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SUBDIFFERENTIAL CALCULUS IN

ASPLUND GENERATED SPACES

by

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Abstract. We extend the definition of the limiting Fréchet subdifferential and the limiting Fréchet normal cone from Asplund spaces to Asplund generated spaces. Then we prove a sum rule, a mean value theorem, and other statements for this concept.

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Let X be a separable, or more generally, a weakly compactly generated Banach space. Then X admits an equivalent Gâteaux smooth norm, and hence X is a weakly trustworthy space. This means that, in X, we have at hand a suitable fuzzy calculus for Gâteaux-like subdifferentials. Unfortunately, such a calculus is not strong enough for handling some problems of nonlinear analysis. A suitable concept for attacking such problems is a Fréchet-like subdifferential. However, the occurrence of such a subdifferential is more or less equivalent with the property of the space to be Asplund. This note is an attempt to go on, beyond the framework of Asplund spaces, and still working with a subdifferential sharing some of the features of the Fréchet-like subdifferential. The point is that, if X is a separable, or more generally, a weakly compactly generated Banach space, we have at hand a better object than just the Gâteaux (sub)differential. Actually, such spaces contain a dense set, which is, when endowed with a suitable norm, an Asplund space. This fact then allows us to define, in Asplund generated spaces, a limiting Fréchet subdifferential and a limiting Fréchet normal cone with properties imitating those of the limiting Fréchet subdifferential and a limiting Fréchet normal cone known from Asplund spaces [MS].

We conjecture that the theory of limiting Fréchet subdifferentials and limiting Fréchet normal cones developed by Mordukhovich and Shao in [MS] for Asplund spaces (see also the books [M1, M2] for a comprehensive theory and applications of these constructions) can be extended to the larger class of Asplund generated spaces. The aim of this note is to support this belief by proving several statements in this vein, which extend those from [MS] to the Asplund generated setting.

Consider a Banach space $(X, \|\cdot\|_X)$ such that there exists an Asplund space $(Y, \|\cdot\|_Y)$, satisfying $Y \subset X$ and $\overline{Y}^{\|\cdot\|_X} = X$, and is such that $\|y\|_X \leq \|y\|_Y$ for every $y \in Y$. Let $i : Y \to X$ be the inclusion mapping; note that i is continuous. The situation described above, that is, the pentad $(X, Y, \|\cdot\|_X, \|\cdot\|_Y, i)$, with Y Asplund, will be called an *Asplund generated scheme.* Note that any Asplund generated space yields an Asplund generated scheme and vice versa. A theory of Asplund generated spaces can be found, e.g., in [F, Chapter 1]. In what follows, we always consider a fixed Asplund generated scheme $(X, Y\|\cdot\|_X, \|\cdot\|_Y, i)$. Let B_X and B_Y denote the unit ball in $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ respectively. The symbol $\rightharpoondown$ will mean the convergence in the weak* topology. Given a function $\varphi : X \to \overline{\mathbb{R}} = (-\infty, +\infty]$, we use the symbol $\varphi_{|Y}$ for the restriction of φ to Y. Thus, $i^*(x^*) = x^*{}_{|Y}$ for $x^* \in X^*$. For $x^* \in X^*$ we define $\|x^*\|_{X^*} = \sup\langle x^*, B_X\rangle$, and for $y^* \in Y^*$ we define

$$\|y^*\|_{Y^*} = \sup\langle y^*, B_Y\rangle \quad \text{and} \quad \|y^*\|_{X^*} = \sup\langle y^*, B_X \cap Y\rangle.$$

Note that if $y^* \in Y^*$ and $\|y^*\|_{X^*} < +\infty$, then y^* is extendable to all of X, i.e., there is $x^* \in X^*$ such that $x^*{}_{|Y} = i^*(x^*) = y^*$ and $\|x^*\|_{X^*} = \|y^*\|_{X^*}$. Given a function $\psi : Y \to \overline{\mathbb{R}}$ and $\overline{y} \in \operatorname{dom}\psi$, the symbol $\partial\psi(\overline{y})$ means the usual *limiting Fréchet subdifferential* of ψ at $\overline{y}$, see e.g., [M1, MS], that is,

$$\partial\psi(\overline{y}) = \big\{\eta \in Y^*; \exists y_n \in Y,\ \exists\varepsilon_n \geq 0,\ \exists\eta_n \in \widehat{\partial}_{\varepsilon_n}\psi(y_n),\ n \in \mathbb{N},$$

$$\text{so that } \|y_n - \overline{y}\|_Y \to 0,\ \varepsilon_n \downarrow 0, \text{ and } \eta_n \rightharpoondown \eta\big\}.$$

Note that $\widehat{\partial}_\varepsilon\psi(y)$ means the Fréchet $\varepsilon-$subdifferential of ψ at y, see [FM, MS]. We also use $\partial^\infty\psi(\overline{y})$ as defined in [M1, MS]. Note that once Y is Asplund (as it always is in this note), then we can replace the ε_n's by 0 in the above formula [MS]. Given a set Ω in a Banach space Z, we put $\delta_\Omega(z) = 0$ if $z \in \Omega$ and $\delta_\Omega(z) = +\infty$ if $z \in Z\backslash\Omega$. Such a δ_Ω is called an *indicator* function of Ω. The *limiting Fréchet normal cone* of Ω at $\overline{z} \in \Omega$ is defined as $N(\overline{z}; \Omega) = \partial\delta_\Omega(\overline{z})$.

Now we are ready to introduce our new concepts.

Definition. Let $(X, Y, \|\cdot\|_X, \|\cdot\|_Y, i)$ be an Asplund generated scheme. Let $\varphi : X \to \overline{\mathbb{R}}$ be a function and $\overline{x} \in \operatorname{dom}\varphi \cap Y$. Then we define

$$\partial_Y\varphi(\overline{x}) = i^{*-1}\big(\partial(\varphi_{|Y})(\overline{x})\big), \quad \partial_Y^\infty\varphi(\overline{x}) = i^{*-1}\big(\partial^\infty(\varphi_{|Y})(\overline{x})\big).$$

Further, given a set $\Omega \subset X$ and $\overline{x} \in \Omega \cap Y$, we define

$$N_Y(\overline{x}; \Omega) = \partial_Y\delta_\Omega(\overline{x}).$$

Definition. Given $\overline{z} \in M \subset Z$, we say that M is *sequentially normally compact* (*SNC*) at $\overline{z}$ if $\|\zeta_n\| \to 0$ whenever $z_n \in Z$, $\varepsilon_n \downarrow 0$, $\zeta_n \in \widehat{\partial}_{\varepsilon_n}\delta_M(z_n)$, $n \in \mathbb{N}$, $\|z_n - \overline{z}\| \to 0$, and $\zeta_n \rightharpoondown 0$. Given a function $\psi : Z \to \overline{\mathbb{R}}$ and $\overline{z} \in \operatorname{dom}\psi$, we say that ψ is *SNC* at $\overline{z}$ if $\operatorname{epi}\psi$ is SNC at the point $(\overline{z}, \psi(\overline{z}))$ in the space $Z \times \mathbb{R}$.

Note that if Z is an Asplund space, we can take $\varepsilon_n = 0$ above; see [MS, Theorem 2.9 (iii)].

Definition. Let $(X, Y, \|\cdot\|_X, \|\cdot\|_Y, i)$ be an Asplund generated scheme. Consider a set $\Omega \subset X$ and $\overline{x} \in \Omega \cap Y$. We say that Ω is Y*–SNC* at $\overline{x}$ if $\Omega \cap Y$ is SNC at $\overline{x}$ (in the space Y). Consider a function $\varphi : X \to \overline{\mathbb{R}}$ and $\overline{x} \in \operatorname{dom}\varphi \cap Y$. We say that φ is Y*–SNC* at $\overline{x}$ if the restriction $\varphi_{|Y}$ is SNC at $\overline{x}$ (in the space Y).

Remarks. Note that, in the special case when $Y = X$, we get the concepts known from Asplund spaces:

$$\partial_X\varphi(\overline{x}) = \partial\varphi(\overline{x}), \quad \partial_X^\infty\varphi(\overline{x}) = \partial^\infty\varphi(\overline{x}), \quad N_X(\overline{x}; \Omega) = N(\overline{x}; \Omega).$$

Also $X-$SNC is just SNC. Moreover, it is straightforward to check that

$$N_Y(\overline{x};\Omega) = i^{*-1}\big(N(\overline{x};\Omega \cap Y)\big),$$

where $N(\overline{x}, \Omega \cap Y)$ is the limiting Fréchet normal cone in the space Y, as it is defined in [MS]. Indeed

$$i^{*-1}(N(\overline{x};\Omega\cap Y)) = i^{*-1}\big(\partial\delta_{\Omega\cap Y}(\overline{x})\big) = i^{*-1}\big(\partial\big((\delta_\Omega)_{|Y}\big)(\overline{x})\big) = \partial_Y\delta_\Omega(\overline{x}) = N_Y(\overline{x};,\Omega).$$

Also, it is not difficult to prove that

$$\partial_Y^\infty\varphi(\overline{x}) = \big\{x^* \in X^*;\ (x^*,0) \in N_{Y\times\mathbb{R}}\big((\overline{x},\varphi(\overline{x}));\ \operatorname{epi}\varphi\big)\big\}.$$

Indeed,

$$\begin{aligned}
\partial_Y^\infty\varphi(\overline{x}) &= i^{*-1}\partial^\infty\big((\varphi_{|Y})(\overline{x})\big)\\
&= i^{*-1}\big\{y^* \in Y^*;\ (y^*,0) \in N((\overline{x},\varphi(\overline{x}));\operatorname{epi}(\varphi_{|Y}))\big\}\\
&= i^{*-1}\big\{y^* \in Y^*;\ (y^*,0) \in N((\overline{x},\varphi(\overline{x}));\operatorname{epi}\varphi\cap(Y\times\mathbb{R}))\big\}\\
&= \big\{i^{*-1}y^*; (i^{*-1}y^*,0) \in (i\times\operatorname{id})^{*-1}\big(N((\overline{x},\varphi(\overline{x}));\operatorname{epi}\varphi\cap(Y\times\mathbb{R}))\big\}\\
&= \big\{x^* \in X^*;\ (x^*,0) \in N_{Y\times\mathbb{R}}\big((\overline{x},\varphi(\overline{x}));\ \operatorname{epi}\varphi\big)\big\}.
\end{aligned}$$

Moreover, we can readily see that

$$x^* \in \partial_Y\varphi(\overline{x}) \Leftrightarrow (x^*,-1) \in N_Y\big((\overline{x},\varphi(\overline{x}));\operatorname{epi}\varphi\big).$$

Further, we can easily verify that a function $\varphi : X \to \overline{\mathbb{R}}$ is $Y-$SNC at $\overline{x} \in \operatorname{dom}\varphi \cap Y$ if and only if the set $\operatorname{epi}\varphi$ is $Y\times\mathbb{R}-$SNC at the point $(\overline{x},\varphi(\overline{x}))$.

Now we are ready to formulate a sum rule in Asplund generated spaces for the new concepts $\partial_Y\varphi$ and $\partial_Y^\infty\varphi$.

Theorem 1. *Let $(X, Y, \|\cdot\|_X, \|\cdot\|_Y, i)$ be an Asplund generated scheme. Consider two lower semicontinuous functions $\varphi_1, \varphi_2 : X \to \overline{\mathbb{R}}$ and $\overline{x} \in \operatorname{dom}\varphi_1 \cap \operatorname{dom}\varphi_2 \cap Y$. Assume that*

(i) φ_j is Y–SNC at $\overline{x}$ for some $j \in \{1,2\}$,

(ii) $\partial_Y^\infty \varphi_1(\overline{x}) \cap -\partial_Y^\infty \varphi_2(\overline{x}) = \{0\}$,

(iii) $i^*\big(\partial_Y \varphi_j(\overline{x})\big) = \partial\big(\varphi_{j|Y}\big)(\overline{x})$ for some $j \in \{1,2\}$, and

(iv) $i^*\big(\partial_Y^\infty \varphi_j(\overline{x})\big) = \partial^\infty\big(\varphi_{j|Y}\big)(\overline{x})$ for some $j \in \{1,2\}$.

Then

$$\partial_Y(\varphi_1+\varphi_2)(\overline{x}) \subset \partial_Y\varphi_1(\overline{x}) + \partial_Y\varphi_2(\overline{x}) \quad \text{and} \quad \partial_Y^\infty(\varphi_1+\varphi_2)(\overline{x}) \subset \partial_Y^\infty\varphi_1(\overline{x}) + \partial_Y^\infty\varphi_2(\overline{x}).$$

Proof. Take any $\xi \in \partial_Y(\varphi_1+\varphi_2)(\overline{x})$ (if it exists; otherwise we are done). Then

$$\xi_{|Y} = i^*(\xi) \in \partial\big((\varphi_1+\varphi_2)_{|Y}\big)(\overline{x}) = \partial\big(\varphi_{1|Y} + \varphi_{2|Y}\big)(\overline{x}) \subset \partial(\varphi_{1|Y})(\overline{x}) + \partial(\varphi_{2|Y})(\overline{x}).$$

Here the inclusion follows from [MS, Theorem 4.1]. Indeed, $\varphi_{j|Y}$ are lower semicontinuous functions on Y. Also, $\partial^\infty(\varphi_{1|Y})(\overline{x}) \cap -\partial^\infty(\varphi_{2|Y})(\overline{x}) = \{0\}$. In fact, take any η in this intersection. Assuming that (iv) holds for, say, $j = 2$, we get that $-\eta = i^*(x^*) = x^*{}_{|Y}$ for some $x^* \in \partial_Y^\infty\varphi_2(\overline{x})$. Then $i^*(-x^*) = \eta \in \partial^\infty(\varphi_{1|Y}(\overline{x})$ and so $-x^* = i^{*-1}(\eta) \in \partial_Y^\infty(\varphi_1)(\overline{x})$. Therefore, by (ii), $x^* = 0$ and so $\eta = 0$. Thus, by [MS, Theorem 4.1], $\xi_{|Y} = \eta_1 + \eta_2$, with suitable $\eta_j \in \partial(\varphi_{j|Y}(\overline{x}),\ j = 1, 2$. But (iii) guarantees that, say, $\eta_1 \in i^*(X^*)$. Then $\eta_2 \in i^*(X^*)$ as well, and so

$$\xi = i^{*-1}(\eta_1) + i^{*-1}(\eta_2) \in i^{*-1}\big(\partial(\varphi_{1|Y}(\overline{x})\big) + i^{*-1}\big(\partial(\varphi_{2|Y}(\overline{x})\big) = \partial_Y\varphi_1(\overline{x}) + \partial_Y\varphi_2(\overline{x}).$$

The second inclusion, dealing with ∂_Y^∞, can be proved analogously. ∎

Remark. If X is an Asplund space itself, then we can put $Y = X$ in Theorem 1 and we thus get exactly [MS, Theorem 4.1] for $n = 2$.

Corollary 1. *Let $(X, Y, \|\cdot\|_X, \|\cdot\|_Y, i)$ be an Asplund generated scheme and consider functions $\varphi_1, \varphi_2 : X \to \overline{\mathbb{R}}$ and a point $\overline{x} \in \operatorname{dom}\varphi_1 \cap \operatorname{dom}\varphi_2 \cap Y$ such that φ_1 is Lipschitzian and φ_2 is lower semicontinuous in a $\|\cdot\|_X-$vicinity of $\overline{x}$. Then*

$$\partial_Y(\varphi_1 + \varphi_2)(\overline{x}) \subset \partial_Y\varphi_1(\overline{x}) + \partial_Y\varphi_2(\overline{x}) \quad \text{and} \quad \partial_Y^\infty(\varphi_1 + \varphi_2)(\overline{x}) \subset \partial_Y^\infty\varphi_1(\overline{x}) + \partial_Y^\infty\varphi_2(\overline{x}).$$

Proof. The Proposition below guarantees that (i)–(iv) in Theorem 1 are satisfied. ∎

Theorem 2. *Let $(X, Y, \|\cdot\|_X, \|\cdot\|_Y, i)$ be an Asplund generated scheme and consider two closed sets $\Omega_1, \Omega_2 \subset X$ and a point $\overline{x} \in \Omega_1 \cap \Omega_2 \cap Y$. Assume that*

(i) Ω_j is $Y-$SNC at $\overline{x}$ for some $j \in \{1, 2\}$,

(ii) $N_Y(\overline{x}; \Omega_1) \cap -N_Y(\overline{x}; \Omega_2) = \{0\}$, and

(iii) $i^*(N_Y(\overline{x}; \Omega_j)) = N(\overline{x}; \Omega_j \cap Y)$ for some $j \in \{1, 2\}$.

Then

$$N_Y(\overline{x}; \Omega_1 \cap \Omega_2) \subset N_Y(\overline{x}, \Omega_1) + N_Y(\overline{x}, \Omega_2).$$

Proof. We observe that, if $\overline{z} \in M \subset Z$, then

$$\partial\delta_M(\overline{z}) = N(\overline{z}, M) = \partial^\infty\delta_M(\overline{z}).$$

Indeed, the first equality here is just the definition, while the second equality can be obtained by some elementary calculation. Thus, in the space Y,

$$\partial\delta_{\Omega_j \cap Y}(\overline{x}) = N(\overline{x}; \Omega_j \cap Y) = \partial^\infty\delta_{\Omega_j \cap Y}(\overline{x}),\ j = 1, 2.$$

And, as $\delta_{\Omega_j \cap Y} = \left(\delta_{\Omega_j}\right)_{|Y}$, we get

$$\partial_Y \delta_{\Omega_j}(\overline{x}) = N_Y(\overline{x}; \Omega_j) = \partial_Y^\infty \delta_{\Omega_j}(\overline{x}).$$

Then our condition (ii) yields immediately the condition (ii) in Theorem 1. (Yes, i^* is injective.) Also, by our condition (iii), we have

$$i^*(\partial_Y \delta_{\Omega_j}(\overline{x})) = i^*(N_Y(\overline{x}; \Omega_j)) = N(\overline{x}; \Omega_j \cap Y) = \partial \delta_{\Omega_j \cap Y}(\overline{x}) = \partial\left(\left(\delta_{\Omega_j}\right)_{|Y}\right)(\overline{x})$$

and thus (iii) in Theorem 1 is verified. And, as $\partial_Y \delta_{\Omega_j}(\overline{x}) = \partial_Y^\infty \delta_{\Omega_j}(\overline{x})$, we got also (iv) in Theorem 1.

Let us check (i) in Theorem 1. Our (i) says that, say, $\Omega_1 \cap Y$ is SNC at $\overline{x}$ (in the space Y). By the fact below, $\delta_{\Omega_1 \cap Y}$ is SNC at $\overline{x}$. But $\delta_{\Omega_1 \cap Y} = (\delta_{\Omega_1})_{|Y}$. Hence, by definition, $\varphi_1 = \delta_{\Omega_1}$ is $Y-$SNC, which is the condition (i) in Theorem 1. We thus verified all the assumptions of Theorem 1. Hence

$$N_Y(\overline{x}; \Omega_1 \cap \Omega_2) = \partial_Y \delta_{\Omega_1 \cap \Omega_2}(\overline{x}) = \partial_Y(\delta_{\Omega_1} + \delta_{\Omega_2})(\overline{x})$$

$$\subset \partial_Y \delta_{\Omega_1}(\overline{x}) + \partial_Y \delta_{\Omega_2}(\overline{x}) = N_Y(\overline{x}; \Omega_1) + N_Y(\overline{x}; \Omega_2).$$

It remains to formulate and prove the following

Fact. *Assume that $\Omega \subset Y$ is SNC at $\overline{y} \in \Omega$. Then the function δ_Ω is SNC at $\overline{y}$.*

In order to prove this, observe that $\operatorname{epi} \delta_\Omega = \Omega \times [0, +\infty)$. So let $(y_n, t_n) \in Y \times \mathbb{R}$, $\varepsilon_n \geq 0$, $(\eta_n, s_n) \in \widehat{\partial}_{\varepsilon_n} \delta_{\Omega \times \mathbb{R}}(y_n, t_n)$, $n \in \mathbb{N}$, $\|y_n - \overline{y}\|_Y \to 0$, $\varepsilon_n \downarrow 0$, and $\eta \rightharpoonup 0$, $s_n \to 0$. We immediately have that $\eta_n \in \widehat{\partial}_{\varepsilon_n} \delta_\Omega(y_n)$, $n \in \mathbb{N}$. And, as Ω was SNC at $\overline{y}$, we get that $\|\eta_n\|_Y \to 0$. We thus proved that $\operatorname{epi} \delta_\Omega$ is SNC at $(\overline{y}, 0)$. And this, by the definition, means that δ_Ω is SNC at $\overline{y}$. ∎

Remark. If X is an Asplund space itself, then we can put $Y = X$ in Theorem 2 and we thus get exactly [MS, Corollary 4.5] for $n = 2$.

Corollary 2. *Let $(X, Y, \|\cdot\|_X, \|\cdot\|_Y, i)$ be an Asplund generated scheme. Consider two closed sets $\Omega_1, \Omega_2 \subset X \times \mathbb{R}$, where $\Omega_1 = \operatorname{epi} \varphi_1$, and $\varphi_1 : X \to \overline{\mathbb{R}}$ is $\|\cdot\|_X$–Lipschitzian in an $\|\cdot\|_X$–vicinity of some $\overline{x} \in Y$. Assume that $(\overline{x}, \varphi_1(\overline{x})) \in \Omega_2$. Then*

$$N_Y(\overline{x}; \Omega_1 \cap \Omega_2) \subset N_Y(\overline{x}, \Omega_1) + N_Y(\overline{x}, \Omega_2).$$

Proof. Use Proposition below and Theorem 2. ∎

Now we prove the following Proposition, which provides a raison d'être for the new concepts ∂_Y and N_Y.

Proposition. *Let $(X, Y, \|\cdot\|_X, \|\cdot\|_Y, i)$ be an Asplund generated scheme. Let $\varphi : X \to \overline{\mathbb{R}}$ be a function, which is $\|\cdot\|_X$–Lipschitzian in a $\|\cdot\|_X$–vicinity of some $\overline{x} \in Y$. Then*

(i) $i^*\big(\partial_Y \varphi(\overline{x})\big) = \partial(\varphi_{|Y})(\overline{x})$ $(\neq \emptyset)$,

(ii) $\partial_Y^\infty \varphi(\overline{x}) = \{0\}$,

(iii) $(i^* \times \text{identity})\big(N_{Y\times\mathbb{R}}((\overline{x}, \varphi(\overline{x}));\ \operatorname{epi} \varphi)\big) = N((\overline{x}, \varphi(\overline{x}));\ \operatorname{epi}(\varphi_{|Y})$ $(\neq \{(0,0)\})$,

(iv) φ *is* Y*–SNC at* $\overline{x}$.

Proof. (i) Take any $\eta \in \partial(\varphi_{|Y})(\overline{x})$. Find $y_n \in Y$, $\varepsilon_n \geq 0$, $\eta_n \in \widehat{\partial}_{\varepsilon_n}(\varphi_{|Y})(y_n)$, $n \in \mathbb{N}$, such that $\|y_n - \overline{x}\|_Y \to 0$, $\varepsilon_n \downarrow 0$, and $\eta_n \to \eta$. Find $\Delta > 0$ so that φ is Lipschitzian on $\{x \in X;\ \|x - \overline{x}\|_X < 2\Delta\}$, with Lipschitz constant $L > 0$, say. Fix any $h \in Y$, with $\|h\|_X = 1$. Fix any $n \in \mathbb{N}$ such that $\|y_n - \overline{x}\|_Y < \Delta$. Then for every $\varepsilon > 0$ there is $\delta(\varepsilon) \in (0, \Delta)$ so that

$$\varphi(y) - \varphi(y_n) \geq \langle \eta_n, y - y_n \rangle \geq -(\varepsilon + \varepsilon_n)\|y - y_n\|_Y$$

whenever $y \in Y$ and $\|y - y_n\|_Y < \delta(\varepsilon)$. Then

$$\langle \eta_n, y - y_n \rangle \le (\varepsilon + \varepsilon_n)\|y - y_n\|_Y + L\|y - y_n\|_X$$

for all $y \in Y$, with $\|y - y_n\|_Y < \delta(\varepsilon)$. For $0 < t < \delta(\varepsilon)/\|h\|_Y$ we have

$$\langle \eta_n, th \rangle \le (\varepsilon + \varepsilon_n)\|th\|_Y + L\|th\|_X,$$

and so $\langle \eta_n, h \rangle \le (\varepsilon+\varepsilon_n)\|h\|_Y + L$. This holds for all $\varepsilon > 0$. Thus $\langle \eta_n, h \rangle \le \varepsilon_n\|h\|_Y + L$, and for $n \to \infty$ we get $\langle \eta, h \rangle = \lim_{n\to\infty}\langle \eta_n, h \rangle \le L$. Therefore $\|\eta\|_{X^*} = \sup\langle \eta, B_X \cap Y \rangle \le L$. Hence η is "extendable" to X and so $\eta \in i^*(X^*)$. We thus proved that $\eta \in i^*(\partial_Y \varphi(\overline{x}))$. The reverse inclusion in (i) holds always.

(ii) This follows from the fact $\partial^\infty(\varphi_{|Y})(\overline{x}) = \{0\}$ as $\varphi_{|Y}$ is $\|\cdot\|_Y$–Lipschitzian. However, because of further purposes and the ease of the reader, we include the proof. So, take any $\xi \in \partial_Y^\infty \varphi(\overline{x})$. Thus $\xi_{|Y} \in \partial^\infty(\varphi_{|Y})(\overline{x})$. Find then $y_n \in Y$, $t_n \in \mathbb{R}$, $(\eta_n, s_n) \in \widehat{N}((y_n, t_n);\ \mathrm{epi}\,(\varphi_{|Y}))$, $n \in \mathbb{N}$, so that $\|y_n - \overline{x}\|_Y \to 0$, $t_n \to \varphi(\overline{x})$, $\eta_n \rightharpoonup \xi_{|Y}$, and $s_n \to 0$. Find $\Delta > 0$ so that φ is Lipschitzian on $\{x \in X;\ \|x - \overline{x}\|_X < 2\Delta\}$, with Lipschitz constant $L > 0$, say. Fix any $h \in Y$, with $\|h\|_X = 1$. Fix any $n \in \mathbb{N}$ such that $\|y_n - \overline{x}\|_Y < \Delta$. Fix any $n \in \mathbb{N}$. (As Y is Asplund, we can take $\varepsilon_n = 0$.) Then

$$\limsup_{y \to y_n, t \to t_n, (y,t) \in \mathrm{epi}\,\varphi_{|Y}} \frac{\langle \eta_n, y - y_n \rangle + s_n(t - t_n)}{\|y - y_n\|_Y + |t - t_n|} \le 0.$$

We can easily check that $t_n > \varphi(y_n)$ implies $(\eta_n, s_n) = (0, 0)$. So, next assume that $t_n = \varphi(y_n)$. Then for every $\varepsilon > 0$ we can find $\delta(\varepsilon) > 0$ so that

$$\langle \eta_n, y - y_n \rangle + s_n(\varphi(y) - \varphi(y_n)) \le \varepsilon\|y - y_n\|_Y + \varepsilon(\varphi(y) - \varphi(y_n))$$

whenever $y \in Y$ and $\|y - y_n\|_Y < \delta(\varepsilon)$. (Yes, $\varphi_{|Y}$ is continuous at y_n.) Then, as φ is Lipschitzian in $\|\cdot\|_X$-norm, with constant L, we have that

$$\langle \eta_n, y - y_n \rangle \le \varepsilon \|y - y_n\|_Y + (\varepsilon + |s_n|)L\|y - y_n\|_X$$

whenever $y \in Y$ and $\|y - y_n\|_Y < \delta(\varepsilon)$. Thus for $0 < t < \delta(\varepsilon)/\|h\|_Y$ we have

$$\langle \eta_n, th \rangle \le \varepsilon \|th\|_Y + (\varepsilon + |s_n|)L\|th\|_X,$$

and so

$$\langle \eta_n, h \rangle \le \varepsilon \|h\|_Y + (\varepsilon + |s_n|)L.$$

And, letting $\varepsilon \downarrow 0$, we get $\langle \eta_n, h \rangle \le |s_n|L$. Therefore $\langle \xi, h \rangle \le \lim_{n\to\infty} \langle \eta_n, h \rangle \le \lim_{n\to\infty} |s_n|L = 0$. This holds for all h belonging to a dense set of the unit sphere of X. Hence $\xi = 0$.

(iii) Take any $(\eta, s) \in N((\overline{x}, \varphi(\overline{x}); \text{ epi}(\varphi_{|Y}))$. Find $y_n \in Y$, $t_n \in R$, $(\eta_n, s_n) \in \widehat{N}((y_n, t_n); \text{epi}(\varphi_{|Y}))$, $n \in \mathbb{N}$, so that $\|y_n - \overline{x}\|_Y \to 0$, $t_n \to \varphi(\overline{x})$, $\eta_n \rightharpoonup \eta$, and $s_n \to s$. Then, exactly as in the proof of (ii), we get that $\|\eta_n\|_{X^*} \le |s_n|L$ for every $n \in \mathbb{N}$. Thus $\|\eta\|_{X^*} \le \limsup_{n\to\infty} \|\eta_n\|_{X^*} \le \limsup_{n\to\infty} |s_n|L = |s|L$, and hence $(\eta, s) \in i^*(N_{Y\times\mathbb{R}}((\overline{x}, \varphi(\overline{x}); \text{ epi}\,\varphi)$. The reverse inclusion holds always.

(iv) We have to prove that $\text{epi}\,\varphi \cap (Y \times \mathbb{R})$, that is, $\text{epi}\,(\varphi_{|Y})$ is SNC at $(\overline{x}, \varphi(\overline{x}))$ in the space $Y \times \mathbb{R}$. So, consider $y_n \in Y$, $t_n \in \mathbb{R}$, $(\eta_n, s_n) \in \widehat{N}((y_n, t_n), \text{epi}\,\varphi_{|Y})$, $n \in \mathbb{N}$, with $\eta_n \rightharpoonup 0$, $s_n \to 0$. Then as in the proof of (ii), we get that $\|\eta_n\|_{X^*} \le |s_n|L$ and hence $\|\eta_n\|_{Y^*} \le \|\eta_n\|_{X^*} \to 0$ as $n \to \infty$. ∎

The next statement is a mean value theorem imitating [MS, Theorem 8.2]. For $a, b \in Y$ we put

$$[a, b] = \{ta + (1-t)b;\ t \in [0, 1]\} \quad \text{and} \quad (a, b) = \{ta + (1-t)b;\ t \in (0, 1)\}.$$

$\xi_n \in X^*$ and that $\|\xi_n\|_{X^*} \leq L$ for all $n \in \mathbb{N}$. We recall that the dual unit ball in X^* is weak* sequentially compact, see, e.g., [F, Chapter 2]. Therefore there exists $\xi \in \partial_Y f(c)$ such that $\langle \xi, b-a \rangle = f(b) - f(a)$.

If (ii) occurs we proceed similarly. We again profit from the $\|\cdot\|_X$-Lipschitz property of $f_{|\Omega}$.

The last statement also follows from [L, Theorem 3.1]. ∎

In Asplund spaces we have a representation formula for the Clarke's subdifferential ∂_C in terms of the Fréchet limit subdifferential $\partial_C\varphi(\overline{x}) = \overline{\text{co}}^*\left[\partial\varphi(\overline{x})\right]$ provided that $\varphi : X \to \overline{\mathbb{R}}$ is Lipschitzian around $\overline{x} \in X$, see [MS, Theorem 8.11]. For the Asplund generated scheme $(X, Y, \|\cdot\|_X, \|\cdot\|_Y, i)$, if $\varphi : X \to \overline{\mathbb{R}}$ is $\|\cdot\|_X-$Lipschitzian around $\overline{x} \in Y$, we have

$$\partial_C\varphi(\overline{x}) \supset \overline{\text{co}}^*\left[\partial_Y\varphi(\overline{x})\right].$$

Indeed, take any $\xi \in \partial_Y\varphi(\overline{x})$. Then $\xi_{|Y} = i^*(\xi) \in \partial(\varphi_{|Y})(\overline{x})$. Find $y_n \in Y$, with $\|y_n - \overline{x}\|_Y \to 0$, $\varepsilon_n \downarrow 0$, and $\eta_n \in \widehat{\partial}_{\varepsilon_n}\varphi_{|Y}(y_n)$ such that $\eta_n \rightharpoonup \xi_{|Y}$. Fix any $k \in Y$ and find $t_n \downarrow 0$ so that $\langle \eta_n, k\rangle - 2\varepsilon_n < \frac{1}{t_n}\big(\varphi(y_n + t_n k) - \varphi(y_n)\big)$. Then

$$\langle \xi, k\rangle \leq \limsup_{n\to\infty} \tfrac{1}{t_n}\big(\varphi(y_n + t_n k) - \varphi(y_n)\big) \leq \varphi^\circ(\overline{x})(k).$$

And, as φ was $\|\cdot\|_X-$Lipschitzian around $\overline{x}$, we get $\langle \xi, h\rangle \leq \varphi^\circ(\overline{x})(h)$ for every $h \in X$. Therefore $\xi \in \partial_C\varphi(\overline{x})$ and the inclusion is proved.

We do not see how to prove the reverse inclusion. Indeed, the Clarke's subdifferential uses in its definition the convergence in the space $(X, \|\cdot\|_X)$ while ∂_Y is defined via the convergence in $(Y, \|\cdot\|_Y)$. Yet there exist expressions for ∂_C, in the Asplund generated setting, in terms of other concepts of $\varepsilon-$subdifferential and even $\varepsilon-$differential, see [FLW].

Remarks. 1° Let X be an Asplund generated space. Then, of course, there are plenty of Y's witnessing for X to be Asplund generated. And we do not know if it impossible to define some reasonable subdifferential not depending on the concrete Y.

2° Let $(X, Y, \|\cdot\|_X, \|\cdot\|_Y, i)$ be an Asplund generated scheme such that X is not Asplund. Let $f : X \to \mathbb{R}$ be a Lipschitz concave function which is nowhere Fréchet differentiable [Ph, Corollary 2.35]. Then $\hat{\partial}_0 f(x) = \emptyset$ for every $x \in X$ while our Proposition guarantees that $\partial_Y f(y)$ is nonempty for every $y \in Y$.

3° In what follows we shall show which Y can be chosen for some concrete Asplund generated spaces X.

(a) Let X be a separable Banach space. Let $\{x_n;\ n \in \mathbb{N}\}$ be a countable dense set in the unit ball of X. Define $T : \ell_2 \to X$ by $T(a_n) = \sum_{n=1}^{\infty} a_n 2^{-n} x_n,\ (a_n) \in \ell_2$; this is a linear bounded mapping with dense range. Since the quotient $\ell_2/T^{-1}(0)$ is again a Hilbert space, we get a linear bounded and injective mapping $S : \ell_2 \to X$ with dense range. Put $Y = S(\ell_2)$ and define $\|y\|_Y = \|S\|^{-1}\|S^{-1}y\|_{\ell_2},\ y \in Y$. Then $(Y, \|\cdot\|_Y)$ is isomorphic with ℓ_2, and hence Asplund. Moreover, $\|y\|_X \leq \|y\|_Y$ for every $y \in Y$.

(b) More generally, let X be a weakly compactly generated Banach space. According to an interpolation theorem, see, e.g., [F, Theorem 1.2.3], X is Asplund generated, with Y even reflexive, witnessing for this. Note that the proof of this theorem is constructive, starting from a given weakly compact subset of X which is linearly dense in it. It should also be noted that the space $C(K)$ of continuous functions on K is weakly compactly generated if and only if K is an Eberlein compact, see, e.g. [F, Theorem 1.2.4].

(c) If μ is a finite measure and $X = L_1(\mu)$, it is enough to take $Y = L_2(\mu)$ and $\|\cdot\|_Y = \|\cdot\|_{L_2}$.

Here, of course $L_2(\mu)$ is Hilbert, hence Asplund space. Note that $L_1(\mu)$ is also weakly compactly generated.

(d) According to Stegall [St], [F, Theorem 1.5.4], a compact space K is homeomorphic to a weak* compact subset of a dual to an Asplund space if and only if $C(K)$ is Asplund generated. In this case, the construction of Y requires more care.

4° We are convinced that our note provides an important tool for extending the nonlinear analysis from the framework of Asplund spaces to a much larger class of Asplund generated spaces. In particular, we do not see any serious obstacle for extending many results from the paper [MS] in this vein.

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