Winter 5-3-2015

No-Arbitrage Option Pricing and the Binomial Asset Pricing Model

Nicholas S. Hurley
Wayne State University, NickHurley2011@yahoo.com

No-Arbitrage Option Pricing and the Binomial Asset Pricing Model

Nicholas Hurley

May 3, 2015

1 Introduction

Financial markets often employ the use of *securities*, which are defined to be any kind of tradable financial asset. Common types of securities include stocks and bonds. A particular type of security, known as a *derivative security* (or simply, a derivative), are financial instruments whose value is derived from another underlying security or asset (such as a stock). A common kind of derivative is an *option*, which is a contract that gives the holder the right but not the obligation to go through with the terms of said contract. An example of an option is the *European Option*, which we will use commonly throughout the following sections:

DEFINITION 1.1 (European Option)**.** *A European call(put) option gives the holder the right, but not the obligation, to buy(sell) an asset at a specified time, t, for a specified price, K.*
The payout of the option is then $\max(S_t - K, 0)$ *(or for a put option,* $\max(K - S_t, 0)$*).*

Because options can be traded - bought and sold, a problem arises on how to value the option (at a particular time, namely when the option is first created). The concept of evaluating an option, typically before the future values of the underlying security are known, is referred to as *option pricing.* The binomial asset pricing model allows us to evaluate options by using a "discrete-time" model of the behavior of the underlying security. While the binomial model is rather simplistic, it does provide a powerful tool in understanding the fundamental aspects of option pricing and no-arbitrage pricing theory.
Before going into any greater detail on the binomial model, there are several important financial terms that will be used:

- **Stock Market:** The stock market is where stocks are traded. Stocks are a type of security that represents partial ownership of a company. Since stocks tend to fluctuate in value they are generally considered a risky asset. A unit of stock is known as a *share.*

- **Short and Long Positions:** An investor dealing with a security such as stocks will be in the long position if he owns shares and will be in the short position if he owes shares. If an investor owns X amount of shares he is said to be long X shares, similarly if he sells X amount of shares ("borrows" them and sells them) he is said to be short X shares. If the investor is in the short position he must eventually buy the stock to repay the broker (individual/firm that organizes buy/sell orders from investors) he bought the shares through. If the stock has decreased in value then the investor will have made a profit (and if it increases, a loss). The concept of the short and long position is not exclusive to stocks, it can also be used with any security, commodity, or option that makes sense. In the case of options, the person who sells the option is in the short position, with the buyer in the long position.

- **Money Market:** The money market includes securities that are practically risk-less. While the potential long-term payoffs of putting money in the money market are small in comparison to that of investing in the stock market, money in the money market accrues interest over time.

- **Interest Rate:** The interest rate, represented by the letter $r \geq 0$, will be defined such that for every dollar put into the money market the investor will receive $(1 + r)^t$ at time t.

- **Portfolio:** A portfolio is simply a collection of securities. A tool that we will use with the binomial model in the proceeding sections is that of a *replicating portfolio.* Creating a replicating portfolio involves investing in both the stock and money markets such that the wealth of the portfolio is equal to the value of the payoff of the option regardless of the behavior of the stock (underlying asset of the option) at each time period. A key component of no-arbitrage option pricing is creating a replicating portfolio.

- **Arbitrage:** A trading strategy that exhibits arbitrage is one that starts with no money, has zero probability of losing money, and positive probability of making money. If arbitrage is possible, then wealth can be generated from nothing. Real markets will occasionally exhibit arbitrage but trading will quickly remove it. The proceeding examples take a look at simple cases of arbitrage and how an asset can be priced to avoid it.

EXAMPLE 1.1. *Consider two markets, Market A and Market B, where in Market A a crate of apples is valued at \$20 a crate and in Market B they are* *valued at \$22 a crate. Taking advantage of a difference in price, an investor looking at these two markets could quickly buy in Market A and sell in Market B - making a guaranteed risk-free profit. Such an event provides an arbitrage opportunity only for quick investors, as trading will quickly eliminate the price difference. Something to note is that it is possible for there to be transaction costs (suppose it costs \$5 per crate to get a crate of apples from Market A to Market B), essentially eliminating any arbitrage opportunity.*

EXAMPLE 1.2 (Hedging and arbitrage)**.** *In the previous example we dealt with a simple difference in market prices, in the following example we discuss what is known as a hedging transaction. A hedging transaction is when an investor has a primary security/portfolio position and establishes a secondary position to counterbalance some or all of the risk of the primary position.*
A simple hedging example is as follows:
Suppose an investor owns a stock originally valued at \$50 *that then increases to* \$60*. If the investor is worried about a future fall in price he can simply sell the stock, but what if he can't or doesn't want to? They could protect the profit by selling short the stock for* \$60*, leaving the investor with an overall gain of* \$10 *while maintaining their established long stock position. Whether the stock rises or falls the two positions offset and the investor is locked in at a* \$10 *profit. While the investor's previously gained profit is essentially guaranteed there is no chance of increasing it, given the new position.*
Now, consider two hedging examples that exhibit an arbitrage opportunity:

1. *Suppose we have a portfolio consisting of a packaged bundle of two stocks - Stock A valued at* \$5 *and Stock B valued at* \$6*. An investor would be able to gain a risk-free profit if there is an imbalance in price of the portfolio and the value of a single share of both stocks. If the price of the portfolio is overvalued at* \$12 *an investor could sell short the portfolio and buy the stocks individually - gaining an initial cash flow of* $\$12 - \$(6+5) = \$1$*. On the other hand, if the price of the portfolio is undervalued at* \$10 *an investor could sell short the two stocks and buy the portfolio - gaining an initial cash flow of* $\$(6+5) - \$10 = \$1$*. Note that the hedge is self-financing and the net future cash flow is zero as the investor owns the stocks to cover the short position. The above instances exhibit an arbitrage opportunity, and like the previous two examples, investors will take advantage causing the stock prices to go up and the price of the portfolio to go down (and vice versa for the second case) until an arbitrage free price for the portfolio is reached (the price of the portfolio equals the price of the two stocks).*

2. *Suppose now that we have two assets and one period (time zero and*

time one). Asset A has a current price of \$1000 *and, in the next period, a future cash flow of* \$1100*, and Asset B has a current price of* \$2000 *and, in the next period, a future cash flow of* \$2250*. Forming a portfolio by buying and selling these two assets separately has an initial cash flow of* \$0 *- the portfolio created is costless. If an investor wanted to construct a portfolio to take advantage of an arbitrage opportunity they could short sell two units of Asset A and buy one unit of Asset B. The initial cash flow is* $-2(\$1000) + 1(\$2000) = \$0$*, and at the end of the period we have an expected cash flow of* $-2(\$1100)+1(\$2250) = \$50$*. This constitutes an arbitrage opportunity since it is costless to construct the portfolio but still produces a positive future cash flow - the investor makes a risk-less profit. Similar to the previous examples, investors would take advantage of such an opportunity as much as they can until the opportunity eventually disappears.*

EXAMPLE 1.3 (No-arbitrage price of a forward contract)**.** *A forward contract, or simply a forward, is a derivative security in which two parties agree to buy or sell an asset at a specified time in the future at a specified price, F, today. There is no cost in entering a forward contract, the price F isn't payed until the specified time period (when the asset is bought/sold). If at time* 0 *the forward contract is created and at time t the asset is traded, then the no-arbitrage price of the forward is:*

$$F = S_0(1 + r)^t.$$

Where S_0 is the time 0 *price of the asset, and r is the interest rate. The payoff of the contract, at time t, is $S_t - F$ for the buyer of the contract (long position) and $F - S_t$ for the seller (short position). If the price of the forward is not $S_0(1 + r)^t$, then there is an arbitrage:*

- *If $F > S_0(1 + r)^t$: At time* 0*, an investor can borrow \$$S_0$ and buy the asset. They can then take the short position on the forward contract. At time t, the asset is sold for the agreed upon price F and the loan from time* 0 *is paid off. The investor is left with a risk-free profit of $F - S_0(1 + r)^t$.*

- *If $F < S_0(1 + r)^t$: At time* 0*, an investor can short sell one unit of the asset and lend \$$S_0$. They can then take the long position on the forward contract. At time t, the asset is bought for the agreed upon price F, the short sold asset is returned, and the loan is recovered (which has since grown to $S_0(1 + r)^t$). The investor is left with a risk-free profit of $S_0(1 + r)^t - F$.*

2 The Binomial Model

The binomial asset-pricing model provides a simple model for understanding the no-arbitrage pricing of options. We start with the simple one-period model and then generalize to a more realistic multi-period model.

2.1 The One-Period Binomial Model

The one-period binomial model has just two times, time zero and time one. At time zero, we have a stock whose price per share S_0 will either increase to $S_1(H)$ or decrease to $S_1(T)$ at time one (all values positive). The H and T represent either a heads or tails, respectively, of an imagined coin toss. The probability of a heads will be denoted as p, and the probability of a tails will be denoted as $q = 1 - p$ (with $0 < p < 1$ and $p + q = 1$). We also introduce two positive numbers u, the up factor, and d, the down factor:

$$u = \frac{S_1(H)}{S_0}, \; d = \frac{S_1(T)}{S_0} \tag{1}$$

where $u > d$ (if $u = d$, the model is uninteresting, if $d > u$ then one could just switch the meaning of u and d). The general one-period binomial model can then be formed:

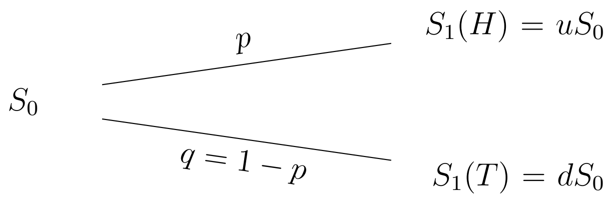

Figure 1: One-period binomial model

In addition to the above figure, there is also the interest rate, $r \geq 0$, to consider. With regards to the one-period binomial model, one dollar invested (borrowed) in the money market at time zero results in a $1 + r$ return (debt) at time one.
In order for the binomial model to be a useful model in evaluating options, it must not exhibit arbitrage. This brings about the following proposition:

PROPOSITION 2.1. *In the binomial model, to rule out arbitrage we must assume:*

$$0 < d < 1 + r < u \tag{2}$$

Proof. The first inequality, $0 < d$, is assumed from the positivity of stock prices. Looking at a one-period binomial model:

- If $d \geq 1 + r$: An investor could, at time zero, borrow $\$S_0$ from the money market and buy one share of stock. The investor's stock is then worth either uS_0 or dS_0 at time one, depending on the coin toss (heads, tails respectively). Even in the worst case scenario, a tails, the investor has at least a stock worth enough to pay back the money market debt ($\$(1 + r)S_0$ at time one) with a positive probability, $p > 0$, of making profit since $u > d \geq 1 + r$.

- If $u \leq 1 + r$: An investor could, at time zero, short sell the stock and invest $\$S_0$ in the money market. The short sold stock is then worth either uS_0 or dS_0 at time one, depending on the coin toss (heads, tails respectively). Even in the worst case scenario, a heads, the investor has at least enough return on the money market loan ($\$(1 + r)S_0$ at time one) to replace the stock with a positive probability, $q = 1 - p > 0$, of making profit since $d < u \leq 1 + r$.

$\square$

Before going through the process of pricing a European call option using the binomial model, there are several principal assumptions we must consider:

- shares of stock can be subdivided for sale or purchase,
- the interest rate for investing is the same as the interest rate for borrowing,
- the purchase price of stock is the same as the selling price,
- the stock can take only two possible values in the next period.

In the general one-period binomial model, we define a derivative security to be a security whose payoff is $V_1(H)$ or $V_1(T)$ depending on the coin toss. For a European call option, the payoff is $V_1 = max(S_1 - K, 0)$ where S_1 is the value of the stock at time one and K is the strike price. To determine V_0, the price of the derivative security at time zero, we construct a replicating portfolio. Suppose we begin with a portfolio that has starting wealth X_0 and we buy Δ_0 shares of stock, our cash position at time zero is then $X_0 - \Delta_0 S_0$. The value of our portfolio (contains stock and money market account) at time one is:

$$X_1 = \Delta_0 S_1 + (1 + r)(X_0 - \Delta_0 S_0) = (1 + r)X_0 + \Delta_0(S_1 - (1 + r)S_0). \quad (3)$$

In order to properly replicate the portfolio we must choose X_0 and Δ_0 such that $X_1(H) = V_1(H)$ and $X_1(T) = V_1(T)$. Note that at time zero we know the values of $V_1(H)$ and $V_1(T)$, we just don't know which one will be realized. Using this requirement and equation (3) we get the following equations:

$$X_0 + \Delta_0(\frac{1}{1+r}S_1(H) - S_0) = \frac{1}{1+r}V_1(H), \tag{4}$$

$$X_0 + \Delta_0(\frac{1}{1+r}S_1(T) - S_0) = \frac{1}{1+r}V_1(T). \tag{5}$$

In order to solve for our two unknowns, X_0 and Δ_0, we multiply equation (4) by $\tilde{p}$ and equation (5) by $\tilde{q}$ (where $\tilde{q} = 1-\tilde{p}$) and then adding them together giving us:

$$X_0 + \Delta_0(\frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)] - S_0) = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)]. \tag{6}$$

If we choose $\tilde{p}$ (and thus $\tilde{q}$) so that:

$$S_0 = \frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)], \tag{7}$$

the terms cancel and we are left with the equation

$$X_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)]. \tag{8}$$

By putting (7) in the following form (doing a little substitution),

$$S_0 = \frac{1}{1+r}[\tilde{p}uS_0 + (1-\tilde{p})dS_0] = \frac{S_0}{1+r}[(u-d)\tilde{p} + d],$$

we can solve for $\tilde{p}$ and $\tilde{q}$:

$$\tilde{p} = \frac{(1+r)-d}{u-d},\ \tilde{q} = \frac{u-(1+r)}{u-d}. \tag{9}$$

To find Δ_0 we can subtract (5) from (4) to get the *delta-hedging formula*:

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}. \tag{10}$$

So, if an investor begins with wealth X_0, given by (8), and buys Δ_0 shares of stock, given by (10), then at time one, they will have a portfolio worth either $V_1(H)$ or $V_1(T)$ depending on the coin toss. We have properly constructed a replicating portfolio and the derivative security at time zero should be priced at

$$V_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)], \tag{11}$$

which is the initial value X_0 (of the replicating portfolio). Any other time zero price would introduce an arbitrage.

The numbers $\tilde{p}$ and $\tilde{q}$ that we arbitrary selected and solved for (equation (9)) are both positive (no arbitrage assumption, equation (2))and sum to one. Because of this, we can think of $\tilde{p}$ as the probability of heads and $\tilde{q}$ as the probability of tails. They are not the actual probabilities, p and q, but instead what we call the *risk-neutral probabilities*. With the real probabilities, the average rate of growth of stock is greater than the rate of growth of an investment in the money market. If this wasn't the case then no one would bother investing in the risky stock market (compared to the money market which has no risk). Therefore, p and q satisfy:

$$S_0 < \frac{1}{1+r}[pS_1(H) + qS_1(T)], \tag{12}$$

unlike $\tilde{p}$ and $\tilde{q}$ that satisfy (7). Therefore, the risk-neutral probabilities assume that investors are neutral about risk - they typically are not, which is why $\tilde{p}$ and $\tilde{q}$ are not the actual probabilities. The risk-neutral probabilities are chosen so that the mean rate of return of any portfolio (comprised of stock and money market accounts) equals the rate of growth of the money market investment. If we consider $\tilde{p}V_1(H) + \tilde{q}V_1(T)$ to be the mean portfolio value under the risk-neutral probabilities (at time one), then rearranging (11) gives us the following relationship

$$r = \frac{[\tilde{p}V_1(H) + \tilde{q}V_1(T)] - V_0}{V_0}, \tag{13}$$

where r (the interest rate) is the rate of return from the money market. Because of this, the equation (11) is called the *risk-neutral pricing formula* for the one-period binomial model. Risk-neutral pricing is described further in section 5.

We illustrate the above replicating portfolio process with a simple example.

EXAMPLE 2.1 (Pricing a European call option)**.** *Consider a European call option that goes through one time period - at time zero the option is purchased and at time one the option is either exercised or it is not. The payoff of the option is then* $max(S_1 - K, 0)$*, where* K*, the strike price, is the agreed upon price paid for the stock. For the following example we have* $S_0 = 2$*,* $u = 2$*,* $d = 1/2$*,* $r = 1/4$*, and* $K = 2.5$*. From this we can easily* *calculate* $S_1(H) = uS_0 = 4$ *and* $S_1(T) = dS_0 = 1$ *and form the following binomial tree:*

$$\begin{tikzcd} & S_1(H) = 4 \\ S_0 = 2 \arrow[ur, no head] \arrow[dr, no head] & \\ & S_1(T) = 1 \end{tikzcd}$$

Figure 2: One-period binomial model using example parameters

So, at time one, if the stock share price increases to $S_1(H)$ *the option will be exercised for a profit of* $S_1(H) - K = 1.5$ *and if it decreases to* $S_1(T)$ *the option will not be exercised - the value of the option is* 0. *In terms of the notation above,* $V_1(H) = 1.5$ *and* $V_1(T) = 0$. *Using equation (9) we can calculate* $\tilde{p}$ *and* $\tilde{q}$*:*

$$\tilde{p} = \frac{(1 + .25) - \frac{1}{2}}{2 - \frac{1}{2}} = \frac{1}{2}, \ \tilde{q} = \frac{2 - (1 + .25)}{2 - \frac{1}{2}} = \frac{1}{2}.$$

In order to replicate the portfolio, we use equation (8) to calculate the initial wealth X_0 *and equation (10) to calculate the number of shares* Δ_0 *to be purchased. So,*

$$X_0 = \frac{1}{1.25}[(.5)(1.5) + (.5)(0)] = 0.6,$$

$$\Delta_0 = \frac{1.5 - 0}{4 - 1} = \frac{1}{2}.$$

Since the value of the option at time zero, V_0, *is equal to the value of the replicating portfolio at time zero,* X_0, *we have* $V_0 = X_0 = 0.6$. *Using equations (4) and (5) we can verify that the values at time one are also equivalent:*

$$V_1(H) = X_1(H) = (1.25)(0.6) + (1.25)(\frac{1}{1.25}(.5)(4) - (.5)(2)) = 1.5,$$

$$V_1(T) = X_1(T) = (1.25)(0.6) + (1.25)(\frac{1}{1.25}(.5)(1) - (.5)(2)) = 0,$$

completing the replicating portfolio for this example.

2.2 The Multi-Period Binomial Model

In the previous section we went through the one-period binomial model where we began with an initial stock price S_0, and at time one it went up by a factor of u or went down with a factor of d, now we move on to the multi-period binomial model. In the multi-period model we assume at time two that the stock price increase or decrease again, also by a factor of u and d respectively. This gives us the following:

$$S_2(HH) = uS_1(H) = u^2S_0,\ S_2(HT) = dS_1(H) = duS_0,$$

$$S_2(TH) = uS_1(T) = udS_0,\ S_2(TT) = dS_1(T) = d^2S_0,$$

This process can be done repeatedly for any number of N times to form an N-period binomial model - creating a binomial tree of stock prices.

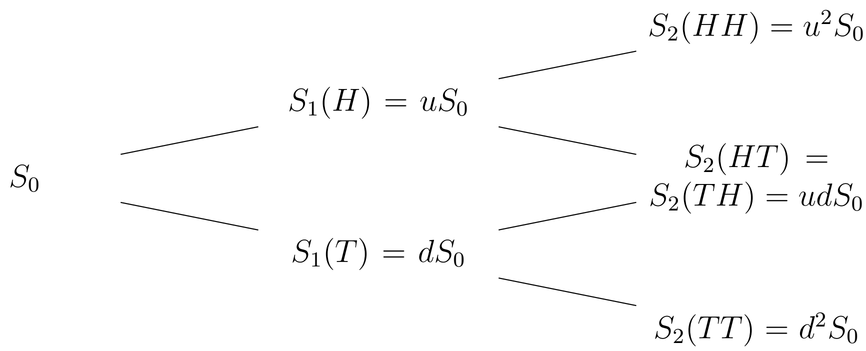

Figure 3: Two-period binomial model

The principal assumptions stated in the previous section hold for the multi-period binomial model as they do for the one-period model. Before going into replicating portfolios in the multi-period binomial model, we need to generalize equation (3) to multiple steps in order to derive the *wealth equation*:

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n), \tag{14}$$

where X_n is the value of the portfolio at time n and Δ_n denotes the number of shares of the stock in the portfolio at time n.
The following theorem is analogous to the replicating portfolio process in the previous section, namely equations (10) and (11).

THEOREM 2.1 (Replication in the multi-period binomial model)**.** *Consider an N-period binomial asset-pricing model, with* $0 < d < 1 + r < u$, *and with*

$$\tilde{p} = \frac{(1+r) - d}{u - d},\ \tilde{q} = \frac{u - (1+r)}{u - d}. \tag{15}$$

Let V_N *be a random variable (a derivative security paying off at time* N*) depending on the first* N *coin tosses* $\omega_1, \omega_2, ..., \omega_N$. *Define recursively backward* *in the sequence of random variables* $V_{N-1}, V_{N-2}, ..., V_0$ *by*

$$V_n(\omega_1\omega_2...\omega_n) = \frac{1}{1+r}[\tilde{p}V_{n+1}(\omega_1\omega_2...\omega_n H) + \tilde{q}V_{n+1}(\omega_1\omega_2...\omega_n T)], \qquad (16)$$

so that each V_n *depends on the first* n *coin tosses* $\omega_1\omega_2...\omega_n$*, where* n *ranges between* $N-1$ *and* 0*. Next define,*

$$\Delta_n(\omega_1\omega_2...\omega_n) = \frac{V_{n+1}(\omega_1\omega_2...\omega_n H) - V_{n+1}(\omega_1\omega_2...\omega_n T)}{S_{n+1}(\omega_1\omega_2...\omega_n H) - S_{n+1}(\omega_1\omega_2...\omega_n T)} \qquad (17)$$

where again n *ranges between* 0 *and* $N-1$*. If we set* $X_0 = V_0$ *and define recursively forward in time the portfolio values* $X_1, X_2, ..., X_N$ *by the wealth equation in (14), then we will have:*

$$X_N(\omega_1\omega_2...\omega_N) = V_N(\omega_1\omega_2...\omega_n) \qquad \forall\, \omega_1\omega_2...\omega_N. \qquad (18)$$

Proof. See pp. 13-14 of [4]. □

DEFINITION 2.1. *For* $n = 1, 2, ..., N$*, the random variable* $V_n(\omega_1\omega_2...\omega_n)$ *in the previous theorem is defined to be the price of the derivative security at time n if the outcomes of the first n tosses are* $\omega_1\omega_2...\omega_n$*. The price of the derivative security at time zero is defined to be* V_0*.*

Note that $\Delta_n(\omega_1\omega_2...\omega_n)$ is the number of shares of stock that should be held at time n. Since Δ_n depends on the first n coin tosses, we say that $\Delta_0, \Delta_1, ..., \Delta_{N-1}$ is an *adapted portfolio process.* What this means is that the number of shares of stock is adjusted at each time period in the replicating portfolio process. The above theorem works by calculating the value of the option, considering an N-period binomial model, at time N (where $N = n+1$) and then working recursively backwards until the value of the option at time zero is known.

3 Complete and Incomplete Markets

In the binomial model, every option can be replicated by a portfolio consisting of the underlying asset (the stock) and the money market account. We say that the binomial model is a complete market.

DEFINITION 3.1. *A market is said to be complete if every contingent claim (derivative) can be replicated by a portfolio consisting of the tradable assets in the market.*

We will now consider a market that has N finite number of tradable assets and, for now, is restricted only to single-period models - only observable at time zero and at time t.
The initial values of the N assets will be represented as a column vector:

$$A_0 = \begin{bmatrix} A_0^1 \\ A_0^2 \\ \vdots \\ A_0^N \end{bmatrix}$$

At time t, the market is in one of a finite number of states $1, 2, ..., n$. We can then construct a $N \times n$ matrix D, where D_{ij} is the value of the ith asset at time t, if the market is in state j. The portfolio can then be represented as a vector θ:

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix} \in \mathbb{R}^N,$$

where θ_i is the quantity of the ith asset in the portfolio. The market value of the portfolio at time zero is the scalar product:

$$A_0 \cdot \theta = A_0^1\theta_1 + A_0^2\theta_2 + ... + A_0^N\theta_N$$

Note: The symbol $'$ represents the operation of taking the transpose of the vector (or matrix).
Then, the value of the portfolio at time t is the vector:

$$D'\theta = \begin{bmatrix} D_{11}\theta_1 + D_{21}\theta_2 + \cdots + D_{N1}\theta_N \\ D_{12}\theta_1 + D_{22}\theta_2 + \cdots + D_{N2}\theta_N \\ \vdots \\ D_{1n}\theta_1 + D_{2n}\theta_2 + \cdots + D_{Nn}\theta_N \end{bmatrix} \in \mathbb{R}^n,$$

where the ith entry is the value of the portfolio if the market is in state i. We can now state a proposition about complete markets using this notation.

PROPOSITION 3.1. *A market consisting of N tradable assets, evolving according to a single period model in which at the end of the time period the market is in one of n possible states, is complete if and only if $N \geq n$ and the rank of the matrix, D, of security prices is n.*
Rank, in linear algebra, is the size of the largest collection of linearly independent columns of a matrix.

Proof. Any contingent claim in our market can be expressed as a vector $v \in \mathbb{R}^n$. A replication for these claims at time t will be a portfolio $\theta = \theta(v) \in \mathbb{R}^N$ for which $D^{'}\theta = v$. Finding such a θ amounts to solving n equations in N unknowns. Thus a replicating portfolio exists for every choice of v if and only if $N \geq n$ and the rank of D is n, as required. $\square$

Note: For a vector $x \in \mathbb{R}^n$ we write $x \geq 0$ if $x = (x_1, ..., x_n)$ and $x_i \geq 0$ for all $i = 1, ..., n$. We write $x > 0$ to mean $x \geq 0, x \neq 0$, in this case x need not be strictly positive in all its coordinates. In this notation, an *arbitrage* is a portfolio $\theta \in \mathbb{R}^N$ with either

$$S_0 \cdot \theta \leq 0,\ D^{'}\theta > 0 \quad \textit{or} \quad S_0 \cdot \theta < 0,\ D^{'}\theta \geq 0. \tag{19}$$

If the value of the portfolio at time zero is less than or equal to 0 and the payoff is nonnegative for all states and strictly positive for some state, then there is an arbitrage. Also, if the value of the portfolio at time zero is strictly negative and the payoff is greater than or equal to 0, there is an arbitrage.
The proceeding example goes through the one-period binomial model using the above market notations.

EXAMPLE 3.1. *As stated previously, a contingent claim/derivative in the binomial model can be replicated by a portfolio consisting of the stock and the money market account. The following vectors represent the single-period binomial model,*

$$A_0 = \begin{bmatrix} 1 \\ S_0 \end{bmatrix},\ \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix},\ A_0 \cdot \theta = \theta_1 + S_0\theta_2,\ D = \begin{bmatrix} (1+r) & (1+r) \\ S_1(H) & S_1(T) \end{bmatrix}$$

where θ_1 is the amount in the money market account and θ_2 is the number of stocks at time zero. Calculating $A_0 \cdot \theta$ gives the value of the derivative (such as a European call option) at time zero (as well as the starting wealth required to replicate). Now, the vector representing the value of the portfolio at time t,

$$D^{'}\theta = \begin{bmatrix} (1+r)\theta_1 + S_1(H)\theta_2 \\ (1+r)\theta_1 + S_1(T)\theta_2 \end{bmatrix}.$$

Setting this vector equal to the payoff vector of the derivative will allow you to solve for θ_1 and θ_2, the portfolio θ can then be used to replicate the derivative's payoff in the final states (construct a replicating portfolio).
Clearly the one-period binomial model is a complete market as $N = 2$ (number of assets) and $n = 2$ (possible final states). Using proposition 3.1, $N \geq n$ and it can be checked (using equation (2)) that $rank(D) = n$.

The market described thus far has only one period, now we briefly turn our attention to a market with multiple periods.

Based on how we've define the above market one would think that a multi-period model would *not* be complete. Suppose for example we have a ten-period binomial model, at time ten we would have $2^{10} = 1024$ final states for our stock price. By proposition 3.1, we would need at least that many independent assets for our market to be complete. This isn't as big of a problem as it may seem as in Theorem 2.1 (replication of the multi-period binomial model) we defined Δ_n to be an adapted portfolio process. What this means is that the replicating portfolio is rebalanced after each time period - using only the two assets (in our case the stock and money market account). The rebalancing can only involve the purchasing more of one asset and the sale of the other asset - no money can be added or taken out. This is known as the *self-financing property* of the replicating portfolio.

So far we've considered markets that are complete, but what about an example of an incomplete market?

4 The Trinomial Model

The one-period trinomial model differs from the binomial model in that the underlying asset, the stock, can take an intermediate price between uS_0 and dS_0, we will refer to this value as mS_0. Note that only $d < m < u$ needs to be true, it may not be the case that $m = 1$. In the binomial model we used H and T to represent an imaginary coin toss, for the trinomial model we introduce M to indicate that the stock at time one took the intermediate path. The following figure shows the one-period trinomial model:

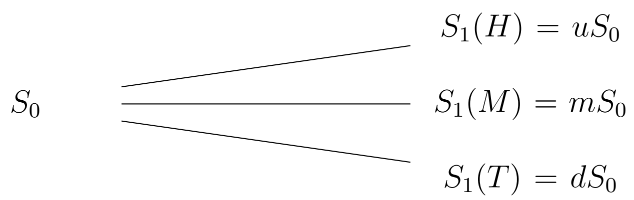

Figure 4: One-period trinomial model

Another major difference between the binomial and trinomial models is that the trinomial model is an incomplete market - the final states of a contingent claim cannot all be replicated by the underlying asset and the money market account. The following example works through it.

EXAMPLE 4.1. *Using the same vector notation as in the last section,*

$$A_0 = \begin{bmatrix} 1 \\ S_0 \end{bmatrix},\ \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix},\ A_0 \cdot \theta = \theta_1 + S_0\theta_2,\ D = \begin{bmatrix} (1+r) & (1+r) & (1+r) \\ S_1(H) & S_1(M) & S_1(T) \end{bmatrix}$$

where again, θ_1 is the amount in the money market account and θ_2 is the number of stocks at time zero. The vector representing the value of the portfolio at time t is,

$$D^{'}\theta = \begin{bmatrix} (1+r)\theta_1 + S_1(H)\theta_2 \\ (1+r)\theta_1 + S_1(M)\theta_2 \\ (1+r)\theta_1 + S_1(T)\theta_2 \end{bmatrix} \tag{20}$$

If you tried to set this vector equal to the payoff vector of a derivative (such as a European call option) you would notice that solving for θ_1 and θ_2 is generally not possible (as it is a system of three linear equations with two unknowns) - thus solving for $A \cdot \theta$ and constructing a replicating portfolio for this model is also not possible.

In the case of the one-period trinomial model, $N = 2$ (number of assets), $n = 3$ (possible final states), and the rank of matrix D obviously can't be greater than 2. Checking proposition 3.1, $N < n$ and $rank(D) < n$ - indicating that the trinomial model is not in fact a complete market. This means that we can't find a portfolio that exactly replicates a derivative under this model (as is) nor can we obtain a unique no-arbitrage price for the derivative.

A "solution" to this problem would be to simply add one additional independent asset to the market in order to make it complete ($N = 3$, and $rank(D) = n$ is possible).

5 Probability Theory and Option Pricing

In the previous sections we've discussed pricing options using the assumptions of the binomial model. Now we will combine these previous concepts with some basic probability theory in order to formulate a more unified theory on option pricing.

5.1 Probability Theory on Coin-Toss Space

We begin with a few basic definitions:

DEFINITION 5.1. *A finite probability space consists of a sample space* Ω *and a probability measure* $\mathbb{P}$*. The sample space* Ω *is a nonempty finite set and the probability measure* $\mathbb{P}$ *is a function that assigns to each element* ω *of* Ω *a number in* $[0,1]$ *so that*

$$\sum_{\omega\in\Omega}\mathbb{P}(\omega)=1. \tag{21}$$

An event is a subset of Ω*, and we define the probability of an event* A *to be*

$$\mathbb{P}(A)=\sum_{\omega\in A}\mathbb{P}(\omega). \tag{22}$$

DEFINITION 5.2. *Let* $(\Omega,\mathbb{P})$ *be a finite probability space. A random variable is a real-valued function defined on* Ω*.*

DEFINITION 5.3. *Let* X *be a random variable defined on a finite probability space* $(\Omega,\mathbb{P})$*. The expectation (or expected value) of* X *is defined to be*

$$\mathbb{E}X=\sum_{\omega\in\Omega}X(\omega)\mathbb{P}(\omega). \tag{23}$$

When we compute the expectation using the risk-neutral probability measure $\tilde{\mathbb{P}}$*, we use the notation*

$$\tilde{\mathbb{E}}X=\sum_{\omega\in\Omega}X(\omega)\tilde{\mathbb{P}}(\omega). \tag{24}$$

In the proceeding sections, the symbol $\mathbb{P}$ *denotes the probability measure associated with the actual probabilities* p *and* q*, while the symbol* $\tilde{\mathbb{P}}$ *denotes the probability measure associated with the risk-neutral probabilities* $\tilde{p}$ *and* $\tilde{q}$*.*

5.2 Conditional Expectation

Conditional expectation is crucial in proving some of the results in the proceeding section.
The following definition is how we will represent the expectation of a random variable X, depending on the first N coin tosses, conditioned on a particular sequence of $n \leq N$ coin tosses. This will allow us to estimate X based on information available at an earlier time.

DEFINITION 5.4. *Let* n *satisfy* $1 \leq n \leq N$*, and let* $\omega_1...\omega_n$ *be given and be fixed. There are* 2^{N-n} *possible continuations* $\omega_{n+1}...\omega_N$ *of the sequence fixed*

$\omega_1 ... \omega_n$. Denote by $\#H(\omega_{n+1}...\omega_N)$ by the number of heads in the continuation $\omega_{n+1}...\omega_N$ and by $\#T(\omega_{n+1}...\omega_N)$ the number of tails. We define

$$\mathbb{E}_n[X](\omega_1...\omega_n) = \sum_{\omega_{n+1}...\omega_N} p^{\#H(\omega_{n+1}...\omega_N)} q^{\#T(\omega_{n+1}...\omega_N)} X(\omega_1...\omega_n\omega_{n+1}...\omega_N) \tag{25}$$

and call $\mathbb{E}_n[X]$ the conditional expectation of X based on the information at time n (under the actual probabilities).
The two extreme cases of conditioning are $\mathbb{E}_0[X]$, the conditional expectation of X based on no information, which reduces to:

$$\mathbb{E}_0[X] = \mathbb{E}X, \tag{26}$$

and $\mathbb{E}_N[X]$, the conditional expectation of X based on knowledge of all N coin tosses, which reduces to:

$$\mathbb{E}_N[X] = X. \tag{27}$$

The fundamental properties of conditional expectation will also be important in proving future results:

THEOREM 5.1 (Fundamental properties of conditional expectations)**.** *Let N be a positive integer, and let X and Y be random variables depending on the first N coin tosses. Let $0 \leq n \leq N$ be given. The following properties hold.*

1. ***Linearity of conditional expectations.*** *For all constants c_1 and c_2, we have*
$$\mathbb{E}_n[c_1X + c_2Y] = c_1\mathbb{E}_n[X] + c_2\mathbb{E}_n[Y]. \tag{28}$$
2. ***Taking out what is known.*** *If X actually depends only on the first n coin tosses, then*
$$\mathbb{E}_n[XY] = X\mathbb{E}_n[Y]. \tag{29}$$
3. ***Iterated conditioning.*** *If $0 \leq n \leq m \leq N$, then*
$$\mathbb{E}_n[\mathbb{E}_m[X]] = \mathbb{E}_n[X]. \tag{30}$$
In particular, $\mathbb{E}[\mathbb{E}_m[X]] = \mathbb{E}X$.
4. ***Independence.*** *If X depends only on tosses $n+1$ through N, then*
$$\mathbb{E}_n[X] = \mathbb{E}X. \tag{31}$$

Proof. See pp. 177-179 of [4]. □

5.3 Martingales and Option Pricing

Using the binomial pricing model described in the previous sections, we chose our risk-neutral probabilities based on the formulas in (9):

$$\tilde{p} = \frac{(1+r)-d}{u-d},\ \tilde{q} = \frac{u-(1+r)}{u-d}.$$

Rearranging these two formulas and using the fact that $\tilde{p}+\tilde{q}=1$ gives:

$$\frac{\tilde{p}u+\tilde{q}d}{1+r} = 1. \tag{32}$$

Consequently, multiplying both sides by $S_n(\omega_1...\omega_n)$ and using the fact that $S_{n+1}(H) = uS_n$ and $S_{n+1}(T) = dS_n$, we have:

$$S_n(\omega_1...\omega_n) = \frac{1}{1+r}\Big[\tilde{p}S_{n+1}(\omega_1...\omega_n H) + \tilde{q}S_{n+1}(\omega_1...\omega_n T)\Big] \tag{33}$$

What this equation means is that the stock price at time n is the discounted weighted average of the two possible stock prices at time $n+1$, using the risk-neutral probabilities as weights. Rewriting the equation using Definition 5.4 gives us:

$$S_n = \frac{1}{1+r}\tilde{\mathbb{E}}_n[S_{n+1}]. \tag{34}$$

If we divide this equation by $(1+r)^n$, we get:

$$\frac{S_n}{(1+r)^n} = \tilde{\mathbb{E}}_n\Big[\frac{S_{n+1}}{(1+r)^{n+1}}\Big]. \tag{35}$$

The term $\frac{1}{(1+r)^{n+1}}$ can be written either inside or outside the conditional expectation because it is constant (property 1 of Theorem 5.1). Also, we refer to $\frac{S_n}{(1+r)^n}$ as the *discounted stock price* since \$1 at time zero is worth $\$(1+r)^n$ at time n. Equation (35) asserts that the risk-neutral probabilities are chosen so that the best estimate, based on the information at time n of the value of the discounted stock price at time $n+1$, is the discounted stock price at time n. A process that satisfies this condition is called a martingale. The formal definition of a martingale is given below:

DEFINITION 5.5. *Consider the binomial asset-pricing model. Now let $M_0, M_1, ..., M_N$ be a sequence of random variables, with each M_n depending only on the first n coin tosses (and M_0 constant). Such a sequence of random variables is called an adapted stochastic process.*

1. *If*
$$M_n = \mathbb{E}_n[M_{n+1}], n = 0, 1, ..., N-1, \tag{36}$$
we say this process is a martingale.

2. *If*
$$M_n \leq \mathbb{E}_n[M_{n+1}], n = 0, 1, ..., N-1,$$
we say the process is a submartingale (even though it may have a tendency to increase);

3. *If*
$$M_n \geq \mathbb{E}_n[M_{n+1}], n = 0, 1, ..., N-1,$$
we say the process is a supermartingale (even though it may have a tendency to decrease).

The following are a few useful properties of martingales:

REMARK 5.1. *The martingale property in (36) of the previous definition is a "one-step-ahead" condition. However, it implies a similar condition for any number of steps. If $M_0, M_1, ..., M_N$ is a martingale and $n \leq N-2$, then the martingale property (36) implies*

$$M_{n+1} = \mathbb{E}_{n+1}[M_{n+2}].$$

Taking conditional expectation on both sides based on the information at time n and using property 3 of Theorem 5.1, we get

$$\mathbb{E}_n[M_{n+1}] = \mathbb{E}_n[\mathbb{E}_{n+1}[M_{n+2}]] = \mathbb{E}_n[M_{n+2}]$$

Because of (36), we have the "two-step-ahead" property

$$M_n = \mathbb{E}_n[M_{n+2}].$$

Iterating this, argument, whenever $0 \leq n \leq m \leq N$, we have the "multistep-ahead" property,
$$M_n = \mathbb{E}_n[M_m]. \tag{37}$$

REMARK 5.2. *The expectation of a martingale is constant over time, i.e., if $M_0, M_1, ..., M_N$ is a martingale, then*

$$M_0 = \mathbb{E}M_n, n = 0, 1, ..., N. \tag{38}$$

Proof. If $M_0, M_1, ..., M_N$ is a martingale, we may take expectations on both sides of (36), using property 3 of Theorem 5.1, and obtain $\mathbb{E}M_n = \mathbb{E}[M_{n+1}]$ for every n. It follows that

$$\mathbb{E}M_0 = \mathbb{E}M_1 = ... = \mathbb{E}M_{N-1} = \mathbb{E}M_N.$$

M_0 isn't random, so $\mathbb{E}M_0 = M_0$, and thus $M_0 = \mathbb{E}M_n, n = 0, 1, ...N$ follows. □

The following theorem formalizes the process at the beginning of the section.

THEOREM 5.2. *Consider the general binomial model with* $0 < d < 1+r < u$. *Let the risk-neutral probabilities be given by*

$$\tilde{p} = \frac{(1+r)-d}{u-d}, \ \tilde{q} = \frac{u-(1+r)}{u-d}.$$

Then, under the risk-neutral measure, the discounted stock price is a martingale, i.e., equation (35) holds at every time n and for every sequence of coin tosses.

Proof. Using Theorem 5.1 and that $\frac{S_{n+1}}{S_n}$ only depends on the $(n+1)$st coin toss ($\frac{S_{n+1}}{S_n}$ takes the value u or d depending on whether the $(n+1)$st coin toss is a heads or tails),

$$\begin{aligned}
\tilde{\mathbb{E}}_n\Big[\frac{S_{n+1}}{(1+r)^{n+1}}\Big] &= \tilde{\mathbb{E}}_n\Big[\frac{S_n}{(1+r)^{n+1}} \cdot \frac{S_{n+1}}{S_n}\Big] \\
&= \frac{S_n}{(1+r)^n}\tilde{\mathbb{E}}_n\Big[\frac{1}{1+r} \cdot \frac{S_{n+1}}{S_n}\Big] \\
&\qquad \text{(Taking out what is known)} \\
&= \frac{S_n}{(1+r)^n} \cdot \frac{1}{1+r}\tilde{\mathbb{E}}\Big[\frac{S_{n+1}}{S_n}\Big] \\
&\qquad \text{(Linearity)} \\
&= \frac{S_n}{(1+r)^n}\frac{\tilde{p}u + \tilde{q}d}{1+r} \\
&\qquad \text{(by (32))} \\
&= \frac{S_n}{(1+r)^n}.
\end{aligned}$$

□

At the end of the binomial section, we discussed how under the risk-neutral probabilities, the average rate of growth of a portfolio consisting of assets in the stock and money markets equals the rate of growth of the money market account. So, as stated previously, the average rate of growth of an investors wealth is equal to the interest rate, r.
This result is explained in the following theorem, which states that the wealth process is also a martingale.

THEOREM 5.3. *Consider the binomial model with N periods.*
Let $\Delta_0, \Delta_1, ..., \Delta_{N-1}$ be an adapted portfolio (as mentioned previously), let X_0 be a real number, and let the wealth process $X_1, ..., X_N$ be generated recursively by (14), the wealth equation:

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n).$$

. Then the discounted wealth process $\frac{X_n}{(1+r)^n}, n = 0, 1, ..., N$, is a martingale under the risk-neutral measure; i.e,

$$\frac{X_n}{(1+r)^n} = \tilde{\mathbb{E}}_n\Big[\frac{X_{n+1}}{(1+r)^{n+1}}\Big], n = 0, 1, ..., N-1. \tag{39}$$

Proof.

$$\begin{aligned}
\tilde{\mathbb{E}}_n\Big[\frac{X_{n+1}}{(1+r)^{n+1}}\Big] &= \tilde{\mathbb{E}}_n\Big[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}} + \frac{X_n - \Delta_n S_n}{(1+r)^n}\Big] \\
&= \tilde{\mathbb{E}}_n\Big[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}}\Big] + \tilde{\mathbb{E}}_n\Big[\frac{X_n - \Delta_n S_n}{(1+r)^n}\Big] \\
&\qquad \text{(Linearity)} \\
&= \Delta_n \tilde{\mathbb{E}}_n\Big[\frac{S_{n+1}}{(1+r)^{n+1}}\Big] + \frac{X_n - \Delta_n S_n}{(1+r)^n} \\
&\qquad \text{(Taking out what is known)} \\
&= \Delta_n \frac{S_n}{(1+r)^n} + \frac{X_n - \Delta_n S_n}{(1+r)^n} \\
&\qquad \text{(Theorem 5.2)} \\
&= \frac{X_n}{(1+r)^n}.
\end{aligned}$$

□

COROLLARY 5.1. *Under the conditions of Theorem 5.3, we have*

$$\tilde{\mathbb{E}}\Big[\frac{X_n}{(1+r)^n}\Big] = X_0, n = 0, 1, ..., N. \tag{40}$$

Proof. The corollary follows from Remark 5.2, the expected value of a martingale cannot change with time and so must always be equal to the time zero value of the martingale, and then applying that fact to the $\tilde{\mathbb{P}}$-martingale $\frac{X_n}{(1+r)^n}, n = 0, 1, ..., N$. $\square$

So, under a risk-neutral measure, the discounted wealth process has constant expectation - it is impossible for it to begin at zero and later be strictly positive with positive probability unless it can also be strictly negative with positive probability.
Theorem 5.3 and its corollary have two important consequences, the first of which is the following proposition:

PROPOSITION 5.1. *There can be no arbitrage in the binomial model.*

Proof. Proof by contradiction. If there were an arbitrage, we could begin with $X_0 = 0$ and find a portfolio process whose corresponding wealth process $X_1, X_2, ..., X_N$ satisfied $X_N(\omega) \geq 0$ for all coin toss sequences ω and $X_N(\omega^*) > 0$ for at least one coin toss sequence ω^*. But then we would have $X_0 = 0$ and $\tilde{\mathbb{E}}\Big[\frac{X_N}{(1+r)^N}\Big] > 0$, which contradicts the corollary. $\square$

This leads to the following:

The First Fundamental Theorem of Asset Pricing
A model has a risk-neutral measure if and only if there is no arbitrage in the model.

The second consequence of Theorem 5.3 is the following version of the *risk-neutral pricing formula.* Let V_N be a random variable (derivative security payoff at time N) depending on the first N coin tosses. Based on Theorem 2.1, we know there is an initial wealth X_0 and a replicating portfolio process $\Delta_0, ..., \Delta_{N-1}$ that generates a wealth process $X_1, ..., X_N$ satisfying $X_N = V_N$, regardless of the coin tosses. Because $\frac{X_n}{(1+r)^n}, n = 0, 1, ..., N$, is a martingale, Remark 5.1 implies:

$$\frac{X_n}{(1+r)^n} = \tilde{\mathbb{E}}_n\Big[\frac{X_N}{(1+r)^N}\Big] = \tilde{\mathbb{E}}_n\Big[\frac{V_N}{(1+r)^N}\Big]. \tag{41}$$

From Definition 2.1, we defined the price of the derivative security at time n to be X_n and denote this price by the symbol V_n. Therefore equation (41) can be rewritten as:

$$\frac{V_n}{(1+r)^n} = \tilde{\mathbb{E}}\Big[\frac{V_N}{(1+r)^N}\Big] \tag{42}$$

or, equivalently,

$$V_n = \tilde{\mathbb{E}}\Big[\frac{V_N}{(1+r)^{N-n}}\Big] \tag{43}$$

This is summarized in the following theorem.

THEOREM 5.4 (Risk-neutral pricing formula)**.** *Consider an N-period binomial asset-pricing model with $0 < d < 1 + r < u$ and with risk-neutral probability measure $\tilde{\mathbb{P}}$. Let V_N be a random variable (the payoff of a derivative security at time N) depending on the coin tosses. Then, for n between 0 and N, the price of the derivative security at time n is given by the risk-neutral pricing formula (43). Furthermore, the discounted price of the derivative security is a martingale under $\tilde{\mathbb{P}}$; i.e,*

$$\frac{V_n}{(1+r)^n} = \tilde{\mathbb{E}}_n\Big[\frac{V_{n+1}}{(1+r)^{n+1}}\Big], n = 0, 1, ..., N-1. \tag{44}$$

The random variables V_n defined above are the same as the random variable V_n defined in Theorem 2.1.

Proof. Let V_N be the payoff at time N of a derivative, and define $V_{N-1}, ..., V_0$ as they are in Theorem 2.1. Then,

$$\begin{aligned}
\tilde{\mathbb{E}}_n\Big[\frac{V_{n+1}}{(1+r)^{n+1}}\Big](\omega_1\omega_2...\omega_n) &= \frac{1}{(1+r)^{n+1}}\tilde{\mathbb{E}}_n[V_{n+1}](\omega_1\omega_2...\omega_n) \\
&= \frac{1}{(1+r)^{n+1}}[\tilde{p}V_{n+1}(\omega_1..\omega_n H) + \tilde{q}V_{n+1}(\omega_1...\omega_n T)] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(Theorem 2.1)} \\
&= \frac{1}{(1+r)^{n+1}}\,(1+r)V_n(\omega_1\omega_2...\omega_n) \\
&= \frac{V_n}{(1+r)^n}(\omega_1\omega_2...\omega_n).
\end{aligned}$$

□

6 Conclusion

Through the use of the binomial asset-pricing model, we have explored the problem of pricing derivatives, namely options. Additionally, we went through the importance of the no-arbitrage assumption in determining the value of an option as well as the process of constructing a replicating portfolio. An introduction to the completeness of markets and the natural relating of option pricing with probability theory gives us further insights into no-arbitrage option pricing. We conclude with the discussion of martingales and a statement of the *First Fundamental Theorem of Asset Pricing.*

References

[1] Avellaneda, M. and Laurence, P. *Quantitative Modeling of Derivative Securities*. Chapman & Hall/CRC, Boca Raton, 2000.

[2] Billingsley, R. *Understanding Arbitrage: An Intuitive Approach to Financial Analysis*. FT Press, 2005.

[3] Etheridge, A. *A Course in Financial Calculus*. Cambridge University Press, Cambridge, 2002.

[4] Shreve, S. *Stochastic Calculus for Finance, Vol. I: The Binomial Asset Pricing Model*. Springer Finance. Springer, New York, 2004.