9-1-2009

Continuous Trace C^*-Algebras, Gauge Groups and Rationalization

John R. Klein
Wayne State University

Claude Schochet
Wayne State University, clsmath@gmail.com

Samuel B. Smith
Saint Joseph's University

CONTINUOUS TRACE C^*-ALGEBRAS, GAUGE GROUPS AND RATIONALIZATION

JOHN R. KLEIN, CLAUDE L. SCHOCHET, AND SAMUEL B. SMITH

Abstract. Let ζ be an n-dimensional complex matrix bundle over a compact metric space X and let A_ζ denote the C^*-algebra of sections of this bundle. We determine the rational homotopy type as an H-space of UA_ζ, the group of unitaries of A_ζ. The answer turns out to be independent of the bundle ζ and depends only upon n and the rational cohomology of X. We prove analogous results for the gauge group and the projective gauge group of a principal bundle over a compact metric space X.

Contents

1. Introduction

We analyze the rational homotopy theory of certain topological groups arising from bundles over a compact metric space X. Our results are motivated by the following situation. Let U_n be the unitary group of $n \times n$ matrices, and let PU_n be the group given by taking the quotient of U_n with its center. Let $\zeta \colon T \to X$ be a principal PU_n-bundle over X, let PU_n act on $M_n = M_n(\mathbb{C})$ by conjugation and let

$$T \times_{PU_n} M_n \to X$$

be the associated n-dimensional complex matrix bundle. Define A_ζ to be the set of continuous sections of the latter. These sections have natural pointwise addition, multiplication, and $*$-operations and give A_ζ the structure of a unital C^*-algebra. The algebra A_ζ is called an *n-homogeneous C^*-algebra* and is the most general

2000 *Mathematics Subject Classification.* 46J05, 46L85, 55P62, 54C35, 55P15, 55P45.

Key words and phrases. continuous trace C^*-algebra, section space, gauge group, projective gauge group, rational H-space, topological group, localization.

The first author is partially supported by the National Science Foundation.

unital *continuous trace* C^*-algebra as studied, for instance, in the book of Raeburn and Williams [16]. Let UA_ζ denote the topological group of unitaries of A_ζ. Our first main result describes the rational homotopy type of UA_ζ.

Recall that, from the point of view of homotopy theory, the simplest groups are the Eilenberg-Mac Lane spaces $K(\pi, n)$ with multiplication given by

$$K(\pi, n) \times K(\pi, n) \simeq K(\pi \times \pi, n) \xrightarrow{K(\text{multiply})} K(\pi, n).$$

Here π is an abelian group and the space $K(\pi, n)$ satisfies $\pi_i(K(\pi, n)) = \pi$ for $i = n$ and $\pi_i(K(\pi, n)) = 0$ for $i \neq n$. Only some of the constructions of a $K(\pi, n)$ yield a bona fide topological group, but all yield an H-space; that is, a space with continuous binary operation and two sided unit. However, this discrepancy is not hard to rectify: up to homotopy all of these H-space structures on Eilenberg-Mac Lane spaces lift to topological group structures in the sense that there is a topological group G and a homotopy equivalence to the given $K(\pi, n)$ which preserves the multiplication up to homotopy.

In fact, the H-space structure on a given Eilenberg-Mac Lane space is unique up to multiplicative equivalence and is *homotopy commutative.* A product $\prod_{j\geq 1} K(\pi_j, j)$ of Eilenberg-Mac Lane spaces also has a preferred H-space structure given by the product of the structures on the factors. This structure, which we refer to as the *standard multiplication,* is also homotopy commutative. However, in this case this structure may not be unique (See [4]).

Given a simply connected CW space X, Sullivan constructed a rationalization map $X \to X_{\mathbb{Q}}$ which has the property that the associated homomorphism on the higher homotopy groups is given by tensoring with the rational numbers ([21]; rationalization is a special case of a more general construction, localization, that can be made for any set of primes). Later, this theory was extended to include nilpotent spaces, i.e., spaces with non-trivial nilpotent fundamental group π having the property that the higher homotopy groups are nilpotent modules over π ([9], [2]).

It is well-known that topological groups are nilpotent spaces, so one can consider the rationalization map $G \to G_{\mathbb{Q}}$ for connected topological groups G (whose underlying space is a CW complex). Since localization commutes with finite products up to homotopy, it follows that $G_{\mathbb{Q}}$ has the structure of an H-space, and furthermore, the rationalization map is an H-map, i.e., it preserves multiplications up to homotopy. This motivates the following: let us call two H-spaces X and Y *rationally H-equivalent* if there is a homotopy equivalence $X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ which is a map of H-spaces.

To state our calculation of the rational homotopy groups of UA_ζ, we introduce some notation. Given $\mathbb{Z}$-graded vector spaces V and W, we grade the tensor product $V \otimes W$ by declaring that $v \otimes w$ has degree $|v| + |w|$. Here $|v|$ denotes the degree of the element $v \in V$. Let $V \widetilde{\otimes} W$ be the effect of considering only tensors with non-negative grading.

Given elements $x_1, \ldots, x_n$ each of homogeneous degree, write $\mathbb{Q}(x_1, \ldots, x_n)$ for the graded vector space with basis $x_1, \ldots, x_n$. Given a topological group G, write $G_\circ$ for the path component of the identity in G. Let $\check{H}^*(X; \mathbb{Q})$ denote the Čech cohomology of a space X with rational coefficients *graded nonpositively* so that $x \in \check{H}^n(X; \mathbb{Q})$ has degree $-n$.

Theorem A. *Let ζ be a principal PU_n bundle over a compact metric space X. Let A_ζ be the associated C^*-algebra, and let UA_ζ be its group of unitaries. Then the rationalization of $(UA_\zeta)_\circ$ is rationally H-equivalent to a product of rational Eilenberg-Mac Lane spaces with the standard multiplication, with degrees and dimensions corresponding to an isomorphism of graded vector spaces*

$$\pi_*\left((UA_\zeta)_\circ\right)\otimes\mathbb{Q} \;\cong\; \check{H}^*(X;\mathbb{Q})\,\widetilde{\otimes}\,\mathbb{Q}(s_1,\ldots,s_n)\,,$$

where the basis element s_i has degree $2i-1$.

Theorem A is a special case of more general calculations of the rational homotopy theory of gauge groups which we now describe. Write $F(X,Y)$ for the (function) space of all continuous maps from X to Y. When G is a topological group, the space $F(X,G)$ is one also with multiplication of functions taken pointwise. In this case, the identity component $F(X,G)_\circ$ is the space of freely nullhomotopic maps.

Theorem B. *Let X be a compact metric space and let G be a connected topological group having the homotopy type of a finite CW complex. Then*

$$\pi_*(F(X,G)_\circ)\otimes\mathbb{Q} \;\cong\; \check{H}^*(X;\mathbb{Q})\,\widetilde{\otimes}\,(\pi_*(G)\otimes\mathbb{Q})\;.$$

Furthermore, $F(X,G)_\circ$ is rationally H-equivalent to a product of Eilenberg-Mac Lane spaces with the standard multiplication, with degrees and dimensions corresponding to the displayed isomorphism.

When X is a finite complex, Theorem B is a consequence of results of Thom [22] and a basic localization result for components of $F(X,Y)$ due to Hilton, Mislin and Roitberg [9]. The result for X finite in this case is described in [12, §4]. Our advance here is the extension of this result to the case when X is compact metric. We deduce Theorem B from an extension of the Hilton-Mislin-Roitberg result to the case X compact metric (Theorem 7.1).

Addendum C. *In Theorem B, the calculation of rational homotopy groups holds for any connected, group-like H-space G. Furthermore, if G is rationally homotopy commutative, then $F(X,G)$ is rationally H-equivalent to a product of Eilenberg-Mac Lane spaces with the standard multiplication.*

The main results of this paper concern extending Theorem B to spaces of sections of certain bundles. Let G be a topological group and let

$$\zeta\colon T\to X$$

be a principal G-bundle. Following [1, §2], we form the associated *adjoint bundle*

$$\mathrm{Ad}(\zeta)\colon T\times_G G^{\mathrm{ad}}\to X$$

where G acts on $G^{\mathrm{ad}}=G$ by conjugation. The *gauge group* $\mathcal{G}(\zeta)$ of ζ is the space of sections of $\mathrm{Ad}(\zeta)$, with group structure defined by pointwise multiplication of sections. Alternatively, $\mathcal{G}(\zeta)$ is the group of G-equivariant bundle automorphisms of ζ that cover the identity map of X.

Theorem D. *Let G be a connected topological group having the homotopy type of a finite CW complex. Let ζ be a principal G-bundle over a compact metric space X. Then there is a rational H-equivalence*

$$\mathcal{G}(\zeta)_\circ \;\simeq_{\mathbb{Q}}\; F(X,G)_\circ\,.$$

Consequently, $\mathcal{G}(\zeta)_\circ$ is rationally homotopy commutative with rational homotopy groups given by the isomorphism appearing in Theorem B.

Again, when X is a finite CW complex this result admits a direct proof. In this case, a result of Gottlieb gives a multiplicative equivalence

$$\mathcal{G}(\zeta) \simeq \Omega_{h_\zeta} F(X, BG),$$

where the right side denotes the loop space of $F(X, BG)$ based at $h_\zeta : X \to BG$, the classifying map for ζ (see Corollary 9.2, [8, th. 1] and [1, prop. 2.4]). The equivalence $\mathcal{G}(\zeta) \simeq_{\mathbb{Q}} F(X, G)$ then follows from the Hilton-Mislin-Roitberg localization result for function spaces mentioned above and basic rational homotopy theory. (See Theorem 5.6 below.) The result in this case was recently, independently obtained by Félix and Oprea at the level of rational homotopy groups [7, th. 3.1]. Another related result here is due to Crabb and Sutherland, who prove the fibrewise rationalization of the bundle $\mathrm{Ad}(\zeta_G)$ is fibre homotopically trivial, where ζ_G is the universal G-bundle [3, prop. 2.2].

The following shows that the homotopy finiteness assumption on G in Theorem D can sometimes be dispensed with.

Addendum E. *Assume G is a topological group such that BG has the rational homotopy type of a group-like H-space. Then the conclusion of Theorem D holds for such G.*

For example, if G is a connected topological group satisfying rational Bott periodicity, then BG has the rational homotopy type of a group-like H-space.

Let

$$PG \ = \ G/Z(G)$$

denote the projectivization of G; i.e., the quotient of G by its center. As the center acts trivially on G^{ad}, one obtains an action of PG on G^{ad}. Given a principal PG-bundle

$$\zeta : T \to X,$$

form the associated *projective adjoint* bundle

$$\mathrm{Pad}(\zeta) : T \times_{PG} G^{\mathrm{ad}} \to X.$$

Define $\mathcal{P}(\zeta)$ to be the topological group of sections of the bundle $\mathrm{Pad}(\zeta)$ with pointwise multiplication again induced by G^{ad}. We call $\mathcal{P}(\zeta)$ the *projective gauge group* of ζ. In Example 3.7 below, we observe that $UA_\zeta \cong \mathcal{P}(\zeta)$ corresponds to the projective adjoint bundle of a principal PU_n-bundle. Theorem A is thus a special case of the following result.

Theorem F. *Let G be a compact connected Lie group. Let ζ be a principal PG-bundle over a compact metric space X. Then there is a rational H-equivalence*

$$\mathcal{P}(\zeta)_\circ \ \simeq_{\mathbb{Q}} \ F(X, G)_\circ \, .$$

Thus $\mathcal{P}(\zeta)_\circ$ is rationally homotopy commutative with rational homotopy groups again given by the isomorphism appearing in Theorem B.

Remark 1.1. Suppose that C is a separable C^*-algebra. Then its unitary group UC (with the usual modification for non-unital C) has the homotopy type of a countable CW complex. Thus so too does $U_\infty C = \varinjlim U_n C$, and the latter is an infinite loop space, by the Bott Periodicity Theorem of R. Wood [24]. Thus $U_\infty C$

satisfies the conditions on G in Addenda C and E. The same is true for UC itself if C is stable. So our results also apply to C^*-algebras constructed similarly to A_ζ but where the initial fibre $M_n(\mathbb{C})$ is replaced by an appropriate C^*-algebra C. We develop these ideas in a subsequent paper.

The paper is organized as follows. In Section 2, we establish our basic conventions for spaces, groups and bundles. In Section 3, we prove various foundational properties of section spaces. In Section 4, we discuss the rationalization of topological groups and the obstruction to homotopy commutativity. In Section 5, we prove preliminary versions of the main theorems for X a finite CW complex, as mentioned above.

By a result of Eilenberg and Steenrod [6], a compact metric space X may be expressed as the inverse limit $\varprojlim_j X_j$ of finite CW complexes. In Section 6, we use this result and the classical works of Dowker [5] and Spanier [19] to identify the homotopy groups of the function space $F(X,Y)$ in terms of the homotopy groups of the approximating function spaces $F(X_j,Y)$. This result is subsequently extended to section spaces. As a consequence, in Section 7, we extend the basic localization result of Hilton-Mislin-Roitberg [9, th. II.3.11] for function spaces from the case X finite CW to the case X compact, metric provided the function space component is a nilpotent space (Theorem 7.1). In Section 8 we deduce Theorems A-F by combining the finite complex case with the results of Section 6.

Acknowledgments. We thank Daniel Isaksen, Gregory Lupton, and J. Peter May for many helpful discussions. We are especially grateful to N. Christopher Phillips for vital assistance given to us. This paper is based in many ways upon our joint work [12].

2. Conventions

This paper brings together results from classical algebraic topology, which is most at home in the category of CW complexes, and functional analysis, which is most at home in the category of compact metric spaces. Many of our technical results deal with extending classical algebraic topology results from finite complexes to compact metric spaces via limit arguments.

We work in the category of compactly generated Hausdorff spaces. Whenever basepoints are required we assume that they are non-degenerate; that is, we assume that the inclusion of the basepoint into the space is a cofibration. If the space is a topological group then we take the identity of the group to be the basepoint. Following the discussion in [23, pp. 20-21], we give the function space $F(X,Y)$ the topology obtained by first taking the compact-open topology and then replacing this with the induced compactly generated topology. In particular, because we are retopologizing products, by a *topological group* we mean a topological group object in the category of compactly generated Hausdorff spaces.

Suppose $A \subset X$ is a subspace. Fixing a map $g\colon A \to Y$, we let $F(X,Y;g)$ denote the subspace of those maps $f\colon X \to Y$ such that f coincides with g on A. In particular, when A is a point, X and Y obtain the structure of based spaces and $F(X,Y;g)$ in this case is just the space of based maps. If $f \in F(X,Y;g)$ is a choice of basepoint, we let $F(X,Y;g)_{(f)}$ denote the path component of f.

An inclusion $A \subset X$ of spaces is a *cofibration* if it satisfies the homotopy extension property. A *Hurewicz fibration* is a map $p\colon E \to B$ satisfying the homotopy lifting problem for all (compactly generated) spaces. A *map of fibrations* $E \to E'$ over B is a map of spaces which commutes with projection to B. One says that p is *fibre homotopy trivial* if there is a space F and a map $q\colon E \to F$ such that $(p,q)\colon E \to B \times F$ is a homotopy equivalence. Given a map $f\colon Y \to B$ we write $f^*(p)\colon Y \times_B E \to Y$ for the pullback fibration (i.e., the fibre product).

An *H-space structure* on a based space X is a map $m\colon X \times X \to X$ whose restriction to $X \times *$ and $* \times X$ is homotopic to the identity as based maps, where $* \in X$ is the basepoint. If an H-space structure on X is understood, we call X an *H-space.* One says that X is *homotopy associative* if the maps $m \circ (m \times \mathrm{id})$ and $m \circ (\mathrm{id} \times m)$ are homotopic. A *homotopy inverse* for X is a map $\iota\colon X \to X$ such that the composites $m \circ (\iota \times \mathrm{id})$ and $m \circ (\mathrm{id} \times \iota)$ are homotopic to the identity. If X comes equipped with a homotopy associative multiplication and a homotopy inverse, then X is said to be *group-like.* If X is group-like then the set of path components $\pi_0(X)$ acquires a group structure.

Nilpotent Spaces. If $(X,*)$ is a based space then its higher homotopy groups $\pi_n(X;*)$ come equipped with an action of the fundamental group $\pi = \pi_1(X,*)$. If X is also a connected CW complex, then we say that X is *nilpotent* if π is a nilpotent group and also the action of π on the higher homotopy groups is nilpotent. The latter condition is equivalent to the statement that each $\pi_n(X;*)$ possesses a finite filtration of π-modules $M_n(i) \subset M_n(i+1) \subset \cdots$ such that the action on the associated graded $M_n(i+1)/M_n(i)$ is trivial. More generally, if X is any based connected space, then we will call X *nilpotent* if X has the homotopy type of a nilpotent CW complex. Topological groups having the homotopy type of a connected CW complex are nilpotent, since the action of π_1 in this case is trivial.

Rationalization. A finitely generated nilpotent group K admits a rationalization, which is a natural homomorphism $K \longrightarrow K_{\mathbb{Q}}$ ([9, §2]). The group $K_{\mathbb{Q}}$ has the property that the self map $x \mapsto x^n$ is a bijection for all integers $n \geq 1$ (i.e., $K_{\mathbb{Q}}$ is uniquely divisible). Furthermore, $K_{\mathbb{Q}}$ is the smallest group having this property in the sense that any homomorphism from K to a group with this property uniquely factors through $K_{\mathbb{Q}}$. When K is abelian, there is a natural isomorphism $K_{\mathbb{Q}} \cong K \otimes \mathbb{Q}$.

A connected based nilpotent space X admits a *rationalization.* This is a nilpotent space $X_{\mathbb{Q}}$ with rational homotopy (and homology) groups, together with a natural map $X \to X_{\mathbb{Q}}$ and a natural map $\ell_X : X \to X_{\mathbb{Q}}$ inducing rationalization on homotopy groups [9, thms. 3A, 3B]. Again, there is a universal property: if Y is a rational space (i.e., a nilpotent space whose homotopy groups are rational), and $f\colon X \to Y$ is a map, then one has a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X_{\mathbb{Q}} \arrow[d, "f_{\mathbb{Q}}"] \\ Y \arrow[r, "\simeq"'] & Y_{\mathbb{Q}}, \end{tikzcd}$$

where the bottom map is a homotopy equivalence since Y is rational. Consequently, f factors uniquely up to homotopy through $X_{\mathbb{Q}}$, and in particular the rationalization of X is uniquely determined up to homotopy equivalence. More generally, we call a map $f\colon X \to Y$ a *rationalization* if Y is rational and the induced map $f_{\mathbb{Q}}\colon X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is a homotopy equivalence. This is equivalent to demanding $f^*\colon [Y,Z] \to [X,Z]$ be an isomorphism of sets for all rational nilpotent spaces Z.

3. Section spaces

Suppose one is given a lifting problem, i.e., a diagram of spaces

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\cap"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dotted] & B \end{tikzcd}$$

such that $A \subset X$ is a cofibration and $E \to B$ is a fibration. Let us denote this lifting problem by $\mathcal{D}$.

Let

$$\Gamma(\mathcal{D})$$

be the space of solutions to the lifting problem, i.e., the space of maps $X \to E$ making the diagram commute. When f is the identity map and A is trivial, then one obtains the space of sections of p (it will be denoted by $\Gamma(p)$ in this instance).

Proposition 3.1. *Let $\mathcal{D}$ be the lifting problem above. Then one has a fibration*

$$F(X,E;g) \xrightarrow{\ p_*\ } F(X,B;p\circ g)$$

whose fibre at f is given by $\Gamma(\mathcal{D})$.

Proof. Here p_* is given by mapping a function $a\colon X \to E$ to $p\circ a\colon X \to B$. The map p_* is a fibration by the exponential law. The fibre over f is clearly $\Gamma(\mathcal{D})$. $\square$

CW structure.

Proposition 3.2. *With respect to the diagram $\mathcal{D}$ above, assume that X is a compact metric space and suppose E and B have the homotopy type of CW complexes. Then the section space $\Gamma(\mathcal{D})$ has the homotopy type of a CW complex.*

Proof. Restriction $f \mapsto f_{|A}$ defines a fibration

$$F(X,E) \to F(A,E)$$

in which both the domain and codomain have the homotopy type of a CW complex (by [14, th. 1]). Apply [18, prop. 3] (or [10, lem. 2.4]) to deduce that the fibre $F(X,E;g)$ of this fibration has the homotopy type of a CW complex. Repeating this argument with the fibration of Proposition 3.1 completes the proof. $\square$

Corollary 3.3. *Let X be a compact metric space and G a topological group having the homotopy type of a CW complex. Let ζ be a principal G-bundle (respectively, principal PG-bundle) over X. Then $\mathcal{G}(\zeta)$ (respectively, $\mathcal{P}(\zeta)$) has the homotopy type of a CW complex.*

Proof. We only prove the case of the adjoint bundle as the other case is proved similarly. The fibre bundle ζ is classified by a map $f\colon X \to BG$ by pulling back the universal principal G-bundle $EG \to BG$ along f. The space of solutions of the lifting problem

$$\begin{tikzcd} & EG \times_G G^{\mathrm{ad}} \arrow[d] \\ X \arrow[ur, dotted] \arrow[r, "f"'] & BG \end{tikzcd}$$

coincides with the section space $\Gamma(\mathrm{Ad}(\zeta))$. Furthermore $EG \times_G G^{\mathrm{ad}}$ and BG have the homotopy type of CW complexes because G does. The proof is completed by applying Proposition 3.2. □

Example 3.4. Suppose that A is a (separable) unital Banach algebra. Then the group of invertibles $GL(A)$ has the homotopy type of a (countable) CW complex. If A is a (separable) unital C^*-algebra then the group of unitaries UA has the homotopy type of a (countable) CW complex.

Here is a proof. As UA is a deformation retraction of $GL(A)$, they both have the same homotopy type. The group $GL(A)$ is an open subset of a Banach space, and any such open set has the homotopy type of a CW complex (cf. [11, cor. IV.5.5]). If A is separable then the open covering involved in the proof of [11, prop. IV.5.4] may be taken to be countable and then an obvious modification of [11, IV.5.5] implies that the CW complex constructed is countable.

Nilpotence.

Proposition 3.5. *With respect to the hypotheses of Proposition 3.2, assume additionally that X has the homotopy type of a CW complex and that E is a connected nilpotent space. Then each component of $\Gamma(\mathcal{D})$ is nilpotent.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} F(X,E;g) \arrow[r] \arrow[d] & F(X,B;p\circ g) \arrow[d] \\ F(X,E) \arrow[r] \arrow[d] & F(X,B) \arrow[d] \\ F(A,E) \arrow[r] & F(A,B) \end{tikzcd}$$

where the horizontal maps are all fibrations. By [9, th. 2.5], each component of $F(X,E)$ is nilpotent. It follows that each component of $F(X,E;g)$ is nilpotent by [9, th. 2.2] applied to the left column of the diagram. After restricting the fibration on the top line to connected components, it follows again by [9, th. 2.2] that each component of $\Gamma(\mathcal{D})$ is nilpotent, since the latter is a fibre by Proposition 3.1. □

Fibrewise Groups.

Definition 3.6. A fibration $p\colon E \to B$ is said to be a *fibrewise group* if it comes equipped with a map $m\colon E \times_B E \to E$, a map $i\colon E \to E$ and a section $e\colon B \to E$, all compatible with projection to B, such that

- m is associative,
- e is a two sided unit for m,
- i is an inverse for m and e

(In other words, (p, m, e) defines a group object in the category of spaces over B.)

If (p, m, e) is a fibrewise group, then the space of sections $\Gamma(p)$ comes equipped with the structure of a topological group with multiplication defined by pointwise multiplication of sections.

Here is a recipe for producing fibrewise groups. Suppose $\zeta : T \to X$ is a principal G-bundle and F is a topological group such that G acts on F through homomorphisms (this means that one has a homomorphism $G \to \mathrm{aut}(F)$, where $\mathrm{aut}(F)$ is the topological group consisting of topological group automorphisms of F). Then the associated fibre bundle

$$T \times_G F \to X$$

is easily observed to be a fibrewise group.

An important special case occurs when F is G^{ad}. We infer that $\mathrm{Ad}(\zeta)$ is a fibrewise group and so the gauge group $\mathcal{G}(\zeta)$ has the structure of a topological group. Similarly, when F is G^{ad} and we let PG act by conjugation, we infer that $\mathcal{P}(\zeta)$ has the structure of a topological group.

Example 3.7. Returning to the C^*-algebra setting, we now show that the group UA_ζ in the Introduction corresponds to a projective gauge group.

Let X be a compact space and let

$$\zeta : T \to X$$

be a principal PU_n-bundle over X with associated C^*-algebra A_ζ. Then there is an natural isomorphism of topological groups

$$UA_\zeta \;\cong\; \mathcal{P}(\zeta).$$

The proof is as follows: passing from M_n to the subspace U_n of unitaries in each fibre of ζ yields a bundle

$$U\zeta : T \times_{PU_n} U_n \to X\,.$$

The sections of this bundle are exactly UA_ζ but it is immediate that the bundle itself is the bundle $\mathrm{Pad}(\zeta)$.

Remark 3.8. Note that if $f : Y \to X$ is continuous then f induces a map of unital $C^\star$-algebras $f^* : A_\zeta \to A_{f^*\zeta}$ and this restricts to a homomorphism of unitary groups $U(A_\zeta) \longrightarrow U(A_{f^*(\zeta)})$. The naturality in the result above is with respect to these maps.

4. Rationalization of topological groups

Now suppose that G is a topological group having the homotopy type of a CW complex. As rationalization commutes with products only up to homotopy, the rationalization of the product structure gives a map

$$G_{\mathbb{Q}} \times G_{\mathbb{Q}} \to G_{\mathbb{Q}}$$

which may fail to be a group structure. It is, however, a group-like H-space.

The map $G \to G_{\mathbb{Q}}$ is a homomorphism of H-spaces in the sense that the diagram

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ \downarrow & & \downarrow \\ G_{\mathbb{Q}} \times G_{\mathbb{Q}} & \longrightarrow & G_{\mathbb{Q}} \end{array}$$

commutes up to homotopy (and the homomorphism is compatible with homotopy associativity).

Homotopy commutativity and rationalization.

Definition 4.1. A group-like H-space X is *homotopy commutative* if the commutator map $[\ ,\ ]: X \times X \to X$ is null homotopic. It is said to be *rationally homotopy commutative* if its rationalization $X_{\mathbb{Q}}$ is homotopy commutative.

The commutator map induces an operation on homotopy groups called the *Samelson product*. After tensoring with the rationals, one obtains a graded Lie algebra structure ([23, chap. X.5]).

Definition 4.2. A homomorphism $X \to Y$ of connected H-spaces of CW type is said to be *rational H-equivalence* if its rationalization $X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is a homotopy equivalence.

Proposition 4.3 (**Scheerer** [17, cor. 1])**.** *Let X and Y be connected group-like H-spaces having the homotopy type of a CW complex. Then there is a rational H-equivalence*

$$X_{\mathbb{Q}} \ \simeq\ Y_{\mathbb{Q}}$$

if and only if there is an isomorphism of Samelson Lie algebras

$$(\pi_*(X) \otimes \mathbb{Q}, [\ ,\ ]) \ \cong\ (\pi_*(Y) \otimes \mathbb{Q}, [\ ,\ ])\ .$$

We observe that the rationalization of such a space X has the homotopy type of a generalized Eilenberg Mac Lane space:

$$X_{\mathbb{Q}} \simeq \prod_{j \geq 1} K(\pi_j(X) \otimes \mathbb{Q}, j). \tag{1}$$

However, as pointed out in the introduction, the multiplication on $X_{\mathbb{Q}}$ need not correspond to the standard multiplication. We may detect when this identification is multiplicative in several ways.

Proposition 4.4 (**cf.** [12, th. 4.25])**.** *Let X be a homotopy associative H-space having the homotopy type of a connected CW complex. Then the following are equivalent:*

(a) *There is a homotopy equivalence*

$$X_{\mathbb{Q}} \ \xrightarrow{\sim}\ (\prod_{j \geq 1} K(\pi_j(X) \otimes \mathbb{Q}, j)$$

which is also an H-map, where the target has the standard multiplication.

(b) *The commutator map $X_{\mathbb{Q}} \times X_{\mathbb{Q}} \to X_{\mathbb{Q}}$ is null homotopic.*

(c) *The Samelson Lie algebra $(\pi_*(X) \otimes \mathbb{Q}, [\ ,\ ])$ is abelian; i.e., $[\ ,\ ] = 0$.*

Corollary 4.5. *Suppose G is a connected topological group such that BG has the rational homotopy type of a loop space, i.e. there is a based space Y and a rational homotopy equivalence $BG \simeq_{\mathbb{Q}} \Omega Y$. Then $G_{\mathbb{Q}}$ is a homotopy commutative H-space and is homotopy equivalent to a product of Eilenberg-Mac Lane spaces with standard multiplication.*

Example 4.6. A topological group G is said to satisfy *rational Bott periodicity* if there is rational homotopy equivalence $BG \simeq_{\mathbb{Q}} \Omega^j G$ for some $j > 0$. Any such G satisfies Corollary 4.5, and is consequently rationally homotopy commutative. In particular, the infinite unitary group U_∞ is rationally homotopy commutative (cf. Remark 1.1).

Example 4.7. Suppose that the topological group G is a direct limit $\varinjlim_n G_n$, where each G_n is rationally homotopy commutative. Then G is rationally homotopy commutative. This gives a second proof that U_∞ is rationally homotopy commutative (cf. Remark 1.1).

We now discuss the various hypotheses on the group G in our main results. Recall from Theorem D we require that G be a topological group of the homotopy type of a finite complex. We first observe that this class includes the connected Lie groups.

Lemma 4.8. *Every connected Lie group G has the homotopy type of a finite CW complex.*

Proof. By [23, th. A.1.2], G has a maximal compact subgroup K, unique up to conjugacy, such that the inclusion $K \subset G$ is a homotopy equivalence. Then K, being a compact Lie group, has the homotopy type of a finite CW complex. □

We will make use of the following results whose proofs are classical.

Proposition 4.9. *Suppose that G is a connected, topological group having the homotopy type of a finite CW complex. Then the following are true:*

(a) *The commutator map $G_{\mathbb{Q}} \times G_{\mathbb{Q}} \to G_{\mathbb{Q}}$ is null homotopic.*
(b) *$\pi_2(G)$ is a finite group.*
(c) *The classifying space BG has the rational homotopy type of a generalized Eilenberg-Mac Lane space and in particular is rationally homotopy equivalent to a loop space.*

Proof. The basic results of Milnor and Moore [15] on the structure of Hopf algebras of characteristic zero imply $H^*(G;\mathbb{Q})$ is an exterior algebra on a finite number of odd degree generators. It follows that $H^*(BG;\mathbb{Q})$ is a polynomial algebra on a finite number of generators of even degree. Represent each generator by a map $x_i\colon BG \to K(\mathbb{Q}, n_i)$. Then the product map $f = \prod x_i\colon BG \to \prod_i K(\mathbb{Q}, n_i)$ gives an isomorphism on rational homotopy groups. This proves (c). Applying the loop space functor to the map f gives (a) by Proposition 4.4.

To compute π_2 we may pass to universal covers and hence assume that the groups are simply connected. Then $H^*(G;\mathbb{Q}) = 0$ in degrees 1 and 2. This implies that $H_2(G;\mathbb{Z})$ is a finite group. The Hurewicz map $\pi_2(G) \to H_2(G;\mathbb{Z})$ is an isomorphism, so $\pi_2(G)$ is a finite group. □

Corollary 4.10. *If G is a connected topological group having the homotopy type of a finite CW complex, then $G_{\mathbb{Q}}$ is a homotopy commutative H-space. In particular, $G_{\mathbb{Q}}$ is homotopy equivalent as an H-space to a product of Eilenberg-Mac Lane spaces with standard multiplication.*

Finally, in Theorem F we restrict to the class of compact Lie groups G. This restriction is chosen to govern the rational homotopy theory of PG as we explain now. First we have the following general fact.

Lemma 4.11. *Let G be a connected Lie group. Then*

(a) *PG is a connected Lie group;*
(b) *the inclusion $Z(G) \to G$ induces a monomorphism*

$$\pi_1(Z(G)) \otimes \mathbb{Q} \to \pi_1(G) \otimes \mathbb{Q}.$$

Proof. PG is a Lie group and it is connected since it is a quotient space of G. By Proposition 4.9, $\pi_2(PG)$ is a finite group which implies the second statement. □

The preceding results imply in particular that G and $PG \times Z(G)$ have the same homotopy type after rationalization. Note, however, that there is no obvious map in either direction. We need to sharpen this identification for our proof of Theorem F. The following classical fact due to E. Cartan explains our restriction there to compact Lie groups.

Proposition 4.12. *Let G be a compact, connected Lie group. Then there is a compact, connected Lie group G_0 such that the following hold.*

(a) *There is a homomorphism $q\colon G_0 \to G$ which is a rational homotopy equivalence.*
(b) *There is a splitting $G_0 \cong P(G_0) \times Z(G_0)$*
(c) *The map q carries $Z(G_0)$ to $Z(G)$ and induces an isomorphism $P(G_0) \cong PG$.*

Proof. By [23, th. A.1.1], G has a finitely-sheeted covering group $q\colon G_0 \to G$ with $G_0 = T^\ell \times G'$, where G' is a simply connected compact Lie group with trivial center and T^ℓ is the product of ℓ-copies of S^1. The results follow directly. □

5. Preliminary results: finite complexes

In this section, we prove Theorems B, D and F when X is a finite CW complex. For Theorem B, the result is a direct consequence of classical work of Thom and a localization theorem for function spaces due to Hilton, Mislin and Roitberg. The proof of Theorem D makes use of Gottlieb's identity [8, th. 1] for the gauge group in addition to the previous ingredients. We deduce Theorem F from Theorem D and Proposition 4.12.

First, we have the famous result of H. Hopf, generalized by Thom [22]:

$$\pi_q(F(X, K(\pi, p))) \;=\; H^{p-q}(X; \pi)\,. \tag{2}$$

Next we have the results [9, th. II.3.11, cor. II.2.6] of Hilton-Mislin-Roitberg on the function space $F(X, Y)$:

Proposition 5.1 (**Hilton-Mislin-Roitberg** [9])**.** *Let X be a finite CW complex and Y be a connected nilpotent space. Then the induced map*

$$F(X, Y) \to F(X, Y_{\mathbb{Q}})$$

is a rationalization map on connected components.

(More precisely, for any map $h\colon X \to Y$, the map $F(X, Y)_{(h)} \to F(X, Y)_{(\ell_Y \circ h)}$ is a rationalization map.)

Proposition 5.2 (cf. [12, th. 4.28])**.** *Let G be a topological group having the homotopy type of a finite connected CW complex. Let X be a finite CW complex. Then the rationalization of $F(X, G)_\circ$ is a homotopy commutative H-space.*

Proof. Recall that $F(X,G)_\circ$ is the component of $F(X,G)$ containing the constant map. By Corollary 4.10, $G_{\mathbb{Q}}$ is homotopy commutative. It follows that $F(X,G_{\mathbb{Q}})_\circ$ is also homotopy commutative. The result now follows from observing that $F(X,G_{\mathbb{Q}})_\circ$ is the rationalization of $F(X,G)_\circ$ by Proposition 5.1. □

Remark 5.3. The preceding result holds in greater generality: for $F(X,G)$ to be rationally homotopy commutative, one only needs to assume that G is a rationally homotopy commutative H-space (see [12, th. 4.10]).

We can now prove Theorem B under the assumption that X is a finite complex as in [12, th. 4.28].

Theorem 5.4 (**Preliminary version of Theorem B**). *Let X be a finite CW complex and let G be a connected topological group having the homotopy type of a finite CW complex. Then*

$$\pi_*(F(X,G)_\circ)\otimes\mathbb{Q} \;\cong\; H^*(X;\mathbb{Q})\,\widetilde{\otimes}\,(\pi_*(G)\otimes\mathbb{Q})\ . \tag{3}$$

Furthermore, $F(X,G)_\circ$ is rationally H-equivalent to a product of Eilenberg-Mac Lane spaces with the standard loop multiplication, with degrees and dimensions corresponding to (3).

Proof. Using Proposition 5.1, we have

$$\pi_*(F(X,G)_\circ)\otimes\mathbb{Q}\cong\pi_*(F(X,G_{\mathbb{Q}})_\circ)$$

and we may compute the latter using the identification (1) and Thom's formula (2). The identification of the rational H-type of $F(X,G)_\circ$ is a direct consequence of Proposition 5.2. □

Remark 5.5. Since the identification (1) only requires a group-like H-space, the homotopy group calculation holds when G is group-like.

The assumption that G is homotopy finite is used only to conclude that $G_{\mathbb{Q}}$ is homotopy commutative via Proposition 5.2. Consequently, the second conclusion of Theorem 5.4 holds when G is a group-like H-space such that $G_{\mathbb{Q}}$ is homotopy commutative.

We next prove Theorem D for X a finite CW complex.

Theorem 5.6 (**Preliminary version of Theorem D**). *Let G be a topological group having the homotopy type of a finite connected CW complex. Let ζ be a principal G-bundle over a finite CW complex X. Then there is a rational H-equivalence*

$$\mathcal{G}(\zeta)_\circ \;\simeq_{\mathbb{Q}}\; F(X,G)_\circ\,.$$

Proof. By [8, th. 1] (see also Corollary 9.2 below), there is a homotopy equivalence of H-spaces

$$\mathcal{G}(\zeta)\;\simeq\;\Omega_h F(X,BG)\,,$$

where the space on the right is the based loop space of the function space $F(X,BG)$ with basepoint given by the classifying map $h\colon X\to BG$ for the bundle ζ. Recall that $F(X,BG)_{(h)}$ denotes the path component of $F(X,BG)$ containing h. By Proposition 5.1 above,

$$F(X,BG)_{(h)}\to F(X,(BG)_{\mathbb{Q}})_{(h')}$$

is a rationalization map, where h' is $\ell_{BG} \circ h$, where $\ell_{BG} \colon BG \to (BG)_{\mathbb{Q}}$ is the rationalization map. In particular, the displayed map is a rational homotopy equivalence.

By Corollary 4.10, $(BG)_{\mathbb{Q}}$ has the homotopy type of a loop space. It follows that all the components of $F(X, (BG)_{\mathbb{Q}})$ have the same homotopy type. Consequently, there is a rational homotopy equivalence

$$F(X, BG)_{(h)} \simeq_{\mathbb{Q}} F(X, (BG)_{\mathbb{Q}})_{\circ}\,.$$

The result now follows by taking the based loop space of both sides. □

Remark 5.7. It is clear from our proof that the finiteness assumption on G was used only to conclude that $(BG)_{\mathbb{Q}}$ has the homotopy type of a loop space. In fact, one sees that the above argument works, without the finiteness assumption on G, at the expense of assuming that $(BG)_{\mathbb{Q}}$ has the structure of a group-like H-space.

While the identity

$$\mathcal{G}(\zeta) \simeq \Omega_h F(X, BG)$$

does extend to more general spaces X (see Corollary 9.2)), the method of proof above is limited by Proposition 5.1. Both the nilpotence result [9, th. 2.5] and the localization result Proposition 5.1 for $F(X, Y)$ require X to be a finite CW complex. In Section 7, we extend the localization result to X compact metric assuming nilpotence. However, the nilpotence of the components of $F(X, Y)$ is not expected to hold for general X.

The last goal of this section is to prove Theorem F when X is a finite complex.

Theorem 5.8 (Preliminary version of Theorem F). *Let G be a compact connected Lie group, and let $\zeta \colon T \to X$ be a principal PG-bundle over a finite CW complex X. Then there is a rational homotopy equivalence of H-spaces*

$$\mathcal{P}(\zeta)_{\circ} \ \simeq_{\mathbb{Q}} \ F(X, G)_{\circ}.$$

Proof. Case 1. Suppose first that G splits as $Z(G) \times P(G)$. Then one has an isomorphism of bundles over X with total spaces

$$(T \times Z(G)) \times_G G^{\mathrm{ad}} \ \cong \ T \times_{PG} G^{\mathrm{ad}}\,.$$

The result follows now by Theorem 5.6 applied to the bundle on the left.

Case 2. This is the general case. By Proposition 4.12, there is a compact Lie group G_0 and a homomorphism $q \colon G_0 \to G$ which is a rational homotopy equivalence. Furthermore, this homomorphism induces an isomorphism $P(G_0) \cong P(G)$ and one also has a splitting $G_0 \cong Z(G_0) \times P(G_0)$.

By Proposition 5.12 below, the evident map

$$Q_0 := EPG \times_{PG} G_0^{\mathrm{ad}} \to EPG \times_{PG} G^{\mathrm{ad}} := Q$$

of fibrewise groups is a rational homotopy equivalence of nilpotent spaces. Let $h \colon X \to BPG$ classify the bundle with total space $T \times_{PG} G^{\mathrm{ad}}$; then h also classifies the bundle with total space $T \times_{PG} G_0^{\mathrm{ad}}$.

Denote the lifting problem

$$\begin{tikzcd} & Q \arrow[d] \\ X \arrow[r, "h"'] \arrow[ur, dotted] & BPG \end{tikzcd}$$

by $\mathcal{D}$. Then the space of lifts $\Gamma(\mathcal{D})$ is the projective gauge group $\mathcal{P}(\zeta)$. Denote the corresponding lifting problem with Q replaced by Q_0 by $\mathcal{D}_0$. Then the map

$$\Gamma(\mathcal{D}_0) \to \Gamma(\mathcal{D})$$

induced by the homomorphism $q\colon G_0 \to G$ is both a rational homotopy equivalence on components and a map of H-spaces (this is by a straightforward induction using the cell structure for X). Similarly, q induces a rational homotopy equivalence of H-spaces $F(X, G_0) \to F(X, G)$.

By Case 1, we also have a rational equivalence of H-spaces

$$\Gamma(\mathcal{D}_0)_\circ \simeq F(X, G_0)_\circ \,.$$

Assembling these three equivalences completes the proof. □

The remainder of this section is devoted to proving Proposition 5.12 used in the proof above. We need some preliminary lemmas.

Lemma 5.9. *Assume G is a compact connected Lie group. Let $E = EG \times_G G^{\mathrm{ad}}$ and $Q = EPG \times_{PG} G^{\mathrm{ad}}$. Then the map*

$$E \to Q$$

induces a surjection on homotopy groups in each degree.

Proof. There is a homotopy fibre sequence

$$BZ(G) \to E \to Q \,.$$

Taking the long exact homotopy sequence, we infer that $\pi_*(E) \to \pi_*(Q)$ is an isomorphism when $* \neq 3$ (here we are using the fact $Z(G)$ is a torus). Consequently, we have an exact sequence

$$0 \to \pi_3(E) \to \pi_3(Q) \to \mathbb{Z}^\ell \to \pi_2(E) \to \pi_2(Q) \to 0$$

where $\ell =$ rank of $Z(G)$.

We can calculate $\pi_3(E)$ using the long exact sequence of the fibration $E \to BG$; one sees (using the fact that $\pi_2(G) = 0$) that it is isomorphic to $\pi_3(G)$. Likewise, we see that $\pi_3(Q)$ is also isomorphic to $\pi_3(G)$ and the homomorphism $\pi_3(E) \to \pi_3(Q)$ is in fact an isomorphism. □

Lemma 5.10. *If $E \to B$ is a fibration of connected spaces having the homotopy type of a CW complex. Assume $\pi_*(E) \to \pi_*(B)$ is surjective in every degree and E is nilpotent. Then B is nilpotent.*

Proof. The quotient of a nilpotent group is again nilpotent, so $\pi_1(B)$ is nilpotent. Furthermore, when $k \geq 2$, we have a short exact sequence of $\pi_1(E)$ modules

$$0 \to \pi_k(F) \to \pi_k(E) \to \pi_k(B) \to 0$$

where F denotes the fibre at the basepoint. The nilpotency of the middle module guarantees that $\pi_k(B)$ is also a nilpotent $\pi_1(E)$-module (see [9, prop. 4.3]). This module structure arises from the homomorphism $\pi_1(E) \to \pi_1(B)$ by restriction. Since this homomorphism is surjective, it follows that $\pi_k(B)$ is a nilpotent $\pi_1(B)$-module. □

Lemma 5.11. *Let G be a compact connected Lie group. Then the space $Q = EP(G) \times_{P(G)} G^{\mathrm{ad}}$ is nilpotent.*

Proof. There is a homotopy fibre sequence

$$BZ(G) \to E \to Q\,,$$

where $E = EG \times_G G^{\mathrm{ad}}$. Then E is homotopy equivalent to LBG, the free loop space of BG (cf. Lemma 9.1). We infer that E is nilpotent by [9, th. 2.5]. Now apply the preceding lemmas. □

Proposition 5.12. *Let G be a compact connected Lie group and let $q\colon G_0 \to G$ be as in Proposition 4.12. Then the map of fibrewise groups*

$$Q_0 := EP(G) \times_{P(G)} G_0^{\mathrm{ad}} \to EP(G) \times_{P(G)} G^{\mathrm{ad}} =: Q$$

is a rational homotopy equivalence of nilpotent spaces.

Proof. Both Q and Q_0 are nilpotent by Lemma 5.11. By applying rationalization to the diagram

$$\begin{tikzcd} G_0^{\mathrm{ad}} \arrow[r] \arrow[d, "q"'] & Q_0 \arrow[r] \arrow[d] & BP(G) \arrow[d, equal] \\ G^{\mathrm{ad}} \arrow[r] & Q \arrow[r] & BP(G) \end{tikzcd}$$

whose rows are fibre sequences, and using the fact that rationalization preserves fibrations ([9, th. 3.12]) we infer that the map $Q_0 \to Q$ is a rational homotopy equivalence. □

6. Limits and function spaces

When X is a compact metric space, a classical result of Eilenberg and Steenrod [6, th. X.10.1] gives an inverse system of finite simplicial (CW) complexes X_j and compatible maps $h_j\colon X \to X_j$ such that the induced map

$$h\colon X \to \varprojlim_j X_j$$

is a homeomorphism. This result and its generalization are at the core of our method for passing from finite complexes to compact metric spaces.

In this and subsequent sections, we consider both direct and inverse limits. Suppose $\{X_j, p_{ij}\}$ is an inverse system of spaces, where $p_{ij}\colon X_i \to X_j$ are maps, $j \le i$. Given compatible maps $h_j\colon X \to X_j$, one has an induced map $h = \varprojlim_j h_j\colon X \to \varprojlim_j X_j$.

We record the following basic result.

Proposition 6.1 (**Eilenberg-Steenrod** [6, th. X.10.1, X.11.9])**.** *Let X be a compact Hausdorff space.*

(a) *There exists an inverse system of finite CW complexes $\{X_j, p_{ij}\}$ and compatible maps $h_j\colon X \to X_j$ inducing a homeomorphism*

$$h = \varprojlim_j h_j\colon X \to \varprojlim_j X_j\,.$$

(b) *Given a map $f\colon X \to Y$ in which Y is a CW complex, there exists an index m and a cellular map $f_m\colon X_m \to Y$ such that the composite*

$$X \xrightarrow{h_m} X_m \xrightarrow{f_m} Y$$

is homotopic to f.

Proposition 6.2. *Under the hypotheses of Proposition 6.1, the map of sets*

$$\varinjlim_j [X_j, Y] \to [X, Y]$$

is a bijection.

Proof. Surjectivity is a direct consequence of Proposition 6.1 (b). Injectivity is a consequence of Spanier's method of proof of [19, th. 13.4]. In Spanier's case, $Y = S^n$ and X has Lebesgue covering dimension at most $2n-2$ and his limit is taken in the category of abelian groups. However, Spanier remarks that the dimension condition can be dropped provided that the limit is taken in the category of sets [19, p. 228]. Furthermore, an inspection of his proof shows that it generalizes without change to Y an arbitrary finite simplicial complex. The argument is then completed by recalling that any finite CW complex has the homotopy type of a simplicial complex. □

We will need to extend this proposition to a certain class of pairs. Suppose now that (X, A) is a pair, where X is a compact Hausdorff space and $A \subset X$ is a closed cofibration. We assume that (X, A) is expressed as an inverse limit of pairs (X_j, A_j) where the latter is a finite CW pair. Such a decomposition exists by the relative version of [6, Ch. X, th. 10.1, 11.9]. As above, write $p_{ij} \colon X_i \to X_j$ for $j \le i$ and $h_j \colon X \to X_j$ for the structure maps. We use the same notation for the restrictions of these maps to A_j and to A, respectively. Let Y be a CW complex, and suppose that one is given a fixed map $g_m \colon A_m \to Y$ for some m and define $g_j \colon A_j \to Y$ for $j > m$ by $g_m \circ p_{jm}$. Define g to be the composite $g_m \circ h_m$. Let

$$[X, Y; g]$$

denote the set of homotopy classes of maps $X \to Y$ which coincide with g on the subspace A (where homotopies are required to be constant on A). Similarly, we have $[X_j, Y; g_j]$ and a map of sets

$$[X_j, Y; g_j] \to [X, Y; g]$$

(for $j \ge m$) which is compatible with the index j.

Lemma 6.3. *Assume there are compatible retractions $r_j \colon X_j \to A_j$ inducing a retraction $r \colon X \to A$. Then the map*

$$\varinjlim_j [X_j, Y; g_j] \to [X, Y; g]$$

is a bijection.

Proof. Let $i \colon A \to X$ and $i_j \colon A_j \to X_j$ be the inclusions, and let $u \colon [X, Y; g] \to [X, Y]$ and $u_j \colon [X_j, Y; g_j] \to [X, Y]$ be the evident maps. For each j, one has a commutative diagram of sets

$$\begin{tikzcd}
{[X_j, Y; g_j]} \arrow[r] \arrow[d, "u_j"'] & {[X, Y; g]} \arrow[d, "u"] \\
{[X_j, Y]} \arrow[r, "h_j^*"] \arrow[d, "i_j^*"'] & {[X, Y]} \arrow[d, "i^*"] \\
{[A_j, Y]} \arrow[r] & {[A, Y]}
\end{tikzcd}$$

where the bottom terms are pointed sets. Furthermore, if $r\colon X \to A$ is a retraction, then $g \circ r$ is a basepoint for $[X, Y]$ making i^* into a split surjection of based sets. The right column is in fact the tail-end of the long exact homotopy sequence of the fibration $F(X, Y) \to F(A, Y)$, which is also equipped with section. It follows from this observation that u is one-to-one. Similarly u_j is one-to-one.

Taking direct limits results in a diagram such that middle and bottom maps are isomorphisms. The rest of the argument follows from an elementary diagram chase, using the fact that u_j and u are one-to-one (we leave the details to the reader). □

Now, let $f_m\colon X_m \to Y$ be a fixed map and define $f_j\colon X_j \to Y$ for $j > m$ by $f_m \circ p_{jm}$. Define f to be the composite $f_m \circ h_m$. Then the map of function spaces $F(X_j, Y) \to F(X, Y)$ sends f_j to f, so we have a map of based spaces that is compatible with the inverse system.

Theorem 6.4. *The inverse system of based spaces above induces an isomorphism of groups*

$$\varinjlim_j \pi_n(F(X_j, Y); f_j) \;\cong\; \pi_n(F(X, Y); f)$$

in all degrees.

Proof. By [13, prop. IX.2], the limit of a direct system of (abelian) groups coincides with the limit taken in the category of sets.

Case 1. $n = 0$. This case is just a reformulation of Proposition 6.2.

Case 2. $n > 0$. Observe that

$$[X \times S^n, Y; f] \;=\; \pi_n(F(X, Y); f)\,,$$

where on the left we are taking homotopy classes of maps $X \times S^n \to Y$ which coincide with f on $X \times * = X$. Note that each inclusion $X_j \times * \subset X_j \times S^n$ is a retract, and these retractions are compatible. The result then follows from Lemma 6.3 with $X \times S^n$ in place of X, $X \times *$ in place of A, $X_j \times S^n$ in place of X_j and $X_j \times *$ in place of A_j. □

Limits and section spaces. Assume that $(X, A) = \varprojlim_j (X_j, A_j)$ as above, where each (X_j, A_j) is a finite CW pair. Suppose that for some index m one is given a lifting problem

$$\begin{tikzcd} A_m \arrow[r, "g_m"] \arrow[d, "\cap"'] & E \arrow[d, "p"] \\ X_m \arrow[r, "f_m"'] \arrow[ur, dotted] & B \end{tikzcd}$$

denoted $\mathcal{D}_m$. Here we assume that $p\colon E \to B$ is a fibration in which E and B have the homotopy type of CW complexes. Using the maps $(X_j, A_j) \to (X_m, A_m)$, we obtain another lifting problem, denoted $\mathcal{D}_j$. Then one has maps $\Gamma(\mathcal{D}_j) \to \Gamma(\mathcal{D}_{j+1})$ for $j \geq m$. Let $\tilde{f}_m\colon X_m \to E$ be any lift. Then we obtain basepoints $\tilde{f}_j \in \mathcal{D}_m$ for $j \geq m$.

Let $f\colon X \to B$ denote the composite of $h_m \circ f_m$ and similarly, let $g\colon A \to E$ be the composite $h_m \circ g_m$, where $h_m\colon (X, A) \to (X_m, A_m)$ is the structure map. Then we get a lifting problem $\mathcal{D}$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\cap"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dotted] & B\,. \end{tikzcd}$$

Let $\tilde{f}\colon X \to E$ be the basepoint of $\mathcal{D}$ determined by $\tilde{f}_m$.

Theorem 6.5. *The map of based sets*

$$\varinjlim_j \pi_n(\Gamma(\mathcal{D}_j); \tilde{f}_j) \to \pi_n(\Gamma(\mathcal{D}); \tilde{f})$$

is an isomorphism in every degree $n \geq 0$, where the direct limit is taken in the category of sets.

Proof. For each n, one has a map of long exact homotopy sequences

$$\begin{tikzcd} \cdots \arrow[r, "\partial"] & \pi_n(\Gamma(\mathcal{D}_j); \tilde{f}_j) \arrow[r, "a_j"] \arrow[d, "c"'] & \pi_n(F(X_j, E); g_j) \arrow[r, "b_j"] \arrow[d, "d"'] & \pi_n(F(X_j, B); f_j) \arrow[r] \arrow[d, "e"] & \cdots \\ \cdots \arrow[r, "\partial"'] & \pi_n(\Gamma(\mathcal{D}); \tilde{f}) \arrow[r, "a"'] & \pi_n(F(X, E); g) \arrow[r, "b"'] & \pi_n(F(X_j, B); f) \arrow[r] & \cdots \end{tikzcd}$$

as given by Proposition 3.1.

To prove surjectivity, let $x \in \pi_n(\Gamma(\mathcal{D}); \tilde{f})$ be any element. By Theorem 6.4, $a(x) = d(y)$ for some y, provided that j is sufficiently large. Then $b_j(y)$ is trivial provided j is large, again by 6.4. It follows that $y = a_j(z)$ for some z. Then $a(c(z) - x) = 0$, so $x = c(z) - \partial u$ for some u. If j is large, one has $u = e(u')$ for some u'. Consequently, $x = c(z - \partial u')$. This establishes surjectivity. A similar diagram chase, which we omit, gives injectivity. □

7. Localization of function spaces revisited

The purpose of this section is to extend the Hilton-Mislin-Roitberg localization result (Proposition 5.1) for function spaces $F(X, Y)$ to the case X compact metric and Y nilpotent CW provided the particular function space component is known, a priori, to be nilpotent.

Suppose that X is a compact metric space and $X = \varprojlim_j X_j$ as above, where each X_j is a finite CW complex. Let Y be a nilpotent space. Let $\ell_Y\colon Y \to Y_{\mathbb{Q}}$ be the rationalization map. Let $f\colon X \to Y$ be a fixed map and consider the connected component $F(X, Y)_{(f)}$ of the function space.

Theorem 7.1. *If $F(X, Y)_{(f)}$ is nilpotent, then the induced map*

$$F(X, Y)_{(f)} \to F(X, Y_{\mathbb{Q}})_{(\ell_Y \circ f)}$$

is a rationalization map.

Proof. By Proposition 6.1, we can assume without loss in generality that f factors as $X \to X_m \to Y$. Let $f_m\colon X_m \to Y$ denote the factorizing map, and define $f_j\colon X_j \to Y$ for $j > m$ to be the composite $p_{jm} \circ f_m$, where $p_{jm}\colon X_j \to X_m$ is the structure map in the inverse system. The approximation $X \cong \varprojlim_j X_j$ gives rise to a commutative diagram

$$\begin{tikzcd}
\varinjlim_j \pi_n(F(X_j,Y);f_j) \arrow[r, "\cong"] \arrow[d] & \pi_n(F(X,Y);f) \arrow[d] \\
\varinjlim_j \pi_n(F(X_j,Y_{\mathbb{Q}});\ell_Y\circ f_j) \arrow[r, "\cong"'] & \pi_n(F(X,Y_{\mathbb{Q}});\ell_Y\circ f)
\end{tikzcd}$$

where the horizontal maps are bijections by Theorem 6.4. Apply the rationalization functor to the diagram and use the fact that rationalization commutes with direct limits. This results in a commutative diagram

$$\begin{tikzcd}
\varinjlim_j \pi_n(F(X_j,Y);f_j)_{\mathbb{Q}} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \pi_n(F(X,Y);f)_{\mathbb{Q}} \arrow[d] \\
\varinjlim_j \pi_n(F(X_j,Y_{\mathbb{Q}});\ell_Y\circ f_j) \arrow[r, "\cong"'] & \pi_n(F(X,Y_{\mathbb{Q}});\ell_Y\circ f)
\end{tikzcd}$$

where the left vertical map is an isomorphism by Proposition 5.1. It follows that the right vertical map is an isomorphism as well. □

8. Proof of the main results

We are now in a position to prove the main theorems in their complete generality.

Proof of Theorem B. Recall we are assuming X is a compact metric space and G is a connected CW topological group having the homotopy type of a finite complex. We need to establish an isomorphism

$$\pi_*(F(X,G)_\circ)\otimes\mathbb{Q} \;\cong\; \check{H}^*(X,\mathbb{Q})\widetilde{\otimes}\big(\pi_*(G)\otimes\mathbb{Q}\big)\,.$$

By Theorem 5.4, the corresponding result holds for X_j a finite CW complex. Write $X = \varprojlim X_j$ as usual for finite complexes X_j. Then for each j we have a natural isomorphism

$$\pi_*(F(X_j,G)_\circ)\otimes\mathbb{Q} \;\cong\; H^*(X_j;\mathbb{Q})\widetilde{\otimes}\big(\pi_*(G)\otimes\mathbb{Q}\big)\,.$$

Take direct limits on both sides and use the fact that

$$\varinjlim(A_j\otimes B) \;\cong\; (\varinjlim A_j)\otimes B$$

for abelian groups to obtain the isomorphism

$$\big(\varinjlim \pi_*(F(X_j,G)_\circ)\big)\otimes\mathbb{Q} \;\cong\; \Big(\varinjlim H^*(X_j;\mathbb{Q})\Big)\widetilde{\otimes}\big(\pi_*(G)\otimes\mathbb{Q}\big)\,.$$

The continuity property of Čech cohomology [6, th. 12.1] implies that

$$\varinjlim H^*(X_j;\mathbb{Q}) \;\cong\; \check{H}^*(X;\mathbb{Q})\,.$$

Then use Theorem 6.4 to identify

$$\varinjlim \pi_*(F(X_j,G)_\circ) \;\cong\; \pi_*(F(X,G)_\circ)$$

which gives the result at the level of homotopy groups.

Finally, use Theorem 7.1 to obtain a homotopy equivalence of H-spaces

$$(F(X,G)_\circ)_{\mathbb{Q}} \;\simeq\; F(X,G_{\mathbb{Q}})_\circ\,.$$

The last part of Theorem B now follows from Propositions 5.2 and 4.4. □

Proof of Addendum C. As was observed in Remark 5.5, the homotopy group calculation of Theorem 5.4 holds when G is a group-like H-space, and the homotopy commutativity holds whenever G is rationally homotopy commutative. The above proof of Theorem B, which uses Theorem 5.4, therefore holds in the stated generality. □

We refocus on the case of the adjoint and projective adjoint bundles. As usual, let X be a compact metric space, and write $X = \varprojlim_j X_j$ for an inverse system of finite complexes X_j. Let

$$\zeta : T \to X$$

be the given principal G-bundle, where G is of CW type. Let $f : X \to BG$ be a classifying map for ζ. By Proposition 6.2, we can assume without loss in generality that f factors as

$$X \to X_m \xrightarrow{f_m} BG$$

for some index m. For $j > m$, define $f_j : X_j \to BG$ by taking the composite of f_m with the map $X_j \to X_m$. This defines a principal G-bundle $\zeta_j : T_j \to X_j$ for each $j \geq m$.

For each j we have a lifting problem

$$\begin{tikzcd} & EG \times_G G^{\mathrm{ad}} \arrow[d, "\mathrm{Ad}(\zeta)"] \\ X_j \arrow[ur, dotted] \arrow[r, "f_j"'] & BG \end{tikzcd}$$

whose space of sections is just the gauge group $\mathcal{G}(\zeta_j)$. Furthermore, one has a direct system of topological groups

$$\mathcal{G}(\zeta_m) \to \mathcal{G}(\zeta_{m+1}) \to \cdots$$

equipped with compatible homomorphisms $\mathcal{G}(\zeta_j) \to \mathcal{G}(\zeta)$. By Theorem 6.5, the homomorphism

$$\varinjlim_j \pi_n(\mathcal{G}(\zeta_j)_\circ) \to \pi_n(\mathcal{G}(\zeta)_\circ)$$

is an isomorphism for $n \geq 0$. A similar statement holds in the projective bundle case. Summarizing, we obtain the following description of the homotopy groups of the gauge group and of the projective gauge group.

Proposition 8.1. *Let X be a compact metric space and suppose $X = \varprojlim_j X_j$ for an inverse system of finite complexes X_j. Then, with notation as above,*

$$\pi_*(\mathcal{G}(\zeta)_\circ) \cong \varinjlim \pi_*(\mathcal{G}(\zeta_j)_\circ) \quad \textit{and} \quad \pi_*(\mathcal{P}(\zeta)_\circ) \cong \varinjlim \pi_*(\mathcal{P}(\zeta_j)_\circ).$$

After rationalization, these become isomorphisms of rational Samelson algebras.

Proof. The only thing we need to prove is the last statement. This follows because the map inducing the isomorphism in each case is induced from maps of H-spaces. They thus induce isomorphisms of rational Samelson Lie algebras. □

Proof of Theorem D. Combining Proposition 8.1 and the preliminary version of Theorem D for finite complexes (Theorem 5.6) with Theorem B one sees that $\mathcal{G}(\zeta)_\circ$ has rational homotopy groups given by Theorem B. Further, since a direct limit of abelian Lie algebras is abelian, we conclude $\mathcal{G}(\zeta)_\circ$ has abelian rational Samelson Lie algebra. This, in turn, implies there exists an H-equivalence $\mathcal{G}(\zeta)_\circ \simeq_{\mathbb{Q}} F(X,G)_\circ$ by Proposition 4.3. □

Proof of Theorem F. The proof is similar to the preceding one. In this case, one combines Proposition 8.1 and the preliminary version of Theorem F for finite complexes (Theorem 5.8) to get that $\mathcal{P}(\zeta)_\circ$ has rational homotopy groups given by Theorem B. The rest of the argument is as in the proof of Theorem D. □

Proof of Addendum E. See Remark 5.7. □

Proof of Theorem A. The proof is a direct consequence of Example 3.7, Theorem F for $G = U(n)$, and the well-known result

$$\pi_*(U(n)) \otimes \mathbb{Q} \;\cong\; \mathbb{Q}(s_1, \ldots, s_n)$$

where $|s_i| = 2i - 1$. □

9. Appendix: on the free loop space

In this section, we sketch a proof of "Gottlieb's identity" for the gauge group used in the proof of Theorem 5.6. While Gottlieb's original proof requires the base space X of the given principal G-bundle to be a finite CW complex, our proof requires only that the bundle be a pullback of the universal principal G-bundle.

Given a space X, let $LX = F(S^1, X)$ be its space of unbased loops. Evaluating loops at their basepoints gives a fibration $LX \to X$. For a topological group G of CW type, let $\xi\colon EG \to BG$ be the universal bundle, and let $\mathrm{Ad}(\xi)\colon EG \times_G G^{\mathrm{ad}} \to BG$ be the associated adjoint bundle. Then the following result is folklore.

Lemma 9.1. *Let G be any topological group of CW type. Then there is a fibrewise homotopy equivalence*

$$L(BG) \;\simeq\; EG \times_G G^{\mathrm{ad}}$$

of fibrewise H-spaces over BG.

Proof. Let $G \times G$ act on G^{ad} by the rule $(g,h)\cdot x = gxh^{-1}$. Then the restriction of this action to the image of the diagonal $\Delta : G \to G \times G$ coincides with the given action of G on G^{ad}. We have a pullback square

$$\begin{array}{ccc}
E(G\times G) \times_G G^{\mathrm{ad}} & \xrightarrow{E\Delta} & E(G\times G)\times_{(G\times G)} G^{\mathrm{ad}} \\
\big\downarrow & & \big\downarrow \\
BG = E(G\times G)/G & \xrightarrow[B\Delta]{} & B(G\times G)
\end{array}$$

in which the vertical maps are fibrations and the horizontal maps are induced by Δ. The space $E(G\times G)\times_{(G\times G)} G^{\mathrm{ad}}$ may be identified with BG. To show this, we first quotient out by the action of the left-hand copy of G in $G \times G$. Since this action is free, we obtain EG. Thus when we take the quotient by the right-hand copy of G we get $EG/G = BG$. It follows that $EG \times_G G^{\mathrm{ad}} = E(G\times G)\times_G G^{\mathrm{ad}}$ is identified with the homotopy pullback of the diagonal of BG with itself. But the latter coincides with the actual pullback of the diagram

$$\begin{tikzcd} & (BG)^I \arrow[d, "p"] \\ BG \arrow[r, "B\Delta"'] & BG \times BG \end{tikzcd}$$

where $(BG)^I = F(I, BG)$ is the free path space of BG, and p is the fibration which evaluates a path at its endpoints. This pullback identically coincides with $L(BG)$. □

Corollary 9.2 (**"Gottlieb's Identity"** [8, th. 1])**.** *Let G be any topological group of CW type. Let $\zeta : T \to X$ be a principal G-bundle induced from the universal principal G-bundle by a map $h_\zeta : X \to BG$. Then there is a homotopy equivalence of H-spaces*

$$\Gamma(\mathrm{Ad}(\zeta)) \;\simeq\; \Omega_{h_\zeta} F(X, BG)\,,$$

where the right side denotes the based loop space of $F(X, BG)$ with loops based at h_ζ.

Proof. $\Gamma(\mathrm{Ad}(\zeta))$ coincides with the space of solutions to the lifting problem

$$\begin{tikzcd} & EG \times_G G^{\mathrm{ad}} \arrow[d] \\ X \arrow[ur, dotted] \arrow[r, "h_\zeta"'] & BG\,. \end{tikzcd}$$

Denote this lifting problem by $\mathcal{D}$. Using Lemma 9.1, we see that $\Gamma(\mathcal{D})$ is homotopy equivalent as an H-space to the space of lifts

$$\begin{tikzcd} & L(BG) \arrow[d] \\ X \arrow[ur, dotted] \arrow[r, "h_\zeta"'] & BG\,. \end{tikzcd}$$

An unraveling of definitions shows that the latter is the space of maps $X \times S^1 \to BG$ whose restriction to $X \times *$ coincides with h_ζ. But this is identical to the space $\Omega_{h_\zeta} F(X, BG)$ by means of the exponential law. □

References

[1] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), 523–615.

[2] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions, and localizations*, Lecture Notes in Mathematics **304**, Springer, 1972.

[3] M. C. Crabb and W. A. Sutherland, *Counting homotopy types of gauge groups*, Proc. London. Math. Soc. (3) **81** (2000), 747–768.

[4] C. J. Curjel, *On the H-space structures of finite complexes*, Comment. Math. Helv. **43** (1968), 1–17.

[5] C. H. Dowker, *Mapping theorems for non-compact spaces*, Amer. J. Math. **69** (1947), 200–242.

[6] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton University Press, Princeton, 1952.

[7] Y. Félix and J. Oprea, *Rational homotopy of gauge groups,* Proc. Amer. Math. Soc. **137** (2009), no. 4, 1519–1527.
[8] D. H. Gottlieb, *On fibre spaces and the evaluation map*, Ann. Math. **87** (1968), 42–55.
[9] P. Hilton, G. Mislin, and J. Roitberg, *Localization of Nilpotent Groups and Spaces*, North-Holland Publishing Co., Amsterdam, 1975, North-Holland Mathematics Studies, No. 15, Notas de Matemática, No. 55 [Notes on Mathematics, No. 55].
[10] P. J. Kahn, *Some function spaces of CW type*, Proc. Amer. Math. Soc. **90**, 599–607.
[11] A. Lundell and S. Weingram, *The Topology of CW Complexes*, University Series in Higher Mathematics, Van Nostrand Reinhold Co., New York, 1969.
[12] G. Lupton, N. C. Phillips, C. L. Schochet, S.B. Smith, *Banach algebras and rational homotopy theory*, Trans. Amer. Math. Soc. **361** (2009), 267–295.
[13] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics vol. 5, Springer, 1998.
[14] J. W. Milnor, *On spaces having the homotopy type of CW-complex*, Trans. Amer. Math. Soc. **90** (1959), 272–280.
[15] J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of. Math. **81** (1965), 211–264.
[16] I. Raeburn and D. P. Williams, *Morita Equivalence and Continuous-Trace C^*-Algebras*, Math. Surveys and Monographs Vol. 60, Amer. Math. Soc., 1998.
[17] H. Scheerer, *On rationalized H- and co-H-spaces. With an appendix on decomposable H- and co-H-spaces*, Manuscripta Math. **51** (1985), 63–87.
[18] R. Schön. *Fibrations over a CWh-base*, Proc. Amer. Math. Soc. **6**2 (1977), 165–166
[19] E. Spanier, *Borsuk's cohomotopy groups*, Ann. Math. **50** (1949), 203–245.
[20] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, Princeton, 1951.
[21] D. P. Sullivan, *Geometric Topology: Localization, Periodicity and Galois Symmetry. The 1970 MIT notes, K*-Monographs in Mathematics, vol. 8. Springer, Dordrecht, 2005.
[22] R. Thom, *L'homologie des espaces fonctionnels*, Colloque de topologie algébrique, Louvain, 1956, Georges Thone, Liège, 1957, pp. 29–39.
[23] G. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics, vol. 61, Springer, 1978.
[24] R. Wood, *Banach algebras and Bott periodicity*, Topology **4**, (1966), 371–389.

Department of Mathematics, Wayne State University, Detroit MI 48202
E-mail address: `klein@math.wayne.edu`

Department of Mathematics, Wayne State University, Detroit MI 48202
E-mail address: `claude@math.wayne.edu`

Department of Mathematics, Saint Joseph's University, Philadelphia PA 19131
E-mail address: `smith@sju.edu`