathcal{S}$-admissible, then $\mathcal{A}$ is $\mathcal{S}$-admissible.*

(c) *If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{S}_B$–$\mathcal{S}_F$–smooth atlases for $\underline{\xi}$ such that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{A}$ is $\mathcal{S}$-admissible, then $\mathcal{B}$ is $\mathcal{S}$-admissible.*

(d) *If φ is an $\mathcal{S}$-admissible chart for $\underline{\xi}$ and U is open in U_φ, then the restriction $\varphi : p^{-1}(U) \to U \times F$ is an $\mathcal{S}$-admissible chart.*

(e) *If φ is a chart for $\underline{\xi}$ and $\mathcal{A}$ an $\mathcal{S}$-admissible atlas, then φ is $\mathcal{S}$-admissible if and only if φ is $\mathcal{S}_B$–$\mathcal{S}_F$–related to each member of $\mathcal{A}$.*

(f) *If $\mathcal{A}$ is an $\mathcal{S}$-admissible atlas and $\mathcal{C}$ any set of $\mathcal{S}$-admissible charts, then $\mathcal{A} \cup \mathcal{C}$ is an $\mathcal{S}$-admissible atlas.* □

Note that the data for a smooth fiber bundle does not include a smooth structure for the total space. However, we will soon see that one is implied which blends in well with the structure.

Suppose $\xi = (F, E, B, p)$ is a fiber bundle and that $\mathcal{S}_B$ and $\mathcal{S}_F$ are smooth structures on F and B, respectively. Suppose $\varphi : p^{-1}(U_\varphi) \to U_\varphi \times F$ is a chart for ξ. Using the construction of Proposition 4.27, we obtain a smooth structure $\varphi^*\big((\mathcal{S}_B \mid U_\varphi) \times \mathcal{S}_F\big)$ on $p^{-1}(U_\varphi)$.

Lemma 4.33. *Let $\xi = (F, E, B, p)$ be a fiber bundle. Suppose $\mathcal{S}_B$ and $\mathcal{S}_F$ are smooth structures on B and F, respectively. Suppose φ and ψ are $\mathcal{S}_B$–$\mathcal{S}_F$–related charts for ξ. Then the smooth structures $\varphi^*\big((\mathcal{S}_B \mid U_\varphi) \times \mathcal{S}_F\big) \mid p^{-1}(U_\varphi \cap U_\psi)$ and $\psi^*\big((\mathcal{S}_B \mid U_\psi) \times \mathcal{S}_F\big) \mid p^{-1}(U_\varphi \cap U_\psi)$ on the manifold $p^{-1}(U_\varphi \cap U_\psi)$ are the same.*

Proof. Let θ be a $\varphi^*\big((\mathcal{S}_B \mid U_\varphi) \times \mathcal{S}_F\big) \mid p^{-1}(U_\varphi \cap U_\psi)$-admissible chart. Since φ and ψ play symmetric roles, it suffices to show that θ is $\psi^*\big((\mathcal{S}_B \mid U_\psi) \times \mathcal{S}_F\big) \mid p^{-1}(U_\varphi \cap U_\psi)$-admissible.

By the construction given in Proposition 4.27, θ is expressible as $\alpha\varphi$ for some $\big((\mathcal{S}_B \,|\, U_\varphi) \times \mathcal{S}_F\big)|p^{-1}(U_\varphi \cap U_\psi)$-admissible chart α. Thus $\theta = \alpha(\varphi\psi^{-1})\psi$ with domains and codomains suitably restricted. By Proposition 4.9(d) and the construction given in Proposition 4.27, we will be done if we show $\alpha(\varphi\psi^{-1})$ is $\mathcal{S}_B \times \mathcal{S}_F$-admissible.

By Proposition 4.23, α is a diffeomorphism with respect to $\mathcal{S}_B \times \mathcal{S}_F$ and the standard smooth structure on the codomain. Since φ and ψ are $\mathcal{S}_B$–$\mathcal{S}_F$–related, $\varphi\psi^{-1}$ is a diffeomorphism with respect to $\mathcal{S}_B \times \mathcal{S}_F$. Thus the composition $\alpha\varphi\psi^{-1}$ is a diffeomorphism with respect to $\mathcal{S}_B \times \mathcal{S}_F$ and the standard smooth structure on the codomain. Thus by Proposition 4.23 and Proposition 4.9(d), $\alpha\varphi\psi^{-1}$ is $\mathcal{S}_B \times \mathcal{S}_F$-admissible. □

Theorem 4.34. *Let $\xi = \big((F, \mathcal{S}_F), E, (B, \mathcal{S}_B), p, \mathcal{S}\big)$ be a smooth fiber bundle. Then there exists a unique smooth structure $\mathcal{S}_E$ for E such that for each $\mathcal{S}$-admissible chart φ for ξ, $\mathcal{S}_E \,|\, p^{-1}(U_\varphi) = \varphi^*\big((\mathcal{S}_B \,|\, U_\varphi) \times \mathcal{S}_F\big)$.*

Proof. For convenience, if $\mathcal{T}$ is a smooth structure for E and φ an $\mathcal{S}$-admissible chart for ξ, let $P(\mathcal{T}, \varphi)$ denote the condition "$\mathcal{T}|p^{-1}(U_\varphi) = \varphi^*\big((\mathcal{S}_B|U_\varphi) \times \mathcal{S}_F\big)$".

Let $\mathcal{A}$ be an $\mathcal{S}$-admissible atlas for ξ. Then $\{p^{-1}(U_\varphi) \mid \varphi \in \mathcal{A}\}$ is an open cover of E. By Lemma 4.33, $\{\varphi^*\big((\mathcal{S}_B|U_\varphi) \times \mathcal{S}_F\big) \mid \varphi \in \mathcal{A}\}$ is compatible. Thus, by Theorem 4.26, there exists a unique smooth structure $\mathcal{T}_\mathcal{A}$ on E such that condition $P(\mathcal{T}_\mathcal{A}, \varphi)$ holds for all $\varphi \in \mathcal{A}$. Thus if there existed a smooth structure $\mathcal{S}_E$ on E satisfying $P(\mathcal{S}_E, \varphi)$ for all $\mathcal{S}$-admissible charts φ, it follows from the above uniqueness of $\mathcal{T}_\mathcal{A}$ that $\mathcal{S}_E = \mathcal{T}_\mathcal{A}$. Uniqueness of $\mathcal{S}_E$ follows.

For existence of $\mathcal{S}_E$, choose any $\mathcal{S}$-admissible atlas $\mathcal{A}$ for ξ and take $\mathcal{S}_E = \mathcal{T}_\mathcal{A}$. It remains to show that for any $\mathcal{S}$-admissible chart ψ for ξ, condition $P(\mathcal{T}_\mathcal{A}, \psi)$ holds. Given such a ψ, let $\mathcal{B} = \mathcal{A} \cup \{\psi\}$. By Proposition 4.32(f), $\mathcal{B}$ is an $\mathcal{S}$-admissible atlas for ξ. The resulting smooth structure $\mathcal{T}_\mathcal{B}$ satisfies condition $P(\mathcal{T}_\mathcal{B}, \varphi)$ for all $\varphi \in \mathcal{B}$, i.e. for ψ and all $\varphi \in \mathcal{A}$. By the uniqueness for $\mathcal{T}_\mathcal{A}$ above, it follows that $\mathcal{T}_\mathcal{A} = \mathcal{T}_\mathcal{B}$, and so $P(\mathcal{T}_\mathcal{A}, \psi)$ holds. □

Example 4.35. In Example 2.11 a fiber bundle $\xi = (I/\partial I, K, I/\partial I, p)$ was constructed, where K is the Klein bottle. An atlas consisting of exactly 2 charts was given. We now proceed to impose a smooth structure on ξ.

We have a homeomorphism $h : I/\partial I \to S^1$ given by $h([t]) = (\cos 2\pi t, \sin 2\pi t)$. Let $\mathcal{S}$ denote the standard smooth structure on S^1 (Example 4.6). Using Proposition 4.27, we obtain a smooth structure $h^*\mathcal{S}$ on $I/\partial I$. Thus the base space and fiber of ξ are now equipped with smooth structures.

The atlas in Example 2.11 depended on a choice of 2 distinct real numbers $c_1 < c_2 \in (0, 1)$. Let $U_i = I/\partial I - \{[c_i]\}$, $i = 1, 2$. The charts $\varphi_i : p^{-1}(U_i) \to U_i \times I/\partial I$ and their inverses $\psi_i : U_i \times I/\partial I \to p^{-1}(U_i)$ are given by

$$\varphi_i([s,t]) = \begin{cases} ([s],[t]) & \text{if } 0 \le s < c_i, \\ ([s],[1-t]) & \text{if } c_i < s \le 1, \end{cases}$$

$$\psi_i([s],[t]) = \begin{cases} [s,t] & \text{if } 0 \le s < c_i, \\ [s,1-t] & \text{if } c_i < s \le 1. \end{cases}$$

It can be checked that the overlap maps

$$\varphi_1\varphi_2^{-1}, \varphi_2\varphi_1^{-1} : (I/\partial I) - \{c_1, c_2\} \to (I/\partial I) - \{c_1, c_2\}$$

are given by

$$\varphi_1\varphi_2^{-1}([s],[t]) = \varphi_2\varphi_1^{-1}([s],[t]) = \begin{cases} ([s],[t]) & \text{if } 0 \le s < c_1, \\ ([s],[1-t]) & \text{if } c_1 < s < c_2, \\ ([s],[t]) & \text{if } c_2 < s \le 1. \end{cases}$$

Writing $q : I \to I/\partial I$ for the quotient map, and setting $V = q\big([0,c_1) \cup (c_2,1]\big)$, $W = q\big((c_1,c_2)\big)$, V and W are disjoint open subsets of $I/\partial I$, and dom $\varphi_1\varphi_2^{-1} =$ dom $\varphi_2\varphi_1^{-1} = V \times (I/\partial I) \cup W \times (I/\partial I)$. The restriction of each overlap map to $V \times (I/\partial I)$ is the identity map, which is smooth regardless of which smooth structure is put on $I/\partial I$. The proof of smoothness of the overlap maps will be complete if we show that the map $f : I/\partial I \to I/\partial I$ given by $f([t]) = [1-t]$ is smooth with respect to $h^*\mathcal{S}$. Define $g : S^1 \to S^1$ by $g(x,y) = (x,-y)$. Then the diagram

$$\begin{array}{ccc} I/\partial I & \xrightarrow{\;h\;} & S^1 \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ I/\partial I & \xrightarrow[\;h\;]{} & S^1 \end{array}$$

commutes. We leave it as an exercise to check that g is smooth with respect to $\mathcal{S}$ (e.g. using the stereographic projection charts). By Proposition 4.27, h is a diffeomorphism with respect to $h^*\mathcal{S}$ and $\mathcal{S}$. Thus, since $f = h^{-1}gh$, it follows that f is smooth with respect to $h^*\mathcal{S}$. Thus $\{\varphi_1, \varphi_2\}$ is an $h^*\mathcal{S}$–$h^*\mathcal{S}$-smooth atlas for ξ, which determines a smooth structure for ξ. Thus, by Theorem 4.34, the Klein bottle K receives a smooth structure.

The final topic of this section deals with smooth structures on connected sums and related matters. Recall Lemmas 2.16 and 2.18.

Theorem 4.36. *Let $(M,\mathcal{S})$ and $(N,\mathcal{T})$ be smooth n-manifolds, $n \ge 1$. Suppose A is open in M, B is open in N, and $h : A \to B$ is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{T}$. Suppose M and N are unpinched by h. Let $q : M \amalg N \to M \cup_h N$ denote the quotient map. Then:*

(a) *The maps $i_M : q(M \times \{1\}) \to M$ and $i_N : q(N \times \{2\}) \to N$ given by $i_M\big(q(x,1)\big) = x$ and $i_N\big(q(y,2)\big) = y$ are homeomorphisms.*

(b) *There exists a unique smooth structure $\mathcal{U}$ on $M \cup_h N$ which satisfies $\mathcal{U} \mid q(M \times \{1\}) = i_M^*\mathcal{S}$ and $\mathcal{U} \mid q(N \times \{2\}) = i_N^*\mathcal{T}$.*

Proof. By Lemma 2.16(b), the map $q(M \times \{1\}) \to M \times \{1\}$ sending $q(x,1)$ to $(x,1)$ is a homeomorphism. i_M is the composition of the above with the projection on the first factor $M \times \{1\} \to M$, which is a homeomorphism. Thus i_M is a homeomorphism, and similarly for i_N.

It follows from Lemmas 2.16 and 2.18 that $M \cup_h N$ is a topological n-manifold, and that $\{q(M \times \{1\}), q(N \times \{2\})\}$ is an open cover of $M \cup_h N$. By Theorem 4.26, we will be done if we show that $\{ i_M^* \mathcal{S}, i_N^* \mathcal{T} \}$ is compatible.

We have $q(M \times \{1\}) \cap q(N \times \{2\}) = q(A \times \{1\}) = q(B \times \{2\})$. Any $i_N^* \mathcal{T} \,|\, q(B \times \{2\})$-admissible chart is a composition

$$i_N^{-1}(\text{dom } \varphi) \xrightarrow{i_N} \text{dom } \varphi \xrightarrow{\varphi} \text{codom } \varphi$$

where φ is $\mathcal{T}$-admissible with $\text{dom } \varphi \subset q(B \times \{2\})$. Since $q(a, 1) = q\big(h(a), 2\big)$ for all $a \in A$, the diagram

$$\begin{array}{ccc} q(A \times \{1\}) & \xrightarrow{i_M} & A \\ \Big\downarrow{=} & & \Big\downarrow{h} \\ q(B \times \{2\}) & \xrightarrow[i_N]{} & B \end{array}$$

commutes. Thus $\varphi i_N = \varphi h i_M$. Since h is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{T}$, and φ is a diffeomorphism with respect to $\mathcal{T}$ and the standard smooth structure on codom φ, it follows that φh is a diffeomorphism with respect to $\mathcal{S}$ and the standard smooth structure on codom φ. Thus $\varphi h i_M$ is $i_M^* \mathcal{S} \,|\, q(A \times \{1\})$-admissible. Similarly, using h^{-1} in place of h, every $i_M^* \mathcal{S} \,|\, q(A \times \{1\})$-admissible chart is $i_N^* \mathcal{T} \,|\, q(B \times \{2\})$-admissible. Thus $i_M^* \mathcal{S} \,|\, q(A \times \{1\}) = i_N^* \mathcal{T} \,|\, q(B \times \{2\})$. □

Example 4.37. Recall the connected sum construction of Example 2.20. Suppose $(M, \mathcal{S})$ and $(N, \mathcal{T})$ are non-empty smooth manifolds, $n \geq 1$. We can choose $\mathcal{S}$-admissible and $\mathcal{T}$-admissible charts $\varphi : \text{dom } \varphi \to E^n$ and $\psi : \text{dom } \psi \to E^n$, respectively. Say $\varphi(P) = 0 = \psi(Q)$. Let $A = \varphi^{-1}\big(E^n_{(0,1/2)}\big) \subset M - \{P\}$, $B = \psi^{-1}\big(E^n_{(0,1/2)}\big) \subset N - \{Q\}$. Recall that $M \# N$ is $(M - \{P\}) \cup_h (N - \{Q\})$ where h is the composition

$$A \xrightarrow{\varphi_1} E^n_{(0,1/2)} \xrightarrow{\alpha_{1/2}} E^n_{(0,1/2)} \xrightarrow{\psi_1^{-1}} B\ .$$

Here φ_1 and ψ_1 are the restrictions of φ and ψ, respectively, and $\alpha_{1/2}$ is given by $\alpha_{1/2}(x) = (\frac{1}{2} - \|x\|)\dfrac{x}{\|x\|}$. $\alpha_{1/2}$ is C^∞ , and equal to its own inverse, and hence is a diffeomorphism with respect to the standard smooth structure on $\mathbf{R}^n$. φ_1 is a diffeomorphismwith respect to $\mathcal{S}$ and the standard smooth structure on $\mathbf{R}^n$. ψ^{-1} is a diffeomorphism with respect to the standard smooth structure on $\mathbf{R}^n$ and $\mathcal{T}$. Thus h is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{T}$.

By Theorem 4.36, $M \# N$ has a smooth structure $\mathcal{U}$ such that $\mathcal{U} \,|\, (M - \{P\}) = \mathcal{S} \,|\, (M - \{P\})$ and $\mathcal{U} \,|\, (N - \{Q\}) = \mathcal{T} \,|\, (N - \{Q\})$.

Exercises for §4

1. Prove that for $n \geq 1$, the atlas for S^n given in Example 2.3 is smooth and that it represents the standard smooth structure (Example 4.6) on S^n.

2. Prove Proposition 4.12.

3. Prove Proposition 4.17.

4. Under the notation and hypotheses of Theorem 4.34, prove that $p : E \to B$ is smooth with respect to $\mathcal{S}_E$ and $\mathcal{S}_B$.

5. Show that the orthogonal group $O(n)$ (see Exercise 7 of §2) admits a smooth structure.

6. Let $(M, \mathcal{S})$, $(N, \mathcal{T})$, and $(Q, \mathcal{U})$ be smooth manifolds. Suppose $f : M \to N$ is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{T}$, and that $g : N \to Q$ is a function. Prove that g is smooth with respect to $\mathcal{T}$ and $\mathcal{U}$ if and only if $gf : M \to Q$ is smooth with respect to $\mathcal{S}$ and $\mathcal{U}$.

7. Let $(M_1, \mathcal{S}_1)$, $(M_2, \mathcal{S}_2)$, $(N_1, \mathcal{T}_1)$, and $(N_2, \mathcal{T}_2)$ be smooth manifolds. Suppose, for $i = 1, 2$, $f_i : M_i \to N_i$ is smooth with respect to $\mathcal{S}_i$ and $\mathcal{T}_i$. Prove that $f_1 \times f_2 : M_1 \times M_2 \to N_1 \times N_2$ is smooth with respect to $\mathcal{S}_1 \times \mathcal{S}_2$ and $\mathcal{T}_1 \times \mathcal{T}_2$.

8. Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds. Let $f : M \to N$ be a constant map. Prove that f is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$.

9. Let V and W be finite-dimensional real vector spaces. Let $\mathcal{S}_V$ and $\mathcal{S}_W$ denote the standard smooth structures on V and W, respectively. Prove that $\mathcal{S}_V \times \mathcal{S}_W$ is the standard smooth structure on $V \times W$.

10. Prove Proposition 4.27.

5. Tangent Spaces and Tangent Maps

Given a smooth n-manifold $(M, \mathcal{S})$ and a point $x \in M$, we will construct below a real n-dimensional vector space $T_x(M, \mathcal{S})$, the *tangent space to M at x with respect to $\mathcal{S}$*, which can be thought of as the best flat approximation to M near x. If M is an open subset of a real n-dimensional vector space V, and $\mathcal{S}$ is the standard smooth structure on M, $T_x(M, \mathcal{S})$ will be canonically identified with V. If $(N, \mathcal{T})$ is another smooth manifold and $f : M \to N$ a smooth map with respect to $\mathcal{S}$ and $\mathcal{T}$, we will construct an $\mathbf{R}$-linear transformation $T_x f : T_x(M, \mathcal{S}) \to T_{f(x)}(N, \mathcal{T})$ which reduces to the derivative $Df(x)$ in case M and N are open subsets of finite-dimensional real vector spaces. $T_x f$ can be thought of as the best linear approximation to f near x.

Definition 5.1. Let $(M, \mathcal{S})$ be a smooth manifold and $x \in M$. Suppose f and g are real-valued functions with dom f and dom g open neighborhoods of x in M, such that f and g are smooth with respect to $\mathcal{S}$ and the standard smooth structure on $\mathbf{R}$. We say *f and g have the same germ at x*, denoted $f \sim_x g$, if there exists a neighborhood U of x in M such that $f(y) = g(y)$ for all $y \in U$.

It is immediate that $\sim_x$ is an equivalence relation on the set of all real-valued functions defined on open neighborhoods of x in M, which are smooth with respect to $\mathcal{S}$ and the standard smooth structure on $\mathbf{R}$.

Definition 5.2. For $(M, \mathcal{S})$, x, and f as above, the $\sim_x$-equivalence class of f is called the *germ of f at x*, and denoted $[f]_x$. We denote by $G_x(M, \mathcal{S})$ the set of all germs at x of smooth real-valued functions defined in neighborhoods of x.

Note that if f and g are real-valued functions defined in neighborhoods of x, and $r \in \mathbf{R}$, we can define $f + g : \text{dom } f \cap \text{dom } g \to \mathbf{R}$, $f \cdot g : \text{dom } f \cap \text{dom } g \to \mathbf{R}$, and $rf : \text{dom } f \to \mathbf{R}$ by $(f+g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x)g(x)$ for all $x \in \text{dom } f \cap \text{dom } g$, and $(rf)(x) = rf(x)$ for all $x \in \text{dom } f$.

Lemma 5.3. *Suppose $(M, \mathcal{S})$ is a smooth manifold and $x \in M$. Let f and g be real-valued functions defined in neighborhoods of x in M, which are smooth with respect to $\mathcal{S}$ and the standard smooth structure on $\mathbf{R}$. Let $r \in \mathbf{R}$. Then $f+g$, $f \cdot g$, and rf are all smooth with respect to $\mathcal{S}$ and the standard smooth structure on $\mathbf{R}$.*

Proof. Write $\mathcal{S}_{\mathbf{R}}$ for the standard smooth structure on $\mathbf{R}$. Let $\Delta : M \to M \times M$, $\alpha : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, and $\mu : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be given by $\Delta(y) = (y, y)$ for all $y \in M$, and $\alpha(r, s) = r + s$, $\mu(r, s) = rs$ for all $(r, s) \in \mathbf{R} \times \mathbf{R}$. Note that $f + g$ is the composition

$$\text{dom } f \cap \text{dom } g \xrightarrow{\Delta} (\text{dom } f \cap \text{dom } g) \times (\text{dom } f \cap \text{dom } g) \xrightarrow{f \times g} \mathbf{R} \times \mathbf{R} \xrightarrow{\alpha} \mathbf{R},$$

$f \cdot g$ is the composition

$$\text{dom } f \cap \text{dom } g \xrightarrow{\Delta} (\text{dom } f \cap \text{dom } g) \times (\text{dom } f \cap \text{dom } g) \xrightarrow{f \times g} \mathbf{R} \times \mathbf{R} \xrightarrow{\mu} \mathbf{R},$$

and rf is the composition

$$\text{dom } f \xrightarrow{\Delta} (\text{dom } f) \times (\text{dom } f) \xrightarrow{c \times f} \mathbf{R} \times \mathbf{R} \xrightarrow{\mu} \mathbf{R},$$

where c is the constant map with value r. For $i = 1, 2$, $\pi_i \Delta = 1_M$ which is smooth with respect to $\mathcal{S}$ and $\mathcal{S}$. Thus, by Proposition 4.17, Δ is smooth with respect to $\mathcal{S}$ and $\mathcal{S} \times \mathcal{S}$. By Exercise 7 of §4, $f \times g$ is smooth with respect to $\mathcal{S} \times \mathcal{S}$ and $\mathcal{S}_{\mathbf{R}} \times \mathcal{S}_{\mathbf{R}}$.

It is an easy exercise to check that any constant map is smooth with respect to any given smooth structures, and so c is smooth with respect to $\mathcal{S}$ and $\mathcal{S}_{\mathbf{R}}$. Thus, by Exercise 7 of §4, $c \times f$ is smooth with respect to $\mathcal{S} \times \mathcal{S}$ and $\mathcal{S}_{\mathbf{R}} \times \mathcal{S}_{\mathbf{R}}$. By Exercise 9 of §4, $\mathcal{S}_{\mathbf{R}} \times \mathcal{S}_{\mathbf{R}}$ is the standard smooth structure on $\mathbf{R} \times \mathbf{R}$. α and μ are clearly smooth with respect to the standard smooth structures. The result now follows easily from Propositions 4.19(a) and 4.18(b). □

If $(M, \mathcal{S})$, x, f, g, and r are as in Lemma 5.3, it is immediate that $[f+g]_x$ and $[f \cdot g]_x$ depend only on $[f]_x$ and $[g]_x$, and that $[rf]_x$ depends only on r and $[f]_x$. Thus we can define operations of addition, multiplication, and scalar multiplication by reals on $G_x(M, \mathcal{S})$ by the rules $[f]_x + [g]_x = [f+g]_x$, $[f]_x[g]_x = [f \cdot g]_x$, and $r[f]_x = [rf]_x$. The proof of the following is easy:

Proposition 5.4. *Let $(M, \mathcal{S})$ be a smooth manifold and $x \in M$. Then $G_x(M, \mathcal{S})$ is an algebra over $\mathbf{R}$ under the above operations.* □

Note that if $f \sim_x g$, then $f(x) = g(x)$.

Definition 5.5. Let $(M, \mathcal{S})$ be a smooth manifold and $x \in M$. A *tangent vector v to $(M, \mathcal{S})$ at x* is a derivation $v : G_x(M, \mathcal{S}) \to \mathbf{R}$, i.e.

(i) v is an $\mathbf{R}$-linear transformation;

(ii) for all $[f]_x$, $[g]_x \in G_x(M, \mathcal{S})$, $v([f]_x[g]_x) = v([f]_x)\, g(x) + f(x) v([g]_x)$.

The *tangent space to $(M, \mathcal{S})$ at x*, denoted $T_x(M, \mathcal{S})$, is the set of all tangent vectors to $(M, \mathcal{S})$ at x.

Example 5.6. Let V be a finite-dimensional vector space and M be an open subset of V. Let S denote the standard smooth structure on M. Let $x \in M$. For $v \in V$, define $\theta_x(v) : G_x(M, \mathcal{S}) \to \mathbf{R}$ by $\theta_x(v)([f]_x) = Df(x)(v)$. The latter is well-defined, for if $f \sim_x g$, then f and g agree on a neighborhood of x and so $Df(x) = Dg(x)$. $\theta_x(v)$ is easily checked to be $\mathbf{R}$-linear. By the Product Rule (Theorem 3.22), for any $[f]_x$, $[g]_x \in G_x(M, \mathcal{S})$, we have

$$\begin{aligned}\theta_x(v)([f]_x[g]_x) &= \theta_x(v)([f \cdot g]_x) = D(f \cdot g)(x)(v)\\ &= \Big(g(x)Df(x) + f(x)Dg(x)\Big)(v)\\ &= g(x)\theta_x(v)([f]_x) + f(x)\theta_x(v)([g]_x)\end{aligned}$$

and so $\theta_x(v) \in T_x(M, \mathcal{S})$. We thus obtain a function $\theta_x : V \to T_x(M, \mathcal{S})$.

In case $V = \mathbf{R}^n$, $\theta_x(v)([f]_x)$ is simply the directional derivative of f at x in the direction of v. (Here, we do not restrict v to being a unit vector as is sometimes done in calculus.) The idea here is that each vector $v \in V$ gives rise to a directional derivative $\theta_x(v)$ defined on germs at x. We will see shortly that, as a consequence of Taylor's Theorem, θ_x is bijective, i.e. there is a one-to-one correspondence between vectors in V and derivations of the algebra of germs at a point in the case of open subsets of V. The derivation concept makes sense in the abstract case, and we adopt it as our definition of tangent vector in general.

For a general smooth manifold $(M, \mathcal{S})$ and $x \in M$, it is easily checked that if v, $w \in T_x(M, \mathcal{S})$ and $r \in \mathbf{R}$, then $v + w$ and rv, defined by $(v+w)([f]_x) = v([f]_x) + w([f]_x)$ and $(rv)([f]_x) = rv([f]_x)$, are in $T_x(M, \mathcal{S})$. We easily obtain:

Proposition 5.7. *If $(M, \mathcal{S})$ is a smooth manifold and $x \in M$, then $T_x(M, \mathcal{S})$ is a real vector space under the above operations.* $\square$

It is not yet obvious that $T_x(M, \mathcal{S})$ is finite-dimensional over $\mathbf{R}$. We will see below that if $(M, \mathcal{S})$ is a smooth n-manifold, then $T_x(M, \mathcal{S})$ is actually n-dimensional over $\mathbf{R}$.

Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds and $f : M \to N$ a smooth map with respect to $\mathcal{S}$ and $\mathcal{T}$. Let $x \in M$. Given $[g]_{f(x)} \in G_{f(x)}(N, \mathcal{T})$ we can form the composition $gf : f^{-1}(\text{dom } g) \to \mathbf{R}$, which is smooth with respect to $\mathcal{S}$ and the standard smooth structure on $\mathbf{R}$. Note that if $g \sim_{f(x)} h$, then $gf \sim_x hf$.

Proposition 5.8. *Suppose $(M, \mathcal{S})$ and $(N, \mathcal{T})$ are smooth manifolds and $f : M \to N$ is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$. Let $x \in M$ and $v \in T_x(M, \mathcal{S})$. Define $T_xf(v) : G_{f(x)}(N, \mathcal{T}) \to \mathbf{R}$ by $T_xf(v)\left([g]_{f(x)}\right) = v([gf]_x)$. Then $T_xf(v) \in T_{f(x)}(N, \mathcal{T})$, and $T_xf : T_x(M, \mathcal{S}) \to T_{f(x)}(N, \mathcal{T})$ is an $\mathbf{R}$-linear transformation.*

Proof. If $[g_1]_{f(x)}$, $[g_2]_{f(x)} \in G_{f(x)}(N, \mathcal{T})$, then

$$\begin{aligned} T_xf(v)\left([g_1]_{f(x)}[g_2]_{f(x)}\right) &= T_xf(v)\left([g_1 \cdot g_2]_{f(x)}\right) = v([(g_1 \cdot g_2)f]_x) \\ &= v([(g_1f) \cdot (g_2f)]_x) = v([(g_1f)]_x[(g_2f)]_x) \\ &= v([(g_1f)]_x)\, g_2(f(x)) + g_1(f(x))v([(g_2f)]_x) \\ &= T_xf(v)\left([g_1]_{f(x)}\right) g_2(f(x)) + g_1(f(x))T_xf(v)\left([g_2]_{f(x)}\right) \end{aligned}$$

and so $T_xf(v)$ satisfies Condition (ii) of Definition 5.5. The check of Condition (i) is similar and shorter. The check that T_xf is $\mathbf{R}$-linear is left as an exercise. $\square$

T_xf is called the *tangent map of f at x*. It is sometimes called the *differential of f at x*.

Proposition 5.9. *Let $(M, \mathcal{S})$, $(N, \mathcal{T})$, and $(Q, \mathcal{U})$ be smooth manifolds. Suppose $f : M \to N$ and $g : N \to Q$ are smooth with respect to $\mathcal{S}$ and $\mathcal{T}$, and $\mathcal{T}$ and $\mathcal{U}$, respectively. Let $x \in M$. Then*

(a) $T_x 1_M = 1_{T_x(M, \mathcal{S})}$.

(b) $T_x(gf) = T_{f(x)}(g)T_x(f)$.

Proof. For any $v \in T_x(M, \mathcal{S})$ and $[h]_x \in G_x(M, \mathcal{S})$, $T_x 1_M(v)([h]_x) = v([h1_M]_x) = v([h]_x)$ and so $T_x 1_M(v) = v$, proving (a).

Let $[h]_{(gf)(x)} \in G_{(gf)(x)}(Q, \mathcal{U})$. Then

$$\begin{aligned} T_x(gf)(v)\left([h]_{(gf)(x)}\right) &= v([hgf]_x) = T_x(f)(v)\left([hg]_{f(x)}\right) \\ &= T_{f(x)}(g)T_x(f)(v)\left([h]_{(gf)(x)}\right), \end{aligned}$$

proving (b). $\square$

Corollary 5.10. *Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds and suppose $f : M \to N$ is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{T}$. Then for each $x \in M$, $T_x f : T_x(M, \mathcal{S}) \to T_{f(x)}(N, \mathcal{T})$ is an* **R***-isomorphism.*

Proof. By hypothesis, $f^{-1} : N \to M$ exists and is smooth with respect to $\mathcal{T}$ and $\mathcal{S}$. Since $f^{-1}f = 1_M$ and $ff^{-1} = 1_N$, it follows from Proposition 5.9 that $T_{f(x)}(f^{-1})T_x(f) = 1_{T_x(M,\mathcal{S})}$ and $T_x(f)T_{f(x)}(f^{-1}) = 1_{T_{f(x)}(N,\mathcal{T})}$. The result now follows. $\square$

Proposition 5.11. **(Local Property for Tangent Spaces)** (a) *Let $(M, \mathcal{S})$ be a smooth manifold. Suppose U is open in M and $x \in U$. Let $i : U \to M$ denote the inclusion map. Then $T_x i : T_x(U, \mathcal{S}|U) \to T_x(M, \mathcal{S})$ is an* **R***-isomorphism.*

(b) *Suppose $(N, \mathcal{T})$ is another smooth manifold and f, $g : M \to N$ are smooth with respect to $\mathcal{S}$ and $\mathcal{T}$. Suppose $x \in M$ is such that f and g agree on some open neighborhood of x in M. Then $T_x f = T_x g$.*

Proof. Since $i = 1_M|_U^M$ (see the notational comment following Definition 2.8), it follows from Proposition 4.19(a) that i is smooth with respect to $\mathcal{S}$ and $\mathcal{S}$. Note that the function $G_x i : G_x(M, \mathcal{S}) \to G_x(U, \mathcal{S}|U)$ given by $G_x i([f]_x) = [fi]_x$ is a bijection.

For any v in the kernel of $T_x i$ we have, for all $[f]_x \in G_x(M, \mathcal{S})$, $0 = T_x i(v)([f]_x) = v([fi]_x) = v\big(G_x i([f]_x)\big)$. Thus, since $G_x i$ is surjective, it follows that $v = 0$. Thus $T_x i$ is injective.

Let $v \in T_x(M, \mathcal{S})$. Since $G_x i$ is bijective, we can form the composition $w = v(G_x i)^{-1} : G_x(U, \mathcal{S}|U) \to \mathbf{R}$. One checks easily that w is a derivation and that $T_x i(w) = v$, establishing the surjectivity of $T_x i$. Thus part (a) is proved.

Let U be an open neighborhood of x in M on which f and g agree, and let $i : U \to M$ denote the inclusion map. Then $fi = gi : U \to N$. Thus

$$
\begin{aligned}
T_x f T_x i &= T_x(fi) && \text{(by Proposition 5.9(b))} \\
&= T_x(gi) \\
&= T_x g T_x i && \text{(by Proposition 5.9(b)).}
\end{aligned}
$$

Thus, since $T_x i$ is an isomorphism by part (a), $T_x f = T_x g$. $\square$

Recall the θ_x of Example 5.6.

Proposition 5.12. *Let V and W be real finite-dimensional vector spaces. Let M and N be open subsets of V and W, respectively, and let $\mathcal{S}$ and $\mathcal{T}$ denote their respective standard smooth structures. Suppose $f : M \to N$ is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$. Then for each $x \in M$, the diagram*

$$
\begin{array}{ccc}
V & \xrightarrow{Df(x)} & W \\
\Big\downarrow{\scriptstyle \theta_x} & & \Big\downarrow{\scriptstyle \theta_{f(x)}} \\
T_x(M, \mathcal{S}) & \xrightarrow[T_x f]{} & T_{f(x)}(N, \mathcal{T})
\end{array}
$$

commutes.

Proof. Let $v \in V$. For each $[g]_{f(x)} \in G_{f(x)}(N, \mathcal{T})$,

$$\begin{aligned}(T_x f)\theta_x(v)([g]_{f(x)}) &= \theta_x(v)([gf]_x) = D(gf)(x)(v) \\ &= Dg\big(f(x)\big)Df(x)(v) \\ &\qquad\text{(by the Chain Rule, Theorem 3.20)} \\ &= \theta_{f(x)}Df(x)(v)([g]_{f(x)}). \quad \square\end{aligned}$$

Theorem 5.13. *Let V be a real finite-dimensional vector space and M an open subset of V. Suppose $x \in M$, and let $\mathcal{S}$ denote the standard smooth structure on M. Then $\theta_x : V \to T_x(M, \mathcal{S})$ is an $\mathbf{R}$-isomorphism.*

Proof. We first consider the case $V = \mathbf{R}^n$. The $\mathbf{R}$-linearity of θ_x is easily checked. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbf{R}^n$. For $1 \leq i \leq n$ and any $[f]_x \in G_x(M, \mathcal{S})$, $\theta_x(e_i)([f]_x) = Df(x)(e_i) = D_i f(x)$. Thus if $\pi_j : M \to \mathbf{R}$, $1 \leq j \leq n$, denotes projection on the j^{th} coordinate, $\theta_x(e_i)([\pi_j]_x) = D_i\pi_j(x) = \delta_{ij}$. Thus if $v = (a_1, \ldots, a_n)$ lies in the kernel of θ_x, we have $0 = \theta_x(v)([\pi_j]_x) = \sum_{i=1}^n a_i\theta_x(e_i)([\pi_j]_x) = \sum_{i=1}^n a_i\delta_{ij} = a_j$ for $1 \leq j \leq n$ and so $v = 0$, establishing the injectivity of θ_x.

To prove surjectivity of θ_x it remains only to check that the $\theta_x(e_i)$, $1 \leq i \leq n$, span $T_x(M, \mathcal{S})$ over $\mathbf{R}$. Let $v \in T_x(M, \mathcal{S})$ be arbitrary. Given $[f]_x \in G_x(M, \mathcal{S})$, choose any convex open neighborhood U of x in dom f. By the First Taylor Theorem (Theorem 3.5), there exist smooth real-valued functions $g_1, \ldots, g_n$ on U such that $g_i(x) = D_i f(x)$ for $1 \leq i \leq n$, and for all $y \in U$,

$$f(y) = f(x) + \sum_{i=1}^n g_i(y)(y_i - x_i),$$

i.e.

$$f = c + \sum_{i=1}^n g_i \cdot (\pi_i - c_i)$$

where c is the constant function with value $f(x)$ and c_i is the constant function with value x_i for $1 \leq i \leq n$. Thus

$$v([f]_x) = v([c]_x) + \sum_{i=1}^n v([g_i]_x)(\pi_i(x) - x_i) + \sum_{i=1}^n g_i(x)v([\pi_i - c_i]_x).$$

Since $v([c]_x) = 0$ (see Exercise 1 in the Exercises for §5), $\pi_i(x) - x_i = 0$, and $g_i(x) = D_i f(x) = \theta_x(e_i)([f]_x)$, we obtain

$$v([f]_x) = \left(\sum_{i=1}^n v([\pi_i - c_i]_x)\theta_x(e_i)\right)([f]_x)$$

and so $v = \sum_{i=1}^n v([\pi_i - c_i]_x)\theta_x(e_i)$, completing the proof in case $V = \mathbf{R}^n$.

For the general case, choose any $\mathbf{R}$-isomorphism $\alpha : V \to \mathbf{R}^n$. Let $\mathcal{T}$ denote the standard smooth structure on $\alpha(M)$. By Proposition 5.12, the diagram

$$\begin{array}{ccc} V & \xrightarrow{D\alpha(x)} & \mathbf{R}^n \\ \downarrow{\scriptstyle \theta_x} & & \downarrow{\scriptstyle \theta_{\alpha(x)}} \\ T_x(M,\mathcal{S}) & \xrightarrow[T_x\alpha]{} & T_{\alpha(x)}(\alpha(M),\mathcal{T}) \end{array}$$

commutes. Since α is a diffeomorphism, $T_x\alpha$ is an $\mathbf{R}$-isomorphism by Corollary 5.10. Since α is an $\mathbf{R}$-linear transformation, $D\alpha(x) = \alpha$, which is an $\mathbf{R}$-isomorphism. $\theta_{\alpha(x)}$ is an $\mathbf{R}$-isomorphism from the $\mathbf{R}^n$ case. It now follows that θ_x is an $\mathbf{R}$-isomorphism. □

The upshot of Theorem 5.13 and Proposition 5.12 is that in the case of open subsets of real finite-dimensional vector spaces with the standard smooth structures, tangent spaces are canonically identified with the containing vector spaces and under this identification, the tangent map of a smooth map at a point is the derivative of that smooth map at that point.

Corollary 5.14. *Let $(M,\mathcal{S})$ be a smooth n-manifold. Then for each $x \in M$, $T_x(M,\mathcal{S})$ is n-dimensional over $\mathbf{R}$.*

Proof. Choose any $\mathcal{S}$-admissible chart φ with $x \in \text{dom } \varphi$. Say codom $\varphi \subset V$ where V is a real n-dimensional vector space. By Proposition 5.11, $T_x(M,\mathcal{S})$ is $\mathbf{R}$-isomorphic to $T_x(\text{dom } \varphi, \mathcal{S}|\text{dom } \varphi)$. Let $\mathcal{T}$ denote the standard smooth structure on codom φ. By Proposition 4.23, $\varphi : \text{dom } \varphi \to \text{codom } \varphi$ is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{T}$. Thus, by Corollary 5.10, $T_x\varphi : T_x(\text{dom } \varphi, \mathcal{S}|\text{dom } \varphi) \to T_{\varphi(x)}(\text{codom } \varphi, \mathcal{T})$ is an $\mathbf{R}$-isomorphism. By Theorem 5.13, $T_{\varphi(x)}(\text{codom } \varphi, \mathcal{T})$ is $\mathbf{R}$-isomorphic to V, and hence n-dimensional over $\mathbf{R}$. □

One consequence of Corollary 5.14 is that non-empty smooth manifolds of different dimensions cannot be diffeomorphic. The corresponding statement for topological manifolds and homeomorphisms is true, but requires some algebraic topology to prove.

Theorem 5.15. **(Inverse Function Theorem)** *Let $(M,\mathcal{S})$ and $(N,\mathcal{T})$ be smooth n-manifolds and $f : M \to N$ a smooth map with respect to $\mathcal{S}$ and $\mathcal{T}$. Suppose for some $x \in M$ the tangent map $T_xf : T_x(M,\mathcal{S}) \to T_{f(x)}(N,\mathcal{T})$ is an $\mathbf{R}$-isomorphism. Then there exist open neighborhoods U of x in M and V of $f(x)$ in N such that $f(U) = V$ and $f : U \to V$ is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{T}$.*

Proof. We can choose $\mathcal{S}$- and $\mathcal{T}$-admissible charts φ and ψ, respectively, with codomains in $\mathbf{R}^n$ such that $x \in \text{dom } \varphi$ and $f(\text{dom } \varphi) \subset \text{dom } \psi$. We have the commutative diagram

$$\begin{array}{ccccc} M & \xleftarrow{i} & \text{dom } \varphi & \xrightarrow{\varphi} & \text{codom } \varphi \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g} \\ N & \xleftarrow[j]{} & \text{dom } \psi & \xrightarrow[\varphi]{} & \text{codom } \psi \end{array}$$

where i, j are the inclusion maps, f' the restriction of f, and $g = \psi f' \varphi^{-1}$. Thus by Proposition 5.9(b) the diagram

$$\begin{array}{ccccc}
T_x(M) & \xleftarrow{T_x i} & T_x(\text{dom } \varphi) & \xrightarrow{T_x \varphi} & T_{\varphi(x)}(\text{codom } \varphi) \\
\Big\downarrow{\scriptstyle T_x f} & & \Big\downarrow{\scriptstyle T_x f'} & & \Big\downarrow{\scriptstyle T_{\varphi(x)} g} \\
T_{f(x)}(N) & \xleftarrow[T_{f(x)} j]{} & T_{f(x)}(\text{dom } \psi) & \xrightarrow[T_{f(x)} \psi]{} & T_{\psi(f(x))}(\text{codom } \psi)
\end{array}$$

commutes where, for notational simplicity, we have suppressed the smooth structures which are understood from context. By the Local Property for Tangent Spaces (Proposition 5.11), $T_x i$ and $T_{f(x)} j$ are **R**-isomorphisms. Since φ and ψ are diffeomorphisms, it follows from Corollary 5.10 that $T_x \varphi$ and $T_{f(x)} \psi$ are **R**-isomorphisms. Thus, since $T_x f$ is an **R**-isomorphism by hypothesis, $T_{\varphi(x)} g$ is an **R**-isomorphism. It now follows, by Proposition 5.12 and Theorem 5.13, that $Dg\big(\varphi(x)\big)$ is an **R**-isomorphism. Thus, by the Inverse Function Theorem (Theorem 3.4), there exist open neighborhoods A of $\varphi(x)$ in codom φ and B of $\psi\big(f(x)\big)$ in codom ψ such that g maps A diffeomorphically onto B. Take $U = \varphi^{-1}(A)$, $V = \psi^{-1}(B)$. □

Exercises for §5

1. Let $(M, \mathcal{S})$ be a smooth manifold, $x \in M$, and $v \in T_x(M, \mathcal{S})$.
(a) If c is a constant real-valued function defined in a neighborhood of x, prove that c is smooth and that $v([c]_x) = 0$.
(b) Prove that for all positive integers n and all $[f]_x \in G_x(M, \mathcal{S})$, $v\big([f]_x^n\big) = n\big(f(x)\big)^{n-1} v([f]_x)$.

2. Suppose $(M, \mathcal{S})$ and $(N, \mathcal{T})$ are smooth manifolds and $x \in M$, $y \in N$. Let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ denote the respective projections. Let $i_y : M \to M \times N$ and $j_x : N \to M \times N$ be given by $i_y(a) = (a, y)$, $j_x(b) = (x, b)$. Prove that i_y and j_x are smooth, and that the functions

$$\alpha : T_{(x,y)}(M \times N, \mathcal{S} \times \mathcal{T}) \to T_x(M, \mathcal{S}) \oplus T_y(N, \mathcal{T}),$$
$$\beta : T_x(M, \mathcal{S}) \oplus T_y(N, \mathcal{T}) \to T_{(x,y)}(M \times N, \mathcal{S} \times \mathcal{T})$$

given by $\alpha(v) = \Big(T_{(x,y)}\pi_1(v), T_{(x,y)}\pi_2(v)\Big)$, $\beta(u, w) = T_x i_y(u) + T_y j_x(w)$ are **R**-isomorphisms, inverse to one-another.

3. Let $f : S^2 \to \mathbf{R}^3$ be given by $f(x, y, z) = (x^2 - y^2, 2xy, z)$.
(a) Show that f is smooth with respect to the standard smooth structures on S^2 and $\mathbf{R}^3$.
(b) Show that $T_{(x,y,z)} f$ has rank 2 for all $(x, y, z) \in S^2$, except for $(0, 0, 1)$ and $(0, 0, -1)$.
(c) Determine the ranks of $T_{(0,0,1)} f$ and $T_{(0,0,-1)} f$.

6. Submanifolds, Regular Values, Immersions and Submersions

Definition 6.1. Let $(M, \mathcal{S})$ be a smooth n-manifold and $X \subset M$. An *$\mathcal{S}$-admissible k-chart for (M, X)*, $0 \le k \le n$, is an $\mathcal{S}$-admissible chart φ for M such that codom φ is an open subset of $V_\varphi \oplus W_\varphi$ where V_φ and W_φ are real vector spaces of dimensions k and $n-k$, respectively, and $\varphi(X \cap \text{dom } \varphi) = V_\varphi \cap \text{codom } \varphi$.

An *$\mathcal{S}$-admissible k-atlas for (M, X)* is a set $\mathcal{A}$ of $\mathcal{S}$-admissible k-charts for (M, X) such that $X \subset \bigcup_{\varphi \in \mathcal{A}} \text{dom } \varphi$.

If φ is an $\mathcal{S}$-admissible k-chart for (M, X), write $\varphi_X : X \cap \text{dom } \varphi \to V_\varphi \cap \text{codom } \varphi$ for the restriction of φ. Note that $X \cap \text{dom } \varphi$ is open in X, $V_\varphi \cap \text{codom } \varphi$ is open in V_φ, and that φ_X is a homeomorphism. Thus if $\mathcal{A}$ is an $\mathcal{S}$-admissible k-atlas for (M, X), it follows that $\{\varphi_X \mid \varphi \in \mathcal{A}\}$ is a k-manifold atlas for X, which we denote by $\mathcal{A}_X$.

Lemma 6.2. *Let $(M, \mathcal{S})$ be a smooth n-manifold and $X \subset M$. Suppose φ and ψ are $\mathcal{S}$-admissible k-charts for (M, X). Then φ_X and ψ_X are smoothly related.*

Proof. We have the commutative diagram

$$\begin{array}{ccccc}
V_\varphi \cap \varphi(\text{dom } \varphi \cap \text{dom } \psi) & \xrightarrow{\psi_X \varphi_X^{-1}} & V_\psi \cap \psi(\text{dom } \varphi \cap \text{dom } \psi) & \xrightarrow{i} & V_\varphi \\
\downarrow{\scriptstyle i_1} & & & & \uparrow{\scriptstyle \pi_1} \\
\varphi(\text{dom } \varphi \cap \text{dom } \psi) & \xrightarrow[\psi\varphi^{-1}]{} & \psi(\text{dom } \varphi \cap \text{dom } \psi) & \xrightarrow[j]{} & V_\psi \oplus W_\psi
\end{array}$$

where π_1 is projection on the first summand, $i_1 : V_\varphi \to V_\varphi \oplus W_\varphi$ is inclusion on the first summand, and i, j are inclusion maps. j is smooth, being an inclusion map of an open subset. Since φ and ψ are smoothly related, $\psi\varphi^{-1}$ is smooth. i_1 and π_1 are $\mathbf{R}$-linear transformations, and hence are smooth. Hence, by commutativity of the above diagram, $i\psi_X\varphi_X^{-1}$ is smooth. Since i is an inclusion map of an open subset, it follows from the Local Property (Proposition 3.21(a)) that $\psi_X\varphi_X^{-1}$ is smooth. Similarly, $\varphi_X\psi_X^{-1}$ is smooth. $\square$

Corollary 6.3. *Let $(M, \mathcal{S})$ be a smooth n-manifold, $X \subset M$, and suppose an $\mathcal{S}$-admissible k-atlas $\mathcal{A}$ for (M, X) exists. Then $\mathcal{A}_X$ is a smooth atlas for X, and the smooth structure on X determined by $\mathcal{A}_X$ depends only on $\mathcal{S}$, and not on the choice of $\mathcal{S}$-admissible k-atlas $\mathcal{A}$.* $\square$

Write $\mathcal{S}|X$ for the smooth structure on X arising from Corollary 6.3. In view of Corollary 6.3, we make the following definition:

Definition 6.4. Let $(M, \mathcal{S})$ be a smooth n-manifold and suppose $X \subset M$ is such that there exists an $\mathcal{S}$-admissible k-atlas for (M, X). We call the resulting smooth k-manifold $(X, \mathcal{S}|X)$ a *k-dimensional smooth submanifold of $(M, \mathcal{S})$*. More briefly, if $\mathcal{S}$ is understood from context, we say that *X is a k-dimensional smooth submanifold of M*.

Example 6.5. Let $(M, \mathcal{S})$ be a smooth n-manifold and suppose U is open in M. Let φ be any $\mathcal{S}$-admissible chart with dom $\varphi \subset U$, and write V_φ for the real

n-dimensional vector space containing codom φ. Take $W_\varphi = 0$ and thus $V_\varphi = V_\varphi \oplus W_\varphi$. Then φ is an $\mathcal{S}$-admissible n-chart for (M, U), and $\varphi_U = \varphi$. It follows that U is an n-dimensional smooth submanifold of M and that our new usage of the notation $\mathcal{S}|U$ coincides with the old usage (i.e. that following Proposition 4.11).

Example 6.6. By convention, the unique map $\mathbf{R}^0 \to \mathbf{R}^0$ is smooth. It follows that every 0-manifold admits a unique smooth structure.

Let $(M, \mathcal{S})$ be a smooth n-manifold and X a discrete subset of M. For each $x \in X$ we can choose an $\mathcal{S}$-admissible chart φ_x such that $X \cap \operatorname{dom} \varphi_x = \{x\}$ and $\varphi_x(x) = 0$. Write W_x for the real n-dimensional vector space which contains codom φ_x, and take $V_x = 0$. Then $V_x \oplus W_x = W_x$, and it follows easily that φ_x is an $\mathcal{S}$-admissible 0-chart for (M, X), and that $\{\varphi_x \mid x \in X\}$ is an $\mathcal{S}$-admissible 0-atlas for (M, X). Thus for each discrete subset of M, is a 0-dimensional smooth submanifold of M.

Example 6.7. Let X be a real n-dimensional vector space and Y a k-dimensional $\mathbf{R}$-linear subspace of X. Let $\mathcal{S}$ be the standard smooth structure on X. We can choose an $(n-k)$-dimensional subspace Z of X such that $X = Y \oplus Z$. Note that $\{1_X\}$ is an $\mathcal{S}$-admissible k-atlas for (X, Y), and so Y is a k-dimensional smooth submanifold of X.

Example 6.8. Let $\mathcal{S}$ denote the standard smooth structure on $\mathbf{R}^{n+1}$, $n \geq 0$. We claim that S^n is an n-dimensional smooth submanifold of $\mathbf{R}^{n+1}$.

Let $Y = \{x \in \mathbf{R}^{n+1} \mid (x_{n+1}+1)^2 > x_1^2 + \cdots + x_n^2\}$. Y is open in $\mathbf{R}^{n+1}$, being the inverse image of $(0, \infty)$ under the continuous map $\mathbf{R}^{n+1} \to \mathbf{R}$ which sends x to $(x_{n+1}+1)^2 - x_1^2 - \cdots - x_n^2$. Geometrically, Y is the interior of the upper solid cone of half-vertex angle $\pi/4$ with vertex at $(0, \ldots, 0, -1)$ and symmetric about the x_{n+1}-axis. We will construct an $\mathcal{S}$-admissible n-atlas $\mathcal{A}$ for $(\mathbf{R}^{n+1}, S^n)$ whose charts all have Y as codomain. We can write $\mathbf{R}^{n+1} = \mathbf{R}^n \oplus \mathbf{R}$, and each $\varphi \in \mathcal{A}$ will have $V_\varphi = \mathbf{R}^n$ and $W_\varphi = \mathbf{R}$.

For $1 \leq i \leq n+1$ let $W_i^+ = \{x \in \mathbf{R}^{n+1} \mid x_i > 0\}$, $W_i^- = \{x \in \mathbf{R}^{n+1} \mid x_i < 0\}$. The W_i^+ and W_i^- are all open in $\mathbf{R}^{n+1}$ and their union is $\mathbf{R}^{n+1} - \{0\}$, which contains S^n. Define $\theta_i^+ : W_i^+ \to Y$ and $\theta_i^- : W_i^- \to Y$ by

$$\theta_i^\pm(x) = (x_1, \ldots, \widehat{x}_i, \ldots, x_{n+1}, \|x\| - 1)$$

where $\|\ \|$ denotes the standard Euclidean norm on $\mathbf{R}^{n+1}$. Define $\rho_i^+ : Y \to W_i^+$ and $\rho_i^- : Y \to W_i^-$ by

$$\rho_i^+(x) = \left(x_1, \ldots, x_{i-1}, \sqrt{(1+x_{n+1})^2 - x_1^2 - \cdots - x_n^2}, x_i, \ldots, x_n\right),$$
$$\rho_i^-(x) = \left(x_1, \ldots, x_{i-1}, -\sqrt{(1+x_{n+1})^2 - x_1^2 - \cdots - x_n^2}, x_i, \ldots, x_n\right).$$

Clearly the θ_i^+, θ_i^-, ρ_i^+, and ρ_i^- are all smooth, and it is easily checked that θ_i^+, ρ_i^+ are inverses of one another, as are θ_i^-, ρ_i^-. Thus the θ_i^+ and θ_i^- are $\mathcal{S}$-admissible charts for $\mathbf{R}^{n+1}$. Note that the last coordinates of $\theta_i^\pm(x)$ are 0 if and only if

$x \in S^n$. Hence the $\theta_i^{\pm}$ are $\mathcal{S}$-admissible n-charts for $(\mathbf{R}^{n+1}, S^n)$ and constitute an $\mathcal{S}$-admissible n-atlas $\mathcal{A}$ for $(\mathbf{R}^{n+1}, S^n)$. Thus S^n is an n-dimensional smooth submanifold of $\mathbf{R}^{n+1}$. Note that the atlas $\mathcal{A}_{S^n}$ for S^n is precisely the atlas of Example 2.3. Thus $\mathcal{S}|S^n$ is the standard smooth structure on S^n.

Lemma 6.9. *Let $(M, \mathcal{S})$ be a smooth manifold and X a k-dimensional smooth submanifold of M. Let $x \in X$ and suppose φ is an $(\mathcal{S}|X)$-admissible chart for X with $x \in \operatorname{dom}\varphi$. Then there exists an $\mathcal{S}$-admissible k-chart ψ for (M, X) such that $x \in \operatorname{dom}\psi$ and ψ_X agrees with φ on some neighborhood of x in X.*

Proof. Let θ be any $\mathcal{S}$-admissible k-chart for (M, X) with $x \in \operatorname{dom}\theta$. By taking restrictions of φ and θ, we can suppose, without loss of generality, that $\operatorname{dom}\varphi = \operatorname{dom}\theta_X$. Let V be the real k-dimensional vector space which contains $\operatorname{codom}\varphi$. Since φ and θ_X are smoothly related, the overlap map $\varphi\theta_X^{-1} : \operatorname{codom}\theta_X \to \operatorname{codom}\varphi$ is a diffeomorphism. Thus $\left(\varphi\theta_X^{-1}\right) \times 1_{W_\theta} : (\operatorname{codom}\theta_X) \times W_\theta \to (\operatorname{codom}\varphi) \times W_\theta$ is a diffeomorphism onto an open subset of $V \times W_\theta$. Take ψ to be the composition $\left(\left(\varphi\theta_X^{-1}\right) \times 1_{W_\theta}\right)\theta$, restricted to $\theta^{-1}\Big((\operatorname{codom}\theta) \cap \left((\operatorname{codom}\theta_X) \times W_\theta\right)\Big)$. □

Proposition 6.10. *Let $(M, \mathcal{S})$ be a smooth manifold, X a k-dimensional smooth submanifold of M, and Y an l-dimensional smooth submanifold of X. Then Y is an l-dimensional smooth submanifold of M, and $\mathcal{S}|Y = (\mathcal{S}|X)|Y$.*

Proof. Note that if ψ is an $\mathcal{S}$-admissible k-chart for (M, X) such that ψ_X is an $(\mathcal{S}|X)$-admissible l-chart for (X, Y), we would have $V_\psi = V_{\psi_X} \oplus W_{\psi_X}$, ψ would be an $\mathcal{S}$-admissible l-chart for (M, Y), and $(\psi_X)_Y = \psi_Y$. Thus it suffices to show that for each $y \in Y$, there exists an $\mathcal{S}$-admissible k-chart ψ for (M, X) with $y \in \operatorname{dom}\psi$ such that ψ_X is an $(\mathcal{S}|X)$-admissible l-chart for (X, Y).

Let $y \in Y$ and choose an arbitrary $(\mathcal{S}|X)$-admissible l-chart φ for (X, Y) with $y \in \operatorname{dom}\varphi$. By Lemma 6.9, there exists an $\mathcal{S}$-admissible k-chart ψ for (M, X) such that ψ_X agrees with φ on a neighborhood of y in X. By restricting φ and ψ we can suppose, without loss of generality, that $\psi_X = \varphi$. □

Theorem 6.11. *Let $(M, \mathcal{S})$ be a smooth manifold and X a k-dimensional smooth submanifold of M. Let $i : X \to M$ denote the inclusion map. Then i is smooth with respect to $\mathcal{S}|X$ and $\mathcal{S}$ and for each $x \in X$, the tangent map $T_x i : T_x(X, \mathcal{S}|X) \to T_x(M, \mathcal{S})$ is injective.*

Proof. To prove that i is smooth it suffices, by Proposition 4.15, to show that whenever φ is an $\mathcal{S}$-admissible k-chart for (M, X), the composition $\varphi i \varphi_X^{-1} : \operatorname{codom}\varphi_X \to \operatorname{codom}\varphi$ is smooth. The latter is a restriction of the inclusion on the first summand $j : V_\varphi \to V_\varphi \oplus W_\varphi$, which is smooth since j is an $\mathbf{R}$-linear transformation. Thus, by the Local Property (Proposition 3.21(a)), $\varphi i \varphi_X^{-1}$ is smooth, establishing the smoothness of i.

Let $x \in X$ and choose an $\mathcal{S}$-admissible k-chart φ for (M, X) such that $x \in \operatorname{dom}\varphi$. By restricting φ we can suppose $\operatorname{codom}\varphi = (\operatorname{codom}\varphi_X) \times U$ where U is open in

W_φ. We have the commutative diagram of inclusion maps

$$\begin{array}{ccc} \text{dom } \varphi_X & \xrightarrow{i'} & \text{dom } \varphi \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_M} \\ X & \xrightarrow[i]{} & M \end{array}$$

which yields, by Proposition 5.9(b), the commutative diagram

$$\begin{array}{ccc} T_x(\text{dom } \varphi_X, \mathcal{S}\,|\,\text{dom } \varphi_X) & \xrightarrow{T_x i'} & T_x(\text{dom } \varphi, \mathcal{S}\,|\,\text{dom } \varphi) \\ {\scriptstyle T_x i_X}\downarrow & & \downarrow{\scriptstyle T_x i_M} \\ T_x(X, \mathcal{S}\,|\,X) & \xrightarrow[T_x i]{} & T_x(M, \mathcal{S}). \end{array}$$

By the Local Property for Tangent Spaces (Proposition 5.11), $T_x i_X$ and $T_x i_M$ are $\mathbf{R}$-isomorphisms. Thus it remains only to prove $T_x i'$ is injective.

We have the commutative diagram

$$\begin{array}{ccccc} \text{dom } \varphi_X & & \xrightarrow{i'} & & \text{dom } \varphi \\ & {\scriptstyle \varphi_X}\searrow & & \swarrow{\scriptstyle \pi\varphi} & \\ & & \text{codom } \varphi_X & & \end{array}$$

where $\pi : (\text{codom } \varphi_X) \times U \to \text{codom } \varphi_X$ is projection on the first factor. π is smooth, being the restriction of the $\mathbf{R}$-linear transformation $\pi : V_\varphi \oplus W_\varphi \to V_\varphi$. Thus, by Proposition 5.9, the diagram

$$\begin{array}{ccccc} T_x(\text{dom } \varphi_X, \mathcal{S}\,|\,\text{dom } \varphi_X) & & \xrightarrow{T_x i'} & & T_x(\text{dom } \varphi, \mathcal{S}\,|\,\text{dom } \varphi) \\ & {\scriptstyle T_x \varphi_X}\searrow & & \swarrow{\scriptstyle T_x(\pi\varphi)} & \\ & & T_{\varphi(x)}(\text{codom } \varphi_X, \mathcal{T}) & & \end{array}$$

commutes where $\mathcal{T}$ is the standard smooth structure on codom φ_X. Since φ_X is a diffeomorphism, it follows from Corollary 5.10 that $T_x\varphi_X$ is an $\mathbf{R}$-isomorphism. Hence, by commutative of the last diagram above, $T_x(\pi\varphi)T_x i'$ is an $\mathbf{R}$-isomorphism. It follows that $T_x i'$ is injective. $\square$

Recall that for topological spaces, if X is a subspace of Y and $g : Z \to X$ is a function where Z is a topological space, then g is continuous if and only if the composition $ig : Z \to Y$ is continuous where $i : X \to Y$ is the inclusion map. We next establish the analogue of this for smooth submanifolds and smooth maps.

Theorem 6.12. *Let $(M, \mathcal{S})$ be a smooth manifold and X a k-dimensional smooth submanifold of M. Let $i : X \to M$ denote the inclusion map. Let $(N, \mathcal{T})$ be a smooth manifold and $g : N \to X$ a function. Then g is smooth with respect to $\mathcal{T}$ and $\mathcal{S}|X$ if and only if $ig : N \to M$ is smooth with respect to $\mathcal{T}$ and $\mathcal{S}$.*

Proof. By Theorem 6.11, i is smooth with respect to $\mathcal{S}|X$ and $\mathcal{S}$. Thus if g is smooth with respect to $\mathcal{T}$ and $\mathcal{S}|X$, it follows from Proposition 4.18(b) that ig is smooth with respect to $\mathcal{T}$ and $\mathcal{S}$.

Conversely, suppose ig is smooth with respect to $\mathcal{T}$ and $\mathcal{S}$. Then, in particular, ig is continuous and so g is continuous. By Proposition 4.15, it remains only to show that for each $y \in N$ there exist $\mathcal{T}$- and $(\mathcal{S}|X)$-admissible charts φ and ψ, respectively, such that $y \in \operatorname{dom} \varphi$, $g(\operatorname{dom} \varphi) \subset \operatorname{dom} \psi$, and $\psi g \varphi^{-1} : \operatorname{codom} \varphi \to \operatorname{codom} \psi$ is smooth. We can choose an $\mathcal{S}$-admissible k-chart θ for (M, X) such that $g(y) \in \operatorname{dom} \theta$ and $\operatorname{codom} \theta = (\operatorname{codom} \theta_X) \times U$ for some open subset U of W_θ. Take $\psi = \theta_X$. Since $g^{-1}(\operatorname{dom} \psi)$ is an open neighborhood of y in N, we can choose a $\mathcal{T}$-admissible chart φ such that $y \in \operatorname{dom} \varphi$ and $g(\operatorname{dom} \varphi) \subset \operatorname{dom} \psi$. Then $\psi g \varphi^{-1} = \pi \theta i g \varphi^{-1}$ where $\pi : (\operatorname{codom} \psi) \times U \to \operatorname{codom} \psi$ is projection on the first factor. Since π, θ, ig and φ^{-1} are all smooth, so is their composition by Proposition 4.18(b). $\square$

We leave the proof of the following as an exercise.

Proposition 6.13. *Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth n-manifolds and $f : M \to N$ a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{T}$. Suppose X is a smooth k-dimensional submanifold of M. Then $f(X)$ is a smooth k-dimensional submanifold of N.* $\square$

Definition 6.14. Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds. An *immersion of* $(M, \mathcal{S})$ *into* $(N, \mathcal{T})$ is a map $f : M \to N$ which is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$ such that for each $x \in M$, $T_x f : T_x(M, \mathcal{S}) \to T_{f(x)}(N, \mathcal{T})$ is injective.

Example 6.15. It is immediate from Theorem 6.11 that if $(X, \mathcal{S} \,|\, X)$ is a smooth submanifold of the smooth manifold $(M, \mathcal{S})$, then the inclusion map $i : X \to M$ is an immersion of $(X, \mathcal{S} \,|\, X)$ into $(M, \mathcal{S})$.

Example 6.16. Let $f : \mathbf{R} \to \mathbf{R}^2$ be given by $f(x) = (x^2, x - x^3)$. f is clearly smooth with respect to $\mathcal{S}$ and $\mathcal{T}$ where the latter are the standard smooth structures. By Proposition 5.12 and Theorem 5.13 we can canonically identify the tangent map $T_x f$ with the derivative $Df(x) : \mathbf{R} \to \mathbf{R}^2$. The latter is

$$\begin{pmatrix} 2x \\ 1 - 3x^2 \end{pmatrix}$$

which has rank 1 for all real x. Thus f is an immersion of $(\mathbf{R}, \mathcal{S})$ into $(\mathbf{R}^2, \mathcal{T})$.

Note that in the above example, $f(-1) = f(1)$ and so f is not globally injective. However, we will see later that every immersion is locally injective. This, as well as the following Proposition, will require the Inverse Function Theorem (Theorem 5.15).

Proposition 6.17. *Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth n-manifolds and $f : M \to N$ an immersion with respect to $\mathcal{S}$ and $\mathcal{T}$ which is bijective. Then f is a diffeomorphism.*

Proof. The only question is the smoothness of f^{-1}. By Proposition 4.19(b) it suffices to check this locally.

Let $y \in N$. By the Inverse Function Theorem (Theorem 5.15), there exist open neighborhoods U of $f^{-1}(y)$ in M and V of y in N such that the restriction $f : U \to V$ is a diffeomorphism with respect to $\mathcal{S}$ and $\mathcal{T}$. In particular, $f^{-1} : V \to U$ is smooth with respect to $\mathcal{T}$ and $\mathcal{S}$. $\square$

Lemma 6.18. *Let U, V and W be finite-dimensional real vector spaces and suppose A, B and C are open subsets of U, V and W, respectively. Suppose $f : A \to W$ and $g : B \to W$ are smooth maps with respect to the standard smooth structures such that for all $(a, b) \in A \times B$, $f(a) + g(b) \in C$. Let $h : A \times B \to C$ be given by $h(a, b) = f(a) + g(b)$. Then h is smooth and $Dh(a, b)(u, v) = Df(a)(u) + Dg(b)(v)$ for all $(a, b) \in A \times B$ and all $(u, v) \in U \times V$.*

Proof. Smoothness of h follows easily from an examination of the coordinate functions of h in terms of those of f and g. Fix $(a, b) \in A \times B$ and define $T : U \times V \to W$ by $T(u, v) = Df(a)(u) + Dg(b)(v)$. Then T is $\mathbf{R}$-linear. By Theorem 3.18 it suffices to show that for some neighborhood N of (a, b) in $A \times B$ and some choice of norms $\|\cdot\|_{U\times V}$ and $\|\cdot\|_W$ for $U \times V$ and W, respectively, there exists a positive constant C such that

$$\|h(u, v) - h(a, b) - T\big((u, v) - (a, b)\big)\|_W \le C\|(u, v) - (a, b)\|^2_{U\times V}$$

for all $(u, v) \in N$. Choose arbitrary norms $\|\cdot\|_U$, $\|\cdot\|_V$, and $\|\cdot\|_W$ for U, V, and W, respectively and define $\|\cdot\|_{U\times V}$ by $\|(u, v)\|_{U\times V} = \max\{\|u\|_U, \|v\|_V\}$. Then $\|\cdot\|_{U\times V}$ is a norm on $U \times V$.

By Theorem 3.17, there exist neighborhoods N_1 of $a \in A$, N_2 of $b \in B$ and positive constants C_1 and C_2 such that

$$\begin{aligned} \|f(u) - f(a) - Df(a)(u - a)\|_W &\le C_1\|u - a\|^2_U \quad \text{for all } u \in N_1, \\ \|g(v) - g(b) - Dg(b)(v - b)\|_W &\le C_2\|v - b\|^2_V \quad \text{for all } v \in N_2. \end{aligned}$$

Take $N = N_1 \times N_2$. Then for all $(u, v) \in N$,

$$\begin{aligned} &\|h(u, v) - h(a, b) - T\big((u, v) - (a, b)\big)\|_W \\ &\quad = \|f(u) + g(v) - f(a) - g(b) - T(u - a, v - b)\|_W \\ &\quad = \|f(u) + g(v) - f(a) - g(b) - Df(u - a) - Dg(b)(v - b)\|_W \\ &\quad \le \|f(u) - f(a) - Df(a)(u - a)\|_W + \|g(v) - g(b) - Dg(v - b)\|_W \\ &\quad \le C_1\|u - a\|^2_U + C_2\|v - b\|^2_V \le (C_1 + C_2)\max\{\|u - a\|^2_U, \|v - b\|^2_V\} \\ &\quad = (C_1 + C_2)\|(u - a, v - b)\|^2_{U\times V} = (C_1 + C_2)\|(u, v) - (a, b)\|^2_{U\times V}. \quad \square \end{aligned}$$

Theorem 6.19. *Let $(M, \mathcal{S})$ be a smooth m-manifold, $(N, \mathcal{T})$ a smooth n-manifold, and suppose $f : M \to N$ is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$. Suppose $x_0 \in M$ is such that $T_{x_0}f$ is injective. Then there exists an open neighborhood A of x_0 in M such that:*

(a) *The restriction f to A is an immersion with respect to $\mathcal{S}$ and $\mathcal{T}$.*
(b) *f is injective on A.*
(c) *$f(A)$ is an m-dimensional smooth submanifold of N with respect to $\mathcal{T}$.*

Proof. We first treat the special case when M and N are open subsets of real vector spaces U and W, respectively, with the standard smooth structures, and that $f(x_0) = 0 \in W$. By Proposition 5.12 and Theorem 5.13, $Df(x_0) : U \to W$ is injective, and so we can write $W = Df(x_0)(U) \oplus V$ for some $(n-m)$-dimensional $\mathbf{R}$-linear subspace V of W. Define $h : M \times V \to W$ by $h(x, v) = f(x) + 1_V(v)$. By Lemma 6.18, $Dh(x_0, 0)(u, v) = Df(x_0)(u) + D1_V(0)(v)$. By Example 3.19, $D1_V(0) = 1_V$, and so $Dh(x_0, 0)(u, v) = Df(x_0)(u) + v$ for all (u, v) in $U \times V$. Thus since $W = Df(x_0)(U) \oplus V$, it follows that $Dh(x_0, 0)$ is onto, and hence is an $\mathbf{R}$-isomorphism since both $U \times V$ and W are n-dimensional over $\mathbf{R}$. By Proposition 5.12 and Theorem 5.13, it follows that $T_{(x_0,0)}h$ is an $\mathbf{R}$-isomorphism and hence, by the Inverse Function Theorem (Theorem 5.15), h maps some neighborhood of $(x_0, 0)$ in $U \times V$ diffeomorphically onto some neighborhood of 0 in W. By restricting to smaller neighborhoods we can suppose there exist open neighborhoods A of x_0 in U, B of 0 in V, and C of 0 in W such that $A \subset M$, $C \subset N$, and h maps $A \times B$ diffeomorphically onto C.

Let $i_0 : A \to A \times B$ and $\pi_1 : A \times B \to A$ be given by $i_0(a) = (a, 0)$ and $\pi_1(a, b) = a$. By Proposition 4.17, i_0 is smooth, and by Example 4.16, π_1 is smooth. We have the commutative diagram

$$\begin{array}{ccccc} & & A & & \\ & \swarrow^{1_A} & \downarrow{\scriptstyle i_0} & \searrow^{f} & \\ A & \xleftarrow[\pi_1]{} & A \times B & \xrightarrow[h]{} & C. \end{array}$$

Since i_0 and h are injective, f is injective on A, proving (b).

By Proposition 5.9(b), for all $a \in A$ the diagram

$$\begin{array}{ccccc} & & T_a(A) & & \\ & \swarrow^{T_a 1_A} & \downarrow{\scriptstyle T_a i_0} & \searrow^{T_a f} & \\ T_a(A) & \xleftarrow[T_{(a,0)}\pi_1]{} & T_{(a,0)}(A \times B) & \xrightarrow[T_{(a,0)}h]{} & T_0(C) \end{array}$$

commutes, where we have omitted notation for the understood standard smooth structures. By Proposition 5.9(a), $T_a 1_A = 1_{T_a(A)}$, and so from commutativity of the left-hand triangle, $T_a i_0$ is injective for all $a \in A$. Since h is a diffeomorphism from $A \times B$ to C, $T_{(a,0)}h$ is an $\mathbf{R}$-isomorphism for all $a \in A$. It follows, from commutativity of the right-hand triangle, that $T_a f$ is injective for all $a \in A$, proving (a).

Note that h^{-1} is an admissible chart for N with domain C, codomain $A \times B$, and that

$$\begin{aligned} h^{-1}(f(A) \cap \text{dom } h^{-1}) &= h^{-1}(f(A)) = A \times \{0\} = (U \times \{0\}) \cap (A \times B) \\ &= (U \times \{0\}) \cap \text{codom } h^{-1} \end{aligned}$$

and so h^{-1} is an admissible m-chart for $(N, f(A))$ whose domain contains $f(A)$. Part (c) follows.

We now consider the general case. We can choose $\mathcal{S}$- and $\mathcal{T}$-admissible charts φ and ψ, respectively, such that $x_0 \in \operatorname{dom} \varphi$ and $f(\operatorname{dom} \varphi) \subset \operatorname{dom} \psi$. Without loss of generality we can suppose $\psi(f(x_0)) = 0$. Let g denote the composition

$$\operatorname{codom} \varphi \xrightarrow{\varphi^{-1}} \operatorname{dom} \varphi \xrightarrow{f} \operatorname{dom} \psi \xrightarrow{\psi} \operatorname{codom} \psi .$$

Write $z_0 = \varphi(x_0)$. Then $g(z_0) = 0$. The diagrams

(*)

$$\begin{array}{ccc} \operatorname{codom} \varphi & \xrightarrow{g} & \operatorname{codom} \psi \\ {\scriptstyle \varphi^{-1}} \downarrow & & \downarrow {\scriptstyle \psi^{-1}} \\ \operatorname{dom} \varphi & \xrightarrow[f]{} & \operatorname{dom} \psi \end{array}$$

and

(**)

$$\begin{array}{ccc} T_{z_0}(\operatorname{codom} \varphi) & \xrightarrow{T_{z_0} g} & T_{g(z_0)}(\operatorname{codom} \psi) \\ {\scriptstyle T_{z_0}\varphi^{-1}} \downarrow & & \downarrow {\scriptstyle T_{g(z_0)}\psi^{-1}} \\ T_{x_0}(\operatorname{dom} \varphi) & \xrightarrow[T_{x_0} f]{} & T_{f(x_0)}(\operatorname{dom} \psi) \end{array}$$

both commute, where in (**) the omitted smooth structures are the evident ones. Since φ^{-1} and ψ^{-1} are diffeomorphisms, it follows that the vertical maps in (**) are $\mathbf{R}$-isomorphisms. Thus, since $T_{x_0} f$ is injective by hypothesis, it follows from (**) that $T_{z_0} g$ is injective. Thus, by the special case proved above, it follows that there exists an open neighborhood A' of z_0 in $\operatorname{codom} \varphi$ such that:

(a′) The restriction of g to A' is an immersion with respect to the standard smooth structures.

(b′) g is injective on A'.

(c′) $g(A')$ is an m-dimensional smooth submanifold of $\operatorname{codom} \psi$.

Take $A = \varphi^{-1}(A')$. By (*) and the fact that φ^{-1} and ψ^{-1} are bijective, part (b) follows. By Proposition 6.13 and the fact that φ^{-1} is a diffeomorphism, part (c) follows.

Let $a \in A$ and write $a' = \varphi(a) \in A'$. We have the commutative diagram

$$\begin{array}{ccc} T_{a'}(A') & \xrightarrow{T_{a'} g} & T_{g(a')}(\operatorname{codom} \psi) \\ {\scriptstyle T_{a'}\varphi^{-1}} \downarrow & & \downarrow {\scriptstyle T_{g(a')}\psi^{-1}} \\ T_a(A) & \xrightarrow[T_a f]{} & T_{f(a)}(\operatorname{dom} \psi) . \end{array}$$

The vertical maps are $\mathbf{R}$-isomorphisms since φ^{-1} and ψ^{-1} are diffeomorphisms. Thus, since $T_{a'} g$ is injective, it follows that $T_a f$ is injective, completing the proof of part (a). $\square$

Example 6.20. A smooth injective map can fail to be an immersion. For example, let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x^3$. Then f is smooth and injective, but $Df(0) = (0)$ and so f is not an immersion.

Theorem 6.21. *Let $(M, \mathcal{S})$ be a compact smooth m-manifold, $(N, \mathcal{T})$ a smooth n-manifold, and $f : M \to N$ an injective immersion with respect to $\mathcal{S}$ and $\mathcal{T}$. Then $f(M)$ is an m-dimensional smooth submanifold of N with respect to $\mathcal{T}$.*

Proof. Let $x \in M$. By Theorem 6.19 there exists an open neighborhood A_x of x in M such that $f(A_x)$ is an m-dimensional smooth submanifold of N with respect to $\mathcal{T}$. Choose a $\mathcal{T}$-admissible m-chart φ_x for $(N, f(A_x))$ with $f(x) \in \text{dom } \varphi_x$. Then there exist real vector spaces V_{φ_x} and W_{φ_x} of dimensions m and $n-m$, respectively, such that codom $\varphi_x \subset V_{\varphi_x} \oplus W_{\varphi_x}$ and $\varphi_x\big(f(A_x) \cap \text{dom } \varphi_x\big) = V_{\varphi_x} \cap \text{codom } \varphi_x$. φ_x need not be a $\mathcal{T}$-admissible m-chart for $(N, f(M))$ since it is possible that for some $y \in M - A_x$, $f(y) \in \text{dom } \varphi_x$ but $\varphi_x(f(y)) \notin V_{\varphi_x}$. We fix this as follows: $M - A_x$ is compact, and so $f(M - A_x)$ is closed in N. Take φ'_x to be the restriction of φ_x to dom $\varphi_x - f(M - A_x)$. Then $x \in \text{dom } \varphi'_x$ and φ'_x is a $\mathcal{T}$-admissible m-chart for $(N, f(M))$. Thus $\{\varphi'_x \mid x \in M\}$ is a $\mathcal{T}$-admissible m-atlas for $(N, f(M))$. □

If the compactness hypothesis in Theorem 6.21 is dropped, the conclusion can fail. For example, the restriction of the immersion in Example 6.16 to $(-\infty, 1)$ is an injective immersion whose image is not a manifold. Much more complicated phenomena are possible. For example, let α be any irrational real number and let $f : \mathbf{R} \to S^1 \times S^1$ be given by $f(t) = \big(e^{2\pi i t}, e^{2\alpha\pi i t}\big)$ where we regard S^1 as the space of complex numbers of absolute value 1. It can be shown that f is an injective immersion whose image is dense in $S^1 \times S^1$.

We next consider analogues of some of the above considerations for the case when the tangent map is surjective. We have the following analogue of Lemma 6.18.

Lemma 6.22. *Let U, V, and W be finite-dimensional real vector spaces, and suppose A, B, and C are open subsets of U, V, and W, respectively. Suppose $f : A \to B$ and $g : A \to C$ are smooth maps with respect to the standard smooth structures. Let $h : A \to B \times C$ be given by $h(a) = \big(f(a), g(a)\big)$ for all $a \in A$. Then h is smooth, and for all $a \in A$, $Dh(a) : U \to V \times W$ is given by $Dh(a)(u) = \big(Df(a)(u), Dg(a)(u)\big)$.*

Proof. Smoothness of h follows from Proposition 4.17. Choose norms $\|\cdot\|_U$, $\|\cdot\|_V$, and $\|\cdot\|_W$ for U, V, and W, respectively, and let $\|\cdot\|_{V\times W}$ be the norm on $V \times W$ given by $\|(v, w)\|_{V\times W} = \max(\|v\|_V, \|w\|_W)$. Fix $a \in A$. By Theorem 3.17 there exists a neighborhood N of a in A and a positive constant C such that for all $u \in N$,

$$\|f(u) - f(a) - Df(a)(u-a)\|_V \le C\|u-a\|_U^2$$

and

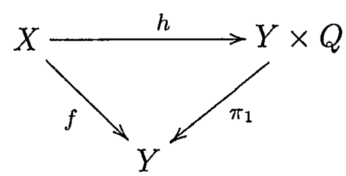
$$\|g(u) - g(a) - Dg(a)(u-a)\|_W \le C\|u-a\|_U^2.$$

Define $T : U \to V \times W$ by $T(u) = \big(Df(a)(u), Dg(a)(u)\big)$ for all $u \in U$. Then T is $\mathbf{R}$-linear and for all $u \in N$,

$$\begin{aligned}
\|h(u)&-h(a) - T(u-a)\|_{V\times W} \\
&= \big\|\big(f(u), g(u)\big) - \big(f(a), g(a)\big) - \big(Df(a)(u-a), Dg(a)(u-a)\big)\big\|_{V\times W} \\
&= \big\|\big(f(u) - f(a) - Df(a)(u-a), g(u) - g(a) - Dg(a)(u-a)\big)\big\|_{V\times W} \\
&= \max\{\|f(u) - f(a) - Df(a)(u-a)\|_V, \|g(u) - g(a) - Dg(a)(u-a)\|_W\} \\
&\le C\|u-a\|_U^2.
\end{aligned}$$

By Theorem 3.18, $T = Dh(a)$. □

Lemma 6.23. *Let $(M, \mathcal{S})$ be a smooth m-manifold, $(N, \mathcal{T})$ a smooth n-manifold, and $f : M \to N$ a smooth map with respect to $\mathcal{S}$ and $\mathcal{T}$. Suppose for some $x_0 \in M$ the tangent map $T_{x_0} f : T_{x_0}(M, \mathcal{S}) \to T_{f(x_0)}(N, \mathcal{T})$ is surjective. Then there exist an open neighborhood X of x_0 in M, an open neighborhood Y of $f(x_0)$ in N such that $f(X) \subset Y$, an open subset Q of some $(m-n)$-dimensional real vector space, and a diffeomorphism $h : X \to Y \times Q$ with respect to $\mathcal{S}$ and $\mathcal{T} \times \mathcal{U}$ (where $\mathcal{U}$ denotes the standard smooth structure on Q) such that the diagram*



commutes.

Proof. We first treat the special case in which M is an open subset of a real m-dimensional vector space U, and N an open subset of a real n-dimensional vector space V (with the standard smooth structures). By Proposition 5.12 and Theorem 5.13, $Df(x_0) : U \to V$ is surjective. Let K denote the kernel of $Df(x_0)$. Then we can write $U = J \oplus K$ for some $\mathbf{R}$-linear subspace J of U. Let $\pi_2 : U \to K$ denote projection on the second factor, and let $h : M \to V \times K$ be given by $h(x) = \big(f(x), \pi_2(x)\big)$. By Lemma 6.22, h is smooth and for all $x \in M$ and $u \in U$, $Dh(x)(u) = \big(Df(x)(u), D\pi_2(x)(u)\big)$. By Example 3.9, $D\pi_2(x) = \pi_2$ for all x, and so $Dh(x)(u) = \big(Df(x)(u), \pi_2(u)\big)$ for all $x \in M$, $u \in U$. We proceed to show that $Dh(x_0)$ is an $\mathbf{R}$-isomorphism. Note that U and $V \times K$ are both m-dimensional over $\mathbf{R}$, and so it suffices to show that $Dh(x_0)$ is injective.

Suppose u lies in the kernel of $Dh(x_0)$. Then $Df(x_0)(u) = 0$ and $\pi_2(u) = 0$. But then $u \in K$ (since $Df(x_0)(u) = 0$), and so $\pi_2(u) = u$, and so $u = 0$. Thus $Dh(x_0)$ is injective, and hence an $\mathbf{R}$-isomorphism.

By the Inverse Function Theorem (Theorem 5.15) there exists an open neighborhood X' of x_0 in M and an open neighborhood Z of $\big(f(x_0), \pi_2(x_0)\big)$ in $V \times K$ such that h maps X' diffeomorphically onto Z. We can choose a product neighborhood $Y \times Q$ of $\big(f(x_0), \pi_2(x_0)\big)$ in $N \times K$ such that $Y \times Q \subset Z$. Then taking $X = X' \cap h^{-1}(Y \times Q)$, the restriction $h : X \to Y \times Q$ fulfills the requirements.

For the general case, choose $\mathcal{S}$- and $\mathcal{T}$-admissible charts φ and ψ, respectively, such that $x_0 \in \operatorname{dom} \varphi$ and $f(\operatorname{dom} \varphi) \subset \operatorname{dom} \psi$. Let $g : \operatorname{codom} \varphi \to \operatorname{codom} \psi$ denote the composition

$$\operatorname{codom} \varphi \xrightarrow{\varphi^{-1}} \operatorname{dom} \varphi \xrightarrow{f} \operatorname{dom} \psi \xrightarrow{\psi} \operatorname{codom} \psi.$$

Since $T_{x_0} f$ is onto, and φ^{-1} and ψ are diffeomorphisms, it follows that $Dg(\varphi(x_0))$ is onto. By the special case treated above, there exist open neighborhoods X' of $\varphi(x_0)$ in $\operatorname{codom} \varphi$, Y' of $g(\varphi(x_0))$ in $\operatorname{codom} \psi$, and an open subset Q of some $m-n$-dimensional real vector space and a diffeomorphism $h' : X' \to Y' \times Q$ such that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{h'} & Y' \times Q \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & Y' & \end{array}$$

commutes. Take $X = \varphi^{-1}(X')$, $Y = \psi^{-1}(Y')$, and h to be the composition

$$X \xrightarrow{\varphi} X' \xrightarrow{h'} Y' \times Q \xrightarrow{\psi^{-1} \times 1_Q} Y \times Q. \quad \square$$

Corollary 6.24. *Let $(M, \mathcal{S})$ be a smooth m-manifold, $(N, \mathcal{T})$ a smooth n-manifold, and $f : M \to N$ a smooth map with respect to $\mathcal{S}$ and $\mathcal{T}$. Suppose for some $x \in M$ the tangent map $T_x f : T_x(M, \mathcal{S}) \to T_{f(x)}(N, \mathcal{T})$ is surjective. Then there exists an open neighborhood of $f(x)$ in N which is contained in the image of f.*

Proof. Let X, Y, Q, and h be the open sets and diffeomorphism which exist by Lemma 6.23. Then the diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \times Q \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & Y & \end{array}$$

commutes. Since $\pi_1 h$ is onto, it follows that Y is contained in the image of f. $\square$

Definition 6.25. Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds and $f : M \to N$ a smooth map with respect to $\mathcal{S}$ and $\mathcal{T}$. A point $y \in f(M)$ is called a *regular value of f with respect to $\mathcal{S}$ and $\mathcal{T}$* if for each $x \in f^{-1}(y)$, $T_x f$ is surjective.

f is said to be a *submersion with respect to $\mathcal{S}$ and $\mathcal{T}$* if each point in $f(M)$ is a regular value of f (or, equivalently, $T_x f$ is surjective for all $x \in M$).

Example 6.26. Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be arbitrary smooth manifolds and $\pi_1 : M \times N \to M$ projection on the first factor. Then π_1 is a submersion with respect to $\mathcal{S} \times \mathcal{T}$ and $\mathcal{S}$. For, by Example 4.16, π_1 is smooth with respect to $\mathcal{S} \times \mathcal{T}$ and $\mathcal{S}$. Suppose $(x_0, y_0) \in M \times N$. Define $\sigma : M \to M \times N$ by $\sigma(x) = (x, y_0)$. It follows from Propositions 4.17 and 4.18(a) and Exercise 9 of §4 that σ is smooth with respect to $\mathcal{S}$ and $\mathcal{S} \times \mathcal{T}$. Note that $\pi_1 \sigma = 1_M$. Thus, by Proposition 5.9, $1_{T_{x_0}(M, \mathcal{S})} = T_{(x_0, y_0)} \pi_1 T_{x_0} \sigma$ and so $T_{(x_0, y_0)} \pi_1$ is surjective.

Similarly, $\pi_2 : M \times N \to N$ is a submersion of with respect to $\mathcal{S} \times \mathcal{T}$ and $\mathcal{T}$.

The following is an immediate corollary of Corollary 6.24.

Corollary 6.27. *Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds and $f : M \to N$ a submersion with respect to $\mathcal{S}$ and $\mathcal{T}$. Then f is locally surjective, i.e. each point in the image of f has a neighborhood in N which is contained in the image of f.* $\square$

Theorem 6.28. *Let $(M, \mathcal{S})$ be a smooth m-manifold, $(N, \mathcal{T})$ a smooth n-manifold, and $f : M \to N$ a smooth map with respect to $\mathcal{S}$ and $\mathcal{T}$. Let y be a regular value of f with respect to $\mathcal{S}$ and $\mathcal{T}$. Then $f^{-1}(y)$ is an $(m-n)$-dimensional smooth submanifold of $(M, \mathcal{S})$.*

Proof. We must show that for each $x \in f^{-1}(y)$, there exists an $\mathcal{S}$-admissible $(m-n)$-chart φ for $\big(M, f^{-1}(y)\big)$ with $x \in \text{dom } \varphi$.

Let $x \in f^{-1}(y)$. Since y is a regular value of f with respect to $\mathcal{S}$ and $\mathcal{T}$, $T_x f : T_x(M, \mathcal{S}) \to T_y(N, \mathcal{T})$ is surjective. Let X, Y, Q, K, and h denote the sets and diffeomorphism which arise from Lemma 6.23. By taking smaller neighborhoods if necessary, we can suppose that $Y = \text{dom } \psi$ for some $\mathcal{T}$-admissible chart ψ. We can choose ψ such that $\psi(y) = 0$. Let φ be the composition

$$X \xrightarrow{h} Y \times Q \xrightarrow{\psi \times 1_Q} (\text{codom } \psi) \times Q.$$

Then φ is an $\mathcal{S}$-admissible chart for M with $x \in \text{dom } \varphi$. We will be done if we check that φ is an $\mathcal{S}$-admissible $(m-n)$-chart for $\big(M, f^{-1}(y)\big)$. It remains only to check that $\varphi\Big(f^{-1}(y) \cap X\Big) = (\{0\} \times K) \cap \text{codom } \varphi$. Note that $(\{0\} \times K) \cap \text{codom } \varphi = \{0\} \times Q$. Since $\pi_1 h(z) = f(z)$ for all $z \in X$ it follows that $h\Big(f^{-1}(y) \cap X\Big) = \{y\} \times Q$, from which the desired conclusion follows easily. $\square$

Example 6.29 Let $f : \mathbf{R}^{n+1} \to \mathbf{R}$ be given by $f(x) = x_1^2 + \cdots + x_{n+1}^2$. Then

$$Df(x) = \begin{pmatrix} 2x_1 & 2x_2 & \ldots & 2x_{n+1}. \end{pmatrix}$$

If $x \neq 0$, $Df(x)$ has rank 1, and hence is surjective. It follows from Proposition 5.12 and Theorem 5.13 that $T_x f$ is surjective for $x \neq 0$, and hence every positive real is a regular value of f. Thus, for each $r > 0$, $f^{-1}(r)$ is an n-dimensional smooth submanifold of $\mathbf{R}^{n+1}$. In particular, this gives an alternate proof that S^n is an n-dimensional smooth submanifold of $\mathbf{R}^{n+1}$. (Compare with Example 6.8.)

Example 6.30. For $1 \leq q \leq n$, the *real Stiefel manifold* $V_q(\mathbf{R}^n)$ is the set of all n-rowed, q-columned real matrices A satisfying $A^*A = I_q$ where A^* denotes the transpose of A and I_q is the $q \times q$ identity matrix. The above condition on A says that each column of A has Euclidean norm 1 (look at the diagonal entries of A^*A), and that distinct columns of A are orthogonal with respect to the standard Euclidean inner product on $\mathbf{R}^n$ (look at the off-diagonal entries of A^*A). Thus $V_q(\mathbf{R}^n)$ can be viewed as the set of all orthonormal q-frames of vectors in $\mathbf{R}^n$. Note that $V_n(\mathbf{R}^n) = O(n)$, the orthogonal group, if we identify matrices with the linear transformations they represent with respect to the standard bases (see Exercise 7 of §2). At the other extreme, $V_1(\mathbf{R}^n) = S^{n-1}$ if we identify $n \times 1$ matrices with points in $\mathbf{R}^n$.

Write $M_{n,q}(\mathbf{R})$ for the set of all real n-rowed, q-columned matrices, a vector space of dimension nq over $\mathbf{R}$. We will show that $V_q(\mathbf{R}^n)$ is an $\Big(nq - q(q+1)/2\Big)$-dimensional smooth submanifold of $M_{n,q}(\mathbf{R})$. Write $\text{Sym}(q)$ for the set of all $q \times q$ symmetric real matrices, a vector space of dimension $q(q+1)/2$ over $\mathbf{R}$. Define

$f : M_{n,q}(\mathbf{R}) \to \mathrm{Sym}(q)$ by $f(X) = X^*X$. Clearly, f is smooth with respect to the standard smooth structures, and $V_q(\mathbf{R}^n) = f^{-1}(I_q)$. Thus our above assertion will follow from Theorem 6.28 if we show that I_q is a regular value of f.

I_q is certainly in the image of f, e.g.

$$f\begin{pmatrix} I_q \\ 0_{n-q,q} \end{pmatrix} = I_q$$

where $0_{n-q,q}$ is the $(n-q) \times q$ 0-matrix. We claim that for each $A \in M_{n,q}(\mathbf{R})$, $Df(A) : M_{n,q}(\mathbf{R}) \to \mathrm{Sym}(q)$ is given by $Df(A)(X) = A^*X + X^*A$ for all $X \in M_{n,q}(\mathbf{R})$. (Compare with Exercise 2 of §3.) For if we let $\alpha : M_{n,q}(\mathbf{R}) \to \mathrm{Sym}(q)$ be given by $\alpha(X) = A^*X + X^*A$ for all $X \in M_{n,q}(\mathbf{R})$, then α is $\mathbf{R}$-linear, and

$$\begin{aligned} f(X) - f(A) - \alpha(X - A) &= X^*X - A^*A - A^*(X - A) - (X - A)^*A \\ &= (X - A)^*(X - A). \end{aligned}$$

Let $\| \|_M$ and $\| \|_S$ denote the sup norms on $M_{n,q}(\mathbf{R})$ and $\mathrm{Sym}(q)$, respectively, i.e the norm of a matrix is the maximum of the absolute values of its entries. For all $B, C \in M_{n,q}(\mathbf{R})$, it is easily checked that $\|B^*C\|_S \le n\|B\|_M\|C\|_M$. It follows that

$$\|f(X) - f(A) - \alpha(X - A)\|_S \le n\|X - A\|_M^2$$

for all $X \in M_{n,q}(\mathbf{R})$. Thus, by Theorem 3.16, $Df(A) = \alpha$.

We next check that for $A \in f^{-1}(I_q)$, $Df(A)$ is surjective. For such an A we have $A^*A = I_q$. Let $B \in \mathrm{Sym}(q)$ be arbitrary. Then $AB/2 \in M_{n,q}(\mathbf{R})$ and we have

$$\begin{aligned} Df(A)(AB/2) &= A^*(AB/2) + (AB/2)^*A = (A^*A)B/2 + (B^*A^*/2)A \\ &= B/2 + B^*/2 && \text{(since } A^*A = I_q\text{)} \\ &= B && \text{(since } B \text{ is symmetric)} \end{aligned}$$

proving that $Df(A)$ is surjective for $A \in f^{-1}(I_q)$. Thus, from Proposition 5.12 and Theorem 5.13, T_Af is surjective for each $A \in f^{-1}(I_q)$, completing the proof that I_q is a regular value of f.

Exercises for §6

1. Let $(M, \mathcal{S})$ be a smooth m-manifold, $(N, \mathcal{T})$ a smooth n-manifold, and $f : M \to N$ a smooth map with respect to $\mathcal{S}$ and $\mathcal{T}$. Let $G(f)$ denote the graph of f, i.e. $G(f) = \{(x, f(x)) \mid x \in M\} \subset M \times N$. Prove that $G(f)$ is an m-dimensional smooth submanifold of $(M \times N, \mathcal{S} \times \mathcal{T})$.

2. Let $\pi : S^n \to P^n(\mathbf{R})$ be the quotient map of Example 2.5. Prove that π is both an immersion and a submersion with respect to the standard smooth structures (Examples 4.6 and 4.7).

3. Let $((F, \mathcal{S}_F), E, (B, \mathcal{S}_B), p, \mathcal{S})$ be a smooth fiber bundle. Prove that $p : E \to B$ is a submersion with respect to $\mathcal{S}_E$ and $\mathcal{S}_B$.

4. Let $f : P^2(\mathbf{R}) \to \mathbf{R}^5$ be given by $f([x, y, z]) = (x^2, y^2, xy, xz, yz)$. Prove that f is an immersion with respect to the standard smooth structures.

5. Let $A = \{(x, y, z, t) \in \mathbf{R}^4 \mid x^2 - y^2 + z^2 = 1 \text{ and } xy - zt = 2\}$. Prove that A is a 2-dimensional smooth submanifold of $\mathbf{R}^4$.

6. Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds. Suppose X is a k-dimensional smooth submanifold of $(M, \mathcal{S})$ and Y an l-dimensional smooth submanifold of $(N, \mathcal{T})$. Prove that $X \times Y$ is a $(k + l)$-dimensional smooth submanifold of $(M \times N, \mathcal{S} \times \mathcal{T})$.

7. Prove Proposition 6.13.

7. Tangent Spaces to Smooth Submanifolds of Euclidean Space

If M is a smooth submanifold of $\mathbf{R}^n$, there is a more intuitive notion of the tangent space to M at a point x in M than the one given in §5, namely the totality of velocity vectors to smooth curves in M at x. In this section we canonically identify the two notions.

Definition 7.1. Let M be a smooth submanifold of $\mathbf{R}^n$ and $x \in M$. A *smooth curve α at x in M* is a smooth map $\alpha : (-\delta, \delta) \to M$ for some $\delta > 0$ such that $\alpha(0) = x$. The *velocity vector of α at x*, denoted $\operatorname{vel}(\alpha)$, is $D(i\alpha)(0)(1) \in \mathbf{R}^n$ where $i : M \to \mathbf{R}^n$ is the inclusion map.

Note that if $\alpha_1, \ldots, \alpha_n$ are the coordinate functions of $i\alpha$, then

$$D(i\alpha)(0) = \begin{pmatrix} \alpha_1'(0) \\ \alpha_2'(0) \\ \vdots \\ \alpha_n'(0) \end{pmatrix}$$

and so $\operatorname{vel}(\alpha) = \big(\alpha_1'(0), \alpha_2'(0), \ldots, \alpha_n'(0)\big)$, which is the usual calculus notion of the velocity vector to a smooth curve in Euclidean space at a point. Let $V_x(M)$ denote the set of all velocity vectors to smooth curves in M at x. We wish to canonically identify $V_x(M)$ with $T_x(M)$ (which we will write instead of $T_x(M, \mathcal{S}|M)$ where $\mathcal{S}$ is the standard smooth structure on $\mathbf{R}^n$).

Lemma 7.2. *Let M be a smooth submanifold of $\mathbf{R}^n$ and $x \in M$. Suppose α and β are smooth curves at x in M. Then:*

(a) *If α and β agree on some neighborhood of* 0, *then $\operatorname{vel}(\alpha) = \operatorname{vel}(\beta)$.*

(b) *If $(-\delta, \delta) \subset (\operatorname{dom} \alpha) \cap (\operatorname{dom} \beta)$, then $\operatorname{vel}(\alpha) = \operatorname{vel}(\beta)$ if and only if $T_0\alpha = T_0\beta : T_0\big((-\delta, \delta)\big) \to T_x(M)$.*

Proof. Let $i : M \to \mathbf{R}^n$ denote the inclusion. If α and β agree on some neighborhood of 0, then by the Local Property (Proposition 3.21(c)), $D(i\alpha)(0) = D(i\beta)(0)$ and it follows that $\operatorname{vel}(\alpha) = \operatorname{vel}(\beta)$, proving part (a).

Suppose $(-\delta, \delta) \subset (\operatorname{dom} \alpha) \cap (\operatorname{dom} \beta)$. By part (a) we can suppose $\operatorname{dom} \alpha = \operatorname{dom} \beta = (-\delta, \delta)$ for purposes of obtaining $\operatorname{vel}(\alpha)$ and $\operatorname{vel}(\beta)$. By the Chain Rule (Theorem 3.3), $\operatorname{vel}(\alpha) = D(i\alpha)(0)(1) = Di(x)D\alpha(0)(1)$. Similarly, $\operatorname{vel}(\beta) = Di(x)D\beta(0)(1)$. It follows from Theorem 6.11, Proposition 5.12, and Theorem 5.13 that $Di(x)$ is injective. Thus, $\operatorname{vel}(\alpha) = \operatorname{vel}(\beta)$ if and only if $D\alpha(0)(1) = D\beta(0)(1)$, i.e. if and only if $D\alpha(0) = D\beta(0)$. By Proposition 5.12 and Theorem 5.13, the latter condition holds if and only if $T_0\alpha = T_0\beta$, proving part (b). □

If $\alpha : (-\delta, \delta) \to M$ is a smooth curve at x in M and $0 < \delta' < \delta$, the diagram

$$\begin{array}{ccc} (-\delta', \delta') & \xrightarrow{\quad j \quad} & (-\delta, \delta) \\ & {\scriptstyle \alpha'} \searrow \qquad \swarrow {\scriptstyle \alpha} & \\ & M & \end{array}$$

commutes where j is the inclusion and α' the restriction of α. Thus, from Propositions 5.9 and 5.12, the diagram

$$\begin{array}{ccccc} \mathbf{R} & \xrightarrow{\theta_0} & T_0\big((-\delta',\delta')\big) & & \\ {\scriptstyle Dj(0)}\downarrow & & \downarrow{\scriptstyle T_0 j} & \searrow^{T_0\alpha'} & \\ & & & & T_x(M) \\ & & & \nearrow_{T_0\alpha} & \\ \mathbf{R} & \xrightarrow[\theta_0]{} & T_0\big((-\delta,\delta)\big) & & \end{array}$$

commutes. Since j is a restriction of $1_{\mathbf{R}}$, it follows from the Local Property (Proposition 3.21(c)) and Example 3.9 that $Dj(0) = 1_{\mathbf{R}}$. Thus we obtain a well-defined function $\varepsilon_{M,x} : V_x(M) \to T_x(M)$ by the rule $\varepsilon_{M,x}\big(\mathrm{vel}\,(\alpha)\big) = T_0\alpha\big(\theta_0(1)\big)$. We wish to show that $\varepsilon_{M,x}$ is a bijection. Note that we have not yet proved that $V_x(M)$ is an $\mathbf{R}$-linear subspace of $\mathbf{R}^n$. There does not seem to be an obvious operation on curves at x in M which induces addition of velocity vectors.

Lemma 7.3. *Let M be a smooth submanifold of $\mathbf{R}^n$ and $x \in M$. Let $i : M \to \mathbf{R}^n$ denote the inclusion map. Then the diagram*

$$\begin{array}{ccc} T_x(M) & \xrightarrow{T_x i} & T_x(\mathbf{R}^n) \\ {\scriptstyle \varepsilon_{M,x}}\uparrow & & \uparrow{\scriptstyle \theta_x} \\ V_x(M) & \xrightarrow[j]{} & \mathbf{R}^n \end{array}$$

commutes where j is the inclusion.

Proof. Let $\alpha : (-\delta, \delta) \to M$ be a smooth curve at x in M. From Proposition 5.12 we have the commutative diagram

$$\begin{array}{ccc} T_0\big((-\delta,\delta)\big) & \xrightarrow{T_0(i\alpha)} & T_x(\mathbf{R}^n) \\ {\scriptstyle \theta_0}\uparrow & & \uparrow{\scriptstyle \theta_x} \\ \mathbf{R} & \xrightarrow[D(i\alpha)(0)]{} & \mathbf{R}^n. \end{array}$$

By Proposition 5.9(b), $T_0(i\alpha) = T_x i T_0\alpha$. Thus

$$\begin{aligned} \theta_x\big(\mathrm{vel}\,(\alpha)\big) &= \theta_x\Big(D(i\alpha)(0)(1)\Big) = T_0(i\alpha)\big(\theta_0(1)\big) \\ &= T_x i\Big(T_0\alpha\big(\theta_0(1)\big)\Big) = T_x i\Big(\varepsilon_{M,x}\big(\mathrm{vel}\,(\alpha)\big)\Big). \end{aligned}$$ □

If M is a smooth submanifold of $\mathbf{R}^m$, N a smooth submanifold of $\mathbf{R}^n$, $f : M \to N$ a smooth map, and α a smooth curve at x in M, then $f\alpha$ is a smooth curve at $f(x)$

in N. We claim that if α and β are curves at x in M such that $\text{vel}(\alpha) = \text{vel}(\beta)$, then $\text{vel}(f\alpha) = \text{vel}(f\beta)$. For by Lemma 7.2 we can suppose $\text{dom}\,\alpha = \text{dom}\,\beta$ and $T_0\alpha = T_0\beta$. By Proposition 5.9(b), $T_0(f\alpha) = (T_xf)(T_0\alpha) = (T_xf)(T_0\beta) = T_0(f\beta)$ and hence $\text{vel}(f\alpha) = \text{vel}(f\beta)$ by Lemma 7.2(b), establishing the above claim. Thus we obtain a well-defined function $V_xf : V_x(M) \to V_{f(x)}(N)$ by the rule $V_xf\big(\text{vel}(\alpha)\big) = \text{vel}(f\alpha)$.

Proposition 7.4. *Suppose M is a smooth submanifold of $\mathbf{R}^m$, N a smooth submanifold of $\mathbf{R}^n$, and $f : M \to N$ a smooth map. Then for each $x \in M$ the diagram*

$$\begin{array}{ccc} V_x(M) & \xrightarrow{V_xf} & V_{f(x)}(N) \\ \downarrow{\scriptstyle \varepsilon_{M,x}} & & \downarrow{\scriptstyle \varepsilon_{N,f(x)}} \\ T_x(M) & \xrightarrow[T_xf]{} & T_{f(x)}(N) \end{array}$$

commutes.

Proof. Let $\alpha : (-\delta, \delta) \to M$ be a smooth curve at x in M. Then

$$\begin{aligned} \varepsilon_{N,f(x)}\Big(V_xf\big(\text{vel}(\alpha)\big)\Big) &= \varepsilon_{N,f(x)}\big(\text{vel}(f\alpha)\big) = T_0(f\alpha)\big(\theta_0(1)\big) \\ &= T_xf\Big(T_0\alpha\big(\theta_0(1)\big)\Big) = T_xf\Big(\varepsilon_{M,x}\big(\text{vel}(\alpha)\big)\Big). \end{aligned}$$

$\square$

Proposition 7.5. *Let M, N, and Q be smooth submanifolds of $\mathbf{R}^m$, $\mathbf{R}^n$, and $\mathbf{R}^q$, respectively, and $x \in M$. Then:*

(a) $V_x1_M = 1_{V_x(M)}$.

(b) If $f : M \to N$ and $g : N \to Q$ are smooth maps, then

$$V_x(gf) = \big(V_{f(x)}g\big)\,(V_xf)\,.$$

Proof. For each smooth curve α at x in M we have $V_x1_M\big(\text{vel}(\alpha)\big) = \text{vel}\big(1_M\alpha\big) = \text{vel}(\alpha)$, proving part (a), and

$$\begin{aligned} V_x(gf)\big(\text{vel}(\alpha)\big) &= \text{vel}(gf\alpha) = V_{f(x)}g\big(\text{vel}(f\alpha)\big) \\ &= V_{f(x)}g\Big(V_xf\big(\text{vel}(\alpha)\big)\Big) \end{aligned}$$

proving part (b). $\square$

Corollary 7.6. *Let M and N be smooth submanifolds of $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively. Suppose $f : M \to N$ is a diffeomorphism. Then for each $x \in M$, $V_xf : V_x(M) \to V_{f(x)}(N)$ is a bijection.*

Proof. By Proposition 7.5, $1_{V_x(M)} = V_x1_M = V_x(f^{-1}f) = V_{f(x)}(f^{-1})V_xf$ and similarly $1_{V_{f(x)}N} = V_{f(x)}(f^{-1})V_xf$. $\square$

Lemma 7.7. *Let M be a smooth submanifold of $\mathbf{R}^n$ and U an open subset of M. Let $j : U \to M$ denote the inclusion map. Then for each $x \in U$, $V_x(U) = V_x(M)$ and $V_x j : V_x(U) \to V_x(M)$ is the identity map.*

Proof. Let $i : M \to \mathbf{R}^n$ denote the inclusion. If α is a smooth curve at x in U, then $V_x j(\operatorname{vel}(\alpha)) = \operatorname{vel}(j\alpha) = D(ij\alpha)(0)(1)$. Since $ij : U \to \mathbf{R}^n$ is the inclusion, $D(ij\alpha)(0)(1) = \operatorname{vel}(\alpha)$, and so $V_x j(\operatorname{vel}(\alpha)) = \operatorname{vel}(\alpha)$ for all smooth curves α at x in U. Thus $V_x(U) \subset V_x(M)$ and $V_x j$ is the inclusion. It remains only to check that $V_x j$ is onto.

Let α be a smooth curve at x in M. Since U is open in M, there exists a $\delta > 0$ such that $(-\delta, \delta) \subset \operatorname{dom} \alpha$ and $\alpha((-\delta, \delta)) \subset U$. Let $\beta : (-\delta, \delta) \to U$ be the restriction of α. Since α and $j\beta$ agree on a neighborhood of 0, it follows from Lemma 7.2(a) that $\operatorname{vel}(\alpha) = \operatorname{vel}(j\beta)$. Since $\operatorname{vel}(j\beta) = V_x j(\operatorname{vel}(\beta))$, the proof is complete. □

Proposition 7.8. *Let M be an open subset of $\mathbf{R}^n$. Then for each $x \in M$, $V_x(M) = \mathbf{R}^n$ and $\varepsilon_{M,x} = \theta_x : V_x(M) \to T_x(M)$. In particular $\varepsilon_{M,x}$ is an $\mathbf{R}$-isomorphism.*

Proof. If $v \in \mathbf{R}^n$, let $\alpha : (-1, 1) \to \mathbf{R}^n$ be given by $\alpha(t) = x + tv$. Then α is a smooth curve at x in $\mathbf{R}^n$, and $\operatorname{vel}(\alpha) = v$, proving that $V_x(\mathbf{R}^n) = \mathbf{R}^n$. Since, by Lemma 7.7, $V_x(M) = V_x(\mathbf{R}^n)$, we have $V_x(M) = \mathbf{R}^n$.

By Lemma 7.3, $(T_x i)\varepsilon_{M,x} = \theta_x : \mathbf{R}^n \to T_x(\mathbf{R}^n)$ where $i : M \to \mathbf{R}^n$ denotes the inclusion. By Proposition 5.12, the diagram

$$\begin{array}{ccc} \mathbf{R}^n & \xrightarrow{Di(x)} & \mathbf{R}^n \\ {\scriptstyle\theta_x}\downarrow & & \downarrow{\scriptstyle\theta_x} \\ T_x(M) & \xrightarrow[T_x i]{} & T_x(\mathbf{R}^n) \end{array}$$

commutes. Since $Di(x) = 1_{\mathbf{R}^n}$ by Proposition 3.21(c) and Example 3.19, it follows that $(T_x i)\theta_x = \theta_x : \mathbf{R}^n \to T_x(\mathbf{R}^n)$. Thus, $(T_x i)\varepsilon_{M,x} = (T_x i)\theta_x$. Since $T_x i$ is injective by Theorem 6.11, the result now follows. □

Theorem 7.9. *Let M be a k-dimensional smooth submanifold of $\mathbf{R}^n$. Then for each $x \in M$, $V_x(M)$ is a k-dimensional $\mathbf{R}$-linear subspace of $\mathbf{R}^n$ and $\varepsilon_{M,x} : V_x(M) \to T_x(M)$ is an $\mathbf{R}$-isomorphism.*

Proof. Let $x \in M$. Choose a diffeomorphism φ from an open neighborhood of x in M to an open set in $\mathbf{R}^k$. By Proposition 7.4 the diagram

$$\begin{array}{ccccc} V_x(M) & \xleftarrow{V_x j} & V_x(\operatorname{dom} \varphi) & \xrightarrow{V_x \varphi} & V_{\varphi(x)}(\operatorname{codom} \varphi) \\ {\scriptstyle\varepsilon_{M,x}}\downarrow & & \downarrow{\scriptstyle\varepsilon_{\operatorname{dom} \varphi, x}} & & \downarrow{\scriptstyle\varepsilon_{\operatorname{codom} \varphi, \varphi(x)}} \\ T_x(M) & \xleftarrow[T_x j]{} & T_x(\operatorname{dom} \varphi) & \xrightarrow[T_x \varphi]{} & T_{\varphi(x)}(\operatorname{codom} \varphi) \end{array}$$

commutes where $j : \operatorname{dom} \varphi \to M$ is the inclusion map. By Lemma 7.7, $V_x j$ is the identity map. $T_x j$ is a bijection by Proposition 5.11. $V_x \varphi$ is a bijection by

Corollary 7.6. $T_x\varphi$ is a bijection by Corollary 5.10. $\varepsilon_{\text{codom }\varphi,\varphi(x)}$ is a bijection by Proposition 7.8. Thus, by commutativity of the above diagram, it follows that $\varepsilon_{M,x}$ is a bijection.

It now follows from Lemma 7.3 that

$$V_x(M) = \theta_x^{-1}\Big((T_x i)\big(T_x(M)\big)\Big)$$

where $i : M \to \mathbf{R}^n$ is the inclusion. Since θ_x is an $\mathbf{R}$-isomorphism (by Theorem 5.13), $T_x i$ is an injective $\mathbf{R}$-homomorphism (by Theorem 6.11), and $T_x(M)$ is a real k-dimensional vector space, it follows that $V_x(M)$ is a k-dimensional $\mathbf{R}$-linear subspace of $\mathbf{R}^n$ and that $\varepsilon_{M,x}$ is $\mathbf{R}$-linear. □

The upshot of Theorem 7.9 and Proposition 7.4 is that for smooth submanifolds of Euclidean spaces, the tangent space $T_x(M)$ is canonically identified with the space $V_x(M)$ of velocity vectors to smooth curves at x in M, and the tangent map $T_x f$ of a smooth map f between two such is identified with the map $V_x f$ which sends the velocity vector of a smooth curve at x to the velocity vector of that curve composed with f at $f(x)$.

Example 7.10. We have seen (Example 6.8) that S^n is an n-dimensional smooth submanifold of $\mathbf{R}^{n+1}$. Let $x \in S^n$ and α a smooth curve at x in S^n. Then for all $t \in \text{dom }\alpha$, $\sum_{i=1}^{n+1} \alpha_i(t)^2 = 1$. Thus, taking derivatives and evaluating at 0, we obtain $\sum_{i=1}^{n+1} \alpha_i(0)\alpha_i'(0) = 0$, which says that $\text{vel}\,(\alpha)$ is orthogonal to x with respect to the standard Euclidean inner product on $\mathbf{R}^{n+1}$. Thus $V_x(S^n)$ is contained in $x^\perp$, the orthogonal complement of x in $\mathbf{R}^{n+1}$. Since both $V_x(S^n)$ and $x^\perp$ are n-dimensional $\mathbf{R}$-linear subspaces of $\mathbf{R}^{n+1}$, it follows that $V_x(S^n) = x^\perp$. Thus, by Theorem 7.9, $T_x(S^n)$ is canonically identified with $x^\perp$ for each $x \in S^n$.

Exercises for §7

1. Suppose M is a smooth submanifold of $\mathbf{R}^m$ and N a smooth submanifold of $\mathbf{R}^n$. Then by Exercise 6 of §6, $M \times N$ is a smooth submanifold of $\mathbf{R}^m \times \mathbf{R}^n$, which we identify with $\mathbf{R}^{m+n}$ in the obvious way. Prove that for each $(x, y) \in M \times N$, $V_{(x,y)}(M \times N) = V_x(M) \times V_y(N)$.

2. For $0 \le k \le n$, the *Grassmannian space* $G_k(\mathbf{R}^n)$ is the set of all k-dimensional $\mathbf{R}$-linear subspaces of $\mathbf{R}^n$. We topologize $G_k(\mathbf{R}^n)$ as follows: Let $\text{Inj}_{\mathbf{R}}(\mathbf{R}^k, \mathbf{R}^n)$ denote the set of all injective $\mathbf{R}$-linear transformations from $\mathbf{R}^k$ to $\mathbf{R}^n$, topologized as a subspace of the finite-dimensional real vector space $\text{Hom}_{\mathbf{R}}(\mathbf{R}^k, \mathbf{R}^n)$. Let $q : \text{Inj}_{\mathbf{R}}(\mathbf{R}^k, \mathbf{R}^n) \to G_k(\mathbf{R}^n)$ be the function given by $q(A) = A(\mathbf{R}^k)$ for each $A \in \text{Inj}_{\mathbf{R}}(\mathbf{R}^k, \mathbf{R}^n)$. Give $G_k(\mathbf{R}^n)$ the quotient topology relative to q, i.e. a subset U of $G_k(\mathbf{R}^n)$ is open in $G_k(\mathbf{R}^n)$ if and only if $q^{-1}(U)$ is open in $\text{Inj}_{\mathbf{R}}(\mathbf{R}^k, \mathbf{R}^n)$.

Let M be a k-dimensional smooth submanifold of $\mathbf{R}^n$. Define $g_M : M \to G_k(\mathbf{R}^n)$ by $g_M(x) = V_x(M)$ for each $x \in M$. Prove that g_M is continuous.

8. Vector Bundles and the Tangent Bundle

Definition 8.1. A *vector bundle* ξ is a fiber bundle (F, E, B, p) such that

(i) F is a real finite-dimensional vector space.

(ii) For each $x \in B$, $p^{-1}(x)$ is equipped with the structure of a real finite-dimensional vector space.

(iii) There exists an atlas $\mathcal{A}$ for ξ such that for each $\varphi \in \mathcal{A}$ and $x \in U_\varphi$, the restriction $\varphi : p^{-1}(x) \to \{x\} \times F$ is an $\mathbf{R}$-isomorphism, where $\{x\} \times F$ is given the obvious vector space structure. Such an $\mathcal{A}$ is called a *linear atlas for* ξ. Any chart φ for ξ satisfying the above condition is called a *linear chart for* ξ.

A vector bundle for which the fiber is n-dimensional over $\mathbf{R}$ is called an *n-plane bundle.* A 1-plane bundle is sometimes called a *line bundle.*

Example 8.2. Let X be any topological space and V any finite-dimensional real vector space. Then the product bundle $(V, X \times V, X, \pi_1)$ is a vector bundle with linear atlas $\{1_{X\times V}\}$, where for each $x \in X$, $\pi_1^{-1}(x) = \{x\} \times V$ is given the obvious real vector space structure.

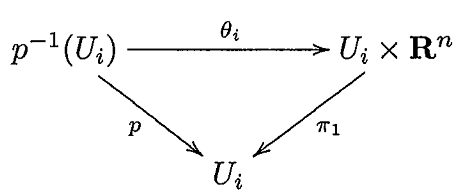
Example 8.3. We show that the canonical line bundle over $P^n(\mathbf{R})$ (Example 2.10) is a line bundle in the sense of Definition 8.1. Using the notation of Example 2.10, for each $[x] \in P^n(\mathbf{R})$, $p_n^{-1}([x]) = \{[x, r] \mid r \in \mathbf{R}\}$. We must first put a real vector space structure on the fiber $p_n^{-1}([x])$. Define addition and scalar multiplication by the rules $[x, r_1] + [x, r_2] = [x, r_1 + r_2]$ and $s[x, r] = [x, sr]$ for all r_1, r_2, and s in $\mathbf{R}$. It is straightforward to check that these operations are well-defined, that $p_n^{-1}([x])$ is a real 1-dimensional vector space under the above operations, and that the $\mathcal{A}_n$ of Example 2.10 is a linear atlas for $(\mathbf{R}, L_n, P^n(\mathbf{R}), p_n)$.

Example 8.4. We show that the fiber bundle $(\mathbf{R}^n, T(S^n), S^n, p)$ of Exercise 8 of §2 is an n-plane bundle. For each $x \in S^n$, $p^{-1}(x) = x^\perp$, the orthogonal complement of x in $\mathbf{R}^{n+1}$ with respect to the standard Euclidean inner product. $x^\perp$ is an n-dimensional $\mathbf{R}$-linear subspace of $\mathbf{R}^{n+1}$, and we choose this real vector space structure as the required real vector space structure on the fiber $p^{-1}(x)$.

For $1 \le i \le n+1$ let $U_i = \{x \in S^n \mid x_i \ne 0\}$. Then $\{U_1, \dots, U_n\}$ is an open cover of S^n. Define $\theta_i : p^{-1}(U_i) \to U_i \times \mathbf{R}^n$ by

$$\theta_i(x, y) = \big(x, (y_1, \dots, \widehat{y}_i, \dots, y_{n+1})\big).$$

θ_i is clearly continuous, the diagram



commutes, and each fiber restriction $\theta_i : p^{-1}(x) \to \{x\} \times \mathbf{R}^n$ is $\mathbf{R}$-linear. Thus, if we show that each θ_i is a homeomorphism, it will follow that $\{\theta_1, \dots, \theta_{n+1}\}$ is a linear atlas for $(\mathbf{R}^n, T(S^n), S^n, p)$. We accomplish the latter task by explicitly finding the inverse of θ_i and observing that it is continuous.

For $1 \le i \le n+1$ let $\eta_i : U_i \times \mathbf{R}^n \to p^{-1}(U_i)$ be given by

$$\eta_i(x,z) = \Big(x, (z_1, \ldots, z_{i-1}, \alpha_i(x,z), z_i, \ldots, z_n)\Big)$$

where $\alpha_i : U_i \times \mathbf{R}^n \to \mathbf{R}$ is given by

$$\alpha_i(x,z) = -x_i^{-1} \sum_{j=1}^{i-1} x_j z_j - x_i^{-1} \sum_{j=i+1}^{n+1} x_j z_{j-1}.$$

η_i is clearly continuous, and one checks that η_i and θ_i are inverses of one another.

Definition 8.5. Let $\xi_1 = (V_1, E_1, B_1, p_1)$ and $\xi_2 = (V_2, E_2, B_2, p_2)$ be vector bundles. A *vector bundle homomorphism* $f : \xi_1 \to \xi_2$ consists of a pair of continuous maps $f_E : E_1 \to E_2$ and $f_B : B_1 \to B_2$ such that

(i) The diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f_E} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow[f_B]{} & B_2 \end{array}$$

commutes.

(ii) From condition (i) it follows that for each $x \in B_1$, f_E maps $p_1^{-1}(x)$ into $p_2^{-1}\big(f_B(x)\big)$. We require that the restriction $f_E : p_1^{-1}(x) \to p_2^{-1}\big(f_B(x)\big)$ be $\mathbf{R}$-linear.

The proof of the following Proposition is immediate.

Proposition 8.6. (a) *Let $\xi = (V, E, B, p)$ be a vector bundle. Then $1_\xi : \xi \to \xi$ given by $1_{\xi_E} = 1_E$, $1_{\xi_B} = 1_B$ is a vector bundle homomorphism.*

(b) *Suppose ξ_i, $i = 1, 2, 3$, are vector bundles and $f : \xi_1 \to \xi_2$, $g : \xi_2 \to \xi_3$ are vector bundle homomorphisms. Then $gf : \xi_1 \to \xi_3$ given by $(gf)_E = g_E f_E$, $(gf)_B = g_B f_B$, is a vector bundle homomorphism.* □

We next consider *smooth* vector bundles. The fiber F is always a real finite-dimensional vector space and the smooth structure $\mathcal{S}_F$ on F will always be the standard smooth structure. Thus we modify the terminology of Definition 4.28 as follows:

Definition 8.7. Let $\xi = (F, E, B, p)$ be a vector bundle and $\mathcal{S}_B$ a smooth structure on B. If φ and ψ are linear charts for ξ, we say φ and ψ are *linearly $\mathcal{S}_B$-related* if φ and ψ are $\mathcal{S}_B$–$\mathcal{S}_F$-related in the sense of Definition 4.28.

A linear atlas $\mathcal{A}$ for ξ is said to be *linearly $\mathcal{S}_B$-smooth* if whenever $\varphi, \psi \in \mathcal{A}$, then φ and ψ are linearly $\mathcal{S}_B$-related.

If $\mathcal{A}$ and $\mathcal{B}$ are linearly $\mathcal{S}_B$-smooth atlases for ξ, we say $\mathcal{A}$ and $\mathcal{B}$ are *linearly $\mathcal{S}_B$-equivalent* if for all $\varphi \in A$ and $\psi \in \mathcal{B}$, φ and ψ are linearly $\mathcal{S}_B$-related.

The following is immediate from Proposition 4.29:

Proposition 8.8. *Let $\xi = (F, E, B, p)$ be a vector bundle and $\mathcal{S}_B$ a smooth structure for B. Then the relation linear $\mathcal{S}_B$-equivalence on the set of all linear $\mathcal{S}_B$-smooth atlases for ξ is an equivalence relation.* □

Definition 8.9. Let $\xi = (F, E, B, p)$ be a vector bundle and suppose $\mathcal{S}_B$ is a smooth structure structure on B. A *linear $\mathcal{S}_B$-smooth structure $\mathcal{S}$ for ξ* is a linear $\mathcal{S}_B$-equivalence class of linear $\mathcal{S}_B$-smooth atlases for ξ.

A linear $\mathcal{S}_B$-smooth atlas for ξ which represents a linear $\mathcal{S}_B$-smooth structure $\mathcal{S}$ for ξ will be said to be *$\mathcal{S}$-admissible.*

A linear chart for ξ which belongs to an $\mathcal{S}$-admissible linear $\mathcal{S}_B$-smooth atlas for ξ will be said to be $\mathcal{S}$-admissible.

Definition 8.10. A *smooth vector bundle* is a quintuple $\big(F, E, (B, \mathcal{S}_B), p, \mathcal{S}\big)$ satisfying:

(i) (F, E, B, p) is a vector bundle.

(ii) $(B, \mathcal{S}_B)$ is a smooth manifold.

(iii) $\mathcal{S}$ is a linear $\mathcal{S}_B$-smooth structure for $(F, E.B, p)$.

If $\xi = \big(F, E, (B, \mathcal{S}_B), p, \mathcal{S}\big)$ is a smooth vector bundle, write $\underset{\sim}{\xi} = (F, E, B, p)$ for its underlying vector bundle.

The following Proposition is the specialization of Proposition 4.32 to smooth vector bundles.

Proposition 8.11. *Let $\xi = \big(F, E, (B, \mathcal{S}_B), p, \mathcal{S}\big)$ be a smooth vector bundle. Then:*

(a) *Any two $\mathcal{S}$-admissible charts are linearly $\mathcal{S}_B$–related.*

(b) *If $\mathcal{A}$ and $\mathcal{B}$ are linear atlases for $\underset{\sim}{\xi}$ such that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B}$ is $\mathcal{S}$-admissible, then $\mathcal{A}$ is $\mathcal{S}$-admissible.*

(c) *If $\mathcal{A}$ and $\mathcal{B}$ are linearly $\mathcal{S}_B$–smooth atlases for $\underset{\sim}{\xi}$ such that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{A}$ is $\mathcal{S}$-admissible, then $\mathcal{B}$ is $\mathcal{S}$-admissible.*

(d) *If φ is a linear $\mathcal{S}$-admissible chart for $\underset{\sim}{\xi}$ and U is open in U_φ, then the restriction $\varphi : p^{-1}(U) \to U \times F$ is a linear $\mathcal{S}$-admissible chart.*

(e) *If φ is a linear chart for $\underset{\sim}{\xi}$ and $\mathcal{A}$ a linear $\mathcal{S}$-admissible atlas, then φ is $\mathcal{S}$-admissible if and only if φ is linearly $\mathcal{S}_B$–related to each member of $\mathcal{A}$.*

(f) *If $\mathcal{A}$ is a linear $\mathcal{S}$-admissible atlas and $\mathcal{C}$ any set of linear $\mathcal{S}$-admissible charts, then $\mathcal{A} \cup \mathcal{C}$ is a linear $\mathcal{S}$-admissible atlas.* □

In particular, if $\big(F, E, (B, \mathcal{S}_B), p, \mathcal{S}\big)$ is a smooth vector bundle, it follows that $\big((F, \mathcal{S}_F), E, (B, \mathcal{S}_B), p, \mathcal{S}\big)$ is a smooth fiber bundle in the sense of Definition 4.31 (where $\mathcal{S}_F$ denotes the standard smooth structure on the vector space F). By Theorem 4.34, there will exist a unique smooth structure $\mathcal{S}_E$ on E such that each $\mathcal{S}$-admissible linear chart $\varphi : p^{-1}(U_\varphi) \to U_\varphi \times F$ is a diffeomorphism with respect to $\mathcal{S}_E \,|\, p^{-1}(U_\varphi)$ and $(\mathcal{S}_B \,|\, U_\varphi) \times \mathcal{S}_F$.

We leave as exercises the verification that the linear atlases in Examples 8.3 and 8.4 above are linearly smooth with respect to the standard smooth structures on the base spaces.

Definition 8.12. Let $\xi_i = \big(V_i, E_i, (B_i, \mathcal{S}_{B_i}), p_i, \mathcal{S}_i\big)$ be smooth vector bundles, $i = 1, 2$. A *smooth vector bundle homomorphism* $f : \xi_1 \to \xi_2$ is a vector bundle homomorphism from (V_1, E_1, B_1, p_1) to (V_2, E_2, B_2, p_2) such that $f_B : B_1 \to B_2$ is smooth with respect to $\mathcal{S}_{B_1}$ and $\mathcal{S}_{B_2}$, and $f_E : E_1 \to E_2$ is smooth with respect to $(\mathcal{S}_1)_E$ and $(\mathcal{S}_2)_E$.

The following analogue of Proposition 8.6 is immediate.

Proposition 8.13. (a) *Let* $\xi = \big(V, E, (B, \mathcal{S}_B), p, \mathcal{S}\big)$ *be a smooth vector bundle. Then* $1_\xi : \xi \to \xi$ *given by* $1_{\xi_E} = 1_E$, $1_{\xi_B} = 1_B$ *is a smooth vector bundle homomorphism.*

(b) *Suppose* ξ_i, $i = 1, 2, 3$, *are smooth vector bundles and* $f : \xi_1 \to \xi_2$, $g : \xi_2 \to \xi_3$ *are smooth vector bundle homomorphisms. Then* $gf : \xi_1 \to \xi_3$ *given by* $(gf)_E = g_E f_E$, $(gf)_B = g_B f_B$ *is a smooth vector bundle homomorphism.* □

Given any m-manifold M, a chart for M with codomain an open subset of $\mathbf{R}^m$ will be called a *Euclidean chart for* M. If φ is any chart for M with codomain an open subset of some m-dimensional real vector space V, and $\alpha : V \to \mathbf{R}^m$ an $\mathbf{R}$-isomorphism, then the composition

$$\text{dom } \varphi \xrightarrow{\ \varphi\ } \text{codom } \varphi \xrightarrow{\ \alpha\ } \alpha(\text{codom } \varphi)$$

is a Euclidean chart for M having the same domain as φ. Moreover, if $\mathcal{S}$ is a smooth structure on M, it is easily seen that φ is $\mathcal{S}$-admissible if and only if the Euclidean chart $\alpha\varphi$ is $\mathcal{S}$-admissible.

It will sometimes be convenient to use a common real m-dimensional vector space to contain the codomains of all charts we consider for m-manifolds, and $\mathbf{R}^m$ will be convenient for that purpose. Given a smooth m-manifold $(M, \mathcal{S})$, we will let $\mathcal{E}(M, \mathcal{S})$ denote the set of all $\mathcal{S}$-admissible Euclidean charts for M. $\mathcal{E}(M, \mathcal{S})$ is an $\mathcal{S}$-admissible atlas for M.

The main goal of this section is to construct for each smooth m-manifold $(M, \mathcal{S})$ a smooth m-plane bundle $\tau_{M,\mathcal{S}} = (\mathbf{R}^m, T(M, \mathcal{S}), (M, \mathcal{S}), p_{M,\mathcal{S}}, \widetilde{\mathcal{S}})$, the *tangent bundle of* $(M, \mathcal{S})$, and, if $(N, \mathcal{T})$ is another smooth manifold and $f : M \to N$ a smooth map with respect to $\mathcal{S}$ and $\mathcal{T}$, a smooth vector bundle homomorphism $\tau f : \tau_{M,\mathcal{S}} \to \tau_{N,\mathcal{T}}$. For $x \in M$, the fiber $p^{-1}_{M,\mathcal{S}}(x)$ will be the tangent space $T_x(M, \mathcal{S})$; τf_B will be f; the restriction of τf_E to $p^{-1}_{M,\mathcal{S}}(x)$ will be the tangent map $T_x f : T_x(M, \mathcal{S}) \to T_{f(x)}(N, \mathcal{T})$.

As a set, $T(M, \mathcal{S})$ is defined to be $\bigcup_{x \in M} T_x(M, \mathcal{S})$, and $p_{M,\mathcal{S}} : T(M, \mathcal{S}) \to M$ is defined by $p_{M,\mathcal{S}}(v) = x$ if $v \in T_x(M, \mathcal{S})$. Thus $p^{-1}_{M,\mathcal{S}}(x) = T_x(M, \mathcal{S})$. Our next task is to put a suitable topology on $T(M, \mathcal{S})$.

Let $\varphi \in \mathcal{E}(M, \mathcal{S})$. Thus codom φ is an open subset of $\mathbf{R}^m$. Write $\mathcal{S}t_\varphi$ for the standard smooth structure on codom φ.

By the Local Property for Tangent Maps (Proposition 5.11), for each $x \in \text{dom } \varphi$, the tangent map $T_x i_\varphi : T_x(\text{dom } \varphi, \mathcal{S} \,|\, \text{dom } \varphi) \to T_x(M, \mathcal{S})$ is an $\mathbf{R}$-isomorphism, where $i_\varphi : \text{dom } \varphi \to M$ denotes the inclusion map. Taking disjoint unions,we get a bijection of sets

$$(1) \qquad T i_\varphi : \bigsqcup_{x \in \text{dom } \varphi} T_x(\text{dom } \varphi, \mathcal{S} \,|\, \text{dom } \varphi) \to \bigsqcup_{x \in \text{dom } \varphi} T_x(M, \mathcal{S}) = p^{-1}_{M,\mathcal{S}}(\text{dom } \varphi).$$

Since $\varphi : \text{dom } \varphi \to \text{codom } \varphi$ is a diffeomorphism with respect to $\mathcal{S} \,|\, \text{dom } \varphi$ and $\mathcal{S}t_\varphi$, for each $x \in \text{dom } \varphi$ the tangent map

$$T_x\varphi : T_x(\text{dom } \varphi, \mathcal{S} \,|\, \text{dom } \varphi) \to T_{\varphi(x)}(\text{codom } \varphi, \mathcal{S}t_\varphi)$$

is an $\mathbf{R}$-isomorphism and so, taking disjoint unions, we obtain a bijection of sets

$$\text{(2)} \qquad T\varphi : \bigcup_{x \in \text{dom } \varphi} T_x(\text{dom } \varphi, \mathcal{S} \,|\, \text{dom } \varphi) \to \bigcup_{x \in \text{dom } \varphi} T_{\varphi(x)}(\text{codom } \varphi, \mathcal{S}t_\varphi).$$

Recall, from Example 5.6, the $\mathbf{R}$-isomorphism $\theta_{\varphi(x)} : \mathbf{R}^m \to T_{\varphi(x)}(\text{codom } \varphi, \mathcal{S}t_\varphi)$ for each $x \in \text{dom } \varphi$. Define

$$\widetilde{\theta}_{\varphi(x)} : \{x\} \times \mathbf{R}^m \to T_{\varphi(x)}(\text{codom } \varphi, \mathcal{S}t_\varphi)$$

by $\widetilde{\theta}_{\varphi(x)}(x, v) = \theta_{\varphi(x)}(v)$. Taking disjoint unions, we get a bijection

$$\text{(3)} \qquad \widetilde{\theta}_\varphi : (\text{dom } \varphi) \times \mathbf{R}^m \to \bigcup_{x \in \text{dom } \varphi} T_{\varphi(x)}(\text{codom } \varphi, \mathcal{S}t_\varphi).$$

From the bijections in (1),(2), and (3) above, we get a bijection of sets

$$\widetilde{\varphi} = \widetilde{\theta}_\varphi^{-1} \circ (T\varphi) \circ (Ti)^{-1} : p_{M,\mathcal{S}}^{-1}(\text{dom } \varphi) \to (\text{dom } \varphi) \times \mathbf{R}^m.$$

Give $(\text{dom } \varphi) \times \mathbf{R}^m$ the product topology. We give $p_{M,\mathcal{S}}^{-1}(\text{dom } \varphi)$ the unique topology for which $\widetilde{\varphi}$ is a homeomorphism. Call this topology the *φ-local topology on* $p_{M,\mathcal{S}}^{-1}(\text{dom } \varphi)$.

Proposition 8.14. *Let $(M, \mathcal{S})$ be a smooth m-manifold. Define $TOP(M, \mathcal{S})$ to be the collection of all subsets A of $T(M, \mathcal{S})$ such that for every $\varphi \in \mathcal{E}(M, \mathcal{S})$, $A \cap p_{M,\mathcal{S}}^{-1}(\text{dom } \varphi)$ is open in the φ-local topology. Then:*

(a) *$TOP(M, \mathcal{S})$ is a topology on $T(M, \mathcal{S})$.*

(b) *If $\varphi \in \mathcal{E}(M, \mathcal{S})$ and U is open in* dom φ*, then $p_{M,\mathcal{S}}^{-1}(U) \in TOP(M, \mathcal{S})$.*

(c) *For each $\varphi \in \mathcal{E}(M, \mathcal{S})$, the subspace topology on $p_{M,\mathcal{S}}^{-1}(\text{dom } \varphi)$ induced by $TOP(M, \mathcal{S})$ is the φ-local topology.*

Proof. The proof of part (a) is immediate.

For any $\psi \in \mathcal{E}(M, \mathcal{S})$ and $V \subset \text{dom } \psi$, note that $\widetilde{\psi}(p_{M,\mathcal{S}}^{-1}(V)) = \psi(V) \times \mathbf{R}^m$. If V is open in dom ψ, then $\psi(V) \times \mathbf{R}^m$ is open in $(\text{codom } \psi) \times \mathbf{R}^m$ and so, since $\widetilde{\psi}$ is a homeomorphism, $p_{M,\mathcal{S}}^{-1}(V)$ is open in the ψ-local topology. In particular, if U is open in dom φ, then $p_{M,\mathcal{S}}^{-1}(U \cap \text{dom } \psi)$ is open in the ψ-local topology, i.e. $p_{M,\mathcal{S}}^{-1}(U) \cap p_{M,\mathcal{S}}^{-1}(\psi)$ is open in the ψ-local topology. It follows that $p_{M,\mathcal{S}}^{-1}(U) \in TOP(M, \mathcal{S})$. Part (b) follows.

Let $\varphi \in \mathcal{E}(M, \mathcal{S})$. By part (b), $p_{M,\mathcal{S}}^{-1}(\text{dom } \varphi) \in TOP(M, \mathcal{S})$. If $A \in TOP(M, \mathcal{S})$, then by definition of $TOP(M, \mathcal{S})$, $A \cap p_{M,\mathcal{S}}^{-1}(\text{dom } \varphi)$ is open in the φ-local topology. Part (c) follows. $\square$

Unless otherwise stated, the topology on $T(M, \mathcal{S})$ will be understood to be $TOP(M, \mathcal{S})$.

Corollary 8.15. *Let $(M, \mathcal{S})$ be a smooth manifold. Then $p_{M,\mathcal{S}} : T(M, \mathcal{S}) \to M$ is continuous.*

Proof. $\{\text{dom } \varphi \mid \varphi \in \mathcal{E}(M, \mathcal{S})\}$ is a basis for the topology on M, and by Proposition 8.14(b), $p_{M,\mathcal{S}}^{-1}(\text{dom } \varphi)$ is open in $T(M, \mathcal{S})$ for all $\varphi \in \mathcal{E}(M, \mathcal{S})$. □

Proposition 8.16. *Let $(M, \mathcal{S})$ be a smooth m-manifold. Then for each $\varphi \in \mathcal{E}(M, \mathcal{S})$:*

(a) *The diagram*

$$\begin{array}{ccc} p_{M,\mathcal{S}}^{-1}(\text{dom } \varphi) & \xrightarrow{\;\;\widetilde{\varphi}\;\;} & (\text{dom } \varphi) \times \mathbf{R}^m \\ & {\scriptstyle p_{M,\mathcal{S}}} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & \text{dom } \varphi & \end{array}$$

commutes.

(b) *For each $x \in \text{dom } \varphi$, the restriction $\widetilde{\varphi} : T_x(M, \mathcal{S}) \to \{x\} \times \mathbf{R}^m$ is an $\mathbf{R}$-isomorphism.*

Proof. Both parts follow from the fact that $\widetilde{\varphi}$ is obtained by taking the disjoint union over $x \in \text{dom } \varphi$ of the compositions

$$T_x(M, \mathcal{S}) \xrightarrow{(T_x i_\varphi)^{-1}} T_x(\text{dom } \varphi, \mathcal{S}|\text{dom } \varphi) \xrightarrow{T_x \varphi} T_{\varphi(x)}(\text{codom } \varphi, \mathcal{S}t_\varphi) \xrightarrow{\widetilde{\theta}^{-1}_{\varphi(x)}} \{x\} \times \mathbf{R}^m$$

and each map in this last composition is an $\mathbf{R}$-isomorphism. □

As a consequence of Proposition 8.14, Corollary 8.15, and Proposition 8.16, $(\mathbf{R}^m, T(M, \mathcal{S}), M, p_{M,\mathcal{S}})$ is an m-plane bundle with linear atlas $\{\widetilde{\varphi} \mid \varphi \in \mathcal{E}(M, \mathcal{S}\}$. To prove that this vector bundle is the underlying vector bundle of a smooth vector bundle, it remains only to prove that any two charts in $\{\widetilde{\varphi} \mid \varphi \in \mathcal{E}(M, \mathcal{S}\}$ are linearly $\mathcal{S}$-related.

Lemma 8.17. *Let $(M, \mathcal{S})$ be a smooth m-manifold and suppose $\varphi, \psi \in \mathcal{E}(M, \mathcal{S})$. Let $g_{\varphi,\psi} : (\text{dom } \varphi \cap \text{dom } \psi) \times \mathbf{R}^m \to (\text{dom } \varphi \cap \text{dom } \psi) \times \mathbf{R}^m$ be given by $g_{\varphi,\psi}(x, y) = \big(x, D(\psi\varphi^{-1})(\varphi(x))(y)\big)$. Then the diagram*

$$\begin{array}{ccc} & \nearrow^{\widetilde{\varphi}} & (\text{dom } \varphi \cap \text{dom } \psi) \times \mathbf{R}^m \\ p_{M,\mathcal{S}}^{-1}(\text{dom } \varphi \cap \text{dom } \psi) & & \big\downarrow {\scriptstyle g_{\varphi,\psi}} \\ & \searrow_{\widetilde{\psi}} & (\text{dom } \varphi \cap \text{dom } \psi) \times \mathbf{R}^m \end{array}$$

commutes.

Proof. If U is open in M and $x \in U$, $T_x(M,\mathcal{S})$ is canonically identified with $T_x(U,\mathcal{S}\,|\,U)$ via the Local Property (Proposition 5.11) and with this identification, the maps $T_x i_\varphi$ and $T_x i_\psi$ appearing in the definitions of $\widetilde{\varphi}$ and $\widetilde{\psi}$ become the identity maps. It follows that $\widetilde{\varphi}$ and $\widetilde{\psi}$ are given as follows: For $u \in p_{M,\mathcal{S}}^{-1}(x)$, $\widetilde{\varphi}(u) = (x,v)$ where $v \in \mathbf{R}^m$ is the unique element such that $\theta_{\varphi(x)}(v) = T_x\varphi(u)$, and $\widetilde{\psi}(u) = (x,w)$ where $w \in \mathbf{R}^m$ is the unique element such that $\theta_{\psi(x)}(w) = T_x\psi(u)$. Thus it remains only to show that for x,u,v as above,

$$\theta_{\psi(x)}\Big(D(\psi\varphi^{-1})((\varphi(x))(v)\Big) = (T_x\psi)(u).$$

By Proposition 5.12 we have the commutative diagram

$$\begin{array}{ccc}
\mathbf{R}^m & \xrightarrow{\quad D(\psi\varphi^{-1})(\varphi(x)) \quad} & \mathbf{R}^m \\
\Big\downarrow{\scriptstyle \theta_{\varphi(x)}} & & \Big\downarrow{\scriptstyle \theta_{\psi(x)}} \\
T_{\varphi(x)}(\text{codom }\varphi \cap \text{codom }\psi) & \xrightarrow[T_{\varphi(x)}(\psi\varphi^{-1})]{} & T_{\psi(x)}(\text{codom }\varphi \cap \text{codom }\psi)\,.
\end{array}$$

Thus

$$\begin{aligned}
\theta_{\psi(x)}\Big(D(\psi\varphi^{-1})((\varphi(x))(v)\Big) &= T_{\varphi(x)}(\psi\varphi^{-1})\big(\theta_{\varphi(x)}(v)\big) \\
&= T_{\varphi(x)}(\psi\varphi^{-1})\big((T_x\varphi)(u)\big) = (T_x\psi)(u). \quad \square
\end{aligned}$$

Corollary 8.18. *Let $(M,\mathcal{S})$ be a smooth m-manifold. Then for any charts $\varphi,\psi \in \mathcal{E}(M,\mathcal{S})$, $\widetilde{\varphi}$ and $\widetilde{\psi}$ are linearly $\mathcal{S}$-related.*

Proof. Since φ and ψ are smoothly related, the overlap map $\psi\varphi^{-1}$ is C^∞. Thus all partial derivatives of the coordinate functions of $\psi\varphi^{-1}$ are C^∞, and so the $g_{\varphi,\psi}$ of Lemma 8.17 is C^∞, i.e. $\widetilde{\psi}\widetilde{\varphi}^{-1}$ is C^∞. $\square$

As a consequence of Proposition 8.14, Corollary 8.15, Proposition 8.16, and Corollary 8.18, $\{\widetilde{\varphi} \mid \varphi \in \mathcal{E}(M,\mathcal{S})\}$ is a linear $\mathcal{S}$-smooth atlas for the vector bundle $(\mathbf{R}^m, T(M,\mathcal{S}), M, p_{M,\mathcal{S}})$. Write $\widetilde{\mathcal{S}}$ for its linear $\mathcal{S}$-smooth equivalence class. Summarizing,

Theorem 8.19. *Let $(M,\mathcal{S})$ be a smooth m-manifold. Then*

$$\big(\mathbf{R}^m, T(M,\mathcal{S}), (M,\mathcal{S}), p_{M,\mathcal{S}}, \widetilde{\mathcal{S}}\big)$$

is a smooth m-plane bundle. $\square$

Definition 8.20. Let $(M,\mathcal{S})$ be a smooth m-manifold. The smooth m-plane bundle of Theorem 8.19 is called the *tangent bundle of* $(M,\mathcal{S})$, and denoted $\tau_{M,\mathcal{S}}$.

It follows from Theorem 4.34 that that there is a unique smooth structure $Tan_{M,\mathcal{S}}$ on $T(M,\mathcal{S})$ such that for each $\varphi \in \mathcal{E}(M,\mathcal{S})$,

$$\widetilde{\varphi} : p_{M,\mathcal{S}}^{-1}(\text{dom }\varphi) \to (\text{dom }\varphi) \times \mathbf{R}^m$$

is a diffeomorphism with respect to $Tan_{M,\mathcal{S}}$ and $\mathcal{S} \times \mathcal{S}t$ where $\mathcal{S}t$ denotes the standard smooth structure on $\mathbf{R}^m$. Thus $\big(T(M,\mathcal{S}), Tan_{M,\mathcal{S}}\big)$ is a smooth $2m$-manifold.

Suppose $(M,\mathcal{S})$ and $(N,\mathcal{T})$ are smooth m- and n-manifolds, respectively, and that $f : M \to N$ is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$. Let $Tf : T(M,\mathcal{S} \to T(N,\mathcal{T})$ be the function obtained by assembling the tangent maps $T_x f$ for $x \in M$, i.e. Tf is the unique function such that for all $x \in M$, the diagram

$$\begin{array}{ccc} T_x(M,\mathcal{S}) & \xrightarrow{T_x f} & T_{f(x)}(N,\mathcal{T}) \\ \downarrow & & \downarrow \\ T(M,\mathcal{S}) & \xrightarrow[Tf]{} & T(N,\mathcal{T}) \end{array}$$

commutes, where the vertical maps are the inclusion maps. Our next task is to show that Tf is smooth with respect to $Tan_{M,\mathcal{S}}$ and $Tan_{N,\mathcal{T}}$.

Lemma 8.21. *Let $(M,\mathcal{S})$ and $(N,\mathcal{T})$ be smooth m- and n-manifolds, respectively, and $f : M \to N$ a smooth map with respect to $\mathcal{S}$ and $\mathcal{T}$. Suppose $\varphi \in \mathcal{E}(M,\mathcal{S})$ and $\psi \in \mathcal{E}(N,\mathcal{T})$ are such that $f(\text{dom }\varphi) \subset \text{dom }\psi$. Then the diagram*

$$\begin{array}{ccccc} p_{M,\mathcal{S}}^{-1}(\text{dom }\varphi) & \xrightarrow{\widetilde{\varphi}} & (\text{dom }\varphi) \times \mathbf{R}^m & \xrightarrow{\varphi \times 1_m} & (\text{codom }\varphi) \times \mathbf{R}^m \\ {\scriptstyle Tf}\downarrow & & & & \downarrow{\scriptstyle l_{f,\varphi,\psi}} \\ p_{N,\mathcal{T}}^{-1}(\text{dom }\psi) & \xrightarrow[\widetilde{\psi}]{} & (\text{dom }\psi) \times \mathbf{R}^n & \xrightarrow[\psi \times 1_n]{} & (\text{codom }\psi) \times \mathbf{R}^n \end{array}$$

commutes, where 1_m and 1_n are the identity maps on $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, and $l_{f,\varphi,\psi}(y,z) = \big(\psi f \varphi^{-1}(y), D(\psi f \varphi^{-1})(y)(z)\big)$.

Proof. Let $x \in \text{dom }\varphi$ and $v \in T_x(M,\mathcal{S})$. Then

$$\begin{aligned} (\psi \times 1_n)\widetilde{\psi} Tf(v) &= (\psi \times 1_n)\widetilde{\psi} T_x f(v) = (\psi \times 1_n)\big(f(x), \theta_{\psi f(x)}^{-1} T_{f(x)}\psi T_x f(v)\big) \\ &= \big(\psi f(x), \theta_{\psi f(x)}^{-1} T_x(\psi f)(v)\big) = \big(\psi f(x), D(\psi f)(x)\theta_x^{-1}(v)\big) \\ &\qquad \text{(by Proposition 5.12)}, \end{aligned}$$

while

$$\begin{aligned} l_{f,\varphi,\psi}(\varphi \times 1_m)\widetilde{\varphi}(v) &= l_{f,\varphi,\psi}(\varphi \times 1_m)\big(x, \theta_{\varphi(x)}^{-1} T_x\varphi(v)\big) \\ &= l_{f,\varphi,\psi}(\varphi \times 1_m)\big(x, D\varphi(x)\theta_x^{-1}(v)\big) \\ &\qquad \text{(by Proposition 5.12)} \\ &= l_{f,\varphi,\psi}\big(\varphi(x), D\varphi(x)\theta_x^{-1}(v)\big) \\ &= \Big(\psi f \varphi^{-1}\varphi(x), D(\psi f \varphi^{-1})\big(\varphi(x)\big) D\varphi(x)\theta_x^{-1}(v)\Big) \\ &= \big(\psi f(x), D(\psi f)(x)\theta_x^{-1}(v)\big) \\ &\qquad \text{(by Theorem 3.20)}, \end{aligned}$$

completing the proof. $\square$

Theorem 8.22. *Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth m- and n-manifolds, respectively, and suppose that $f : M \to N$ is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$. Then $Tf : T(M, \mathcal{S}) \to T(N, \mathcal{T})$ is smooth with respect to $Tan_{M,\mathcal{S}}$ and $Tan_{N,\mathcal{T}}$.*

Proof. By Proposition 4.19(b) it suffices to show that whenever $\varphi \in \mathcal{E}(M, \mathcal{S})$ and $\psi \in \mathcal{E}(N, \mathcal{T})$ are such that $f(\text{dom}\ \varphi) \subset \text{dom}\ \psi$, then the restriction $Tf : p_{M,\mathcal{S}}^{-1}(\text{dom}\ \varphi) \to p_{N,\mathcal{T}}^{-1}(\text{dom}\ \psi)$ is smooth. Since the horizontal maps in the diagram of Lemma 8.21 are diffeomorphisms, it suffices, by Lemma 8.21 to check that $l_{f,\varphi,\psi}$ is C^∞. By smoothness of f, it follows that $\psi f \varphi^{-1}$ is C^∞. Since partial derivatives of C^∞ functions are C^∞, $D(\psi f \varphi^{-1})$ is C^∞. It follows that $l_{f,\varphi,\psi}$ is C^∞. □

Summarizing Definition 8.20 through 8.22 we have:

Theorem 8.23. *Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth m- and n-manifolds, respectively, and suppose that $f : M \to N$ is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$. Then $\tau f : \tau_{M,\mathcal{S}} \to \tau_{N,\mathcal{T}}$ given by $(\tau f)_E = Tf : T(M, \mathcal{S}) \to T(N, \mathcal{T})$, $(\tau f)_B = f : M \to N$ is a smooth vector bundle homomorphism.* □

As a consequence of Proposition 5.9 we have:

Proposition 8.24. *Let $(M, \mathcal{S})$, $(N, \mathcal{T})$, and $(Q, \mathcal{U})$ be smooth manifolds. Suppose $f : M \to N$, $g : N \to Q$ are smooth maps with respect to $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{T}$ and $\mathcal{U}$, respectively. Then:*

(a) $\tau 1_M = 1_{\tau_{M,\mathcal{S}}}$.

(b) $\tau(gf) = (\tau g)(\tau f)$. □

If f is a smooth map, then Theorem 8.22 yields that Tf is smooth, and so we can consider $T(Tf)$. We proceed to obtain some information about latter.

Lemma 8.25. *Suppose U is open in $\mathbf{R}^m$, and V open in $\mathbf{R}^n$. Let St_U and St_V denote the standard smooth structures on U and V, respectively. Suppose $f : U \to V$ is smooth with respect to St_U and St_V. Then:*

(a) *The diagram*

$$\begin{array}{ccc} U \times \mathbf{R}^m & \xrightarrow{\widehat{T}f} & V \times \mathbf{R}^n \\ \uparrow{\scriptstyle \widetilde{1_U}} & & \uparrow{\scriptstyle \widetilde{1_V}} \\ T(U, St_U) & \xrightarrow[Tf]{} & T(V, St_V) \end{array}$$

commutes, where $\widehat{T}f(x, y) = \big(f(x), Df(x)(y)\big)$, and $\widetilde{1_U}$, $\widetilde{1_V}$ are the diffeomorphisms arising from $1_U \in \mathcal{E}(U, St_U)$, $1_V \in \mathcal{E}(V, St_V)$, respectively, by the construction preceding Proposition 8.14.

(b) *For all $(x, y) \in U \times \mathbf{R}^m$,*

$$D\widehat{T}f(x, y) = \begin{pmatrix} Df(x) & 0 \\ * & Df(x) \end{pmatrix}$$

for some $n \times m$ matrix $*$ depending on x and y.

Proof. Part (a) follows immediately from Proposition 5.12 and the construction preceding Proposition 8.14.

Writing $f_1, \ldots, f_n$ for the coordinate functions of f,

$$\widehat{T}f(x,y) = \Big(f_1(x), \ldots, f_n(x), \textstyle\sum_i D_i f_1(x) y_i, \ldots, \sum_i D_i f_n(x) y_i\Big)$$

as a function of the $2m$ variables $x_1, \ldots, x_m, y_1, \ldots, y_m$. Part (b) now follows from Definition 3.2. $\square$

Theorem 8.26. *Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds and suppose $f : M \to N$ is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$. Suppose $x \in M$ is such that $T_x f : T_x(M, \mathcal{S}) \to T_{f(x)}(N, \mathcal{T})$ has rank r. Then for all $y \in T_x(M, \mathcal{S})$, the rank of $T_y(Tf) : T_y(T(M, \mathcal{S}), Tan_{M,\mathcal{S}}) \to T_{Tf(y)}(T(N, \mathcal{T}), Tan_{N,\mathcal{T}})$ is $\geq 2r$.*

Proof. We first treat the case when M and N are open in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, with their standard smooth structures. By Proposition 5.12, the diagram

$$\begin{array}{ccc} \mathbf{R}^m & \xrightarrow{Df(x)} & \mathbf{R}^n \\ {\scriptstyle \theta_x}\downarrow & & \downarrow{\scriptstyle \theta_{f(x)}} \\ T_x(M, \mathcal{S}) & \xrightarrow[T_x f]{} & T_{f(x)}(N, \mathcal{T}) \end{array}$$

commutes. Since, by Theorem 5.13, the vertical maps are $\mathbf{R}$-isomorphisms, the ranks of $T_x f$ and $Df(x)$ are the same, and so $Df(x)$ has rank r.

We can write $\widetilde{1_M}(y) = (x, z)$ for some $z \in \mathbf{R}^m$. Applying Lemma 8.25(a) with $(M, \mathcal{S})$ and $(N, \mathcal{T})$ replacing (U, St_U) and (V, St_V), respectively, it follows from the fact that $\widetilde{1_M}$ and $\widetilde{1_N}$ are diffeomorphisms, that rank $T_y(Tf) =$ rank $T_{(x,z)}(\widehat{T}f)$. By Lemma 8.25(b), the latter has rank ≥ 2 rank $Df(x)$, completing the proof of the special case when M and N are open subsets of finite-dimensional vector spaces.

For the general case, choose $\varphi \in \mathcal{E}(M, \mathcal{S})$ and $\psi \in \mathcal{E}(N, \mathcal{T})$ such that $x \in$ dom φ and $f(\text{dom } \varphi) \subset$ dom ψ. Write $\mathcal{S}_\varphi$ for $\mathcal{S} \,|\,$dom φ and $\mathcal{T}_\psi$ for $\mathcal{T} \,|\,$dom ψ. Note that $T(\text{dom } \varphi, \mathcal{S}_\varphi) = p_{M,\mathcal{S}}^{-1}(\text{dom } \varphi)$ and hence, by continuity of $p_{M,\mathcal{S}}$, $T(\text{dom } \varphi, \mathcal{S}_\varphi)$ is open in $T(M, \mathcal{S})$. Similarly, $T(\text{dom } \psi, \mathcal{T}_\psi)$ is open in $T(N, \mathcal{T})$. Write $f_1 :$ dom $\varphi \to$ dom ψ for the restriction of f. By the Local Property, $T_x f = T_x f_1$ where we identify $T_x(\text{dom } \varphi, \mathcal{S}_\varphi)$ with $T_x(M, \mathcal{S})$ and $T_{f(x)}(\text{dom } \psi, \mathcal{T}_\psi)$ with $T_{f(x)}(N, \mathcal{T})$ via the maps induced by the inclusions $i_\varphi :$ dom $\varphi \to M$ and $i_\psi :$ dom $\psi \to N$. Thus rank $T_x f =$ rank $T_x f_1$.

Since $T(\text{dom } \varphi, \mathcal{S}_\varphi)$ is open in $T(M, \mathcal{S})$ and $T(\text{dom } \psi, \mathcal{T}_\psi)$ is open in $T(N, \mathcal{T})$, $T_y(Tf) = T_y(Tf_1)$. Thus it remains only to show that rank $T_y(Tf_1) \geq 2$ rank $T_x f_1$.

Since $\varphi :$ dom $\varphi \to$ codom φ and $\psi :$ dom $\psi \to$ codom ψ are bijective, there exists a unique function $f_2 :$ codom $\varphi \to$ codom ϕ such that the diagram

$$\begin{array}{ccc} \text{dom } \varphi & \xrightarrow{f_1} & \text{dom } \psi \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \text{codom } \varphi & \xrightarrow[f_2]{} & \text{codom } \psi \end{array}$$

commutes. Write $\mathcal{S}t_\varphi$ and $\mathcal{S}t_\psi$ for the standard smooth structures on codom φ and codom ψ, respectively. Since φ and ψ are diffeomorphisms and f_1 is smooth, it follows that f_2 is smooth, and that the diagrams

$$\begin{array}{ccc} T_x(\text{dom } \varphi, \mathcal{S}_\varphi) & \xrightarrow{T_x f_1} & T_{f_1(x)}(\text{dom } \psi, \mathcal{T}_\psi) \\ {\scriptstyle T_x\varphi}\downarrow & & \downarrow{\scriptstyle T_{f_1(x)}\psi} \\ T_{\varphi(x)}(\text{codom } \varphi, \mathcal{S}t_\varphi) & \xrightarrow[T_{\varphi(x)} f_2]{} & T_{f_2\varphi(x)}(\text{codom } \psi, \mathcal{S}t_\psi) \end{array} ,$$

$$\begin{array}{ccc} T(\text{dom } \varphi, \mathcal{S}_\varphi) & \xrightarrow{T f_1} & T(\text{dom } \psi, \mathcal{T}_\psi) \\ {\scriptstyle T\varphi}\downarrow & & \downarrow{\scriptstyle T\psi} \\ T(\text{codom } \varphi, \mathcal{S}t_\varphi) & \xrightarrow[T f_2]{} & T(\text{codom } \psi, \mathcal{S}t_\psi) \end{array}$$

Since the vertical maps in the first diagram are $\mathbf{R}$-isomorphisms, it follows that rank $T_x f_1 =$ rank $T_{\varphi(x)} f_2$.

Applying T to the second diagram, we obtain

$$T_{Tf_1(y)}(T\psi) \circ T_y(Tf_1) = T_{T\varphi(y)}(Tf_2) \circ T_y(T\varphi).$$

Since φ and ψ are diffeomorphisms, $T_{Tf_1(y)}(T\psi)$ and $T_y(T\varphi)$ are $\mathbf{R}$-isomorphisms, and so rank $T_y(Tf_1) =$ rank $T_{T\varphi(y)}(Tf_2)$.

By the special case above, rank $T_{T\varphi(y)}(Tf_2) \geq 2$ rank $(T_{\varphi(x)} f_2)$. The assertion follows. $\square$

Corollary 8.27. *Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds, and suppose $f : M \to N$ is an immersion with respect to $\mathcal{S}$ and $\mathcal{T}$. Then $Tf : T(M, \mathcal{S}) \to T(N, \mathcal{T})$ is an immersion with respect to $Tan_{M,\mathcal{S}}$ and $Tan_{N,\mathcal{T}}$.*

Proof. Say M is a k-manifold. Then for all $x \in M$, rank $T_x f = k$. It follows from Theorem 8.26 that for all $y \in T(M, \mathcal{S})$, rank $T_y(Tf) \geq 2k$. Since $\dim_{\mathbf{R}} T_y\big(T(M, \mathcal{S})\big) = 2k$ for all $y \in T(M, \mathcal{S})$, we must have rank$\,, T_y(Tf) = 2k$. Thus, since $T(M, k)$ is a $2k$-manifold, Tf is an immersion. $\square$

We next consider the interaction between smooth submanifolds and the tangent bundle.

Lemma 8.28. *Let $(M, \mathcal{S})$ be a smooth m-manifold and $\varphi \in \mathcal{E}(M, \mathcal{S})$. Then:*

(a) *The composition*

$$p_{M,\mathcal{S}}^{-1}(\text{dom } \varphi) \xrightarrow{\tilde{\varphi}} (\text{dom } \varphi) \times \mathbf{R}^m \xrightarrow{\varphi \times 1_m} (\text{codom } \varphi) \times \mathbf{R}^m$$

is a $Tan_{M,\mathcal{S}}$-admissible chart, where 1_m denotes the identity map on $\mathbf{R}^m$.

(b) *Let X be a k-dimensional smooth submanifold of $(M, \mathcal{S})$, and write Y for the image of $Ti : T(X, \mathcal{S} \,|\, X) \to T(M, \mathcal{S})$ where $i : X \to M$ denotes the inclusion map. Suppose $\varphi \in \mathcal{E}(M, \mathcal{S})$ is an $\mathcal{S}$-admissible k-chart for (M, X). Then the $Tan_{M,\mathcal{S}}$-admissible chart constructed in part (a) is a $Tan_{M,\mathcal{S}}$-admissible $2k$-chart for $\big(T(M, \mathcal{S}), Y\big)$.*

Proof. Since $p_{M,\mathcal{S}}^{-1}(\text{dom }\varphi)$ is open in $T(M,\mathcal{S})$, $(\text{codom }\varphi)\times\mathbf{R}^m$ is open in the real $2m$-dimensional vector space $\mathbf{R}^m\times\mathbf{R}^m$, and both $\widetilde{\varphi}$ and φ are diffeomorphisms, part (a) is immediate.

Regard $\mathbf{R}^k$ as a subspace of $\mathbf{R}^m$ in the standard way. We will restrict to $\mathcal{S}$-admissible k-charts φ for (M,X) satisfying $\varphi(X\cap\text{dom }\varphi)=\mathbf{R}^k\cap\text{codom }\varphi$. For such a φ, write $\varphi_X : X\cap\text{dom }\varphi\to\mathbf{R}^k\cap\text{codom }\varphi$ for the restriction of φ. Recall from §6 that $\{\varphi_X \mid \varphi\in\mathcal{E}(M,\mathcal{S})\}$ is a smooth atlas for X which represents the smooth structure $\mathcal{S}\,|\,X$.

Let $i : X\to M$ denote the inclusion map. Applying Lemma 8.21 to i,φ_X, and φ, we get the commutative diagram

$$\begin{array}{ccccc}
p_{X,\mathcal{S}|X}^{-1}(\text{dom }\varphi_X) & \xrightarrow{\widetilde{\varphi}_X} & (\text{dom }\varphi_X)\times\mathbf{R}^k & \xrightarrow{\varphi_X\times 1_k} & (\text{codom }\varphi_X)\times\mathbf{R}^k \\
\downarrow{\scriptstyle Ti} & & & & \downarrow{\scriptstyle l_{i,\varphi_X,\varphi}} \\
p_{M,\mathcal{S}}^{-1}(\text{dom }\varphi) & \xrightarrow[\widetilde{\varphi}]{} & (\text{dom }\varphi)\times\mathbf{R}^m & \xrightarrow[\varphi\times 1_m]{} & (\text{codom }\varphi)\times\mathbf{R}^m .
\end{array}$$

Write ψ for the $Tan_{M,\mathcal{S}}$-admissible chart arising from part (a). Then Ti maps $p_{X,\mathcal{S}|X}^{-1}(\text{dom }\varphi_X)$ onto $Y\cap\text{dom }\psi$. $\mathbf{R}^k\times\mathbf{R}^k$ is a $2k$-dimensional $\mathbf{R}$-linear subspace of $\mathbf{R}^m\times\mathbf{R}^m$, and it is easily checked that $l_{i,\varphi_X,\varphi}$ is the restriction of the inclusion map $\mathbf{R}^k\times\mathbf{R}^k\to\mathbf{R}^m\times\mathbf{R}^m$. Since $\text{codom }\varphi_X=\mathbf{R}^m\cap\text{codom }\varphi$, it follows from commutativity of the above diagram that $\psi(Y\cap\text{dom }\psi)=(\mathbf{R}^k\times\mathbf{R}^k)\cap\text{codom }\psi$. □

Theorem 8.29. *Suppose X is a k-dimensional smooth submanifold of a smooth m-manifold $(M,\mathcal{S})$. Let $i : X\to M$ denote the inclusion map. Then $Ti : T(X,\mathcal{S}|X)\to T(M,\mathcal{S})$ maps $T(X,\mathcal{S}|X)$ diffeomorphically onto a $2k$-dimensional smooth submanifold of $T(M,\mathcal{S})$ with respect to the smooth structures $Tan_{X,\mathcal{S}\,|\,X}$ and $Tan_{M,\mathcal{S}}$.*

Proof. Let Y denote the image of Ti. By Lemma 8.28(b), Y is a $2k$-dimensional smooth submanifold of $\big((T(M,\mathcal{S}),Tan_{M,\mathcal{S}}\big)$. Since i is an immersion by Theorem 6.11, it follows from Corollary 8.27 that Ti is an immersion. Since Ti maps $T(X,\mathcal{S}|X)$ bijectively onto the smooth manifold Y, the result is now immediate from Proposition 6.17. □

We return now to some general notions concerning vector bundles.

Definition 8.30. Let ξ and η be vector bundles with the same base space B. A vector bundle homomorphism $f:\xi\to\eta$ is called a *vector bundle isomorphism from ξ to η* if $f_B=1_B$ and f_E is a homeomorphism. If a vector bundle isomorphism from ξ to η exists, we say *ξ is isomorphic to η*.

If ξ and η are smooth vector bundles, a *smooth vector bundle isomorphism from ξ to η* is a vector bundle isomorphism $f:\xi\to\eta$ such that f_E is a diffeomorphism. If a smooth vector bundle isomorphism from ξ to η exists, we say *ξ is smoothly isomorphic to η*.

The proof of the following is easy and left as an exercise.

Proposition 8.31. (a) *Let ξ be a (smooth) vector bundle. Then 1_ξ is a (smooth) vector bundle isomorphism.*

(b) *Let $f : \xi \to \eta$ be a (smooth) vector bundle isomorphism. Then $f^{-1} : \eta \to \xi$ given by $(f^{-1})_B = 1_B$ (where B is the base space of both η and ξ) and $(f^{-1})_E = (f_E)^{-1}$ is a (smooth) vector bundle isomorphism.*

(c) *The composition of two (smooth) vector bundle isomorphisms is a (smooth) vector bundle isomorphism.* □

Corollary 8.32. *Let B be a topological space (smooth manifold). Then the relation "is (smoothly) isomorphic to" is an equivalence relation on the class of all (smooth) vector bundles with nase space B.* □

Definition 8.33. Let $\xi = (V, E, (M, \mathcal{S}_M), p, \mathcal{S})$ be a smooth vector bundle. A *smooth section σ of ξ* is a smooth map $\sigma : M \to E$ such that $p\sigma = 1_M$. Write $\Gamma(\xi)$ for the set of all smooth sections of ξ.

Proposition 8.34. *Let $\xi = (F, E, (M, \mathcal{S}_M), p, \mathcal{S})$ be a smooth vector bundle. Let $\sigma_1, \sigma_2 \in \Gamma(\xi)$ and $k \in \mathbf{R}$. Define $\sigma_1 + \sigma_2 : M \to E$ and $k\sigma_1 : M \to E$ by $(\sigma_1 + \sigma_2)(x) = \sigma_1(x) + \sigma_2(x)$, $(k\sigma_1)(x) = k\sigma_1(x)$ for all $x \in M$, where the $+$ and scalar multiplication on the right-hand sides are those in the fiber $p^{-1}(x)$. Then $\sigma_1 + \sigma_2$ and $k\sigma_1$ are both in $\Gamma(\xi)$.*

Proof. The only question is the smoothness of $\sigma_1 + \sigma_2$ and $k\sigma_1$. It suffices to check this locally. Since the U_φ, as φ ranges over the linear $\mathcal{S}$-admissible charts for (F, E, M, p) form an open cover of M, and that each such φ is a diffeomorphism, it suffices to check that for each such φ, the compositions

$$U_\varphi \xrightarrow{\sigma_1+\sigma_2} p^{-1}(U_\varphi) \xrightarrow{\varphi} U_\varphi \times F \tag{1}$$

and

$$U_\varphi \xrightarrow{k\sigma_1} p^{-1}(U_\varphi) \xrightarrow{\varphi} U_\varphi \times F \tag{2}$$

are both smooth.

Using the fact that φ is $\mathbf{R}$-linear on fibers, one checks that the composition in (1) equals the composition

$$U_\varphi \xrightarrow{\Delta} U_\varphi \times U_\varphi \xrightarrow{\sigma_1 \times \sigma_2} p^{-1}(U_\varphi) \times p^{-1}(U_\varphi) \xrightarrow{\varphi \times \varphi} (U_\varphi \times F) \times (U_\varphi \times F)$$

$$\xrightarrow{\pi} U_\varphi \times F \times F \xrightarrow{1_{U_\varphi} \times \text{add}} U_\varphi \times F$$

where Δ is the diagonal map, π is given by $\pi\big((u_1, f_1), (u_2, f_2)\big) = (u_1, f_1, f_2)$, and add $: F \times F \to F$ is given by add$(f_1, f_2) = f_1 + f_2$. Since all the maps in the above composition are smooth, it follows that the composition in (1) is smooth, and hence $\sigma_1 + \sigma_2$ is smooth.

Similarly, the composition in (2) equals the composition

$$U_\varphi \xrightarrow{\sigma_1} p^{-1}(U_\varphi) \xrightarrow{\varphi} U_\varphi \times F \xrightarrow{1_{U_\varphi} \times m_k} U_\varphi \times F$$

where $m_k : F \to F$ is given by $m_k(u) = ku$. All the maps in this composition are smooth, and hence so is $k\sigma_1$. □

Proposition 8.35. *Let ξ be a smooth vector bundle. Then under the operations of Proposition 8.34, $\Gamma(\xi)$ is a real vector space.* □

Proof. Say $\xi = (F, E, (M, \mathcal{S}_M), p, \mathcal{E})$. The only detail that needs checking is that $\Gamma(\xi)$ is non-empty. Define $0 : M \to E$ by $0(x) = 0_x$ for all $x \in M$ where 0_x is the 0-element in the fiber $p^{-1}(x)$. Let φ be any linear $\mathcal{S}$-admissible chart for (F, E, M, p). Consider the composition

$$(1) \qquad U_\varphi \xrightarrow{0} p^{-1}(U_\varphi) \xrightarrow{\varphi} U_\varphi \times F.$$

We have $\pi_1\varphi 0 = 1_{U_\varphi}$ and $\pi_2\varphi 0 =$ constant map with value 0, both of which are smooth. Hence the composition in (1) is smooth, and so $0 \in \Gamma(\xi)$. □

Except for the trivial cases where the base space or fiber is 0-dimensional, $\Gamma(\xi)$ is infinite-dimensional over $\mathbf{R}$. For example, if ξ is the product $\mathbf{R}$-bundle over a smooth manifold M, $\Gamma(\xi)$ can be identified with the real vector space of all smooth real-valued functions defined on M.

Definition 8.36. A smooth vector bundle is *trivial* if it is smoothly isomorphic to a product vector bundle.

A smooth manifold is *parallelizable* if its tangent bundle is trivial.

Proposition 8.37. *Let $\xi = (F, E, (M, \mathcal{S}_M), p, \mathcal{S})$ be a smooth n-plane bundle. The following three conditions are equivalent:*

(i) *ξ is trivial.*

(ii) *ξ admits n smooth sections $\sigma_1, \ldots, \sigma_n$ such that for each $x \in M$, the elements $\sigma_1(x), \ldots, \sigma_n(x)$ are linearly independent in $p^{-1}(x)$.*

(iii) *ξ admits a linear $\mathcal{S}$-admissible atlas with exactly one chart.*

Proof. Suppose (i) holds. We then have a fiber-preserving diffeomorphism $f : M \times F \to E$ such that for each $x \in M$, the restriction $f : \{x\} \times F \to p^{-1}(x)$ is an $\mathbf{R}$-isomorphism. Choose any $\mathbf{R}$-basis $v_1, \ldots, v_n$ for F. For $1 \le i \le n$ let $\tau_i : M \to M \times F$ be given by $\tau_i(x) = (x, v_i)$. Each τ_i is clearly smooth. Define $\sigma_i = f\tau_i : M \to E$ for $1 \le i \le n$. It is easily checked that the σ_i satisfy the condition of (ii).

Suppose (ii) holds. Then for each $x \in M$, $\sigma_1(x), \ldots, \sigma_n(x)$ is an $\mathbf{R}$-basis for the fiber $p^{-1}(x)$. Choose any $\mathbf{R}$-basis $v_1, \ldots, v_n$ for F. Define $\varphi : E \to M \times F$ as follows: For each $x \in M$, the restriction of φ to $p^{-1}(x)$ is the $\mathbf{R}$-isomorphism onto $\{x\} \times F$ which carries $\sigma_i(x)$ to v_i for $1 \le i \le n$. Condition (iii) will follow if we show that φ is a linear $\mathcal{S}$-admissible chart for (F, E, M, p). Clearly, φ is bijective, the diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & M \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & M & \end{array}$$

commutes, and φ is an $\mathbf{R}$-isomorphism on each fiber. Thus, to prove (iii), it remains only to show that for each linear $\mathcal{S}$-admissible chart ψ for (F, E, M, p), the compositions

$$U_\psi \times F \xrightarrow{\psi^{-1}} p^{-1}(U_\psi) \xrightarrow{\phi} U_\psi \times F$$

and

$$U_\psi \times F \xrightarrow{\varphi^{-1}} p^{-1}(U_\psi) \xrightarrow{\psi} U_\psi \times F$$

are smooth. (Since ψ is a diffeomorphism, smoothness, and hence continuity, of φ will be a consequence of the above.) Note that $\pi_1\varphi\psi^{-1} = \pi_1\psi\varphi^{-1} = 1_{U_\psi}$, and so it remains only to check smoothness of $\pi_2\varphi\psi^{-1}$ and $\pi_2\psi\varphi^{-1}$.

For $1 \leq i \leq n$ let $\rho_i : F \to \mathbf{R}$ denote the i^{th} coordinate map with respect to the basis $v_1, \ldots, v_n$, i.e. $\rho_i\left(\sum_{j=1}^n t_j v_j\right) = t_i$. It suffices to check the smoothness of $\rho_i\pi_2\varphi\psi^{-1}$ and $\rho_i\pi_2\psi\varphi^{-1}$ for $1 \leq i \leq n$.

For all $(x, v) \in U_\psi \times F$ and $1 \leq i \leq n$ we have

$$\begin{aligned} \rho_i\pi_2\psi\varphi^{-1}(x,v) &= \rho_1\pi_2\psi\varphi^{-1}\left(x, \sum_{j=1}^n \rho_j(v)v_j\right) = \rho_i\pi_2\psi\left(\sum_{j=1}^n \rho_j(v)\sigma_j(x)\right) \\ &\qquad \text{(by the } \mathbf{R}\text{-linearity of } \varphi^{-1} \text{ on fibers)} \\ &= \rho_i\pi_2\left(\sum_{j=1}^n \rho_j(v)\psi\sigma_j(x)\right) \\ &\qquad \text{(by the } \mathbf{R}\text{-linearity of } \psi \text{ on fibers)} \\ &= \rho_i\pi_2\left(x, \sum_{j=1}^n \rho_j(v)\sum_{k=1}^n \rho_k\big(\pi_2\psi\sigma_j(x)\big)v_k\right) \\ &= \rho_i\left(\sum_{j=1}^n\sum_{k=1}^n \rho_j(v)\rho_k\big(\pi_2\psi\sigma_j(x)\big)v_k\right) \\ &= \sum_{j=1}^n \rho_j(v)\rho_i\big(\pi_2\psi\sigma_j(x)\big) = \sum_{j=1}^n \rho_j\big(\pi_2(x,v)\big)\rho_i\big(\pi_2\psi\sigma_j\pi_1(x,v)\big) \end{aligned}$$

and so

$$\rho_i\pi_2\psi\varphi^{-1} = \sum_{j=1}^n (\rho_j\pi_2)\cdot(\rho_i\pi_2\psi\sigma_j\pi_1).$$

Since the $\rho_j\pi_2$ and $\rho_i\pi_2\psi\sigma_j\pi_1$ are all smooth, the smoothness of $\rho_i\pi_2\psi\varphi^{-1}$ follows from the smoothness of the addition and multiplication maps $\mathbf{R}\times\mathbf{R}\to\mathbf{R}$.

Since $\sigma_1(x), \ldots, \sigma_n(x)$ is an $\mathbf{R}$-basis for $p^{-1}(x)$ for each $x \in U_\psi$, we can write

$$\psi^{-1}(x, v_i) = \sum_{j=1}^{n} a_{ji}(x)\sigma_j(x)$$

for $1 \leq i \leq n$ for unique functions $a_{ji} : U_\psi \to \mathbf{R}$. We check next that the a_{ji} are smooth.

Since ψ^{-1} is an $\mathbf{R}$-isomorphism on fibers, it follows that for each $x \in U_\psi$, $\psi^{-1}(x, v_1), \ldots, \psi^{-1}(x, v_n)$ is an $\mathbf{R}$-basis for $p^{-1}(x)$. Thus there exist functions $b_{ji} : U_\psi \to \mathbf{R}$ such that

$$\sigma_i(x) = \sum_{j=1}^{n} b_{ji}(x)\psi^{-1}(x, v_j),$$

and

$$\begin{pmatrix} a_{11}(x) & \ldots & a_{1n}(x) \\ \vdots & \ddots & \vdots \\ a_{n1}(x) & \ldots & a_{nn}(x) \end{pmatrix} = \begin{pmatrix} b_{11}(x) & \ldots & b_{1n}(x) \\ \vdots & \ddots & \vdots \\ b_{n1}(x) & \ldots & b_{nn}(x) \end{pmatrix}^{-1}.$$

Thus, since the $a_{ji}(x)$ are rational functions of the $b_{kl}(x)$ with non-vanishing denominators, it suffices to show that the b_{ji} are all smooth.

We have

$$\rho_j\pi_2\psi\sigma_i(x) = \rho_j\pi_2\psi\Big(\sum_{k=1}^{n} b_{ki}(x)\psi^{-1}(x, v_k)\Big) = \rho_j\pi_2\psi\psi^{-1}\Big(x, \sum_{k=1}^{n} b_{ki}(x)v_k\Big)$$

(since ψ^{-1} is $\mathbf{R}$-linear on fibers)

$$= \rho_j\Big(\sum_{k=1}^{n} b_{ki}(x)v_k\Big) = b_{ji}(x)$$

and so $b_{ji} = \rho_j\pi_2\psi\sigma_i$ which is smooth, establishing the smoothness of the a_{ji}.

We have, for each $(x, v) \in U_\psi \times F$ and $1 \leq i \leq n$,

$$\rho_i\pi_2\varphi\psi^{-1}(x, v) = \rho_i\pi_2\varphi\psi^{-1}\Big(x, \sum_{j=1}^{n} \rho_j(v)v_j\Big) = \rho_i\pi_2\varphi\Big(\sum_{j=1}^{n} \rho_j(v)\psi^{-1}(x, v_j)\Big)$$

(since ψ^{-1} is $\mathbf{R}$-linear on fibers)

$$= \rho_i\pi_2\varphi\Big(\sum_{j=1}^{n} \rho_j(v)\sum_{k=1}^{n} a_{kj}(x)\sigma_k(x)\Big)$$

$$= \rho_i\pi_2\Big(x, \sum_{j=1}^{n}\sum_{k=1}^{n} \rho_j(v)a_{kj}(x)v_k\Big) = \rho_i\Big(\sum_{j=1}^{n}\sum_{k=1}^{n} \rho_j(v)a_{kj}(x)v_k\Big)$$

$$= \sum_{j=1}^{n} \rho_j(v)a_{ij}(x) = \sum_{j=1}^{n} \rho_j\pi_2(x, v)a_{ij}\pi_1(x, v)$$

and so

$$\rho_i \pi_2 \varphi \psi^{-1} = \sum_{j=1}^{n} (\rho_j \pi_2) \cdot (a_{ij} \pi_1),$$

which is smooth, completing the proof that condition (ii) implies condition (iii).

Suppose condition (iii) holds. Then there exists a linear $\mathcal{S}$-admissible chart $\varphi : E \to M \times F$. It is immediate that φ is the map on total spaces for a smooth vector bundle isomorphism from ξ to the product bundle over M with fiber F. Thus condition (i) holds. □

The proof of the following proposition is easy.

Proposition 8.38. *Let $f : \xi \to \eta$ be a smooth vector bundle isomorphism. Then:*

(a) *If σ is a smooth section of ξ, then $f_E \sigma$ is a smooth section of η.*

(b) *The function $\Gamma(\xi) \to \Gamma(\eta)$ which sends σ to $f_E \sigma$ is a bijection.*

(c) *If $\sigma_1, \ldots, \sigma_n \in \Gamma(\xi)$, then for each x in the base space of ξ, $\sigma_1(x), \ldots, \sigma_n(x)$ are linearly independent if and only if $f_E \sigma_1(x), \ldots, f_E \sigma_n(x)$ are linearly independent.* □

Definition 8.39. Let $(M, \mathcal{S})$ be a smooth manifold. A *smooth vector field on* $(M, \mathcal{S})$ is a smooth section of the tangent bundle $\tau_{M,\mathcal{S}}$.

Thus by Proposition 8.37, a smooth m-manifold $(M, \mathcal{S})$ is parallelizable if and only if $(M, \mathcal{S})$ admits m smooth vector fields which are linearly independent at each point of M. Note that each open subset of a real finite-dimensional vector space is parallelizable since, by Example 4.5 and Definition 8.20, its tangent bundle admits an admissible atlas with exactly one chart.

It can be shown, using algebraic topology, that if $n \geq 2$ is even, every smooth vector field on S^n must be 0 at some point of S^n. If n is odd, S^n admits a smooth vector field which is nowhere 0 (see Exercises for §8). Using some rather heavy machinery from algebraic topology, it has been proved that for $n \geq 1$, S^n is parallelizable if and only if $n = 1, 3,$ or 7.

Exercises for §8

1. Let (F_1, E_1, B_1, p_1) and (F_2, E_2, B_2, p_2) be vector bundles. Show that $(F_1 \times F_2, E_1 \times E_2, B_1 \times B_2, p_1 \times p_2)$ admits a vector bundle structure.

2. Let $\xi = (F, E, B, p)$ be an n-plane bundle and $f : X \to B$ a continuous map. Define $f^*E = \{(x, e) \in X \times E \mid f(x) = p(e)\}$. Define $q : f^*E \to X$ by $q(x, e) = x$.

(a) Prove that $f^*\xi = (F, f^*E, X, q)$ admits the structure of an n-plane bundle.

(b) Define $\widetilde{f} : f^*E \to E$ by $\widetilde{f}(x, e) = e$. Prove that $\widetilde{f}$ and f constitute a vector bundle homomorphism from $f^*\xi$ to ξ.

3. A *Lie group* is a smooth manifold $(G, \mathcal{S})$ such that G is a topological group for which the multiplication map $G \times G \to G$ is smooth with respect to $\mathcal{S} \times \mathcal{S}$ and $\mathcal{S}$, and the inversion map $G \to G$ is smooth with respect to $\mathcal{S}$ and $\mathcal{S}$. For example, $GL_n(\mathbf{R})$ and $O(n)$ are Lie groups under matrix multiplication; $\mathbf{R}^n$ is a Lie group under addition.

Let $(G, \mathcal{S})$ be a Lie group. For each $g \in G$, let $L_g : G \to G$ be given by $L_g(x) = gx$.

(a) Prove that L_g is a smooth map.

(b) Write e for the identity element of G. For each fixed $v \in T_e(G, \mathcal{S})$ define $\sigma_v : G \to T(G, \mathcal{S})$ by $\sigma_v(g) = TL_g(v)$ for each $g \in G$. Prove that σ_v is a smooth vector field on $(G, \mathcal{S})$. (The main point here is the smoothness of σ_v.)

(c) Prove that $(G, \mathcal{S})$ is parallelizable.

4. (a) Show that if $n \geq 1$ is odd, S^n admits a nowhere-zero smooth vector field.

(b) Show that if $n \equiv 3 \pmod 4$, $n \geq 3$, then S^n admits 3 smooth vector fields σ_1, σ_2, σ_3 such that for all $x \in S^n$, $\sigma_1(x), \sigma_2(x), \sigma_3(x)$ are linearly independent over $\mathbf{R}$. In particular, S^3 is parallelizable.

5. The purpose of this problem is to illustrate the relation between smooth vector fields and differential equations. Recall the following theorem from ordinary differential equations:

Existence Theorem. *Let $g_1, \ldots, g_n$ be smooth real-valued functions defined on an open subset U of $\mathbf{R}^n$, and let $a \in U$. Then there exists a $\delta > 0$ and a smooth map $f = (f_1, \ldots, f_n) : (-\delta, \delta) \to U$ such that for all $t \in (-\delta, \delta)$ and $1 \leq i \leq n$,*

$$f_i'(t) = g_i\big(f_1(t), \ldots, f_n(t)\big) \quad \textit{and} \quad f_i(0) = a_i.$$

Now let $(M, \mathcal{S})$ be a smooth manifold and suppose $\sigma : M \to T(M, \mathcal{S})$ is a smooth vector field. A smooth map $\alpha : (a, b) \to M$, where (a, b) is an open interval, is called an *integral curve for* σ if for each $t \in (a, b)$, $T_t\alpha\big(\theta_t(1)\big) = \sigma\big(\alpha(t)\big)$. Prove that for each $x_0 \in M$, there exists an integral curve $\alpha : (-\delta, \delta) \to M$ for σ (for some $\delta > 0$) such that $\alpha(0) = x_0$.

6. Prove that the atlas $\mathcal{A}_n$ of Example 8.3 is linearly smooth.

7. Prove that the atlas of Example 8.4 is linearly smooth.

9. Categories and Functors

Definition 9.1. A *category* C consists of:

(i) A class $\mathrm{Ob}\,(C)$ whose members are called the *objects of* C.

(ii) For each ordered pair (X, Y) of objects of C, a set $C(X, Y)$ called the *set of morphisms in* C *from* X *to* Y. If $\alpha \in C(X, Y)$, we call X the *domain of* α (denoted dom α) and Y the *codomain of* α (denoted codom α). We also write $\alpha : X \to Y$ or $X \xrightarrow{\alpha} Y$ to denote the statement that $\alpha \in C(X, Y)$.

(iii) For each ordered triple (X, Y, Z) of objects of C, a function

$$C(X,Y) \times C(Y,Z) \to C(X,Z).$$

We denote the image of (α, β) under this function by $\beta\alpha$ and call it the *composition of* α *and* β.

(iv) We require that whenever $\alpha \in C(W, X)$, $\beta \in C(X, Y)$, and $\gamma \in C(Y, Z)$, then $\gamma(\beta\alpha) = (\gamma\beta)\alpha$.

(v) We require that for each object X of C there exist a morphism $1_X \in C(X, X)$ with the property that $\alpha 1_X = \alpha$ whenever $\alpha \in C(X, Y)$ and $1_X\beta = \beta$ whenever $\beta \in C(Y, X)$ for all objects Y of C.

Example 9.2. The category of sets, *Set*, is as follows: $\mathrm{Ob}\,(Set)$ consists of all sets. If X, $Y \in \mathrm{Ob}\,(Set)$, then $Set(X, Y)$ is the set of all functions from X to Y. The composition of morphisms is the usual composition of functions.

Example 9.3. The category of topological spaces, *Top*, is as follows: $\mathrm{Ob}\,(Top)$ consists of all topological spaces. If X, $Y \in \mathrm{Ob}\,(Top)$, then $Top(X, Y)$ is the set of all continuous maps from X to Y. The composition of morphisms is the usual composition of functions.

A variant of this, important for homotopy theory, is the category of *based* or *pointed* topological spaces Top_*. $\mathrm{Ob}\,(Top_*)$ consists of all ordered pairs (X, x_0) where X is a topological space and $x_0 \in X$. If (X, x_0), $(Y, y_0) \in \mathrm{Ob}\,(Top_*)$, then $Top_*\big((X, x_0), (Y, y_0)\big)$ is the set of all continuous maps $f : X \to Y$ such that $f(x_0) = y_0$. The composition of morphisms is the usual composition of functions.

An important related example is the based homotopy category Hty_*. The objects of Hty_* are the same as the objects of Top_*. However, $Hty_*\big((X, x_0), (Y, y_0)\big)$ *does not consist of functions* . A morphism from (X, x_0) to (Y, y_0) is a based homotopy class of morphisms in $Top_*\big((X, x_0), (Y, y_0)\big)$. The composition of morphisms is given by taking based homotopy classes of the usual composition of representative maps.

Example 9.4. The category of smooth manifolds, *Sm*, is as follows: $\mathrm{Ob}\,(Sm)$ consists of all smooth manifolds. If $(M, \mathcal{S})$ and $(N, \mathcal{T})$ are smooth manifolds, then $Sm\big((M, \mathcal{S}), (N, \mathcal{T})\big)$ is the set of all maps $f : M \to N$ which are smooth with respect to $\mathcal{S}$ and $\mathcal{T}$. The composition of morphisms is the usual composition of functions.

Example 9.5. The category of groups *Gr* is as follows: $\mathrm{Ob}\,(Gr)$ consists of all groups. If X, $Y \in \mathrm{Ob}\,(Gr)$, then $Gr(X, Y)$ is the set of all group homomorphisms from X to Y. The composition of morphisms is the usual composition of functions.

Example 9.6. The category of abelian groups Ab is as follows: $\mathrm{Ob}\,(Ab)$ consists of all abelian groups. If X, $Y \in \mathrm{Ob}\,(Ab)$, then $Ab(X,Y)$ is the set of all group homomorphisms from X to Y. The composition of morphisms is the usual composition of functions.

More generally, let R be a ring. We can form the category of left R-modules $R\backslash Mod$ (respectively, the category of right R-modules Mod/R), where the objects are left (respectively, right) R-modules and the morphisms are left (respectively, right) R-homomorphisms. The composition of morphisms is the usual composition of functions. If F is a field, write $VS_F = F\backslash Mod$, the category of vector spaces over F.

Example 9.7. The category of vector bundles $Vect$ is as follows: $\mathrm{Ob}\,(Vect)$ consists of all vector bundles. If ξ and η are vector bundles, $Vect(\xi,\eta)$ consists of all vector bundle homomorphisms from ξ to η as defined in Definition 8.5. The composition of morphisms is the composition of vector bundle homomorphisms as given in Proposition 8.6(b).

We can also form the category of smooth vector bundles $SmVect$ whose objects are smooth vector bundles (Definition 8.10), whose morphisms are smooth vector bundle homomorphisms (Definition 8.12) and with composition of morphisms as in Proposition 8.13(b).

Commutative diagrams are frequently used to express equality of compositions of morphisms.

Definition 9.8. Let C and D be categories. A *covariant functor from C to D* consists of:

(i) A rule which assigns to each object X of C an object FX of D.

(ii) A rule which assigns to each morphism $\alpha : X \to Y$ in C a morphism $F\alpha : FX \to FY$ in D.

(iii) We require that whenever $\alpha \in C(X,Y)$ and $\beta \in C(Y,Z)$, then $F(\beta\alpha) = (F\beta)(F\alpha)$.

(iv) We require that for each object X in C, $F(1_X) = 1_{F(X)}$.

We will sometimes write $F : C \to D$ to denote the statement that F is a functor (covariant as defined above, or contravariant as defined later) from C to D.

Example 9.9. Theorem 8.23 and Proposition 8.24 can be summarized by saying that τ is a covariant functor from the category of smooth manifolds to the category of smooth vector bundles.

Example 9.10. The rule which assigns to each based topological space (X, x_0) its fundamental group $\pi_1(X, x_0)$ and to each based continuous map $f : (X, x_0) \to (Y, y_0)$ the induced homomorphism $\pi_1 f = f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a covariant functor $\pi_1 : Top_* \to Gr$.

Example 9.11. If G is a group, let ΓG denote the commutator subgroup of G. If $f : G \to H$ is a group homomorphism, then $f(\Gamma G) \subset \Gamma H$ and so, by restriction, we get a group homomorphism $\Gamma f : \Gamma G \to \Gamma H$. It is easily seen that we obtain a covariant functor $\Gamma : Gr \to Gr$.

Since ΓG is normal in G, we can form the quotient group $G/\Gamma G$, which is abelian and which we now denote by AG. A group homomorphism f as above induces, on passing to quotients, a group homomorphism $Af : AG \to AH$. It is easily checked that we obtain a covariant functor $A : Gr \to Ab$, the *abelianization functor.*

Example 9.12. For any category C we have the identity functor 1_C on C given by $1_C X = X$ for all objects X of C and $1_C \alpha = \alpha$ for all morphisms α in C.

Definition 9.13. Let C and D be categories. A *contravariant functor from C to D* consists of:

(i) A rule which assigns to each object X of C an object FX of D.

(ii) A rule which assigns to each morphism $\alpha : X \to Y$ in C a morphism $F\alpha : FY \to FX$ in D (note the direction reversal).

(iii) We require that whenever $\alpha \in C(X, Y)$ and $\beta \in C(Y, Z)$, then $F(\beta\alpha) = (F\alpha)(F\beta)$.

(iv) We require that for each object X in C, $F(1_X) = 1_{F(X)}$.

Example 9.14. Let F be a field. The rule which assigns to each vector space V over F its dual space $V^* = \mathrm{Hom}_F(V, F)$ and to each F-linear transformation $\alpha : V \to W$ its dual map $\alpha^* : W^* \to V^*$ is a contravariant functor from VS_F to VS_F. We leave the proof of the following proposition as an exercise.

Proposition 9.15. *Let C, D, and E be categories and suppose $F : C \to D$, $G : D \to E$ are functors (of either type). Then $GF : C \to E$ given by the rules $(GF)(X) = G(F(X))$ for each $X \in Ob(C)$ and $(GF)(\alpha) = G(F(\alpha))$ for each $\alpha \in C(X, Y)$ is a functor. GF is covariant if F and G are either both covariant or both contravariant, and contravariant if one is covariant and the other contravariant.* □

Exercises for §9

1. Prove Proposition 9.15.

2. Let C and D be categories and $F, G : C \to D$ covariant functors. A *natural transformation T from F to G* is a rule which assigns to each object X of C a morphism $TX : FX \to GX$ in D such that whenever $\alpha : X \to Y$ is a morphism in C, the diagram

$$\begin{array}{ccc} FX & \xrightarrow{F\alpha} & FY \\ {\scriptstyle TX}\downarrow & & \downarrow{\scriptstyle TY} \\ GX & \xrightarrow[G\alpha]{} & GY \end{array}$$

commutes.

(a) Let K be a field and $F : VS_K \to VS_K$ the double dual functor, i.e. $FV = (V^*)^*$ for all vector spaces V over K, and $F\alpha = (\alpha^*)^*$ for all K-linear transformations α. For each vector space V over K, define $TV : V \to (V^*)^*$ by the rule $(TV)(v)(\lambda) = \lambda(v)$ for all $v \in V$ and $\lambda \in V^*$. Show that T is a natural transformation from the identity functor 1_{VS_K} to F.

(b) For each group G, let $TG : G \to G/\Gamma G$ denote the natural projection, where the notation is as in Example 9.11. Let $I : Ab \to Gr$ denote the inclusion functor from the category of abelian groups to the category of groups, i.e. $IG = G$ for each abelian group G and $I\alpha = \alpha$ for each homomorphism of abelian groups α. Show that T is a natural transformation from 1_{Gr} to IA where A is the abelianization functor.

3. Let C be a category. The *opposite category to* C, denoted C^{op}, is defined as follows: $\mathrm{Ob}\,(C^{\mathrm{op}}) = \mathrm{Ob}\,(C)$; if X, $Y \in \mathrm{Ob}\,(C^{\mathrm{op}})$, then $C^{\mathrm{op}}(X,Y) = C(Y,X)$. We define a composition law $\cdot$ for C^{op} as follows: If $\alpha \in C^{\mathrm{op}}(X,Y)$ and $\beta \in C^{\mathrm{op}}(Y,Z)$, then $\beta \cdot \alpha = \alpha\beta$ where the composition on the right-hand side is the composition in the category C. For each $X \in \mathrm{Ob}\,(C^{\mathrm{op}})$ define $1_X \in C^{\mathrm{op}}(X,X)$ to be the same as the $1_X \in C(X,X)$.

(a) Show that C^{op} is a category.

(b) Show that $Op : C \to C^{\mathrm{op}}$ given by the rules $Op(X) = X$ for all $X \in \mathrm{Ob}\,(X)$ and $Op(\alpha) = \alpha$ for all morphisms α in C is a contravariant functor from C to C^{op}.

(c) Show that $(C^{\mathrm{op}})^{\mathrm{op}} = C$.

4. Let C and D be categories. Define $C \times D$ as follows:

$$\mathrm{Ob}\,(C \times D) = \mathrm{Ob}\,(C) \times \mathrm{Ob}\,(D);$$

if (X,Y), $(X',Y') \in \mathrm{Ob}\,(C \times D)$, then

$$(C \times D)\big((X,Y),(X',Y')\big) = C(X,X') \times D(Y,Y').$$

Define a composition law

$$(C \times D)\big((U,V),(W,X)\big) \times (C \times D)\big((W,X),(Y,Z)\big) \to (C \times D)\big((U,V),(Y,Z)\big)$$

by sending $\big((\alpha,\beta),(\alpha',\beta')\big)$ to $(\alpha'\alpha,\beta'\beta)$. For $(X,Y) \in \mathrm{Ob}\,(C \times D)$ define $1_{(X,Y)} = \big(1_X, 1_Y\big)$. Show that $C \times D$ is a category.

5. Let K be a field. Define $F : VS_K^{\mathrm{op}} \times VS_K \to VS_K$ as follows: For vector spaces X and Y over K, $F(X,Y) = \mathrm{Hom}_K(X,Y)$. If $\alpha : X' \to X$ and $\beta : Y \to Y'$ are K-linear transformations, then

$$F(\alpha,\beta) : \mathrm{Hom}_K(X,Y) \to \mathrm{Hom}_K(X',Y')$$

is given by $F(\alpha,\beta)(\gamma) = \beta\gamma\alpha$. Show that F is a covariant functor.

10. Exterior Algebra

This section deals with some multilinear algebra which will be needed in the sequel. Throughout this section, all vector spaces and linear transformations are assumed to be in the category $VS_{\mathbf{R}}$ of vector spaces over $\mathbf{R}$.

Definition 10.1 Let V be a vector space and k a positive integer. Write V^k for the k-fold cartesian product

$$\underbrace{V \times \cdots \times V}_{k}.$$

A function $f : V^k \to \mathbf{R}$ is said to be *k-linear* if for each $1 \le i \le k$ and each fixed $(k-1)$-tuple $(v_1, \ldots, v_{k-1})$ of vectors in V, the function $V \to \mathbf{R}$ which sends x to $f(v_1, \ldots, v_{i-1}, x, v_i, \ldots, v_{k-1})$ is a linear transformation.

Thus a 1-linear map $V \to \mathbf{R}$ is simply a linear transformation from V to $\mathbf{R}$. However, if $k > 1$, a k-linear map is *not* a linear transformation. For example, if $f : V \times V \to \mathbf{R}$ is 2-linear, it is not true (except for trivial cases) that $f\Big((w,x) + (y,z)\Big) = f(w,x) + f(y,z)$. In fact,

$$\begin{aligned} f\Big((w,x)+(y,z)\Big) &= f(w+y, x+z) = f(w, x+z) + f(y, x+z) \\ &= f(w,x) + f(w,z) + f(y,x) + f(y,z) \end{aligned}$$

using additivity of f in each variable separately. (Tensor products could be introduced here to pass from k-linear maps to linear maps, but this will not be necessary for what follows.)

For $k \ge 1$ let Σ_k denote the k^{th} symmetric group, i.e. the group of permutations of $\{1, \ldots, k\}$. Recall that for each $\sigma \in \Sigma_k$ the *sign of σ*, denoted $\operatorname{sgn}(\sigma)$, is defined to be 1 if σ is expressible as a composition of an even number of transpositions, and -1 if σ is expressible as a composition of an odd number of transpositions.

Definition 10.2. A k-linear map $f : V^k \to \mathbf{R}$ is said to be *alternating* if for all $(v_1, \ldots, v_k) \in V^k$ and all $\sigma \in \Sigma_k$,

$$f\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big) = \operatorname{sgn}(\sigma) f(v_1, \ldots, v_k).$$

We will refer to such a map as an *alternating k-linear map on V*.

Example 10.3. Let $V = \mathbf{R}^n$ and regard the members of V as column vectors. For $(v_1, \ldots, v_n) \in V^n$, let

$$\begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix}$$

denote the $n \times n$ matrix whose columns are $v_1, \ldots, v_n$. Define $f : V^n \to \mathbf{R}$ by

$$f(v_1, \ldots, v_n) = \det \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix}.$$

Then f is an alternating n-linear map on V.

Example 10.4. Every $\mathbf{R}$-linear map $f : V \to \mathbf{R}$ is an alternating 1-linear map on V.

Example 10.5. Let $f : V^k \to \mathbf{R}$ be k-linear. Define $Af : V^k \to \mathbf{R}$ by

$$Af(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \operatorname{sgn}(\sigma) f\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big).$$

It is easy to check that Af is an alternating k-linear map.

The factor $\frac{1}{k!}$ is not essential to get an alternating map. However, by including it we have the nice property that a k-linear map f is alternating if and only if $Af = f$.

Definition 10.6. The set of all alternating k-linear maps $V^k \to \mathbf{R}$ is called the *k^{th} exterior power of V, and denoted $\Lambda^k(V)$.*

Caution: There are a number of different notational conventions used in various treatments of exterior algebra. In particular, the notation $\Lambda^k(V)$ means something different in some treatments; what we have denoted by $\Lambda^k(V)$ is sometimes denoted $\Lambda^k(V^*)$ elsewhere. There are also different definitions of the wedge product (defined below) in common use. It is usually straightforward to translate from one convention to another. Sometimes numerical constants have to be thrown in to get a correct translation. This should be borne in mind when reading other sources. The notation and conventions used here are consistent with those used in R. Abraham, J.E. Marsden & T. Ratiu, *Manifolds, Tensor Analysis, and Applications.*

Note that $\Lambda^k(V)$ is a real vector space under ordinary addition of real-valued functions, and ordinary multiplication of real-valued functions by real constants.

Example 10.7. $\Lambda^1(V) = V^*$, the dual space of V.

Definition 10.8. $\Lambda^0(V) = \mathbf{R}$ for all V.

Definition 10.8 fits in well with the above scheme. If we think of V^0 as the one element set consisting of the unique 0-tuple (), every function $V^0 \to \mathbf{R}$ is 0-linear and alternating, and we identify each such function with its image in $\mathbf{R}$. We leave the proof of the following proposition as an exercise.

Proposition 10.9. *Let $\alpha : V \to W$ be a real linear transformation. Let $f \in \Lambda^k(W)$. Then the composition*

$$V^k \xrightarrow{\alpha^k} W^k \xrightarrow{f} \mathbf{R}$$

where

$$\alpha^k(v_1, \ldots, v_k) = \big(\alpha(v_1), \ldots, \alpha(v_k)\big),$$

lies in $\Lambda^k(V)$. Moreover, if we denote the above composition by $\Lambda^k(\alpha)(f)$, the function

$$\Lambda^k(\alpha) : \Lambda^k(W) \to \Lambda^k(V)$$

is a real linear transformation. □

Note that when $k = 0$, $\alpha^0 = 1_{\{()\}}$ and so $\Lambda^0(\alpha) = 1_{\mathbf{R}}$ for all linear transformations α.

We leave the proof of the following proposition as an exercise.

Proposition 10.10. *For each $k \geq 0$, the rule which assigns to each real vector space V its k^{th} exterior power $\Lambda^k(V)$ and to each real linear transformation α the linear transformation $\Lambda^k(\alpha)$ is a contravariant functor $\Lambda^k : VS_{\mathbf{R}} \to VS_{\mathbf{R}}$.*

Λ^1 coincides with the dual space functor.

Λ^0 is the constant functor $\Lambda^0(V) = \mathbf{R}$ for all V, $\Lambda^0(\alpha) = 1_{\mathbf{R}}$ for all α. $\square$

Note that if $f : V^j \to \mathbf{R}$ is j-linear and $g : V^k \to \mathbf{R}$ is k-linear, then $f \cdot g : V^{j+k} \to \mathbf{R}$ given by

$$(f \cdot g)(v_1, \ldots, v_{j+k}) = f(v_1, \ldots, v_j)g(v_{j+1}, \ldots, v_{j+k})$$

is $(j + k)$-linear.

Definition 10.11. Let $f \in \Lambda^j(V)$, $g \in \Lambda^k(V)$ where $j, k \geq 0$. Define $f \wedge g \in \Lambda^{j+k}(V)$ by

$$f \wedge g = \frac{(j+k)!}{j!k!} A(f \cdot g).$$

$f \wedge g$ is called the *wedge product of f and g*.

Thus, if $f \in \Lambda^j(V)$ and $g \in \Lambda^k(V)$, then

$$(f \wedge g)(v_1, \ldots, v_{j+k}) = \frac{1}{j!k!} \sum_{\sigma \in \Sigma_{j+k}} \operatorname{sgn}(\sigma) f\big(v_{\sigma(1)}, \ldots, v_{\sigma(j)}\big) g\big(v_{\sigma(j+1)}, \ldots, v_{\sigma(j+k)}\big).$$

For example, let $f \in \Lambda^2(V)$, $g \in \Lambda^1(V)$. We have

$$\Sigma_3 = \{\, 1,\ (1\ 2),\ (1\ 3),\ (2\ 3),\ (1\ 2\ 3),\ (1\ 3\ 2) \,\}$$

and

$$\begin{aligned} &\operatorname{sgn}(1) = \operatorname{sgn}(1\ 2\ 3) = \operatorname{sgn}(1\ 3\ 2) = 1, \\ &\operatorname{sgn}(1\ 2) = \operatorname{sgn}(1\ 3)\operatorname{sgn}(2\ 3) = -1. \end{aligned}$$

Thus

$$\begin{aligned} (f \wedge g)(v_1, v_2, v_3) &= \frac{1}{2!1!}\Big(f(v_1, v_2)g(v_3) - f(v_2, v_1)g(v_3) - f(v_3, v_2)g(v_1) \\ &\quad - f(v_1, v_3)g(v_2) + f(v_2, v_3)g(v_1) + f(v_3, v_1)g(v_2)\Big) \\ &= \frac{1}{2}\Big(f(v_1, v_2)g(v_3) + f(v_1, v_2)g(v_3) + f(v_2, v_3)g(v_1) \\ &\quad - f(v_1, v_3)g(v_2) + f(v_2, v_3)g(v_1) - f(v_1, v_3)g(v_2)\Big) \\ &= f(v_1, v_2)g(v_3) + f(v_2, v_3)g(v_1) - f(v_1, v_3)g(v_2). \end{aligned}$$

Quite generally, the alternating character of f and g results in a reduction from $(j + k)!$ terms to $\dfrac{(j+k)!}{j!k!}$ terms in the expression for $(f \wedge g)(v_1, \ldots, v_{j+k})$, as well as a disappearance of the numerical factors, which motivates the reason for the numerical factors in Example 10.5 and Definition 10.11. We proceed to make this precise.

A permutation $\sigma \in \Sigma_{j+k}$ is called a (j,k)-*shuffle* if

$$\sigma(1) < \sigma(2) < \cdots < \sigma(j) \text{ and } \sigma(j+1) < \sigma(j+2) < \cdots < \sigma(j+k).$$

Let Shuf (j,k) denote the set of (j,k)-shuffles. For example,

$$\text{Shuf}\,(2,1) = \{\,1,\ (2\ 3),\ (1\ 2\ 3)\}.$$

In general, Shuf (j,k) is in one-to-one correspondence with the set of j element subsets of $\{1,\ldots,j+k\}$ as follows: If $S = \{s_1,\ldots,s_j\}$ with $1 \le s_1 < s_2 < \cdots < s_j \le j+k$, we associate with S the unique (j,k)-shuffle σ_S satisfying $\sigma_S(i) = s_i$ for $1 \le i \le j$.

Lemma 10.12. *Let* $\theta : \Sigma_j \times \Sigma_k \to \Sigma_{j+k}$ *be given as follows: For* $\tau \in \Sigma_j$ *and* $\mu \in \Sigma_k$,

$$\theta(\tau,\mu)(i) = \tau(i) \quad \text{if } 1 \le i \le j,$$
$$\theta(\tau,\mu)(j+i) = j + \mu(i) \quad \text{if } 1 \le i \le k.$$

Then:

(a) $\operatorname{sgn}\big(\theta(\tau,\mu)\big) = \operatorname{sgn}(\tau)\operatorname{sgn}(\mu)$.

(b) *Let* $\varphi : \text{Shuf}\,(j,k) \times \Sigma_j \times \Sigma_k \to \Sigma_{j+k}$ *be given by* $\varphi(\sigma,\tau,\mu) = \sigma\theta(\tau,\mu)$. *Then* φ *is a bijection.*

Proof. Note that the sets $\{1,\ldots,j\}$ and $\{j+1,\ldots,j+k\}$ are both invariant under $\theta(\tau,\mu)$, the restriction of $\theta(\tau,\mu)$ to $\{1,\ldots,j\}$ is τ, and that the diagram

$$\begin{array}{ccc} \{1,\ldots,k\} & \xrightarrow{+j} & \{j+1,\ldots,j+k\} \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\theta(\tau,\mu)} \\ \{1,\ldots,k\} & \xrightarrow[+j]{} & \{j+1,\ldots,j+k\} \end{array}$$

commutes. Thus if τ is a product of r transpositions and μ is a product of s transpositions, it follows that $\theta(\tau,\mu)$ is a product of $r+s$ transpositions, proving part (a).

Since the cardinalities of Shuf (j,k), Σ_j, and Σ_k are, respectively, $\dfrac{(j+k)!}{j!k!}$, $j!$, and $k!$, it follows that the cardinality of Shuf $(j,k) \times \Sigma_j \times \Sigma_k$ equals the cardinality of Σ_{j+k}. Thus, part (b) will follow if we show that φ is injective.

Suppose $\varphi(\sigma_1,\tau_1,\mu_1) = \varphi(\sigma_2,\tau_2,\mu_2)$. Then $\sigma_1^{-1}\sigma_2 = \theta(\tau_1,\mu_1)\theta(\tau_2,\mu_2)^{-1}$. Since the sets $\{1,\ldots,j\}$ and $\{j+1,\ldots,j+k\}$ are both invariant under the $\theta(\tau_i,\mu_i)$ for $i = 1, 2$, it follows that these sets are also invariant under $\sigma_1^{-1}\sigma_2$. We claim that $\sigma_1^{-1}\sigma_2$ is order-preserving (and hence, the identity). For if not, there would either exist p, q such that $1 \le p < q \le j$ with $(\sigma_1^{-1}\sigma_2)(p) > (\sigma_1^{-1}\sigma_2)(q)$ or p, q such that $j+1 \le p < q \le j+k$ with $(\sigma_1^{-1}\sigma_2)(p) > (\sigma_1^{-1}\sigma_2)(q)$. Suppose the first of these occurred. Since σ_1 is order-preserving on $\{1,\ldots,j\}$ (since it is a (j,k)-shuffle) it follows that $\sigma_1\big((\sigma_1^{-1}\sigma_2(p)\big) > \sigma_1\big((\sigma_1^{-1}\sigma_2(q)\big)$, i.e. $\sigma_2(p) > \sigma_2(q)$, contradicting the fact that σ_2 is order-preserving on $\{1,\ldots,j\}$. Similarly, the second possibility above is impossible. Thus $\sigma_1 = \sigma_2$, and hence $\theta(\tau_1,\mu_1) = \theta(\tau_2,\mu_2)$. Clearly, θ is injective, and so $(\tau_1,\mu_1) = (\tau_2,\mu_2)$. □

Proposition 10.13. *Let $f \in \Lambda^j(V)$, $g \in \Lambda^k(V)$. Then for all $v_1, \ldots, v_{j+k} \in V$,*

$$(f \wedge g)(v_1, \ldots, v_{j+k}) = \sum_{\sigma \in \mathrm{Shuf}\,(j,k)} \mathrm{sgn}\,(\sigma) f\big(v_{\sigma(1)}, \ldots, v_{\sigma(j)}\big) g\big(v_{\sigma(j+1)}, \ldots, v_{\sigma(j+k)}\big).$$

Proof. For notational convenience, abbreviate $\big(v_{\rho(1)}, \ldots, v_{\rho(l)}\big)$ by $\big(v|_{\rho(1)}^{\rho(l)}\big)$. Thus we must show

$$(f \wedge g)(v_1, \ldots, v_{j+k}) = \sum_{\sigma \in \mathrm{Shuf}\,(j,k)} \mathrm{sgn}\,(\sigma) f\big(v|_{\sigma(1)}^{\sigma(j)}\big) g\big(v|_{\sigma(j+1)}^{\sigma(j+k)}\big).$$

Write $S = \mathrm{Shuf}\,(j,k) \times \Sigma_j \times \Sigma_k$. We have

$$
\begin{aligned}
(f \wedge g)(v_1, \ldots, v_{j+k}) &= \frac{1}{j!k!} \sum_{\rho \in \Sigma_{j+k}} \mathrm{sgn}\,(\rho) f\big(v|_{\rho(1)}^{\rho(j)}\big) g\big(v|_{\rho(j+1)}^{\rho(j+k)}\big) \\
&= \frac{1}{j!k!} \sum_{(\sigma,\tau,\mu) \in S} \mathrm{sgn}\,\big(\sigma\theta(\tau,\mu)\big) f\big(v|_{\sigma\theta(\tau,\mu)(1)}^{\sigma\theta(\tau,\mu)(j)}\big) g\big(v|_{\sigma\theta(\tau,\mu)(j+1)}^{\sigma\theta(\tau,\mu)(j+k)}\big) \\
&\hspace{20em} \text{(by Lemma 10.12(b))} \\
&= \frac{1}{j!k!} \sum_{(\sigma,\tau,\mu) \in S} \mathrm{sgn}\,(\sigma)\, \mathrm{sgn}\,(\tau)\, \mathrm{sgn}\,(\mu) f\big(v|_{\sigma\tau(1)}^{\sigma\tau(j)}\big) g\big(v|_{\sigma(j+\mu(1))}^{\sigma(j+\mu(k))}\big) \\
&\hspace{20em} \text{(by Lemma 10.12(a))} \\
&= \frac{1}{j!k!} \sum_{(\sigma,\tau,\mu) \in S} \mathrm{sgn}\,(\sigma)\, \mathrm{sgn}\,(\tau)\, \mathrm{sgn}\,(\mu)\, \mathrm{sgn}\,(\tau) f\big(v|_{\sigma(1)}^{\sigma(j)}\big) \mathrm{sgn}\,(\mu) g\big(v|_{\sigma(j+1)}^{\sigma(j+k)}\big) \\
&\hspace{16em} \text{(since } f \text{ and } g \text{ are alternating)} \\
&= \frac{1}{j!k!} \sum_{(\sigma,\tau,\mu) \in S} \mathrm{sgn}\,(\sigma) f\big(v|_{\sigma(1)}^{\sigma(j)}\big) g\big(v_{\sigma(j+1)}^{\sigma(j+k)}\big) \\
&= \frac{1}{j!k!} \sum_{\sigma \in \mathrm{Shuf}\,(j,k)} j!k!\, \mathrm{sgn}\,(\sigma) f\big(v|_{\sigma(1)}^{\sigma(j)}\big) g\big(v|_{\sigma(j+1)}^{\sigma(j+k)}\big) \\
&\hspace{14em} \text{(since the cardinality of } \Sigma_j \times \Sigma_k \text{ is } j!k!) \\
&= \sum_{\sigma \in \mathrm{Shuf}\,(j,k)} \mathrm{sgn}\,(\sigma) f\big(v|_{\sigma(1)}^{\sigma(j)}\big) g\big(v|_{\sigma(j+1)}^{\sigma(j+k)}\big). \qquad \square
\end{aligned}
$$

Theorem 10.14. *Let V be a real vector space. Suppose $f \in \Lambda^i(V)$, $g \in \Lambda^j(V)$, $h \in \Lambda^k(V)$, and $r \in \mathbf{R}$. Then:*

(a) *$f \wedge (g \wedge h) = (f \wedge g) \wedge h$.*
(b) *$g \wedge f = (-1)^{ij} f \wedge g$.*
(c) *If $j = k$, then $f \wedge (g + h) = f \wedge g + f \wedge h$.*
(d) *$(rf) \wedge g = f \wedge (rg) = r(f \wedge g)$.*
(e) *$1 \wedge f = f \wedge 1 = f$ where $1 \in \Lambda^0(V) = \mathbf{R}$.*

Proof. To prove part (a), define an *(i,j,k)-shuffle* to be a permutation $\sigma \in \Sigma_{i+j+k}$ such that

$$
\begin{aligned}
&\sigma(1) < \sigma(2) < \cdots < \sigma(i), \\
&\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(i+j), \text{ and} \\
&\sigma(i+j+1) < \sigma(i+j+2) < \cdots < \sigma(i+j+k).
\end{aligned}
$$

Let $\mathrm{Shuf}\,(i,j,k)$ denote the set of all (i,j,k)-shuffles. We have bijections

$$\varphi : \mathrm{Shuf}\,(i, j+k) \times \mathrm{Shuf}\,(j,k) \to \mathrm{Shuf}\,(i,j,k)$$

and

$$\psi : \mathrm{Shuf}\,(i+j, k) \times \mathrm{Shuf}\,(i,j) \to \mathrm{Shuf}\,(i,j,k)$$

given by

$$\begin{aligned} \varphi(\tau,\omega)(q) &= \tau(q) \quad \text{if } 1 \le q \le i, \\ \varphi(\tau,\omega)(i+q) &= \tau\big(i+\omega(q)\big) \quad \text{if } 1 \le q \le j+k, \end{aligned}$$

and

$$\begin{aligned} \psi(\rho,\mu)(q) &= \rho\mu(q) \quad \text{if } 1 \le q \le i+j, \\ \psi(\rho,\mu)(i+j+q) &= \rho(i+j+q) \quad \text{if } 1 \le q \le k. \end{aligned}$$

Note that $\mathrm{sgn}\,\big(\varphi(\tau,\omega)\big) = \mathrm{sgn}\,(\tau)\,\mathrm{sgn}\,(\omega)$ and $\mathrm{sgn}\,\big(\psi(\rho,\mu)\big) = \mathrm{sgn}\,(\rho)\,\mathrm{sgn}\,(\mu)$. In the summations below, σ runs over $\mathrm{Shuf}\,(i,j,k)$, τ runs over $\mathrm{Shuf}\,(i,j+k)$, ω runs over $\mathrm{Shuf}\,(j,k)$, ρ runs over $\mathrm{Shuf}\,(i+j,k)$, and μ runs over $\mathrm{Shuf}\,(i,j)$. We employ the abbreviated notation used in the proof of Proposition 10.13. For all $v_1, \ldots, v_{i+j+k}$ we have, using Proposition 10.13,

$$\begin{aligned} \big(f \wedge (g \wedge h)\big)\big(v|_1^{i+j+k}\big) &= \sum_\tau \mathrm{sgn}\,(\tau) f\big(v|_{\tau(1)}^{\tau(i)}\big)(g \wedge h)\big(v|_{\tau(i+1)}^{\tau(i+j+k)}\big) \\ &= \sum_\tau \mathrm{sgn}\,(\tau) f\big(v|_{\tau(1)}^{\tau(i)}\big) \sum_\omega \mathrm{sgn}\,(\omega) g\big(v|_{\tau(i+\omega(1))}^{\tau(i+\omega(j))}\big) h\big(v|_{\tau(i+\omega(j+1))}^{\tau(i+\omega(j+k))}\big) \\ &= \sum_{\tau,\omega} \mathrm{sgn}\,\big(\varphi(\tau,\omega)\big) f\big(v|_{\varphi(\tau,\omega)(1)}^{\varphi(\tau,\omega)(i)}\big) g\big(v|_{\varphi(\tau,\omega)(i+1)}^{\varphi(\tau,\omega)(i+j)}\big) h\big(v|_{\varphi(\tau,\omega)(i+j+1)}^{\varphi(\tau,\omega)(i+j+k)}\big) \\ &= \sum_\sigma \mathrm{sgn}\,(\sigma) f\big(v|_{\sigma(1)}^{\sigma(i)}\big) g\big(v|_{\sigma(i+1)}^{\sigma(i+j)}\big) h\big(v|_{\sigma(i+j+1)}^{\sigma(i+j+k)}\big) \\ &= \sum_{\rho,\mu} \mathrm{sgn}\,\big(\psi(\rho,\mu)\big) f\big(v|_{\psi(\rho,\mu)(1)}^{\psi(\rho,\mu)(i)}\big) g\big(v|_{\psi(\rho,\mu)(i+1)}^{\psi(\rho,\mu)(i+j)}\big) h\big(v|_{\psi(\rho,\mu)(i+j+1)}^{\psi(\rho,\mu)(i+j+k)}\big) \\ &= \sum_\rho \mathrm{sgn}\,(\rho) \Big(\sum_\mu \mathrm{sgn}\,(\mu) f\big(v|_{\rho\mu(1)}^{\rho\mu(i)}\big) g\big(v|_{\rho\mu(i+1)}^{\rho\mu(i+j)}\big) \Big) h\big(v|_{\rho(i+j+1)}^{\rho(i+j+k)}\big) \\ &= \sum_\rho \mathrm{sgn}\,(\rho)(f \wedge g)\big(v|_{\rho(1)}^{\rho(i+j)}\big) h\big(v|_{\rho(i+j+1)}^{\rho(i+j+k)}\big) \\ &= \big((f \wedge g) \wedge h\big)\big(v|_1^{i+j+k}\big), \end{aligned}$$

proving part (a).

To prove part (b), let τ be the (j,i)-shuffle characterized by the property $\tau(q) = i+q$ for $1 \le q \le j$ (and thus $\tau(j+q) = q$ for $1 \le q \le i$). Note that τ is expressible as the composition of ij transpositions (successively, move each of $j+1$, $j+2$, $\ldots$, $j+i$ to the left past 1, 2, $\ldots$, j). Thus $\mathrm{sgn}\,(\tau) = (-1)^{ij}$. Note also that if σ is an (i,j)-shuffle, then $\sigma\tau$ is a (j,i)-shuffle and that the map $\mathrm{Shuf}\,(i,j) \to \mathrm{Shuf}\,(j,i)$ which sends σ to $\sigma\tau$ is a bijection. In the summations below, σ runs over Shuf(i,j) and ρ runs over Shuf(j,i). We have

$$\begin{aligned}(g\wedge f)(v|_1^{i+j}) &= \sum_\rho \operatorname{sgn}(\rho) g(v|_{\rho(1)}^{\rho(j)}) f(v|_{\rho(j+1)}^{\rho(j+i)}) \\ &= \sum_\sigma \operatorname{sgn}(\sigma\tau) g(v|_{\sigma\tau(1)}^{\sigma\tau(j)}) f(v|_{\sigma\tau(j+1)}^{\sigma\tau(j+i)}) \\ &= \sum_\sigma \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma) g(v|_{\sigma(i+1)}^{\sigma(i+j)}) f(v|_{\sigma(1)}^{\sigma(i)}) \\ &= \operatorname{sgn}(\tau)\sum_\sigma \operatorname{sgn}(\sigma) f(v|_{\sigma(1)}^{\sigma(i)}) g(v|_{\sigma(i+1)}^{\sigma(i+j)}) \\ &= (-1)^{ij}(f\wedge g)(v|_1^{i+j}),\end{aligned}$$

proving part (b).

The proofs of parts (c) and (d) are easy exercises. Part (e) is immediate from the observation that $1\cdot f = f\cdot 1 = f$. $\square$

Write $\Lambda(V) = \bigoplus_k \Lambda^k(V)$. Theorem 10.14 can be summarized by the statement that $\Lambda(V)$ is a *graded algebra over* $\mathbf{R}$ *with unit*, and is *commutative* (in the graded sense). $\Lambda(V)$ is called the *exterior algebra* or *Grassmann algebra on* V^*.

As a consequence of the graded commutativity property (Theorem 10.14(b)) it follows that if k is odd, then for all $f\in\Lambda^k(V)$, $f\wedge f = -f\wedge f$ and so $f\wedge f = 0$. For k even, $f\wedge f$ is generally not 0.

The proof of the following proposition is easy and left as an exercise.

Proposition 10.15. *Let $\alpha: V\to W$ be an $\mathbf{R}$-linear transformation. Then for all $f\in\Lambda^i(W)$ and $g\in\Lambda^j(W)$, $\Lambda^{i+j}(\alpha)(f\wedge g) = \Lambda^i(\alpha)(f)\wedge\Lambda^j(\alpha)(g)$.* $\square$

Propositions 10.15 and 10.9, together with the remark following the latter, can be paraphrased by saying that the $\Lambda^k(\alpha)$ constitute a *homomorphism of commutative graded algebras with unit over* $\mathbf{R}$. We can form the category $CGA_{\mathbf{R}}$ of commutative graded algebras with unit over $\mathbf{R}$. Then Propositions 10.15 and 10.10 state that Λ is a contravariant functor from $VS_{\mathbf{R}}$ to $CGA_{\mathbf{R}}$.

Lemma 10.16. *Let $k\geq 1$ and suppose $f\in\Lambda^{k-1}(V)$, $g\in\Lambda^1(V)$. Then for all $v_1,\dots,v_k\in V$,*

$$(f\wedge g)(v_1,\dots,v_k) = \sum_{i=1}^k (-1)^{k-i} f(v_1,\dots,\widehat{v}_i,\dots,v_k)g(v_i).$$

Proof. For $1\leq i\leq k$, let σ_i denote the $(k-1,1)$-shuffle determined by $\sigma_i(k) = i$. Then Shuf$(k-1,1) = \{\sigma_1,\dots,\sigma_k\}$. Note that σ_i is a composition of $k-i$ transpositions (move k to the left past $i, i+1,\dots,k-1$) and so $\operatorname{sgn}(\sigma_i) = (-1)^{k-i}$ for each i. Thus, by Proposition 10.13,

$$\begin{aligned}(f\wedge g)(v_1,\dots,v_k) &= \sum_{i=1}^k \operatorname{sgn}(\sigma_i) f(v_{\sigma_i(1)},\dots,v_{\sigma_i(k-1)}) g(v_{\sigma_i(k)}) \\ &= \sum_{i=1}^k (-1)^{k-i} f(v_1,\dots,\widehat{v}_i,\dots,v_k)g(v_i). \qquad \square\end{aligned}$$

Theorem 10.17. *Let $k \geq 1$ and suppose $f_1, \ldots, f_k \in \Lambda^1(V)$. Then for all $v_1, \ldots, v_k \in V$,*

$$(f_1 \wedge \cdots \wedge f_k)(v_1, \ldots, v_k) = \sum_{\sigma \in \Sigma_k} \operatorname{sgn}(\sigma) f_1\big(v_{\sigma(1)}\big) \cdots f_k\big(v_{\sigma(k)}\big).$$

Proof. Proceed by induction on k. The result is trivial for $k = 1$. Suppose $k > 1$ and that the result holds for $f_1 \wedge \cdots \wedge f_{k-1}$. By Lemma 10.16,

$$(*)\ \ (f_1 \wedge \cdots \wedge f_k)(v_1, \ldots, v_k) = \sum_{i=1}^{k} (-1)^{k-i} (f_1 \wedge \cdots \wedge f_{k-1})(v_1, \ldots, \widehat{v_i}, \ldots, v_k) f_k(v_i).$$

For $1 \leq i \leq k$ let $X_i = \{\sigma \in \Sigma_k \mid \sigma(k) = i\}$. Then Σ_k is the disjoint union of the X_i. For $1 \leq i \leq k$ we have bijections $\varepsilon_i : \Sigma_{k-1} \to X_i$ given by

$$\varepsilon_i(\tau) = \begin{cases} \tau(j) & \text{if } j \leq k-1 \text{ and } \tau(j) < i, \\ \tau(j) + 1 & \text{if } j \leq k-1 \text{ and } \tau(j) \geq i, \\ i & \text{if } j = k. \end{cases}$$

Thus, $\varepsilon_i(\tau)$ is given by first moving $1, \ldots, k-1$ via τ, and then moving k to the left past $i, i+1, \ldots, k-1$. Thus $\operatorname{sgn}(\varepsilon_i(\tau)) = (-1)^{k-i} \operatorname{sgn}(\tau)$.

For a fixed i between 1 and k, let

$$w_j = \begin{cases} v_j & \text{if } 1 \leq j < i, \\ v_{j+1} & \text{if } i \leq j \leq k-1. \end{cases}$$

Then, by the inductive hypothesis,

$$\begin{aligned} (f_1 \wedge \cdots \wedge f_{k-1})(v_1, \ldots, \widehat{v_i}, \ldots, v_k) &= (f_1 \wedge \cdots \wedge f_{k-1})(w_1, \ldots, w_{k-1}) \\ &= \sum_{\tau \in \Sigma_{k-1}} \operatorname{sgn}(\tau) f_1\big(w_{\tau(1)}\big) \cdots f_{k-1}\big(w_{\tau(k-1)}\big) \\ &= \sum_{\tau \in \Sigma_{k-1}} \operatorname{sgn}(\tau) f_1\big(v_{\varepsilon_i(\tau)(1)}\big) \cdots f_{k-1}\big(v_{\varepsilon_i(\tau)(k-1)}\big) \end{aligned}$$

and so

$$\begin{aligned} &(-1)^{k-i} (f_1 \wedge \cdots \wedge f_{k-1})(v_1, \ldots, \widehat{v_i}, \ldots, v_k) f_k(v_i) \\ &\quad = \sum_{\tau \in \Sigma_{k-1}} (-1)^{k-i} \operatorname{sgn}(\tau) f_1\big(v_{\varepsilon_i(\tau)(1)}\big) \cdots f_{k-1}\big(v_{\varepsilon_i(\tau)(k-1)}\big) f_k(v_i) \\ &\quad = \sum_{\tau \in \Sigma_{k-1}} \operatorname{sgn}\big(\varepsilon_i(\tau)\big) f_1\big(v_{\varepsilon_i(\tau)(1)}\big) \cdots f_{k-1}\big(v_{\varepsilon_i(\tau)(k-1)}\big) f_k\big(v_{\varepsilon_i(\tau)(k)}\big) \\ &\quad = \sum_{\sigma \in X_i} \operatorname{sgn}(\sigma) f_1\big(v_{\sigma(1)}\big) \cdots f_k\big(v_{\sigma(k)}\big). \end{aligned}$$

Thus, by $(*)$,

$$\begin{aligned} (f_1 \wedge \cdots \wedge f_k)(v_1, \ldots, v_k) &= \sum_{i=1}^{k} \sum_{\sigma \in X_i} \operatorname{sgn}(\sigma) f_1\big(v_{\sigma(1)}\big) \cdots f_k\big(v_{\sigma(k)}\big) \\ &= \sum_{\sigma \in \Sigma_k} \operatorname{sgn}(\sigma) f_1\big(v_{\sigma(1)}\big) \cdots f_k\big(v_{\sigma(k)}\big), \end{aligned}$$

completing the induction. □

Theorem 10.18. *Suppose V is finite-dimensional over* $\mathbf{R}$, *and* $\{ f_1, \ldots, f_n \}$ *is an* $\mathbf{R}$*-basis of* $\Lambda^1(V) = V^*$. *Then for* $k \geq 0$,

$$\{ f_{i_1} \wedge f_{1_2} \wedge \cdots \wedge f_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n \}$$

is an $\mathbf{R}$*-basis of* $\Lambda^k(V)$.

Proof. Let $\{ e_1, \ldots, e_n \}$ be the basis of V dual to $\{ f_1, \ldots, f_n \}$, i.e. $f_i(e_j) = \delta_{ij}$ for all i, j. Let

$$X = \{ (i_1, \ldots, i_k) \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n \}.$$

For $I = (i_1, \ldots, i_k) \in X$, write $f_I = f_{i_1} \wedge \cdots \wedge f_{i_k}$ and $e_I = \left(e_{i_1}, \ldots, e_{i_k}\right)$. It follows from Theorem 10.17 that for $I, J \in X$,

$$f_I\left(e_J\right) = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases}$$

From the k-linearity and alternating properties, it follows that if $g, h \in \Lambda^k(V)$, then $g = h$ if and only if $g\left(e_I\right) = h\left(e_I\right)$ for all $I \in X$.

To prove that $\{ f_I \mid I \in X \}$ spans $\Lambda^k(V)$, let $g \in \Lambda^k(V)$ and consider $h = \sum_{I \in X} g\left(e_I\right) f_I$. For each $J \in X$ we have $h\left(e_J\right) = \sum_{I \in X} g\left(e_I\right) f_I\left(e_J\right) = g\left(e_J\right)$, and so $g = h$. Thus $\{ f_I \mid I \in X \}$ spans $\Lambda^k(V)$.

To prove that $\{ f_I \mid I \in X \}$ is linearly independent, suppose $r_I \in \mathbf{R}$ are such that $\sum_{I \in X} r_I f_I = 0$. Let $J \in X$. Then $0 = \sum_{I \in X} r_I f_I\left(e_J\right) = r_J$. □

Corollary 10.19. *Let V be n-dimensional over* $\mathbf{R}$. *Then* $\Lambda^k(V) = 0$ *for* $k > n$ *and for* $0 \leq k \leq n$, $\Lambda^k(V)$ *is* $\binom{n}{k}$*-dimensional over* $\mathbf{R}$. *In particular,* $\Lambda^n(V)$ *is* 1*-dimensional over* $\mathbf{R}$. □

Theorem 10.20. *Suppose V is n-dimensional over* $\mathbf{R}$ *and let* $\alpha : V \to V$ *be an* $\mathbf{R}$*-linear transformation. Then* $\Lambda^n(\alpha) : \Lambda^n(V) \to \Lambda^n(V)$ *is given by multiplication by* $\det(\alpha)$.

Proof. Choose any basis $\{ e_1, \ldots, e_n \}$ of V and let $\{ f_1, \ldots, f_n \}$ be the basis of V^* dual to the above. By Theorem 10.18, $\Lambda^n(V)$ is 1-dimensional, spanned by $f_1 \wedge \cdots \wedge f_n$. By Propositions 10.15 and 10.10,

$$\begin{aligned} \Lambda^n(\alpha)(f_1 \wedge \cdots \wedge f_n) &= \Lambda^1(\alpha)(f_1) \wedge \cdots \wedge \Lambda^1(\alpha)(f_n) \\ &= \alpha^*(f_1) \wedge \cdots \wedge \alpha^*(f_n). \end{aligned}$$

Let $A = (a_{i,j})$ be the matrix of α with respect to $\{ e_1, \ldots, e_n \}$. Then the matrix of α^* with respect to $\{ f_1, \ldots, f_n \}$ is the transpose of A, and so $\alpha^*(f_i) = \sum_j a_{i,j} f_j$. Thus

$$\begin{aligned} \alpha^*(f_1) \wedge \cdots \wedge \alpha^*(f_n) &= \Big(\sum_j a_{1,j} f_j\Big) \wedge \cdots \wedge \Big(\sum_j a_{n,j} f_j\Big) \\ &= \sum_{j_1, \ldots, j_n} a_{1,j_1} \cdots a_{n,j_n} f_{j_1} \wedge \cdots \wedge f_{j_n}. \end{aligned}$$

Since $f \wedge f = 0$ for all $f \in \Lambda^1(V)$, it follows from graded commutativity of the wedge product (Theorem 10.14(b)) that $f_{j_1} \wedge \cdots \wedge f_{j_n} = 0$ unless $(j_1, \ldots, j_n)$ is a permutation of $(1, \ldots, n)$. Thus,

$$\Lambda^n(\alpha)(f_1 \wedge \cdots \wedge f_n) = \sum_{\sigma \in \Sigma_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} f_{\sigma(1)} \wedge \cdots \wedge f_{\sigma(n)}.$$

From Theorem 10.14(b) it follows that

$$f_{\sigma(1)} \wedge \cdots \wedge f_{\sigma(n)} = \operatorname{sgn}(\sigma) f_1 \wedge \cdots \wedge f_n$$

and so

$$\begin{aligned}\Lambda^n(\alpha)(f_1 \wedge \cdots \wedge f_n) &= \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} f_1 \wedge \cdots \wedge f_n \\ &= \det(\alpha)\, f_1 \wedge \cdots \wedge f_n. \quad \square\end{aligned}$$

For each $k \geq 0$ and real vector spaces V and W we have a function

$$\Lambda^k : \operatorname{Hom}_{\mathbf{R}}(V, W) \to \operatorname{Hom}_{\mathbf{R}}\left(\Lambda^k(W), \Lambda^k(V)\right)$$

which sends α to $\Lambda^k(\alpha)$. In general, Λ^k is *not* a linear transformation (e.g. usually $\det(\alpha+\beta) \neq \det(\alpha)+\det(\beta)$). If V and W are both finite-dimensional over $\mathbf{R}$, then both $\operatorname{Hom}_{\mathbf{R}}(V, W)$ and $\operatorname{Hom}_{\mathbf{R}}\left(\Lambda^k(W), \Lambda^k(V)\right)$ are finite-dimensional real vector spaces, and hence have their standard smooth structures.

Theorem 10.21. *If V and W are finite-dimensional real vector spaces, then for each $k \geq 0$, the function*

$$\Lambda^k : \operatorname{Hom}_{\mathbf{R}}(V, W) \to \operatorname{Hom}_{\mathbf{R}}\left(\Lambda^k(W), \Lambda^k(V)\right)$$

is smooth with respect to the standard smooth structures.

Proof. Let $\{ v_1, \ldots, v_m \}$ and $\{ w_1, \ldots, w_n \}$ be $\mathbf{R}$-bases for V and W, respectively, and write $\{ v_1^*, \ldots, v_m^* \}$ and $\{ w_1^*, \ldots, w_n^* \}$ for the respective bases dual to these for V^* and W^*. Let

$$\begin{aligned} X &= \{ (i_1, \ldots, i_k) \mid 1 \leq i_1 < \cdots < i_k \leq m \}, \\ Y &= \{ (j_1, \ldots, j_k) \mid 1 \leq j_1 < \cdots < j_k \leq n \}. \end{aligned}$$

For $I = (i_1, \ldots, i_k) \in X$ write $v_I^* = v_{i_1}^* \wedge \cdots \wedge v_{i_k}^*$ and similarly define w_J^* for $J \in Y$. Then by Theorem 10.18, $\{ v_I^* \mid I \in X \}$ and $\{ w_J^* \mid J \in Y \}$ are $\mathbf{R}$-bases for $\Lambda^k(V)$ and $\Lambda^k(W)$, respectively. Using the above bases on V and W we identify $\operatorname{Hom}_{\mathbf{R}}(V, W)$ with the space of real $n \times m$ matrices. Similarly, using the above bases on $\Lambda^k(W)$ and $\Lambda^k(V)$, $\operatorname{Hom}_{\mathbf{R}}\left(\Lambda^k(W), \Lambda^k(V)\right)$ is identified with the space of real $X \times Y$-matrices (i.e. the space of real-valued functions on $X \times Y$).

Let $\alpha : V \to W$ be a real linear transformation with matrix $(a_{j,i})$, i.e. $\alpha(v_i) = \sum_j a_{j,i} w_j$. Let $(A_{I,J})$ be the matrix of $\Lambda^k(\alpha)$, i.e. $\Lambda^k(\alpha)(w_J^*) = \sum_{I \in X} A_{I,J} v_I^*$ for all $J \in Y$. It suffices to show that the $A_{I,J}$ are smooth functions of the $a_{j,i}$. We will see, in fact, that they are polynomial functions in the $a_{j,i}$.

Let $J = (j_1, \ldots, j_k) \in Y$. We have, by Proposition 10.15,

$$\Lambda^k(\alpha)(w_J^*) = \alpha^*(w_{j_1}^*) \wedge \cdots \wedge \alpha^*(w_{j_k}^*) = \Big(\sum_i a_{j_1,i} v_i^*\Big) \wedge \cdots \wedge \Big(\sum_i a_{j_k,i} v_i^*\Big).$$

If $I = (i_1, \ldots, i_k) \in X$, the contribution to the v_I^* term in this last expression is

$$\sum_{\sigma \in \Sigma_k} a_{j_1,i_{\sigma(1)}} v_{i_{\sigma(1)}}^* \wedge \cdots \wedge a_{j_k,i_{\sigma(k)}} v_{i_{\sigma(k)}}^* = \sum_{\sigma \in \Sigma_k} \operatorname{sgn}(\sigma) a_{j_1,i_{\sigma(1)}} \cdots a_{j_k,i_{\sigma(k)}} v_I^*$$

and so $A_{I,J} = \sum_{\sigma \in \Sigma_k} \operatorname{sgn}(\sigma) a_{j_1,i_{\sigma(1)}} \cdots a_{j_k,i_{\sigma(k)}}$, which is a polynomial function in the $a_{j,i}$. □

It is interesting to note that the $A_{I,J}$ above is the determinant of the $k \times k$ submatrix

$$\begin{pmatrix} a_{j_1,i_1} & a_{j_1,i_2} & \cdots & a_{j_1,i_k} \\ a_{j_2,i_1} & a_{j_2,i_2} & \cdots & a_{j_2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j_k,i_1} & a_{j_k,i_2} & \cdots & a_{j_k,i_k} \end{pmatrix}$$

of $(a_{j,i})$.

Exercises for §10

1. Let V be a real vector space and suppose $f_1, \ldots, f_k \in V^*$.

(a) Prove that $f_1, \ldots, f_k$ are linearly independent over $\mathbf{R}$ if and only if $f_1 \wedge \cdots \wedge f_k \neq 0$.

(b) Suppose $f_1, \ldots, f_k$ and $g_1, \ldots, g_k$ are two linearly independent k-tuples in V^*. Prove that the subspace of V^* spanned by $f_1, \ldots, f_k$ is the same as the subspace spanned by $g_1, \ldots, g_k$ if and only if $f_1 \wedge \cdots \wedge f_k$ is a scalar multiple of $g_1 \wedge \cdots \wedge g_k$.

2. Let V be a real vector space (not necessarily finite-dimensional over $\mathbf{R}$), and $\alpha : V \to V$ an $\mathbf{R}$-linear transformation. Suppose $\lambda_1, \ldots, \lambda_k$ are distinct real eigenvalues of α. Prove that $\lambda_1 \lambda_2 \cdots \lambda_k$ is an eigenvalue of $\Lambda^k(\alpha)$.

3. Let V and W be finite-dimensional real vector spaces and $\alpha : V \to W$ an $\mathbf{R}$-linear transformation. Prove that the rank of α is the largest positive integer k such that $\Lambda^k(\alpha) : \Lambda^k(W) \to \Lambda^k(V)$ is not the 0-map.

11. Exterior Powers of Smooth Vector Bundles

Definition 11.1. Let $Q : VS_{\mathbf{R}} \to VS_{\mathbf{R}}$ be a contravariant functor (recall $VS_{\mathbf{R}}$ denotes the category of real vector spaces). Q is said to be a *smooth contravariant functor* if QV is finite-dimensional over $\mathbf{R}$ whenever V is, and if V and W are both finite-dimensional over $\mathbf{R}$, the function $Q : \mathrm{Hom}_{\mathbf{R}}(V, W) \to \mathrm{Hom}_{\mathbf{R}}(QW, QV)$ sending α to $Q\alpha$ is smooth with respect to the standard smooth structures.

Example 11.2. It follows from Theorem 10.21 that for each $k \geq 0$, Λ^k is a smooth contravariant functor.

Suppose $Q : VS_{\mathbf{R}} \to VS_{\mathbf{R}}$ is a smooth contravariant functor and that

$$\xi = \big(V, E, (M, \mathcal{S}_M), p, \mathcal{S}\big)$$

is a smooth vector bundle. We wish to construct a new smooth vector bundle

$$Q\xi = \big(QV, QE, (M, \mathcal{S}_M), p_Q, \mathcal{S}_Q\big)$$

where, as a set, $QE = \coprod_{x \in M} Q\big(p^{-1}(x)\big)$ and $p_Q : QE \to M$ sends the points in $p^{-1}(x)$ to x. Thus $p_Q^{-1}(x) = Q\big(p^{-1}(x)\big)$. The idea is to construct linear charts as follows: Let $\varphi : p^{-1}(U_\varphi) \to U_\varphi \times V$ be an $\mathcal{S}$-admissible linear chart. For each $x \in U_\varphi$, let $\varphi_x : p^{-1}(x) \to V$ denote the composition

$$p^{-1}(x) \xrightarrow{\ \varphi\ } \{x\} \times V \xrightarrow{\ \pi_2\ } V.$$

Then each φ_x is an $\mathbf{R}$-isomorphism, and hence $Q\varphi_x : QV \to Q\big(p^{-1}(x)\big)$ is an $\mathbf{R}$-isomorphism. Define $\varphi^Q : p_Q^{-1}(U_\varphi) \to U_\varphi \times QV$ by $\varphi^Q(y) = \big(x, (Q\varphi_x)^{-1}(y)\big)$ whenever $y \in p_Q^{-1}(x) = Q\big(p^{-1}(x)\big)$.

Proposition 11.3. *Let $(M, \mathcal{S})$ be a smooth manifold and V, W finite-dimensional real vector spaces. Suppose $f : M \times V \to W$ is a function such that for each $x \in M$, the function $f_x : V \to W$ given by $f_x(v) = f(x, v)$ is an $\mathbf{R}$-linear transformation. Define $f^\sharp : M \to \mathrm{Hom}_{\mathbf{R}}(V, W)$ by $f^\sharp(x) = f_x$. Then f is smooth if and only if $f^\sharp$ is smooth.*

Proof. Choose $\mathbf{R}$-bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ for V and W, respectively. For each $x \in M$, let $\big(a_{j,i}(x)\big)$ be the matrix of f_x with respect to the above bases. Thus for all $x \in M$

$$f(x, v_i) = f_x(v_i) = \sum_j a_{j,i}(x) w_j.$$

The $a_{j,i} : M \to \mathbf{R}$ are the coordinate functions of $f^\sharp$ with respect to basis $\{\varepsilon_{i,j}\}$ where $\varepsilon_{i,j} : V \to W$ is given by $\varepsilon_{i,j}(v_k) = \delta_{i,k} w_j$. Thus $f^\sharp$ is smooth if and only if the $a_{j,i}$ are all smooth.

Suppose f is smooth. For $1 \leq i \leq m$, let $\iota_i : M \to M \times V$ be given by $\iota_i(x) = (x, v_i)$, and for $1 \leq i \leq n$ let $\pi_j : W \to \mathbf{R}$ be given by $\pi_j\Big(\sum_q s_q w_q\Big) = s_j$. Then the ι_i and π_j are all smooth, and hence each $\pi_j f \iota_i$ is smooth, i.e. each $a_{j,i}$ is smooth. Thus $f^\sharp$ is smooth.

Suppose, conversely, $f^\sharp$ is smooth. Note that f is the composition

$$M \times V \xrightarrow{f^\sharp \times 1_V} \mathrm{Hom}_{\mathbf{R}}(V, W) \times V \xrightarrow{\mathrm{eval}} W$$

where eval : $\mathrm{Hom}_{\mathbf{R}}(V, W) \times V \to W$ is given by $\mathrm{eval}(\alpha, v) = \alpha(v)$. The map eval is smooth since its coordinate functions with respect to the w_k are polynomial functions in the coordinates with respect to the $\varepsilon_{i,j}$ and w_k. The smoothness of f follows. □

Lemma 11.4. *Let $Q : VS_{\mathbf{R}} \to VS_{\mathbf{R}}$ be a smooth contravariant functor, V, W finite-dimensional real vector spaces, $(M, \mathcal{S}_M)$ a smooth manifold, and $f : M \times V \to W$ a smooth map such that for each $x \in M$, $f_x : V \to W$ given by $f_x(v) = f(x, v)$ is $\mathbf{R}$-linear. Let $g : M \times QW \to QV$ be given by $g(x, y) = Qf_x(y)$. Then g is smooth.*

Proof. By Proposition 11.3, it suffices to show that $g^\sharp : M \to \mathrm{Hom}_{\mathbf{R}}(QW, QV)$ is smooth. It is easily checked that $g^\sharp$ is the composition

$$M \xrightarrow{f^\sharp} \mathrm{Hom}_{\mathbf{R}}(V, W) \xrightarrow{Q} \mathrm{Hom}_{\mathbf{R}}(QW, QV).$$

$f^\sharp$ is smooth by Proposition 11.3, and Q is smooth by hypothesis. □

Lemma 11.5. *Let $Q : VS_{\mathbf{R}} \to VS_{\mathbf{R}}$ be a smooth contravariant functor and $\xi = (V, E, (M, \mathcal{S}_M), p, \mathcal{S})$ a smooth vector bundle. Let φ and ψ be linear $\mathcal{S}$-admissible charts. Then $\psi^Q(\varphi^Q)^{-1}$ is a self-diffeomorphism of $(U_\varphi \cap U_\psi) \times QV$ where φ^Q and ψ^Q are as in the paragraph following Example 11.2.*

Proof. Since the restrictions of φ and ψ to fibers are $\mathbf{R}$-isomorphisms and Q is a functor, it follows that the restrictions of φ^Q and ψ^Q to fibers are $\mathbf{R}$-isomorphisms. Thus φ^Q and ψ^Q are both bijections, so it remains only to check the smoothness of $\psi^Q(\varphi^Q)^{-1}$ (the smoothness of $\varphi^Q(\psi^Q)^{-1}$ will then follow by symmetry).

Write $U = U_\varphi \cap U_\psi$. Since φ and ψ are linearly $\mathcal{S}_M$-related charts, it follows that $\varphi\psi^{-1} : U \times V \to U \times V$ is a diffeomorphism, $\pi_1\varphi\psi^{-1} = \pi_1$, and for each $x \in U$, $(\pi_2\varphi\psi^{-1})_x : V \to V$ (using the notation of Lemma 11.4) is an $\mathbf{R}$-isomorphism. Thus by Lemma 11.4, the map $g : U \times QV \to QV$ given by $g(x, y) = Q\big((\pi_2\varphi\psi^{-1})_x\big)(y)$ is smooth. Now $(\pi_2\varphi\psi^{-1})_x = \varphi_x\psi_x^{-1}$ and so $Q\big((\pi_2\varphi\psi^{-1})_x\big) = Q(\varphi_x\psi_x^{-1}) = Q(\psi_x^{-1})Q\varphi_x$.

We have

$$\psi^Q(\varphi^Q)^{-1}(x, y) = \psi^Q\big(Q\varphi_x(y)\big) = \big(x, (Q\psi_x)^{-1}Q\varphi_x(y)\big) = \big(x, g(x, y)\big)$$

and so both $\pi_1\psi^Q(\varphi^Q)^{-1}$ and $\pi_2\psi^Q(\varphi^Q)^{-1}$ are smooth, completing the proof. □

Theorem 11.6. *Let $Q : VS_{\mathbf{R}} \to VS_{\mathbf{R}}$ be a smooth contravariant functor and $\xi = (V, E, (M, \mathcal{S}_M), p, \mathcal{S})$ be a smooth vector bundle. Let QE, $p_Q : QE \to M$, and φ^Q for each linear $\mathcal{S}$-admissible chart φ be as in the paragraph following Example 11.2. Then:*

(a) *Let $\mathcal{T}_S^Q$ denote the set of all subsets A of QE such that $\varphi^Q\big(A \cap p_Q^{-1}(U_\varphi)\big)$ is open in $U_\varphi \times QV$ for each linear $\mathcal{S}$-admissible chart φ. Then $\mathcal{T}_S^Q$ is a topology on QE.*

(b) *Using the topology* $\mathcal{T}_{\mathcal{S}}^Q$ *on* QE, $p_Q : QE \to M$ *is continuous.*

(c) *Using the topology* $\mathcal{T}_{\mathcal{S}}^Q$ *on* QE, φ^Q *is a homeomorphism for each linear* $\mathcal{S}$*-admissible chart.*

(d) *The set of all* φ^Q, *as* φ *runs over the linear* $\mathcal{S}$*-admissible charts, is a linearly* $\mathcal{S}_M$*-smooth atlas for* (QV, QE, M, p_Q).

Proof. Part (a) is immediate.

Suppose U is open in M. For each admissible φ, $\varphi^Q(p_Q^{-1}(U) \cap p_Q^{-1}(U_\varphi)) = (U \cap U_\varphi) \times QV$, which is open in $U_\varphi \times QV$. Thus $p_Q^{-1}(U)$ is open in QE, proving part (b).

Let φ be a linear $\mathcal{S}$-admissible chart. Clearly, φ^Q is a bijection and an open map. Thus it remains only to prove that φ^Q is continuous to establish part (c). Let X be open in $U_\varphi \times QV$. For each linear $\mathcal{S}$-admissible chart ψ,

$$(\varphi^Q)^{-1}(X) \cap p_Q^{-1}(U_\psi) = (\varphi^Q)^{-1}\Big(X \cap \big((U_\varphi \cap U_\psi) \times QV\big)\Big)$$

and so

$$\psi^Q\Big((\varphi^Q)^{-1}(X) \cap p_Q^{-1}(U_\psi)\Big) = \psi^Q(\varphi^Q)^{-1}\Big(X \cap \big((U_\varphi \cap U_\psi) \times QV\big)\Big).$$

Since, by Lemma 11.5, $\psi^Q\Big((\varphi^Q)\Big)^{-1}$ is a self-homeomorphism (in fact, a diffeomorphism) of $(U_\varphi \cap U_\psi) \times QV$, and $X \cap \big((U_\varphi \cap U_\psi) \times QV\big)$ is open in $(U_\varphi \cap U_\psi) \times QV$, it follows that $\psi^Q(\varphi^Q)^{-1}\Big(X \cap \big((U_\varphi \cap U_\psi) \times QV\big)\Big)$ is open in $(U_\varphi \cap U_\psi) \times QV$, and hence open in $U_\psi \times QV$, proving part (c).

Part (d) is an immediate consequence of Lemma 11.5. □

Corollary 11.7. *Let* $Q : V\mathcal{S}_{\mathbf{R}} \to V\mathcal{S}_{\mathbf{R}}$ *be a smooth contravariant functor, and* $\xi = \big(V, E, (M, \mathcal{S}_M), p, \mathcal{S}\big)$ *a smooth vector bundle. Then* $Q\xi = \big(QV, QE, (M, \mathcal{S}_M), \mathcal{S}_Q\big)$ *is a smooth vector bundle where* $\mathcal{S}_Q$ *is the smooth equivalence class of the linear* $\mathcal{S}_B$*-smooth atlas* $\{\varphi^Q \mid \varphi\ \mathcal{S}\text{-admissible}\}$. □

In particular, for each smooth vector bundle ξ and $k \geq 0$ we obtain a smooth vector bundle $\Lambda^k\xi$, the k^{th} *exterior power of* ξ.

Proposition 11.8. *Suppose* C *is a finite-dimensional real vector space and* Q *the constant contravariant functor with value* C, *i.e.* $QV = C$ *for all real vector spaces* V *and* $Q\alpha = 1_C$ *for all* $\mathbf{R}$*-linear maps* α. *Then:*

(a) Q *is a continuous contravariant functor.*

(b) *If* $\xi = \big(V, E, (M, \mathcal{S}_M, p, \mathcal{S}\big)$ *is a smooth vector bundle, then* $Q\xi$ *is the product bundle with fiber* C *and the identity map on* $M \times C$ *is a linearly* $\mathcal{S}_Q$*-admissible chart.*

Proof. For all finite-dimensional real vector spaces V and W, $Q : \mathrm{Hom}_{\mathbf{R}}(V, W) \to \mathrm{Hom}_{\mathbf{R}}(C, C)$ is the constant map with value 1_C, which is smooth.

For ξ as above, $QE = \coprod_{x \in M} Q(p^{-1}(x)) = \coprod_{x \in M} C = M \times C$ (the index coordinate for the disjoint union is needed here) and for each linearly $\mathcal{S}$-admissible chart φ,

$p_Q^{-1}(U_\varphi) = U_\varphi \times C$ and φ^Q is the identity map on $U_\varphi \times C$. It follows that the identity map $1_{M\times C} : QE \to M \times C$ is a diffeomorphism where QE is given the smooth structure arising from $\mathcal{S}_Q$ and $M \times C$ the product smooth structure. $\square$

In particular, for any smooth vector bundle ξ, $\Lambda^0\xi$ is the product bundle with fiber $\mathbf{R}$.

Exercises for §11

1. If $Q : VS_{\mathbf{R}} \to VS_{\mathbf{R}}$ is a *covariant* functor, we can define what it means for Q to be smooth. Just make the obvious modifications in Definition 11.1. State and prove analogues of Lemmas 11.4 and 11.5, Theorem 11.6, and Corollary 11.7 for smooth covariant functors.

2. Show that if $Q : VS_{\mathbf{R}} \to VS_{\mathbf{R}}$ is a smooth covariant functor, then if $\xi = \big(V, X, (M, \mathcal{S}), p, \mathcal{D}\big)$ and $\eta = \big(W, Y, (N, \mathcal{T}), q, \mathcal{E}\big)$ are smooth vector bundles and $f : \xi \to \eta$ a smooth homomorphism of vector bundles, then there is a smooth homomorphism of vector bundles $Qf : Q\xi \to Q\eta$ (where the latter vector bundles are as constructed in Problem 1) such that $Qf_B = f_B$ and for each $x \in M$, the restriction $Qf_E : p_Q^{-1}(x) = Q\big(p^{-1}(x)\big) \to q_Q^{-1}\big(f_B(x)\big) = Q\Big(q^{-1}\big(f_B(x)\big)\Big)$ is Qf_x where $f_x : p^{-1}(x) \to q^{-1}\big(f_B(x)\big)$ is the restriction of f_E. Show that with these constructions, Q yields a covariant functor from the category of smooth vector bundles to itself.

(The latter fails for smooth contravariant functors Q since there is no reasonable way to define Q on morphisms.)

As remarked at the end of Problem 2 of §11, a smooth contravariant functor $Q : VS_{\mathbf{R}} \to VS_{\mathbf{R}}$ does *not* yield a contravariant functor from the category of smooth vector bundles $SmVect$ to itself since there is no reasonable way to specify Q on the morphisms of $SmVect$. However, Q *does* behave well with respect to smooth sections and smooth vector bundle homomorphisms, as we shall see next.

Theorem 12.1. *Let $\xi = \big(V, X, (M, \mathcal{S}), p, \mathcal{D}\big)$ and $\eta = \big(W, Y, (N, \mathcal{T}), q, \mathcal{E}\big)$ be smooth vector bundles and $f : \xi \to \eta$ a smooth vector bundle homomorphism. For each $x \in M$ let $f_x : p^{-1}(x) \to q^{-1}\big(f_B(x)\big)$ denote the restriction of f_E. Let $Q : VS_{\mathbf{R}} \to VS_{\mathbf{R}}$ be a smooth contravariant functor. Let $\sigma : N \to QY$ be a smooth section of $Q\eta$. Define $f^*\sigma : M \to QX$ by $(f^*\sigma)(x) = (Qf_x)\Big(\sigma\big(f_B(x)\big)\Big)$. Then $f^*\sigma$ is a smooth section of $Q\xi$.*

Proof. Clearly, $pf^*\sigma = 1_M$, so the only question is the smoothness of $f^*\sigma$. It suffices to check this locally. Thus it suffices to check that whenever φ is a linear $\mathcal{D}$-admissible chart and ψ an linear $\mathcal{E}$-admissible chart such that $f_B(U_\varphi) \subset U_\psi$, then the restriction $f^*\sigma : U_\varphi \to p_Q^{-1}(U_\varphi)$ is smooth. Since φ^Q is a diffeomorphism, it suffices to show that for φ and ψ as above, the composition

$$U_\varphi \xrightarrow{f^*\sigma} p_Q^{-1}(U_\varphi) \xrightarrow{\varphi^Q} U_\varphi \times QV$$

is smooth. Since $\pi_1\varphi^Q f^*\sigma = 1_{U_\varphi}$ which is smooth, it remains only to check smoothness of $\pi_2\varphi^Q f^*\sigma$.

$\pi_2\psi f_E \varphi^{-1} : U_\varphi \times V \to W$ is smooth and is $\mathbf{R}$-linear on fibers. Thus, by Lemma 11.4, the map $g : U_\varphi \times QW \to QV$ given by $g(x, z) = Q\big((\pi_2\psi f_E\varphi^{-1})_x\big)(z)$ is smooth. We will be done if we show that $\pi_2\varphi^Q f^*\sigma$ is equal to the composition

$$U_\varphi \xrightarrow{\Delta} U_\varphi \times U_\varphi \xrightarrow{1_{U_\varphi}\times f_B} U_\varphi \times U_\psi \xrightarrow{1_{U_\varphi}\times\sigma} U_\varphi \times q_Q^{-1}(U_\psi)$$

$$\xrightarrow{1_{U_\varphi}\times\psi^Q} U_\varphi \times U_\psi \times QW \xrightarrow{\pi_1\times 1_{QW}} U_\varphi \times QW \xrightarrow{\quad g\quad} QV$$

since all maps in this composition are smooth.

For each $x \in U_\varphi$ we have

$$\begin{aligned}(\pi_2\varphi^Q f^*\sigma)(x) &= \pi_2\varphi^Q\bigg((Qf_x)\Big(\sigma\big(f_B(x)\big)\Big)\bigg) = \pi_2\bigg(x, (Q\varphi_x^{-1})(Qf_x)\Big(\sigma\big(f_B(x)\big)\Big)\bigg)\\ &= Q(f_x\varphi_x^{-1})\Big(\sigma\big(f_B(x)\big)\Big)\end{aligned}$$

while

$$g\big(\pi_1 \times 1_{QW}\big)\big(1_{U_\varphi} \times \psi^Q\big)\big(1_{U_\varphi} \times \sigma\big)\big(1_{U_\varphi} \times f_B\big)\Delta(x) = g\big(\pi_1 \times 1_{QW}\big)\big(x, \psi^Q\sigma f_B(x)\big)$$

$$
\begin{aligned}
&= g(\pi_1 \times 1_{QW})\Big(x, f_B(x), Q(\psi^{-1}_{f_B(x)})\big(\sigma(f_B(x))\big)\Big)\\
&= g\Big(x, Q(\psi^{-1}_{f_B(x)})\big(\sigma(f_B(x))\big)\Big)\\
&= Q\big((\pi_2\psi f_E\varphi^{-1})_x\big)\Big(Q(\psi^{-1}_{f_B(x)})\Big)\big(\sigma(f_B(x))\big)\\
&= Q\big(\psi_{f_B(x)} f_x \varphi_x^{-1}\big)Q(\psi^{-1}_{f_B(x)})\big(\sigma(f_B(x))\big)\\
&= Q\big(\psi^{-1}_{f_B(x)}\psi_{f_B(x)} f_x \varphi_x^{-1}\big)\big(\sigma(f_B(x))\big) = Q(f_x\varphi_x^{-1})\big(\sigma(f_B(x))\big),
\end{aligned}
$$

completing the proof. □

$f^*\sigma$ is sometimes called the section *induced from* σ *by* f or the *pull-back of* σ *via* f.

Theorem 12.2. *Let* $Q : VS_{\mathbf{R}} \to VS_{\mathbf{R}}$ *be a smooth contravariant functor. Then the rules which assign to each smooth vector bundle* ξ *the space of smooth sections* $\Gamma(Q\xi)$, *and to each smooth vector bundle homomorphism* $f : \xi \to \eta$ *the function* $f^* : \Gamma(Q\eta) \to \Gamma(Q\xi)$, *constitute a contravariant functor from* $SmVect$ *to* $VS_{\mathbf{R}}$.

Proof. We must check the following:

(i) If $f : \xi \to \eta$ is a smooth vector bundle homomorphism, then $f^* : \Gamma(Q\eta) \to \Gamma(Q\xi)$ is $\mathbf{R}$-linear.

(ii) If $f : \xi \to \eta$ and $g : \eta \to \rho$ are smooth vector bundle homomorphisms, then $(gf)^* = f^*g^* : \Gamma(Q\rho) \to \Gamma(Q\xi)$.

(iii) $1_\xi^* = 1_{\Gamma(Q\xi)}$.

Let $f : \xi \to \eta$ be a smooth vector bundle homomorphism. Suppose σ, $\tau \in \Gamma(Q\eta)$ and $k \in \mathbf{R}$. Then for each x in the base space of ξ,

$$
\begin{aligned}
f^*(\sigma + k\tau)(x) &= (Qf_x)\Big((\sigma + k\tau)\big(f_B(x)\big)\Big) = Qf_x\Big(\sigma\big(f_B(x)\big) + k\tau\big(f_B(x)\big)\Big)\\
&= Qf_x\Big(\sigma\big(f_B(x)\big) + kQf_x\Big(\tau\big(f_B(x)\big)\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } Qf_x \text{ is } \mathbf{R}\text{-linear)}\\
&= f^*\sigma(x) + kf^*\tau(x) = (f^*\sigma + kf^*\tau)(x)
\end{aligned}
$$

and so $f^*(\sigma + k\tau) = f^*\sigma + kf^*\tau$, proving (i).

Suppose $f : \xi \to \eta$ and $g : \eta \to \rho$ are smooth vector bundle homomorphisms. For each $\sigma \in \Gamma(Q\rho)$ and x in the base space of ξ we have

$$
\begin{aligned}
(gf)^*(\sigma)(x) &= Q\big((gf)_x\big)\big(\sigma(gf)_B(x)\big) = Q\big(g_{f_B(x)}f_x\big)\Big(\sigma\Big(g_B\big(f_B(x)\big)\Big)\Big)\\
&= Q(f_x)Q\big(g_{f_B(x)}\big)\Big(\sigma\Big(g_B\big(f_B(x)\big)\Big)\Big) = (Qf_x)\Big((g^*\sigma)\big(f_B(x)\big)\Big)\\
&= f^*(g^*\sigma)(x)
\end{aligned}
$$

and so $(gf)^*(\sigma) = f^*g^*\sigma$, proving (ii).

For any smooth vector bundle ξ with projection p and any $\sigma \in \Gamma(Q\xi)$ we have, for each x in the base space of ξ,

$$(1_\xi^*\sigma)(x) = Q\big((1_\xi)_x\big)\Big(\sigma\big((1_\xi)_B(x)\big)\Big) = Q\Big(1_{p^{-1}(x)}\Big)\big(\sigma(x)\big) = \sigma(x)$$

and so $1_\xi^*\sigma = \sigma$, proving (iii). □

Definition 12.3. Let $(M, \mathcal{S})$ be a smooth manifold and $k \geq 0$. A smooth section of $\Lambda^k\tau_{M,\mathcal{S}}$ is called a *k-form on* $(M, \mathcal{S})$. The real vector space of all k-forms on $(M, \mathcal{S})$ is denoted $\Omega^k(M, \mathcal{S})$. A *differential form on* $(M, \mathcal{S})$ is a k-form on $(M, \mathcal{S})$ for some $k \geq 0$.

If $(M, \mathcal{S})$ and $(N, \mathcal{T})$ are smooth manifolds and $f : M \to N$ is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$ (sometimes expressed by writing $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ is smooth), the tangent bundle functor τ yields a smooth vector bundle homomorphism $\tau f : \tau_{M,\mathcal{S}} \to \tau_{N,\mathcal{T}}$. If σ is a k-form on $(N, \mathcal{T})$, we obtain, from Theorem 12.1, a k-form $(\tau f)^*\sigma$ on $(M, \mathcal{S})$. We abbreviate $(\tau f)^*\sigma$ by $f^*\sigma$.

Composing the covariant functor $\tau : Sm \to SmVect$ with the contravariant functor $\Gamma\Lambda^k : SmVect \to VS_{\mathbf{R}}$ of Theorem 12.2 (with $Q = \Lambda^k$), we obtain:

Corollary 12.4. *For each $k \geq 0$, the rules which assign to each smooth manifold $(M, \mathcal{S})$ its vector space of k-forms $\Omega^k(M, \mathcal{S})$, and to each smooth map $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ the* **R***-linear transformation $f^* : \Omega^k(N, \mathcal{T}) \to \Omega^k(M, \mathcal{S})$, constitute a contravariant functor from the category of smooth manifolds Sm to the category of real vector spaces $VS_{\mathbf{R}}$.* □

We next proceed to show that the wedge product operation, performed fiberwise, gives rise to a wedge product operation on the smooth sections of the exterior powers of a smooth vector bundle, in particular on differential forms.

Theorem 12.5. *Let $\xi = \big(V, E, (M, \mathcal{S}), p, \mathcal{E}\big)$ be a smooth vector bundle. Let i, $j \geq 0$ and suppose $\sigma \in \Gamma(\Lambda^i\xi)$, $\tau \in \Gamma(\Lambda^j\xi)$. Then $\sigma \wedge \tau : M \to \Lambda^{i+j}E$ given by $(\sigma \wedge \tau)(x) = \sigma(x) \wedge \tau(x)$ for all $x \in M$ is in $\Gamma(\Lambda^{i+j}\xi)$.*

Proof. The only question is the smoothness of $\sigma \wedge \tau$. It suffices to check this locally. Thus it suffices to check that for each linear $\mathcal{E}$-admissible chart φ, the restriction of $\sigma \wedge \tau$ to U_φ is smooth.

For any $k \geq 0$ write p_k for the projection map of the vector bundle $\Lambda^k\xi$, and φ^k instead of φ^{Λ^k} for the linear Λ^kV-bundle chart for $\Lambda^k\xi$ as constructed in §11. Since the map $\wedge : \Lambda^iV \times \Lambda^jV \to \Lambda^{i+j}V$ given by $\wedge(a, b) = a \wedge b$ is **R**-bilinear, its coordinate functions with respect to any choice of bases are homogeneous quadratic polynomials in the coordinates, and hence $\wedge$ is a smooth map. The result will now follow if we check that the restriction of $\sigma \wedge \tau$ to U_φ is the composition

$$U_\varphi \xrightarrow{\Delta} U_\varphi \times U_\varphi \xrightarrow{\sigma\times\tau} p_i^{-1}(U_\varphi) \times p_j^{-1}(U_\varphi) \xrightarrow{\varphi^i\times\varphi^j} U_\varphi \times \Lambda^iV \times U_\varphi \times \Lambda^jV$$
$$\xrightarrow{1_{U_\varphi\times\Lambda^iV}\times\pi_2} U_\varphi \times \Lambda^iV \times \Lambda^jV \xrightarrow{1_{U_\varphi}\times\wedge} U_\varphi \times \Lambda^{i+j}V \xrightarrow{(\varphi^{i+j})^{-1}} p_{i+j}^{-1}(U_\varphi)$$

since all maps in this composition are smooth.

Any $x \in U_\varphi$ is sent, by the above composition, to

$$(\varphi^{i+j})^{-1}(1_{U_\varphi} \times \wedge)(1_{U_\varphi \times \Lambda^i V} \times \pi_2)\Big(\varphi^i(\sigma(x)), \varphi^j(\tau(x))\Big)$$

$$= (\varphi^{i+j})^{-1}(1_{U_\varphi} \times \wedge)(1_{U_\varphi \times \Lambda^i V} \times \pi_2)\Big(x, (\Lambda^i\varphi_x)^{-1}(\sigma(x)), x, (\Lambda^j\varphi_x)^{-1}(\tau(x))\Big)$$

(by definition of φ^k)

$$= (\varphi^{i+j})^{-1}(1_{U_\varphi} \times \wedge)\Big(x, (\Lambda^i\varphi_x)^{-1}(\sigma(x)), (\Lambda^j\varphi_x)^{-1}(\tau(x))\Big)$$

$$= (\varphi^{i+j})^{-1}\Big(x, (\Lambda^i\varphi_x)^{-1}(\sigma(x)) \wedge (\Lambda^j\varphi_x)^{-1}(\tau(x))\Big)$$

$$= (\Lambda^{i+j}\varphi_x)\Big((\Lambda^i\varphi_x)^{-1}(\sigma(x)) \wedge (\Lambda^j\varphi_x)^{-1}(\tau(x))\Big)$$

(by definition of φ^k)

$$= (\Lambda^i\varphi_x)(\Lambda^i\varphi_x)^{-1}(\sigma(x)) \wedge (\Lambda^j\varphi_x)(\Lambda^j\varphi_x)^{-1}(\tau(x))$$

(by Proposition 10.15)

$$= \sigma(x) \wedge \tau(x) = (\sigma \wedge \tau)(x). \qquad \square$$

The following is immediate from Theorem 10.14 and Proposition 10.15:

Proposition 12.6. *Let ξ be a smooth vector bundle. Suppose $\sigma \in \Gamma(\Lambda^i\xi)$, $\tau \in \Gamma(\Lambda^j\xi)$, $\mu \in \Gamma(\Lambda^k\xi)$, and $c \in \mathbf{R}$. Then:*

(a) $\sigma \wedge (\tau \wedge \mu) = (\sigma \wedge \tau) \wedge \mu$.

(b) $\tau \wedge \sigma = (-1)^{ij}\sigma \wedge \tau$.

(c) *If $j = k$, then $\sigma \wedge (\tau + \mu) = \sigma \wedge \tau + \sigma \wedge \mu$.*

(d) $(c\sigma) \wedge \tau = \sigma \wedge (c\tau) = c(\sigma \wedge \tau)$.

(e) $1 \wedge \sigma = \sigma \wedge 1 = \sigma$ *where $1 \in \Gamma(\Lambda^0\xi)$ is the section given by $1(x) = (x, 1)$ for each x in the base space of ξ.*

(f) *Suppose η is another smooth vector bundle and $f : \eta \to \xi$ is a smooth vector bundle homomorphism. Then $f^*(\sigma \wedge \tau) = f^*(\sigma) \wedge f^*(\tau)$.* $\square$

Corollary 12.7. *Let $(M, \mathcal{S})$ be a smooth manifold. Suppose $\sigma \in \Omega^i(M, \mathcal{S})$, $\tau \in \Omega^j(M, \mathcal{S})$, $\mu \in \Omega^k(M, \mathcal{S})$, and $c \in \mathbf{R}$. Then:*

(a) $\sigma \wedge (\tau \wedge \mu) = (\sigma \wedge \tau) \wedge \mu$.

(b) $\tau \wedge \sigma = (-1)^{ij}\sigma \wedge \tau$.

(c) *If $j = k$, then $\sigma \wedge (\tau + \mu) = \sigma \wedge \tau + \sigma \wedge \mu$.*

(d) $(c\sigma) \wedge \tau = \sigma \wedge (c\tau) = c(\sigma \wedge \tau)$.

(e) $1 \wedge \sigma = \sigma \wedge 1 = \sigma$ *where $1 \in \Omega^0(M, \mathcal{S})$ is the 0-form given by $1(x) = (x, 1)$ for each $x \in M$.*

(f) *Suppose $(N, \mathcal{T})$ is another smooth manifold and $f : N \to M$ is a smooth map with respect to $\mathcal{T}$ and $\mathcal{S}$. Then $f^*(\sigma \wedge \tau) = f^*(\sigma) \wedge f^*(\tau)$.* $\square$

Proposition 12.8. *Let $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ be a smooth map and σ a k-form on $(N, \mathcal{T})$. Let $x \in M$ and $v_1, \ldots, v_k \in T_x(M, \mathcal{S})$. Then*

$$(f^*\sigma)(x)(v_1, \ldots, v_k) = \sigma\big(f(x)\big)\big(T_x f(v_1), \ldots, T_x f(v_k)\big).$$

Proof. From the definition of $f^*\sigma$ (following Definition 12.3) we have

$$(f^*\sigma)(x) = \Lambda^k(T_x f)\Big(\sigma\big(f(x)\big)\Big).$$

From the definition of the **R**-homomorphism $\Lambda^k(T_x f)$ (Proposition 10.9),

$$\Lambda^k(T_x f)\Big(\sigma\big(f(x)\big)\Big)(v_1, \ldots, v_k) = \sigma\big(f(x)\big)\big(T_x f(v_1), \ldots, T_x f(v_k)\big). \qquad \square$$

Note that for any smooth vector bundle ξ with base space M, a smooth section of $\Lambda^0\xi$ is a smooth map $M \to M \times \mathbf{R}$ which sends x to $\big(x, \sigma(x)\big)$ where $\sigma : M \to \mathbf{R}$ is smooth. Thus $\Gamma(\Lambda^0\xi)$ can be identified with the real vector space of smooth real-valued maps on M. In particular, $\Omega^0(M, \mathcal{S})$ is identified with $C^\infty(M, \mathcal{S})$, the **R**-algebra of smooth real-valued maps on M.

We can also view the identification of $\Omega^0(M, \mathcal{S})$ with $C^\infty(M, \mathcal{S})$ as follows: Associate to $\sigma \in \Omega^0(M, \mathcal{S})$ the real-valued function on M which sends x to $\sigma(x)\big(()\big)$ where $()$ is the unique 0-tuple of elements of $T_x(M, \mathcal{S})$ (see the remarks following Definition 10.8).

Proposition 12.9. *Let $(M, \mathcal{S})$ be a smooth manifold. Identify $\Omega^0(M, \mathcal{S})$ with $C^\infty(M, \mathcal{S})$ by the rule which assigns to each $\sigma \in \Omega^0(M, \mathcal{S})$ the real-valued function which sends x to $\sigma(x)()$. Then:*

(a) *If $\sigma \in \Omega^0(M, \mathcal{S}) = C^\infty(M, \mathcal{S})$ and $\tau \in \Omega^i(M, \mathcal{S})$, then for all $x \in M$, $(\sigma \wedge \tau)(x) = \sigma(x)\tau(x)$.*

(b) *If $f : (N, \mathcal{T}) \to (M, \mathcal{S})$ is a smooth map, then for any $\sigma \in \Omega^0(M, \mathcal{S}) = C^\infty(M, \mathcal{S})$, the pull-back $f^*\sigma \in \Omega^0(N, \mathcal{T}) = C^\infty(N, \mathcal{T})$ is the composition $\sigma f : N \to \mathbf{R}$.*

Proof. Let $x \in M$. By definition (see Theorem 12.5), $(\sigma \wedge \tau)(x) = \sigma(x) \wedge \tau(x)$. By Definition 10.11, $\sigma(x) \wedge \tau(x) = A\big(\sigma(x) \cdot \tau(x)\big)$. Note that for any i-tuple $v_1, \ldots, v_i$ in $T_x(M, \mathcal{S})$,

$$\big(\sigma(x) \cdot \tau(x)\big)(v_1, \ldots, v_i) = \sigma(x)()\tau(x)(v_1, \ldots, v_i).$$

Since $\sigma(x)$ is 0-linear and $\tau(x)$ is alternating, it follows easily that $\sigma(x) \cdot \tau(x)$ is alternating and hence $A\big(\sigma(x) \cdot \tau(x)\big) = \sigma(x) \cdot \tau(x)$. Thus

$$\big(\sigma(x) \wedge \tau(x)\big)(v_1, \ldots, v_i) = \sigma(x)()\tau(x)(v_1, \ldots, v_i)$$

and so $\sigma(x) \wedge \tau(x) = \sigma(x)()\tau(x)$, proving part (a).

If $f : (N, \mathcal{T}) \to (M, \mathcal{S})$ is smooth and $y \in N$, then by Proposition 12.8, $(f^*\sigma)(y)() = \sigma\big(f(y)\big)()$, yielding part (b). $\square$

We next look at differential forms for open subsets of Euclidean spaces. The reason for calling these gadgets "differential forms" will become apparent from this case. The standard smooth structures are assumed throughout, and we suppress notation for these.

If U is an open subset of $\mathbf{R}^m$, the standard smooth structure on U is represented by the one-chart atlas $\{1_U\}$ and consequently the tangent bundle of U admits a one-chart admissible atlas $\{\widetilde{1_U}\}$ (see the material preceding Proposition 8.14). Thus $\widetilde{1_U} : T(U) \to U \times \mathbf{R}^m$ is given by $\widetilde{1_U}(v) = \big(x, \theta_x^{-1}(v)\big)$ for $v \in T_x(U)$ where θ_x is defined in Example 5.6. The following is an immediate consequence of Proposition 5.12.

Proposition 12.10. *Let U be open in $\mathbf{R}^m$, V open in $\mathbf{R}^n$, and $f : U \to V$ a smooth map. Define $\widetilde{T}f : U \times \mathbf{R}^m \to V \times \mathbf{R}^n$ by $\widetilde{T}f(x,y) = \big(f(x), Df(x)(y)\big)$. Then the diagram*

$$\begin{array}{ccc} T(U) & \xrightarrow{\;Tf\;} & T(V) \\ {\scriptstyle \widetilde{1_U}}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{1_V}} \\ U \times \mathbf{R}^m & \xrightarrow[\widetilde{T}f]{} & V \times \mathbf{R}^n \end{array}$$

commutes. □

Thus, the tangent map Tf is identified with $\widetilde{T}f$ via the diffeomorphisms $\widetilde{1_U}$ and $\widetilde{1_V}$.

For $i \geq 0$ let $\widetilde{1_U}^{\,i} : \Lambda^i T(U) \to U \times \Lambda^i(\mathbf{R}^m)$ denote the chart for $\Lambda^i \tau_U$ arising from $\widetilde{1_U}$ by the construction in §11. Thus $\widetilde{1_U}^{\,i}$ is a diffeomorphism which is $\mathbf{R}$-linear on fibers. Note that each i-form σ on U has the form $\sigma(x) = \big(\widetilde{1_U}^{\,i}\big)^{-1}\big(x, \widetilde{\sigma}(x)\big)$ for a unique smooth map $\widetilde{\sigma} : U \to \Lambda^i(\mathbf{R}^m)$, and conversely any such smooth $\widetilde{\sigma}$ yields an i-form by the above formula.

Notation 12.11. Let $e_1, \ldots, e_m$ denote the standard basis of $\mathbf{R}^m$, and $e_1^*, \ldots, e_m^*$ the basis dual to this for $(\mathbf{R}^m)^* = \Lambda^1(\mathbf{R}^m)$. Let U be open in $\mathbf{R}^m$. For $1 \leq i \leq m$, we denote by dx_i the 1-form on U given by $dx_i(x) = \big(\widetilde{1_U}^{\,1}\big)^{-1}(x, e_i^*)$.

If $I = (i_1, \ldots, i_k)$ where $1 \leq i_j \leq m$ for each j, let e_I^* denote $e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* \in \Lambda^k(\mathbf{R}^m)$ and dx_I the k-form given by $dx_I(x) = \big(\widetilde{1_U}^{\,k}\big)^{-1}(x, e_I^*)$.

Lemma 12.12. *Let U be open in $\mathbf{R}^m$ and $I = (i_1, \ldots, i_k)$ where $1 \leq i_j \leq m$ for each j. Then $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Omega^k(U)$.*

Proof. Recall that $\big(\widetilde{1_U}^{\,k}\big)^{-1}$ is given by $\big(\widetilde{1_U}^{\,k}\big)^{-1}(x,y) = \Lambda^k(\widetilde{1_U})_x(y)$. Thus,

$$\begin{aligned} dx_I(x) &= \Lambda^k(\widetilde{1_U})_x(e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*) \\ &= \Lambda^1(\widetilde{1_U})_x(e_{i_1}^*) \wedge \cdots \wedge \Lambda^1(\widetilde{1_U})_x(e_{i_k}^*) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Proposition 10.15)} \\ &= dx_{i_1}(x) \wedge \cdots \wedge dx_{i_k}(x) = \big(dx_{i_1} \wedge \cdots \wedge dx_{i_k}\big)(x). \end{aligned}$$ □

Proposition 12.13. *For $1 \leq k \leq m$ let $X_{m,k}$ denote the set of all sequences $I = (i_1, \ldots, i_k)$ such that $1 \leq i_1 < \cdots < i_k \leq m$. Let U be open in $\mathbf{R}^m$. Then each k-form on U can be expressed uniquely in the form*

$$\sum_{I \in X_{m,k}} f_I \, dx_I$$

where the $f_I \in C^\infty(U)$.

Proof. This follows immediately from the fact that $\{\, e_I^* \mid I \in X_{m,k} \,\}$ is an $\mathbf{R}$-basis for $\Lambda^k(\mathbf{R}^m)$ (by Theorem 10.18) and so every smooth map $\widetilde{\sigma} : U \to \Lambda^k(\mathbf{R}^m)$ can be expressed uniquely in the form $\widetilde{\sigma}(x) = \sum_{I \in X_{m,k}} f_I(x) e_I^*$ for $x \in U$ where the $f_I : U \to \mathbf{R}$ are smooth. □

Theorem 12.14. *Suppose U is open in $\mathbf{R}^m$, V open in $\mathbf{R}^n$, and $f : U \to V$ is smooth. Then for $1 \le i \le n$,*

$$f^*(dx_i) = \sum_{j=1}^{m} D_j f_i \, dx_j$$

where f_i is the i^{th} coordinate function of f.

Proof. Let $x \in U$. By Proposition 12.10 the diagram

$$\begin{array}{ccc} T_x(U) & \xrightarrow{T_x f} & T_{f(x)}(V) \\ {\scriptstyle (\widetilde{1_U})_x}\big\downarrow & & \big\downarrow{\scriptstyle (\widetilde{1_V})_{f(x)}} \\ \mathbf{R}^m & \xrightarrow[Df(x)]{} & \mathbf{R}^n \end{array}$$

commutes, and the vertical maps are $\mathbf{R}$-isomorphisms. Write $\sigma_1 = f^*(dx_i)$, $\sigma_2 = \sum_{j=1}^m D_j f_i \, dx_j$. It suffices to check that for $1 \le q \le m$,

$$\sigma_1(x)\big((\widetilde{1_U})_x^{-1}(e_q)\big) = \sigma_2(x)\big((\widetilde{1_U})_x^{-1}(e_q)\big).$$

We have

$$\begin{aligned} \sigma_1(x)\big((\widetilde{1_U})_x^{-1}(e_q)\big) &= f^*(dx_i)(x)\big((\widetilde{1_U})_x^{-1}(e_q)\big) \\ &= dx_i\big(f(x)\big)\big((T_x f)(\widetilde{1_U})_x^{-1}(e_q)\big) \\ &\qquad\qquad\qquad\qquad \text{(by Proposition 12.8)} \\ &= dx_i\big(f(x)\big)\big((\widetilde{1_V})_{f(x)}^{-1} Df(x)(e_q)\big) \\ &\qquad\qquad \text{(by commutativity of the above diagram)} \\ &= (\widetilde{1_V^1})^{-1}\big(f(x), e_i^*\big)\big((\widetilde{1_V})_{f(x)}^{-1} Df(x)(e_q)\big) \\ &\qquad\qquad\qquad\qquad \text{(by definition of } dx_i) \\ &= \Lambda^1\big((\widetilde{1_V})_{f(x)}\big)(e_i^*)\big((\widetilde{1_V})_{f(x)}^{-1} Df(x)(e_q)\big) \\ &\qquad\qquad\qquad\qquad \text{(by definition of } \widetilde{1_V^1}) \\ &= e_i^*\big((\widetilde{1_V})_{f(x)}(\widetilde{1_V})_{f(x)}^{-1} Df(x)(e_q)\big) \\ &\qquad\qquad\qquad \text{(since } \Lambda^1 \text{ is the dual space functor)} \\ &= e_i^*\big(Df(x)(e_q)\big) = e_i^*\left(\sum_{j=1}^{n} D_q f_j(x) \, e_j\right) = D_q f_i(x), \end{aligned}$$

while

$$\sigma_2(x)\big((\widetilde{1_U})_x^{-1}(e_q)\big) = \left(\sum_{j=1}^{m} D_j f_i(x) \, dx_j(x)\right)\big((\widetilde{1_U})_x^{-1}(e_q)\big)$$

$$= \sum_{j=1}^{m} D_j f_i(x)(\widetilde{1_U}^{1})^{-1}(x, e_j^*)\big((\widetilde{1_U})_x^{-1}(e_q)\big)$$
(by definition of dx_j)

$$= \sum_{j=1}^{m} D_j f_i(x)\Lambda^1\big((\widetilde{1_U})_x\big)(e_j^*)\big((\widetilde{1_U})_x^{-1}(e_q)\big)$$
(by definition of $\widetilde{1_U}^{1}$)

$$= \sum_{j=1}^{m} D_j f_i(x)(e_j^*)\big((\widetilde{1_U})_x(\widetilde{1_U})_x^{-1}(e_q)\big)$$
(since Λ^1 is the dual space functor)

$$= \sum_{j=1}^{m} D_j f_i(x)e_j^*(e_q) = D_q f_i(x). \qquad \square$$

Thus, for smooth maps between open subsets of Euclidean spaces, Proposition 12.9(b) and Theorem 12.14 tell us how to pull back 0-forms and 1-forms. Thus, since pull-backs preserve wedge products (Corollary 12.7(f)), Propositions 12.9(a) and 12.13 and Lemma 12.12 enable us to pull back arbitrary differential forms in the case of smooth maps between open subsets of Euclidean space.

The Local Property for Tangent Spaces (Proposition 5.11) yields the following:

Proposition 12.15. *Suppose $(M, \mathcal{S})$ and $(N, \mathcal{T})$ are smooth manifolds, $x \in M$ and $f, g : (M, \mathcal{S}) \to (N, \mathcal{T})$ smooth maps which agree in some neighborhood of x in M. Then for each k-form σ on $(N, \mathcal{T})$, $(f^*\sigma)(x) = (g^*\sigma)(x)$.*

Proof. Say U is an open neighborhood of x in M on which f and g agree. Let $i : U \to M$ denote the inclusion. Then $fi = gi$.

It suffices to show that whenever $v_1, \ldots, v_k \in T_x(M, \mathcal{S})$, then

$$(f^*\sigma)(x)(v_1, \ldots, v_k) = (g^*\sigma)(x)(v_1, \ldots, v_k).$$

By Proposition 5.11, $T_x i : T_x(U, \mathcal{S}|U) \to T_x(M, \mathcal{S})$ is an $\mathbf{R}$-isomorphism and so each v_i can be written $T_x i(w_i)$ for some $w_i \in T_x(U, \mathcal{S}|U)$. Thus

$$\begin{aligned}(f^*\sigma)(x)(v_1, \ldots, v_k) &= (f^*\sigma)(x)\big(T_x i(w_1), \ldots, T_x i(w_k)\big)\\ &= \sigma\big(f(x)\big)\big((T_x f)(T_x i)(w_1), \ldots, (T_x f)(T_x i)(w_k)\big)\\ &\qquad \text{(by Proposition 12.8)}\\ &= \sigma\big((fi)(x)\big)\big(T_x(fi)(w_1), \ldots, T_x(fi)(w_k)\big)\\ &\qquad \text{(by Proposition 5.9(b)).}\end{aligned}$$

Similarly, $(g^*\sigma)(x)(v_1, \ldots, v_k) = \sigma\big((gi)(x)\big)\big(T_x(gi)(w_1), \ldots, T_x(gi)(w_k)\big)$. The result now follows since $fi = gi$. $\square$

Exercises for §12

1.(a) Let V be a finite-dimensional real vector space. Prove that for each $k \geq 0$, the function

$$f : \Lambda^k(V) \times V^k \to \mathbf{R}$$

given by $f(\alpha, v_1, \ldots, v_k) = \alpha(v_1, \ldots, v_k)$ is smooth.

(b) Let σ be a k-form on the smooth manifold $(M, \mathcal{S})$ and suppose

$$v_1, \ldots, v_k : M \to T(M, \mathcal{S})$$

are smooth vector fields. Define $f : M \to \mathbf{R}$ by

$$f(x) = \sigma(x)\big(v_1(x), \ldots, v_k(x)\big).$$

Prove that f is smooth.

2. Write $x_i : \mathbf{R}^n \to \mathbf{R}$ for the i^{th} coordinate function, i.e. $x_i(t_1, \ldots, t_n) = t_i$. Let $f : \mathbf{R}^3 \to \mathbf{R}^2$ be given by $f(x, y, z) = (xy^2, x + z^2)$. Calculate $f^*(x_1 x_2 \wedge dx_1 \wedge dx_2)$. Express your answer in the form $\alpha \wedge dx_2 \wedge dx_3 + \beta \wedge dx_1 \wedge dx_3 + \gamma \wedge dx_1 \wedge dx_2$ where $\alpha, \beta, \gamma : \mathbf{R}^3 \to \mathbf{R}$ are smooth.

3. Let $f : \mathbf{R} \to S^1$ be given by $f(t) = (\cos t, \sin t)$ and let $i : S^1 \to \mathbf{R}^2$ denote the inclusion map.

(a) Show that f is an immersion.

(b) Write x, y instead of x_1, x_2. Let $\omega = i^*(y \wedge dx - x \wedge dy)$. Show that for each $z \in S^1$, $\omega(z) \neq 0$.

(c) Let $\sigma = i^*(y \wedge dx + x \wedge dy)$. Find all $z \in S^1$ for which $\sigma(z) = 0$.

13. Exterior Differentiation, Exact and Closed Forms, and de Rham Cohomology

For each smooth manifold $(M, \mathcal{S})$ and each $k \geq 0$ we will define a function $d : \Omega^k(M, \mathcal{S}) \to \Omega^{k+1}(M, \mathcal{S})$ called the *exterior derivative*. In contrast to wedge products and pull-backs of smooth sections, which exist for exterior powers of arbitrary smooth vector bundles (not just tangent bundles), exterior differentiation is special to differential forms. We will first define d in the case of open subsets of Euclidean space and establish properties of d in this case. We will then extend the definition and properties to the general case by means of charts.

Definition 13.1. Suppose U is open in $\mathbf{R}^m$. If $f \in \Omega^0(U) = C^\infty(U)$ the *exterior derivative of* f, denoted df, is defined by

$$df = \sum_{i=1}^{m} D_i f \wedge dx_i.$$

If $\omega = \sum_{I \in X_{m,k}} f_I \wedge dx_I \in \Omega^k(U)$, the *exterior derivative of* ω, denoted $d\omega$, is defined by

$$d\omega = \sum_{I \in X_{m,k}} df_I \wedge dx_I \in \Omega^{k+1}(U).$$

Example 13.2. For $1 \leq i \leq m$ let $x_i : U \to \mathbf{R}$ denote the i^{th} coordinate map (as is commonly done in calculus). Then

$$dx_i = \sum_{j=1}^{m} D_j x_i \, dx_j = \sum_{j=1}^{m} \delta_{ij} \, dx_j = dx_i$$

where the extreme left d is the exterior derivative operator and the other dx_k are as defined in Notation 12.11. Thus Notation 12.11 is consistent with Definition 13.1 if we use x_i to denote the i^{th} coordinate map. If, say, $t_1, \ldots, t_m$ were used to denote the coordinate maps in some example, then $dt_1, \ldots, dt_n$ would be used in place of $dx_1, \ldots, dx_n$ in Notation 12.11.

Proposition 13.3. *Let U be open in $\mathbf{R}^n$. Then for each $k \geq 0$, $d : \Omega^k(U) \to \Omega^{k+1}(U)$ is an $\mathbf{R}$-linear transformation.*

Proof. The result is immediate from the $\mathbf{R}$-linearity of the operators D_i and the $\mathbf{R}$-bilinearity of $\wedge$. □

Lemma 13.4. *Let U be open in $\mathbf{R}^m$, $k \geq 0$, and $I = (i_1, \ldots, i_k)$ where $1 \leq i_j \leq m$ for $1 \leq j \leq k$. (We do not assume $I \in X_{m,k}$ here. If $k = 0$, then I is the empty sequence and we make the convention that $dx_I = 1 \in \Omega^0(U)$.) Then for all $f \in \Omega^0(U)$, $d(f \wedge dx_I) = df \wedge dx_I$.*

Proof. The result is trivial if the i_j are not distinct, for it then follows from Lemma 12.12 and Corollary 12.7(b) that $dx_I = 0$ and so both sides are 0. The result is also trivial if $k = 0$ since $1 \in \Omega^0(U)$ is an identity element for the wedge product (Corollary 12.7(e)).

Suppose $k > 0$ and the i_j are distinct. Let $\sigma \in \Sigma_k$ be the permutation such that $J = \big(i_{\sigma(1)}, \ldots, i_{\sigma(k)}\big) \in X_{m,k}$. By Lemma 12.12 and Corollary 12.7(b), $dx_I = \operatorname{sgn}(\sigma)\, dx_J$ and so

$$\begin{aligned} d(f \wedge dx_I) &= d\big(f \wedge \operatorname{sgn}(\sigma)\, dx_J\big) = \operatorname{sgn}(\sigma)\, d(f \wedge dx_J) && \text{(by Proposition 13.3)} \\ &= \operatorname{sgn}(\sigma)\, df \wedge dx_J && \text{(by Definition 13.1)} \\ &= df \wedge \operatorname{sgn}(\sigma)\, dx_J = df \wedge dx_I. \qquad \square \end{aligned}$$

Lemma 13.5. *Let U be open in $\mathbf{R}^m$ and $I = (i_1, \ldots, i_k)$ where $k \geq 1$ and $1 \leq i_j \leq m$ for each j. Then $d(dx_I) = 0$.*

Proof. $dx_I = 1 \wedge dx_I$ and so by Lemma 13.4, $d(dx_I) = d(1) \wedge dx_I = 0 \wedge dx_I = 0$. $\square$

Proposition 13.6. *Let U be open in $\mathbf{R}^m$. Let $\omega \in \Omega^i(U)$ and $\rho \in \Omega^j(U)$, i, $j \geq 0$. Then*

$$d(\omega \wedge \rho) = (d\omega) \wedge \rho + (-1)^i \omega \wedge (d\rho).$$

Proof. Since both sides are $\mathbf{R}$-bilinear in ω and ρ, it suffices to prove the result when $\omega = f \wedge dx_I$ and $\rho = g \wedge dx_J$ where f, $g \in \Omega^0(U) = C^\infty(U)$ and $I \in X_{m,i}$, $J \in X_{m,j}$. Then

$$\omega \wedge \rho = f \wedge dx_I \wedge g \wedge dx_J = f \wedge g \wedge dx_I \wedge dx_J = (f \wedge g) \wedge dx_{IJ}$$

where IJ is the sequence obtained by juxtaposing I and J. (Note that g and dx_I commute since g is a 0-form.) Thus,

$$\begin{aligned} d(\omega \wedge \rho) &= d\big((f \wedge g) \wedge dx_{IJ}\big) = d(f \wedge g) \wedge dx_{IJ} && \text{(by Lemma 13.4)} \\ &= \sum_{k=1}^{m} D_k(f \wedge g) \wedge dx_k \wedge dx_{IJ} && \text{(by Definition 13.1)} \\ &= \sum_{k=1}^{m} \Big((D_k f) \wedge g + f \wedge (D_k g)\Big) \wedge dx_k \wedge dx_{IJ} \end{aligned}$$

(by the product rule since for 0-forms, $\wedge$ is ordinary multiplication of real-valued functions)

$$\begin{aligned} &= \sum_{k=1}^{m} (D_k f) \wedge g \wedge dx_k \wedge dx_I \wedge dx_J + \sum_{k=1}^{m} f \wedge (D_k g) \wedge dx_k \wedge dx_I \wedge dx_J \\ &= \sum_{k=1}^{m} (D_k f) \wedge dx_k \wedge dx_I \wedge g \wedge dx_J + \sum_{k=1}^{m} (-1)^i f \wedge dx_I \wedge (D_k g) \wedge dx_k \wedge dx_J \end{aligned}$$

(by graded commutativity (Corollary 12.7(b)); g and $D_k g$ are 0-forms, dx_k a 1-form, and dx_I an i-form)

$$\begin{aligned} &= df \wedge dx_I \wedge g \wedge dx_J + (-1)^i f \wedge dx_I \wedge dg \wedge dx_J \\ &= (d\omega) \wedge \rho + (-1)^i \omega \wedge (d\rho). \qquad \square \end{aligned}$$

Proposition 13.7. *Let U be open in $\mathbf{R}^m$. Then for all $k \geq 0$ and all $\omega \in \Omega^k(U)$, $d(d\omega) = 0$.*

Proof. By $\mathbf{R}$-linearity of d it suffices to treat the case $\omega = f \wedge dx_I$ where $f \in \Omega^0(U)$ and I is a sequence of length k. We have

$$\begin{aligned} d\big(d(f \wedge dx_I)\big) &= d\big((df) \wedge dx_I)\big) \\ &\qquad\qquad \text{(by Lemma 13.4)} \\ &= d(df) \wedge dx_I - d(dx_I) \\ &\qquad\qquad \text{(by Proposition 13.6)} \\ &= d(df) \wedge dx_I \\ &\qquad\qquad \text{(by Lemma 13.5)} \end{aligned}$$

and so it remains only to show $d(df) = 0$ for all $f \in \Omega^0(U)$.

We have

$$d(df) = d\Big(\sum_{i=1}^{m} D_i f \wedge dx_i\Big) = \sum_{i=1}^{m} d(D_i f) \wedge dx_i$$

(by Definition 13.1, Proposition 13.3, and Lemma 13.4)

$$= \sum_{i=1}^{m} \Big(\sum_{j=1}^{m} D_j D_i f \wedge dx_j\Big) \wedge dx_i$$

$$= \sum_{i=1}^{m} D_i D_i f \wedge dx_i \wedge dx_i + \sum_{1 \leq i < j \leq m} \Big(D_j D_i f \wedge dx_j \wedge dx_i + D_i D_j f \wedge dx_i \wedge dx_j\Big).$$

Since $dx_i \wedge dx_i = 0$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$ for all i, j by Corollary 12.7(b), and $D_i D_j f = D_j D_i f$ by Advanced Calculus, the result follows. □

Remark 13.8. The $\mathbf{R}$-isomorphism $\iota : \mathbf{R}^m \to (\mathbf{R}^m)^*$ determined by $\iota(e_i) = e_i^*$ for each i yields, for each open subset U of $\mathbf{R}^m$, a smooth vector bundle isomorphism $\iota_U : \tau_U \to \Lambda^1 \tau_U$ with map of total spaces $(\iota_U)_E$ the composition

$$T(U) \xrightarrow{\widetilde{1_U}} U \times \mathbf{R}^m \xrightarrow{1_U \times i} U \times (\mathbf{R}^m)^* \xrightarrow{(\widetilde{1_U}^1)^{-1}} \Lambda^1 T(U).$$

Thus, ι_U induces an $\mathbf{R}$-isomorphism $\Psi_U : \Gamma(\tau_U) \to \Omega^1(U)$, i.e. from smooth vector fields on U to 1-forms on U. (For general smooth manifolds $(M, \mathcal{S})$, construction of such an isomorphism $\Gamma(\tau_{M,\mathcal{S}}) \to \Omega^1(M, \mathcal{S})$ requires an additional piece of structure on M called a *Riemannian metric.*)

Note also that for $1 \leq i \leq m$, $\Lambda^i(\mathbf{R}^m)$ and $\Lambda^{m-i}(\mathbf{R}^m)$ have the same dimension over $\mathbf{R}$. In fact we choose an explicit isomorphism, traditionally denoted

$$* : \Lambda^i(\mathbf{R}^m) \to \Lambda^{m-i}(\mathbf{R}^m),$$

called the *Hodge $*$-operator*, as follows: For each $I \in X_{m,i}$ let $I' \in X_{m,m-i}$ denote the increasing sequence complementary to I and let sgn (I) denote the sign of the

$(i, m-i)$-shuffle which sends $1, \ldots, i$ to the respective entries of I. Then $*$ is the $\mathbf{R}$-isomorphism determined by $*(e_I^*) = \operatorname{sgn}(I)\, e_{I'}^*$ for each $I \in X_{m,i}$. It is easily checked that the composition

$$\Lambda^i(\mathbf{R}^m) \xrightarrow{\ *\ } \Lambda^{m-i}(\mathbf{R}^m) \xrightarrow{\ *\ } \Lambda^i(\mathbf{R}^m)$$

is multiplication by $(-1)^{i(m-i)}$.

For example, in case $m = 3$ we have

$$\begin{array}{rlrl} *1 = & e_1^* \wedge e_2^* \wedge e_3^*, & *(e_1^* \wedge e_2^* \wedge e_3^*) = & 1, \\ *e_1^* = & e_2^* \wedge e_3^*, & *(e_2^* \wedge e_3^*) = & e_1^*, \\ *e_2^* = & -e_1^* \wedge e_3^*, & *(e_1^* \wedge e_3^*) = & -e_2^*, \\ *e_3^* = & e_1^* \wedge e_2^*, & *(e_1^* \wedge e_2^*) = & e_3^*. \end{array}$$

More generally the Hodge $*$-operator can be defined on the exterior powers of any oriented finite-dimensional real inner product space V; replace the e_i^* by the duals of any orthonormal basis in the given orientation of V. The resulting $*$-operator can be shown to depend only on the inner product and orientation of V, and not on the choice of orthonormal basis.

For U open in $\mathbf{R}^m$, the Hodge $*$-operator as defined above yields $\mathbf{R}$-isomorphisms

$$* : \Omega^i(U) \to \Omega^{m-i}(U)$$

as follows: If $\omega \in \Omega^i(U)$, then $*\omega$ is the composition

$$U \xrightarrow{\ \omega\ } \Lambda^i T(U) \xrightarrow{\widetilde{1_U}^i} U \times \Lambda^i(\mathbf{R}^m) \xrightarrow{1_U \times *} U \times \Lambda^{m-i}(\mathbf{R}^m) \xrightarrow{(\widetilde{1_U}^{m-i})^{-1}} \Lambda^{m-i}(U).$$

This composition is smooth since $* : \Lambda^i(\mathbf{R}^m) \to \Lambda^{m-i}(\mathbf{R}^m)$ is $\mathbf{R}$-linear, and so certainly smooth.

We can form the composition

$$C^\infty(U) = \Omega^0(U) \xrightarrow{\ d\ } \Omega^1(U) \xrightarrow{\Psi_U^{-1}} \Gamma(\tau_U).$$

One can check that this composition sends any smooth real-valued f on U to $\operatorname{grad} f$, the *gradient* of f.

In case $m = 3$, we can form the composition

$$\Gamma(\tau_U) \xrightarrow{\Psi_U} \Omega^1(U) \xrightarrow{\ d\ } \Omega^2(U) \xrightarrow{\ *\ } \Omega^1(U) \xrightarrow{\Psi_U^{-1}} \Gamma(\tau_U).$$

One can check that this composition sends any smooth vector field σ on U to $\operatorname{curl} \sigma$, the *curl* of σ.

For general m we can form the composition

$$\Gamma(\tau_U) \xrightarrow{\Psi_U} \Omega^1(U) \xrightarrow{\ *\ } \Omega^{m-1}(U) \xrightarrow{\ d\ } \Omega^m(U) \xrightarrow{\ *\ } \Omega^0(U) = C^\infty(U).$$

One can check that this composition sends any smooth vector field σ on U to $\operatorname{div}\sigma$, the *divergence* of σ.

Thus, in case $m = 3$, for any $f \in C^\infty(U)$,

$$\operatorname{curl}(\operatorname{grad} f) = \Psi_U^{-1} * d\,\Psi_U\big(\Psi_U^{-1} d(f)\big) = \Psi_U^{-1} * d(df) = 0$$

by Proposition 13.7.

For any $\sigma \in \Gamma(\tau_U)$,

$$\operatorname{div}(\operatorname{curl}\sigma) = *d * \Psi_U\big(\Psi_U^{-1} * d * (\sigma)\big) = *d * *d\,\Psi_U(\sigma) = *d\big(d(\Psi_U(\sigma)\big) = 0$$

by Proposition 13.7. Thus the fact that the composition of any two maps in the sequence

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U)$$

is 0 yields the classical vector analysis facts that $\operatorname{curl}(\operatorname{grad} f) = 0$ and $\operatorname{div}(\operatorname{curl}\sigma) = 0$ for any smooth real-valued f and any smooth vector field σ on an open subset of $\mathbf{R}^3$. Conversely, it is easily checked that the latter implies $dd = 0$ for open subsets of $\mathbf{R}^3$.

Proposition 13.9. *Let U be open in $\mathbf{R}^m$, V open in $\mathbf{R}^n$, and $f : U \to V$ a smooth map. Then for all $k \geq 0$ the diagram*

$$\begin{array}{ccc} \Omega^k(V) & \xrightarrow{d} & \Omega^{k+1}(V) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \Omega^k(U) & \xrightarrow[d]{} & \Omega^{k+1}(U) \end{array}$$

commutes.

Proof. We first consider the case $k = 0$. Let $g \in \Omega^0(V) = C^\infty(V)$. We have

$$\begin{aligned}
f^*d(g) &= f^*\left(\sum_{i=1}^n D_i g \wedge dx_i\right) && \text{(by Definition 13.1)} \\
&= \sum_{i=1}^n f^*(D_i g) \wedge f^*(dx_i) && \text{(by Corollary 12.7(f))} \\
&= \sum_{i=1}^n \big((D_i g)f\big) \wedge \left(\sum_{j=1}^m (D_j f_i) \wedge dx_j\right) \\
& && \text{(by Proposition 12.9(b) and Theorem 12.14)} \\
&= \sum_{j=1}^m \left(\sum_{i=1}^n \Big((D_i g)f\Big) \cdot D_j f_i\right) \wedge dx_j = \sum_{j=1}^m \big(D_j(gf)\big) \wedge dx_j \\
& && \text{(by the Chain Rule)} \\
&= d(gf) = d(f^*g),
\end{aligned}$$

the latter by Definition 13.1 and Proposition 12.9(b), establishing the result for $k = 0$.

We next show that whenever $1 \le i \le n$ and $I \in X_{n,i}$, then $d\big(f^*(dx_I)\big) = 0$. For the case $i = 1$, writing x_j for the j^{th}, coordinate function on U, $1 \le j \le n$, we have by Example 13.2, $f^*(dx_j) = d(f^*x_j)$ by the $k = 0$ case of the Proposition proved above, and so $d\big(f^*(dx_j)\big) = dd(f^*x_j) = 0$ by Proposition 13.7. Now let $i > 1$ and suppose inductively that $d\big(f^*(dx_J)\big) = 0$ for all $J \in X_{n,i-1}$. For $I \in X_{n,i}$ we can write $dx_I = dx_j \wedge dx_J$ for some $(j) \in X_{n,1}$ and $J \in X_{n,i-1}$. Then

$$\begin{aligned} d\big(f^*(dx_I)\big) &= d\big(f^*(dx_j \wedge dx_J)\big) = d\big(f^*(dx_j) \wedge f^*(dx_J)\big) \\ &\qquad\qquad\qquad\qquad\qquad \text{(by Corollary 12.7(f))} \\ &= d\big(f^*(dx_j)\big) \wedge f^*(dx_J) - f^*(dx_j) \wedge d\big(f^*(dx_J)\big) \\ &\qquad\qquad\qquad\qquad\qquad \text{(by Proposition 13.6)} \\ &= 0 \end{aligned}$$

since both $d\big(f^*(dx_j)\big)$ and $d\big(f^*(dx_J)\big)$ are 0 by the induction hypothesis, completing proof of the claim.

To complete the proof of the Proposition, since f^* and d are $\mathbf{R}$-linear it suffices to prove that $d(f^*\omega) = f^*(d\omega)$ whenever $\omega = g \wedge dx_I$ for some $g \in \Omega^0(V)$ and $I \in X_{n,k}$, $k \ge 1$. We have

$$\begin{aligned} d(f^*\omega) &= d\big(f^*(g \wedge dx_I)\big) = d\big(f^*(g) \wedge f^*(dx_I)\big) \\ &\qquad\qquad\qquad\qquad\qquad \text{(by Corollary 12.7(f))} \\ &= d\big(f^*(g)\big) \wedge f^*(dx_I) + f^*(g) \wedge d\big(f^*(dx_I)\big) \\ &\qquad\qquad\qquad\qquad\qquad \text{(by Proposition 13.6)} \\ &= d\big(f^*(g)\big) \wedge f^*(dx_I) \\ &\qquad\qquad\qquad\qquad\qquad \text{(since } d\big(f^*(dx_I)\big) = 0\text{)} \\ &= f^*(dg) \wedge f^*(dx_I) \qquad\qquad \text{(by the } k = 0 \text{ case)} \\ &= f^*(dg \wedge dx_I) \qquad\qquad \text{(by Corollary 12.7(f))} \\ &= f^*\big(d(g \wedge dx_I)\big) \qquad\qquad \text{(by Definition 13.1)} \\ &= f^*(d\omega), \end{aligned}$$

completing the proof. □

Our next task is to extend exterior differentiation to general smooth manifolds by locally transferring the Euclidean space case by means of charts.

Lemma 13.10. *Let $(M, \mathcal{S})$ be an m-dimensional smooth manifold. Let φ and ψ be $\mathcal{S}$-admissible charts whose codomains are open subsets of $\mathbf{R}^m$. Then for any k-form ω on* dom $\varphi \cap$ dom ψ,

$$\varphi^* d\big((\varphi^{-1})^*\omega\big) = \psi^* d\big((\psi^{-1})^*\omega\big).$$

Proof. By Theorem 12.2, $(\psi\varphi^{-1})^* = (\varphi^{-1})^*\psi^*$ and $(\varphi^{-1})^* = (\varphi^*)^{-1}$. Thus it suffices to show that

$$(\psi\varphi^{-1})^* d\big((\psi^{-1})^*\omega\big) = d\big((\varphi^{-1})^*\omega\big)$$

for all k-forms ω on dom $\varphi \cap$ dom ψ. The overlap map

$$\psi\varphi^{-1} : \varphi(\text{dom } \varphi \cap \text{dom } \psi) \to \psi(\text{dom } \varphi \cap \text{dom } \psi)$$

is a smooth map between open subsets of $\mathbf{R}^m$. Hence, by Proposition 13.9, the diagram

$$\begin{array}{ccc} \Omega^k\big(\psi(\text{dom } \varphi \cap \text{dom } \psi)\big) & \xrightarrow{(\psi\varphi^{-1})^*} & \Omega^k\big(\varphi(\text{dom } \varphi \cap \text{dom } \psi)\big) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ \Omega^{k+1}\big(\psi(\text{dom } \varphi \cap \text{dom } \psi)\big) & \xrightarrow[(\psi\varphi^{-1})^*]{} & \Omega^{k+1}\big(\varphi(\text{dom } \varphi \cap \text{dom } \psi)\big) \end{array}$$

commutes. Thus for any k-form ω on dom $\varphi \cap$ dom ψ,

$$\begin{aligned} (\psi\varphi^{-1})^* d\big((\psi^{-1})^*\omega\big) &= d\big((\psi\varphi^{-1})^*(\psi^{-1})^*\omega\big) \\ &= d\big((\psi^{-1}\psi\varphi^{-1})^*\omega\big) \qquad \text{(by Theorem 12.2)} \\ &= d\big((\varphi^{-1})^*\omega\big). \qquad \square \end{aligned}$$

Recall, from Proposition 5.11 and Theorem 8.29, that if $(M, \mathcal{S})$ is a smooth manifold, U open in M, and $i : U \to M$ the inclusion map, then Ti maps $T(U, \mathcal{S}|U)$ diffeomorphically onto $p_{M,\mathcal{S}}^{-1}(U)$. We would now like to be able to identify $\Lambda^k T(U, \mathcal{S}|U)$ with $p_k^{-1}(U)$ where p_k is the projection map for $\Lambda^k \tau_{M,\mathcal{S}}$. Because of the fact that Λ^k does not yield a functor $SmVect \to SmVect$, $\Lambda^k f$ does not have an immediate meaning for smooth maps f. However, we shall see below that in the case of i as above, $\Lambda^k i$ does make sense and yields the desired diffeomorphism. The argument works for general smooth contravariant functors.

Lemma 13.11. *Let $Q : VS_{\mathbf{R}} \to VS_{\mathbf{R}}$ be a smooth contravariant functor, $(M, \mathcal{S})$ a smooth m-manifold, and U an open subset of M. Let $i : U \to M$ denote the inclusion map. Write p_Q for the projection map of the smooth vector bundle $Q\tau_{M,\mathcal{S}}$, and $Qi : p_Q^{-1}(U) \to QT(U, \mathcal{S}|U)$ for the function whose restriction to $p_Q^{-1}(x)$ for x in U is $QT_x i : QT_x(M, \mathcal{S}) \to QT_x(U, \mathcal{S}|U)$. Then Qi is a diffeomorphism.*

Proof. By Proposition 5.11, $T_x i : T_x(U, \mathcal{S}|U) \to T_x(M, \mathcal{S})$ is an $\mathbf{R}$-isomorphism for each $x \in U$. Thus, since Q is a functor, each $QT_x i$ is an $\mathbf{R}$-isomorphism. It follows easily that Qi is a bijection. Let q denote the projection map for $Q\tau_{U,\mathcal{S}|U}$. To complete the proof it suffices to check that for each $\mathcal{S}|U$-admissible chart φ with codomain an open subset of $\mathbf{R}^m$, the restriction $Qi : p_Q^{-1}(\text{dom } \varphi) \to q^{-1}(\text{dom } \varphi)$ is a diffeomorphism. Note that $i\varphi$ is an $\mathcal{S}$-admissible chart and the diagram

$$\begin{array}{ccc} p_Q^{-1}(\text{dom } \varphi) & \xrightarrow{\quad Qi \quad} & q^{-1}(\text{dom } \varphi) \\ & {\scriptstyle \widetilde{i\varphi}^Q}\searrow \quad \swarrow{\scriptstyle \widetilde{\varphi}^Q} & \\ & \text{dom } \varphi \times Q\mathbf{R}^m & \end{array}$$

commutes where $\widetilde{i\varphi}^Q$ and $\widetilde{\varphi}^Q$ are the linear $Q\mathbf{R}^m$-bundle charts for $Q\tau_{M,\mathcal{S}}$ and $Q\tau_{U,\mathcal{S}|U}$ derived from the charts $i\varphi$ and φ, respectively, by the construction in §11. Since $\widetilde{i\varphi}^Q$ and $\widetilde{\varphi}^Q$ are diffeomorphisms, the result follows. □

Lemma 13.12. *Let $(M, \mathcal{S})$ be a smooth manifold, U an open subset of M, and $i_U : U \to M$ the inclusion. Then:*

(a) *For any k-form ω on $(M, \mathcal{S})$, the restriction of ω to U is $(\Lambda^k i_U)^{-1} i_U^* \omega$.*

(b) *If ω and ρ are k-forms on $(M, \mathcal{S})$, then ω and ρ agree on U if and only if $i_U^*\omega = i_U^*\rho$.*

Proof. By definition of $i_U^*\omega$, $i_U^*\omega(x) = \Lambda^k(T_x i_U)\big(\omega(x)\big)$ for each $x \in U$. Thus, by definition of $\Lambda^k i_U$, $(\Lambda^k i_U)^{-1}\big(i_U^*\omega(x)\big) = \big(\Lambda^k(T_x i_U)\big)^{-1}\Lambda^k(T_x i_U)\big(\omega(x)\big) = \omega(x)$, proving part (a).

If $i_U^*\omega = i_U^*\rho$, then application of $(\Lambda^k i_U)^{-1}$ to both sides and part (a) yield that the restrictions of ω and ρ to U agree, proving part (b). □

Theorem 13.13. *Let $(M, \mathcal{S})$ be a smooth m-manifold and ω a k-form on $(M, \mathcal{S})$. Define $d\omega : M \to \Lambda^{k+1}T(M, \mathcal{S})$ as follows: For each $\mathcal{S}$-admissible chart φ with codomain an open subset of $\mathbf{R}^m$ define the restriction of $d\omega$ to* dom φ *to be*

$$(\Lambda^{k+1} i_\varphi)^{-1} \varphi^* d\big((\varphi^{-1})^* i_\varphi^* \omega\big)$$

where i_φ : dom $\varphi \to M$ is the inclusion and $\Lambda^{k+1} i$ is as in Lemma 13.11 with $Q = \Lambda^{k+1}$. Then $d\omega$ is a $(k+1)$-form on M.

Proof. If φ and ψ are $\mathcal{S}$-admissible charts whose codomains are open subsets of $\mathbf{R}^m$, it follows from the Local Property for Tangent Spaces (Proposition 5.11(b)) that $T_x i_\varphi = T_x i_\psi$ for all $x \in$ dom $\varphi \cap$ dom ψ and hence $\Lambda^{k+1} i_\varphi$ and $\Lambda^{k+1} i_\psi$ agree on fibers over points in dom $\varphi \cap$ dom ψ. By Proposition 12.15, $i_\varphi^*\omega$ and $i_\psi^*\omega$ agree on points of dom $\varphi \cap$ dom ψ. Thus by Lemma 13.10,

$$\varphi^* d\big((\varphi^{-1})^* i_\varphi^* \omega\big) = \psi^* d\big((\psi^{-1})^* i_\psi^* \omega\big).$$

It follows that $d\omega$ is well-defined.

Since each $\varphi^* d\big((\varphi^{-1})^* i_\varphi^* \omega\big)$ is smooth and, by Lemma 13.11, $(\Lambda^{k+1} i_\varphi)^{-1}$ is smooth, it follows that the restriction of $d\omega$ to dom φ is smooth for each $\mathcal{S}$-admissible φ as above. Thus, by the Local Property, $d\omega$ is smooth. Clearly, $(d\omega)(x) \in \Lambda^{k+1}T_x(M, \mathcal{S})$ for each $x \in M$, completing the proof. □

Corollary 13.14. *Let $(M, \mathcal{S})$ be a smooth m-manifold and ω a k-form on $(M, \mathcal{S})$. Then for each $\mathcal{S}$-admissible chart φ with codomain contained in $\mathbf{R}^m$,*

$$i_\varphi^*(d\omega) = \varphi^* d\big((\varphi^{-1})^* i_\varphi^* \omega\big).$$

Conversely, if ρ is a $(k+1)$-form on $(M, \mathcal{S})$ such that

$$i_\varphi^* \rho = \varphi^* d\big((\varphi^{-1})^* i_\varphi^* \omega\big)$$

for a collection of $\mathcal{S}$-admissible charts φ as above whose domains cover M, then $\rho = d\omega$.

Proof. By Lemma 13.12 and the definition of $d\omega$ given by Theorem 13.13,

$$(\Lambda^{k+1} i_\varphi)^{-1} i_\varphi^*(d\omega) = (\Lambda^{k+1} i_\varphi)^{-1} \varphi^* d\big((\varphi^{-1})^* i_\varphi^* \omega\big)$$

for all φ as above. Since each $\Lambda^{k+1} i_\varphi$ is a diffeomorphism by Lemma 13.11, it follows that $d\omega$ has the stated property.

Suppose ρ is a $(k+1)$-form on $(M, \mathcal{S})$ such that for a collection of φ as above whose domains cover M,

$$i_\varphi^* \rho = \varphi^* d\big((\varphi^{-1})^* i_\varphi^* \omega\big).$$

Thus $i_\varphi^* \rho = i_\varphi^*(d\omega)$ for all φ in a collection of charts whose domains cover M, and and so, by Lemma 13.12(b), ρ and $d\omega$ agree on all members of some open cover of M. □

Theorem 13.15. *Let $(M, \mathcal{S})$ be a smooth m-manifold. Then:*

(a) *If M is an open subset of $\mathbf{R}^m$, then for each $k \geq 0$ the $d : \Omega^k(M) \to \Omega^{k+1}(M)$ as given by Theorem 13.13 agrees with that given by Definition 13.1.*

(b) *If $x \in M$, $k \geq 0$, and $\omega, \rho \in \Omega^k(M, \mathcal{S})$ are such that ω and ρ agree in some open neighborhood of x in M, then $d\omega(x) = d\rho(x)$.*

(c) *For each $k \geq 0$, $d : \Omega^k(M, \mathcal{S}) \to \Omega^{k+1}(M, \mathcal{S})$ is $\mathbf{R}$-linear.*

(d) *If $(N, \mathcal{T})$ is a smooth n-manifold and $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ a smooth map, then for all $k \geq 0$ the diagram*

$$\begin{array}{ccc} \Omega^k(N, \mathcal{T}) & \xrightarrow{f^*} & \Omega^k(M, \mathcal{S}) \\ \downarrow d & & \downarrow d \\ \Omega^{k+1}(N, \mathcal{T}) & \xrightarrow[f^*]{} & \Omega^{k+1}(M, \mathcal{S}) \end{array}$$

commutes.

(e) *For any $\omega \in \Omega^k(M, \mathcal{S})$, $d(d\omega) = 0$.*

(f) *If $\omega \in \Omega^i(M, \mathcal{S})$ and $\rho \in \Omega^j(M, \mathcal{S})$, then*

$$d(\omega \wedge \rho) = (d\omega) \wedge \rho + (-1)^i \omega \wedge (d\rho).$$

Proof. Part (a) follows immediately from Corollary 13.14 by use of the chart 1_M.

Let x, ω, and ρ be as in part (b). We can choose an $\mathcal{S}$-admissible chart φ with codomain contained in $\mathbf{R}^m$ such that $x \in \operatorname{dom} \varphi$, and ω and ρ agree agree on $\operatorname{dom} \varphi$. It then follows from Lemma 13.12(b) that $i_\varphi^* \omega = i_\varphi^* \rho$. Thus by Corollary 13.14, $i_\varphi^*(d\omega) = i_\varphi^*(d\rho)$. Thus, by Lemma 13.12(b), $d\omega$ and $d\rho$ agree on $\operatorname{dom} \varphi$, proving part (b).

It suffices to check part (c) locally. For each $\mathcal{S}$-admissible chart φ with codomain contained in $\mathbf{R}^m$, note that i_φ^*, $(\varphi^{-1})^*$, φ^*, and the d for open subsets of $\mathbf{R}^m$ are all $\mathbf{R}$-linear, and that $(\Lambda^{k+1} i_\varphi)^{-1}$, is fiberwise $\mathbf{R}$-linear. Part (c) follows.

To prove part (d) it suffices to show (by Corollary 13.14) that whenever φ and ψ are $\mathcal{S}$-admissible and $\mathcal{T}$-admissible charts with codomains contained in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, such that $f(\operatorname{dom} \varphi) \subset \operatorname{dom} \psi$, then for all k-forms ω on $(N, \mathcal{T})$,

(1) $$i_\varphi^* f^*(d\omega) = \varphi^* d\big((\varphi^{-1})^* i_\varphi^* f^* \omega\big).$$

Let $g : \operatorname{dom} \varphi \to \operatorname{dom} \psi$ denote the restriction of f. We have the commutative diagram

(2)
$$\begin{array}{ccccc}
M & \xleftarrow{\;i_\varphi\;} & \operatorname{dom} \varphi & \xleftarrow{\;\varphi^{-1}\;} & \operatorname{codom} \varphi \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \psi g \varphi^{-1}} \\
N & \xleftarrow[\;i_\psi\;]{} & \operatorname{dom} \psi & \xleftarrow[\;\psi^{-1}\;]{} & \operatorname{codom} \psi.
\end{array}$$

By Proposition 13.9 the diagram

(3)
$$\begin{array}{ccc}
\Omega^k(\operatorname{codom} \varphi) & \xleftarrow{\;(\psi g \varphi^{-1})^*\;} & \Omega^k(\operatorname{codom} \psi) \\
{\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle d} \\
\Omega^{k+1}(\operatorname{codom} \varphi) & \xleftarrow[\;(\psi g \varphi^{-1})^*\;]{} & \Omega^{k+1}(\operatorname{codom} \psi)
\end{array}$$

commutes. Thus

$$\begin{aligned}
\varphi^* d\big((\varphi^{-1})^* i_\varphi^* f^* \omega\big) &= \varphi^* d\big((\psi g \varphi^{-1})^* (\psi^{-1})^* i_\psi^* \omega\big) && \text{(by (2))} \\
&= \varphi^* (\psi g \varphi^{-1})^* d\big((\psi^{-1})^* i_\psi^* \omega\big) && \text{(by (3))} \\
&= (\psi g \varphi^{-1} \varphi)^* d\big((\psi^{-1})^* i_\psi^* \omega\big) = g^* \psi^* d\big((\psi^{-1})^* i_\psi^* \omega\big) \\
&= g^* i_\psi^* (d\omega) && \text{(by Corollary 13.14)} \\
&= i_\varphi^* f^* (d\omega) && \text{(by (2))}
\end{aligned}$$

which establishes (1), completing the proof of part (d).

For any $\omega \in \Omega^k(M, \mathcal{S})$ and any $\mathcal{S}$-admissible chart φ with codomain contained in $\mathbf{R}^m$ we have

$$\begin{aligned}
i_\varphi^*(dd\omega) &= \varphi^* d\big((\varphi^{-1})^* i_\varphi^* d\omega\big) \\
&= \varphi^* dd\big((\varphi^{-1})^* i_\varphi^* \omega\big) && \text{(by part (d))} \\
&= 0
\end{aligned}$$

by Proposition 13.7 since $(\varphi^{-1})^* i_\varphi^* \omega$ is a is a k-form on codom φ, an open subset of $\mathbf{R}^m$. Thus $i_\varphi^*(0) = \varphi^* d\big((\varphi^{-1})^* i_\varphi^* d\omega\big)$ for all φ as above, and so, by Corollary 13.14, $d(d\omega) = 0$, proving part (e).

Let ω and ρ be as in part (f), and φ any $\mathcal{S}$-admissible chart with codomain contained in $\mathbf{R}^m$. Then

$$\begin{aligned}
i_\varphi^*\big(d(\omega \wedge \rho)\big) &= \varphi^* d\big((\varphi^{-1})^* i_\varphi^* (\omega \wedge \rho)\big) && \text{(by Corollary 13.14)} \\
&= \varphi^* d\big((i_\varphi \varphi^{-1})^* (\omega \wedge \rho)\big) = \varphi^* d\big((i_\varphi \varphi^{-1})^* \omega \wedge (i_\varphi \varphi^{-1})^* \rho\big) \\
& && \text{(by Corollary 12.7(f))} \\
&= \varphi^* \Big(d\big((i_\varphi \varphi^{-1})^* \omega\big) \wedge (i_\varphi \varphi^{-1})^* \rho + (-1)^i (i_\varphi \varphi^{-1})^* \omega \wedge d\big((i_\varphi \varphi^{-1})^* \rho\big)\Big)
\end{aligned}$$

(by Proposition 13.6, since $(i_\varphi\varphi^{-1})^*\omega$ and $(i_\varphi\varphi^{-1})^*\rho$ are differential forms on codom φ, an open subset of $\mathbf{R}^m$)

$$= \varphi^*\Big((i_\varphi\varphi^{-1})^*d\omega \wedge (i_\varphi\varphi^{-1})^*\rho + (-1)^i(i_\varphi\varphi^{-1})^*\omega \wedge (i_\varphi\varphi^{-1})^*d\rho\Big) \qquad \text{(by part (d))}$$

$$= \varphi^*(i_\varphi\varphi^{-1})^*\big((d\omega \wedge \rho + (-1)^i\omega \wedge (d\rho)\big) \qquad \text{(by Corollary 12.7(f))}$$

$$= (i_\varphi\varphi^{-1}\varphi)^*\big((d\omega \wedge \rho + (-1)^i\omega \wedge (d\rho)\big)$$

$$= i_\varphi^*\big((d\omega) \wedge \rho + (-1)^i\omega \wedge (d\rho)\big).$$

Part (f) now follows from Corollary 13.14. □

Definition 13.16. Let $(M, \mathcal{S})$ be a smooth manifold and $k \geq 0$. A k-form ω on $(M, \mathcal{S})$ is said to be *closed* if $d\omega = 0$. If $k > 0$, ω is said to be *exact* if $\omega = d\rho$ for some $(k-1)$-form ρ on $(M, \mathcal{S})$.

The set of all closed k-forms on $(M, \mathcal{S})$ is denoted $Z^k(M, \mathcal{S})$. The set of all exact k-forms on $(M, \mathcal{S})$ is denoted $B^k(M, \mathcal{S})$. (By convention, $B^0(M, \mathcal{S}) = 0$.) Closed forms are sometimes called *de Rham cocycles.* Exact forms are sometimes called *de Rham coboundaries.*

Proposition 13.17. *Let $(M, \mathcal{S})$ be a smooth manifold. Then for each $k \geq 0$, $Z^k(M, \mathcal{S})$ and $B^k(M, \mathcal{S})$ are* **R***-linear subspaces of $\Omega^k(M, \mathcal{S})$ and $B^k(M, \mathcal{S}) \subset Z^k(M, \mathcal{S})$.*

Proof. Since $Z^k(M, \mathcal{S})$ and $B^k(M, \mathcal{S})$ are the kernel and image, respectively, of the appropriate **R**-linear transformation d, it follows that they are **R**-linear subspaces of $\Omega^k(M, \mathcal{S})$. Since $d(d\omega) = 0$ for all k-forms ω by Theorem 13.15(e), every exact form is closed, i.e. $B^k(M, \mathcal{S}) \subset Z^k(M, \mathcal{S})$ for all $k \geq 0$. □

Example 13.18. Let $U = \mathbf{R}^2 - \{0\}$ and

$$\omega = \frac{-y}{x^2 + y^2} \wedge dx + \frac{x}{x^2 + y^2} \wedge dy.$$

We leave it as an exercise (see Exercises for §13) to show that ω is a closed 1-form on U, but is not exact.

Definition 13.19. Let $(M, \mathcal{S})$ be a smooth manifold. For $k \geq 0$, the k^{th} *de Rham cohomology group of* $(M, \mathcal{S})$, denoted $H^k_{dR}(M, \mathcal{S})$, is the quotient real vector space $Z^k(M, \mathcal{S})/B^k(M, \mathcal{S})$.

The $H^k_{dR}(M, \mathcal{S})$ are real vector spaces. The smooth structure $\mathcal{S}$ on M plays an essential role in their definition. Thus, it may come as a surprise that the $H^k_{dR}(M, \mathcal{S})$ depend, up to isomorphism, only on the *topology* of M, and not on the smooth structure $\mathcal{S}$. This is by no means obvious. For M compact, the $H^k_{dR}(M, \mathcal{S})$ actually turn out to be finite-dimensional over **R**. In algebraic topology, singular cohomology groups $H^k(X; \mathbf{R})$ with coefficients in **R** are defined for arbitrary topological spaces X, which depend only on the topology of X (in fact, only on the homotopy type of

X). There is an important result, known as the *de Rham Theorem*, which asserts that for M compact, $H^k_{dR}(M,\mathcal{S}) \cong H^k(M;\mathbf{R})$.

If ω is a closed k-form on $(M,\mathcal{S})$, we denote by $[\omega]$ the de Rham cohomology class of ω, i.e. the image of ω under the natural projection

$$Z^k(M,\mathcal{S}) \to Z^k(M,\mathcal{S})/B^k(M,\mathcal{S}) = H^k_{dR}(M,\mathcal{S}).$$

Proposition 13.20. *Let $(M,\mathcal{S})$ be a smooth m-manifold, and $Comp(M)$ the set of connected components of M. For each $C \in Comp(M)$ let $\chi_C : M \to \mathbf{R}$ denote the characteristic function of C, i.e.*

$$\chi_C(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{if } x \notin C. \end{cases}$$

Then $\{[\chi_C] \mid C \in Comp(M)\}$ is an $\mathbf{R}$-basis for $H^0_{dR}(M,\mathcal{S})$.

Proof. Since $B^0(M,\mathcal{S}) = 0$ we must show that $\{\chi_C \mid C \in Comp(M)\}$ is an $\mathbf{R}$-basis for $Z^0(M,\mathcal{S})$. Note that $\{\chi_C \mid C \in Comp(M)\}$ is an $\mathbf{R}$-basis for the space of locally constant real-valued functions on M. Thus we must show that if $f \in \Omega^0(M,\mathcal{S}) = C^\infty(M,\mathcal{S})$, then $df = 0$ if and only if f is locally constant.

By Lemma 13.12, $df = 0$ if and only if for each $\mathcal{S}$-admissible chart φ with codomain contained in $\mathbf{R}^m$, $i^*_\varphi(df) = 0$. Since $\varphi : \operatorname{dom}\varphi \to \operatorname{codom}\varphi$ is a diffeomorphism, the latter holds if and only if $(\varphi^{-1})^* i^*_\varphi(df) = 0$, i.e. if and only if $d\big((\varphi^{-1})^* i^*_\varphi f\big) = 0$, i.e. if and only if $d(f i_\varphi \varphi^{-1}) = 0$. Since

$$d(f i_\varphi \varphi^{-1}) = \sum_{j=1}^{m} D_j(f i_\varphi \varphi^{-1}) \wedge dx_j,$$

the latter holds if and only if $D_j(f i_\varphi \varphi^{-1}) = 0$ for $1 \le j \le m$, i.e. if and only if $f i_\varphi \varphi^{-1}$ is constant, i.e. if and only if f is constant on $\operatorname{dom}\varphi$. ☐

Thus, in particular, $H^0_{dR}(M,\mathcal{S})$ depends only on the topology of M.

Proposition 13.21. *Let $(M,\mathcal{S})$ and $(N,\mathcal{T})$ be smooth manifolds and $f : (M,\mathcal{S}) \to (N,\mathcal{T})$ a smooth map. Then:*
(a) *For each $k \ge 0$, $f^*\big(Z^k(N,\mathcal{T})\big) \subset Z^k(M,\mathcal{S})$.*
(b) *For each $k \ge 0$, $f^*\big(B^k(N,\mathcal{T})\big) \subset B^k(M,\mathcal{S})$.*

Proof. Both parts are immediate from the fact that for any differential form ω on $(N,\mathcal{T})$, $d(f^*\omega) = f^*(d\omega)$ by Theorem 13.15(d). ☐

In view of Proposition 13.21 we can make the following definition:

Definition 13.22. Let $f : (M,\mathcal{S}) \to (N,\mathcal{T})$ be a smooth map. For each $k \ge 0$, denote by $H^k_{dR} f : H^k_{dR}(N,\mathcal{T}) \to H^k_{dR}(M,\mathcal{S})$ the $\mathbf{R}$-homomorphism induced by $f^* : Z^k(N,\mathcal{T}) \to Z^k(M,\mathcal{S})$, i.e. for each closed k-form ω on $(N,\mathcal{T})$, $H^k_{dR} f([\omega]) = [f^*\omega]$.

Proposition 13.23. *For each $k \geq 0$, the rules which assign to each smooth manifold $(M, \mathcal{S})$ its k^{th} de Rham cohomology group $H^k_{dR}(M, \mathcal{S})$ and to each smooth map $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ the* **R***-linear transformation $H^k_{dR}f : H^k_{dR}(N, \mathcal{T}) \to H^k_{dR}(M, \mathcal{S})$, constitute a contravariant functor $H^k_{dR} : Sm \to VS_{\mathbf{R}}$.*

Proof. Suppose $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ and $g : (N, \mathcal{T}) \to (Q, \mathcal{U})$ are smooth maps. Then for any closed k-form ω on $(Q, \mathcal{U})$ we have

$$\begin{aligned} H^k_{dR}(gf)([\omega]) &= [(gf)^*(\omega)] = [f^*(g^*\omega)] \qquad \text{(by Corollary 12.4)} \\ &= H^k_{dR}f([g^*\omega]) = H^k_{dR}f\big(H^k_{dR}g([\omega])\big), \end{aligned}$$

and

$$H^k_{dR}1_Q([\omega]) = [1_Q^*\omega] = [\omega]$$

by Corollary 12.4. ☐

We next show that the wedge product of forms induces a wedge product operation on de Rham cohomology classes.

Lemma 13.24. *Let $(M, \mathcal{S})$ be a smooth manifold. Suppose ω and ρ are closed i- and j-forms, respectively, on $(M, \mathcal{S})$. Then $\omega \wedge \rho$ is a closed $(i+j)$-form on $(M, \mathcal{S})$ and the de Rham cohomology class $[\omega \wedge \rho]$ of $\omega \wedge \rho$ depends only on $[\omega]$ and $[\rho]$.*

Proof. By Theorem 13.15(f), $d(\omega \wedge \rho) = (d\omega) \wedge \rho + (-1)^i \omega \wedge (d\rho) = 0$ since $d\omega = 0$ and $d\rho = 0$, proving that $\omega \wedge \rho$ is a closed $(i+j)$-form. If $[\omega'] = [\omega]$ and $[\rho'] = [\rho]$, then we can write $\omega' = \omega + d\alpha$ and $\rho' = \rho + d\beta$ for some $(i-1)$- and $(j-1)$-forms α and β, respectively. (By convention, k-forms for negative k are all 0, so if i or j is 0, the corresponding α or β is 0.) Then

$$\begin{aligned} \omega' \wedge \rho' &= \omega \wedge \rho + \omega \wedge (d\beta) + (d\alpha) \wedge \rho + (d\alpha) \wedge (d\beta) \\ &= \omega \wedge \rho + (-1)^i d(\omega \wedge \beta) + d(\alpha \wedge \rho) + d\big(\alpha \wedge (d\beta)\big) \end{aligned}$$

by Theorem 13.15(f) since $d\omega$, $d\rho$, and $d(d\beta)$ are all 0. Thus $[\omega' \wedge \rho'] = [\omega \wedge \rho]$. ☐

In view of Lemma 13.24 we can make the following definition:

Definition 13.25. Let $(M, \mathcal{S})$ be a smooth manifold and suppose $a \in H^i_{dR}(M, \mathcal{S})$, $b \in H^j_{dR}(M, \mathcal{S})$. We define $a \wedge b \in H^{i+j}_{dR}(M, \mathcal{S})$ as follows: If $a = [\omega]$ and $b = [\rho]$ where ω and ρ are closed forms, then $a \wedge b = [\omega \wedge \rho]$.

The following is an immediate consequence of Corollary 12.7:

Theorem 13.26. *Let $(M, \mathcal{S})$ be a smooth manifold. Suppose $a \in H^i_{dR}(M, \mathcal{S})$, $b \in H^j_{dR}(M, \mathcal{S})$, and $c \in H^k_{dR}(M, \mathcal{S})$ where $i, j, k \geq 0$. Then:*

(a) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

(b) $a \wedge b = (-1)^{ij} b \wedge a$.

(c) *If $j = k$, then $a \wedge (b + c) = a \wedge b + a \wedge c$.*

(d) *If $r \in \mathbf{R}$, then $(ra) \wedge b = a \wedge (rb) = r(a \wedge b)$.*

(e) *If $M \neq \emptyset$, then $[1] \wedge a = a \wedge [1] = a$ where $1 \in \Omega^0(M, \mathcal{S})$ is the constant function with value 1.*

(f) *If* $f:(N,\mathcal{T})\to(M,\mathcal{S})$ *is a smooth map, then*

$$H_{dR}^{i+j}f(a\wedge b)=H_{dR}^{i}f(a)\wedge H_{dR}^{j}f(b). \qquad \square$$

Writing $H_{dR}^*(M,\mathcal{S})=\bigoplus_{k\geq 0}H_{dR}^k(M,\mathcal{S})$, $H_{dR}^*(M,\mathcal{S})$ is a graded algebra over $\mathbf{R}$ (with unit if $M\neq\emptyset$), which is associative and commutative (in the graded sense). H_{dR}^* is a contravariant functor from the smooth category to the category of graded algebras over $\mathbf{R}$.

It turns out that the above graded ring structure on $H_{dR}^*(M,\mathcal{S})$ depends, up to isomorphism, only on the topology of M, and not on the smooth structure $\mathcal{S}$.

If V is a real m-dimensional vector space, then by Theorem 10.18, $\Lambda^kV=0$ if $k>m$. Thus, for a smooth m- manifold $(M,\mathcal{S})$, $\Omega^k(M,\mathcal{S})=0$ for $k>m$, and hence $H_{dR}^k(M,\mathcal{S})=0$ for $k>m$. This, combined with Proposition 13.20, yields:

Corollary 13.27. *Let* P *be a one point space. Then*

$$H_{dR}^k(P)\cong\begin{cases}\mathbf{R} & \text{if } k=0,\\ 0 & \text{if } k\neq 0.\end{cases} \qquad \square$$

Exercises for §13

1. Verify the claims made in Example 13.18.

2. (a) Let $(M,\mathcal{S})$ be a smooth manifold and $\omega\in\Omega^{2k}(M,\mathcal{S})$. Prove that $\omega\wedge d\omega$ is exact.

   (b) Give an explicit example of a 1-form ω such that $\omega\wedge d\omega$ is not closed.

3. A *real cochain complex* (C,δ) consists of:

   (i) A sequence $\{C^k\mid k\geq 0\}$ of real vector spaces. (For convenience we also set $C^k=0$ if $k<0$.)

   (ii) A sequence of $\mathbf{R}$-homomorphisms $\delta:C^k\to C^{k+1}$.

   (iii) We require that for each k, the composition

$$C^k\xrightarrow{\delta}C^{k+1}\xrightarrow{\delta}C^{k+2}$$

is the zero map.

Thus, if $(M,\mathcal{S})$ is a smooth manifold, the *de Rham complex* $(\Omega(M,\mathcal{S}),d)$ given by $\Omega(M,\mathcal{S})^k=\Omega^k(M,\mathcal{S})$, with d being exterior differentiation, is an example of a real cochain complex.

If (C,δ) is a real cochain complex, define $Z^k(C,\delta)$ to be the kernel of $\delta:C^k\to C^{k+1}$ and $B^k(C,\delta)$ the image of $\delta:C^{k-1}\to C^k$. Members of $Z^k(C,\delta)$ are called *k-cocycles of* (C,δ). Members of $B^k(C,\delta)$ are called *k-coboundaries of* (C,δ).

It is immediate from the condition $\delta\delta=0$ that $B^k(C,\delta)\subset Z^k(C,\delta)$ for all k, and hence we can form the quotient $Z^k(C,\delta)/B^k(C,\delta)$, which is denoted $H^k(C,\delta)$ and called the k^{th} *cohomology group* (or *module*, or *vector space*) of (C,δ).

If (C, δ) and (D, ε) are real cochain complexes, a *real cochain map* $f : (C, \delta) \to (D, \varepsilon)$ consists of a sequence of $\mathbf{R}$-homomorphisms $f^k : C^k \to D^k$ such that for each k, the diagram

$$\begin{array}{ccc} C^k & \xrightarrow{\ \delta\ } & C^{k+1} \\ {\scriptstyle f^k}\big\downarrow & & \big\downarrow{\scriptstyle f^{k+1}} \\ D^k & \xrightarrow[\ \varepsilon\ ]{} & D^{k+1} \end{array}$$

commutes.

For example, if $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ is a smooth map, the $\mathbf{R}$-homomorphisms $f^* : \Omega^k(M, \mathcal{S}) \to \Omega^k(N, \mathcal{T})$ constitute a real cochain map from the de Rham complex of $(N, \mathcal{T})$ to the de Rham complex of $(M, \mathcal{S})$.

(a) Show that by taking the real cochain complexes as objects and real cochain maps as morphisms, we obtain a category $CoChain_{\mathbf{R}}$, the *category of real cochain complexes.*

(b) Given a real cochain map $f : (C, \delta) \to (D, \varepsilon)$, show that for each k there is a well-defined $\mathbf{R}$-homomorphism $H^k f : H^k(C, \delta) \to H^k(D, \varepsilon)$ given by $H^k f([z]) = [f^k(z)]$ for each $z \in Z^k(C, \delta)$ where $[z]$ denotes the coset of z modulo coboundaries.

(c) Show that for each k the rules which assign to each real cochain complex (C, δ) the real vector space $H^k(C, \delta)$, and to each real cochain map f the $\mathbf{R}$-homomorphism $H^k f$, constitute a covariant functor $H^k : CoChain_{\mathbf{R}} \to VS_{\mathbf{R}}$.

4. Let (C, δ) and (D, ε) be real cochain complexes and $f, g : (C, \delta) \to (D, \varepsilon)$ real cochain maps. A *cochain homotopy T from f to g* consists of a sequence of $\mathbf{R}$-homomorphisms $T^k : C^k \to D^{k-1}$ such that for each k,

$$\varepsilon T^k + T^{k+1}\delta = f^k - g^k : C^k \to D^k.$$

We say *f is cochain homotopic to g* (denoted $f \simeq g$) if there exists a cochain homotopy from f to g.

(a) Show that $\simeq$ is an equivalence relation on $CoChain\big((C, \delta), (D, \varepsilon)\big)$.

(b) Prove that if $f \simeq g$, then $H^k f = H^k g$ for all k.

5. Let $(M, \mathcal{S})$ be a smooth manifold and X a smooth submanifold of $(M, \mathcal{S})$. X is said to be a *smooth retract of* $(M, \mathcal{S})$ if there exists a smooth map $r : M \to X$ such that $r(x) = x$ for all $x \in X$. Such an r is called a *smooth retraction of M onto X*. Prove that if r is a smooth retraction of M onto X, then for each $k \geq 0$, $H^k_{dR} r : H^k_{dR}(X, \mathcal{S}|X) \to H^k_{dR}(M, \mathcal{S})$ is injective.

14. Integration of Forms and Stokes' Theorem

In this section we take up the integration of differential forms over special kinds of parametrized domains call *smooth cubical chains*. The integrals are *oriented integrals* in that they depend on the orientation of the parametrization. The graded commutativity of wedge products appropriately deals with changes of orientation. We will prove a version of Stokes' Theorem for these integrals, which may be viewed as a generalization of the Fundamental Theorem of Calculus. We will then use Stokes' Theorem to prove that certain de Rham cohomology classes are non-zero.

I denotes the closed unit interval $[0,1]$ and for $k>0$, I^k is the k-fold cartesian product $\underbrace{I\times\cdots\times I}_{k}$, contained in $\mathbf{R}^k$. By convention we take $I^0=\mathbf{R}^0=\{0\}$. Note that for $k>0$, I^k is *not* a manifold.

Definition 14.1. Let $(M,\mathcal{S})$ be a smooth manifold and $k\geq 0$. A *smooth k-cube σ in $(M,\mathcal{S})$* is a function $\sigma: I^k\to M$ which is the restriction of some smooth map $\widetilde{\sigma}: U\to M$ where U is some open subset of $\mathbf{R}^k$ which contains I^k.

Since I^0 is open in $\mathbf{R}^0$ (in fact, equals $\mathbf{R}^0$) every function $\sigma: I^0\to M$ is a smooth 0-cube in M. We can identify the smooth 0-cubes in M with the points of M. Note that if $k>0$, the image of a smooth k-cube need *not* be a k-dimensional set since we are allowing *arbitrary* maps on I^k which have smooth extensions (an extreme case would be a constant k-cube). "Nice" smooth k-cubes, e.g. those which are restrictions of immersions or embeddings of open neighborhoods of I^k into M, are of primary geometric interest, but for functorial reasons we are forced to include singular and degenerate smooth k-cubes. For example, if $f:(M,\mathcal{S})\to(N,\mathcal{T})$ is a smooth map and σ is a smooth k-cube in $(M,\mathcal{S})$, then $f\sigma$ is a smooth k-cube in $(N,\mathcal{T})$. However even if σ is "nice", $f\sigma$ may be quite singular and degenerate.

If X is any set, we can form the *real vector space* $\mathbf{R}X$ *on* X as follows: Elements of $\mathbf{R}X$ are formal sums $\sum_{x\in X} r_x x$ where the r_x are real and all but finitely many are 0. Addition and scalar multiplication in $\mathbf{R}X$ is defined in the obvious way, and it is easy to check that $\mathbf{R}X$ is a real vector space with these operations. If $x_0\in X$, we identify x_0 with the element $\sum_{x\in X} r_x x$ where

$$r_x=\begin{cases}1 & \text{if } x=x_0,\\ 0 & \text{if } x\neq x_0.\end{cases}$$

X is then an $\mathbf{R}$-basis for $\mathbf{R}X$.

Definition 14.2. Let $(M,\mathcal{S})$ be a smooth manifold and $k\geq 0$. Denote by $Q_k(M,\mathcal{S})$ the real vector space on the set of all smooth k-cubes in $(M,\mathcal{S})$. Members of $Q_k(M,\mathcal{S})$ are called *smooth cubical k-chains in* $(M,\mathcal{S})$, and $Q_k(M,\mathcal{S})$ the k^{th} cubical chain space of $(M,\mathcal{S})$.

Smooth cubical k-chains are going to be the domains of integration of k-forms. If $\beta: I^k\to\mathbf{R}$ is continuous, then the k-fold integral

$$\int_{I^k}\beta\, dV_k$$

exists in both the Riemann and Lebesgue senses, and the two agree. Moreover, by Fubini's Theorem,

$$\int_{I^k} \beta \, dV_k = \int_0^1 \cdots \int_0^1 \beta(x_1, \ldots, x_k) \, dx_{\alpha(1)} \cdots dx_{\alpha(k)},$$

a k-fold iteration of single integrals, for any permutation $\alpha \in \Sigma_k$. The latter can be evaluated using the Fundamental Theorem of Calculus.

For the case $k = 0$ we interpret $\int_{I^0} \beta \, dV_0$ to be $\beta(0)$.

Let σ be a smooth k-cube in $(M, \mathcal{S})$ and $\widetilde{\sigma}$ a smooth extension of σ to an open neighborhood U of I^k in $\mathbf{R}^k$. If ω is a k-form on $(M, \mathcal{S})$, then $\widetilde{\sigma}^*\omega$ is a k-form on U and hence, by Proposition 12.13, we can write $\widetilde{\sigma}^*\omega = \beta_{\widetilde{\sigma},\omega} \wedge dx_1 \wedge \cdots \wedge dx_k$ for a unique smooth real-valued map $\beta_{\widetilde{\sigma},\omega}$ on U. If $\widehat{\sigma} : V \to M$ is another smooth extension of σ, then $\widetilde{\sigma}$ and $\widehat{\sigma}$ agree on $(0,1)^k$ and so by Proposition 12.15, $\widetilde{\sigma}^*\omega$ and $\widehat{\sigma}^*\omega$ agree on $(0,1)^k$. Hence $\beta_{\widetilde{\sigma},\omega}$ and $\beta_{\widehat{\sigma},\omega}$ agree on $(0,1)^k$. Thus, since $(0,1)^k$ is dense in I^k, it follows by continuity that $\beta_{\widetilde{\sigma},\omega}$ and $\beta_{\widehat{\sigma},\omega}$ agree on I^k. Thus we can unambiguously write $\sigma^*\omega$ for the restriction to I^k of $\widetilde{\sigma}^*\omega$ for any smooth extension $\widetilde{\sigma}$ of σ, and $\sigma^*\omega = \beta_{\sigma,\omega} \wedge dx_1 \wedge \cdots \wedge dx_k$ for a unique real-valued $\beta_{\sigma,\omega}$ on I^k which has a smooth extension to an open neighborhood of I^k in $\mathbf{R}^k$.

Definition 14.3. Let $(M, \mathcal{S})$ be a smooth manifold and σ a smooth k-cube in $(M, \mathcal{S})$. Let ω be a k-form on $(M, \mathcal{S})$. We define $\int_\sigma \omega$ to be

$$\int_{I^k} \beta_{\sigma,\omega} \, dV_k.$$

If $c = \sum_{i=1}^q r_i \sigma_i \in Q_k(M, \mathcal{S})$, where the σ_i are smooth k-cubes in $(M, \mathcal{S})$ and the $r_i \in \mathbf{R}$, we define

$$\int_c \omega = \sum_{i=1}^q r_i \int_{\sigma_i} \omega.$$

Note that no measures or metrics on M are required to define $\int_c \omega$.

Proposition 14.4. *Let $(M, \mathcal{S})$ be a smooth manifold and $k \geq 0$. Then the real-valued function $\int_c \omega$ for $\omega \in \Omega^k(M, \mathcal{S})$ and $c \in Q_k(M, \mathcal{S})$ is $\mathbf{R}$-bilinear in the arguments ω and c.*

Proof. $\mathbf{R}$-linearity in c for fixed ω is immediate.

Let σ be a fixed smooth k-cube in $(M, \mathcal{S})$, and $\widetilde{\sigma}$ any smooth extension of σ. Then for any $\omega, \mu \in \Omega^k(M, \mathcal{S})$ and $r \in \mathbf{R}$, it follows from Corollary 12.4 that

$$\widetilde{\sigma}^*(\omega + r\mu) = \widetilde{\sigma}^*\omega + r\widetilde{\sigma}^*\mu.$$

Hence

$$\beta_{\sigma,\omega+r\mu} = \beta_{\sigma,\omega} + r\beta_{\sigma,\mu}.$$

Thus

$$\int_\sigma (\omega + r\mu) = \int_{I^k} \beta_{\sigma,\omega+r\mu}\, dV_k = \int_{I^k} (\beta_{\sigma,\omega} + r\beta_{\sigma,\mu})\, dV_k$$
$$= \int_{I^k} \beta_{\sigma,\omega}\, dV_k + r\int_{I^k} \beta_{\sigma,\mu}\, dV_k = \int_\sigma \omega + r\int_\sigma \mu$$

and so $\int_\sigma \omega$ is an $\mathbf{R}$-linear function of ω for any fixed smooth k-cube σ in $(M, \mathcal{S})$. The $\mathbf{R}$-linearity of $\int_c \omega$ in ω for any fixed $c \in Q_k(M, \mathcal{S})$ now follows easily. □

Let $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ be a smooth map and σ a smooth k-cube in $(M, \mathcal{S})$. Note that if $\widetilde{\sigma}$ is any smooth extension of σ, then $f\widetilde{\sigma}$ is a smooth extension of $f\sigma$, and so $f\sigma$ is a smooth k-cube in $(N, \mathcal{T})$. Since the set of smooth k-cubes on $(M, \mathcal{S})$ is an $\mathbf{R}$-basis for $Q_k(M, \mathcal{S})$, we can make the following definition:

Definition 14.5. Let $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ be a smooth map. Then for each $k \geq 0$, $f_* : Q_k(M, \mathcal{S}) \to Q_k(N, \mathcal{T})$ is the unique $\mathbf{R}$-homomorphism which sends any smooth k-cube σ in $(M, \mathcal{S})$ to the smooth k-cube $f\sigma$ in $(N, \mathcal{T})$.

The proof of the following proposition is easy and left as an exercise.

Proposition 14.6. *For each $k \geq 0$ the rules which assign, to each smooth manifold $(M, \mathcal{S})$ the real vector space $Q_k(M, \mathcal{S})$, and to each smooth map $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ the $\mathbf{R}$-homomorphism $f_* : Q_k(M, \mathcal{S}) \to Q_k(N, \mathcal{T})$, constitute a covariant functor from Sm to $VS_{\mathbf{R}}$.* □

Proposition 14.7. *Let $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ be a smooth map. Then for all $c \in Q_k(M, \mathcal{S})$ and $\omega \in \Omega^k(N, \mathcal{T})$,*

$$\int_c f^*\omega = \int_{f_*c} \omega.$$

Proof. Both sides are $\mathbf{R}$-linear in c, and so it suffices to check the case $c = \sigma$ for σ a general smooth k-cube in $(M, \mathcal{S})$.

Let $\widetilde{\sigma}$ be any smooth extension of σ. Then $f\widetilde{\sigma}$ is a smooth extension of $f_*\sigma$. Since $(f\widetilde{\sigma})^*\omega = \widetilde{\sigma}^* f^*\omega$ by Corollary 12.4, it follows that $\beta_{f_*\sigma,\omega} = \beta_{\sigma,f^*\omega}$. Hence

$$\int_{f_*\sigma} \omega = \int_{I^k} \beta_{f_*\sigma,\omega}\, dV_k = \int_{I^k} \beta_{\sigma,f^*\omega}\, dV_k = \int_\sigma f^*\omega. \qquad \square$$

Definition 14.8. Let $(M, \mathcal{S})$ be a smooth manifold and σ a smooth k-cube in $(M, \mathcal{S})$, $k \geq 1$. For $1 \leq i \leq k$ define $\sigma_0^i, \sigma_1^i : I^{k-1} \to M$ by

$$\sigma_0^i(t_1, \ldots, t_{k-1}) = \sigma(t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1}),$$
$$\sigma_1^i(t_1, \ldots, t_{k-1}) = \sigma(t_1, \ldots, t_{i-1}, 1, t_i, \ldots, t_{k-1}).$$

The σ_ε^i, $\varepsilon = 0, 1$, $1 \leq i \leq k$ are called the *faces of σ*.

Note that if σ is a smooth k-cube in $(M,\mathcal{S})$, $k \geq 1$, then each face of σ is a smooth $(k-1)$-cube in $(M,\mathcal{S})$. For let $\widetilde{\sigma} : U \to M$ be a smooth extension of σ, U open in $\mathbf{R}^k$. For $\varepsilon = 0, 1$ and $1 \leq i \leq k$ let $j^i_\varepsilon : \mathbf{R}^{k-1} \to \mathbf{R}^k$ be the map given by

$$j^i_\varepsilon(t_1, \ldots, t_{k-1}) = (t_1, \ldots, t_{i-1}, \varepsilon, t_i, \ldots, t_{k-1}).$$

The j^i_ε are smooth, and $\widetilde{\sigma} j^i_\varepsilon : \left(j^i_\varepsilon\right)^{-1}(U) \to M$ is a smooth extension of σ^i_ε.

Definition 14.9. Let $(M,\mathcal{S})$ be a smooth manifold and $k \geq 1$. The k^{th} *cubical boundary map for* $(M,\mathcal{S})$ is the unique $\mathbf{R}$-homomorphism $\partial : Q_k(M,\mathcal{S}) \to Q_{k-1}(M,\mathcal{S})$ defined on smooth k-cubes σ in $(M,\mathcal{S})$ by

$$\partial(\sigma) = \sum_{i=1}^{k} (-1)^i \left(\sigma^i_0 - \sigma^i_1\right).$$

Proposition 14.10. *Let $(M,\mathcal{S})$ be a smooth manifold. Then for $k \geq 2$ and all $c \in Q_k(M,\mathcal{S})$, $\partial(\partial c) = 0$.*

Proof. Since ∂ is $\mathbf{R}$-linear, it suffices to treat the case $c = \sigma$, a smooth k-cube in $(M,\mathcal{S})$. An easy check shows that for $1 \leq j < i \leq k$, $\varepsilon = 0$ or 1, $\eta = 0$ or 1,

$$(\sigma^i_\varepsilon)^j_\eta = (\sigma^j_\eta)^{i-1}_\varepsilon. \tag{1}$$

Thus

$$\begin{aligned}
\partial(\partial\sigma) &= \partial\left(\sum_{i=1}^{k}(-1)^i\left(\sigma^i_0 - \sigma^i_1\right)\right) = \sum_{i=1}^{k}(-1)^i\left(\partial(\sigma^i_0) - \partial(\sigma^i_1)\right) \\
&= \sum_{i=1}^{k}(-1)^i \sum_{j=1}^{k-1}(-1)^j\left((\sigma^i_0)^j_0 - (\sigma^i_0)^j_1 - (\sigma^i_1)^j_0 + (\sigma^i_1)^j_1\right) \\
&= \sum_{1\leq i\leq j\leq k-1} (-1)^{i+j}\left((\sigma^i_0)^j_0 - (\sigma^i_0)^j_1 - (\sigma^i_1)^j_0 + (\sigma^i_1)^j_1\right) \\
&\quad + \sum_{1\leq j<i\leq k} (-1)^{i+j}\left((\sigma^i_0)^j_0 - (\sigma^i_0)^j_1 - (\sigma^i_1)^j_0 + (\sigma^i_1)^j_1\right) \\
&= S_1 + S_2
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \sum_{1\leq i\leq j\leq k-1} (-1)^{i+j}\left((\sigma^i_0)^j_0 - (\sigma^i_0)^j_1 - (\sigma^i_1)^j_0 + (\sigma^i_1)^j_1\right), \\
S_2 &= \sum_{1\leq j<i\leq k} (-1)^{i+j}\left((\sigma^i_0)^j_0 - (\sigma^i_0)^j_1 - (\sigma^i_1)^j_0 + (\sigma^i_1)^j_1\right).
\end{aligned}$$

By (1),

$$S_2 = \sum_{1\leq j<i\leq k} (-1)^{i+j}\left((\sigma^j_0)^{i-1}_0 - (\sigma^j_0)^{i-1}_1 - (\sigma^j_1)^{i-1}_0 + (\sigma^j_1)^{i-1}_1\right).$$

Note that each term in this last summation occurs exactly once in the summation S_1 with the opposite sign. $\square$

Proposition 14.11. *Let $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ be a smooth map. Then for all $k \geq 1$, the diagram*

$$\begin{array}{ccc} Q_k(M,\mathcal{S}) & \xrightarrow{f_*} & Q_k(N,\mathcal{T}) \\ \partial \downarrow & & \downarrow \partial \\ Q_{k-1}(M,\mathcal{S}) & \xrightarrow[f_*]{} & Q_{k-1}(N,\mathcal{T}) \end{array}$$

commutes.

Proof. Since all maps in the above diagram are **R**-homomorphisms, it is sufficient to check that for each smooth k-cube σ in $(M, \mathcal{S})$, $\partial(f_*\sigma) = f_*(\partial\sigma)$.

Note that for $1 \leq i \leq k$ and $\varepsilon = 0, 1$,

$$(f_*\sigma)^i_\varepsilon = f_*(\sigma^i_\varepsilon).$$

Thus,

$$\begin{aligned} f_*(\partial\sigma) &= f_*\left(\sum_{i=1}^{k}(-1)^i\left(\sigma^i_0 - \sigma^i_1\right)\right) = \sum_{i=1}^{k}(-1)^i\left(f_*(\sigma^i_0) - f_*(\sigma^i_1)\right) \\ &= \sum_{i=1}^{k}(-1)^i\left((f_*\sigma)^i_0 - (f_*\sigma)^i_1\right) = \partial(f_*\sigma). \qquad \square \end{aligned}$$

Theorem 14.12. (Generalized Stokes' Theorem) *Let $(M, \mathcal{S})$ be a smooth manifold, $k \geq 0$, $\omega \in \Omega^k(M, \mathcal{S})$, and $c \in Q_{k+1}(M, \mathcal{S})$. Then*

$$\int_c d\omega = \int_{\partial c} \omega.$$

Proof. Since both sides are **R**-linear in c, it suffices to treat the case $c = \sigma$, a smooth $(k+1)$-cube in $(M, \mathcal{S})$. Let $\widetilde{\sigma}$ be a smooth extension of σ to an open set U in $\mathbf{R}^{k+1}$. By Proposition 12.13 we can write

$$\widetilde{\sigma}^*\omega = \sum_{i=1}^{k+1} \psi_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{k+1}$$

where the ψ_i are smooth real-valued functions on U. By Theorem 13.15(d) and Definition 13.1 we have

$$\begin{aligned} \widetilde{\sigma}^*(d\omega) = d(\widetilde{\sigma}^*\omega) &= \sum_{i=1}^{k+1} d\psi_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{k+1} \\ &= \sum_{i=1}^{k+1}\sum_{j=1}^{k+1} D_j\psi_i \wedge dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{k+1}. \end{aligned}$$

By Corollary 12.7(b),

$$dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{k+1} = \begin{cases} 0 & \text{if } j \neq i, \\ (-1)^{i-1} dx_1 \wedge \cdots \wedge dx_{k+1} & \text{if } j = i. \end{cases}$$

Thus

$$\widetilde{\sigma}^*(d\omega) = \left(\sum_{i=1}^{k+1} (-1)^{i-1} D_i \psi_i \right) \wedge dx_1 \wedge \cdots \wedge dx_{k+1},$$

i.e.

$$\beta_{\sigma, d\omega} = \sum_{i=1}^{k+1} (-1)^{i-1} D_i \psi_i.$$

Thus

$$\begin{aligned}
\int_\sigma d\omega &= \int_{I^{k+1}} \left(\sum_{i=1}^{k+1} (-1)^{i-1} D_i \psi_i \right) dV_{k+1} \\
&= \sum_{i=1}^{k+1} (-1)^{i-1} \underbrace{\int_0^1 \cdots \int_0^1}_{k+1} D_i \psi_i \, dx_1 \cdots dx_{k+1} \\
&= \sum_{i=1}^{k+1} (-1)^{i-1} \underbrace{\int_0^1 \cdots \int_0^1}_{k+1} D_i \psi_i \, dx_i \, dx_1 \cdots \widehat{dx_i} \cdots dx_{k+1} \\
&\qquad\qquad \text{(by Fubini's Theorem)} \\
&= \sum_{i=1}^{k+1} (-1)^{i-1} \underbrace{\int_0^1 \cdots \int_0^1}_{k} \Big(\psi_i(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{k+1}) \\
&\qquad - \psi_i(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{k+1}) \Big) \, dx_1 \cdots \widehat{dx_i} \cdots dx_{k+1} \\
&\qquad\qquad \text{(by the Fundamental Theorem of Calculus)} \\
&= \sum_{i=1}^{k+1} (-1)^{i} \underbrace{\int_0^1 \cdots \int_0^1}_{k} \Big(\psi_i(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{k+1}) \\
&\qquad - \psi_i(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{k+1}) \Big) \, dx_1 \cdots \widehat{dx_i} \cdots dx_{k+1} \\
&= \sum_{i=1}^{k+1} (-1)^{i} \underbrace{\int_0^1 \cdots \int_0^1}_{k} \Big(\psi_i(y_1, \ldots, y_{i-1}, 0, y_i, \ldots, y_k) \\
&\qquad - \psi_i(y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_k) \Big) \, dy_1 \cdots dy_k.
\end{aligned}$$

On the other hand,

$$\int_{\partial\sigma} \omega = \sum_{i=1}^{k+1} (-1)^i \left(\int_{\sigma_0^i} \omega - \int_{\sigma_1^i} \omega \right).$$

Thus we will be done if we show that for $1 \leq i \leq k+1$ and $\varepsilon = 0, 1$,

$$(1) \qquad \int_{\sigma_\varepsilon^i} \omega = \underbrace{\int_0^1 \cdots \int_0^1}_{k} \psi_i(y_1, \ldots, y_{i-1}, \varepsilon, y_i, \ldots, y_k)\, dy_1 \cdots dy_k.$$

Let $j_\varepsilon^i : \mathbf{R}^k \to \mathbf{R}^{k+1}$ be as in the paragraph preceding Definition 14.9. Then $\widetilde{\sigma} j_\varepsilon^i$ is a smooth extension of σ_ε^i, and

$$\begin{aligned}(\widetilde{\sigma} j_\varepsilon^i)^*(\omega) &= (j_\varepsilon^i)^*(\widetilde{\sigma}^*\omega) = (j_\varepsilon^i)^*\left(\sum_{j=1}^{k+1} \psi_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k+1}\right) \\ &= \sum_{j=1}^{k+1} (j_\varepsilon^i)^*(\psi_j) \wedge (j_\varepsilon^i)^*(dx_1) \wedge \cdots \wedge \widehat{(j_\varepsilon^i)^*(dx_j)} \wedge \cdots \wedge (j_\varepsilon^i)^*(dx_{k+1}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Corollary 12.7(f)).}\end{aligned}$$

Writing $y_1, \ldots, y_k$ for the coordinate functions on $\mathbf{R}^k$ we have, by Theorem 12.14,

$$(j_\varepsilon^i)^*(dx_q) = \sum_{s=1}^{k} D_s(j_\varepsilon^i)_q\, dy_s = \begin{cases} dy_q & \text{if } 1 \leq q \leq i-1, \\ 0 & \text{if } q = i, \\ dy_{q-1} & \text{if } i+1 \leq q \leq k+1, \end{cases}$$

and $(j_\varepsilon^i)^*(\psi_j) = \psi_j j_\varepsilon^i$ by Proposition 12.9(b). Thus, the only possible non-zero contribution to the above summation for $(\widetilde{\sigma} j_\varepsilon^i)^*(\omega)$ occurs when $(j_\varepsilon^i)^*(dx_i)$ is omitted, i.e. when $j = i$. Thus,

$$(\widetilde{\sigma} j_\varepsilon^i)^*(\omega) = \psi_i j_\varepsilon^i \wedge dy_1 \wedge \cdots \wedge dy_k$$

and so $\beta_{\sigma_\varepsilon^i, \omega} = \psi_i j_\varepsilon^i$. Thus

$$\begin{aligned}\int_{\sigma_\varepsilon^i} \omega &= \int_{I^k} \psi_i j_\varepsilon^i\, dV_k \\ &= \underbrace{\int_0^1 \cdots \int_0^1}_{k} \psi_i(y_1, \ldots, y_{i-1}, \varepsilon, y_i, \ldots, y_k)\, dy_1 \cdots dy_k,\end{aligned}$$

establishing (1). □

We will shortly use the Generalized Stokes' Theorem to prove the non-triviality of certain de Rham cohomology groups. In order to facilitate this we first make some definitions.

Definition 14.13. Let $(M, \mathcal{S})$ be a smooth manifold and $k \geq 0$. We denote the image of $\partial : Q_{k+1}(M, \mathcal{S}) \to Q_k(M, \mathcal{S})$ by $B_k(M, \mathcal{S})$, and the kernel of $\partial : Q_k(M, \mathcal{S}) \to Q_{k-1}(M, \mathcal{S})$ by $Z_k(M, \mathcal{S})$. (By convention, $Q_i(M, \mathcal{S}) = 0$ if $i < 0$

and so $Z_0(M,\mathcal{S}) = Q_0(M,\mathcal{S})$.) Members of $Z_k(M,\mathcal{S})$ are called *smooth cubical k-cycles in* $(M,\mathcal{S})$, and members of $B_k(M,\mathcal{S})$ are called *smooth cubical k-boundaries in* $(M,\mathcal{S})$.

It is immediate from Proposition 14.10 that $B_k(M,\mathcal{S}) \subset Z_k(M,\mathcal{S})$ and so, in analogy with our formation of de Rham cohomology groups, we can form the quotient $Z_k(M,\mathcal{S})/B_k(M,\mathcal{S})$. The latter is called the k^{th} *unnormalized real homology group of* $(M,\mathcal{S})$ and denoted $H_k^U(M,\mathcal{S})$. These are related to, but not equal to, the *normalized* homology groups usually used in algebraic topology. Later we will describe the modification needed to obtain the normalized homology groups and indicate the relation between the latter and the de Rham cohomology groups (the *de Rham Theorem*). For the present we will simply use the terminology of cycles and boundaries, along with the Generalized Stokes' Theorem, to obtain information about de Rham cohomology.

Corollary 14.14. *Let* $(M,\mathcal{S})$ *be a smooth manifold. Suppose* ω *is a closed k-form on* $(M,\mathcal{S})$ *and c a smooth cubical k-cycle in* $(M,\mathcal{S})$. *Then:*

(a) *If* ω *is exact, then* $\int_c \omega = 0$.

(b) *If c is a smooth cubical k-boundary in* $(M,\mathcal{S})$, *then* $\int_c \omega = 0$.

Proof. If ω is exact, then $\omega = d\rho$ for some $(k-1)$-form ρ on $(M,\mathcal{S})$. By the Generalized Stokes' Theorem,

$$\int_c \omega = \int_c d\rho = \int_{\partial c} \rho = 0$$

since $\partial c = 0$, proving part (a).

If $c \in B_k(M,\mathcal{S})$, then $c = \partial e$ for some $e \in Q_{k+1}(M,\mathcal{S})$. By the Generalized Stokes' Theorem,

$$\int_c \omega = \int_{\partial e} \omega = \int_e d\omega = 0$$

since $d\omega = 0$, proving part (b). ☐

Corollary 14.15. *Let* $(M,\mathcal{S})$ *be a smooth manifold. Suppose* ω *is a closed k-form on* $(M,\mathcal{S})$ *and c a smooth cubical k-cycle in* $(M,\mathcal{S})$ *such that* $\int_c \omega \neq 0$. *Then* ω *is not an exact form on* $(M,\mathcal{S})$ *and c is not a smooth cubical k-boundary in* $(M,\mathcal{S})$. *In particular,* $H^k_{dR}(M,\mathcal{S})$ *and* $H_k^U(M,\mathcal{S})$ *are both non-zero.* ☐

Theorem 14.16. *Suppose* $n \geq 1$ *and let*

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i}{\|x\|^n} \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \in \Omega^{n-1}(\mathbf{R}^n - \{0\})$$

where $\|x\|$ *denotes the Euclidean norm of x. Then* ω *is a closed* $(n-1)$*-form on* $\mathbf{R}^n - \{0\}$ *which is not exact. In particular,* $H^{n-1}_{dR}(\mathbf{R}^n - \{0\}) \neq 0$.

Proof. ω is clearly an $(n-1)$ form on $\mathbf{R}^n - \{0\}$. We first check that ω is closed. By Definition 13.2,

$$\begin{aligned} d\omega &= \sum_{i=1}^{n} (-1)^{i-1} d\left(\frac{x_i}{\|x\|^n}\right) \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \sum_{j=1}^{n} D_j\left(\frac{x_i}{\|x\|^n}\right) \wedge dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n. \end{aligned}$$

By Corollary 12.7(b),

$$dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n = \begin{cases} (-1)^{i-1} dx_1 \wedge \cdots \wedge dx_n & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and so

$$d\omega = \sum_{i=1}^{n} D_i\left(\frac{x_i}{\|x\|^n}\right) \wedge dx_1 \wedge \cdots \wedge dx_n.$$

Thus, to show $d\omega = 0$, it remains only to check that

$$\sum_{i=1}^{n} D_i\left(\frac{x_i}{\|x\|^n}\right) = 0.$$

By Calculus, for $1 \leq i \leq n$ we have

$$D_i\left(\frac{x_i}{\|x\|^n}\right) = \frac{\|x\|^2 - n x_i^2}{\|x\|^{n+2}}$$

and so

$$\begin{aligned} \sum_{i=1}^{n} D_i\left(\frac{x_i}{\|x\|^n}\right) &= \sum_{i=1}^{n} \frac{\|x\|^2 - n x_i^2}{\|x\|^{n+2}} \\ &= \frac{n\|x\|^2}{\|x\|^{n+2}} - \frac{n}{\|x\|^{n+2}} \sum_{i=1}^{n} x_i^2 = 0, \end{aligned}$$

completing the check that $d\omega = 0$.

To show that ω is not exact it suffices, by Corollary 14.15, to show that for some smooth cubical $(n-1)$-cycle c in $\mathbf{R}^n - \{0\}$, $\int_c \omega \neq 0$.

Let $\sigma : I^n \to \mathbf{R}^n$ be given by $\sigma(x) = x - (\frac{1}{2}, \ldots, \frac{1}{2})$. σ is a smooth n-cube in $\mathbf{R}^n$. In fact, the function given by the same formula on all of $\mathbf{R}^n$ is a smooth extension of σ. Note that for $1 \leq i \leq n$ and $\varepsilon = 0, 1$, σ_ε^i has image contained in $\mathbf{R}^n - \{0\}$. Thus if we write $f : \mathbf{R}^n - \{0\} \to \mathbf{R}^n$ for the inclusion map, we can write $\sigma_\varepsilon^i = f\tau_{\varepsilon,i}$ where the $\tau_{\varepsilon,i}$ are smooth $(n-1)$-cubes in $\mathbf{R}^n - \{0\}$. Explicitly,

$$\tau_{\varepsilon,i}(x_1, \ldots, x_{n-1}) = (x_1 - \tfrac{1}{2}, \ldots, x_{i-1} - \tfrac{1}{2}, \varepsilon - \tfrac{1}{2}, x_i - \tfrac{1}{2}, \ldots, x_{n-1} - \tfrac{1}{2}).$$

Let

$$c = \sum_{i=1}^{n} (-1)^i (\tau_{0,i} - \tau_{1,i}) \in Q_{n-1}(\mathbf{R}^n - \{0\}).$$

Note that

$$f_*(c) = \sum_{i=1}^{n} (-1)^i (f\tau_{0,i} - f\tau_{1,i}) = \sum_{i=1}^{n} (-1)^i (\sigma_0^i - \sigma_1^i) = \partial\sigma.$$

Thus, by Propositions 14.11 and 14.10, $f_*(\partial c) = \partial(f_* c) = \partial(\partial c) = 0$. Since f is injective, it follows easily that $f_* : Q_{n-1}(\mathbf{R}^n - \{0\}) \to Q_{n-1}(\mathbf{R}^n)$ is injective since distinct smooth $(n-1)$- cubes are sent by f_* to distinct smooth $(n-1)$-cubes and the latter are linearly independent over $\mathbf{R}$. Thus $\partial c = 0$, i.e. c is a smooth cubical $(n-1)$-cycle in $\mathbf{R}^n - \{0\}$. We will be done if we show $\int_c \omega \neq 0$.

For $1 \le i \le n$ and $\varepsilon = 0, 1$, let $\widetilde{\tau}_{\varepsilon,i} : \mathbf{R}^{n-1} \to \mathbf{R}^n - \{0\}$ be the map given by the same formula as that for $\tau_{\varepsilon,i}$. $\widetilde{\tau}_{\varepsilon,i}$ is a smooth extension of $\tau_{\varepsilon,i}$. For $1 \le i \le n$ let $g_i : \mathbf{R}^n - \{0\} \to \mathbf{R}$ be given by $g_i(x) = \dfrac{x_i}{\|x\|^n}$. Then by Corollary 12.7(f),

$$\widetilde{\tau}^*_{\varepsilon,i}\omega = \sum_{j=1}^{n} (-1)^{j-1} \widetilde{\tau}^*_{\varepsilon,i}(g_j) \wedge \widetilde{\tau}^*_{\varepsilon,i}(dx_1) \wedge \cdots \wedge \widehat{\widetilde{\tau}^*_{\varepsilon,i}(dx_j)} \wedge \cdots \wedge \widetilde{\tau}^*_{\varepsilon,i}(dx_n).$$

We have, by Theorem 12.14,

$$\widetilde{\tau}^*_{\varepsilon,i}(dx_j) = \sum_{k=1}^{n-1} D_k(\widetilde{\tau}_{\varepsilon,i})_j \wedge dx_k.$$

Since

$$\widetilde{\tau}_{\varepsilon,i}(x_1, \ldots, x_{n-1}) = \begin{cases} x_j - \frac{1}{2} & \text{if } 1 \le j \le i-1, \\ \varepsilon - \frac{1}{2} & \text{if } j = i, \\ x_{j-1} - \frac{1}{2} & \text{if } i+1 \le j \le n, \end{cases}$$

we obtain

$$D_k(\widetilde{\tau}_{\varepsilon,i})_j = \begin{cases} 1 & \text{if } 1 \le j \le i-1 \text{ and } k = j, \\ 1 & \text{if } i+1 \le j \le n \text{ and } k = j-1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\widetilde{\tau}^*_{\varepsilon,i}(dx_j) = \begin{cases} dx_j & \text{if } 1 \le j \le i-1, \\ 0 & \text{if } j = i, \\ dx_{j-1} & \text{if } i+1 \le j \le n. \end{cases}$$

Thus the only possible non-zero contribution in the above summation for $\widetilde{\tau}^*_{\varepsilon,i}\omega$ occurs when $\widetilde{\tau}^*_{\varepsilon,i}(dx_i)$ is omitted, i.e. when $j = i$. Thus,

$$\widetilde{\tau}^*_{\varepsilon,i}\omega = (-1)^{i-1} \widetilde{\tau}^*_{\varepsilon,i}(g_i) \wedge dx_1 \wedge \cdots \wedge dx_{n-1}.$$

Since $\widetilde{\tau}_{\varepsilon,i}^{*}(g_i) = g_i\widetilde{\tau}_{\varepsilon,i}$ by Proposition 12.9(b), it follows that

$$\beta_{\tau_{\varepsilon,i},\omega} = (-1)^{i-1} g_i \tau_{\varepsilon,i},$$

and so

$$\int_{\tau_{\varepsilon,i}} \omega = (-1)^{i-1} \int_{I^{n-1}} g_i \tau_{\varepsilon,i}\, dV_{n-1}.$$

Thus

$$\begin{aligned}
\int_c \omega &= \sum_{i=1}^{n} (-1)^i \Big(\int_{\tau_{0,i}} \omega - \int_{\tau_{1,i}} \omega \Big) \\
&= \sum_{i=1}^{n} (-1)^i (-1)^{i-1} \Big(\int_{I^{n-1}} g_i \tau_{0,i}\, dV_{n-1} - \int_{I^{n-1}} g_i \tau_{1,i}\, dV_{n-1} \Big) \\
&= \sum_{i=1}^{n} \int_{I^{n-1}} \Big(g_i \tau_{1,i} - g_i \tau_{0,i} \Big)\, dV_{n-1}.
\end{aligned}$$

For all $x = (x_1, \ldots, x_{n-1}) \in I^{n-1}$,

$$\begin{aligned}
g_i\tau_{1,i}(x) - g_i\tau_{0,i}(x) &= g_i(x_1 - \tfrac{1}{2}, \ldots, x_{i-1} - \tfrac{1}{2}, \tfrac{1}{2}, x_i - \tfrac{1}{2}, \ldots, x_{n-1} - \tfrac{1}{2}) \\
&\quad - g_i(x_1 - \tfrac{1}{2}, \ldots, x_{i-1} - \tfrac{1}{2}, -\tfrac{1}{2}, x_i - \tfrac{1}{2}, \ldots, x_{n-1} - \tfrac{1}{2}) \\
&= \frac{\frac{1}{2}}{\|\tau_{0,i}(x)\|^n} - \frac{-\frac{1}{2}}{\|\tau_{1,i}(x)\|^n} = \frac{1}{\|\tau_{1,i}(x)\|^n} > 0
\end{aligned}$$

since $\|\tau_{0,i}(x)\| = \|\tau_{1,i}(x)\|$. Thus $\displaystyle\int_{I^{n-1}} \Big(g_i\tau_{1,i} - g_i\tau_{0,i} \Big)\, dV_{n-1} > 0$ for $1 \leq i \leq n$ and so $\displaystyle\int_c \omega > 0$, completing the proof. □

Theorem 14.16 will later be used, in conjunction with some additional results on de Rham cohomology, to prove a purely topological result known as the Brouwer Fixed-Point Theorem.

In the remainder of this section we look at the relation between the Generalized Stokes' Theorem and classical vector analysis results, including the classical Stokes' Theorem.

Lemma 14.17. *Let U and V be open subsets of $\mathbf{R}^n$ and $f : U \to V$ a smooth map. Then*

$$f^*(dx_1 \wedge \cdots \wedge dx_n) = \det(Df) \wedge dx_1 \wedge \cdots \wedge dx_n.$$

Proof. By Corollary 12.7(f) and Theorem 12.14,

$$f^*(dx_1 \wedge \cdots \wedge dx_n) = \Big(\sum_{j=1}^{n} D_j f_1 \wedge dx_j \Big) \wedge \cdots \wedge \Big(\sum_{j=1}^{n} D_j f_n \wedge dx_j \Big).$$

By the same calculation as used in the proof of Theorem 10.20, the latter is $\det(Df) \wedge dx_1 \wedge \cdots \wedge dx_n$. □

Recall, by Remark 13.8, that if U is an open subset of $\mathbf{R}^n$, smooth vector fields on U are in one-to-one correspondence with 1-forms on U. Explicitly, if F is a smooth vector field on U with component functions $F_1, \ldots, F_n$, i.e. the F_i are smooth real-valued functions on U and $F(x) = \widetilde{1_U}^{-1}\left(x, \sum_{i=1}^n F_i(x)e_i\right)$, we associate with F the 1-form $\Psi(F) = \sum_{i=1}^n F_i \wedge dx_i$. F as above is classically written $\sum_{i=1}^n F_i e_i$ or, in case $n = 3$, $F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$.

For U as above, a subset X of U is a *k-patch* if it is the image of a smooth k-cube σ in U such that σ has a smooth extension $\widetilde{\sigma}$ which is a diffeomorphism onto a k-dimensional smooth submanifold of U. Call such a σ a *parametrization* of the k-patch X. Two parametrizations σ,τ of X determine the *same orientation of* X if $D(\widetilde{\tau}^{-1}\widetilde{\sigma})$ has positive determinant at all points of I^k. An *oriented k-patch X in U* is a k-patch in U with a choice of orientation class of parametrizations. Any representative of the latter is called an *orientation-preserving parametrization of* X.

In the special case of an n-patch X in U, there is a natural orientation, namely the class of parametrizations σ for which $\det(D\widetilde{\sigma}) > 0$ at all points of I^n.

Let $f : U \to \mathbf{R}$ be smooth and X an n-patch in U. By the classical change of variables formula for multiple integrals,

$$\int_X f\, dV_n = \int_{I^n} (f\sigma) \cdot |\det(D\widetilde{\sigma})|\, dV_n$$

where the dot denotes point-wise multiplication. If X is given the natural orientation and σ is orientation-preserving, then the absolute value bars in the above formula can be dropped. It follows from Lemma 14.17, Corollary 12.7(f), and Proposition 12.9 that

$$(f\widetilde{\sigma}) \cdot \det(D\widetilde{\sigma}) \wedge dx_1 \wedge \cdots \wedge dx_n = \widetilde{\sigma}^*(f \wedge dx_1 \wedge \cdots \wedge dx_n) = \widetilde{\sigma}^*(*f)$$

where $*$ denotes the Hodge star operator. Thus,

Observation 14.18. *Let U be open in $\mathbf{R}^n$, $f : U \to \mathbf{R}$ a smooth map, X an n-patch in U, and $\sigma : I^n \to U$ a parametrization of X which preserves the natural orientation of X. Then*

$$\int_X f\, dV_n = \int_\sigma *f. \qquad \square$$

1-patches and 2-patches will be called *curve* and *surface patches*, respectively. If C is an oriented curve patch in U and F a smooth vector field on U, the *line integral of the tangential component of F along C*, denoted $\int_C F \cdot \mathbf{T}\, ds$ (which we will not define here) is studied in calculus and found to be computable as follows: If σ is any orientation-preserving parametrization of C, then

$$\int_C F \cdot \mathbf{T}\, ds = \int_I \left(\sum_{i=1}^n (F_i\widetilde{\sigma}) \cdot \widetilde{\sigma}_i'\right) dV_1$$

where $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_n$ are the coordinate functions of $\widetilde{\sigma}$ and the dot on the right denotes point-wise multiplication. From Corollary 12.7(f) and Theorem 12.14,

$$\widetilde{\sigma}^*\big(\Psi(F)\big) = \widetilde{\sigma}^*\Big(\sum_{i=1}^{n} F_i \wedge dx_i\Big) = \sum_{i=1}^{n} F_i\widetilde{\sigma} \wedge \widetilde{\sigma}^*(dx_i) = \sum_{i=1}^{n} (F_i\widetilde{\sigma}) \cdot \widetilde{\sigma}_i' \wedge dx_1.$$

Thus

Observation 14.19. *Let U be open in $\mathbf{R}^n$, C an oriented curve patch in U, and F a smooth vector field on U. Then for any orientation-preserving parametrization σ of C,*

$$\int_C F \cdot \mathbf{T}\, ds = \int_\sigma \Psi(F). \qquad \square$$

Now suppose U is open in $\mathbf{R}^2$, C an oriented curve patch in U, and F a smooth vector field on U. The *line integral of the normal component of F along C*, denoted $\int_C F \cdot \mathbf{n}\, ds$, is studied in calculus and found to be computable as follows: If σ is any orientation-preserving parametrization of C, then

$$\int_C F \cdot \mathbf{n}\, ds = \int_I \Big((F_1\widetilde{\sigma}) \cdot \widetilde{\sigma}_2' - (F_2\widetilde{\sigma}) \cdot \widetilde{\sigma}_1'\Big)\, dV_1.$$

From Corollary 12.7(f) and Theorem 12.14,

$$\begin{aligned}\widetilde{\sigma}^*\big({*\Psi(F)}\big) &= \widetilde{\sigma}^*(F_1 \wedge dx_2 - F_2 \wedge dx_1) = (F_1\widetilde{\sigma}) \wedge \widetilde{\sigma}^*(dx_2) - (F_2\widetilde{\sigma}) \wedge \widetilde{\sigma}^*(dx_1) \\ &= \Big((F_1\widetilde{\sigma}) \cdot \widetilde{\sigma}_2' - (F_2\widetilde{\sigma}) \cdot \widetilde{\sigma}_1'\Big) \wedge dx_1.\end{aligned}$$

Thus

Observation 14.20. *Let U be open in $\mathbf{R}^2$, C an oriented curve patch in U, and F a smooth vector field on U. Then for any orientation-preserving parametrization σ of C,*

$$\int_C F \cdot \mathbf{n}\, ds = \int_\sigma *\Psi(F). \qquad \square$$

Now suppose U is open in $\mathbf{R}^3$ and $\mathcal{S}$ is a smooth oriented surface patch in U. Let F be a smooth vector field on U. The *surface integral of the normal component of F along $\mathcal{S}$*, denoted $\int_{\mathcal{S}} F \cdot \mathbf{n}\, dS$, is studied in calculus and is computable as follows: If σ is any orientation-preserving parametrization of $\mathcal{S}$, then

$$\int_{\mathcal{S}} F \cdot \mathbf{n}\, dS = \int_{I^2} \Big((F_1\widetilde{\sigma}) \cdot J_{2,3}(\widetilde{\sigma}) - (F_2\widetilde{\sigma}) \cdot J_{1,3}(\widetilde{\sigma}) + (F_3\widetilde{\sigma}) \cdot J_{1,2}(\widetilde{\sigma})\Big)\, dV_2$$

where

$$J_{i,j}(\widetilde{\sigma}) = \det\begin{pmatrix} D_1\widetilde{\sigma}_i & D_1\widetilde{\sigma}_j \\ D_2\widetilde{\sigma}_i & D_2\widetilde{\sigma}_j \end{pmatrix}.$$

From Corollary 12.7(f) and Theorem 12.14,

$$\begin{aligned}
\widetilde{\sigma}^*\big(*\Psi(F)\big) &= \widetilde{\sigma}^*\Big(*(F_1 \wedge dx_1 + F_2 \wedge dx_2 + F_3 \wedge dx_3)\Big) \\
&= \widetilde{\sigma}^*(F_1 \wedge dx_2 \wedge dx_3 - F_2 \wedge dx_1 \wedge dx_3 + F_3 \wedge dx_1 \wedge dx_2) \\
&= (F_1\widetilde{\sigma}) \wedge \widetilde{\sigma}^*(dx_2) \wedge \widetilde{\sigma}^*(dx_3) - (F_2\widetilde{\sigma}) \wedge \widetilde{\sigma}^*(dx_1) \wedge \widetilde{\sigma}^*(dx_3) \\
&\qquad + (F_3\widetilde{\sigma}) \wedge \widetilde{\sigma}^*(dx_1) \wedge \widetilde{\sigma}^*(dx_2) \\
&= \Big((F_1\widetilde{\sigma}) \cdot J_{2,3}(\widetilde{\sigma}) - (F_2\widetilde{\sigma}) \cdot J_{1,3}(\widetilde{\sigma}) + (F_3\widetilde{\sigma}) \cdot J_{1,2}(\widetilde{\sigma})\Big) \wedge dx_1 \wedge dx_2.
\end{aligned}$$

Thus

Observation 14.21. *Let U be open in $\mathbf{R}^3$, $\mathcal{S}$ an oriented surface patch in U, and F a smooth vector field on U. Then for any orientation-preserving parametrization σ of $\mathcal{S}$,*

$$\int_{\mathcal{S}} F \cdot \mathbf{n}\, dS = \int_\sigma *\Psi(F). \qquad \square$$

Suppose U is open in $\mathbf{R}^n$ and let σ be a smooth k-cube in U which has a smooth extension to a diffeomorphism onto a smooth k-dimensional submanifold of U. Write X_σ for the image of σ. Thus X_σ is a k-patch in U, and we orient it by choosing the orientation containing σ. Define ∂X_σ to be the union of the images of the faces σ^i_ε, $1 \le i \le k$, $\varepsilon = 0, 1$. Thus $\partial X_\sigma = X_\sigma - \sigma\big((0,1)^k\big)$. ∂X_σ is the union of the $(k-1)$-patches parametrized by the faces of σ. Write $X^i_{\sigma,\varepsilon} = \sigma^i_\varepsilon(I^{k-1})$. We orient $X^i_{\sigma,\varepsilon}$ as follows: the parametrization σ^i_ε is orientation-preserving if σ^i_ε occurs with coefficient $+1$ in the expression for $\partial\sigma$, and orientation-reversing otherwise. These are called the *induced orientations on the* $X^i_{\sigma,\varepsilon}$. The following is then a consequence of Observations 14.19, 14.20, and 14.21:

Proposition 14.22. *With notation as above, let F be a smooth vector field on U. Then:*

(a) *If $k = 2$, then*

$$\int_{\partial X_\sigma} F \cdot \mathbf{T}\, ds = \int_{\partial\sigma} \Psi(F).$$

(b) *If $n = k = 2$, then*

$$\int_{\partial X_\sigma} F \cdot \mathbf{n}\, ds = \int_{\partial\sigma} *\Psi(F).$$

(c) *If $n = k = 3$, then*

$$\int_{\partial X_\sigma} F \cdot \mathbf{n}\, dS = \int_{\partial\sigma} *\Psi(F). \qquad \square$$

We proceed now to apply the Generalized Stokes' Theorem (Theorem 14.12) to each of the cases of Proposition 14.22.

If $n = 3$ and $k = 2$, then

$$\begin{aligned}
\int_{\partial X_\sigma} F \cdot \mathbf{T}\, ds &= \int_{\partial\sigma} \Psi(F) && \text{(by Proposition 14.22(a))} \\
&= \int_\sigma d\Psi(F) && \text{(by Theorem 14.12)} \\
&= \int_\sigma *\big(*d\Psi(F)\big) = \int_\sigma *\Psi(\operatorname{curl} F) && \text{(by Remark 13.8)} \\
&= \int_{X_\sigma} \operatorname{curl} F \cdot \mathbf{n}\, dS && \text{(by Observation 14.21).}
\end{aligned}$$

Thus we obtain the classical Stokes' Theorem.

If $k = n = 2$, then

$$\begin{aligned}
\int_{\partial X_\sigma} F \cdot \mathbf{n}\, ds &= \int_{\partial\sigma} *\Psi(F) && \text{(by Proposition 14.22(b))} \\
&= \int_\sigma d * \Psi(F) && \text{(by Theorem 14.12)} \\
&= \int_\sigma *\big(*d * \Psi(F)\big) = \int_\sigma *\Psi(\operatorname{div} F) && \text{(by Remark 13.8)} \\
&= \int_{X_\sigma} \operatorname{div} F\, dV_2 && \text{(by Observation 14.18).}
\end{aligned}$$

Thus we obtain the classical Green's Theorem.

If $k = n = 3$, then

$$\begin{aligned}
\int_{\partial X_\sigma} F \cdot \mathbf{n}\, dS &= \int_{\partial\sigma} *\Psi(F) && \text{(by Proposition 14.22(c))} \\
&= \int_\sigma d * \Psi(F) && \text{(by Theorem 14.12)} \\
&= \int_\sigma *\big(*d * \Psi(F)\big) = \int_\sigma *\Psi(\operatorname{div} F) && \text{(by Remark 13.8)} \\
&= \int_{X_\sigma} \operatorname{div} F\, dV_3 && \text{(by Observation 14.18).}
\end{aligned}$$

Thus we obtain the classical Divergence Theorem.

Exercises for §14

1. Let $(M, \mathcal{S})$ be a smooth manifold, $\omega \in \Omega^p(M, \mathcal{S})$, $\rho \in \Omega^q(M, \mathcal{S})$, and $c \in Q_{p+q+1}(M, \mathcal{S})$. Prove the following *integration by parts* formula:

$$\int_c (d\omega) \wedge \rho = \int_{\partial c} \omega \wedge \rho - (-1)^p \int_c \omega \wedge (d\rho).$$

2. Let $(M, \mathcal{S})$ be a smooth manifold. Suppose ω is a k-form on M.
   (a) Suppose $\int_\sigma \omega = 0$ for every smooth k-cube σ in $(M, \mathcal{S})$. Prove that $\omega = 0$.
   (b) Prove that ω is closed if and only if $\int_{\partial c} \omega = 0$ for all $c \in Q_{k+1}(M, \mathcal{S})$.
3. Let σ, $\tau : I^2 \to S^2$ be given as follows:

$$\sigma(x, y) = (\sin \pi y \, \cos 2\pi x, \sin \pi y \, \sin 2\pi x, \cos \pi y),$$
$$\tau(x, y) = (\sin \pi y, 0, \cos \pi y).$$

Let $c = \sigma - \tau \in Q_2(S^2)$.
   (a) Prove that c is a smooth cubical 2-cycle in S^2.
   (b) Let $i : S^2 \to \mathbf{R}^3$ denote the inclusion map. Let $\omega = i^*(z \wedge dx \wedge dy)$. Calculate $\int_c \omega$.
   (c) What can you conclude from (a) and (b) about $H^2_{dR}(S^2)$? Explain.
4. Let $\sigma : I^2 \to S^1 \times S^1$ be given by

$$\sigma(x, y) = \big((\cos 2\pi x, \sin 2\pi x), (\cos 2\pi y, \sin 2\pi y)\big).$$

   (a) Prove that σ is a smooth cubical 2-cycle in $S^1 \times S^1$.
   (b) Let $i : S^1 \to \mathbf{R}^2$ denote the inclusion map. Let $\omega = i^*(x \wedge dy)$. Let $\rho = \pi_1^*\omega \wedge \pi_2^*\omega$ where $\pi_1, \pi_2 : S^1 \times S^1 \to S^1$ are the projections on the first and second factors, respectively. Calculate $\int_\sigma \rho$.
   (c) What can you conclude from (a) and (b) about $H^2_{dR}(S^1 \times S^1)$? Explain.
5. Let σ, $\tau : I \to S^1 \times S^1$ be given by

$$\sigma(x) = \big((\cos 2\pi x, \sin 2\pi x), (0, 1)\big),$$
$$\tau(x) = \big((\cos 2\pi x, \sin 2\pi x), (0, -1)\big).$$

Prove that for every closed 1-form ω on $S^1 \times S^1$,

$$\int_\sigma \omega = \int_\tau \omega.$$

15. Smooth Homotopy Invariance of de Rham Cohomology

In this section we introduce a smooth version of homotopy between smooth maps and prove that smoothly homotopic maps induce the very same homomorphisms in de Rham cohomology. As a consequence, smooth manifolds of the same smooth homotopy type have isomorphic de Rham cohomology groups. This will greatly facilitate obtaining information about de Rham cohomology groups of certain manifolds (in particular, $\mathbf{R}^n$ and S^n). This cohomology information will then be used, in conjunction with analytic and topological arguments, to deduce the purely topological Brouwer Fixed-Point Theorem.

Definition 15.1. Suppose $f, g : (M, \mathcal{S}) \to (N, \mathcal{T})$ are smooth maps. We say f *is smoothly homotopic to* g (denoted $f \simeq g$) if a smooth map $h : M \times \mathbf{R} \to N$ exists such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in M$. Such an h is called a *smooth homotopy from* f *to* g. We write $f \simeq_h g$ to denote the statement "h is a smooth homotopy from f to g".

For topological homotopy, one uses I instead of $\mathbf{R}$. This will not do for our purposes since I is not a manifold. We could take the approach that we followed earlier for smooth cubes and replace $\mathbf{R}$ by an open interval (depending on h) which contains I. Our approach is equivalent to this since any open interval is diffeomorphic to $\mathbf{R}$, and has the slight notational advantage that the homotopy parameter space $\mathbf{R}$ is the same for all smooth homotopies.

Example 15.2. For an arbitrary smooth map $f : M \to \mathbf{R}^n$, let $h : M \times \mathbf{R} \to \mathbf{R}^n$ be given by $h(x, t) = (1 - t)f(x)$. Then h is a smooth homotopy from f to the constant map with value 0.

We wish to show that the relation $\simeq$ on the set of smooth maps from $(M, \mathcal{S})$ to $(N, \mathcal{T})$ is an equivalence relation. The usual topological homotopy argument for this works fine for the reflexive and symmetric properties, but fails for the transitive property since the topological pasting construction used there could destroy smoothness. The following lemma will allow us to replace the usual topological pasting construction by a smooth pasting construction.

Lemma 15.3. *Let a and b be real numbers with $a < b$. Then there exist smooth maps $\alpha_{a,b} : \mathbf{R} \to \mathbf{R}$ and $\beta_{a,b} : \mathbf{R} \to \mathbf{R}$ such that:*

(i) *$\alpha_{a,b}(x) = 0$ if $x \leq a$ or $x \geq b$, and $\alpha_{a,b}(x) > 0$ for $a < x < b$.*

(ii) *$\beta_{a,b}(x) = 0$ for $x \leq a$, $\beta_{a,b}(x) = 1$ for $x \geq b$, and $\beta_{a,b}(x)$ is strictly increasing for $a < x < b$.*

Proof. Let $f : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-x^{-2}} & \text{if } x > 0. \end{cases}$$

It is elementary to check that f is smooth everywhere (*including* 0) and that $f(x) > 0$ for $x > 0$. Take

$$\alpha_{a,b}(x) = f(x - a)f(b - x)$$

and

$$\beta_{a,b}(x) = \frac{\displaystyle\int_a^x \alpha_{a,b}(t)\,dt}{\displaystyle\int_a^b \alpha_{a,b}(t)\,dt}. \qquad \square$$

Proposition 15.4. *Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds. Then $\simeq$ is an equivalence relation on the set of all smooth maps from $(M, \mathcal{S})$ to $(N, \mathcal{T})$.*

Proof. Let $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ be smooth. Let $h : M \times \mathbf{R} \to N$ be the composition

$$M \times \mathbf{R} \xrightarrow{\pi_1} M \xrightarrow{f} N.$$

Then $f \simeq_h f$, and so $\simeq$ is reflexive.

Suppose $f, g : (M, \mathcal{S}) \to (N, \mathcal{T})$ are smooth and $f \simeq_h g$. Let $\eta : \mathbf{R} \to \mathbf{R}$ be given by $\eta(t) = 1 - t$. Then $g \simeq_k f$ where k is the composition

$$M \times \mathbf{R} \xrightarrow{1_M \times \eta} M \times \mathbf{R} \xrightarrow{h} N.$$

Thus $\simeq$ is symmetric.

Suppose $f, g, h : (M, \mathcal{S}) \to (N, \mathcal{T})$ are smooth and $f \simeq_j g$, $g \simeq_k h$. Let $\varphi, \psi : M \times \mathbf{R} \to N$ be the compositions

$$M \times \mathbf{R} \xrightarrow{1_M \times \beta_{0,1/3}} M \times \mathbf{R} \xrightarrow{j} N$$

and

$$M \times \mathbf{R} \xrightarrow{1_M \times \beta_{2/3,1}} M \times \mathbf{R} \xrightarrow{k} N,$$

respectively. Then φ and ψ are smooth. Note that φ and ψ agree on $M \times (\frac{1}{3}, \frac{2}{3})$. In fact, for $\frac{1}{3} < t < \frac{2}{3}$, $\varphi(x,t) = j(x,1) = g(x)$, $\psi(x,t) = k(x,0) = g(x)$. Thus we have a well-defined map $q : M \times \mathbf{R} \to N$ given by

$$q(x,t) = \begin{cases} \varphi(x,t) & \text{if } t < \frac{2}{3}, \\ \psi(x,t) & \text{if } t > \frac{1}{3}. \end{cases}$$

Since the restrictions of q to the open sets $M \times (-\infty, \frac{2}{3})$ and $M \times (\frac{1}{3}, \infty)$ are the restrictions of φ and ψ, respectively, which are smooth, it follows from Proposition 4.19(b) that q is smooth. Note that $f \simeq_q h$, and so $\simeq$ is transitive. $\square$

Proposition 15.5. *Let $f, g : (M, \mathcal{S}) \to (N, \mathcal{T})$ be smoothly homotopic maps. Then:*

(a) *For any smooth map $h : (Q, \mathcal{U}) \to (M, \mathcal{S})$, $fh \simeq gh$.*
(b) *For any smooth map $j : (N, \mathcal{T}) \to (Q, \mathcal{U})$, $jf \simeq jg$.*

Proof. Say $f \simeq_k g$. Then $k(h \times 1_{\mathbf{R}})$ is a smooth homotopy from fh to gh, and jk is a smooth homotopy from jf to jg. $\square$

Corollary 15.6. *Let $f_1, f_2 : (M, \mathcal{S}) \to (N, \mathcal{T})$ and $g_1, g_2 : (N, \mathcal{T}) \to (Q, \mathcal{U})$ be smooth maps such that $f_1 \simeq f_2$ and $g_1 \simeq g_2$. Then $g_1 f_1 \simeq g_2 f_2$.*

Proof. By Proposition 15.5, $g_1 f_1 \simeq g_1 f_2$ and $g_1 f_2 \simeq g_2 f_2$. The result now follows from Proposition 15.4. □

Definition 15.7. A smooth map $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ is called a *smooth homotopy equivalence* if there exists a smooth map $g : (N, \mathcal{T}) \to (M, \mathcal{S})$ such that $fg \simeq 1_N$ and $gf \simeq 1_M$. In this case we say that g is a *smooth homotopy inverse to f*, and that $(M, \mathcal{S})$ and $(N, \mathcal{T})$ are *smoothly homotopy equivalent.* We write $(M, \mathcal{S}) \simeq (N, \mathcal{T})$ to denote the statement that $(M, \mathcal{S})$ and $(N, \mathcal{T})$ are smoothly homotopy equivalent.

Proposition 15.8. *$\simeq$ is an equivalence relation on the class of all smooth manifolds.*

Proof. For any smooth manifold $(M, \mathcal{S})$, 1_M is a smooth homotopy equivalence from $(M, \mathcal{S})$ to itself, so $\simeq$ is reflexive.

Symmetry of $\simeq$ is immediate from the definition.

Suppose $(M, \mathcal{S}) \simeq (N, \mathcal{T})$ and $(N, \mathcal{T}) \simeq (Q, \mathcal{U})$. Say $f : (M, \mathcal{S}) \to (N, \mathcal{T})$, $g : (N, \mathcal{T}) \to (Q, \mathcal{U})$ are smooth homotopy equivalences with respective smooth homotopy inverses h and k. Thus $gf \simeq 1_M$, $fg \simeq 1_N$, $kh \simeq 1_N$, and $hk \simeq 1_Q$. Then, using Corollary 15.6 and Proposition 15.4,

$$(hf)(gk) = h(fg)k \simeq h 1_N k = hk \simeq 1_Q$$

and similarly $(gk)(hf) \simeq 1_M$. Thus hf is a smooth homotopy equivalence from $(M, \mathcal{S})$ to $(N, \mathcal{T})$, and so $\simeq$ is transitive. □

Example 15.9. Every diffeomorphism is a smooth homotopy equivalence.

Example 15.10. For any $m, n \geq 0$, $\mathbf{R}^m$ is smoothly homotopy equivalent to $\mathbf{R}^n$. For let $f : \mathbf{R}^m \to \mathbf{R}^n$ and $g : \mathbf{R}^n \to \mathbf{R}^m$ be arbitrary smooth maps (e.g. constant maps). By Example 15.2, both gf and $1_{\mathbf{R}^m}$ are smoothly homotopic to the constant map with value 0, and so $gf \simeq 1_{\mathbf{R}^m}$. Similarly, $fg \simeq 1_{\mathbf{R}^n}$. In particular, $\mathbf{R}^n \simeq \mathbf{R}^0$ for all $n \geq 0$.

Example 15.11. Suppose $n > 0$. Let $i : S^{n-1} \to \mathbf{R}^n - \{0\}$ denote the standard inclusion, and let $g : \mathbf{R}^n - \{0\} \to S^{n-1}$ be given by $g(x) = \dfrac{x}{\|x\|}$. Then g and i are smooth with respect to the standard smooth structures, and $gi = 1_{S^{n-1}}$. Let $h : \big(\mathbf{R}^n - \{0\}\big) \times \mathbf{R} \to \mathbf{R}^n - \{0\}$ be given by

$$h(x,t) = \frac{x}{1 + \beta_{0,1}(t)\big(\|x\| - 1\big)}.$$

Then h is a smooth homotopy from $1_{\mathbf{R}^n - \{0\}}$ to ig. Thus i and g are smooth homotopy equivalences.

As stated earlier, we want show that smoothly homotopic maps induce the very same homomorphisms in de Rham cohomology. The next lemma reduces this task to verifying a special case.

Lemma 15.12. *Let $(M,\mathcal{S})$ be a smooth manifold. Let $i_0, i_1 : M \to M \times \mathbf{R}$ be the inclusions given by $i_j(x) = (x,j)$, $j = 0,1$. Suppose $(M,\mathcal{S})$ has the property that*

$$H^k_{dR} i_0 = H^k_{dR} i_1 : H^k_{dR}(M \times \mathbf{R}, \mathcal{S} \times \mathcal{S}_{\mathbf{R}}) \to H^k_{dR}(M,\mathcal{S})$$

for all $k \geq 0$, where $\mathcal{S}_{\mathbf{R}}$ denotes the standard smooth structure on $\mathbf{R}$. Then for any smooth manifold $(N,\mathcal{T})$ and smooth maps $f,g : (M,\mathcal{S}) \to (N,\mathcal{T})$ such that $f \simeq g$,

$$H^k_{dR} f = H^k_{dR} g : H^k_{dR}(N,\mathcal{T}) \to H^k_{dR}(M,\mathcal{S})$$

for all $k \geq 0$.

Proof. Let h be a smooth homotopy from f to g. Then $f = hi_0$, $g = hi_1$ and so by Proposition 13.23,

$$H^k_{dR} f = H^k_{dR}(hi_0) = H^k_{dR} i_0 H^k_{dR} h = H^k_{dR} i_1 H^k_{dR} h = H^k_{dR}(hi_1) = H^k_{dR} g. \qquad \square$$

Thus, to prove that smoothly homotopic maps induce the same de Rham cohomology homomorphisms, it remains only to show that every smooth manifold $(M,\mathcal{S})$ satisfies the hypothesis of Lemma 15.12. We will make use of the result of Exercise 4 of §13 to show this, i.e. show that the cochain maps $i_0^*, i_1^* : \Omega^*(M \times \mathbf{R}, \mathcal{S} \times \mathcal{S}_{\mathbf{R}}) \to \Omega^*(M,\mathcal{S})$ are cochain homotopic.

The concept of cochain homotopy can be motivated by its dual concept, "chain homotopy" which has a geometric motivation as follows: Suppose $f,g : (M,\mathcal{S}) \to (N,\mathcal{T})$ are smooth maps and h is a smooth homotopy from f to g. For each smooth k-cube σ in $(M,\mathcal{S})$ let $T^k\sigma : I^{k+1} \to N$ be the composition

$$I^{k+1} = I^k \times I \subset I^k \times \mathbf{R} \xrightarrow{\sigma \times 1_{\mathbf{R}}} M \times \mathbf{R} \xrightarrow{h} N.$$

$T^k\sigma$ is a smooth $(k+1)$-cube in $(N,\mathcal{T})$. Let $T^k : Q_k(M,\mathcal{S}) \to Q_{k+1}(N,\mathcal{T})$ be the $\mathbf{R}$-homomorphism extending the above construction. Writing $f_k, g_k : Q_k(M,\mathcal{S}) \to Q_k(N,\mathcal{T})$ instead of f_*, g_* as earlier, it is straightforward to check that $\partial T^k + T^{k-1}\partial = g_k - f_k$ for all k. Thus the T^k constitute a "chain homotopy" from the chain map f_* to the chain map g_*. "Chain homotopy" plays a role in the category of chain complexes analogous to that of smooth homotopy in the category *Sm*. Having motivated the concept of chain homotopy in the category of chain complexes, cochain homotopy is motivated by the fact that it is the dual concept for the category of cochain complexes.

We carry out the task of showing that all smooth manifolds satisfy the hypothesis of Lemma 15.12 in two stages: We first show that in the case of open subsets of Euclidean spaces, i_0^* and i_1^* are "naturally" cochain homotopic. We next extend this to the general case via charts.

Lemma 15.13. *There is a construction which assigns to each open subset U of $\mathbf{R}^n$ a cochain homotopy $T^U : \Omega^*(U \times \mathbf{R}) \to \Omega^*(U)$ from i_0^* to i_1^* such that if $f : U \to V$ is a smooth map where U, V are open in $\mathbf{R}^n$, then the diagram*

$$\begin{array}{ccc} \Omega^k(V \times \mathbf{R}) & \xrightarrow{T_k^V} & \Omega^{k-1}(V) \\ {\scriptstyle (f \times 1_{\mathbf{R}})^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \Omega^k(U \times \mathbf{R}) & \xrightarrow[T_k^U]{} & \Omega^{k-1}(U) \end{array}$$

commutes for all k.

Proof. Write $x_1, \ldots, x_n$ for the coordinate functions in $\mathbf{R}^n$ and t for the coordinate function in $\mathbf{R}$. For each $k > 0$ we have $\Omega^k(U \times \mathbf{R}) = \Omega_0^k(U \times \mathbf{R}) \oplus \Omega_1^k(U \times \mathbf{R})$ where $\Omega_0^k(U \times \mathbf{R})$ consists of all $\sum_{I \in X_{n,k}} f_I \wedge dx_I$ where each $f_I : U \times \mathbf{R} \to \mathbf{R}$ is smooth, and $\Omega_1^k(U \times \mathbf{R})$ consists of all $\sum_{J \in X_{n,k-1}} g_J \wedge dx_J \wedge dt$ where each $g_J : U \times \mathbf{R} \to \mathbf{R}$ is smooth. If $\omega \in \Omega^k(U \times \mathbf{R})$ we will write $\omega = \omega_0 + \omega_1$ where $\omega_i \in \Omega_i^k(U \times \mathbf{R})$, $i = 0, 1$.

If $g : U \times \mathbf{R} \to \mathbf{R}$ is smooth, define $\overline{g} : U \to \mathbf{R}$ by $\overline{g}(x) = \displaystyle\int_0^1 g(x,t)\, dt$. Then $\overline{g}$ is smooth. For $k > 0$ we define $T_k^U : \Omega^k(U \times \mathbf{R}) \to \Omega^{k-1}(U)$ as follows: If ω is a k-form on $U \times \mathbf{R}$ and $\omega_1 = \sum_{J \in X_{n,k-1}} g_J \wedge dx_J \wedge dt$, then $T_k^U(\omega) = (-1)^{k-1} \sum_{J \in X_{n,k-1}} \overline{g}_J \wedge dx_J$. (Thus $T_k^U(\omega)$ depends only on ω_1.) We also define $T_0^U = 0$.

Note that for *any* $(k-1)$-tuple J of integers between 1 and n (not necessarily an increasing $(k-1)$-tuple), if $g : U \times \mathbf{R} \to \mathbf{R}$ is smooth, then $T_k^U(g \wedge dx_J \wedge dt) = (-1)^{k-1}\overline{g} \wedge dx_J$. For if a repeat occurs in J, both sides are 0; if a repeat does not occur, the permutation required to bring the entries of J into increasing order must be applied to both sides.

We proceed to calculate $dT_k^U(\omega) + T_{k+1}^U d(\omega)$. If $f : U \times \mathbf{R} \to \mathbf{R}$ is a smooth map we will write $D_t f$ instead of $D_{n+1} f$ for the partial derivative with respect to t to emphasize the special role played here by the last coordinate. Say

$$\omega = \sum_{I \in X_{n,k}} f_I \wedge dx_I + \sum_{J \in X_{n,k-1}} g_J \wedge dx_J \wedge dt$$

where the $f_I, g_J : U \times \mathbf{R} \to \mathbf{R}$ are smooth. Then

$$dT_k^U(\omega) = (-1)^{k-1} \sum_{J \in X_{n,k-1}} d\overline{g}_J \wedge dx_J$$

and so

$$dT_k^U(\omega) = (-1)^{k-1} \sum_{J \in X_{n,k-1}} \sum_{i=1}^n D_i \overline{g}_J \wedge dx_i \wedge dx_J. \tag{1}$$

We have

$$\begin{aligned} d\omega &= \sum_{I \in X_{n,k}} df_I \wedge dx_I + \sum_{J \in Xn,k-1} dg_J \wedge dx_J \wedge dt \\ &= \sum_{I \in X_{n,k}} \left(\left(\sum_{i=1}^n D_i f_I \wedge dx_i \right) + D_t f_I \wedge dt \right) \wedge dx_I \\ &\quad + \sum_{J \in Xn,k-1} \sum_{i=1}^n D_i g_J \wedge dx_i \wedge dx_J \wedge dt \end{aligned}$$

since in the second summation, $D_t g_J \wedge dt \wedge dx_J \wedge dt = 0$ for all J. Thus

$$\begin{aligned}(d\omega)_1 &= \sum_{I\in X_{n,k}} D_t f_I \wedge dt \wedge dx_I + \sum_{J\in Xn,k-1} \sum_{i=1}^{n} D_i g_J \wedge dx_i \wedge dx_J \wedge dt \\ &= \sum_{I\in X_{n,k}} (-1)^k D_t f_I \wedge dx_I \wedge dt + \sum_{J\in Xn,k-1} \sum_{i=1}^{n} D_i g_J \wedge dx_i \wedge dx_J \wedge dt.\end{aligned}$$

Thus

$$\begin{aligned}T_{k+1}^U(d\omega) &= (-1)^k \sum_{I\in X_{n,k}} (-1)^k \overline{D_t f}_I \wedge dx_I \\ &\qquad + (-1)^k \sum_{J\in Xn,k-1} \sum_{i=1}^{n} \overline{D_i g}_J \wedge dx_i \wedge dx_J \\ &= \sum_{I\in X_{n,k}} \overline{D_t f}_I \wedge dx_I + (-1)^k \sum_{J\in Xn,k-1} \sum_{i=1}^{n} \overline{D_i g}_J \wedge dx_i \wedge dx_J.\end{aligned}$$

By the Fundamental Theorem of Calculus we have, for each $I \in X_{n,k}$,

$$\begin{aligned}\overline{D_t f}_I(x) &= \int_0^1 D_t f_I(x,t)\,dt = f_I(x,1) - f_I(x,0) \\ &= \big(f_I i_1 - f_I i_0\big)(x).\end{aligned}$$

We have, for each $J \in X_{n,k-1}$ and $1 \le i \le n$,

$$\overline{D_i g}_J(x) = \int_0^1 D_i g_J(x,t)\,dt = D_i\bigg(\int_0^1 g_J(x,t)\,dt\bigg) = D_i \bar{g}_J(x).$$

Thus

$$\begin{aligned}T_{k+1}^U(d\omega) &= \sum_{I\in X_{n,k}} \big(f_I i_1 - f_I i_0\big) \wedge dx_I \\ &\qquad + (-1)^k \sum_{J\in X_{n,k-1}} \sum_{i=1}^{n} D_i \bar{g}_J \wedge dx_i \wedge dx_J.\end{aligned} \tag{2}$$

Thus, by (1) and (2),

$$dT_k^U(\omega) + T_{k+1}^U d(\omega) = \sum_{I\in X_{n,k}} \big(f_I i_1 - f_I i_0\big) \wedge dx_I. \tag{3}$$

We next calculate $i_1^*(\omega) - i_0^*(\omega)$. Since $i_j(x_1, \ldots, x_n) = (x_1, \ldots, x_n, j)$ for $j = 0, 1$, it follows that $i_j^*(dx_i) = dx_i$ for $1 \le i \le n$ and $i_j^*(dt) = d(j) = 0$. Thus, for $I \in X_{n,k}$ we have $i_j^*(dx_I) = dx_I$, and so

$$i_j^*(\omega) = i_j^*\bigg(\sum_{I\in X_{n,k}} f_I \wedge dx_I\bigg) + i_j^*\bigg(\sum_{J\in X_{n,k-1}} g_J \wedge dx_J \wedge dt\bigg)$$

$$= \sum_{I \in X_{n,k}} (i_j^* f_I) \wedge i_j^*(dx_I) + i_j^* \Bigg(\sum_{J \in X_{n,k-1}} g_J \wedge dx_J \Bigg) \wedge i_j^*(dt)$$
$$= \sum_{I \in X_{n,k}} f_I i_j \wedge dx_I.$$

Thus

$$i_1^*(\omega) - i_0^*(\omega) = \sum_{I \in X_{n,k}} \big(f_I i_1 - f_I i_0\big) \wedge dx_I. \tag{4}$$

From (3) and (4), it follows that T^U is a cochain homotopy from i_0^* to i_1^*.

Now suppose $f : U \to V$ is smooth where U and V are open in $\mathbf{R}^n$. Let $\pi_U : U \times \mathbf{R} \to U$ and $\pi_V : V \times \mathbf{R} \to V$ denote the projections on the first factor. We claim

$$\pi_U^*(dx_i) = dx_i \text{ for } 1 \le i \le n, \tag{5}$$
$$\pi_V^*(dx_i) = dx_i \text{ for } 1 \le i \le n, \tag{6}$$
$$(f \times 1_{\mathbf{R}})^*(dx_i) = \pi_U^* f^*(dx_i) \text{ for } 1 \le i \le n, \tag{7}$$
$$(f \times 1_{\mathbf{R}})^*(dt) = dt, \text{ and} \tag{8}$$
$$(f \times 1_{\mathbf{R}})^* \text{ carries } \Omega_0^k(V \times \mathbf{R}) \text{ into } \Omega_0^k(U \times \mathbf{R}). \tag{9}$$

(5) and (6) are immediate since $\pi_U(x_1, \ldots, x_n, t) = (x_1, \ldots, x_n)$ and similarly for π_V.

From (6) and commutativity of the diagram

$$\begin{array}{ccc} U \times \mathbf{R} & \xrightarrow{f \times 1_{\mathbf{R}}} & V \times \mathbf{R} \\ {\scriptstyle \pi_U}\big\downarrow & & \big\downarrow{\scriptstyle \pi_V} \\ U & \xrightarrow[f]{} & V \end{array}$$

we obtain, for $1 \le i \le n$,

$$(f \times 1_{\mathbf{R}})^*(dx_i) = (f \times 1_{\mathbf{R}})^* \pi_V^*(dx_i) = \pi_U^* f^*(dx_i),$$

establishing (7).

(8) is immediate since $(f \times 1_{\mathbf{R}})(x_1, \ldots, x_n, t) = \big(f(x_1, \ldots, x_n, t), t\big)$.

To prove (9) it suffices to check that for each smooth $h : V \times \mathbf{R} \to \mathbf{R}$ and $I \in X_{n,k}$, $(f \times 1_{\mathbf{R}})^*(h \wedge dx_I) \in \Omega_0^k(U \times \mathbf{R})$. By (7) we have

$$(f \times 1_{\mathbf{R}})^*(h \wedge dx_I) = h(f \times 1_{\mathbf{R}}) \wedge (f \times 1_{\mathbf{R}})^*(dx_I) = h(f \times 1_{\mathbf{R}}) \wedge \pi_U^* f^*(dx_I).$$

Since $f^*(dx_i) = \sum_{j=1}^n (D_j f_i) \wedge dx_j$ for $1 \le i \le n$, it follows from (5) that

$$\pi_U^* f^*(dx_i) = \sum_{j=1}^n (D_j f_i) \pi_U \wedge dx_j$$

from which it follows easily that $(f \times 1_{\mathbf{R}})^*(h \wedge dx_I)$ has the form $\sum_{J \in X_{n,k}} f_J \wedge dx_J$ where the $f_J : U \times \mathbf{R} \to \mathbf{R}$ are smooth, proving (9).

It remains only to show

$$f^* T_k^V(\omega) = T_k^U (f \times 1_{\mathbf{R}})^*(\omega) \text{ for all } \omega \in \Omega^k(V \times \mathbf{R}). \tag{10}$$

Since T_k^U and T_k^V are 0 on $\Omega_0^k(U \times \mathbf{R})$ and $\Omega_0^k(V \times \mathbf{R})$, respectively, it follows from (9) that both $f^* T_k^V(\omega)$ and $T_k^U(f \times 1_{\mathbf{R}})^*(\omega)$ are 0 for $\omega \in \Omega_0^k(V \times \mathbf{R})$. Thus it remains only to show that (10) holds for all $\omega \in \Omega_1^k(V \times \mathbf{R})$. It is sufficient to treat the case $\omega = g \wedge dx_J \wedge dt$ where $g : V \times \mathbf{R} \to \mathbf{R}$ is smooth and $J \in X_{n,k-1}$.

We have $f^* T_k^V(\omega) = (-1)^{k-1} f^*(\overline{g} \wedge dx_J) = (-1)^{k-1} \overline{g} f \wedge f^*(dx_J)$. Thus if we write

$$f^*(dx_J) = \sum_{I \in X_{n,k-1}} f_I \wedge dx_I \tag{11}$$

where each $f_I : U \to \mathbf{R}$ is smooth, then

$$f^* T_k^V(\omega) = (-1)^{k-1} \sum_{I \in X_{n,k-1}} \overline{g} f \wedge f_I \wedge dx_I. \tag{12}$$

On the other hand,

$$\begin{aligned}
(f \times 1_{\mathbf{R}})^*(\omega) &= g(f \times 1_{\mathbf{R}}) \wedge (f \times 1_{\mathbf{R}})^*(dx_J) \wedge (f \times 1_{\mathbf{R}})^*(dt) \\
&= g(f \times 1_{\mathbf{R}}) \wedge \pi_U^* f^*(dx_J) \wedge dt \qquad \text{(by (7) and (8))} \\
&= g(f \times 1_{\mathbf{R}}) \wedge \pi_U^* \Big(\sum_{I \in X_{n,k-1}} f_I \wedge dx_I \Big) \wedge dt \qquad \text{(by (11))} \\
&= \sum_{I \in X_{n,k-1}} g(f \times 1_{\mathbf{R}}) \wedge f_I \pi_U \wedge \pi_U^*(dx_I) \wedge dt \\
&= \sum_{I \in X_{n,k-1}} g(f \times 1_{\mathbf{R}}) \wedge f_I \pi_U \wedge dx_I \wedge dt \qquad \text{(by (5))}.
\end{aligned}$$

Hence

$$T_k^U (f \times 1_{\mathbf{R}})^*(\omega) = (-1)^{k-1} \sum_{I \in X_{n,k-1}} \overline{g(f \times 1_{\mathbf{R}}) \wedge f_I \pi_U} \wedge dx_I.$$

Thus, by (12) the proof will be complete if we check that for all $I \in X_{n,k-1}$,

$$\overline{g} f \wedge f_I = \overline{g(f \times 1_{\mathbf{R}}) \wedge f_I \pi_U}.$$

For all $x \in U$ we have

$$\begin{aligned}
\overline{g(f \times 1_{\mathbf{R}}) \wedge f_I \pi_U}(x) &= \int_0^1 \Big(g(f \times 1_{\mathbf{R}}) \wedge f_I \pi_U \Big)(x, t)\, dt \\
&= \int_0^1 g\big(f(x), t\big) f_I(x)\, dt = \Big(\int_0^1 g\big(f(x), t\big)\, dt \Big) f_I(x) \\
&= \overline{g}\big(f(x)\big) f_I(x) = (\overline{g} f \wedge f_I)(x),
\end{aligned}$$

completing the proof. □

Corollary 15.14. *There is a construction which assigns to each smooth manifold M which is diffeomorphic to an open subset of $\mathbf{R}^n$, a cochain homotopy $T^M : \Omega^*(M \times \mathbf{R}) \to \Omega^*(M)$ from i_0^* to i_1^* such that if $f : M \to N$ is a smooth map where N is diffeomorphic to an open subset of $\mathbf{R}^n$, then the diagram*

$$\begin{array}{ccc} \Omega^k(N \times \mathbf{R}) & \xrightarrow{T_k^N} & \Omega^{k-1}(N) \\ {\scriptstyle (f\times 1_{\mathbf{R}})^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \Omega^k(M \times \mathbf{R}) & \xrightarrow[T_k^M]{} & \Omega^{k-1}(M) \end{array}$$

commutes for all k.

Proof. Choose any diffeomorphism $\alpha : M \to U$ where U is open in $\mathbf{R}^n$. Since $\alpha \times 1_{\mathbf{R}}$ is a diffeomorphism, $(\alpha \times 1_{\mathbf{R}})^* : \Omega^k(U \times \mathbf{R}) \to \Omega^k(M \times \mathbf{R})$ is an $\mathbf{R}$-isomorphism for all k. Define $T_k^M : \Omega^k(M \times \mathbf{R}) \to \Omega^{k-1}(M)$ to be the composition

$$\Omega^k(M \times \mathbf{R}) \xrightarrow{(\alpha\times 1_{\mathbf{R}})^{*-1}} \Omega^k(U \times \mathbf{R}) \xrightarrow{T_k^U} \Omega^{k-1}(U) \xrightarrow{\alpha^*} \Omega^{k-1}(M)$$

where T_k^U is provided by Lemma 15.13. Write $i_\varepsilon^M : M \to M \times \mathbf{R}$ and $i_\varepsilon^U : U \to U \times \mathbf{R}$ to distinguish the inclusions i_0 and i_1 for the spaces M and U. For $\varepsilon = 0, 1$ the diagram

(1)
$$\begin{array}{ccc} M & \xrightarrow{i_\varepsilon^M} & M \times \mathbf{R} \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha\times 1_{\mathbf{R}}} \\ U & \xrightarrow[i_\varepsilon^U]{} & U \times \mathbf{R} \end{array}$$

commutes. We then have, for all k,

$$\begin{aligned} dT_k^M + T_{k-1}^M d &= d\alpha^* T_k^U (\alpha \times 1_{\mathbf{R}})^{*-1} + \alpha^* T_{k-1}^U (\alpha \times 1_{\mathbf{R}})^{*-1} d \\ &= \alpha^* dT_k^U (\alpha \times 1_{\mathbf{R}})^{*-1} + \alpha^* T_{k-1}^U d(\alpha \times 1_{\mathbf{R}})^{*-1} \\ &\qquad\qquad \text{(by Theorem 13.15(d))} \\ &= \alpha^* \big(dT_k^U + T_{k-1}^U d\big)(\alpha \times 1_{\mathbf{R}})^{*-1} \\ &= \alpha^* \big(i_1^{U*} - i_0^{U*}\big)(\alpha \times 1_{\mathbf{R}})^{*-1} \\ &\qquad\qquad \text{(by Lemma 15.13)} \\ &= \alpha^* \big(\alpha^{*-1} i_1^{M*} - \alpha^{*-1} i_0^{M*}\big) \qquad \text{(by (1))} \\ &= i_1^{M*} - i_0^{M*} \end{aligned}$$

and so T^M is a cochain homotopy from i_0^{M*} to i_1^{M*}. (We could, at this point, verify that the above T^M is independent of the choice of α, but we will not explicitly need this to complete the proof.)

Choose any diffeomorphism $\beta : N \to V$ where V is open in $\mathbf{R}^n$, and construct T^N as above using β. Since $\beta f \alpha^{-1} : U \to V$ is a smooth map between open subsets of $\mathbf{R}^n$, it follows from Lemma 15.13 that for each k, the diagram

$$\begin{array}{ccc} \Omega^k(V \times \mathbf{R}) & \xrightarrow{T_k^V} & \Omega^{k-1}(V) \\ \big((\beta f \alpha^{-1}) \times 1_{\mathbf{R}}\big)^* \downarrow & & \downarrow (\beta f \alpha^{-1})^* \\ \Omega^k(U \times \mathbf{R}) & \xrightarrow[T_k^U]{} & \Omega^{k-1}(U) \end{array} \tag{2}$$

commutes. Thus

$$\begin{aligned} f^* T_k^N &= f^* \beta^* T_k^V (\beta \times 1_{\mathbf{R}})^{*-1} = \alpha^* \alpha^{*-1} f^* \beta^* T_k^V (\beta \times 1_{\mathbf{R}})^{*-1} \\ &= \alpha^* (\beta f \alpha^{-1})^* T_k^V (\beta^{-1} \times 1_{\mathbf{R}})^* \\ &= \alpha^* T_k^U \big((\beta f \alpha^{-1}) \times 1_{\mathbf{R}}\big)^* (\beta^{-1} \times 1_{\mathbf{R}})^* && \text{(by (2))} \\ &= \alpha^* T_k^U \big((\beta^{-1} \beta f \alpha^{-1}) \times 1_{\mathbf{R}}\big)^* = \alpha^* T_k^U \big((f \times 1_{\mathbf{R}})(\alpha^{-1} \times 1_{\mathbf{R}})\big)^* \\ &= \alpha^* T_k^U (\alpha \times 1_{\mathbf{R}})^{*-1} (f \times 1_{\mathbf{R}})^* = T_k^M (f \times 1_{\mathbf{R}})^*. \qquad \square \end{aligned}$$

Theorem 15.15. *Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds. Suppose $f, g : (M, \mathcal{S}) \to (N, \mathcal{T})$ are smooth maps which are smoothly homotopic. Then for all $k \geq 0$,*

$$H_{dR}^k f = H_{dR}^k g : H_{dR}^k(N, \mathcal{T}) \to H_{dR}^k(M, \mathcal{S}).$$

Proof. Let $i_0^M, i_1^M : M \to M \times \mathbf{R}$ denote the inclusions given by $i_\varepsilon^M(x) = (x, \varepsilon)$, $\varepsilon = 0, 1$. From Lemma 15.12 and Exercise 4 of §13 it suffices to show that the cochain maps $i_0^{M*}, i_1^{M*} : \Omega^*(M \times \mathbf{R}) \to \Omega^*(M)$ are cochain homotopic. Our strategy is to use Corollary 15.14 to construct such cochain homotopies locally, and to show that these local constructions are compatible on overlaps.

Say M is n-dimensional and let $\mathcal{O}$ be the collection of all open subsets of M which are diffeomorphic to open subsets of $\mathbf{R}^n$. For each $U \in \mathcal{O}$ let $T^U : \Omega^*(U \times \mathbf{R}) \to \Omega^*(U)$ denote the cochain homotopy from i_0^{U*} to i_1^{U*} provided by Corollary 15.14, and let $j_U : U \to M$ denote the inclusion. For $k \geq 0$ let $p_k : \Lambda^k(M, \mathcal{S}) \to M$ denote the projection. For $k \geq 1$ and $\omega \in \Omega^k(M \times \mathbf{R})$ let $S_k^U(\omega) : U \to p_{k-1}^{-1}(U)$ denote the composition

$$U \xrightarrow{T_k^U\left((j_U \times 1_{\mathbf{R}})^* \omega\right)} \Lambda^{k-1}(U) \xrightarrow{(\Lambda^{k-1} j_U)^{-1}} p_{k-1}^{-1}(U)$$

where $\Lambda^{k-1} j_U$ is the diffeomorphism of Lemmas 13.11 and 13.12. Then $S_k^U(\omega)$ is smooth, and $p_{k-1} S_k^U(\omega) = 1_U$. Thus, if we show that whenever $U, V \in \mathcal{O}$ then $S_k^U(\omega)$ and $S_k^V(\omega)$ agree on $U \cap V$, we would have a $(k-1)$-form $T_k^M(\omega)$ on M whose restriction to each $U \in \mathcal{O}$ is $S_k^U(\omega)$. To show this, it suffices to check that whenever W is open in U, then $S_k^W(\omega)$ is the restriction of $S_k^U(\omega)$ to W (for then the restrictions of $S_k^U(\omega)$ and $S_k^V(\omega)$ to $U \cap V$ would both be $S_k^{U \cap V}(\omega)$).

Let $i : W \to U$ denote the inclusion. Then $j_W = j_U i$. By Corollary 15.14, the diagram

$$(1)\qquad \begin{array}{ccc} \Omega^k(U \times \mathbf{R}) & \xrightarrow{T_k^U} & \Omega^{k-1}(U) \\ {\scriptstyle (i\times 1_{\mathbf{R}})^*}\big\downarrow & & \big\downarrow{\scriptstyle i^*} \\ \Omega^k(W \times \mathbf{R}) & \xrightarrow[T_k^W]{} & \Omega^{k-1}(W) \end{array}$$

commutes. Thus

$$\begin{aligned} S_k^W(\omega) &= (\Lambda^{k-1} j_W)^{-1} T_k^W\left((j_W \times 1_{\mathbf{R}})^*\omega\right) \\ &= \left(\Lambda^{k-1}(j_U i)\right)^{-1} T_k^W\left(\left((j_U \times 1_{\mathbf{R}})(i \times 1_{\mathbf{R}})\right)^*\omega\right) \\ &= (\Lambda^{k-1} j_U)^{-1}(\Lambda^{k-1} i)^{-1} T_k^W\left((i \times 1_{\mathbf{R}})^*(j_U \times 1_{\mathbf{R}})^*\omega\right) \\ &= (\Lambda^{k-1} j_U)^{-1}(\Lambda^{k-1} i)^{-1} i^* T_k^U\left((j_U \times 1_{\mathbf{R}})^*\omega\right) \qquad \text{(by (1)).} \end{aligned}$$

Note that $(\Lambda^{k-1} i)^{-1} i^* T_k^U\left((j_U \times 1_{\mathbf{R}})^*\omega\right)$ is the restriction of $T_k^U\left((j_U \times 1_{\mathbf{R}})^*\omega\right)$ to W by Lemma 13.12(a). Thus $S_k^W(\omega)$ is the restriction of $(\Lambda^{k-1} j_U)^{-1} T_k^U\left((j_U \times 1_{\mathbf{R}})^*\omega\right)$ to W, i.e. the restriction of $S_k^U(\omega)$ to W. Hence we have a well-defined $(k-1)$-form $T_k^M(\omega)$ on M whose restriction to U is $S_k^U(\omega)$ for each $U \in \mathcal{O}$ and each $\omega \in \Omega^k(M \times \mathbf{R})$.

Since T_k^U and $(j_U \times 1_{\mathbf{R}})^*$ are $\mathbf{R}$-linear and $\Lambda^{k-1} j_U$ is fiberwise $\mathbf{R}$-linear for each $U \in \mathcal{O}$, it follows that $T_k^M : \Omega^k(M \times \mathbf{R}) \to \Omega^{k-1}(M)$ is $\mathbf{R}$-linear for each k.

It remains only to check that $dT^M + T^M d = i_1^{M*} - i_0^{M*}$. By Lemma 13.12(b) it suffices to check that for all $U \in \mathcal{O}$ and all k,

$$(2)\qquad j_U^*\left(dT_k^M + T_{k-1}^M d\right) = j_U^*\left(i_1^{M*} - i_0^{M*}\right)$$

where the right-hand side is restricted to k-forms.

From commutativity of

$$\begin{array}{ccc} U & \xrightarrow{j_U} & M \\ {\scriptstyle i_\varepsilon^U}\big\downarrow & & \big\downarrow{\scriptstyle i_\varepsilon^M} \\ U \times \mathbf{R} & \xrightarrow[j_U \times 1_{\mathbf{R}}]{} & M \times \mathbf{R} \end{array}$$

for $\varepsilon = 0, 1$ we have

$$(3)\qquad j_U^*\left(i_1^{M*} - i_0^{M*}\right) = \left(i_1^{U*} - i_0^{U*}\right)(j_U \times 1_{\mathbf{R}})^*.$$

Recall that for $\rho \in \Omega^k(M, \mathcal{S})$ and $x \in U$, $(j_U^*\rho)(x) = (\Lambda^k T_x j_U)\left(\rho(x)\right)$, and that $\Lambda^k j_U$ denotes the map whose restriction to the fiber over x is $\Lambda^k T_x j_U$. It follows that for each $\omega \in \Omega^k(M \times \mathbf{R})$ and $x \in U$,

$$j_U^* T_k^M(\omega)(x) = j_U^*(\Lambda^{k-1} j_U)^{-1} T_k^U\left((j_U \times 1_{\mathbf{R}})^*(\omega)\right)(x)$$

$$= (\Lambda_U^{k-1} T_x j_U)(\Lambda_U^{k-1} T_x j_U)^{-1} T_k^U \big((j_U \times 1_{\mathbf{R}})^*(\omega)\big)(x)$$
$$= T_k^U \big((j_U \times 1_{\mathbf{R}})^*(\omega)\big)(x)$$

and so

$$j_U^* T_k^M = T_k^U (j_U \times 1_{\mathbf{R}})^* \tag{4}$$

for all k. Thus

$$\begin{aligned} j_U^*\big(dT_k^M + T_{k-1}^M d\big) &= dj_U^* T_k^M + j_U^* T_{k-1}^M d \\ &\qquad \text{(by Theorem 13.15(d))} \\ &= dT_k^U (j_U \times 1_{\mathbf{R}})^* + T_{k-1}^U (j_U \times 1_{\mathbf{R}})^* d \\ &\qquad \text{(by (4))} \\ &= \big(dT_k^U + T_{k-1}^U d\big)(j_U \times 1_{\mathbf{R}})^* \\ &\qquad \text{(by Theorem 13.15(d))} \\ &= \big(i_1^{U*} - i_0^{U*}\big)(j_U \times 1_{\mathbf{R}})^* \\ &\qquad \text{(by Corollary 15.14)} \end{aligned}$$

and so we are done by (2) and (3). □

Corollary 15.16. *If $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ is a smooth homotopy equivalence, then $H_{dR}^k f : H_{dR}^k(N, \mathcal{T}) \to H_{dR}^k(M, \mathcal{S})$ is an isomorphism for all k.*

Proof. Let $g : (N, \mathcal{T}) \to (M, \mathcal{S})$ be a smooth homotopy inverse to f. Since $gf \simeq 1_M$ it follows from Theorem 15.16 that $H_{dR}^k(gf) = H_{dR}^k(1_M)$. Thus, since H_{dR}^k is a contravariant functor, $(H_{dR}^k f)(H_{dR}^k g) = 1_{H_{dR}^k(M,\mathcal{S})}$. Similarly, $(H_{dR}^k g)(H_{dR}^k f) = 1_{H_{dR}^k(N,\mathcal{T})}$. Thus $H_{dR}^k f$ and $H_{dR}^k g$ are inverses of one another. □

Corollary 15.17. *For all $n \geq 0$, $H_{dR}^k(\mathbf{R}^n) = 0$ for $k > 0$, and $H_{dR}^0(\mathbf{R}^n) \cong \mathbf{R}$.*

Proof. By Example 15.10, $\mathbf{R}^n$ is smoothly homotopy equivalent to $\mathbf{R}^0$. The assertion now follows from Corollaries 15.16 and 13.27. □

Corollary 15.18. *Suppose $n > 0$ and let $i : S^{n-1} \to \mathbf{R}^n - \{0\}$ denote the inclusion map. Then for each k, $H_{dR}^k i : H_{dR}^k(\mathbf{R}^n - \{0\}) \to H_{dR}^k(S^{n-1})$ is an isomorphism. In particular, $H_{dR}^{n-1}(S^{n-1}) \neq 0$.*

Proof. This follows immediately from Corollary 15.16, Example 15.11, and Theorem 14.16. □

Theorem 15.19. (Brouwer Fixed-Point Theorem) *For $n \geq 1$ let D^n denote the closed unit disk in $\mathbf{R}^n$, i.e. $D^n = \{\, x \in \mathbf{R}^n \mid \|x\| = 1 \,\}$ where $\|\ \|$ denotes the standard Euclidean norm. Suppose $f : D^n \to D^n$ is continuous. Then there exists at least one $x \in D^n$ such that $f(x) = x$.*

Proof. We proceed by contradiction. Suppose $f : D^n \to D^n$ were a continuous map such that $f(x) \neq x$ for each $x \in D^n$. Define $r : D^n \to S^{n-1}$ as follows: Given

$x \in D^n$, $r(x)$ is the intersection of S^{n-1} with the ray which begins at $f(x)$ and passes through x. Explicitly, if we write

$$\lambda(x) = \frac{-x \cdot \big(x - f(x)\big) + \sqrt{\Big(x \cdot \big(x - f(x)\big)\Big)^2 + \|x - f(x)\|^2(1 - \|x\|^2)}}{\|x - f(x)\|^2}$$

where $\cdot$ denotes the standard Euclidean inner product in $\mathbf{R}^n$, then

$$r(x) = \big(1 + \lambda(x)\big)x - \lambda(x)f(x).$$

Note that $r(x) = x$ if $x \in S^{n-1}$ (for $x \cdot \big(x - f(x)\big) > 0$ if $\|x\| = 1$). If $n = 1$ we have a contradiction since D^1 is connected, r is onto, but S^0 is not connected.

Suppose $n > 1$. r extends to a continuous map $g : [-2, 2]^n \to S^{n-1}$ by defining $g(x) = \dfrac{x}{\|x\|}$ if $\|x\| > 1$. By the Stone-Weierstrass Theorem, the coordinate functions of g can be uniformly approximated by polynomial functions in n variables. Thus there exists a polynomial map $P : [-2, 2]^n \to \mathbf{R}^n$ such that $\|P(x) - g(x)\| < \frac{1}{2}$ for all $x \in [-2, 2]^n$. Since $\|g(x)\| = 1$ for each $x \in [-2, 2]^n$ we must have $P(x) \neq 0$ for each $x \in [-2, 2]^n$. By restriction we obtain a smooth map $Q : (-2, 2)^n \to \mathbf{R}^n - \{0\}$ with the property that $\|Q(x) - x\| < \frac{1}{2}$ for all $x \in S^{n-1}$ (since $g(x) = x$ for all $x \in S^{n-1}$). In particular, for each $x \in S^{n-1}$, the entire line segment joining x and $Q(x)$ is contained in $\mathbf{R}^n - \{0\}$.

Let $i : S^{n-1} \to \mathbf{R}^n - \{0\}$ and $j : S^{n-1} \to (-2, 2)^n$ denote the inclusion maps. Define $h : S^{n-1} \times \mathbf{R} \to \mathbf{R}^n - \{0\}$ by

$$h(x, t) = \beta_{0,1}(t)x + \big(1 - \beta_{0,1}(t)\big)Q(x).$$

h is a smooth homotopy from Qj to i. Thus, by Theorem 15.15, $H^{n-1}_{dR}(Qj) = H^{n-1}_{dR}i$. By Corollary 15.18, $H^{n-1}_{dR}i \neq 0$, and so $H^{n-1}_{dR}(Qj) \neq 0$. Since H^{n-1}_{dR} is a contravariant functor, $H^{n-1}_{dR}(Qj) = (H^{n-1}_{dR}j)(H^{n-1}_{dR}Q)$, and so we must have $H^{n-1}_{dR}j \neq 0$. Since $(-2, 2)$ is diffeomorphic to $\mathbf{R}$ (e.g. the map $(-2, 2) \to \mathbf{R}$ which sends t to $\tan(\pi t/4)$ is a diffeomorphism), it follows that $(-2, 2)^n$ is diffeomorphic to $\mathbf{R}^n$ and hence, by Corollary 15.17, $H^{n-1}_{dR}\big((-2, 2)^n\big) = 0$ (since $n > 1$). Thus $H^{n-1}_{dR}j : H^{n-1}_{dR}\big((-2, 2)^n\big) \to H^{n-1}_{dR}(S^{n-1})$ is the 0-homomorphism, a contradiction. $\square$

Exercises for §15

1. Let $f : S^n \to S^n$ be given by $f(x) = -x$ for all $x \in S^n$.

   (a) Prove that if n is even, then f is *not* smoothly homotopic to the identity map on S^n.

   (b) If n is odd, find an explicit smooth homotopy from f to 1_{S^n}.

2. Let $(M, \mathcal{S})$ be a smooth manifold such that $H^k_{dR}(M, \mathcal{S}) \neq 0$ for at least one $k > 0$. Let $f : M \times M \to M \times M$ be given by $f(x, y) = (y, x)$ for all $(x, y) \in M \times M$. Prove that f is not smoothly homotopic to the identity map on $M \times M$.

16. Paracompactness, Smooth Partitions of Unity, and Piecing of Local Sections

This section is concerned with some technicalities concerning the construction of smooth sections of smooth vector bundles by suitably piecing together local sections with certain desirable properties. This material will be subsequently used to study orientations of manifolds and Riemannian metrics on manifolds. It will sometimes be necessary to restrict ourselves to paracompact manifolds for these constructions.

Recall that a topological space is *paracompact* if it is Hausdorff and every open covering of X has a locally finite open refinement which covers X (i.e. given any open cover $\mathcal{O}$ of X, there exists an open cover $\mathcal{U}$ of X such that each member of $\mathcal{U}$ is contained in a member of $\mathcal{O}$, and each point of x has an open neighborhood which meets only finitely many members of $\mathcal{U}$). Every metric space is paracompact, and every compact Hausdorff space is paracompact. If X is Hausdorff and is a finite union of open subspaces each of which is paracompact, then X is paracompact. The class of paracompact manifolds thus includes all compact manifolds (more generally, all manifolds admitting a finite atlas), and all submanifolds of Euclidean space. Practically all manifolds of mathematical or physical importance are paracompact. Recall that every paracompact space is normal.

Definition 16.1. Let X be a topological space and $f : X \to \mathbf{R}$ a continuous map. The *support of* f, denoted $\operatorname{supp} f$, is the closure in X of $f^{-1}(\mathbf{R} - \{0\})$.

Definition 16.2. Let $(M, \mathcal{S})$ be a smooth manifold and $\mathcal{O}$ a locally finite open cover of M. A *smooth partition of unity on* $(M, \mathcal{S})$ *subordinate to* $\mathcal{O}$ is a collection $\{ f_A \mid A \in \mathcal{O} \}$ of smooth real-valued functions on M such that:

(i) $\operatorname{supp} f_A \subset A$ for all $A \in \mathcal{O}$.

(ii) $f_A(x) \geq 0$ for all $A \in \mathcal{O}$ and all $x \in M$.

(iii) For each $x \in M$, $\sum_{A \in \mathcal{O}} f_A(x) = 1$. (*Note:* This last sum is finite since, by the local finiteness of $\mathcal{O}$, x lies in the supports of only finitely many of the f_A.)

Lemma 16.3. *Let $(M, \mathcal{S})$ be a smooth n-manifold, U open in M, and $a \in U$. Then there exists a smooth map $f : M \to \mathbf{R}$ such that:*

(i) $\operatorname{supp} f \subset U$.

(ii) $f(x) \geq 0$ *for all* $x \in M$.

(iii) $f(a) > 0$.

Proof. Let $B = \{ y \in \mathbf{R}^n \mid \|y\| < 1 \}$ and $\frac{1}{2}\overline{B} = \{ y \in \mathbf{R}^n \mid \|y\| \leq \frac{1}{2} \}$ where $\| \ \|$ denotes the standard Euclidean norm on $\mathbf{R}^n$. There exists an open neighborhood N of a contained in U and a diffeomorphism $g : N \to B$ with $g(a) = 0$. Define $f : M \to \mathbf{R}$ by

$$f(x) = \begin{cases} \beta_{0,1}(1 - 4\|g(x)\|^2) & \text{if } x \in N, \\ 0 & \text{otherwise} \end{cases}$$

where $\beta_{0,1}$ is as in Lemma 15.3. The restriction of f to N is smooth, and f is identically 0 on $M - g^{-1}(\frac{1}{2}\overline{B})$. Since $\frac{1}{2}\overline{B}$ is compact and g is a homeomorphism, $g^{-1}(\frac{1}{2}\overline{B})$ is compact and hence $M - g^{-1}(\frac{1}{2}\overline{B})$ is open in M. Therefore, since $M = U \cup \big(M - g^{-1}(\frac{1}{2}\overline{B})\big)$ it follows, by the Local Property, that f is smooth. Note that $f(a) = 1$, $\operatorname{supp} f = g^{-1}(\frac{1}{2}\overline{B}) \subset U$, and $f(x) \geq 0$ for all $x \in M$. $\square$

Lemma 16.4. *Let $(M, \mathcal{S})$ be a smooth manifold, U open in M, and K a compact subset of U. Then there exists a smooth map $f : M \to \mathbf{R}$ such that:*
(i) $\operatorname{supp} f \subset U$.
(ii) $f(x) \geq 0$ *for all* $x \in M$.
(iii) $f(a) > 0$ *for all* $a \in K$.

Proof. By Lemma 16.3, for each $a \in K$ there exists a smooth function $f_a : M \to \mathbf{R}$ such that $\operatorname{supp} f_a \subset U$, $f_a(x) \geq 0$ for all $x \in M$, and $f_a(a) > 0$. Thus $K \subset \bigcup_{a \in K} f_a^{-1}\big((0,\infty)\big)$. Since each $f_a^{-1}\big((0,\infty)\big)$ is open in M, it follows from the compactness of K that there exist finitely many points $a_1, \dots, a_r$ in K such that

$$K \subset f_{a_1}^{-1}\big((0,\infty)\big) \cup \cdots \cup f_{a_r}^{-1}\big((0,\infty)\big).$$

Define $f : M \to \mathbf{R}$ by $f(x) = \sum_{i=1}^r f_{a_i}(x)$. Then f is smooth. Note that $f(x) \geq 0$ for all $x \in M$ since each $f_{a_i}(x) \geq 0$, and $\operatorname{supp} f = \bigcup_{i=1}^{r} \operatorname{supp} f_{a_i} \subset U$. If $x \in K$, then $x \in f_{a_i}^{-1}\big((0,\infty)\big)$ for at least one i and for such i, $f_{a_i}(x) > 0$, whence $f(x) > 0$. □

Theorem 16.5. *Let $(M, \mathcal{S})$ be a smooth manifold which is normal (e.g. if M is paracompact). Let $\mathcal{O}$ be a locally finite open cover of M by sets whose closures in M are compact. Then there exists a smooth partition of unity on M subordinate to $\mathcal{O}$.*

Proof. By the Shrinking Lemma there exists an open cover $\{ U_A \mid A \in \mathcal{O} \}$ of M such that $\overline{U}_A \subset A$ for each $A \in \mathcal{O}$. For each $A \in \mathcal{O}$, $\overline{U}_A$ is compact since $\overline{A}$ is compact. By Lemma 16.4 there exists, for each $A \in \mathcal{O}$, a smooth map $f_A : M \to \mathbf{R}$ such that

(1) $\qquad \operatorname{supp} f_A \subset A,$

(2) $\qquad f_A(x) \geq 0$ for all $x \in M$, and

(3) $\qquad f_A(a) > 0$ for all $a \in \overline{U}_A$.

Define $f : M \to \mathbf{R}$ by $f(x) = \sum_{A \in \mathcal{O}} f_A(x)$. By the local finiteness of $\mathcal{O}$ and (1), each $x \in M$ is contained in only finitely many of the $\operatorname{supp} f_A$ and so this last sum is finite for each x. Moreover, given $x \in M$, there exists an open neighborhood N_x of x in M which meets only finitely many members of $\mathcal{O}$, say $A_1, \dots, A_r$. Then for all $y \in N_x$, $f(y) = \sum_{i=1}^r f_{A_i}(y)$, a finite sum of smooth real-valued functions, and hence the restriction of f to N_x is smooth. By the Local Property, f is smooth. Note also that since $\{ U_A \mid A \in \mathcal{O} \}$ covers M, it follows from (2) and (3) that $f(x) > 0$ for all $x \in M$. For each $A \in \mathcal{O}$ define $g_A : M \to \mathbf{R}$ by

$$g_A(x) = \frac{f_A(x)}{f(x)}.$$

Each g_A is smooth and $\operatorname{supp} g_A = \operatorname{supp} f_A \subset A$. $\{ g_A \mid A \in \mathcal{O} \}$ is the required smooth partition of unity on $(M, \mathcal{S})$ subordinate to $\mathcal{O}$. □

Let ξ be a smooth vector bundle. Recall, from Proposition 8.34, that $\Gamma(\xi)$ is a real vector space under fiberwise sum and scalar multiplication. We also observed, in Proposition 12.9(b), that for each smooth manifold $(M, \mathcal{S})$, $\Omega^k(M, \mathcal{S})$ is a module over $C^\infty(M, \mathcal{S})$ via fiberwise multiplication. We could have observed earlier (and will now formally observe) that in general, $\Gamma(\xi)$ is a module over $C^\infty(M, \mathcal{S})$ via fiberwise multiplication.

Proposition 16.6. *Let $\xi = \big(F, E, (M, \mathcal{S}_M), p, \mathcal{S}\big)$ be a smooth vector bundle, $\sigma \in \Gamma(\xi)$, and $f \in C^\infty(M, \mathcal{S}_M)$. Define $f \cdot \sigma : M \to E$ by $(f \cdot \sigma)(x) = f(x)\sigma(x)$ for each $x \in M$. Then $f \cdot \sigma \in \Gamma(\xi)$.*

Proof. The only question is the smoothness of $f \cdot \sigma$. It suffices to check that for each $\mathcal{S}$-admissible linear chart φ, the composition

$$U_\varphi \xrightarrow{f\cdot\sigma} p^{-1}(U_\varphi) \xrightarrow{\varphi} U_\varphi \times F$$

is smooth. One checks that this composition equals the composition

$$\begin{aligned} U_\varphi \xrightarrow{\Delta} U_\varphi \times U_\varphi \xrightarrow{f\times\sigma} \mathbf{R} \times p^{-1}(U_\varphi) &\xrightarrow{1_{\mathbf{R}}\times\varphi} \mathbf{R} \times U_\varphi \times F \\ &\xrightarrow{\tau\times 1_F} U_\varphi \times \mathbf{R} \times F \xrightarrow{1_{U_\varphi}\times \mathrm{scal}} U_\varphi \times F \end{aligned} \tag{1}$$

where $\tau : \mathbf{R} \times U_\varphi \to U_\varphi \times \mathbf{R}$ interchanges factors and scal : $\mathbf{R} \times F \to F$ is the scalar multiplication map $\mathrm{scal}\,(r, v) = rv$. Since all maps in (1) are smooth, the Proposition follows. □

Note that if $\xi = \big(F, E, (M, \mathcal{S}_M), p, \mathcal{S}\big)$ is a smooth vector bundle and A is open in M, we obtain a smooth vector bundle $\xi|A = \big(F, p^{-1}(A), (A, \mathcal{S}_M|A), p_A, \mathcal{S}_A\big)$, the *restriction of ξ to A*, where $p_A : p^{-1}(A) \to A$ is the restriction of p, and the $\mathcal{S}_A$-admissible linear charts are all $\mathcal{S}$-admissible $\varphi : p^{-1}(U_\varphi) \to U_\varphi \times F$ with $U_\varphi \subset A$.

Lemma 16.7. *Let $\xi = \big(F, E, (M, \mathcal{S}_M), p, \mathcal{S}\big)$ be a smooth vector bundle. Let A be open in M and suppose $\sigma \in \Gamma(\xi|A)$. Let $f : M \to \mathbf{R}$ be a smooth map with* $\operatorname{supp} f \subset A$. *Define $f \cdot \sigma : M \to E$ by*

$$(f \cdot \sigma)(x) = \begin{cases} f(x)\sigma(x) & \text{if } x \in A, \\ 0_x & \text{if } x \notin A \end{cases}$$

where 0_x is the zero element of the fiber $p^{-1}(x)$. Then $f \cdot \sigma \in \Gamma(\xi)$.

Proof. The only question is the smoothness of $f \cdot \sigma$. The restriction of $f \cdot \sigma$ to A is smooth by Proposition 16.6. The restriction of $f \cdot \sigma$ to $M - \operatorname{supp} f$ is the zero-section in $\Gamma(\xi|(M - \operatorname{supp} f)$, which is smooth. Since $\{A, M - \operatorname{supp} f\}$ is an open cover of M, the smoothness of $f \cdot \sigma$ follows. □

Theorem 16.8. (Piecing Theorem) *Let $\xi = \big(F, E, (M, \mathcal{S}_M), p, \mathcal{S}\big)$ be a smooth vector bundle. Let $\mathcal{O}$ be a locally finite open cover of M, and $\{f_A \mid A \in \mathcal{O}\}$ a smooth partition of unity subordinate to $\mathcal{O}$. Suppose we are given, for each $A \in \mathcal{O}$, a smooth section $\sigma_A \in \Gamma(\xi|A)$. Define $\sigma : M \to E$ by*

$$\sigma(x) = \sum_{A\in\mathcal{O}} (f_A \cdot \sigma_A)(x).$$

Then $\sigma \in \Gamma(\xi)$.

Proof. Note, by the local finiteness of $\mathcal{O}$ and the hypothesis on the supports of the f_A, that for each $x \in M$, only finitely many of the $f_A(x)$ can be non-zero, and so the summation for $\sigma(x)$ is finite. The only question is the smoothness of σ. It suffices to check this locally. Each $x \in M$ has an open neighborhood N_x in M which meets only finitely many members of $\mathcal{O}$, say $A_1, \ldots, A_r$. The restriction of σ to N_x is then

$$\sum_{i=1}^{r} \left(f_{A_i}|N_x\right) \cdot \left(\sigma_{A_i}|(N_x \cap A_i)\right)$$

which is smooth by Lemma 16.7 and Theorem 8.34. □

Exercises for §16

1. Let $(M, \mathcal{S})$ be a paracompact smooth manifold. Suppose $X \subset U$ where X is closed in M and U is open in M. Prove that there exists a smooth map $f : M \to \mathbf{R}$ such that:
   (i) $\operatorname{supp} f \subset U$.
   (ii) $f(x) \geq 0$ for all $x \in M$.
   (iii) $f(x) > 0$ for all $x \in C$.

2. Let $(M, \mathcal{S})$ be a smooth paracompact manifold and $\xi = \big(F, E, (M, \mathcal{S}_M), p, \mathcal{S}\big)$ a smooth vector bundle. Suppose for some open cover $\mathcal{O}$ of M (not necessarily locally finite), there exists a family of smooth sections $\{\sigma_A \in \Gamma(\xi|A) \mid A \in \mathcal{O}\}$ such that:
   (i) For each $A \in \mathcal{O}$, $\sigma_A(x) \neq 0_x$ for each $x \in A$.
   (ii) Whenever $A, B \in \mathcal{O}$ and $x \in A \cap B$, $\sigma_A(x)$ is a positive real multiple of $\sigma_B(x)$.
   Prove that ξ admits a smooth section which is nowhere 0.

17. Orientations

Let V be a real n-dimensional vector space, $0 < n < \infty$. The intuitive idea of an orientation of V is as follows: If $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ are ordered bases of V, we say $(v_1, \ldots, v_n)$ *is similarly oriented to* $(w_1, \ldots, w_n)$ if one can pass continuously, through ordered bases, from $(v_1, \ldots, v_n)$ to $(w_1, \ldots, w_n)$. (This can be made precise.) One sees easily that *"is similarly oriented to"* is an equivalence relation on the set of all ordered bases of V, and one defines an *orientation of* V to be an equivalence class of ordered bases of V with respect to the above equivalence relation. It will be technically convenient to give a different, though equivalent, definition of orientation below. We will use the fact that $\mathbf{R} - \{0\}$ has two components, from which it will follow that each V as above has exactly two orientations. Recall, from Corollary 10.19, that $\Lambda^n(V)$ is 1-dimensional over V, and hence $\Lambda^n(V) - \{0\}$ has exactly two components.

Definition 17.1. Let V be a real n-dimensional vector space, $0 < n < \infty$. An *orientation* $\mathcal{O}$ *of* V is a choice of component of $\Lambda^n(V) - \{0\}$.

If $\mathcal{O}$ is an orientation of V, an ordered basis $(f_1, \ldots, f_n)$ of V^* is an $\mathcal{O}$*-basis of* V^* if $f_1 \wedge \cdots \wedge f_n \in \mathcal{O}$. An ordered basis $(v_1, \ldots, v_n)$ of V is an $\mathcal{O}$*-basis of* V if the dual basis $(v_1^*, \ldots, v_n^*)$ is an $\mathcal{O}$-basis of V^*.

An *oriented real vector space* is a pair $(V, \mathcal{O})$ where V is a non-zero finite-dimensional vector space and $\mathcal{O}$ is an orientation of V.

Thus each n-dimensional real vector space, $0 < n < \infty$, admits exactly two orientations. Each non-zero element $\alpha \in \Lambda^n(V)$ determines an orientation of V, namely the component of α in $\Lambda^n(V) - \{0\}$.

Proposition 17.2. *Let $(V, \mathcal{O})$ be an oriented real n-dimensional vector space. Let $(v_1, \ldots, v_n)$ and $(f_1, \ldots, f_n)$ be $\mathcal{O}$-bases of V and V^*, respectively. Let $(w_1, \ldots, w_n)$ and $(g_1, \ldots, g_n)$ be arbitrary bases of V and V^*, respectively. Then:*

(i) *$(w_1, \ldots, w_n)$ is an $\mathcal{O}$-basis for V if and only if the determinant of the $\mathbf{R}$-linear transformation $V \to V$ which sends v_i to w_i for $1 \le i \le n$ is positive.*

(ii) *$(g_1, \ldots, g_n)$ is an $\mathcal{O}$-basis for V^* if and only if the determinant of the $\mathbf{R}$-linear transformation $V^* \to V^*$ which sends f_i to g_i for $1 \le i \le n$ is positive.*

Proof. Let $f : V \to V$ denote the $\mathbf{R}$-linear transformation of (i). By Theorem 10.20, $\Lambda^n f : \Lambda^n(V) \to \Lambda^n(V)$ is given by multiplication by $\det(f)$. Since

$$(\Lambda^n f)(w_1^* \wedge \cdots \wedge w_n^*) = v_1^* \wedge \cdots \wedge v_n^*,$$

it follows that

$$v_1^* \wedge \cdots \wedge v_n^* = \det(f)\, w_1^* \wedge \cdots \wedge w_n^*.$$

Since $v_1^* \wedge \cdots \wedge v_n^* \in \mathcal{O}$, it follows that $w_1^* \wedge \cdots \wedge w_n^* \in \mathcal{O}$ if and only if $\det(f) > 0$, proving part (i).

Let $g : V^* \to V^*$ denote the $\mathbf{R}$-linear transformation of part (ii). Let $(f_1^*, \ldots, f_n^*)$ and $(g_1^*, \ldots, g_n^*)$ be the ordered bases of V whose dual bases are $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$, respectively. Then $(f_1^*, \ldots, f_n^*)$ is an $\mathcal{O}$-basis of V, and $(g_1^*, \ldots, g_n^*)$ will be an $\mathcal{O}$-basis of V if and only if $(g_1, \ldots, g_n)$ is an $\mathcal{O}$-basis of V^*.

Let $f : V \to V$ be the $\mathbf{R}$-linear transformation which sends f_i^* to g_i^*, $1 \le i \le n$. Then $(f^{-1})^* = g$ and so $\det(g) = \det(f^{-1}) = \big(\det(f)\big)^{-1}$. Thus $\det(g) > 0$ if and only if $\det(f) > 0$. By part (i), $\det(f) > 0$ if and only if $(g_1^*, \ldots, g_n^*)$ is an $\mathcal{O}$-basis of V. Part (ii) now follows. □

Definition 17.3. Let $(M, \mathcal{S})$ be a smooth n-manifold, $n > 0$. An *orientation form on* $(M, \mathcal{S})$ is an n-form ω on $(M, \mathcal{S})$ such that $\omega(x) \neq 0$ for all $x \in M$. $(M, \mathcal{S})$ is said to be *orientable* if an orientation form on $(M, \mathcal{S})$ exists.

If ω is an orientation form on $(M, \mathcal{S})$, then for each $x \in M$, $\omega(x)$ is a non-zero element of $\Lambda^n\big(T_x(M, \mathcal{S})\big)$, and hence determines an orientation of $T_x(M, \mathcal{S})$. Thus an orientation form on $(M, \mathcal{S})$ yields simultaneous orientations of all the tangent spaces $T_x(M, \mathcal{S})$ in a "coherent manner". We will see below that not all smooth manifolds are orientable.

Example 17.4. Let U be an open subset of $\mathbf{R}^n$, $n > 0$. Then $dx_1 \wedge \cdots \wedge dx_n$ is an orientation form on U. For recall, by Notation 12.11, for $x \in U$ and $i, j \in \{1, \ldots, n\}$,

$$\begin{aligned} dx_i(x)\big(\widetilde{1_U}^{-1}(x, e_j)\big) &= (\widetilde{1_U}^{1})^{-1}(x, e_i^*)\big(\widetilde{1_U}^{-1}(x, e_j)\big) = \Lambda^1(\widetilde{1_U})_x(e_i^*)\big((\widetilde{1_U})_x^{-1}(e_j)\big) \\ &= e_i^*\big((\widetilde{1_U})_x(\widetilde{1_U})_x^{-1}(e_j)\big) = e_i^*(e_j) = \delta_{ij}. \end{aligned}$$

Thus by Theorem 10.17,

$$\begin{aligned} &(dx_1 \wedge \cdots \wedge dx_n)(x)\big(\widetilde{1_U}^{-1}(x, e_1), \ldots, \widetilde{1_U}^{-1}(x, e_n)\big) \\ &\qquad = dx_1(x)\big(\widetilde{1_U}^{-1}(x, e_1)\big) \cdots dx_n(x)\big(\widetilde{1_U}^{-1}(x, e_n)\big) = 1 \end{aligned}$$

and so $(dx_1 \wedge \cdots \wedge dx_n)(x) \neq 0$.

Example 17.5. By Exercise 3(b) of §12, the form $i^*(y \wedge dx - x \wedge dy)$ is an orientation form on S^1 where $i : S^1 \to \mathbf{R}^2$ denotes the inclusion map.

Proposition 17.6. *Suppose $(M, \mathcal{S})$ and $(N, \mathcal{T})$ are smooth n-manifolds and $f : (M, \mathcal{S}) \to (N, \mathcal{T})$ an immersion. Suppose ω is an orientation form on $(N, \mathcal{T})$. Then $f^*\omega$ is an orientation form on $(M, \mathcal{S})$.*

Proof. Let $x \in M$. Then $\omega\big(f(x)\big) \neq 0$ and so there exist $v_1, \ldots, v_n \in T_{f(x)}(N, \mathcal{T})$ such that $\omega\big(f(x)\big)(v_1, \ldots, v_n) \neq 0$. Since f is an immersion and both M and N are n-dimensional, the tangent map $T_x f : T_x(M, \mathcal{S}) \to T_{f(x)}(N, \mathcal{T})$ is an isomorphism. Thus for $1 \leq i \leq n$ we have $v_i = T_x f(u_i)$ for some $u_i \in T_x(M, \mathcal{S})$. We then have

$$\begin{aligned} f^*\omega(x)(u_1, \ldots, u_n) &= \Lambda^n(T_x f)\Big(\omega\big(f(x)\big)\Big)(u_1, \ldots, u_n) \\ &= \omega\big(f(x)\big)\big(T_x f(u_1), \ldots, T_x f(u_n)\big) = \omega\big(f(x)\big)(v_1, \ldots, v_n) \neq 0 \end{aligned}$$

and hence $f^*\omega(x) \neq 0$ for each $x \in M$. □

Corollary 17.7. *Let $(M, \mathcal{S})$ be an orientable smooth manifold and U an open subset of M. Then $(U, \mathcal{S}|U)$ is orientable.*

Proof. The inclusion map $i : U \to M$ is an immersion by Theorem 6.11. □

Theorem 17.8. *Let $(M, \mathcal{S})$ be a smooth n-manifold, $n > 0$. If $(M, \mathcal{S})$ is orientable, there exists an $\mathcal{S}$-admissible atlas $\mathcal{A} \subset \mathcal{E}(M, \mathcal{S})$ such that whenever $\varphi, \psi \in \mathcal{A}$, $\det D(\psi\varphi^{-1})(x) > 0$ for all $x \in \varphi(\text{dom } \varphi \cap \text{dom } \psi)$.*

If M is paracompact, the converse holds.

Proof. Suppose ω is an orientation form on $(M, \mathcal{S})$. Suppose $\varphi \in \mathcal{E}(M, \mathcal{S})$ has a connected codomain. Let $i_\varphi : \text{dom } \varphi \to M$ denote the inclusion map. By Proposition 17.6, $i_\varphi^*\omega$ is an orientation form on dom φ and $(\varphi^{-1})^* i_\varphi^*\omega$ an orientation form on codom φ. We can write $(\varphi^{-1})^* i_\varphi^* \omega = g_\varphi \wedge dx_I$, where $I = (1, 2, \ldots, n)$, for a unique smooth map $g_\varphi : \text{codom } \varphi \to \mathbf{R}$. Since $(g_\varphi \wedge dx_I)(x)$ is non-zero for each $x \in \text{codom } \varphi$ we must have $g_\varphi(x) \neq 0$ for all $x \in \text{codom } \varphi$. Thus, by the connectedness of codom φ, it follows that either $g_\varphi(x) > 0$ for all $x \in \text{codom } \varphi$ (in which case we will say φ is *orientation-preserving*) or $g_\varphi(x) < 0$ for all $x \in \text{codom } \varphi$ (in which case we will say φ is *orientation-reversing*). The proof in the first direction will be complete if we establish the following two statements:

(i) Whenever $\varphi, \psi \in \mathcal{E}(M, \mathcal{S})$ have connected codomains, then for all $x \in \varphi(\text{dom } \varphi \cap \text{dom } \psi)$, $D(\psi\varphi^{-1})(x) > 0$.

(ii) M admits an $\mathcal{S}$-admissible atlas consisting of orientation-preserving charts whose codomains are connected open subsets of $\mathbf{R}^n$.

Let φ and ψ be as in (i). We have the commutative diagram

$$\begin{array}{ccccccc}
M & \xleftarrow{\;i_\varphi\;} & \text{dom } \varphi & \xleftarrow{\;\varphi^{-1}\;} & \text{codom } \varphi & \xleftarrow{\;j_\varphi\;} & \varphi(\text{dom } \varphi \cap \text{dom } \psi) \\
{\scriptstyle =}\big\downarrow & & & & & & \big\downarrow{\scriptstyle \psi\varphi^{-1}} \\
M & \xleftarrow[\;i_\psi\;]{} & \text{dom } \psi & \xleftarrow[\;\psi^{-1}\;]{} & \text{codom } \psi & \xleftarrow[\;j_\psi\;]{} & \psi(\text{dom } \varphi \cap \text{dom } \psi)
\end{array}$$

where j_φ and j_ψ are the inclusion maps. Thus

$$j_\varphi^*(\varphi^{-1})^* i_\varphi^* \omega = (\psi\varphi^{-1})^* j_\psi^* (\psi^{-1})^* i_\psi^* \omega. \tag{1}$$

Since

$$j_\varphi^*(\varphi^{-1})^* i_\varphi^* \omega = j_\varphi^*(g_\varphi \wedge dx_I) = g_\varphi j_\varphi \wedge dx_I$$

and

$$\begin{aligned}
(\psi\varphi^{-1})^* j_\psi^* (\psi^{-1})^* i_\psi^* \omega &= (\psi\varphi^{-1})^* j_\psi^* (g_\psi \wedge dx_I) \\
&= (\psi\varphi^{-1})^* (g_\psi j_\psi \wedge dx_I) \\
&= g_\psi j_\psi \psi\varphi^{-1} \wedge (\psi\varphi^{-1})^*(dx_I) \\
&= g_\psi j_\psi \psi\varphi^{-1} \wedge \det D(\psi\varphi^{-1}) \wedge dx_I,
\end{aligned}$$

the last equality following from Lemma 14.17, it follows from (1) that for all $x \in \varphi(\text{dom } \varphi \cap \text{dom } \psi)$,

$$g_\varphi(x) = g_\psi\Big(\psi\big(\varphi^{-1}(x)\big)\Big) \det D(\psi\varphi^{-1})(x). \tag{2}$$

Since $g_\varphi(x) > 0$ and $g_\psi\Bigl(\psi\bigl(\varphi^{-1}(x)\bigr)\Bigr) > 0$, it follows from (2) that $D(\psi\varphi^{-1})(x) > 0$, proving (i).

To prove (ii), note first that an $\mathcal{S}$-admissible atlas whose codomains are connected open subsets of $\mathbf{R}^n$ exists. It suffices to show that each orientation-reversing chart in such an atlas can be replaced by an orientation-preserving one having the same domain.

Suppose φ is an $\mathcal{S}$-admissible orientation-reversing chart with codomain a connected open subset of $\mathbf{R}^n$. Choose any $\mathbf{R}$-linear transformation $L : \mathbf{R}^n \to \mathbf{R}^n$ having negative determinant and let $\overline{\varphi} = L^{-1}\varphi$. Then $\overline{\varphi}$ is an $\mathcal{S}$-admisible chart having the same domain as φ. Thus $i_{\overline{\varphi}} = i_\varphi$ and we have

$$\begin{aligned}(\overline{\varphi}^{-1})^* i_{\overline{\varphi}}^* \omega &= \bigl((L^{-1}\varphi)^{-1}\bigr)^* i_\varphi^* \omega = L^*(\varphi^{-1})^* i_\varphi^* \omega \\ &= L^*(g_\varphi \wedge dx_I) = g_\varphi L \wedge L^*(dx_I) = g_\varphi L \wedge \det L \wedge dx_I,\end{aligned}$$

the last equality by Lemma 14.17. Thus

$$g_{\overline{\varphi}}(x) = (\det L) g_\varphi\bigl(L(x)\bigr) > 0$$

for all $x \in \operatorname{codom} \overline{\varphi}$, completing the proof of (ii) and the proof in the first direction.

Now suppose M is paracompact and that $\mathcal{A} \subset \mathcal{E}(M, \mathcal{S})$ is an $\mathcal{S}$-admissible atlas such that whenever $\varphi, \psi \in \mathcal{A}$, $\det D(\psi\varphi^{-1})(x) > 0$ for all $x \in \varphi(\operatorname{dom} \varphi \cap \operatorname{dom} \psi)$. We make use of the result of Exercise 2 of §16. Thus it suffices to show that there exists a family $\{\sigma_\varphi \in \Gamma\bigl(\Lambda^n(\tau_{M,\mathcal{S}})|\operatorname{dom} \varphi\bigr) \mid \varphi \in \mathcal{A}\}$ such that for each $\varphi \in \mathcal{A}$ and $y \in \operatorname{dom} \varphi$, $\sigma_\varphi(y) \neq 0_y$, and that whenever $\varphi, \psi \in \mathcal{A}$ and $y \in \operatorname{dom} \varphi \cap \operatorname{dom} \psi$, $\sigma_\varphi(y)$ is a positive real multiple of $\sigma_\psi(y)$.

Let $p : \Lambda^n T(M, \mathcal{S}) \to M$ denote the projection map for $\Lambda^n \tau_{M,\mathcal{S}}$. Let $I = (1, 2, \ldots, n)$ as above. For $\varphi \in \mathcal{A}$ let $\omega_\varphi = \varphi^*(dx_I) \in \Omega^n(\operatorname{dom} \varphi)$. Let $i_\varphi : \operatorname{dom} \varphi \to M$ denote the inclusion and take σ_φ to be the composition $(\Lambda^n i_\varphi)^{-1}\omega_\varphi$ where $\Lambda^n i_\varphi : p^{-1}(\operatorname{dom} \varphi) \to \Lambda^n T(\operatorname{dom} \varphi)$ is as in Lemma 13.11. Since each $\varphi : \operatorname{dom} \varphi \to \operatorname{codom} \varphi$ is a diffeomorphism (and hence an immersion), it follows from Proposition 17.6 and Example 17.4 that each ω_φ is an orientation form on $(\operatorname{dom} \varphi, \mathcal{S}|\operatorname{dom} \varphi)$. Since each $\Lambda^n i_\varphi$ is a fiber-preserving diffeomorphism whose restrictions to fibers are $\mathbf{R}$-isomorphisms, it follows that $\sigma_\varphi \in \Gamma\bigl(\Lambda^n(\tau_{M,\mathcal{S}})|\operatorname{dom} \varphi\bigr)$ and that $\sigma_\varphi(y)$ is non-zero whenever $\varphi \in \mathcal{A}$ and $y \in \operatorname{dom} \varphi$.

Suppose $\varphi, \psi \in \mathcal{A}$ and let $i : \operatorname{dom} \varphi \cap \operatorname{dom} \psi \to M$ denote the inclusion. By Proposition 5.11, $T_y i : T_y(\operatorname{dom} \varphi \cap \operatorname{dom} \psi) \to T_y(M)$ is an $\mathbf{R}$-isomorphism for each $y \in \operatorname{dom} \varphi \cap \operatorname{dom} \psi$, and so $\Lambda^n(T_y i) : \Lambda^n T_y(M) \to \Lambda^n T_y(\operatorname{dom} \varphi \cap \operatorname{dom} \psi)$ is an $\mathbf{R}$-isomorphism for each such y. Consequently we will be done if we show that $\Lambda^n(T_y i)\bigl(\sigma_\varphi(y)\bigr)$ and $\Lambda^n(T_y i)\bigl(\sigma_\psi(y)\bigr)$ are positive real multiples of one another whenever $y \in \operatorname{dom} \varphi \cap \operatorname{dom} \psi$.

Let $k_\varphi : \operatorname{dom} \varphi \cap \operatorname{dom} \psi \to \operatorname{dom} \varphi$ and $k_\psi : \operatorname{dom} \varphi \cap \operatorname{dom} \psi \to \operatorname{dom} \psi$ denote the inclusion maps. Then $i_\varphi k_\varphi = i_\psi k_\psi = i$ and so

$$\begin{aligned}\Lambda^n(T_y i)\bigl(\sigma_\varphi(y)\bigr) &= \Lambda^n\bigl((T_y i_\varphi)(T_y k_\varphi)\bigr)\bigl(\sigma_\varphi(y)\bigr) \\ &= \Lambda^n(T_y k_\varphi)\Lambda^n(T_y i_\varphi)(T_y i_\varphi)^{-1}\omega_\varphi(y) \\ &= \Lambda^n(T_y k_\varphi)\omega_\varphi(y) = (k_\varphi^* \omega_\varphi)(y) = \bigl(k_\varphi^* \varphi^*(dx_I)\bigr)(y)\end{aligned}$$

and similarly $\Lambda^n(T_y i)(\sigma_\psi(y)) = (k_\psi^*\psi^*(dx_I))(y)$. Thus it remains only to show that $(k_\varphi^*\varphi^*(dx_I))(y)$ and $(k_\psi^*\psi^*(dx_I))(y)$ are positive real multiples of one another whenever $\varphi, \psi \in \mathcal{A}$ and $y \in \text{dom } \varphi \cap \text{dom } \psi$.

Writing $X = \text{dom } \varphi \cap \text{dom } \psi$ we have the commutative diagram

$$\begin{array}{ccccccc}
\text{dom } \varphi & \xleftarrow{k_\varphi} & X & \xrightarrow{=} & X & \xrightarrow{k_\psi} & \text{dom } \psi \\
\varphi \big\downarrow & & \varphi' \big\downarrow & & \big\downarrow \psi' & & \big\downarrow \psi \\
\text{codom } \varphi & \xleftarrow[j_\varphi]{} & \varphi(X) & \xrightarrow[\psi\varphi^{-1}]{} & \psi(X) & \xrightarrow[j_\psi]{} & \text{codom } \psi
\end{array}$$

where φ' and ψ' are the respective restrictions of φ and ψ. Thus

$$\begin{aligned}
k_\psi^*\psi^*(dx_I) &= (\psi')^* j_\psi^*(dx_I) = (\psi')^*(dx_I) \\
&= (\varphi')^*(\psi\varphi^{-1})^*(dx_I) = (\varphi')^*\big(\det(\psi\varphi^{-1}) \wedge dx_I\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 14.17)} \\
&= \big(\det(\psi\varphi^{-1})\big)\varphi' \wedge (\varphi')^*(dx_I) = \big(\det(\psi\varphi^{-1})\big)\varphi' \wedge (\varphi')^* j_\varphi^*(dx_I) \\
&= \big(\det(\psi\varphi^{-1})\big)\varphi' \wedge k_\varphi^*\varphi^*(dx_I)
\end{aligned}$$

and so

$$\big(k_\psi^*\psi^*(dx_I)\big)(y) = \det(\psi\varphi^{-1})\big(\varphi(y)\big)\big(k_\varphi^*\varphi^*(dx_I)\big)(y).$$

Since $\det(\psi\varphi^{-1})\big(\varphi(y)\big) > 0$ by hypothesis, we are done. □

Corollary 17.9. *Let $(M, \mathcal{S})$ be a smooth n-manifold, $n > 0$, which admits a two chart $\mathcal{S}$-admissible atlas $\{\varphi, \psi\}$ with codomains open subsets of $\mathbf{R}^n$ such that* $\text{dom } \varphi \cap \text{dom } \psi$ *is connected. Then $(M, \mathcal{S})$ is orientable.*

Proof. dom φ and dom ψ are both paracompact, being homeomorphic to subsets of $\mathbf{R}^n$. Thus M, being a finite union of open paracompact subspaces, is paracompact.

Since $\text{dom } \varphi \cap \text{dom } \psi$ is connected, $\det D(\psi\varphi^{-1})$ is either strictly positive or strictly negative on $\varphi(\text{dom } \varphi \cap \text{dom } \psi)$. If strictly positive, the result follows from Theorem 17.8. If strictly negative, replace φ by $\overline{\varphi}$ as in the proof of Theorem 17.8. Then $\{\overline{\varphi}, \psi\}$ is an $\mathcal{S}$-admissible atlas satisfying the hypothesis of Theorem 17.8. □

Corollary 17.10. *For $n \geq 2$, S^n, with its standard smooth structure, is orientable.*

Proof. The atlas given in Example 4.6 consisting of the stereographic projection charts satisfies the hypotheses of Corollary 17.9. □

Theorem 17.11. *Let $(M, \mathcal{S})$ be a smooth n-manifold, $n > 0$, which admits an $\mathcal{S}$-admissible atlas with exactly two charts φ, ψ whose codomains are connected open subsets of $\mathbf{R}^n$ such that*

$$\det D(\psi\varphi^{-1}) : \varphi(\text{dom } \varphi \cap \text{dom } \psi) \to \mathbf{R}$$

assumes both positive and negative values (and thus $\varphi(\text{dom } \varphi \cap \text{dom } \psi)$ *is necessarily disconnected). Then $(M, \mathcal{S})$ is not orientable.*

Proof. We proceed by contradiction. Suppose ω is an orientation form on $(M, \mathcal{S})$. With g_φ and g_ψ as in the proof of Theorem 17.8, the same argument given there yields that g_φ and g_ψ do not change sign on codom φ and codom ψ, respectively, and that formula (2) in the above proof holds for all $x \in \varphi(\text{dom } \varphi \cap \text{dom } \psi)$. Since $\det D(\psi\varphi^{-1})$ does change sign on $\varphi(\text{dom } \varphi \cap \text{dom } \psi)$, formula (2) yields a contradiction. □

Theorem 17.12. *Suppose $n > 0$. Then $P^n(\mathbf{R})$ is orientable if n is odd, and non-orientable if n is even. Moreover, if n is even, the open subset $\{ [x] \in P^n(\mathbf{R}) \mid x_1 \neq 0 \text{ or } x_2 \neq 0 \}$ is non-orientable.*

Proof. We start with the smooth atlas for $P^n(\mathbf{R})$ given in Example 4.7. The charts $\theta_1, \ldots, \theta_{n+1}$ of that atlas all have E^n, the open unit ball in $\mathbf{R}^n$, as codomain and so these codomains are all connected. For $1 \leq i \leq n+1$ let $\varphi_i : V_i \to E^n$ be given by $\varphi_i = (-1)^i \theta_i$. Then $\{ \varphi_1, \ldots, \varphi_{n+1} \}$ is also an admissible atlas for $P^n(\mathbf{R})$ with respect to the standard smooth structure on $P^n(\mathbf{R})$. It follows from Example 4.7 that for $1 \leq j < i \leq n+1$, the overlap map $\varphi_j\varphi_i^{-1}$ has $\{ y \in E^n \mid y_j \neq 0 \}$ as its domain and is given by

$$\varphi_j\varphi_i^{-1}(y) = (-1)^{i+j} \frac{y_j}{|y_j|} (y_1, \ldots, \widehat{y}_j, \ldots, y_{i-1}, \sqrt{1 - \|y\|^2}, y_i, \ldots, y_n).$$

Thus, for $1 \leq j < i \leq n+1$ and all $y \in \{ y \in E^n \mid y_j \neq 0 \}$,

$$D(\varphi_j\varphi_i^{-1})(y) = (-1)^{i+j} \frac{y_j}{|y_j|} \begin{pmatrix} I_{j-1} & 0 & 0 \\ * & A & * \\ 0 & 0 & I_{n-i+1} \end{pmatrix}$$

where A is the $(i-j) \times (i-j)$ matrix given by

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ a_j & a_{j+1} & a_{j+2} & \ldots & a_{i-1} \end{pmatrix}$$

where $a_k = -y_k/\sqrt{1 - \|y\|^2}$ for $j \leq k \leq i-1$. It follows that for all i, j, y as above

$$\begin{aligned} \det D(\varphi_j\varphi_i^{-1})(y) &= (-1)^{n(i+j)} \left(\frac{y_j}{|y_j|}\right)^n \det A \\ &= (-1)^{n(i+j)} \left(\frac{y_j}{|y_j|}\right)^n (-1)^{i+j-1} \left(\frac{-y_j}{\sqrt{1 - \|y\|^2}}\right) \\ &= \frac{(-1)^{(n+1)(i+j)} y_j^{n+1}}{|y_j|^n \sqrt{1 - \|y\|^2}}. \end{aligned}$$

If n is odd, it follows that $\det D(\varphi_j\varphi_i^{-1})(y) > 0$ whenever $1 \leq j < i \leq n+1$ and all $y \in \text{dom } (\varphi_j\varphi_i^{-1})$. For $1 \leq j < i \leq n+1$ and all $y \in \text{dom } (\varphi_i\varphi_j^{-1})$,

$$D(\varphi_i\varphi_j^{-1})(y) = \left(D(\varphi_j\varphi_i^{-1})(\varphi_i\varphi_j^{-1}(y))\right)^{-1}$$

and so it follows that $\det D(\varphi_i\varphi_j^{-1})(y) > 0$ whenever $1 \leq j < i \leq n+1$ and all $y \in \text{dom }(\varphi_j\varphi_i^{-1})$. Since $D(\varphi_i\varphi_i^{-1})(y) = I_n$ which has determinant 1, it now follows from Theorem 17.8 that $P^n(\mathbf{R})$ is orientable if $n \geq 1$ is odd.

If $n \geq 2$ is even, y_j^{n+1} is positive if y_j is positive, and negative if y_j is negative. It follows, from the formula above, that whenever $1 \leq j < i \leq n+1$, $\det D(\varphi_j\varphi_i^{-1})(y)$ assumes both positive and negative values. It now follows, from Theorem 17.11, that the open subset $V_i \cup V_j$ is non-orientable for $1 \leq j < i \leq n+1$, and hence $P^n(\mathbf{R})$ is non-orientable by Corollary 17.7. $\square$

Note that $P^2(\mathbf{R}) - \{\,[0,0,1]\,\} = V_1 \cup V_2$ and hence is non-orientable. Topologically, $P^2(\mathbf{R}) - \{\,[0,0,1]\,\}$ is an open Möbius strip.

Theorem 17.13. *Let $(M, \mathcal{S})$ be a smooth n-manifold, $n > 0$, with M paracompact. Then $\big(T(M, \mathcal{S}), Tan_{M,\mathcal{S}}\big)$ is orientable.*

Proof. We leave it as an exercise to show that the total space of any smooth vector bundle with a paracompact base space is paracompact. In particular, $T(M, \mathcal{S})$ is paracompact. Let p denote the projection map for $\tau_{M,\mathcal{S}}$. It follows from Theorem 8.19 that whenever $\varphi \in \mathcal{E}(M, \mathcal{S})$, the map

$$\widetilde{\varphi} : p^{-1}(\text{dom } \varphi) \to \text{codom } \varphi \times \mathbf{R}^n$$

given by

$$\widetilde{\varphi}(v) = \Big(\varphi\big(p(v)\big), \theta^{-1}_{\varphi(p(v))} T_{p(v)}\varphi(v)\Big)$$

is a $Tan_{M,\mathcal{S}}$-admissible chart. If $\varphi, \psi \in \mathcal{E}(M, \mathcal{S})$, it follows from Proposition 5.12 that the overlap map

$$\widetilde{\psi}\widetilde{\varphi}^{-1} : \varphi(\text{dom } \varphi \cap \text{dom } \psi) \times \mathbf{R}^n \to \psi(\text{dom } \varphi \cap \text{dom } \psi) \times \mathbf{R}^n$$

is given by

$$\widetilde{\psi}\widetilde{\varphi}^{-1}(x, y) = \big(\psi\varphi^{-1}(x), D(\psi\varphi^{-1}(x)(y)\big)$$

and so

$$D(\widetilde{\psi}\widetilde{\varphi}^{-1})(x, y) = \begin{pmatrix} D(\psi\varphi^{-1})(x) & 0 \\ * & D(\psi\varphi^{-1})(x) \end{pmatrix}$$

from which it follows that

$$\det D(\widetilde{\psi}\widetilde{\varphi}^{-1})(x, y) = \Big(\det D(\psi\varphi^{-1})(x)\Big)^2 > 0.$$

The result now follows from Theorem 17.8. $\square$

We next associate with each smooth manifold $(M, \mathcal{S})$ of positive dimension a two-sheeted covering space over M called the *orientation covering of* $(M, \mathcal{S})$. The fiber over each x in M will consist of the two orientations of the tangent space $T_x(M, \mathcal{S})$. We will see that $(M, \mathcal{S})$ is orientable if and only if its orientation covering admits a section. It will then be possible to use theorems on covering spaces to deduce facts about existence or non-existence of orientations for particular smooth manifolds. We now proceed to the definitions.

Let $(M, \mathcal{S})$ be an n-manifold, $n > 0$. Give $\mathcal{E}(M, \mathcal{S})$ the discrete topology. Let $X(M, \mathcal{S}) = \{\,(x, \varphi) \in M \times \mathcal{E}(M, \mathcal{S}) \mid x \in \text{dom } \varphi\,\}$, topologized as a subspace of $M \times \mathcal{E}(M, \mathcal{S})$. Define a relation $\sim$ on $X(M, \mathcal{S})$ by $(x, \varphi) \sim (y, \psi)$ if and only if $x = y$ and $\det D(\psi\varphi^{-1})\big(\varphi(x)\big) > 0$.

Lemma 17.14. *$\sim$ is an equivalence relation on $X(M, \mathcal{S})$.*

Proof. For any $(x, \varphi) \in X(M, \mathcal{S})$,

$$\det D(\varphi\varphi^{-1})\big(\varphi(x)\big) = \det I_n = 1$$

and so $\sim$ is reflexive.

Suppose $(x, \varphi) \sim (x, \psi)$. By the chain rule,

$$D(\varphi\psi^{-1})\big(\psi(x)\big)D(\psi\varphi^{-1})\big(\varphi(x)\big) = I_n$$

and thus $\det D(\psi\varphi^{-1})\big(\varphi(x)\big)$ and $\det D(\varphi\psi^{-1})\big(\psi(x)\big)$ have the same sign. Hence $\sim$ is symmetric.

Suppose $(x, \varphi) \sim (x, \psi)$ and $(x, \psi) \sim (x, \theta)$. By the chain rule,

$$D(\theta\varphi^{-1})\big(\varphi(x)\big) = D(\theta\psi^{-1})\big(\psi(x)\big)D(\psi\varphi^{-1})\big(\varphi(x)\big)$$

and so

$$\det D(\theta\varphi^{-1})\big(\varphi(x)\big) = \det D(\theta\psi^{-1})\big(\psi(x)\big)\det D(\psi\varphi^{-1})\big(\varphi(x)\big).$$

Thus, since $\det D(\theta\psi^{-1})\big(\psi(x)\big)$ and $\det D(\psi\varphi^{-1})\big(\varphi(x)\big)$ are both positive, so is $\det D(\theta\varphi^{-1})\big(\varphi(x)\big)$, proving transitivity. □

For $(x, \varphi) \in X(M, \mathcal{S})$ write $[x, \varphi]$ for the $\sim$-equivalence class of (x, φ).

Definition 17.15. Let $(M, \mathcal{S})$ be a smooth n-manifold, $n > 0$. The *orientation space of* $(M, \mathcal{S})$ is the quotient space $X(M, \mathcal{S})/\sim$ and denoted $\mathcal{O}(M, \mathcal{S})$. The map $p : \mathcal{O}(M, \mathcal{S}) \to M$ given by $p[x, \varphi] = x$ is called the *orientation covering of* $(M, \mathcal{S})$.

We proceed to show that the orientation covering of a smooth manifold is indeed a covering map.

Lemma 17.16. *Let $(M, \mathcal{S})$ be a smooth n-manifold, $n > 0$. For $\varphi \in \mathcal{E}(M, \mathcal{S})$ let $s_\varphi : \operatorname{dom}\varphi \to \mathcal{O}(M, \mathcal{S})$ be given by $s_\varphi(x) = [x, \varphi]$. Then s_φ is continuous.*

Proof. s_φ is the composition

$$\operatorname{dom}\varphi \xrightarrow{f} X(M, \mathcal{S}) \xrightarrow{q} \mathcal{O}(M, \mathcal{S})$$

where f is given by $f(x) = (x, \varphi)$ and q is the quotient map. f is clearly continuous and hence s_φ is. □

Theorem 17.17. *Let $(M, \mathcal{S})$ be a smooth n-manifold, $n > 0$. Then:*

(a) *The orientation covering $p : \mathcal{O}(M, \mathcal{S}) \to M$ is a covering map.*
(b) *For each $x \in M$, $p^{-1}(x)$ contains exactly two points.*
(c) *For each $\varphi \in \mathcal{E}(M, \mathcal{S})$, $\operatorname{dom}\varphi$ is evenly covered by p.*

Proof. We have the commutative diagram

$$\begin{array}{ccc} X(M, \mathcal{S}) & \xrightarrow{q} & \mathcal{O}(M, \mathcal{S}) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow[=]{} & M \end{array}$$

where $\pi_1(x,\varphi) = x$ for all $(x,\varphi) \in X(M,\mathcal{S})$. Since π_1 is continuous and q is a quotient map, it follows that p is continuous. p is clearly onto. Choose any $\mathbf{R}$-linear transformation $L : \mathbf{R}^n \to \mathbf{R}^n$ such that $\det L = -1$ and for $\varphi \in \mathcal{E}(M,\mathcal{S})$ let $\widetilde{\varphi} = L^{-1}\varphi$ as in the proof of Theorem 17.8. Then $\widetilde{\varphi} \in \mathcal{E}(M,\mathcal{S})$, dom $\widetilde{\varphi} =$ dom φ, and for all $\psi \in \mathcal{E}(M,\mathcal{S})$ and $y \in$ dom $\varphi \cap$ dom ψ,

$$(1) \qquad \det D(\psi\varphi^{-1})(\varphi(y)) = -\det D(\psi\widetilde{\varphi}^{-1})(\widetilde{\varphi}(y)), \text{ and}$$

$$(2) \qquad \det D(\widetilde{\varphi}\varphi^{-1})(\varphi(y)) = -1.$$

For $\varphi \in \mathcal{E}(M,\mathcal{S})$ let $V_\varphi = \{\, [y,\varphi] \mid y \in \text{dom } \varphi \,\} \subset \mathcal{O}(M,\mathcal{S})$. We will be done if we show that for each $\varphi \in \mathcal{O}(M,\mathcal{S})$:

(i) V_φ is open in $\mathcal{O}(M,\mathcal{S})$.

(ii) $p^{-1}(\text{dom } \varphi) = V_\varphi \cup V_{\widetilde{\varphi}}$.

(iii) $V_\varphi \cap V_{\widetilde{\varphi}} = \emptyset$.

(iv) p maps each of V_φ and $V_{\widetilde{\varphi}}$ homeomorphically onto dom φ.

We have

$$q^{-1}(V_\varphi) = \{\, (y,\psi) \mid y \in \text{dom } \varphi \cap \text{dom } \psi \text{ and } \det D(\psi\varphi^{-1})(\varphi(y)) > 0 \,\}.$$

Let $(y,\psi) \in q^{-1}(V_\varphi)$ and write U for the component of dom $\varphi \cap$ dom ψ which contains y. Then U is open in M. Since $\det D(\psi\varphi^{-1}) : \varphi(U) \to \mathbf{R}$ is continuous, never 0, positive at $\varphi(y)$, and $\varphi(U)$ is connected, it follows that $\det D(\psi\varphi^{-1}(\varphi(z)) > 0$ for all $z \in U$. Hence $(y,\psi) \in U \times \{\,\psi\,\} \subset q^{-1}(V_\varphi)$. Since $\mathcal{E}(M,\mathcal{S})$ has the discrete topology, $U \times \{\,\psi\,\}$ is open in $M \times \mathcal{E}(M,\mathcal{S})$, and hence in $X(M,\mathcal{S})$. Thus $q^{-1}(V_\varphi)$ is open in $X(M,\mathcal{S})$. Since q is a quotient map, (i) now follows.

Trivially, $V_\varphi \cup V_{\widetilde{\varphi}} \subset p^{-1}(\text{dom } \varphi)$. Suppose $[y,\psi] \in p^{-1}(\text{dom } \varphi)$. Then $y \in$ dom $\varphi \cap$ dom ψ. By (1), one of $\det D(\psi\varphi^{-1})(\varphi(y))$, $\det D(\psi\widetilde{\varphi}^{-1})(\widetilde{\varphi}(y))$ must be positive and so either $(y,\psi) \sim (y,\varphi)$ or $(y,\psi) \sim (y,\widetilde{\varphi})$, i.e. either $[y,\psi] \in V_\varphi$ or $[y,\psi] \in V_{\widetilde{\varphi}}$, proving (ii).

For each $y \in$ dom φ, $(y,\varphi) \nsim (y,\widetilde{\varphi})$ by (2) and so (iii) follows.

Note that $s_\varphi(\text{dom } \varphi) = V_\varphi$ where s_φ is as in Lemma 17.16. Thus, by restriction of the codomain, we obtain a continuous map $s_\varphi : \text{dom } \varphi \to V_\varphi$. It is immediate the the compositions

$$\text{dom } \varphi \xrightarrow{s_\varphi} V_\varphi \xrightarrow{p} \text{dom } \varphi,$$
$$V_\varphi \xrightarrow{p} \text{dom } \varphi \xrightarrow{s_\varphi} V_\varphi$$

are the respective identity maps, proving (iv). □

Theorem 17.18. *Let $(M,\mathcal{S})$ be a smooth manifold, $n > 0$. If $(M,\mathcal{S})$ is orientable, then the orientation covering $p : \mathcal{O}(M,\mathcal{S}) \to M$ admits a section. The converse holds if M is paracompact.*

Proof. Suppose $(M,\mathcal{S})$ is orientable. By Theorem 17.8 M admits an $\mathcal{S}$-admissible atlas $\mathcal{A}$ such that for all $\varphi,\psi \in \mathcal{A}$ and $x \in$ dom $\varphi \cap$dom ψ, $\det D(\psi\varphi^{-1})(\varphi(x)) > 0$. Define $s : M \to \mathcal{O}(M,\mathcal{S})$ as follows: If $x \in M$, choose any $\varphi \in \mathcal{A}$ such that $x \in$ dom φ and define $s(x) = [x,\varphi]$. s is well-defined for if ψ is another chart in $\mathcal{A}$ with $x \in \text{dom } \psi$, $(x, \varphi) \sim (x, \psi)$ since $\det D(\psi\varphi^{-1})(\varphi(x)) > 0$. Clearly, $ps = 1_M$, so it remains only to check the continuity of s. It suffices to check that for each $\varphi \in \mathcal{A}$, the restrictions of s to dom φ is continuous. The latter restriction is precisely s_φ, which is continuous by Lemma 17.16.

Conversely, suppose M is paracompact and that a section $s : M \to \mathcal{O}(M, \mathcal{S})$ exists. Take

$$\mathcal{A} = \{\, \varphi \in \mathcal{E}(M, \mathcal{S}) \mid \text{dom } \varphi \text{ is connected and } s(x) = [x, \varphi] \text{ for some } x \in \text{dom } \varphi \,\}.$$

Then $\mathcal{A}$ is an $\mathcal{S}$-admissible atlas for $(M, \mathcal{S})$. By Theorem 17.8 we will be done if we show that for all $\varphi, \psi \in \mathcal{A}$ and $x \in \text{dom } \varphi \cap \text{dom } \psi$, $\det D(\psi\varphi^{-1})(\varphi(x)) > 0$.

We first prove that for each $\varphi \in \mathcal{A}$ the restriction of s to dom φ is s_φ as given in Lemma 17.16. By definition of $\mathcal{A}$ there exists an $x \in \text{dom } \varphi$ such that $s(x) = [x, \varphi]$. By definition of s_φ we also have $s_\varphi(x) = [x, \varphi]$. Thus $s|\text{dom } \varphi$ and s_φ are lifts to $\mathcal{O}(M, \mathcal{S})$ of the inclusion map $i : \text{dom } \varphi \to M$ which agree at the point x. Since dom φ is connected, it follows from the Uniqueness of Liftings Property for covering spaces that s and s_φ agree at all points of dom φ.

Now let $\varphi, \psi \in \mathcal{A}$ and $x \in \text{dom } \varphi \cap \text{dom } \psi$. Then by the above, $s(x) = s_\varphi(x) = s_\psi(x)$, i.e. $[x, \varphi] = [x, \psi]$. Thus $(x, \varphi) \sim (x, \psi)$ and so by definition of $\sim$, $\det D(\psi\varphi^{-1})(\varphi(x)) > 0$. □

Corollary 17.19. *Let $(M, \mathcal{S})$ be a smooth n-manifold, $n > 0$, such that M is paracompact and simply connected. Then $(M, \mathcal{S})$ is orientable.*

Proof. Since M is simply connected and locally pathwise connected (every topological manifold is locally pathwise connected) it follows from the General Lifting Theorem for Covering Spaces that *every* covering map $E \to M$ admits a section. In particular the orientation covering admits a section. □

We give one more example of a general class of smooth manifolds which are automatically orientable, namely *complex analytic manifolds.* Every vector space V over the complex numbers $\mathbf{C}$ has an underlying real vector space structure obtained simply by restricting the scalars to $\mathbf{R}$. If V is n-dimensional over $\mathbf{C}$, then V is $2n$-dimensional over $\mathbf{R}$.

Definition 17.20. Let M be a topological $2n$-manifold. A *complex analytic atlas for M* is a manifold atlas $\mathcal{A}$ for M consisting of charts whose codomains are open subsets of $\mathbf{C}^n$ such that whenever $\varphi, \psi \in \mathcal{A}$, the overlap map $\psi\varphi^{-1}$ is complex analytic.

Two complex analytic atlases $\mathcal{A}$ and $\mathcal{B}$ for M are *analytically equivalent* if whenever $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{B}$, the overlap maps $\psi\varphi^{-1}$ and $\varphi\psi^{-1}$ are complex analytic.

The same proofs given for smoothly equivalent atlases yield that the relation *analytically equivalent to* is an equivalence relation on the set of all complex analytic atlases for M.

Definition 17.21. A *complex analytic manifold* is a pair $(M, \mathcal{C})$ consisting of a topological manifold M and an analytic equivalence class $\mathcal{C}$ of complex analytic atlases for M.

Every complex analytic map, regarded as a map betwween open sets in the underlying real vector spaces, is smooth and hence any complex analytic manifold has an underlying smooth structure. The converse for $2n$-manifolds is far from true; most $2n$-manifolds do not admit any complex analytic structure. For example, S^{2n} for $n \geq 1$ does *not* admit a complex analytic structure except for the case $n = 1$ and possibly $n = 3$ (to the best of my knowledge the case of S^6 has not yet been settled).

Much of the smooth theory has a complex analytic analogue. A notable exception is partitions of unity. Complex analytic partitions of unity do not exist, except in trivial cases. Complex analytic sections of complex analytic vector bundles are much more scarse than smooth sections.

Example 17.22. A slight variant of the stereographic projection charts (Example 2.4) yields a complex analytic atlas for S^2. Let $U = S^2 - \{(0,0,1)\}$, $V = S^2 - \{(0,0,-1)\}$, and define $\varphi : U \to \mathbf{C}$, $\psi : V \to \mathbf{C}$ by

$$\varphi(a,b,c) = (1-c)^{-1}(a+bi),$$
$$\psi(a,b,c) = (1+c)^{-1}(a-bi).$$

One can check that both φ and ψ are homeomorphisms and that both overlap maps $\mathbf{C} - \{0\} \to \mathbf{C} - \{0\}$ send z to z^{-1}, which is complex analytic.

Example 17.23. For $n \geq 0$, *complex projective n-space* $P^n(\mathbf{C})$ is the quotient space obtained from $\mathbf{C}^{n+1} - \{0\}$ by identifying each $(w_1, \ldots, w_{n+1}) \in \mathbf{C}^{n+1} - \{0\}$ with $(zw_1, \ldots, zw_{n+1})$ whenever $z \in \mathbf{C} - \{0\}$. We will show that $P^n(\mathbf{C})$ admits the structure of a complex analytic manifold of complex dimension n (real dimension $2n$.)

We write $[w_1, \ldots, w_{n+1}]$ for the image of $(w_1, \ldots, w_{n+1})$ under the quotient map $q : \mathbf{C}^{n+1} - \{0\} \to P^n(\mathbf{C})$. We first check that $P^n(\mathbf{C})$ is Hausdorff. Let $f : P^n(\mathbf{C}) \to \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^{n+1}, \mathbf{C}^{n+1})$ be given by

$$f([z_1, \ldots, z_{z+1}]) = \frac{1}{\sum_{j=1}^{n+1} |z_j|^2} \begin{pmatrix} \overline{z}_1 z_1 & \overline{z}_1 z_2 & \cdots & \overline{z}_1 z_{n+1} \\ \overline{z}_2 z_1 & \overline{z}_2 z_2 & \cdots & \overline{z}_2 z_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{z}_{n+1} z_1 & \overline{z}_{n+1} z_2 & \cdots & \overline{z}_{n+1} z_{n+1} \end{pmatrix}$$

where $\overline{z}$ denotes the complex conjugate of z. f is well-defined since for each $z \in \mathbf{C} - \{0\}$, $(z_1, \ldots, z_{n+1}) \in \mathbf{C}^{n+1} - \{0\}$, and $k, l \in \{1, \ldots, n+1\}$,

$$\frac{\overline{zz_k} zz_l}{\sum_{j=1}^{n+1} |zz_j|^2} = \frac{\overline{z}z\overline{z}_k z_l}{|z|^2 \sum_{j=1}^{n+1} |z_j|^2} = \frac{\overline{z}_k z_l}{\sum_{j=1}^{n+1} |z_j|^2}.$$

f is continuous since the composition fq is clearly continuous. Note that for each $x = [z_1, \ldots, z_{n+1}] \in P^n(\mathbf{C})$, at least one row of the matrix $f(x)$ is non-zero and each such row is a non-zero complex multiple of $(z_1, \ldots, z_{n+1})$. Thus x is recoverable from $f(x)$, i.e. f is injective. Since $\mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^{n+1}, \mathbf{C}^{n+1})$ is Hausdorff, it follows that $P^n(\mathbf{C})$ is Hausdorff.

For $1 \leq j \leq n+1$ let $V_j = \{[w_1, \ldots, w_{n+1}] \in P^n(\mathbf{C}) \mid w_j \neq 0\}$. It is easily checked that $\{V_1, \ldots, V_{n+1}\}$ is an open cover of $P^n(\mathbf{C})$. For $1 \leq j \leq n+1$ let $\varphi_j : V_j \to \mathbf{C}^n$ and $\psi_j : \mathbf{C}^n \to V_j$ be given by

$$\varphi_j([w_1, \ldots, w_{n+1}] = w_j^{-1}(w_1, \ldots, \widehat{w}_j, \ldots, w_{n+1}),$$
$$\psi_j(z_1, \ldots, z_n) = [z_1, \ldots, z_{j-1}, 1, z_j, \ldots, z_n].$$

It is straightforward to check that φ_j and ψ_j are both continuous and inverses of one another. Thus $\{\varphi_1, \ldots, \varphi_{n+1}\}$ is a real $2n$-manifold atlas for M. The overlap maps are easily checked to be as follows:

$$\varphi_j \varphi_k^{-1}(z_1, \ldots, z_n) = \begin{cases} z_j^{-1}(z_1, \ldots, \widehat{z}_j, \ldots, z_{k-1}, 1, z_k, \ldots, z_n) & \text{if } j < k, \\ z_{j-1}^{-1}(z_1, \ldots, z_{k-1}, 1, z_k, \ldots, \widehat{z}_{j-1}, \ldots, z_n) & \text{if } j > k \end{cases}$$

which are complex analytic on their domains. Thus $P^n(\mathbf{C})$ admits a complex analytic structure.

If V is a complex vector space let $\rho : \mathrm{Hom}_{\mathbf{C}}(V, V) \to \mathrm{Hom}_{\mathbf{R}}(V, V)$ be the function which assigns to each complex linear transformation $V \to V$ its underlying real linear transformation. It is immediate that ρ is a homomorphism of algebras over $\mathbf{R}$. If $B = (b_1, \ldots, b_n)$ is an ordered $\mathbf{C}$-basis of V, define $\rho(B) = (b_1, ib_1, b_2, ib_2, \ldots, b_n, ib_n)$. $\rho(B)$ is an ordered $\mathbf{R}$-basis of V. If $z = a + bi$ where a and b are real, define

$$\rho(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

More generally, if

$$A = \begin{pmatrix} z_{11} & \ldots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \ldots & z_{nn} \end{pmatrix}$$

where the $z_{ij} \in \mathbf{C}$, define

$$\rho(A) = \begin{pmatrix} \rho(z_{11}) & \ldots & \rho(z_{1n}) \\ \vdots & \ddots & \vdots \\ \rho(z_{n1}) & \ldots & \rho(z_{nn}) \end{pmatrix}.$$

We leave the proof of the following lemma as an exercise.

Lemma 17.23. *Let V be a complex finite-dimensional vector space and $f : V \to V$ a complex linear transformation. Let B be any ordered basis of V and A the matrix of f relative to B. Then the matrix of $\rho(f)$ relative to $\rho(B)$ is $\rho(A)$.* □

If V is a complex finite-dimensional vector space we have the two determinant functions

$$\det_{\mathbf{C}} : \mathrm{Hom}_{\mathbf{C}}(V, V) \to \mathbf{C}$$

and

$$\det_{\mathbf{R}} : \mathrm{Hom}_{\mathbf{R}}(V, V) \to \mathbf{R}.$$

Lemma 17.24. *Let V be a complex finite-dimensional vector space. Then for each $f \in \mathrm{Hom}_{\mathbf{C}}(V, V)$,*

$$\det_{\mathbf{R}}\big(\rho(f)\big) = |\det_{\mathbf{C}}(f)|^2.$$

Proof. Since $\mathbf{C}$ is algebraically closed, we can choose an ordered $\mathbf{C}$-basis B of V such that the matrix A of f relative to B is triangular. Then $\det_{\mathbf{C}}(f) = z_1 \cdots z_n$ where the z_j are the diagonal entries of A. By Lemma 17.23, the matrix of $\rho(f)$ relative to $\rho(B)$ is $\rho(A)$ from which it follows that

$$\det_{\mathbf{R}}\big(\rho(f)\big) = \det_{\mathbf{R}}\big(\rho(z_1)\big) \cdots \det_{\mathbf{R}}\big(\rho(z_n)\big).$$

The result now follows since $\det_{\mathbf{R}}\big(\rho(z)\big) = |z|^2$ for all $z \in \mathbf{C}$. □

Corollary 17.25. *For any $n \times n$ matrix A with complex entries, $\det \rho(A) \geq 0$.* □

Theorem 17.26. *Let $(M, \mathcal{C})$ be a complex analytic manifold of real dimension $2n$, $n > 0$, with M paracompact. Let $\mathcal{S}$ be the smooth structure on M determined by $\mathcal{C}$. Then $(M, \mathcal{S})$ is orientable.*

Proof. Let $\mathcal{A}$ be an $\mathcal{S}$-admissible complex analytic atlas. By Theorem 17.8 it suffices to show that for all $\varphi, \psi \in \mathcal{A}$ and $x \in \varphi(\text{dom } \varphi \cap \text{dom } \psi)$, $\det D(\psi\varphi^{-1})(x) > 0$.

$\psi\varphi^{-1}$ is a complex analytic function between open sets in $\mathbf{C}^n$. Using real coordinates we can write $\psi\varphi^{-1} = (u_1, v_1, \dots, u_n, v_n)$ where the u_k, v_k are smooth real-valued functions of the real variables $x_1, y_1, \dots, x_n, y_n$ and satisfy the Cauchy-Riemann equations

$$\frac{\partial u_k}{\partial x_j} = \frac{\partial v_k}{\partial y_j}, \quad \frac{\partial u_k}{\partial y_j} = -\frac{\partial v_k}{\partial x_j}$$

for $1 \leq k \leq n$, $1 \leq j \leq n$. We have

$$D(\psi\varphi^{-1}) = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{pmatrix}$$

where

$$A_{kj} = \begin{pmatrix} \dfrac{\partial u_k}{\partial x_j} & \dfrac{\partial u_k}{\partial y_j} \\ \dfrac{\partial v_k}{\partial x_j} & \dfrac{\partial v_k}{\partial y_j} \end{pmatrix}.$$

By the Cauchy-Riemann equations,

$$A_{kj} = \begin{pmatrix} \dfrac{\partial u_k}{\partial x_j} & -\dfrac{\partial v_k}{\partial x_j} \\ \dfrac{\partial v_k}{\partial x_j} & \dfrac{\partial u_k}{\partial x_j} \end{pmatrix}.$$

Thus, for each $x \in \varphi(\text{dom } \varphi \cap \text{dom } \psi)$, $D(\psi\varphi^{-1})(x)$ lies in the image of ρ and hence has non-negative determinant by Corollary 17.25. Since $\psi\varphi^{-1}$ is a diffeomorphism, the latter determinant can never be 0 and hence must be strictly positive. □

The last topic of this section is the Poincaré Duality Theorem for de Rham cohomology. The proof will not be given.

Definition 17.27. Let $(M, \mathcal{S})$ be a smooth n-manifold, $n > 0$, and suppose ω is an orientation form on $(M, \mathcal{S})$. A smooth n-cube σ in $(M, \mathcal{S})$ is said to be *ω-preserving* if $\sigma^*\omega = g_\sigma \wedge dx_1 \wedge \cdots \wedge dx_n$ where $g_\sigma : I^n \to \mathbf{R}$ has a smooth extension to an open subset of $\mathbf{R}^n$ and $g_\sigma(x) > 0$ for all $x \in I^n$.

An *ω-fundamental cycle C on* $(M, \mathcal{S})$ is a smooth cubical n-cycle of the form $C = \sum_{i=1}^q \sigma_i$ where each σ_i is ω-preserving.

For $(M, \mathcal{S})$ and ω as above, it is known that an ω-fundamental cycle for $(M, \mathcal{S})$ exists if and only if M is compact. This is not easy to prove, and we will not attempt a proof here. The idea, for compact M, is to "tile" the manifold by cubes which intersect in faces. The proof of the following proposition, however, is easy.

Proposition 17.28. *Let $(M, \mathcal{S})$ be a smooth n-manifold, $n > 0$. Suppose $(M, \mathcal{S})$ has an orientation form ω and that an ω-fundamental cycle C on $(M, \mathcal{S})$ exists. Then $H^n_{dR}(M, \mathcal{S}) \neq 0$. In fact, $[\omega] \neq 0$.*

Proof. Since every n-form on a smooth n-manifold is closed, ω is closed. Thus we will be done if we show that ω is not exact.

Write $C = \sum_{i=1}^q \sigma_i$ where each σ_i is ω-preserving. We have

$$\int_C \omega = \sum_{i=1}^q \int_{\sigma_i} \omega = \sum_{i=1}^q \int_{I^n} g_{\sigma_i} \, dV_n > 0$$

since each g_{σ_i} is strictly positive on I^n. If $\omega = d\alpha$ for some $(n-1)$-form α on $(M, \mathcal{S})$ we would have, by the Generalized Stokes' Theorem (Theorem 14.12)

$$\int_C \omega = \int_C d\alpha = \int_{\partial C} \alpha = \int_0 \alpha = 0$$

since C is a cycle, a contradiction. □

If $(M, \mathcal{S})$ is a smooth n-manifold and X an arbitrary smooth cubical n-cycle on $(M, \mathcal{S})$, then whenever $\alpha \in \Omega^i(M, \mathcal{S})$ and $\beta \in \Omega^{n-i}(M, \mathcal{S})$ are closed forms, we obtain the real number $\int_X \alpha \wedge \beta$. The latter depends only on the de Rham cohomology classes of these closed forms, for if μ and ν are $(i-1)$- and $(n-i-1)$-forms, respectively, then

$$(\alpha + d\mu) \wedge (\beta + d\nu) = \alpha \wedge \beta + d(\mu \wedge \beta \pm \alpha \wedge \nu + \mu \wedge d\nu)$$

and

$$\int_X d(\mu \wedge \beta \pm \alpha \wedge \nu + \mu \wedge d\nu) = \int_{\partial X} (\mu \wedge \beta \pm \alpha \wedge \nu + \mu \wedge d\nu) = 0$$

since $\partial X = 0$. We thus obtain a well-defined $\mathbf{R}$-bilinear map

$$D_X : H^i_{dR}(M, \mathcal{S}) \times H^{n-i}_{dR}(M, \mathcal{S}) \to \mathbf{R}$$

given by $D_X([\alpha], [\beta]) = \int_X \alpha \wedge \beta$.

Theorem 17.29. (Poincaré Duality Theorem) *Suppose $(M, \mathcal{S})$ is a compact smooth orientable n-manifold, $n > 0$, and ω an orientation form on $(M, \mathcal{S})$. Then:*

(a) *The $H^i_{dR}(M, \mathcal{S})$ are all finite-dimensional over $\mathbf{R}$.*

(b) *An ω-fundamental cycle C on $(M, \mathcal{S})$ exists.*

(c) *For each i, the map $D_C : H^i_{dR}(M, \mathcal{S}) \times H^{n-i}_{dR}(M, \mathcal{S}) \to \mathbf{R}$ is a dual pairing, i.e. its adjoints*

$$\varepsilon : H^i_{dR}(M, \mathcal{S}) \to \mathrm{Hom}_{\mathbf{R}}(H^{n-i}_{dR}(M, \mathcal{S}), \mathbf{R})$$

and

$$\eta : H^{n-i}_{dR}(M, \mathcal{S}) \to \mathrm{Hom}_{\mathbf{R}}(H^i_{dR}(M, \mathcal{S}), \mathbf{R})$$

given by $\varepsilon(a)(b) = \eta(b)(a) = D_C(a,b)$ *are* **R**-*isomorphisms.*

Exercises for §17

1. Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be orientable smooth manifolds. Prove that $(M\times N, \mathcal{S}\times\mathcal{T})$ is orientable.

2. Let ω be the orientation form on S^1 of Example 17.5. Find an explicit ω-fundamental cycle on S^1.

3. Let $(M, \mathcal{S})$ and $(N, \mathcal{T})$ be smooth manifolds of dimensions m and n, respectively. Suppose $f : M \to N$ is smooth with respect to $\mathcal{S}$ and $\mathcal{T}$, and that there exists an m-form ω on $(N, \mathcal{T})$ such that $f^*\omega$ is an orientation form on $(M, \mathcal{S})$. Prove that f must be an immersion.

4. Let $(M, \mathcal{S})$ be a paracompact connected smooth manifold whose fundamental group is finite of odd order. Prove that $(M, \mathcal{S})$ must be orientable.

18. Riemannian Metrics

Recall that if V is a real vector space, an *inner product g on V* is an $\mathbf{R}$-bilinear map $g : V \times V \to \mathbf{R}$ which is *symmetric* (i.e. $g(v, u) = g(u, v)$ for all $u, v \in V$), and *positive-definite* (i.e. $g(v, v) > 0$ for all non-zero $v \in V$). The standard dot product in $\mathbf{R}^n$ is the motivating example. Roughly speaking, a *Riemannian metric* on a smooth vector bundle is a simultaneous choice of inner products on the fibers such that these inner products "vary smoothly" from fiber to fiber. A Riemannian metric on the tangent bundle of a smooth manifold $(M, \mathcal{S})$ is called a *Riemannian metric on* $(M, \mathcal{S})$ and a *Riemannian manifold* is a pair consisting of a smooth manifold and a Riemannian metric on it. Riemannian manifolds are the principal objects of study in Differential Geometry.

The objects of this section are two-fold:

(1) To show that every smooth vector bundle with a paracompact base space admits a Riemannian metric.

(2) To show that for smooth manifolds $(M, \mathcal{S})$, the following three conditions are equivalent:

(i) $(M, \mathcal{S})$ admits a Riemannian metric.

(ii) M is paracompact.

(iii) M is metrizable.

If V is a real vector space we write $S(V)$ for the real vector space of all symmetric $\mathbf{R}$-bilinear maps $V \times V \to \mathbf{R}$. We write $P(V)$ for the set of positive-definite elements in $S(V)$, i.e. the set of inner products on V. If $f : V \to W$ is an $\mathbf{R}$-linear transformation, we write $S(f) : S(W) \to S(V)$ for the function which sends any symmetric $\mathbf{R}$-bilinear map $\alpha : W \times W \to \mathbf{R}$ to the composition

$$V \times V \xrightarrow{f \times f} W \times W \xrightarrow{\alpha} \mathbf{R}.$$

In terms of elements, $S(f)(\alpha)(v_1, v_2) = \alpha\big(f(v_1), f(v_2)\big)$. $S(f)$ is easily seen to be $\mathbf{R}$-linear. Moreover, if f is injective, $S(f)$ carries $P(W)$ into $P(V)$.

Proposition 18.1. *S is a smooth contravariant functor from $VS_{\mathbf{R}}$ to $VS_{\mathbf{R}}$.*

Proof. It is an easy exercise to verify that $S : VS_{\mathbf{R}} \to VS_{\mathbf{R}}$ is a contravariant functor.

Let V be a finite-dimensional real vector space with $\mathbf{R}$-basis $B = \{ v_1, \dots, v_m \}$. For $i, j \in \{ 1, \dots, m \}$ let $s(v_i, v_j) : V \times V \to \mathbf{R}$ be given by

$$s(v_i, v_j)\left(\sum_{k=1}^{m} a_k v_k, \sum_{k=1}^{m} b_k v_k\right) = a_i b_j + a_j b_i.$$

One checks easily that $s(v_i, v_j) \in S(V)$ and that $S(B) = \{ s(v_i, v_j) \mid 1 \le i \le j \le m \}$ is an $\mathbf{R}$-basis for $S(V)$.

Suppose W is another finite-dimensional real vector space with $\mathbf{R}$-basis $C = \{ w_1, \dots, w_n \}$, and let $f : V \to W$ be $\mathbf{R}$-linear. To show that S is a smooth functor it suffices to check that the entries of the matrix of $S(f)$ with respect to the bases $S(B)$ and $S(C)$ are smooth functions of the matrix entries of f with respect to B

and C. The latter are, in fact, homogeneous quadratic polynomial functions; it is straightforward to check that if

$$f(v_i) = \sum_{k=1}^{n} a_{ki} w_k, \quad 1 \le i \le m,$$

then

$$S(f)\big(s(w_i, w_j)\big) = \sum_{1 \le k \le l \le m} A^{ij}_{kl} s(v_k, v_l), \quad 1 \le i \le j \le n,$$

where

$$A^{ij}_{kl} = \begin{cases} a_{ik}a_{jl} + a_{jk}a_{il} & \text{if } k < l, \\ a_{ik}a_{jk} & \text{if } k = l. \end{cases} \qquad \square$$

Thus given any smooth vector bundle $\xi = \big(V, E, (M, \mathcal{S}_M), p, \mathcal{S}\big)$ we obtain, by the construction of §11 and Corollary 11.7, a smooth vector bundle

$$S(\xi) = \big(S(V), SE, (M, \mathcal{S}_M), p_S, \mathcal{S}_S\big).$$

For each $x \in M$ the fiber over x in $S(\xi)$ is $S\big(p^{-1}(x)\big)$, the real vector space of all symmetric real-valued $\mathbf{R}$-bilinear maps on the fiber over x in ξ.

Definition 18.2. Let $\xi = \big(V, E, (M, \mathcal{S}_M), p, \mathcal{S}\big)$ be a smooth vector bundle. A *Riemannian metric on* ξ is a smooth section g of $S(\xi)$ such that for each $x \in M$, $g(x) \in P\big(p^{-1}(x)\big)$, i.e. $g(x)$ is an inner product on the fiber over x in ξ.

A *Riemannian metric* g *on* $(M, \mathcal{S})$, where $(M, \mathcal{S})$ is a smooth manifold, is a Riemannian metric on the tangent bundle of $(M, \mathcal{S})$. In this case the triple $(M, \mathcal{S}, g)$ is called a *Riemannian manifold.*

Definition 18.3. Let V be a real vector space. A *convex cone in* V is a non-empty subset C of V such that whenever $v_1, \ldots, v_n$ is a finite collection in C and $r_1, \ldots, r_n$ are non-negative real numbers which are not all 0, then $\sum_{i=1}^{n} r_i v_i \in C$.

Example 18.4. Let V be any real vector space and $v_1, \ldots, v_n$ any $\mathbf{R}$-linearly independent collection in V. Let C be the set of all non-trivial real linear combinations of the v_i with non-negative coefficients. Then C is a convex cone in V.

Example 18.5. For any real vector space V, $P(V)$ is a convex cone in $S(V)$.

Proposition 18.6. *Let* $\xi = \big(V, E, (M, \mathcal{S}_M), p, \mathcal{S}\big)$ *be a smooth vector bundle with* M *paracompact. Suppose we are given, for each* $x \in M$, *a convex cone* $C_x \subset p^{-1}(x)$. *Suppose for some open cover* $\mathcal{O}$ *of* M *(not necessarily locally finite) there exists a family of smooth sections* $\{\sigma_A \in \Gamma(\xi|A) \mid A \in \mathcal{O}\}$ *such that for all* $A \in \mathcal{O}$ *and* $x \in A$, $\sigma_A(x) \in C_x$. *Then* ξ *admits a smooth section* σ *such that* $\sigma(x) \in C_x$ *for each* $x \in M$.

Proof. It is trivial that any open refinement of $\mathcal{O}$ satisfies the stated properties of $\mathcal{O}$. Since any open cover of a topological manifold has an open refinement whose members have compact closure in the manifold, we can suppose, without loss of generality, that each member of $\mathcal{O}$ has compact closure in M. Since M is paracompact, we can further assume, without loss of generality, that $\mathcal{O}$ is locally finite. By Theorem 16.5, there exists a smooth partition of unity $\{ f_A \mid A \in \mathcal{O} \}$ subordinate to $\mathcal{O}$. By the Piecing Theorem (Theorem 16.8), $\sigma : M \to E$ given by

$$\sigma(x) = \sum_{A \in \mathcal{O}} (f_A \cdot \sigma_A)(x)$$

is a smooth section of ξ. If $x \in M$ and $A_1, \ldots, A_r$ are the members of $\mathcal{O}$ which contain x, then

$$\sigma(x) = \sum_{i=1}^{r} f_{A_i}(x) \sigma_{A_i}(x).$$

Since the $\sigma_{A_i}(x)$ are all in C_x, and the $f_{A_i}(x)$ are all non-negative and not all 0, it follows that $\sigma(x) \in C_x$ since C_x is a convex cone. □

Exercise 2 of §16 follows from Proposition 18.6 by taking C_x to be the set of all positive real multiples of $\sigma_A(x)$ whenever $x \in A \in \mathcal{O}$.

Theorem 18.7. *Every smooth vector bundle with a paracompact base space admits a Riemannian metric. In particular, every paracompact smooth manifold admits a Riemannian metric.*

Proof. Let $\xi = \big(V, E, (M, \mathcal{S}_M), p, \mathcal{S}\big)$ be a smooth vector bundle with M paracompact. For each $x \in M$, take $C_x = P\big(p^{-1}(x)\big)$. Let $\mathcal{A}$ be any $\mathcal{S}$-admissible linear V-bundle atlas for ξ. Recall, from Corollary 11.7, that each $\varphi \in \mathcal{A}$ yields an $\mathcal{S}_S$-admissible linear $S(V)$-bundle chart φ^S whose inverse

$$(\varphi^S)^{-1} : U_\varphi \times S(V) \to p_S^{-1}(U_\varphi)$$

is given by $(\varphi^S)^{-1}(x, \alpha) = S(\varphi_x)(\alpha)$ where $\varphi_x : p^{-1}(x) \to V$ is the composition

$$p^{-1}(x) \xrightarrow{\varphi} \{ x \} \times V \xrightarrow{\pi_2} V.$$

Choose any $\alpha \in P(V)$ and define, for each $\varphi \in \mathcal{A}$, $\sigma_\varphi : U_\varphi \to p_S^{-1}(U_\varphi)$ by $\sigma_\varphi(x) = (\varphi^S)^{-1}(x, \alpha)$ for each $x \in U_\varphi$. Each σ_φ is a smooth section of $S(\xi)|U_\varphi$. Since, for each $x \in U_\varphi$, φ_x is an $\mathbf{R}$-isomorphism, it follows that $S(\varphi_x)(\alpha) \in P\big(p^{-1}(x)\big)$, i.e. $\sigma_\varphi(x) \in P\big(p^{-1}(x)\big)$ for each $x \in U_\varphi$. The result now follows by applying Proposition 18.6 to the open cover $\{ U_\varphi \mid \varphi \in \mathcal{A} \}$ and the family of local sections $\{ \sigma_\varphi \mid \varphi \in \mathcal{A} \}$. □

We have thus fulfilled the first aim of this section. The remainder is concerned with the second aim.

Proposition 18.8. *Let $\xi = \big(V, E, (M, \mathcal{S}_M), p, \mathcal{S}\big)$ be a smooth vector bundle and $\gamma \in \Gamma\big(S(\xi)\big)$. Define $f : E \to \mathbf{R}$ by $f(v) = \gamma\big(p(v)\big)(v, v)$. Then f is smooth.*

Proof. It suffices to check that for each $\mathcal{S}$-admissible linear V-bundle chart φ, the restriction of f to $p^{-1}(U_\varphi)$ is smooth.

Choose any $\mathbf{R}$-basis $\{ v_1, \ldots, v_n \}$ of V. For φ as above, $x \in U_\varphi$, and $v \in p^{-1}(U_\varphi)$ we can write

$$\varphi^S\big(\gamma(x)\big) = \Big(x, \sum_{1 \le i \le j \le n} c_{ij}(x) s(v_i, v_j)\Big),$$

$$\varphi(v) = \left(p(v), \sum_{k=1}^{n} a_k(v) v_k\right)$$

where the $c_{ij} : U_\varphi \to \mathbf{R}$ and $a_k : p^{-1}(U_\varphi) \to \mathbf{R}$ are smooth by the smoothness of $\varphi^S \gamma$ and φ, respectively. Then for $x \in U_\varphi$, and $v \in p^{-1}(x)$,

$$S(\varphi_x)^{-1}\big(\gamma(x)\big) = \sum_{1 \le i \le j \le n} c_{ij}(x) s(v_i, v_j)$$
$$\varphi_x(v) = \sum_{k=1}^{n} a_k(v) v_k$$

where $\varphi_x : p^{-1}(x) \to V$ is the fiber isomorphism arising from φ. Then

$$\begin{aligned}
f(v) &= \gamma\big(p(v)\big)(v, v) \\
&= \left(S(\varphi_x) S(\varphi_x)^{-1}\right)\left(\gamma\big(p(v)\big)\right)\left(\varphi_x^{-1}\varphi_x(v), \varphi_x^{-1}\varphi_x(v)\right) \\
&= S(\varphi_x)^{-1}\left(\gamma\big(p(v)\big)\right)\left(\varphi_x \varphi_x^{-1}\varphi_x(v), \varphi_x \varphi_x^{-1}\varphi_x(v)\right) \\
&= S(\varphi_x)^{-1}\left(\gamma\big(p(v)\big)\right)\left(\varphi_x(v), \varphi_x(v)\right) \\
&= \left(\sum_{1 \le i \le j \le n} c_{ij}\big(p(v)\big) s(v_i, v_j)\right)\left(\sum_{k=1}^{n} a_k(v) v_k, \sum_{k=1}^{n} a_k(v) v_k\right) \\
&= \sum_{1 \le i \le j \le n} 2 c_{ij}\big(p(v)\big) a_i(v) a_j(v)
\end{aligned}$$

which is smooth. □

Definition 18.9. Let $(M, \mathcal{S}, g)$ be a Riemannian manifold. For $x \in M$ and and $v \in T_x(M, \mathcal{S})$, the *g-norm of* v, denoted $\|v\|_g$, is the non-negative real number $\sqrt{g(x)(v, v)}$.

As a corollary of Proposition 18.8 we have

Corollary 18.10. *Let* $(M, \mathcal{S}, g)$ *be a Riemannian manifold. Then the function* $T(M, \mathcal{S}) \to \mathbf{R}$ *which sends* v *to* $\|v\|_g^2$ *is smooth.* □

Definition 18.11. Let $(M, \mathcal{S})$ be a smooth manifold and $x, y \in M$. A *1-chain from* x *to* y *in* $(M, \mathcal{S})$ is a smooth cubical 1-chain in $(M, \mathcal{S})$ of the form $\sum_{i=1}^{r} \sigma_i$ where the σ_i are smooth 1-cubes in $(M, \mathcal{S})$ such that

$$\begin{aligned}
&\sigma_1(0) = x, \\
&\sigma_i(1) = \sigma_{i+1}(0) \quad \text{for } 1 \le i \le r-1, \\
&\sigma_r(1) = y.
\end{aligned}$$

Proposition 18.12. *Let $(M, \mathcal{S})$ be a smooth n-manifold with M connected. Then for all $x, y \in M$, there exists a 1-chain from x to y in $(M, \mathcal{S})$.*

Proof. The proof is a smooth variant of the standard proof that any connected and locally path-connected space is path-connected. Fix x in M and let S be the set of all $z \in M$ such that a 1-chain from x to z in $(M, \mathcal{S})$ exists. We must show that $S = M$. This will follow from the connectedness of M if we show that S is non-empty and both open and closed in M.

Let $\sigma : I \to M$ be the constant map with value x. Then σ is a 1-chain in $(M, \mathcal{S})$ from x to x and so $x \in S$. Thus $S \neq \emptyset$.

To show that S is both open and closed in M it suffices to show that for each $y \in \overline{S}$, there exists an open neighborhood N of y in M such that $N \subset S$. Suppose $y \in \overline{S}$. Choose any $\mathcal{S}$-admissible chart φ such that $y \in \text{dom } \varphi$ and $\text{codom } \varphi = E^n$, the open unit ball in $\mathbf{R}^n$. Since every neighborhood of y in M meets S, there exists a point $w \in S \cap \text{dom } \varphi$. Let C be any 1-chain in $(M, \mathcal{S})$ from x to u. Let $z \in \text{dom } \varphi$ be arbitrary. Define $\sigma : I \to B^n$ by

$$\sigma(t) = (1-t)\varphi(u) + t\varphi(z).$$

If $\widehat{\sigma} : \mathbf{R} \to \mathbf{R}^n$ is given by the same formula as the one above for σ, then $\widehat{\sigma}$ is continuous and so $\widehat{\sigma}^{-1}(E^n)$ is an open neighborhood of I in $\mathbf{R}$ and restriction of $\widehat{\sigma}$ yields a smooth extension $\widetilde{\sigma} : \widehat{\sigma}^{-1}(E^n) \to E^n$ of σ. Let $\tau : I \to M$ be the composition

$$I \xrightarrow{\sigma} E^n \xrightarrow{\varphi^{-1}} \text{dom } \varphi \xrightarrow{i} M$$

where i is the inclusion map. Then τ is a 1-chain in $(M, \mathcal{S})$ from u to z, and $C + \tau$ is a 1-chain in $(M, \mathcal{S})$ from x to z. Thus $z \in S$. Thus $\text{dom } \varphi \subset S$, completing the proof. $\square$

Recall, as a special case of Theorem 8.19, that if (a, b) is an open interval, then the identity map on (a, b), regarded as an admissible manifold chart for the standard smooth structure on (a, b), yields a tangent bundle chart

$$\widetilde{1_{(a,b)}} : T\big((a, b)\big) \to (a, b) \times \mathbf{R}.$$

We thus obtain a smooth vector field $u : (a, b) \to T\big((a, b)\big)$ given by $u(t) = \widetilde{1_{(a,b)}}^{-1}(t, 1) = \theta_t(1)$.

If $a \leq a' < b' \leq b$ and $i : (a', b') \to (a, b)$ denotes the inclusion, it follows from Proposition 5.12 and the fact that $Di(t) = 1_{\mathbf{R}}$ for all $t \in (a', b')$, that the diagram

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{=} & \mathbf{R} \\ {\scriptstyle \theta_t}\big\downarrow & & \big\downarrow{\scriptstyle \theta_t} \\ T_t\big((a', b')\big) & \xrightarrow[Ti]{} & T_t\big((a', b')\big) \end{array}$$

commutes. It follows that the diagram

$$\begin{array}{ccc} T\big((a', b')\big) & \xrightarrow{Ti} & T\big((a, b)\big) \\ {\scriptstyle u}\big\uparrow & & \big\uparrow{\scriptstyle u} \\ (a', b') & \xrightarrow[i]{} & (a, b) \end{array}$$

commutes. Thus if $(M, \mathcal{S})$ is a smooth manifold, $\sigma : I \to M$ a smooth 1-cube in $(M, \mathcal{S})$, and $\widetilde{\sigma} : (a, b) \to M$ a smooth extension of σ, the restriction of the composition

$$(a,b) \xrightarrow{u} T\big((a,b)\big) \xrightarrow{T\widetilde{\sigma}} T(M, \mathcal{S})$$

to $(0, 1)$ depends only on σ and not on the choice of smooth extension $\widetilde{\sigma}$. By continuity of the above composition, it follows that the restriction of the above composition to $[0, 1]$ depends only on σ. For $t \in [0, 1]$ we will write $\sigma'(t)$ for the image of t under the above composition. If g is a Riemannian metric on $(M, \mathcal{S})$, it follows from Corollary 18.10 that the function $I \to \mathbf{R}$ sending t to $\|\sigma'(t)\|_g$ is continuous.

Definition 18.13. Let $(M, \mathcal{S}, g)$ be a Riemannian manifold and $\sigma : I \to M$ a smooth 1-cube in $(M, \mathcal{S})$. We define

$$l_g(\sigma) = \int_0^1 \|\sigma'(t)\|_g \, dt.$$

We call $l_g(\sigma)$ the *g-length of* σ.

More generally, if $C = \sum_{i=1}^r \sigma_i$ is a 1-chain in $(M, \mathcal{S})$ between two points of M where the σ_i are smooth 1-cubes in $(M, \mathcal{S})$, we define the *g-length of* C, $l_g(C)$, to be $\sum_{i=1}^r l_g(\sigma_i)$.

Definition 18.14. Let $(M, \mathcal{S}, g)$ be a connected Riemannian manifold. For $x, y \in M$ let $C(x, y)$ denote the set of all 1-chains in $(M, \mathcal{S})$ from x to y. The *g-distance from x to y*, denoted $d_g(x, y)$, is defined to be

$$d_g(x, y) = \inf\{\, d_g(C) \mid C \in C(x, y) \,\}.$$

We are aiming to prove that d_g is a metric on M, and that the resulting metric topology is the given topology on M.

We apply Theorem 12.1 to the smooth contravariant functor S. A smooth vector bundle homomorphism $f : \xi \to \eta$ induces a map of smooth sections $f^* : \Gamma\big(S(\eta)\big) \to \Gamma\big(S(\xi)\big)$. In particular, if $f : (N, \mathcal{T}) \to (M, \mathcal{S})$ is a smooth map, the map of smooth vector bundles $\tau f : \tau_{N,\mathcal{T}} \to \tau_{M,\mathcal{S}}$ yields $f^* : \Gamma\big(S(\tau_{M,\mathcal{S}})\big) \to \Gamma\big(S(\tau_{N,\mathcal{T}})\big)$. Explicitly, if $\sigma : M \to ST(M, \mathcal{S})$ is a smooth section of $S(\tau_{N,\mathcal{T}})$, then for $x \in N$ and $u, v \in T_x(N, \mathcal{T})$,

$$(f^*\sigma)(x)(u, v) = \sigma\big(f(x)\big)\big(T_x f(u), T_x f(v)\big).$$

In general, $f^*\sigma$ will not be a Riemannian metric, even if σ is. However we have the following proposition.

Proposition 18.15. *Let $(M, \mathcal{S}, g)$ be a Riemannian manifold, $(N, \mathcal{T})$ a smooth manifold, and $f : (N, \mathcal{T}) \to (M, \mathcal{S})$ an immersion. Then f^*g is a Riemannian metric on $(N, \mathcal{T})$.*

Proof. Let $x \in M$ and u a non-zero vector in $T_x(N, \mathcal{T})$. Then $(f^*g)(x)(u, u) = g\big(f(x)\big)\big(T_x f(u), T_x f(u)\big)$. Since f is an immersion, $T_x f$ is injective and so $T_x f(u) \neq 0$. Thus $g\big(f(x)\big)\big(T_x f(u), T_x f(u)\big) > 0$ since $g\big(f(x)\big)$ is positive-definite. □

Proposition 18.16. *Let $(M, \mathcal{S}, g)$ be a Riemannian manifold, $(N, \mathcal{T})$ a smooth manifold, and $f : (N, \mathcal{T}) \to (M, \mathcal{S})$ an immersion. Let $x, y \in N$ and $C \in C(x, y)$. Then $f_*C \in C\big(f(x), f(y)\big)$ and $l_{f^*g}(C) = l_g(f_*C)$.*

Proof. It is immediate that $f_*C \in C\big(f(x), f(y)\big)$. To prove the statement about the lengths, it suffices to show that for each smooth 1-cube σ in $(N, \mathcal{T})$ and $t \in I$,

$$\|(f_*\sigma)'(t)\|_g^2 = \|\sigma'(t)\|_{f^*g}^2.$$

Let $\widetilde{\sigma}$ be any smooth extension of σ. Then $f\widetilde{\sigma}$ is a smooth extension of $f_*\sigma$. For $t \in I$,

$$(f_*\sigma)'(t) = T_t(f\widetilde{\sigma})\big(u(t)\big) = (T_{\sigma(t)}f)(T_t\widetilde{\sigma})\big(u(t)\big) = (T_{\sigma(t)}f)\big(\sigma'(t)\big).$$

Thus,

$$\begin{aligned} \|(f_*\sigma)'(t)\|_g^2 &= g\Big(f\big(\sigma(t)\big)\Big)\Big((f_*\sigma)'(t), (f_*\sigma)'(t)\Big) \\ &= g\Big(f\big(\sigma(t)\big)\Big)\Big((T_{\sigma(t)}f)\big(\sigma'(t)\big), (T_{\sigma(t)}f)\big(\sigma'(t)\big)\Big) \\ &= (f^*g)\big(\sigma(t)\big)\big(\sigma'(t), \sigma'(t)\big) = \|\sigma'(t)\|_{f^*g}^2. \qquad \square \end{aligned}$$

Lemma 18.17. *Let $(M, \mathcal{S}, g)$ be a Riemannian manifold and $\sigma : I \to M$ a smooth 1-cube in $(M, \mathcal{S})$. Let $\lambda : \mathbf{R} \to \mathbf{R}$ be given by $\lambda(t) = At + B$ where both B and $A + B$ are in I (and hence $\lambda(I) \subset I$). Then $\sigma\lambda : I \to M$ is a smooth 1-cube in $(M, \mathcal{S})$ and*

$$l_g(\sigma\lambda) = \begin{cases} 0 & \text{if } A = 0, \\ \dfrac{A}{|A|} \displaystyle\int_B^{A+B} \|\sigma'(t)\|_g \, dt & \text{if } A \neq 0. \end{cases}$$

Proof. It is elementary to prove that λ has a fixed point x_0 in I (or alternatively, for an over-kill proof, one can invoke the Brouwer Fixed-Point Theorem). Note that for all $t \in \mathbf{R}$,

$$\begin{aligned} |\lambda(t) - x_0| &= |\lambda(t) - \lambda(x_0)| = |At + B - (Ax_0 + B)| \\ &= |A||t - x_0| \leq |t - x_0| \end{aligned}$$

since $|A| \leq 1$. It follows that every interval containing x_0 is mapped into itself by λ. In particular, if $\widetilde{\sigma} : (a, b) \to M$ is a smooth extension of σ where (a, b) is an open interval containing I, λ maps (a, b) to itself. $\widetilde{\sigma}\lambda : (a, b) \to M$ is a smooth extension of $\sigma\lambda$ and so $\sigma\lambda$ is a smooth 1-cube in $(M, \mathcal{S})$. By Proposition 5.12, for each $t \in (a, b)$ the diagram

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{D\lambda(t)} & \mathbf{R} \\ {\scriptstyle \theta_t}\big\downarrow & & \big\downarrow{\scriptstyle \theta_{\lambda(t)}} \\ T_t\big((a, b)\big) & \xrightarrow[T_t\lambda]{} & T_{\lambda(t)}\big((a, b)\big) \end{array}$$

commutes. Since $D\lambda(t)$ is multiplication by A, it follows that $(T_t\lambda)\big(u(t)\big) = Au\big(\lambda(t)\big)$. Thus

$$\begin{aligned}(\sigma\lambda)'(t) &= T_t(\widetilde{\sigma}\lambda)\big(u(t)\big) = T_{\lambda(t)}(\widetilde{\sigma})(T_t\lambda)\big(u(t)\big) \\ &= T_{\lambda(t)}(\widetilde{\sigma})\Big(Au\big(\lambda(t)\big)\Big) = A\sigma'\big(\lambda(t)\big)\end{aligned}$$

and so

$$l_g(\sigma\lambda) = \int_0^1 \|A\sigma'\big(\lambda(t)\big)\|_g\,dt = |A| \int_0^1 \|\sigma'\big(\lambda(t)\big)\|_g\,dt.$$

If $A = 0$ we are done. If $A \neq 0$, the change of variable $s = \lambda(t)$ yields

$$l_g(\sigma\lambda) = |A| \int_{\lambda(0)}^{\lambda(1)} \|\sigma'(s)\|_g \frac{1}{A}\,ds = \frac{A}{|A|} \int_B^{A+B} \|\sigma'(s)\|_g\,ds. \qquad \square$$

Lemma 18.18. *Let $(M, \mathcal{S}, g)$ be a connected Riemannian manifold. Then for all $x, y, z \in M$:*
(a) $d_g(x,x) = 0$.
(b) $d_g(x,y) = d_g(y,x)$.
(c) $d_g(x,z) \leq d_g(x,y) + d_g(y,z)$.

Proof. Let σ be the constant smooth 1-cube with value x. Then $\sigma \in C(x,x)$. Taking $\lambda : \mathbf{R} \to \mathbf{R}$ to be the 0 map, we have $\sigma = \sigma\lambda$. It follows from Lemma 18.17 that $l_g(\sigma) = l_g(\sigma\lambda) = 0$. Part (a) now follows.

If σ is a smooth 1-cube in $(M, \mathcal{S})$, let $\overline{\sigma} : I \to M$ denote the reverse of σ, i.e. $\overline{\sigma}(t) = \sigma(1-t)$ for all $t \in I$. Then $\overline{\sigma}$ is also a smooth 1-cube in $(M, \mathcal{S})$. If $C = \sum_{i=1}^r \sigma_i \in C(x,y)$, define $\overline{C} = \sum_{i=1}^r \overline{\sigma}_{r-i+1}$. Then $\overline{C} \in C(y,x)$. Since $\overline{\overline{C}} = C$, the function sending C to $\overline{C}$ is a bijection from $C(x,y)$ to $C(y,x)$. Thus part (b) will follow if we show that $l_g(\overline{C}) = l_g(C)$ for all $C \in C(x,y)$. It is sufficient to check that $l_g(\overline{\sigma}) = l_g(\sigma)$ for all smooth 1-cubes σ in $(M, \mathcal{S})$.

Let $\tau : \mathbf{R} \to \mathbf{R}$ be given by $\tau(t) = 1 - t$. Then $\overline{\sigma} = \sigma\tau$. Thus, by Lemma 18.17,

$$l_g(\overline{\sigma}) = l_g(\sigma\tau) = -\int_1^0 \|\sigma'(t)\|_g\,dt = \int_0^1 \|\sigma'(t)\|_g\,dt = l_g(\sigma),$$

completing the proof of part (b).

If $C_1 \in C(x,y)$ and $C_2 \in C(y,z)$, note that $C_1 + C_2 \in C(x,z)$. Thus, for all such C_1 and C_2,

$$d_g(x,z) \leq l_g(C_1 + C_2) = l_g(C_1) + l_g(C_2).$$

Thus

$$\begin{aligned}d_g(x,z) &\leq \inf\{\, l_g(C_1) + l_g(C_2) \mid C_1 \in C(x,y), C_2 \in C(y,z)\,\} \\ &= d_g(x,y) + d_g(y,z),\end{aligned}$$

proving part (c). $\square$

Thus, to conclude that d_g is a metric, it remains only to show that for $x \neq y$ in M, $d_g(x,y) > 0$.

Lemma 18.19. *Let $(M, \mathcal{S}, g)$ be a Riemannian manifold and $\sigma : I \to M$ a smooth 1-cube. Then for each $a \in I$ there exist smooth 1-cubes σ_1 and σ_2 in $(M, \mathcal{S})$ such that:*

(i) $\sigma_1 \in C(\sigma(0), \sigma(a))$ *and* $\sigma_2 \in C(\sigma(a), \sigma(1))$.
(ii) $l_g(\sigma_1) + l_g(\sigma_2) = l_g(\sigma)$.
(iii) $\sigma_1(I) = \sigma([0, a])$ *and* $\sigma_2(I) = \sigma([a, 1])$.

Proof. If $a = 0$, take σ_1 to be the constant 1-cube with value $\sigma(0)$ and $\sigma_2 = \sigma$. If $a = 1$, take $\sigma_1 = \sigma$ and σ_2 to be the constant 1-cube with value $\sigma(1)$.

Suppose $0 < a < 1$. Let $\tau_1, \tau_2 : \mathbf{R} \to \mathbf{R}$ be given by $\tau_1(t) = at$, $\tau_2(t) = a+(1-a)t$. Take $\sigma_1 = \sigma\tau_1$, $\sigma_2 = \sigma\tau_2$. It is immediate that condition (i) and (iii) are satisfied. By Lemma 18.17,

$$l_g(\sigma_1) = \int_0^a \|\sigma'(t)\|_g \, dt, \quad l_g(\sigma_2) = \int_a^1 \|\sigma'(t)\|_g \, dt$$

and so

$$l_g(\sigma_1) + l_g(\sigma_2) = \int_0^1 \|\sigma'(t)\|_g \, dt = l_g(\sigma). \qquad \square$$

If $C = \sum_{i=1}^r \sigma_i$ is a 1-chain in $(M, \mathcal{S})$ where the σ_i are smooth 1-cubes in $(M, \mathcal{S})$, and $z \in M$, we will say C *passes through* z if for some i between 1 and r and $t \in I$, $\sigma_i(t) = z$.

Corollary 18.20. *Let $(M, \mathcal{S}, g)$ be a Riemannian manifold, $x, y \in M$, and $C \in C(x, y)$. Suppose C passes through z. Then there exist $C_1 \in C(x, z)$, $C_2 \in C(z, y)$ such that $l_g(C_1) + l_g(C_2) = l_g(C)$.*

Proof. Say $C = \sum_{i=1}^r \sigma_i$ where the σ_i are smooth 1-cubes in $(M, \mathcal{S})$, and $z = \sigma_q(a)$. By Lemma 18.19, there exist smooth 1-cubes $\tau_1 \in C(\sigma_q(0), z)$ and $\tau_2 \in C(z, \sigma_q(1))$ such that $l_g(\sigma_q) = l_g(\tau_1) + l_g(\tau_2)$. Take $C_1 = \sum_{i=1}^{q-1} \sigma_i + \tau_1$, $C_2 = \tau_2 + \sum_{i=q+1}^r \sigma_i$. $\square$

We next examine Riemannian metrics and their resulting length functions on open subsets of $\mathbf{R}^n$. We identify $S(\mathbf{R}^n)$ with vector space of real symmetric $n \times n$ matrices as follows: If A is such a matrix, then for $x, y \in \mathbf{R}^n$ (regarded as column matrices), the symmetric $\mathbf{R}$-bilinear map $A : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is given by $A(x, y) = x^* A y$ (matrix multiplication) where x^* denotes the transpose of x and we identify the 1×1 real matrices with $\mathbf{R}$. It is well-known from linear algebra that the real symmetric bilinear map A is positive-definite if and only if all the eigenvalues of the matrix A are strictly positive. Let U be open in $\mathbf{R}^n$ and $\widetilde{1_U} : T(U) \to U \times \mathbf{R}^n$ the linear $\mathbf{R}^n$-bundle chart of Theorem 8.19 arising from the manifold chart 1_U. Explicitly, $\widetilde{1_U}^{-1}(x, v) = \theta_x(v)$ where θ_x is as in Example 5.6. We thus obtain a linear $S(\mathbf{R}^n)$-bundle chart $\widetilde{1_U}^S : ST(U) \to U \times S(\mathbf{R}^n)$ given by $(\widetilde{1_U}^S)^{-1}(x, \alpha) = S(\theta_x^{-1})(\alpha)$. Clearly, the smooth sections of $S(\tau_U)$ are in one-to-one correspondence with the smooth maps $f : U \to S(\mathbf{R}^n)$; each such f yields the smooth section $g_f : U \to ST(U)$ given by $g_f(x) = (\widetilde{1_U}^S)^{-1}(x, f(x)) = S(\theta_x^{-1})(f(x))$. g_f will be a Riemannian metric on U if and only if for all $x \in U$, $f(x) \in P(\mathbf{R}^n)$, the set of positive-definite symmetric $\mathbf{R}$-bilinear maps on $\mathbf{R}^n$. We will say that such an f is *positive definite*.

Lemma 18.21. *Let U be open in $\mathbf{R}^n$ and $f : U \to S(\mathbf{R}^n)$ a smooth map. Then for all $x \in U$ and $v, w \in T_x(U)$, $g_f(x)(v,w) = \theta_x^{-1}(v)^* f(x)\theta_x^{-1}(w)$.*

Proof. We have

$$\begin{aligned} g_f(x)(v,w) &= S(\theta_x^{-1})\big(f(x)\big)(v,w) = f(x)\big(\theta_x^{-1}(v), \theta_x^{-1}(w)\big) \\ &= \theta_x^{-1}(v)^* f(x)\theta_x^{-1}(w). \qquad \square \end{aligned}$$

Lemma 18.22. *Let U be open in $\mathbf{R}^n$ and $f : U \to S(\mathbf{R}^n)$ a positive-definite smooth map. Let $\sigma : I \to U$ be a smooth 1-cube, and $\widetilde{\sigma}$ a smooth extension of σ. Then for all $t \in I$,*

$$\|\sigma'(t)\|_{g_f}^2 = \sum_{i,j} f_{ij}\big(\widetilde{\sigma}(t)\big)\widetilde{\sigma}_i'(t)\widetilde{\sigma}_j'(t)$$

where $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n$ are the component functions of $\widetilde{\sigma}$ and f_{ij} is the ij^{th} matrix component function of f for $i, j \in \{1, \ldots, n\}$.

Proof. By Lemma 18.21,

$$\|\sigma'(t)\|_{g_f}^2 = \theta_{\sigma(t)}^{-1}\big(\sigma'(t)\big)^* f(x)\theta_{\sigma(t)}^{-1}\big(\sigma'(t)\big).$$

We have

$$\begin{aligned} \theta_{\sigma(t)}^{-1}\big(\sigma'(t)\big) &= \theta_{\sigma(t)}^{-1}(T_t\widetilde{\sigma})\big(u(t)\big) = \theta_{\sigma(t)}^{-1}(T_t\widetilde{\sigma})\theta_t(1) \\ &= D\widetilde{\sigma}(t)(1) \\ & \qquad\qquad\qquad\qquad \text{(by Proposition 5.12)} \\ &= \begin{pmatrix} \widetilde{\sigma}_1'(t) \\ \vdots \\ \widetilde{\sigma}_n'(t) \end{pmatrix}. \end{aligned}$$

The result now follows. $\square$

Definition 18.23. Let U be open in $\mathbf{R}^n$. The *standard Riemannian metric on* U is g_e where $e : U \to S(\mathbf{R}^n)$ is the constant map with value I_n. We write $\| \ \|_e$ instead of $\| \ \|_{g_e}$.

Thus, by Lemma 18.21, for $x \in U$ and $v, w \in T_x(U)$, $g_e(v,w)$ is the standard Euclidean inner product of $\theta_x^{-1}(v)$ and $\theta_x^{-1}(w)$, and $\|v\|_e$ is the standard Euclidean norm of $\theta_x^{-1}(v)$.

Lemma 18.24. *Let $\lambda_{\max}, \lambda_{\min} : P(\mathbf{R}^n) \to \mathbf{R}$ be given by*

$$\begin{aligned} \lambda_{\max}(A) &= \sup\{\, A(x,x) \mid x \in S^{n-1} \,\}, \\ \lambda_{\min}(A) &= \inf\{\, A(x,x) \mid x \in S^{n-1} \,\}. \end{aligned}$$

Then $\lambda_{\max}$ and $\lambda_{\min}$ are continuous and strictly positive on $P(\mathbf{R}^n)$.

Proof. Since S^{n-1} is compact and $A(x,x)$ is a continuous function of x for fixed $A \in P(\mathbf{R}^n)$, the above sup and inf exist and are, in fact, achieved at points of S^{n-1}. We first claim that $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ are, respectively, the largest and the smallest of the eigenvalues of A. Let $\langle\ ,\ \rangle$ and $\|\ \|$ denote the standard Euclidean inner product and norm, respectively. By the Spectral Theorem, $\mathbf{R}^n$ has an orthonormal basis consisting of eigenvectors for the matrix A and, since A is positive-definite, the eigenvalues of A are strictly positive. Let λ_M and λ_m denote, respectively, the maximum and minimum of these eigenvalues. Let $x \in S^{n-1}$. We can write $x = \sum_{i=1}^r x_i$ where the x_i are mutually orthogonal eigenvectors for A. Then $Ax = \sum_{i=1}^r \lambda_i x_i$ where the λ_i are the corresponding eigenvalues of A. Then $A(x,x) = \langle Ax, x\rangle = \sum_{i=1}^r \lambda_i \|x_i\|^2$. Thus, since $\sum_{i=1}^r \|x_i\|^2 = \|x\|^2 = 1$,

$$\lambda_m = \sum_{i=1}^r \lambda_m \|x_i\|^2 \le \langle Ax, x\rangle \le \sum_{i=1}^r \lambda_M \|x_i\|^2 = \lambda_M.$$

Hence, $\lambda_m \le \lambda_{\min}(A)$ and $\lambda_{\max}(A) \le \lambda_M$. On the other hand, if λ is any eigenvalue of A and $x \in S^{n-1}$ a corresponding eigenvector of A, $A(x,x) = \langle Ax, x\rangle = \langle \lambda x, x\rangle = \lambda\|x\|^2 = \lambda$ and so every eigenvalue of A occurs as a value of $A(x,x)$ for $x \in S^{n-1}$. In particular, λ_m and λ_M occur as values, and so $\lambda_{\min}(A) \le \lambda_m$, $\lambda_{\max}(A) \ge \lambda_M$, establishing the above claim.

Let $\nu : S(\mathbf{R}^n) \to \mathbf{R}$ be given by $\nu(A) = \sup\{\, \|Ax\| \mid x \in S^{n-1}\,\}$, the operator norm of A. ν is a norm, in the sense of Definition 1.1, on the finite-dimensional real vector space $S(\mathbf{R}^n)$, and hence by Theorem 1.5 and Definition 1.7, the standard topology on $S(\mathbf{R}^n)$ coincides with the metric topology derived from the norm ν. It is then immediate that ν is continuous. By a Spectral Theorem argument similar to the one above (which we leave as an exercise), $\nu(A)$ is the maximum of the absolute values of the eigenvalues of A. Thus, $\lambda_{\max}$ is the restriction of ν to $P(\mathbf{R}^n)$, and hence $\lambda_{\max}$ is continuous.

Each matrix A in $P(\mathbf{R}^n)$ is invertible, $A^{-1} \in P(\mathbf{R}^n)$, and the eigenvalues of A^{-1} are the reciprocals of those of A. Thus

$$\lambda_{\min}(A) = \frac{1}{\lambda_{\max}(A^{-1})}.$$

Continuity of $\lambda_{\min}$ now follows. □

Lemma 18.25. *Let U be open in $\mathbf{R}^n$ and g an arbitrary Riemannian metric on U. Then there exist continuous functions $\alpha, \beta : U \to \mathbf{R}$, depending on g, such that:*

(i) *For all $x \in U$, $\alpha(x) > 0$ and $\beta(x) > 0$.*

(ii) *For all $x \in U$ and $v \in T_x(U)$,*

$$\alpha(x)\|v\|_e \le \|v\|_g \le \beta(x)\|v\|_e.$$

Proof. Say $g = g_f$ where $f : U \to S(\mathbf{R}^n)$ is smooth and positive-definite. For $x \in U$ and all non-zero $v \in T_x(U)$,

$$\|v\|_g^2 = f(x)(\theta_x^{-1}(v), \theta_x^{-1}(v)) = \|\theta_x^{-1}(v)\|^2 f(x)\left(\frac{\theta_x^{-1}(v)}{\|\theta_x^{-1}(v)\|}, \frac{\theta_x^{-1}(v)}{\|\theta_x^{-1}(v)\|}\right).$$

Thus

$$\|\theta_x^{-1}(v)\|^2\lambda_{\min}(f(x)) \le \|v\|_g^2 \le \|\theta_x^{-1}(v)\|^2\lambda_{\max}(f(x))$$

and so

$$\sqrt{\lambda_{\min}(f(x))}\,\|v\|_e \le \|v\|_g \le \sqrt{\lambda_{\max}(f(x))}\,\|v\|_e$$

for all non-zero $v \in T_x(U)$. The above inequalities also hold trivially for $v = 0$. The result now follows from Lemma 18.24 by taking $\alpha(x) = \sqrt{\lambda_{\min}(f(x))}$ and $\beta(x) = \sqrt{\lambda_{\max}(f(x))}$. □

Lemma 18.26. *Let U be open in $\mathbf{R}^n$, $x, y \in U$, and $C \in C(x,y)$. Then $l_e(C) \ge \|y - x\|$ where $\|\ \|$ is the standard Euclidean norm on $\mathbf{R}^n$.*

Proof. We first treat the case $C = \sigma$, a smooth 1-cube in U from x to y. It suffices to show that for each $\varepsilon > 0$, $l_e(\sigma) \ge \|y - x\| - \varepsilon$.

Let such an ε be given. Let $\widetilde{\sigma}$ be a smooth extension of σ. It follows from Lemma 18.22 and Definition 18.13 that

$$l_e(\sigma) = \int_0^1 \sqrt{\textstyle\sum_{i=1}^n \left(\widetilde{\sigma}_i'(t)\right)^2}\,dt. \tag{1}$$

Let $\delta > 0$ be arbitrary. By uniform continuity of the $\widetilde{\sigma}_i'$ we can choose a partition

$$0 = t_0 < t_1 < \cdots < t_q = 1$$

of I such that for $1 \le i \le n$ and $1 \le j \le q$, whenever $a, b \in [t_{j-1}, t_j]$, then $|\widetilde{\sigma}_i'(b) - \widetilde{\sigma}_i'(a)| < \delta$. By the Mean-Value Theorem, each $\widetilde{\sigma}_i'$ assumes the value

$$\frac{\sigma_i(t_j) - \sigma_i(t_{j-1})}{t_j - t_{j-1}}$$

at some points of the interval (t_{j-1}, t_j) for $1 \le j \le q$ (usually different points in (t_{j-1}, t_j) for different i). Thus for $1 \le i \le n$ and $1 \le j \le q$,

$$\left|\frac{\sigma_i(t_j) - \sigma_i(t_{j-1})}{t_j - t_{j-1}} - \widetilde{\sigma}_i'(t)\right| < \delta$$

for all $t \in [t_{j-1}, t_j]$. Thus, by continuity of $\sqrt{x_1^2 + \cdots + x_n^2}$ in $x_1, \ldots, x_n$, we can choose δ sufficiently small so that for $1 \le j \le q$,

$$\sqrt{\textstyle\sum_{i=1}^n \left(\widetilde{\sigma}_i'(t)\right)^2} \ge \sqrt{\sum_{i=1}^n \left(\frac{\sigma_i(t_j) - \sigma_i(t_{j-1})}{t_j - t_{j-1}}\right)^2} - \varepsilon.$$

Thus

$$\int_{t_{j-1}}^{t_j} \sqrt{\textstyle\sum_{i=1}^n \left(\widetilde{\sigma}_i'(t)\right)^2}\,dt \ge \left(\sqrt{\sum_{i=1}^n \left(\frac{\sigma_i(t_j) - \sigma_i(t_{j-1})}{t_j - t_{j-1}}\right)^2} - \varepsilon\right)(t_j - t_{j-1})$$

$$= \sqrt{\sum_{i=1}^{n} \left(\sigma_i(t_j) - \sigma_i(t_{j-1})\right)^2} - \varepsilon(t_j - t_{j-1})$$
$$= \|\sigma(t_j) - \sigma(t_{j-1})\| - \varepsilon(t_j - t_{j-1}).$$

Thus, by (1),

$$\begin{aligned} l_e(\sigma) &= \sum_{j=1}^{q} \int_{t_{j-1}}^{t_j} \sqrt{\sum_{i=1}^{n} \left(\tilde{\sigma}_i'(t)\right)^2}\, dt \\ &\geq \sum_{j=1}^{q} \Big(\|\sigma(t_j) - \sigma(t_{j-1})\| - \varepsilon(t_j - t_{j-1}) \Big) \\ &= \sum_{j=1}^{q} \|\sigma(t_j) - \sigma(t_{j-1})\| - \varepsilon \\ &\geq \|\sigma(t_q) - \sigma(t_0)\| - \varepsilon = \|y - x\| - \varepsilon, \end{aligned}$$

this last inequality by the triangle inequality for the norm $\| \ \|$. This establishes the lemma in the special case when C consists of a single smooth 1-cube.

For the general case, let $C = \sum_{i=1}^{r} \sigma_i \in C(x, y)$ where the σ_i are smooth 1-cubes. Write $P_0 = x = \sigma_1(0)$, $P_i = \sigma_{i-1}(1) = \sigma_i(0)$ for $1 \leq i \leq r-1$, and $P_r = y = \sigma_r(1)$. Then, by the special case proven above,

$$l_e(C) = \sum_{i=1}^{r} l_e(\sigma_i) \geq \sum_{i=1}^{r} \|P_i - P_{i-1}\| \geq \|P_r - P_0\| = \|y - x\|,$$

again by the triangle inequality for $\| \ \|$. □

Lemma 18.27. *Let $(M, \mathcal{S}, g)$ be a connected Riemannian manifold, and $x \in M$. Then:*

(a) *Given any $a > 0$, there exists an open neighborhood N of x in M such that for all $y \in N$, $d_g(x, y) < a$.*

(b) *Given any open neighborhood U of x in M, there exists a positive constant b such that for all $y \in M - U$, $d_g(x, y) \geq b$.*

Proof. Choose an $\mathcal{S}$-admissible chart φ such that $x \in \operatorname{dom} \varphi$, $\operatorname{codom} \varphi = E^n$, the open unit ball in $\mathbf{R}^n$, and $\varphi(x) = 0$. For $0 < r < 1$ write

$$\begin{aligned} rE^n &= \{ w \in \mathbf{R}^n \mid \|w\| < r \}, \\ rD^n &= \{ w \in \mathbf{R}^n \mid \|w\| \leq r \}, \text{ and} \\ rS^{n-1} &= \{ w \in \mathbf{R}^n \mid \|w\| = r \}. \end{aligned}$$

Let f be the composition

$$E^n \xrightarrow{\varphi^{-1}} \operatorname{dom} \varphi \xrightarrow{i} M.$$

f is an immersion and so, by Proposition 18.15, f^*g is a Riemannian metric on E^n. Moreover, by Proposition 18.16, for each 1-chain C in B^n, $l_{f^*g}(C) = l_g(f_*C)$.

It follows from Lemma 18.25 and the compactness of $\frac{1}{2}D^n$ that there exists a positive constant $\beta_{\max}$ such that for all $w \in \frac{1}{2}D^n$ and all $v \in T_w(E^n)$, $\|v\|_{f^*g} \leq \beta_{\max}\|v\|_e$. Thus, for any 1-chain C comprised of smooth 1-cubes having images contained in $\frac{1}{2}D^n$, $l_{f^*g}(C) \leq \beta_{\max}\, l_e(C)$. Let $r = \min\left\{\frac{1}{2}, \frac{a}{2\beta_{\max}}\right\}$ and $N = \varphi^{-1}(rE^n)$. Suppose $y \in N$. Let $\sigma : I \to E^n$ be given by $\sigma(t) = t\varphi(y)$. Then σ is a smooth 1-cube from 0 to $\varphi(y)$ with image contained in $\frac{1}{2}D^n$, and so $l_{f^*g}(\sigma) \leq \beta_{\max}\, l_e(\sigma)$. Since $\|\widetilde{\sigma}'(t)\|_e = \|\varphi(y)\|$ for all $t \in I$ and any smooth extension $\widetilde{\sigma}$ of σ, it follows that

$$l_e(\sigma) = \int_0^1 \|\varphi(y)\|\, dt = \|\varphi(y)\|.$$

Thus

$$l_{f^*g}(\sigma) \leq \beta_{\max}\, \|\varphi(y)\| \leq \beta_{\max}\, r \leq \frac{a}{2} < a$$

and thus $l_g(f_*\sigma) = l_{f^*g}(\sigma) < a$. Since $f_*\sigma \in C(x,y)$, it follows that $d_g(x,y) < a$. Thus part (a) is proved.

For part (b), we can suppose that the φ above is chosen so that dom $\varphi \subset U$. Write $V = \varphi^{-1}(\frac{1}{2}E^n)$, $X = \varphi^{-1}(\frac{1}{2}S^n)$. Note that V and $M - \overline{V}$ constitute a separation of $M - X$. Thus, if $y \in M - V$ (in particular, if $y \in M - U$), any 1-chain C from x to y must pass through a point of X. If $C = \sum_{i=1}^{r} \sigma_i$ where the σ_i are smooth 1-cubes in M, let j be the smallest index for which $\sigma_j(I)$ meets X. From the compactness of X and I, and continuity of σ_j, it follows that there is a smallest $t_0 \in I$ for which $\sigma_j(t_0) \in X$. Let $z = \sigma_j(t_0)$. It follows from Lemma 18.19 that there exist 1-chains C_1 and C_2 from x to z and z to y, respectively, in $(M, \mathcal{S})$ such that the images of the smooth 1-cubes comprising C_1 are all contained in $\varphi^{-1}(\frac{1}{2}D^n)$, and $l_g(C) = l_g(C_1) + l_g(C_2)$. We can write $C_1 = f_*C'$ for some 1-chain C' in E^n from 0 to $\varphi(z)$ such that the images of the smooth 1-cubes comprising C' are all contained in $\frac{1}{2}D^n$. By compactness of $\frac{1}{2}D^n$ and Lemma 18.25, there exists a positive constant $\alpha_{\min}$ such that for all $w \in \frac{1}{2}D^n$ and $v \in T_w(E^n)$, $\|v\|_{f^*g} \geq \alpha_{\min}\, \|v\|_e$. It follows that $l_{f^*g}(C') \geq \alpha_{\min}\, l_e(C')$. By Proposition 18.26, $l_e(C') \geq \|\varphi(z) - 0\| = \frac{1}{2}$ since $\varphi(z) \in \frac{1}{2}S^{n-1}$. Thus

$$l_g(C) \geq l_g(C_1) = l_g(f_*C') = l_{f^*g}(C') \geq \frac{\alpha_{\min}}{2}$$

and so $d_g(x,y) \geq \frac{\alpha_{\min}}{2}$. Part (b) now follows with $b = \frac{\alpha_{\min}}{2}$. $\square$

Theorem 18.28. *Let $(M, \mathcal{S}, g)$ be a connected Riemannian manifold. Then d_g is a metric on M, and the metric topology arising from d_g coincides with the given topology on M.*

Proof. By Lemma 18.18, to show that d_g is a metric on M, it remains only to show that whenever x and y are distinct points of M, then $d_g(x,y) > 0$. Suppose $x \neq y$ in M. We can choose an open neighborhood U of x in M such that $y \notin U$. By Lemma 18.27(b), there exist a positive constant b such that for all $z \in M - U$, $d_g(x,z) \geq b$. In particular, $d_g(x,y) \geq b > 0$, completing the proof that d_g is a metric on M.

In the remainder of the proof, by "open in M" we will mean open in the given topology on M. For $x \in M$ and $r > 0$, write

$$B_g(x,r) = \{ y \in M \mid d_g(x,y) < r \}.$$

We must show:

(i) If U is open in M and $x \in U$, then there exists an $r > 0$ such that $B_g(x,r) \subset U$.

(ii) For each $x \in M$ and $r > 0$, there exists an open neighborhood N of x in M such that $N \subset B_g(x,r)$.

Let U be open in M and $x \in U$. By Lemma 18.27(b), there exists a positive constant b such that $d_g(x,y) \geq b$ for all $y \in M - U$. It follows that $B_g(x,b) \subset U$. Condition (i) now follows.

Let $x \in M$ and $r > 0$ be given. Applying Lemma 18.27(a) with $a = r$, there exists an open neighborhood N of x in M such that for all $y \in N$, $d_g(x,y) < r$, i.e. $N \subset B_g(x,r)$. This completes the proof. □

Theorem 18.29. *Let $(M, \mathcal{S})$ be a smooth manifold. Then the following conditions are equivalent:*

(i) *M is paracompact.*

(ii) *$(M, \mathcal{S})$ admits a Riemannian metric.*

(iii) *M is metrizable.*

Proof. (i) implies (ii) by Theorem 18.7.

Suppose $(M, \mathcal{S})$ admits a Riemannian metric g. Let $\{ M_\alpha \mid \alpha \in J \}$ be the components of M. Each M_α is open in M (an easily proved property of topological manifolds) and so each M_α is a smooth submanifold of M. Thus if $i_\alpha : M_\alpha \to M$ denotes the inclusion, $i_\alpha^* g$ is a Riemannian metric on M_α. Thus, by Theorem 18.28, each M_α is metrizable. Thus, since the M_α are open in M, it follows easily that M is metrizable. Thus (ii) implies (iii).

The implication (iii) implies (i) follows from the general topological theorem that every metric space is paracompact. □

19. The de Rham Theorem

In this final section we describe the connection between de Rham cohomology, which is defined on the smooth category, and functors of algebraic topology which are defined on the topological category. In particular we state, without proof, the de Rham theorem which yields, as a consequence, the fact that the de Rham cohomology of a paracompact smooth manifold depends only on the homeomorphism type (in fact, only on the homotopy type) of the manifold.

Definition 19.1. Let $(M, \mathcal{S})$ be a smooth manifold and $\sigma : I^k \to M$ a smooth k-cube in $(M, \mathcal{S})$, $k \geq 1$. σ is said to be *end-degenerate* if σ has a smooth extension $\widetilde{\sigma}$ such that $\widetilde{\sigma}(t_1, \ldots, t_k)$ is independent of t_k. We say σ is *end-essential* if it is not end-degenerate.

We define all smooth 0-cubes to be end-essential.

Proposition 19.2. *Let σ be an end-degenerate smooth k-cube in $(M, \mathcal{S})$. Then for any k-form ω on $(M, \mathcal{S})$, $\int_\sigma \omega = 0$.*

Proof. Let $\widetilde{\sigma} : U \to M$ be a smooth extension of σ where $U = (a, b)^k$ for some open interval (a, b) containing I. Let $V = (a, b)^{k-1} \subset \mathbf{R}^{k-1}$ and let $\pi : U \to V$, $i : V \to U$, and $\widetilde{\tau} : V \to M$ be given by $\pi(t_1, \ldots, t_k) = (t_1, \ldots, t_{k-1})$, $i(t_1, \ldots, t_{k-1}) = (t_1, \ldots, t_{k-1}, 0)$, and $\widetilde{\tau} = \widetilde{\sigma} i$. All these maps are smooth and the diagram

$$\begin{array}{ccc} U & \xrightarrow{\widetilde{\sigma}} & M \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ V & \xrightarrow[\widetilde{\tau}]{} & M \end{array}$$

commutes. Thus $\widetilde{\sigma}^*(\omega) = \pi^* \widetilde{\tau}^*(\omega)$. Since V is a smooth $(k-1)$-manifold, it follows that the k-form $\widetilde{\tau}^*(\omega)$ on V is 0, and hence $\widetilde{\sigma}^*(\omega) = 0$. The assertion now follows. $\square$

For $k \geq 1$, let $D_k(M, \mathcal{S})$ denote the $\mathbf{R}$-subspace of $Q_k(M, \mathcal{S})$ spanned by all the end-degenerate smooth k-cubes in $(M, \mathcal{S})$. We also define $D_0(M, \mathcal{S}) = 0$.

Proposition 19.3. *Let $(M, \mathcal{S})$ be a smooth manifold. Then for all $k \geq 0$,*

$$\partial(D_k(M, \mathcal{S})) \subset D_{k-1}(M, \mathcal{S}).$$

Proof. Let $\sigma : I^k \to M$ be an end-degenerate smooth k-cube in $(M, \mathcal{S})$. It is easily checked that for $\varepsilon = 0, 1$ and $1 \leq i \leq k-1$, σ^i_ε is end-degenerate and that $\sigma^k_0 = \sigma^k_1$. Thus

$$\partial(\sigma) = \sum_{i=1}^{k-1} (-1)^i (\sigma^i_0 - \sigma^i_1) \in D_{k-1}(M, \mathcal{S}). \qquad \square$$

As a consequence of Proposition 19.3, ∂ induces $\mathbf{R}$-homomorphisms

$$\partial : Q_k(M, \mathcal{S})/D_k(M, \mathcal{S}) \to Q_{k-1}(M, \mathcal{S})/D_{k-1}(M, \mathcal{S})$$

such that the diagram

$$\begin{array}{ccc} Q_k(M,\mathcal{S})/D_k(M,\mathcal{S}) & \xrightarrow{\partial} & Q_{k-1}(M,\mathcal{S})/D_{k-1}(M,\mathcal{S}) \\ \downarrow & & \downarrow \\ Q_k(M,\mathcal{S})/D_k(M,\mathcal{S}) & \xrightarrow[\partial]{} & Q_{k-1}(M,\mathcal{S})/D_{k-1}(M,\mathcal{S}) \end{array}$$

commutes where the vertical maps are the natural projections. It follows immediately from Proposition 14.10 that for $k \geq 2$ the composition

$$Q_k(M,\mathcal{S})/D_k(M,\mathcal{S}) \xrightarrow{\partial} Q_{k-1}(M,\mathcal{S})/D_{k-1}(M,\mathcal{S}) \xrightarrow{\partial} Q_{k-2}(M,\mathcal{S})/D_{k-2}(M,\mathcal{S})$$

is the 0-homomorphism.

Definition 19.4. Let $(M,\mathcal{S})$ be a smooth manifold. For $k \geq 0$ the k^{th} *normalized smooth cubical real chain space of* $(M,\mathcal{S})$ is $Q_k(M,\mathcal{S})/D_k(M,\mathcal{S})$ and denoted $C_k(M,\mathcal{S})$. The chain complex $\big(C(M,\mathcal{S}),\partial\big)$ is called the *normalized smooth cubical real complex of* $(M,\mathcal{S})$. The k^{th} homology group of this chain complex is called the k^{th} *real smooth homology group of* $(M,\mathcal{S})$ and denoted $H_k^{\mathcal{S}m}(M,\mathcal{S};\mathbf{R})$.

Note that a purely topological analogue of the chain complex $\big(C(M,\mathcal{S}),\partial\big)$ can be given. Let X be *any* topological space. For $k \geq 0$, a *singular k-cube in X* is a continuous map $\sigma : I^k \to X$. We denote by $Q_k(X;\mathbf{R})$ the real vector space on the set of all singular k-cubes in X. Faces of singular k-cubes and boundary maps are defined just as in the smooth case, and we obtain a chain complex $\big(Q(X;\mathbf{R}),\partial\big)$, the *unnormalized real singular cubical complex of* X. Note that if P is a one-point space, there is a unique singular k-cube in P for each $k \geq 0$, and the boundary map ∂ is the 0-homomorphism in each dimension. Thus the homology of the above chain complex, the *unnormalized real singular cubical homology of a point*, is isomorphic to $\mathbf{R}$ *in each non-negative dimension.* This is unsatisfactory from a geometric point of view; it is desirable to have a non-trivial homology theory which reflects geometric properties of spaces. In particular, if X is an n-manifold or a space built out of cells of dimension $\leq n$, we want a non-trivial homology theory which is trivial on such spaces in dimensions $> n$. This can be achieved by factoring out end-degenerate singular k-cubes, i.e. singular k-cubes which are independent of the last coordinate, for $k \geq 1$. Just as in the smooth case, we define $D_k(X;\mathbf{R})$ to be the real subspace of $Q_k(X;\mathbf{R})$ spanned by the end-degenerate singular k-cubes in X for $k > 0$, $D_0(X;\mathbf{R}) = 0$, and define $C_k(X;\mathbf{R}) = Q_k(X;\mathbf{R})/D_k(X;\mathbf{R})$. Just as in the smooth case, ∂ induces an $\mathbf{R}$-homomorphism $\partial : C_k(X;\mathbf{R}) \to C_{k-1}(X;\mathbf{R})$ for all $k \geq 1$ and we obtain a chain complex $\big(C(X;\mathbf{R}),\partial\big)$.

Definition 19.5. If X is a topological space, the chain complex $\big(C(X;\mathbf{R}),\partial\big)$ described above is called the *real normalized singular cubical complex of* X. The k^{th} homology group of this chain complex is called the k^{th} *singular homology group of X with real coefficients*, and denoted $H_k(X;\mathbf{R})$.

Note that for a one-point space P, $C_k(P;\mathbf{R}) = 0$ for $k > 0$ and $C_0(P;\mathbf{R}) \cong \mathbf{R}$. It follows that $H_k(P;\mathbf{R}) = 0$ for $k \neq 0$, $H_0(P;\mathbf{R}) \cong \mathbf{R}$.

The $H_k(X;\mathbf{R})$, as given in Definition 19.5, are the real singular homology groups of interest in algebraic topology. A number of variants are possible, and are important. The first variant is to replace $\mathbf{R}$ by a different commutative ring R, and the real vector space $Q_k(X;\mathbf{R})$ by the free R-module on the set of singular k-cubes in X. We obtain, in this way, $H_k(X;R)$, the k^{th} singular homology group with coefficients in R. Particular coefficient rings in addition to $\mathbf{R}$ that have proved important for algebraic topology are the rings of rational numbers, integers, integers modulo a prime, and integers localized at a set of primes.

Another variant is to factor out by cubes which are independent of any one of the variables instead of just the end-degenerate ones. The resulting homology theory is isomorphic to the singular homology described above. Another variant is to use singular simplices instead of singular cubes. This approach has the advantage that it yields the "correct" homology groups without any normalization. The cubical approach, however, has the advantage of being more convenient for dealing with homotopy properties of homology and for the study of the homology of product spaces and, more generally, fiber bundles.

If $(M,\mathcal{S})$ is a smooth manifold, we have inclusions $Q_k(M,\mathcal{S}) \subset Q_k(M;\mathbf{R})$ and $D_k(M,\mathcal{S}) \subset D_k(M;\mathbf{R})$ which are compatible with the boundary map, and hence we obtain a chain map

$$i_M : \big(C(M,\mathcal{S}),\partial\big) \to \big(C(M;\mathbf{R}),\partial\big)$$

which induces homomorphisms in homology

$$H_k i_M : H_k^{\mathcal{S}m}(M,\mathcal{S};\mathbf{R}) \to H_k(M;\mathbf{R}).$$

We omit the proof of the following theorem. Its proof is not particularly hard, but does require some machinery.

Theorem 19.6. *Let $(M,\mathcal{S})$ be a smooth manifold. Then for all $k \geq 0$,*

$$H_k i_M : H_k^{\mathcal{S}m}(M,\mathcal{S};\mathbf{R}) \to H_k(M;\mathbf{R})$$

is an isomorphism. □

Let $\pi : Q_k(M,\mathcal{S}) \to Q_k(M,\mathcal{S})/D_k(M,\mathcal{S}) = C_k(M,\mathcal{S})$ denote the natural projection. If $c \in Q_k(M,\mathcal{S})$, we will say *c is a k-cycle modulo end-degeneracy for $(M,\mathcal{S})$* if $\pi(c)$ is a k-cycle for the chain complex $\big(C(M,\mathcal{S}),\partial\big)$, and write $[c] \in H_k^{\mathcal{S}m}(M,\mathcal{S};\mathbf{R})$ for the homology class of $\pi(c)$. Since π is onto, each member of $H_k^{\mathcal{S}m}(M,\mathcal{S};\mathbf{R})$ is representable in the form $[c]$ for some k-cycle modulo end-degeneracy for $(M,\mathcal{S})$. The proof of the following lemma is an easy algebraic exercise.

Lemma 19.7. *Let $(M,\mathcal{S})$ be a smooth manifold and $k \geq 0$. Then:*

(a) *If $c \in Q_k(M,\mathcal{S})$, then c is a k-cycle modulo end-degeneracy for $(M,\mathcal{S})$ if and only if $\partial c \in D_{k-1}(M,\mathcal{S})$.*

(b) *If c, c' are k-cycles modulo end-degeneracy for $(M,\mathcal{S})$, then $[c] = [c']$ if and only if there exist $x \in Q_{k+1}(M,\mathcal{S})$ and $y \in D_k(M,\mathcal{S})$ such that $c' = c + \partial x + y$.* □

Corollary 19.8. *Let $(M,\mathcal{S})$ be a smooth manifold, ω a closed k-form on $(M,\mathcal{S})$, and c a k-cycle modulo end-degeneracy. Then the real number $\int_c \omega$ depends only on $[\omega] \in H^k_{dR}(M,\mathcal{S})$ and $[c] \in H^{\mathcal{S}m}_k(M,\mathcal{S};\mathbf{R})$.*

Proof. If $[\omega'] = [\omega]$ and $[c'] = [c]$, then we can write $\omega' = \omega + d\rho$ for some $\rho \in \Omega^{k-1}(M,\mathcal{S})$ and $c' = c + \partial x + y$ for some $x \in Q_{k+1}(M,\mathcal{S})$ and $y \in D_k(M,\mathcal{S})$. Thus

$$\begin{aligned}\int_{c'} \omega' &= \int_{c+\partial x+y} (\omega + d\rho) \\ &= \int_c \omega + \int_{\partial x} \omega + \int_y \omega + \int_c d\rho + \int_{\partial x} d\rho + \int_y d\rho.\end{aligned}$$

Since $y \in D_k(M,\mathcal{S})$, it follows from Proposition 19.2 that $\int_y \omega = \int_y d\rho = 0$. By the Generalized Stokes' Theorem (Theorem 14.12),

$$\int_{\partial x} \omega = \int_x d\omega, \qquad \int_{\partial x} d\rho = \int_x dd\rho, \qquad \int_c d\rho = \int_{\partial c} \rho.$$

The first two of these integrals are 0 since $d\omega = dd\rho = 0$. Since $\partial c \in D_{k-1}(M,\mathcal{S})$, it follows from Proposition 19.2 that the third of the above integrals is 0. Thus

$$\int_{c'} \omega' = \int_c \omega. \qquad \square$$

Corollary 19.8 thus yields a well-defined $\mathbf{R}$-bilinear map

$$Int : H^k_{dR}(M,\mathcal{S}) \times H^{\mathcal{S}m}_k(M,\mathcal{S};\mathbf{R}) \to \mathbf{R}$$

given by

$$Int([\omega],[c]) = \int_c \omega$$

whenever ω is a closed k-form on $(M,\mathcal{S})$ and c is a k-cycle modulo end-degeneracy for $(M,\mathcal{S})$.

Theorem 19.9. (The de Rham Theorem) *Let $(M,\mathcal{S})$ be a smooth paracompact manifold and suppose that $H_i(M;\mathbf{R})$ is finite-dimensional over $\mathbf{R}$ for all i. Then for all $k \geq 0$,*

$$Int : H^k_{dR}(M,\mathcal{S}) \times H^{\mathcal{S}m}_k(M,\mathcal{S};\mathbf{R}) \to \mathbf{R}$$

is a dual pairing. $\square$

The hypothesis of Theorem 19.9 is satisfied by all compact smooth manifolds as well as many non-compact ones. Since, by Theorem 19.6, $H^{\mathcal{S}m}_k(M,\mathcal{S};\mathbf{R})$ is isomorphic to $H_k(M;\mathbf{R})$, the de Rham Theorem yields an isomorphism

$$H^k_{dR}(M,\mathcal{S}) \cong \mathrm{Hom}_{\mathbf{R}}(H_k(M;\mathbf{R}),\mathbf{R})$$

for paracompact $(M, \mathcal{S})$ whose real singular homology is finite-dimensional over $\mathbf{R}$. In particular, for such manifolds the de Rham cohomology groups are independent, up to isomorphism, of the smooth structure.

Singular cohomology groups $H^k(X; R)$ for topological spaces are also defined in algebraic topology. For the case of real coefficients, $H^k(X; \mathbf{R})$ is isomorphic to $\mathrm{Hom}_{\mathbf{R}}\big(H_k(X; \mathbf{R}), \mathbf{R}\big)$. Thus the de Rham theorem yields that the de Rham cohomology groups and the singular cohomology groups with real coefficients are isomorphic for paracompact smooth manifolds with finite-dimensional real singular homology. The machinery of algebraic topology is sufficiently well-developed so that calculation of singular homology and cohomology is usually quite feasible.