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Natural Superconvergent Points of Triangular Finite Elements

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Abstract. In this work, we analytically identify natural superconvergent points of function values and gradients for triangular elements. Both the Poisson equation and the Laplace equation are discussed for polynomial finite element spaces (with degrees up to 8) under four different mesh patterns. Our results verify computer findings of [2], especially, we confirm that the computed data have 9 digits of accuracy with an exception of one pair (which has 8-7 digits of accuracy). In addition, we demonstrate that the function value superconvergent points predicted by the symmetry theory [14] are the only superconvergent points for the Poisson equation. Finally, we provide function value superconvergent points for the Laplace equation, which are not reported elsewhere in the literature.

Key Words. Finite element method, natural superconvergence, triangular elements.

AMS Subject Classification. 65N30, 65N15

1. INTRODUCTION

Natural superconvergent points are special points where the convergence of numerical approximations exceeds the possible global rate without any post-processing. The investigation regarding the finite element superconvergence has a long history since 70's [6]. For the literature, the reader is referred to books [3, 4, 5, 9, 12, 17, 20] and references therein.

In mid-90's, Babuška et al. developed a “computer-based proof” [2] that systematically predicted derivative superconvergent points for the Laplace equation, the Poisson equation, and linear elasticity equations. They considered four mesh patterns of triangular elements and three families of rectangular elements of degree \( n \), \( 1 \leq n \leq 7 \). Their investigation reduced the problem of finding superconvergent points to the problem of finding intersections of certain polynomial contours. The actual superconvergent points were located by computer programs without explicitly constructing those polynomials, and 10 digits were provided in their reported data [2, 3]. Later, Zhang proposed an analytic approach which constructs explicitly the needed polynomials through an orthogonal decomposition under local rectangular and brick meshes [18, 19]. His result confirmed that ten digits reported by computer findings are correct up to rounding with only one exception (8-accurate digits).

A parallel analytic approach for triangular meshes are much more involved and tedious, which will be the main object of the current investigation. We consider the Laplace and Poisson equations on the four triangular mesh patterns used in the computer-based proof. By a special orthogonal decomposition, we explicitly construct those polynomials from which the superconvergent points are located. Our results verify that the computed data

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for triangular elements in [2] have 9 digits of accuracy except one pair (with 8-7 accurate digits). In addition, we report for the first time, superconvergent points for function values of the Laplace equation.

Another systematic way to find superconvergent points in mid-90's is the symmetry theory developed by Schatz et al. [14]. This theory predicts that superconvergence occurs at local mesh symmetry points for a large class of 2nd-order elliptic problems of any dimension. For odd order elements (linear, cubic, etc.), superconvergence happens to derivatives: and for even order elements (quadratic and so on), it is for function values. A by-product of our current work is to confirm that for the Poisson equation under triangular meshes, the mesh symmetry points are "almost" all superconvergent points.

An outline of this paper is as follows: Section 2 contains the main theorems which will reduce the problem of finding superconvergent points to that of finding intersections of certain polynomial contours. In Section 3, superconvergent points of function values and gradients of both the Poisson equation and the Laplace equation are provided for four patterns of triangular meshes. The main idea is illustrated in the beginning of Section 3.1 for the regular pattern. The detailed superconvergent points are provided in Sections 3.1-3.4, for four mesh patterns, respectively.

2. THEORETICAL SETTING

We shall outline the process by Babuška et al. [2] in finding derivative superconvergent points. Here we follow the description provided by Wahlbin [17].

The main hypotheses in [2] are: (a) there is no roundoff error; (b) the mesh is locally translation invariant; (c) the solution is sufficiently smooth locally and the pollution error is under control. Throughout this paper, we assume hypotheses (b) and (c). However, the hypothesis (a) is no longer needed, since the explicit expressions of involved polynomials are provided. Another advantage of this method is that it is easily repeated.

Let \( \Omega \in \mathbb{R}^2 \) be a bounded domain. Denote in general a square centered at \( x = (x_1, x_2) \) of side \( 2h \) as \( c(x, h) = \{ y = (y_1, y_2) \in \Omega \mid |y_i - x_i| \leq h, \ i = 1, 2 \} \). Consider the local natural superconvergent points near \( x^0 \in \Omega \). Let \( \Omega_1 = c(x^0, H) \) and \( \Omega_0 = c(x^0, 2H) \) be two squares in \( \Omega \) with \( H = h^\delta, \ 0 < \delta < 1 \), such that the \( 2h \)-periodic extensions of the master cell fit them exactly. Assume that a finite element approximation \( u_h \in S_h(\Omega_0) \) of \( u \), the solution of a Poisson equation, satisfies

\[
D(u - u_h, v) = 0, \quad \forall v \in S_h^{comp}(\Omega_0),
\]

where \( D(w, v) = \int \nabla w \cdot \nabla v, \ S_h^{comp}(\Omega_0) \) is the finite element subspace which has compact support on \( \Omega_0 \). We shall assume that

\[
\|u - u_h\|_{L^\infty(\Omega_0)} \leq Ch^{n+1-L},
\]

with \( L \geq 0 \) and \( L + \delta < 1 \). This assumption implies that pollution effects from outside of the domain \( \Omega_0 \) have been properly controlled and the error loss is of order \( h^L \). Moreover, we may assume that

\[
\|u - u_h\|_{W^{-1}_\infty(\Omega_0)} \leq Ch^{n+2-\Lambda},
\]

with \( \Lambda \geq 0 \) and \( \Lambda + \delta < 1 \).
Under various conditions given in [15], we have

**Lemma 2.1.** Let \( u \) and \( u_h \) satisfy (2.1). Then for each \( s \geq 0 \) and \( 1 \leq q \leq \infty \) there exists a constant \( C \) independent of \( u, u_h, h, H, \) and \( x^0 \) such that

\[
\|u - u_h\|_{W^2_q(\Omega_1)} \leq C \min_{v \in S_h} (|u - v|_{W^2_q(\Omega_1)} + H^{-1}\|u - v\|_{L^\infty(\Omega_0)})
+ CH^{-1-s-2/q}\|u - u_h\|_{W^{-s}_q(\Omega_0)}.
\]

The corresponding result for the error in function values for \( u - u_h \) is also found in [15].

**Lemma 2.2.** Let \( u \) and \( u_h \) satisfy (2.1). Then for each \( s \geq 0 \) and \( 1 \leq q \leq \infty \) there exists a constant \( C \) such that

\[
\|u - u_h\|_{L^\infty(\Omega_1)} \leq C \left( \frac{H}{h} \right)^{\tilde{n}} \min_{v \in S_h} \|u - v\|_{L^\infty(\Omega_0)} + CH^{s-2/q}\|u - u_h\|_{W^{-s}_q(\Omega_0)}.
\]

Here \( \tilde{n} = 1 \) if \( n = 1 \), \( \tilde{n} = 0 \) otherwise.

Let \( Q \) be the \((n+1)\)th order Taylor expansion of \( u \) at \( x^0 \). Then

\[
\|u - Q\|_{W^2_q(\Omega_0)} \leq CH^{n+2-s}, \quad \text{for} \quad 0 \leq s \leq n + 2.
\]

Interpolate it into \( S_h(\Omega_0) \) to form \( I_h Q \). Then set \( \rho = Q - I_h Q \). The key observation in [2] is that \( \rho \) is \( 2h \)-periodic. Let \( S_h^{\pi}(c(x^0, h)) \) denote the \( 2h \)-periodic functions in \( S_h(c(x^0, h)) \), and define \( PP(\rho) \in S_h^{\pi}(c(x^0, h)) \) by

\[
\int_{c(x^0, h)} (\rho - PP(\rho)) = 0; \quad DC_{c(x^0, h)}(\rho - PP(\rho), v) = 0, \quad \forall v \in S_h^{\pi}(c(x^0, h)).
\]

Denote \( H^{1,\pi}(\Omega_0) \) the \( 2h \)-periodic functions in \( H^1(\Omega_0) \). Then from [2, 17], \( \rho \in H^{1,\pi}(\Omega_0) \).

The following lemma is also found in [2, 17].

**Lemma 2.3.** For all \( \varphi \in H^{1,\pi}(\Omega_0) \), we have

\[
D(\varphi - PP(\varphi), v) = 0, \quad \forall v \in S_h^{\text{comp}}(\Omega_0).
\]

Now put \( \psi = \rho - PP(\rho) \), by Lemma 2.1, we have [2, 17]

**Theorem 2.1.**

\[
\frac{\partial}{\partial x_i}(u - u_h)(x) = \frac{\partial \psi}{\partial x_i}(x) + R_i(x), \quad i = 1, 2, \quad x \in \Omega_1,
\]

where

\[
\|R_i\|_{L^\infty(\Omega_1)} \leq C (h^{n+\delta} + h^{n+1-L-\delta}),
\]

provided \( L + \delta < 1 \).

**Remark 2.1.** Theorem 2.1 states that the major part of the finite element approximation error in the derivatives can be measured by \( \frac{\partial \psi}{\partial x_i}(x) \), since the remainder is of an order \( \min(\delta, 1-L-\delta) \) higher than the global convergence rate. □

In this work, we establish an analogue of Theorem 2.1 for the error in function values.
Theorem 2.2.

\[(u - u_h)(x) = \psi(x) + R_u(x), \quad x \in \Omega_1,\]

where

\[\|R_u\|_{L_\infty(\Omega_1)} \leq C \left( \ln \frac{1}{h} \right)^\frac{n}{2} h^{n+1+\delta} + Ch^{n+2-\Lambda-\delta},\]

provided \(\Lambda + \delta < 1\).

Proof. Let \(N_h Q\) be the Neumann projection of \(Q\) into \(S_h(\Omega_0)\), i.e.

\[\int_{\Omega_0} (Q - N_h Q) = 0; \quad D(Q - N_h Q, v) = 0, \quad \forall v \in S_h(\Omega_0). \tag{2.6}\]

Then we can write

\[u - u_h = (Q - N_h Q) + [(u - Q) - (u_h - N_h Q)]. \tag{2.7}\]

We denote \(R_Q = [(u - Q) - (u_h - N_h Q)]\).

From (2.1) and (2.6) we get

\[D(R_Q, v) = 0, \quad \forall v \in S_h^{\text{comp}}(\Omega_0).\]

By Lemma 2.2,

\[\|R_Q\|_{L_\infty(\Omega_1)} \leq C \left( \ln \frac{1}{h} \right)^\frac{n}{2} \min_{v \in S_h} \|(u - Q) - v\|_{L_\infty(\Omega_0)} + CH^{-1}\|R_Q\|_{W_{\infty}^{-1}(\Omega_0)}. \tag{2.8}\]

Letting \(v\) be the interpolation of \(u - Q\) into \(S_h(\Omega_0)\). From assumption (2.4),

\[\|(u - Q) - v\|_{L_\infty(\Omega_0)} \leq Ch^{n+1}\|u - Q\|_{W_{\infty}^{-1}(\Omega_0)} \leq Ch^{n+1}H. \tag{2.9}\]

By assumption (2.3)

\[\|R_Q\|_{W_{\infty}^{-1}(\Omega_0)} \leq \|u - u_h\|_{W_{\infty}^{-1}(\Omega_0)} + \|Q - N_h Q\|_{W_{\infty}^{-1}(\Omega_0)} \leq Ch^{n+2-\Lambda}. \tag{2.10}\]

Hence (2.8) - (2.10) give

\[\|R_Q\|_{L_\infty(\Omega_1)} \leq C \left( \ln \frac{1}{h} \right)^\frac{n}{2} h^{n+1}H + CH^{-1}h^{n+2-\Lambda}. \tag{2.11}\]

Recall that \(\rho = Q - I_h Q\) and \(\psi = \rho - PP(\rho)\). Rewrite

\[Q - N_h Q = \psi + [Q - N_h Q - \psi].\]

Notice that

\[Q - N_h Q - \psi = I_h Q + PP(\rho) - N_h Q \in S_h(\Omega_0). \tag{2.12}\]

From Lemma 2.3 and (2.6), we get

\[D(Q - N_h Q - \psi, v) = 0, \quad \forall v \in S_h^{\text{comp}}(\Omega_0).\]
From Lemma 2.2 and (2.12),
\[ \| Q - N_h Q - \psi \|_{L_\infty(\Omega_1)} \leq C H^{-1} \| Q - N_h Q - \psi \|_{W_\infty^{-1}(\Omega_0)}. \]  
(2.13)

By assumption (2.3)
\[ \| Q - N_h Q \|_{W_\infty^{-1}(\Omega_0)} \leq C h^{n+2-\Lambda}. \]  
(2.14)

By a duality argument,
\[ \| \psi \|_{W_\infty^{-1}(\Omega_0)} \leq \| \rho \|_{W_\infty^{-1}(\Omega_0)} \leq C h^{n+2} \| Q \|_{W_\infty^{n+1}(\Omega_0)} \leq C h^{n+2-\Lambda}. \]  
(2.15)

Therefore, from (2.13) - (2.15),
\[ \| Q - N_h Q - \psi \|_{L_\infty(\Omega_1)} \leq C H^{-1} h^{n+2-\Lambda}. \]  
(2.16)

Finally, we set \( R_u = (u - u_h) - \psi \). From (2.7), (2.11), and (2.16), we get
\[ \| R_u \|_{L_\infty(\Omega_1)} \leq C \left( \ln \frac{1}{h} \right)^n h^{n+1} H + C H^{-1} h^{n+2-\Lambda} \]
\[ \leq C \left( \ln \frac{1}{h} \right)^n h^{n+1+\delta} + C h^{n+2-\Lambda-\delta}, \]  
(2.17)

and the theorem follows. \( \square \)

Remark 2.2. Theorem 2.2 indicates that \( \psi \) gives the main part of the error in function values. The convergence rate of the remainder is of an order \( \min(\delta, 1 - \Lambda - \delta) \) higher than the global rate. \( \square \)

Remark 2.3. By Remarks 2.1, 2.2, the task of finding superconvergent points can be narrowed down to a master cell, or equivalently to the reference cell \( K = [-1, 1]^2 \). And the superconvergent points of derivatives and function values are those points \( x \) where \( \frac{\partial \psi}{\partial x}(x) = 0 \) and \( \psi(x) = 0 \), respectively. Thus, the task of identifying superconvergent points is equivalent to finding the critical points of some periodic polynomials \( \psi \) of degree \( n + 1 \) on the reference cell \( K \) such that \( \psi \notin V_n(K) \), and
\[ \int_K \psi = 0; \quad \int_K \nabla \psi \cdot \nabla v = 0, \forall v \in V_n^v(K), \]
where \( V_n(K) \) and \( V_n^v(K) \) are the images of \( S_h(c(x^0, h)) \) and \( S_h^v(c(x^0, h)) \), respectively. In another word, \( V_n(K) \) and \( V_n^v(K) \) are the finite element local space and the periodic finite element local space on the reference cell \( K \), respectively. \( \square \)

3. SUPERCONVERGENT POINTS FOR PERIODIC MESHES OF TRIANGLES

For periodic uniform triangular local mesh, we consider four patterns: Regular pattern, Chevron pattern, Union Jack pattern, and Criss-Cross pattern (see Figure 1). These patterns were discussed by Babuška, et al. in [2].

Clearly, a mesh in any one of these four patterns is a local translation invariant. Hence, the assumption (b) in Section 2 is satisfied. In addition, we require the assumption (c) from now on.
In the following, we study the superconvergent points of function values and derivatives for the Poisson and Laplace equations. Their solutions are approximated by finite element spaces with polynomials and harmonic polynomials, respectively.

For each pattern, the finite element local space \( V_n(\hat{K}) \) is the space of continuous piecewise polynomials on \( \hat{K} \); the periodic finite element local space \( V^*_n(\hat{K}) \) is the space of periodic continuous piecewise polynomials on \( \hat{K} \). \( V_n(\hat{K}) \) and \( V^*_n(\hat{K}) \) are both subspaces of \( C^0(\hat{K}) \). Structures of \( V_n(\hat{K}) \) and \( V^*_n(\hat{K}) \) associated with different mesh patterns are different.

We shall show the study of the regular pattern in details, and present only the main results for the other three patterns.

For finite elements with various mesh patterns and degrees, we locate the superconvergent points for solutions of (i) the Poisson equation; and (ii) the Laplace equation. Based on results in Section 2, we set \( u \) to be \( Q \) for (i) the class of general polynomials \( x^{n+1}, x^n y, \ldots, y^{n+1} \) (for the Poisson equation); and (ii) the class of harmonic polynomials \( Re(z^{n+1}) \) and \( Im(z^{n+1}) \) (for the Laplace equation).

In the context, \( \square^R, \square^C, \square^UJ \), and \( \square^{CC} \) will be used to denote the object \( \square \) defined in the regular, Chevron, Union Jack, and the Criss-Cross patterns, respectively.

3.1. REGULAR PATTERN

3.1.1. Preliminaries and Theorems

Set the reference cell \( \hat{K} = [-1,1]^2 \). Partition \( \hat{K} \) into two triangular elements and denote them as \( \hat{T}_1 = \{(x,y) \in \hat{K} | x \geq y\} \), \( \hat{T}_2 = \{(x,y) \in \hat{K} | x \leq y\} \) (see Figure 2). In the following, we denote \( \hat{T}_i \) the \( i \)th element, \( n_i \) the \( i \)th node, and \( i_{ij} \) the side connecting \( n_i \) and \( n_j \).

Define \( P_n^w(\hat{K}) \) the space of continuous piecewise polynomials of degree not greater than \( n \) on \( \hat{K} \). That is, for any \( f \in P_n^w(\hat{K}) \), \( f \) is continuous on \( \hat{K} \); \( f|_{\hat{T}_1}, f|_{\hat{T}_2} \) are polynomials of degree (\( \leq n \)). Let \( PP_n^w(\hat{K}) \) be the space of periodic continuous piecewise polynomials of degree not greater than \( n \) on \( \hat{K} \). In other words, if \( f \in PP_n^w(\hat{K}) \), then \( f \in P_n^w(\hat{K}) \), and \( f(x,1) = f(x,-1) \), \( f(1,y) = f(-1,y) \). Denote \( P_n(\hat{K}) \) the space of polynomials of degree not greater than \( n \) on \( \hat{K} \).

From the definitions, we conclude that, in general, \( P_n(\hat{K}) \subset P_n^w(\hat{K}) \) and \( PP_n^w(\hat{K}) \subset P_n^w(\hat{K}) \). Moreover, the finite element local space \( V_n(\hat{K}) \) is \( P_n^w(\hat{K}) \), and the periodic finite element local space \( V^*_n(\hat{K}) \) is \( PP_n^w(\hat{K}) \). We shall use these two sets of notations alternatively. However, \( V_n \) and \( V^*_n \) are preferred when we consider finite element approximation.

Define \( \Phi_{n+1}(\hat{K}) \) the subspace of \( PP_n^{w+1}(\hat{K}) \) that consists of functions \( \psi \), which can be
decomposed into $P_{n+1}(\hat{K})$ and $P^w_n(\hat{K})$, such that

$$\int_{\hat{K}} \psi = 0; \quad \int_{\hat{K}} \nabla \psi \cdot \nabla v = 0, \quad \forall v \in PP^w_n(\hat{K}).$$

(3.1)

In other words, for any $\psi \in \Phi_{n+1}(\hat{K})$, we have $\psi = \chi + r$ satisfying (3.1), where $\chi \in P_{n+1}(\hat{K})$ and $r \in P^w_n(\hat{K})$.

**Lemma 3.1.** $\dim \Phi_{n+1}(\hat{K}) = n + 2$.

**Proof.** Suppose $\psi_i \in \Phi_{n+1}(\hat{K})$ with $\psi_i = \chi_i + r_i$, where $\chi_i \in P_{n+1}(\hat{K})$ and $r_i \in P^w_n(\hat{K})$, $i = 1, 2$. Set $\delta = \psi_1 - \psi_2$. Clearly, $\delta$ is periodic. Also $\delta = r_1 - r_2$, which implies $\delta \in P^w_n(\hat{K})$. Thus, $\delta \in PP^w_n(\hat{K})$. From (3.1), we conclude that

$$\int_{\hat{K}} \delta = 0; \quad \int_{\hat{K}} \nabla \delta \cdot \nabla v = 0, \quad \forall v \in PP^w_n(\hat{K}).$$

But this happens only when $\delta \equiv 0$. Therefore, the dimension of $\Phi_{n+1}(\hat{K})$ is the same as the dimension of the space of monomials of degree $n + 1$, which is $n + 2$. $\square$

From Theorems 2.1, 2.2, and Remark 2.3, we get

**Theorem 3.1.** (i) Function value superconvergent points of $V_n(\hat{K})$ for the Poisson equation are the intersections of the contours

$$\{ \psi = 0 \mid \psi \in \Phi_{n+1}(\hat{K}) \}.$$

(ii) Derivative superconvergent points of $V_n(\hat{K})$ along the $x$-direction for the Poisson equation are the intersections of the contours

$$\{ \frac{\partial \psi}{\partial x} = 0 \mid \psi \in \Phi_{n+1}(\hat{K}) \}.$$

Similar result holds on the $y$-direction.

Applying Theorems 2.1 and 2.2 to the case of harmonic functions (solutions of the Laplace equation) yields the following theorem.
Theorem 3.2. (i) Function value superconvergent points of $V_n(\hat{K})$ for the Laplace equation are the intersections of the contours

$$\psi_{n+1}^{Re} = 0 \quad \text{and} \quad \psi_{n+1}^{Im} = 0,$$

with

$$\psi_{n+1}^{Re} = Re(z^{n+1}) - r_n, \quad \psi_{n+1}^{Im} = Im(z^{n+1}) - s_n,$$

where

$$\psi_{n+1}^{Re}, \psi_{n+1}^{Im} \in \Phi_{n+1}(\hat{K}); \quad r_n, s_n \in V_n(\hat{K}); \quad z = x + iy.$$

(ii) Derivative superconvergent points of $V_n(\hat{K})$ along the x-direction for the Laplace equation are the intersections of the contours

$$\frac{\partial \psi_{n+1}^{Re}}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \psi_{n+1}^{Im}}{\partial x} = 0.$$

Similar result holds on the y-direction.

Proof. Let $u$ be any harmonic function. The $(n + 1)^{th}$ order Taylor expansion of $u$ at the center of $\hat{K}$ is of the form $\alpha Re(z^{n+1}) + \beta Im(z^{n+1}) + t_n$, where $\alpha, \beta \in \mathbb{R}$ and $t_n \in P_n(\hat{K})$. Therefore, the results follow from Theorems 2.1 and 2.2. \(\Box\)

Remark 3.1. Theorems 3.1 and 3.2 indicate that to locate superconvergent points of function values and derivatives for the Poisson and Laplace equations, we need to specify spaces $\Phi_{n+1}(\hat{K})$. We next determine a basis for each $\Phi_{n+1}(\hat{K})$ via an orthogonal decomposition of $PP_n(\hat{K})$. \(\Box\)

3.1.2. Orthogonal Decomposition of $PP_n(\hat{K})$ and Construction of $\Phi_{n+1}(\hat{K})$

In current work, a set of hierarchic basis functions are used for $V_n(\hat{K})$. The basis functions can be organized into three categories: nodal shape functions, side modes and internal modes (see [16, Chapter 6]). There are 4 nodal shape functions, which are denoted as $\nu_i, \ i = 1, \ldots, 4$.

$$\nu_1 = \begin{cases} \frac{1}{2} (1 - x) & \text{in } \hat{T}_1, \\ \frac{1}{2} (1 - y) & \text{in } \hat{T}_2; \end{cases} \ \nu_2 = \begin{cases} \frac{1}{2} (x - y) & \text{in } \hat{T}_1, \\ 0 & \text{in } \hat{T}_2; \end{cases}$$

$$\nu_3 = \begin{cases} \frac{1}{2} (1 + y) & \text{in } \hat{T}_1, \\ \frac{1}{2} (1 + x) & \text{in } \hat{T}_2; \end{cases} \ \nu_4 = \begin{cases} 0 & \text{in } \hat{T}_1, \\ \frac{1}{2} (y - x) & \text{in } \hat{T}_2. \end{cases}$$

Clearly, we have

$$\nu_1(-x, -y) = \nu_3(x, y), \quad \text{and} \quad \nu_2(-x, -y) = \nu_4(x, y).$$

Side modes and internal modes are constructed from the nodal shape functions in the following means. Let $p_k$ be the Legendre polynomial of degree $k$ on $[-1, 1]$. Define

$$\phi_0(x) = 1, \quad \phi_1(x) = x,$$

$$\phi_k(x) = \int_{-1}^{x} p_{k-1}(t) dt, \quad k = 2, 3, \ldots$$

(3.4)
For $k \geq 2$, since $\varphi_k(x)$ are polynomials which vanish at $x = \pm 1$, the term $1 - x^2$ can be factored out. We define $\varphi_k(x)$ so that

$$\varphi_k(x) = \frac{1}{4}(1 - x^2)\varphi_{k-2}(x), \quad k = 2, 3, \ldots$$

(3.5)

The subscripts of $\phi_k$ and $\varphi_k$ indicate the orders (degrees) of the polynomials.

Denote $\zeta^1_{k \ell}$ the $k$th order side mode along the side $i_\ell$. Define the side modes associated with side $i_1$ as

$$\zeta^1_{k \ell} = \nu_1 \nu_2 \varphi_{k-2}(\nu_2 - \nu_1), \quad k = 2, \ldots, n.$$  

(3.6)

The other side modes are defined analogously. There are $n - 1$ side modes on each side.

Denote $\iota^1_{k \ell}$, the $j$th internal mode on $\tilde{T}_1$ of order $k$. In the first element $\tilde{T}_1$, the internal modes are defined as

$$\iota^1_{k \ell} = \nu_1 \nu_2 \nu_3 x^{j-1}y^{k-j-2}, \quad k = 3, \ldots, n; \quad j = 1, \ldots, k - 2.$$  

(3.7)

The definition of the internal modes in $\tilde{T}_2$ is similar. There are $(n - 1)(n - 2)/2$ internal modes in each element.

Notice that $V_n(\hat{K}) = P_n^w(\hat{K})$, the dimension of $P_n^w(\hat{K})$ can be decided from that of $V_n(\hat{K})$. Sum up the numbers of nodal shape functions, side modes and internal modes, we have

$$\dim P_n^w(\hat{K}) = 4 + 5(n - 1) + 2 \frac{(n-1)(n-2)}{2} = (n+1)^2.$$

(3.8)

A basis for $PP_n^w(\hat{K})$ may be constructed from the hierarchic basis functions of $P_n^w(\hat{K})$. In fact, the sum of the four nodal shape functions $\nu_1 + \nu_2 + \nu_3 + \nu_4$ is a periodic basis function. The sums of the same order side modes along the opposite boundary sides are periodic basis functions. The side modes along the interior sides and all of the internal modes are automatically periodic. To simplify notations, in the context, we denote $\zeta^d = \zeta^1_{k \ell}$, $\zeta^h = \zeta^1_{k \ell} + \zeta^3_{k \ell}$, $\zeta^v = \zeta^1_{k \ell} + \zeta^4_{k \ell}$, which represent the $k$th order diagonal, horizontal, and vertical periodic side modes. Also denote $\iota_{k \ell}^{\pm} = \iota_{k \ell}^1 \pm \iota_{k \ell}^2$, which are referred as $k$th order internal modes of plus/minus type.

An example of the construction of $PP_n^w(\hat{K})$ is given for case $n = 3$ (see FIGURE 3).

$$\nu = \nu_1 + \nu_2 + \nu_3 + \nu_4 \equiv 1,$$

$$\zeta^h = \zeta^1_{23} + \zeta^3_{23}, \quad \zeta^v = \zeta^2_{23} + \zeta^4_{23}, \quad \zeta^d = \zeta^1_{23},$$

$$\zeta^h = \zeta^1_{32} + \zeta^3_{32}, \quad \zeta^v = \zeta^2_{32} + \zeta^4_{32}, \quad \zeta^d = \zeta^3_{32},$$

$$\iota^1_{3,1}, \quad \iota^2_{3,1}.$$  

(3.9)

The dimension of $PP_n^w(\hat{K})$ can be determined by deleting 3 nodal freedoms and $2(n - 1)$ side mode freedoms from $\dim P_n^w(\hat{K})$. Hence,

$$\dim PP_n^w(\hat{K}) = \dim P_n^w(\hat{K}) - 3 - 2(n-1) = n^2.$$

(3.10)

Lemma 3.2. (i) $\phi_{k+1}(-x) = (-1)^{k+1}\phi_{k+1}(x)$ for $k = 1, 2, \ldots$;

(ii) $\varphi_k(-x) = (-1)^k \varphi_k(x)$ for $k = 0, 1, \ldots$;
(iii) $c_t^k(-x, -y) = (-1)^k c_t^k(x, y)$ for $t = h, v, d,$ and $k = 2, \ldots, n$;
(iv) $i_{k,j}^1(-x, -y) = (-1)^{k-1} i_{k,j}^2(x, y)$ for $k = 3, \ldots, n,$ and $j = 1, \ldots, k - 2$.
Moreover, $i_{k,j}^t(-x, -y) = (-1)^{k-1} i_{k,j}^t(x, y),$ and $i_{k,j}^k(-x, -y) = (-1)^k i_{k,j}^k(x, y)$.

Proof. (i) Since $\phi_{k+1}(1) = 0$ and $p_k(-x) = (-1)^k p_k(x)$ for any $k$, the assertion follows from definition (3.4).
(ii) By (3.5) and result of part (i), the desired result follows.
(iii) This is a consequence of definition (3.6), properties (3.3), and result of part (ii).
(iv) Definition (3.7) and properties (3.3) give (iv). □

Lemma 3.2 states that the periodic basis functions are either even or odd.

Further, we consider the orthogonal decomposition of $PP_n^w(K)$ under the Laplace operator. Towards this end, we define

$$\Psi_{n+1}(K) = \{ u \in PP_{n+1}^w(K) \mid \int_{\Omega_1 \cup \Omega_2} \nabla u \cdot \nabla v = 0 , \forall v \in PP_n^w(K) \}. \quad (3.11)$$

Compare (3.11) with (3.1). Clearly, $\Phi_{n+1}(K) \subset \Psi_{n+1}(K)$, in general.

By the Gram-Schmidt process, we can decompose $PP_n^w(K)$ into

$$PP_n^w(K) = PP_0^w(K) \oplus \Psi_2(K) \oplus \cdots \oplus \Psi_{n-1}(K) \oplus \Psi_n(K). \quad (3.12)$$

Note that $PP_{n}^w(K) = \text{Span}\{1\}$ and $\Psi_1(K) = \{0\}$. The dimension of $\Psi_{n+1}(K)$ can be determined as

$$\dim \Psi_{n+1}(K) = \dim PP_{n+1}^w(K) - \dim PP_n^w(K) = 2n + 1, \quad n = 1, 2, 3, \ldots \quad (3.13)$$

The first two spaces of $\Psi_{n+1}(K)$ ($n = 1, 2$) can be expressed as

$$\Psi_2(K) = \text{Span}\{ \phi_2(x), c_2^v, \phi_2(y) \};$$
$$\Psi_3(K) = \text{Span}\{ \phi_3(x), \pm c_3^g + 18 c_3^{d,1} + 2 c_3^2 + c_3^h + c_3^v + c_3^d, \}$$
$$16 c_3^{d,1} + 4 c_3^{d,1} + c_3^2 + c_3^v + c_3^d, \phi_3(y) \}. $$
The space \( \Phi_{n+1}(\hat{K}) \) can be constructed from \( \Psi_{n+1}(\hat{K}) \). For instance, for \( n = 1, 2 \), we get

\[
\Phi_2(\hat{K}) = \text{Span} \{ \phi_2(x) + \frac{1}{3}, \phi_2(y) + \frac{1}{3} \};
\]

\[
\Phi_3(\hat{K}) = \text{Span} \{ \phi_3(x), \phi_2(x)y + \zeta_2^{12} - \zeta_3^{43}, x\phi_2(y) - \zeta_2^{23} + \zeta_4^{14}, \phi_3(y) \}.
\]

By the Gram-Schmidt process, we can obtain more spaces of \( \psi_{n+1}(\hat{K}) \), and thus more spaces of \( \Phi_{n+1}(\hat{K}) \). Here, we give two more \( \Phi_{n+1}(\hat{K}) \) (for \( n = 3, 4 \)) without the associated \( \Psi_{n+1}(\hat{K}) \).

\[
\Phi_4(\hat{K}) = \text{Span} \{ \phi_4(x) + r_3^0, \phi_3(x)y + \zeta_3^{12} - \zeta_3^{43} + r_3^3, \phi_2(x)\phi_2(y) + r_3^2, x\phi_3(y) - \zeta_3^{23} + \zeta_4^{14} + r_3^3, \phi_4(y) + r_3^3 \},
\]

where

\[
\begin{align*}
r_3^0 &= r_3^3 = \frac{1}{14} + \frac{3}{14} (\zeta_3^{2} + \zeta_3^{2}),
&+ \frac{3}{14} \zeta_3^{2} + \frac{15}{14} \zeta_3^{1},
\end{align*}
\]

\[
\begin{align*}
r_3^3 &= r_3^1 = \frac{1}{14} + \frac{3}{14} (\zeta_3^{2} + \zeta_3^{2}) - \frac{1}{14} \zeta_3^{2} - \frac{3}{14} \zeta_3^{1},
\end{align*}
\]

\[
\begin{align*}
r_3^3 &= r_3^2 = \frac{1}{18} + \frac{2}{18} (\zeta_3^{2} + \zeta_3^{2}) - \frac{1}{6} \zeta_3^{2} - \frac{1}{6} \zeta_3^{3},
\end{align*}
\]

\[
\begin{align*}
\Phi_5(\hat{K}) = \text{Span} \{ \phi_5(x) + r_4^0, \phi_4(x)y + \zeta_4^{12} - \zeta_4^{43} + r_4^1, \phi_3(x)\phi_2(y) + r_4^2, \phi_2(x)\phi_3(y) + r_4^3, x\phi_4(y) - \zeta_4^{23} + \zeta_5^{14} + r_4^4, \phi_5(y) + r_4^5 \},
\]

where

\[
\begin{align*}
r_4^0 &= \frac{5}{24} \zeta_3^{3} + \frac{7}{24} \zeta_3^{6} - \frac{5}{36} \zeta_3^{2} - \frac{3}{36} \zeta_3^{3,1} + \frac{245}{72} \zeta_4^{1,1} + \frac{35}{72} \zeta_4^{2},
\end{align*}
\]

\[
\begin{align*}
r_4^1 &= \frac{55}{60} \zeta_3^{3} + \frac{25}{60} \zeta_3^{6} + \frac{13}{60} \zeta_3^{2} + \frac{52}{60} \zeta_3^{3,1} - \frac{235}{72} \zeta_4^{1,1} + \frac{125}{72} \zeta_4^{2},
\end{align*}
\]

\[
\begin{align*}
r_4^2 &= \frac{5}{12} \zeta_3^{3} + \frac{1}{12} \zeta_3^{6} - \frac{20}{36} \zeta_3^{2} - \frac{5}{12} \zeta_3^{3,1} - \frac{43}{18} \zeta_3^{1,1} + \frac{5}{18} \zeta_3^{2,1},
\end{align*}
\]

\[
\begin{align*}
r_4^3 &= \frac{5}{12} \zeta_3^{3} + \frac{1}{12} \zeta_3^{6} - \frac{20}{36} \zeta_3^{2} - \frac{5}{12} \zeta_3^{3,1} - \frac{43}{18} \zeta_3^{1,1} + \frac{5}{18} \zeta_3^{2,1},
\end{align*}
\]

\[
\begin{align*}
r_4^4 &= \frac{25}{60} \zeta_3^{3} + \frac{55}{60} \zeta_3^{6} + \frac{13}{60} \zeta_3^{2} - \frac{52}{60} \zeta_3^{3,1} + \frac{125}{72} \zeta_3^{1,1} - \frac{235}{72} \zeta_3^{2,1},
\end{align*}
\]

\[
\begin{align*}
r_4^5 &= \frac{5}{72} \zeta_3^{3} + \frac{5}{72} \zeta_3^{6} - \frac{5}{36} \zeta_3^{2} + \frac{35}{72} \zeta_3^{3,1} + \frac{35}{72} \zeta_3^{2,1} + \frac{245}{72} \zeta_3^{2,1}.
\end{align*}
\]

For information of cases \( n = 5, \ldots, 8 \), the reader is referred to [13].

We shall study the structures of functions in \( \Phi_{n+1}(\hat{K}) \) for general \( n \) (\( n > 2 \)). By the definition of \( \Phi_{n+1}(\hat{K}) \) in (3.1), every function in \( \Phi_{n+1}(\hat{K}) \) can be decomposed into a part in \( P_{n+1}(\hat{K}) \) and a part in \( P_{n+1}^w(\hat{K}) \). As shown in Lemma 3.1, the dominating part in \( P_{n+1}(\hat{K}) \) has a basis \( \{(\phi_{n+1-j}(x)\phi_j(y))\}_{j=1}^{n+1} \). The remaining term in \( P_{n+1}^w(\hat{K}) \) insures that the function lies in \( \Phi_{n+1}(\hat{K}) \). We need to determine the patterns of these remaining terms.

From the above description, \( \Phi_{n+1}(\hat{K}) \) can be constructed from periodic basis functions. We know that \( \phi_{n+1-j}(x)\phi_j(y) \) are in \( PP_{n+1}(\hat{K}) \), except for \( j = 1 \) and \( n \). We shall modify \( \phi_{n}(x)y \) and \( x\phi_{n}(y) \). Since the restriction of \( \zeta_4^{12} \) on \( l_{12} \) is \( \phi_{n}(x) \); and similar situation happens on the other boundary sides of \( \hat{K} \), we conclude that \( \phi_{n}(x)y + \zeta_4^{12} - \zeta_4^{23} \) and \( x\phi_{n}(y) - \zeta_4^{23} + \zeta_4^{14} \) vanish on the boundary of \( \hat{K} \), and hence are in \( PP_{n+1}^w(\hat{K}) \).

After getting these modified \( (n+1)^{th} \) order periodic polynomials, we denote all the remaining terms as \( r_n^i \). Since \( \Phi_{n+1}(\hat{K}) \subset PP_{n+1}^w(\hat{K}) \), it is clear that \( r_n^i \in PP_{n+1}^w(\hat{K}) \). Then we have the following lemma.
Lemma 3.3. For each $\psi \in \Phi_{n+1}(\hat{K})$, $\psi(-x, -y) = (-1)^{n+1}\psi(x, y)$. Moreover, we have $\nabla \psi(-x, -y) = (-1)^{n}\nabla \psi(x, y)$.

**Proof.** It is straightforward to check that the lemma holds for $n = 1, 2$. We assume $n > 2$ in the following. It is sufficient to show the symmetric properties hold for basis functions.

For $j = 0, \ldots, n + 1, j \neq 1, n$, the $j$th basis function in $\Phi_{n+1}(\hat{K})$ is written as

$$B_{n+1}^j = \phi_{n+1-j}(x)\phi_j(y) + r_n^j.$$ 

By definition (3.1), for all $\nu \in PP_n^w(\hat{K})$,

$$\int_{\hat{K}} \nabla B_{n+1}^j \cdot \nabla \nu = \int_{\hat{K}} \nabla (\phi_{n+1-j}(x)\phi_j(y)) \cdot \nabla \nu + \int_{\hat{K}} \nabla r_n^j \cdot \nabla \nu = 0.$$ 

From Lemma 3.2, we have

$$\nabla (\phi_{n+1-j}(-x)\phi_j(-y)) = (-1)^{n}\nabla (\phi_{n+1-j}(x)\phi_j(y)).$$

If $n$ is even, pick $\nu$ even. Then $\nabla \nu$ is odd, and $\nabla (\phi_{n+1-j}(x)\phi_j(y)) \cdot \nabla \nu$ is odd. We have

$$\int_{\hat{K}} \nabla (\phi_{n+1-j}(x)\phi_j(y)) \cdot \nabla \nu = 0,$$

and hence

$$\int_{\hat{K}} \nabla r_n^j \cdot \nabla \nu = 0.$$ 

Note that $r_n^j$ is in $PP_n^w(\hat{K})$, which has basis functions either even or odd. Let $\nu$ run through all even basis functions, we conclude that $r_n^j$ is odd, just as $\phi_{n+1-j}(-x)\phi_j(-y)$. Therefore, $B_{n+1}^j$ is odd, as desired. Similarly, if $n$ is odd, we shall conclude that $B_{n+1}^j$ is even.

For $j = 1$ and $n$, the desired results shall follow if $\zeta_{12}^m - \zeta_{43}^m$ and $-\zeta_{23}^m + \zeta_{14}^m$ have proper symmetry properties. Namely, if $n$ is even, both of them are odd; if $n$ is odd, both of them are even. These follow from (3.3), (3.6), and Lemma 3.2. $\square$

**Remark 3.2.** Lemma 3.3 reveals that, to locate superconvergent points in $\hat{K}$, we need to consider only the situations in $\hat{T}_1$. The superconvergent points in $\hat{T}_2$ are symmetric to those in $\hat{T}_1$ about the origin. $\square$

Lemma 3.2 also induces general expressions of basis functions in $\Phi_{n+1}(\hat{K})$.

**Theorem 3.3.** For all $n > 2$, we have

$$\Phi_{n+1}(\hat{K}) = \text{Span} \{ \phi_{n+1}(x) + r_n^0, \phi_n(x)y + \zeta_{12}^m - \zeta_{43}^m + r_n^1, \phi_{n+1-j}(x)\phi_j(y) + r_n^j \ (j = 2, \ldots, n-1), x\phi_n(y) - \zeta_{23}^m + \zeta_{14}^m + r_n^n, \phi_{n+1}(y) + r_n^{n+1} \},$$

where $r_n^j$ satisfy the following property:

(i) For even order $r_n^j$, only odd order side modes, odd order internal modes of minus type, and even order internal modes of plus type are involved in the expressions;

(ii) For odd order $r_n^j$, only even order side modes, odd order internal modes of plus type, and even order internal modes of minus type are involved in the expressions.

**Proof.** Note that any function in $\Phi_{n+1}(\hat{K})$ is expressed in terms of periodic side modes and internal modes. This is a direct result from Lemmas 3.2 and 3.3. $\square$

**Remark 3.3.** As we have seen above, the expressions of basis functions in $\Phi_{n+1}(\hat{K})$ obtained from the orthogonal decomposition reveal the structure of them (in terms of periodic basis
functions). This may provide us an analytical way to find superconvergent points. However, this approach is quite complicated, especially when \( n \) is large.

In fact, if we are interested only in the expressions (not the structures) of the basis functions in \( \Phi_{n+1}(\hat{K}) \), then we can simplify the process by solving

\[
\int_{\hat{K}} \nabla(u - I_h u - z) \cdot \nabla v = 0, \quad \forall v \in V_n^*(\hat{K});
\]

\[
\int_{\hat{K}} (u - I_h u - z) = 0,
\]

where \( u \) are monomials of degree \( n + 1 \), \( I_h u \) is an interpolation of \( u \) in \( V_n(\hat{K}) \), and \( z \) is a periodic finite element approximation of \( u - I_h u \). Then \( u - I_h u - z \) will serve as a periodic basis function in \( \Phi_{n+1}(\hat{K}) \) corresponding to \( u \).

The approach works as the following. An interpolation \( I_h u \) in \( V_n(\hat{K}) \) are determined from the conditions:

(i) \( I_h u(\pm 1, \pm 1) = u(\pm 1, \pm 1) \);

(ii) \( \int_{l} (u - I_h u) s_j ds = 0, \quad j = 0, 1, \cdots, n - 2, \)

along each side \( l \) in \( \hat{K} \); and

(iii) \( \int_{\hat{T}} (u - I_h u) x^j y^k dxdy = 0, \quad j, k \geq 0, \quad j + k = 0, 1, \cdots, n - 3, \)

on each element \( \hat{T} \) in \( \hat{K} \).

After \( I_h u \) is determined, the periodic finite element approximation \( z \) of \( u - I_h u \) can be achieved by solving (3.14) and (3.15). Here, the periodic basis functions of \( V_n^*(\hat{K}) \) are described above.

For instance, consider the periodic basis function corresponding to \( x^3 y \) in \( \Phi_4(\hat{K}) \). This is a case for \( n = 3 \) (see FIGURE 3). First interpolate \( x^3 y \) in \( V_3(\hat{K}) \). The restriction of the interpolation \( I_h(x^3 y) \) on \( \hat{T}_i \) is a polynomial of degree \( \leq 3 \); i.e.

\[
I_h(x^3 y)|_{\hat{T}_i} = \sum_{0 \leq j + k \leq 3} c_{j,k}^i x^j y^k, \quad j, k \geq 0, \quad i = 1, 2.
\]

Substituting \( I_h(x^3 y)|_{\hat{T}_i} \) in conditions (3.16)-(3.18), we have a system of equations for \( c_{j,k}^i \)'s on each \( \hat{T}_i \). Use \( \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} \) to denote the piecewise polynomial \( \begin{pmatrix} f_1(x, y) \end{pmatrix} \) in \( \hat{T}_1 \). With help of a symbolic computation software (MAPLE), we can solve the systems and obtain

\[
I_h(x^3 y) = \begin{pmatrix} -x^3 + x^2 y + x^2 + \frac{1}{6} x y + \frac{1}{6} x - \frac{1}{6} y - \frac{1}{6} \\ x^3 - x^2 y + x^2 + \frac{1}{6} x y - \frac{1}{6} x + \frac{1}{6} y - \frac{1}{6} \end{pmatrix}.
\]

Next, we use the periodic basis functions constructed in (3.9) to solve the periodic finite element approximation problem (3.14) (see FIGURE 3), and have

\[
z^* = \begin{pmatrix} \frac{5}{14} x^2 y - \frac{5}{14} x y^2 + \frac{3}{14} x^2 - \frac{3}{14} x y + \frac{3}{14} y^2 - \frac{3}{14} x + \frac{3}{14} y + \frac{1}{2} \\ \frac{5}{14} x^2 y + \frac{3}{14} x y^2 - \frac{3}{14} x^2 + \frac{3}{14} x y - \frac{3}{14} x - \frac{3}{14} y + \frac{1}{2} \end{pmatrix}.
\]
Here, $\psi^* = x^3 y - I_h(x^3 y) - z^*$ is not in $\Phi_4(\bar{K})$ yet, since its integral over $\bar{K}$ is not 0. Note that a constant function is periodic. We can let $z = z^* + c$ and solve (3.15) for $c$. It turns out that $c = -\frac{2}{3}$. Put $z = z^* - \frac{2}{3}$. Then, we obtain $\psi = x^3 y - I_h(x^3 y) - z \in \Phi_4(\bar{K})$ corresponding to $x^3 y$, which is

$$\psi = \left( x^3 y + x^3 - \frac{19}{14} x^2 y + \frac{5}{13} xy^2 - \frac{17}{14} x^2 + \frac{3}{13} y^2 - \frac{3}{14} y^2 + \frac{1}{14} x - \frac{1}{14} y + \frac{16}{105} \right).$$

This systematic process is more suitable for computer implementation than the orthogonal decomposition process. Once the code is set, we can feed the program with different $(n + 1)^{th}$ order polynomials to get the corresponding functions in $\Phi_{n+1}(K)$. In particular, this process offers us an approach for $\psi_{n+1}^{Re}$ and $\psi_{n+1}^{Im}$. Letting $u$ be the harmonic polynomial $Re(z^{n+1})$ or $Im(z^{n+1})$, the process will yield $\psi_{n+1}^{Re}$ or $\psi_{n+1}^{Im}$, respectively. □

3.1.3. Superconvergent Points

After specification of basis functions of $\Phi_{n+1}(\bar{K})$, we are ready to locate superconvergent points for the Poisson equation.

Theorem 3.4. Consider superconvergence for the Poisson equation on $l_{12}$.
(i) For odd $n$, the midpoint of $l_{12}$ is $x$-derivative superconvergent point;
(ii) If $n = 2$, the two Gaussian points on $l_{12}$ are $x$-derivative superconvergent points;
(iii) For even $n$, the endpoints and midpoint of $l_{12}$ are function value superconvergent points.

Similar results hold on the other sides. In particular, we refer to tangential derivative superconvergence on each side, including the diagonal side.

Proof. Clearly, $\phi_k^{12}|_{l_{12}} = \phi_k(x)$, which will cancel $\phi_k(x)y|_{l_{12}}$ since $y = -1$ on $l_{12}$. The other side modes and all internal modes are identically 0 on $l_{12}$. Note that $\phi_j(y) = 0$ on $l_{12}$. Therefore, restricted on $l_{12}$, the basis functions of $\Phi_{n+1}(K)$ are classified in two types: $\phi_{n+1}(x) + r_n^i$ and $r_n^i$ for $j = 1, \ldots, n + 1$.

(i) Suppose $n = 2m + 1$. From Theorem 3.3,

$$r_{2m+1}^j|_{l_{12}} = c_0^j + \sum_{i=1}^{m} c_{2i}^j \phi_{2i}|_{l_{12}} = c_0^j + \sum_{i=1}^{m} c_{2i}^j \phi_{2i}(x),$$

where $c_{2i}^j$s are constants, $j = 0, \ldots, 2m + 2$. Also, $\phi_{n+1}(x) = \phi_{2(m+1)}(x)$. Thus, the basis functions of $\Phi_{n+1}(K)$ restricted on $l_{12}$ are all linear combinations of even order $\phi_k(x)$s, whose derivatives are odd order Legendre polynomials, which have a common zero at the midpoint of $l_{12}$.

(ii) The superconvergence is verified directly.

(iii) Suppose $n = 2m$. From Theorem 3.3,

$$r_{2m}^j|_{l_{12}} = \sum_{i=2}^{m} c_i^j \phi_{2i-1}|_{l_{12}} = \sum_{i=2}^{m} c_i^j \phi_{2i-1}(x),$$

where $c_i^j$s are constants, $j = 0, \ldots, 2m + 1$. Also, $\phi_{n+1}(x) = \phi_{2m+1}(x)$. Therefore, the basis functions of $\Phi_{n+1}(K)$ are all linear combinations of odd order $\phi_k(x)$s, which have common zeros at two endpoints and midpoint of $l_{12}$. □

1If $m = 0$, then $r_n^i$s are constants.
Theorem 3.4 predicts superconvergence at some specific points on the element edges. Whether these are all superconvergent points should be justified by Theorems 3.1 and 3.2. Namely, we need to determine the intersections of the contours induced from the basis functions of $\Phi_{n+1}(K)$, which are solutions of a system of polynomial equations. When $n$ is small (e.g., $n = 1$ or 2), the system can be solved analytically. When $n$ is large, this can be done accurately with the help of computation softwares (MATLAB, MAPLE) and numerical methods (e.g., Newton's method), which are available in many reference books (e.g., [8]). We have the following results for $n$ up to 8.

Proposition 3.1. For the regular mesh, superconvergent points of function values for the Poisson equation in $T_1$ are:

(i) If $n$ is odd, there is no superconvergent point;

(ii) If $n$ is even, the vertices and midpoints of edges are the only superconvergent points.

Proposition 3.2. For the regular mesh, superconvergent points of $\frac{\partial u}{\partial x}$ (or $\frac{\partial u}{\partial y}$) for the Poisson equation in $T_1$ are:

(i) If $n$ is odd, the midpoint of $l_{12}$ (or $l_{23}$) is the only superconvergent point;

(ii) If $n = 2$, the two Gaussian points on $l_{12}$ (or $l_{23}$) are the only superconvergent points;

(iii) If $n$ is even and greater than 2, there is no superconvergent point.

Remark 3.4. Theorem 3.4 coincides with the results from the symmetry principles [14, 17], which are sufficient. Propositions 3.1 and 3.2 are conclusive; i.e., they indicate that there are no other superconvergent points. Proposition 3.2 also agrees with the corresponding results in [2]. Note that the case $n = 2$ was reported much earlier [1, 20]. □

Now, we consider the Laplace equation. We first determine $\psi_{R_1}^{Re}$ and $\psi_{R_1}^{Im}$. These functions can be obtained from basis functions of $\Phi_{n+1}(K)$ by adding the periodic polynomials corresponding to terms in $Re(z^{n+1})$ and $Im(z^{n+1})$, respectively; they can also be derived from the process described in Remark 3.1. For $n = 1, \ldots, 4$, we have

$$\psi_{R_1}^{Re} = x^2 - y^2,$$
$$\psi_{R_1}^{Im} = \left(\begin{array}{c} xy + x - y - \frac{2}{3} \\ xy - x + y - \frac{2}{3} \end{array}\right);$$
$$\psi_{R_1}^3 = \left(\begin{array}{c} x^3 - 3xy^2 - 3xy + 3y^2 - x + 3y \\ x^3 - 3xy^2 + 3xy - 3y^2 - x + 3y \end{array}\right),$$
$$\psi_{R_1}^3 = \left(\begin{array}{c} y^3 - 3x^2y + 3xy^2 - 3x^2 + y + 3x \\ y^3 - 3x^2y - 3xy + 3x^2 - y + 3x \end{array}\right);$$
$$\psi_{R_1}^4 = \left(\begin{array}{c} x^4 - 6x^2y^2 + y^4 - 6x^2y + 6xy^2 - 2x^2 + 8xy - 2y^2 + 2x - 2y - \frac{8}{15} \\ x^4 - 6x^2y^2 + y^4 + 6x^2y - 6xy^2 - 2x^2 + 8xy - 2y^2 - 2x + 2y - \frac{8}{15} \end{array}\right),$$
$$\psi_{R_1}^4 = \left(\begin{array}{c} x^3y - x^3 + x^3 - x^2y - xy^2 + y^3 - x^2 + y^2 \\ x^3 - x^3y + x^3 + x^2y - xy^2 + y^3 - x^2 + y^2 \end{array}\right);$$
$$\psi_{R_1}^5 = \left(\begin{array}{c} x^5 - 10x^3y^2 + 5y^4 + 155x^3y + \frac{25}{12} x^2y^2 + \frac{65}{12} xy^3 - 5y^4 - \frac{135}{28} x^3 + \frac{75}{4} x^2y \\ -\frac{85}{28} x^3 - \frac{65}{12} xy^2 + \frac{25}{2} x^2 - \frac{95}{14} xy + \frac{15}{8} y^2 - \frac{17}{9} x + \frac{20}{21} y \end{array}\right),$$
$$\psi_{R_1}^5 = \left(\begin{array}{c} x^5 - 10x^3y^2 + 5y^4 + 155x^3y + \frac{25}{12} x^2y^2 - \frac{65}{12} xy^3 + 5y^4 - \frac{135}{28} x^3 + \frac{75}{4} x^2y \\ -\frac{85}{28} x^3 - \frac{65}{12} xy^2 - \frac{25}{2} x^2 + \frac{95}{14} xy - \frac{15}{8} y^2 - \frac{17}{9} x + \frac{20}{21} y \end{array}\right).$$
By Theorem 3.2 (i), the function value superconvergent points are the intersections of the contours $\psi_{n+1}^{Re} = 0$ and $\psi_{n+1}^{Im} = 0$. For instance, when $n = 1$, the superconvergent points in $T_1$ can be obtained by solving

\[
\begin{align*}
x^2 - y^2 &= 0, \\
xy + x - y - \frac{2}{3} &= 0,
\end{align*}
\]
in $T_1$. From the first equation, we have $x = y$ or $x = -y$. Substituting these into the second equation, we obtain $(\pm \sqrt{3}, \pm \sqrt{3})$ and $(1 - \frac{\sqrt{3}}{3}, -1 + \frac{\sqrt{3}}{3})$ in $T_1$, which are desired superconvergent points.

When $n = 2$, we need to solve

\[
\begin{align*}
x^3 - 3xy^2 - 3xy + 3y^2 - x + 3y &= 0, \\
y^3 - 3yx^2 + 3yx - 3x^2 - y + 3x &= 0,
\end{align*}
\]
or equivalently, to solve

\[
\begin{align*} (x - 1)(x^2 - 3y^2 + x - 3y) &= 0, \\
(y + 1)(y^2 - 3x^2 - y + 3x) &= 0.
\end{align*}
\]

It is straightforward to verify that the solutions (superconvergent points) in $T_1$ are the vertices, the midpoints of edges, and $(\frac{1}{4} \pm \frac{\sqrt{3}}{4}, \frac{1}{4} \pm \frac{\sqrt{7}}{4})$. 

\[
\psi_{n+1}^{Re} \quad \text{and} \quad \psi_{n+1}^{Im} \quad \text{for} \quad n = 5, \ldots, 8 \quad \text{are provided in} \quad [13].
\]
For each $n$, we need to solve a pair of polynomial equations. When $n$ is large, numerical methods are required. Although how to efficiently solve a polynomial equation system is an interesting problem, it is not our focus. Here, we simply use the Newton method. With

**Table 1(a). Function Value Superconvergent Points for the Regular Pattern**

<table>
<thead>
<tr>
<th></th>
<th>$n=1$</th>
<th>$n=2$</th>
<th>$n=3$</th>
<th>$n=4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.81649685809277260</td>
<td>-1.0000000000000000</td>
<td>-0.519329623592281</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>2</td>
<td>-0.81649685809277260</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>3</td>
<td>0.4226497308103742</td>
<td>0.0000000000000000</td>
<td>0.519329623592281</td>
<td>-0.9511897312118490</td>
</tr>
<tr>
<td>4</td>
<td>-0.4226497308103742</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>5</td>
<td>0.81649685809277260</td>
<td>1.0000000000000000</td>
<td>-0.519329623592281</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>6</td>
<td>0.81649685809277260</td>
<td>-1.0000000000000000</td>
<td>-0.519329623592281</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>7</td>
<td>-0.4114378277661476</td>
<td>1.0000000000000000</td>
<td>0.519329623592281</td>
<td>-0.9511897312118490</td>
</tr>
<tr>
<td>8</td>
<td>-0.4114378277661476</td>
<td>0.519329623592281</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>9</td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td>10</td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td>11</td>
<td>0.9114378277661476</td>
<td>0.9173868533105418</td>
<td>1.0000000000000000</td>
<td>1.0000000000000000</td>
</tr>
<tr>
<td>12</td>
<td>0.9114378277661476</td>
<td>0.9173868533105418</td>
<td>1.0000000000000000</td>
<td>1.0000000000000000</td>
</tr>
<tr>
<td>13</td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td>14</td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
</tbody>
</table>
help of MAPLE and MATLAB, we obtain function value superconvergent points for \( n = 1, \ldots, 8 \), which are listed in TABLE 1 with 16 digits of accuracy. In FIGURE 4 and FIGURE 5, we plot the contours for cases \( n = 1, \ldots, 8 \) by MATLAB.
By Theorem 3.2 (ii), the $x$-derivative superconvergent points are the intersections of the contours $\frac{\partial \psi_{n+1}^{Re}}{\partial x} = 0$ and $\frac{\partial \psi_{n+1}^{Im}}{\partial x} = 0$. As an example, we consider the case for $n = 2$. The $x$-derivative superconvergent points in $T_1$ are the solutions of the system

$$\begin{align*}
\frac{\partial \psi_{n+1}^{Re}|_{T_1}}{\partial x} &= 3x^2 - 3y^2 - 3y - 1 = 0, \\
\frac{\partial \psi_{n+1}^{Im}|_{T_1}}{\partial x} &= -3(y + 1)(2x - 1) = 0.
\end{align*}$$

Solve this system, we obtain four superconvergent points: $(\pm \frac{\sqrt{3}}{3}, -1)$ and $(\frac{1}{2}, -\frac{1}{2} \pm \frac{\sqrt{6}}{6})$.

<p>| Table 2(a). Derivative Superconvergent Points for the Regular Pattern (in $T_1$, $n = 1, \ldots, 4$) |</p>
<table>
<thead>
<tr>
<th>n = 1</th>
<th>n = 2</th>
<th>n = 3</th>
<th>n = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.5773502691896258</td>
<td>-1.0000000000000000</td>
<td>-0.975626047562738165</td>
</tr>
<tr>
<td>2</td>
<td>0.5773502691896258</td>
<td>0.0000000000000000</td>
<td>0.4357538487328842</td>
</tr>
<tr>
<td>3</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>0.4357538487328842</td>
</tr>
<tr>
<td>4</td>
<td>0.5000000000000000</td>
<td>1.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td>5</td>
<td>-0.908298420048538630</td>
<td>-1.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td>6</td>
<td>-0.0917017069301370</td>
<td>-0.82582296332329942</td>
<td>-0.0000000000000000</td>
</tr>
<tr>
<td>7</td>
<td>0.7276352521634410</td>
<td>-0.7609910954779083</td>
<td>-0.3809408103995712</td>
</tr>
<tr>
<td>8</td>
<td>0.3048132073344000</td>
<td>-0.3648132073349060</td>
<td>-0.55977653529966</td>
</tr>
<tr>
<td>9</td>
<td>0.5106905181904221</td>
<td>-0.9856261245779083</td>
<td>-0.55977653529966</td>
</tr>
<tr>
<td>10</td>
<td>0.2804554307532021</td>
<td>0.0392284719195946</td>
<td>0.8126724617779161</td>
</tr>
<tr>
<td>11</td>
<td>0.57424399395274366</td>
<td>0.8126724617779161</td>
<td>0.57424399395274366</td>
</tr>
</tbody>
</table>
TABLE 2(b). Derivative Superconvergent Points for the Regular Pattern (in $T_1$, $n = 5, \ldots, 8$)

<table>
<thead>
<tr>
<th>$n = 5$</th>
<th>$n = 6$</th>
<th>$n = 7$</th>
<th>$n = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>-1.0000000000000000</td>
<td>-0.7829941074859182</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>$2$</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>$3$</td>
<td>-0.6546536707079771</td>
<td>-0.2929200000000000</td>
<td>-0.6874190011562877</td>
</tr>
<tr>
<td>$4$</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>$5$</td>
<td>0.0000000000000000</td>
<td>0.2929200000000000</td>
<td>0.4887571135712024</td>
</tr>
<tr>
<td>$6$</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>$7$</td>
<td>0.6546536707079771</td>
<td>0.2929200000000000</td>
<td>0.4887571135712024</td>
</tr>
<tr>
<td>$8$</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>$9$</td>
<td>-0.0718949366554462</td>
<td>-0.4296804190843017</td>
<td>0.87439190011562877</td>
</tr>
<tr>
<td>$10$</td>
<td>-0.5342407051608834</td>
<td>-0.4296804190843017</td>
<td>0.87439190011562877</td>
</tr>
<tr>
<td>$11$</td>
<td>-0.6850147521433846</td>
<td>-0.4296804190843017</td>
<td>0.87439190011562877</td>
</tr>
<tr>
<td>$12$</td>
<td>-0.5340670245483939</td>
<td>-0.4296804190843017</td>
<td>0.87439190011562877</td>
</tr>
<tr>
<td>$13$</td>
<td>-0.221793001015251</td>
<td>-0.4296804190843017</td>
<td>0.87439190011562877</td>
</tr>
<tr>
<td>$14$</td>
<td>0.4712058509661711</td>
<td>0.1479480038944324</td>
<td>0.9765399749278833</td>
</tr>
<tr>
<td>$15$</td>
<td>0.3851935361634937</td>
<td>0.031990009685319</td>
<td>-0.9442318434680794</td>
</tr>
<tr>
<td>$16$</td>
<td>0.8667183981855447</td>
<td>0.5300096814348669</td>
<td>-0.3200582852380828</td>
</tr>
<tr>
<td>$17$</td>
<td>0.4160309104060867</td>
<td>0.4069980068538466</td>
<td>-0.8204988182282370</td>
</tr>
<tr>
<td>$18$</td>
<td>0.9113920897170500</td>
<td>0.4101295910636911</td>
<td>-0.5269703699170518</td>
</tr>
<tr>
<td>$19$</td>
<td>0.5542101472898229</td>
<td>0.0485573784465860</td>
<td>-0.8982852824639150</td>
</tr>
<tr>
<td>$20$</td>
<td>0.1284564918329218</td>
<td>0.3738017041726830</td>
<td>-0.2244710257141005</td>
</tr>
</tbody>
</table>

When $n = 3$, the superconvergent points are the solutions of the system

\[
\begin{align*}
\frac{\partial \psi_4^{Re_{T_1}}}{\partial x} & = 4x^3 - 12xy^2 - 12xy + 6y^2 - 4x + 8y + 2 = 0, \\
\frac{\partial \psi_4^{Re_{T_1}}}{\partial x} & = (y + 1)(3x^2 - y^2 - 2x) = 0.
\end{align*}
\]

From the second equation, we have $y + 1 = 0$ or $3x^2 - y^2 - 2x = 0$. If we substitute $y = -1$ into the first equation, the system can be easily solved. However, when substituting $y = \pm \sqrt{3x^2 - 2x}$ into the first equation, we have

\[(x - 1)(256x^5 - 416x^4 + 173x^3 + 21x^2 - 17x - 1) = 0,
\]

provided $3x^2 - 2x \geq 0$. To solve a polynomial equation of degree 5, numerical methods are required.

Following a similar process for function value superconvergent points, the $x$-derivative superconvergent points can be located in $K$. In TABLE 2, the superconvergent points in $T_1$
are summarized for \( n = 1, \ldots, 8 \) with 16 digits of accuracy. Since the cases for \( n = 1, 2 \) are trivial, we plot only the contours for cases \( n = 3, \ldots, 8 \) by MATLAB in FIGURE 6.

**Remark 3.5.** In order to compare our results with data in [2]², where the master cell is \( \kappa = [0, 1]^2 \), we map \( \hat{K} \) to \( \kappa \) by affine mapping \( x^\kappa = (x^\hat{K} + 1)/2 \) and \( y^\kappa = (y^\hat{K} + 1)/2 \). Note that the nodal shape functions defined in \( \hat{K} \) are mapped to those defined in \( \kappa \) under the same affine mapping. We then conclude that \( \Phi_{n+1}(\hat{K}) \) is transferred to \( \Phi_{n+1}(\kappa) \) because of its construction. Since affine mappings preserve zeros for derivatives, the derivative superconvergence points in \( \hat{K} \) are mapped to those in \( \kappa \). We found that the first 9 digits of the superconvergent points listed in [2] are the same as those obtained here³. \( \square \)

By Theorem 3.2 (ii), we can also determine the \( y \)-derivative superconvergent points. It can be shown that

\[
\frac{\partial \psi_{n+1}^R(-y,x)}{\partial y} = (-1)^{k+1} \frac{\partial \psi_{n}^R(x,y)}{\partial x}, \quad \frac{\partial \psi_{n+1}^L(-y,x)}{\partial y} = (-1)^{k} \frac{\partial \psi_{n}^L(x,y)}{\partial x};
\]

and

\[
\frac{\partial \psi_{2k+1}^R(-y,x)}{\partial y} = \frac{\partial \psi_{2k+1}^R(x,y)}{\partial x}, \quad \frac{\partial \psi_{2k+1}^L(-y,x)}{\partial y} = \frac{\partial \psi_{2k+1}^L(x,y)}{\partial x} \quad \text{for} \quad k = 1, 2, \ldots
\]

Therefore, the \( y \)-derivative superconvergent points for the Laplace equation turn out to be the symmetry points of the \( x \)-derivative superconvergent points about \( y = -x \).

### 3.2. CHEVRON PATTERN

We observe that a period occupies only half of the square in the Chevron pattern. Therefore, we set \( \hat{K} = [-1, 1] \times [0, 1] \) here. Partition \( \hat{K} \) into four triangular elements (see FIGURE 7).

![Figure 7: Partition of \( \hat{K} \) for the Chevron Pattern](image)

We still use \( P_n^w(\hat{K}) \) and \( PP_n^w(\hat{K}) \) to denote the counterpart spaces defined in Section 3.1. The definitions of the spaces are adjusted accordingly due to the changing of the reference cell.

The hierarchic basis functions are used for \( V_n(\hat{K}) \). Let \( \nu_i \) be the linear nodal shape function corresponding to vertex \( n_i \), which are defined in (3.20). The side modes and internal modes are defined as in (3.6) and (3.7), respectively.

The periodic basis functions are constructed from \( \nu_i, \xi_{k,j}^i \) and \( \xi_{k,j}^i \) similar to those in Section 3.1. Notice that \( \nu_1 + \nu_3 + \nu_4 + \nu_6 \) and \( \nu_2 + \nu_5 \) are two periodic basis functions in this case.

---

²In [2], TABLE I, superconvergent points are given for \( n \) up to 7.
³Some of the 10th decimal places in [2] are not accurate in the "round-off" sense.
\[ \nu_1 = \begin{cases} 
-x & \text{in } \hat{T}_1, \\
1 - y & \text{in } \hat{T}_2, \\
0 & \text{in } \hat{T}_3, \hat{T}_4; 
\end{cases} \quad \nu_4 = \begin{cases} 
-x + y - 1 & \text{in } \hat{T}_2, \\
0 & \text{in } \hat{T}_1, \hat{T}_3, \hat{T}_4; 
\end{cases} \\
\nu_2 = \begin{cases} 
1 + x - y & \text{in } \hat{T}_1, \\
1 - x - y & \text{in } \hat{T}_3, \hat{T}_4; 
\end{cases} \quad \nu_5 = \begin{cases} 
y & \text{in } \hat{T}_1, \\
y & \text{in } \hat{T}_3, \\
-x + 1 & \text{in } \hat{T}_4; 
\end{cases} \\
\nu_3 = \begin{cases} 
x & \text{in } \hat{T}_4, \\
1 - y & \text{in } \hat{T}_1, \hat{T}_2; 
\end{cases} \quad \nu_6 = \begin{cases} 
x + y - 1 & \text{in } \hat{T}_4, \\
0 & \text{in } \hat{T}_1, \hat{T}_2, \hat{T}_3. 
\end{cases} 
\]

It is straightforward to verify that
\[ \dim P_n^w(K) = (2n + 1)(n + 1), \quad \dim PP_n^w(K) = 2n^2. \] (3.21)

Theorems 3.1 and 3.2 still hold for the Chevron mesh. To determine the space \( \Phi_{n+1}(K) \), we can either make an orthogonal decomposition of \( PP_n^w(K) \), or solve a periodic finite element approximation problem, as described in Remark 3.1.

If the first approach is adopted, we need to define \( \Psi_n(K) \) as in (3.11). The difference here is that the integral domain consists of 4 elements instead of 2. By the Gram-Schmidt process, we obtain a list of \( \Psi_n(K) \), where \( PP_n^w(K) \) can be decomposed as in (3.12). Based on \( \Psi_n(K) \), we are able to construct \( \Phi_n(K) \).

Remark 3.6. From the geometrical point of view, the left side of the \( y \)-axis is one patch of the regular mesh. In fact, under the corresponding affine mapping \( x_{Rg} = 2x_{Ch} + 1 \) and \( y_{Rg} = 2y_{Ch} - 1 \), the shape functions on \( \hat{T}_1 \) and \( \hat{T}_2 \) defined in (3.20) are mapped to the ones of the regular pattern. Similar situation happens to the right side of the \( y \)-axis. We thus conclude that similar symmetry results as in Lemma 3.3 hold for the Chevron pattern. Hence, we need to work only on superconvergent points in the first element. Superconvergent points in the other elements can be obtained by symmetry. \( \square \)

In the following, basis functions of \( \Phi_{n+1}(K) \) in \( \hat{T}_1 \) are provided for \( n = 1, \ldots, 4 \). For cases of \( n = 5, \ldots, 8 \), the reader is referred to [13]. To simplify notations, let \( B_{n+1} \) be the \( j \)th basis function of \( \Phi_{n+1}(K) \). Denote \( \theta_n \), the column vector consists of all nodal shape functions, side modes and internal modes of order \( \leq n \) in \( T_1 \). The length of \( \theta_n \) is \( (n + 1)(n + 2)/2 \). In particular, we assign \( \nu_1, \nu_2 \) and \( \nu_5 \) the first 3 entries of \( \theta_n \); assign the side modes \( \psi_{25}^{12}, \psi_{51}^{25} \) and \( \psi_{51}^{51} \) as the \((k(k + 1)/2 + 1)^{th}\) to \((k(k + 1)/2 + 3)^{th}\) entries, \( k = 2, \ldots, n \); and assign the internal modes \( \psi_{k,1}^{1} \) to \( \psi_{k,k-2}^{1} \) as the \((k(k + 1)/2 + 4)^{th}\) to \((k(k + 1)/2 + 2)^{th}\) entries, \( k = 3, \ldots, n \). For instance,
\[ \theta_1 = [\nu_1, \nu_2, \nu_5]^T, \]
\[ \theta_2 = [\theta_1, \psi_{25}^{12}, \psi_{51}^{25}, \psi_{51}^{51}]^T, \]
\[ \theta_3 = [\theta_2, \psi_{51}^{25}, \psi_{51}^{51}, \psi_{51}^{51}, \psi_{4,1}^{1}, \psi_{4,2}^{1}]^T, \]
\[ \theta_4 = [\theta_3, \psi_{4,1}^{51}, \psi_{4,2}^{51}, \psi_{4,1}^{1}, \psi_{4,2}^{1}]^T. \] (3.22)

Then, in element \( \hat{T}_1 \), the basis functions of \( \Phi_{n+1}(K) \) can be expressed as
By Theorem 3.1, to locate the superconvergent points of the Poisson equation, we may choose $B_{n+1}$ as vs. Then we can verify the following results for $n$ up to 8.

**Proposition 3.3.** Consider element $T_1$ of the Chevron mesh. For the Poisson equation, the function value superconvergent points are:

(i) If $n$ is odd, there is no superconvergent point;

(ii) If $n$ is even, the midpoints of sides $l_{12}$ and $l_{15}$ are the only superconvergent points.

**Proposition 3.4.** Consider element $T_1$ of the Chevron mesh. For the Poisson equation, the $x$-derivative superconvergent points are:

(i) If $n$ is odd, the midpoint of side $l_{12}$ is the only superconvergent point;

(ii) If $n$ is even, there is no superconvergent point.

The $y$-derivative superconvergent points are:

(i) If $n = 1$, the midpoint of side $l_{25}$ is the only superconvergent point;

(ii) If $n > 1$, there is no superconvergent point.

**Remark 3.7.** Similar as in the regular pattern, superconvergence for the Chevron pattern at symmetry points can be shown by symmetric properties of periodic basis functions and Legendre polynomials. However, Propositions 3.3 and 3.4 are conclusive. Moreover, the result of $y$-derivative superconvergence for $n = 1$ is not at a symmetry point.

**Remark 3.8.** The basis functions of $\Phi_{n+1}(K)$ obtained from the orthogonal decomposition reveal the structures. However, there are more elements involved in the Chevron pattern than in the regular pattern, and the expressions of the basis functions here are even more complicated. Thus, we may use the second approach: solving a periodic finite element approximation problem.
Next, we turn to harmonic polynomials (for the Laplace equation). With the help of basis functions of \( \Phi_{n+1}(K) \), we can determine \( \psi_{n+1}^{Re} \) and \( \psi_{n+1}^{Im} \). In particular, in the element \( T_1 \), we have, for instance

\[
\psi_{n+1}^{Re} = B_1^1 - B_2^1 = x^2 - y^2 + x + y,
\]
\[
\psi_{n+1}^{Im} = B_2^1 = xy;
\]
\[
\psi_{n+1}^{Re} = B_3^1 - 3B_3^3 = x^3 - 3xy^2 + \frac{3}{2}x^2 + \frac{3}{2}xy + \frac{1}{2}x,
\]
\[
\psi_{n+1}^{Im} = B_3^1 - 3B_3^3 = y^3 - 3x^2y - \frac{3}{2}y^2 - \frac{3}{2}xy + \frac{1}{2}y;
\]
\[
\psi_{n+1}^{Re} = B_4^1 - 6B_4^3 + B_4^5 = x^4 - 6x^2y^2 + y^4 + 2x^3 + 3x^2y - 3xy^2 - 2y^3 + x^2 + 2xy + y^2 - \frac{1}{30},
\]
\[
\psi_{n+1}^{Im} = B_4^1 - B_4^4 = x^3y - xy^3 + x^2y + xy^2;
\]
\[
\psi_{n+1}^{Re} = B_5^1 - 10B_5^3 + 5B_5^5
\]
\[
= x^5 - 10x^3y^2 + 5xy^4 + \frac{5}{2}x^4 + \frac{85}{24}x^3y - \frac{35}{24}x^2y^2 - \frac{175}{24}xy^3 + \frac{85}{24}x^3 + \frac{15}{4}x^2y + \frac{10}{3}xy^2 + \frac{15}{24}y^2 - \frac{15}{24}xy + \frac{1}{4}x,
\]
\[
\psi_{n+1}^{Im} = B_5^1 - 10B_5^4 + 5B_5^2
\]
\[
= y^5 - 10y^3x^2 + 5y^4x + \frac{5}{2}y^4 + \frac{85}{24}xy^3 - \frac{35}{24}x^2y^2 + \frac{85}{24}x^3y + \frac{85}{24}y^3 + \frac{15}{4}xy^2 + \frac{10}{3}x^2y - \frac{15}{24}y^2 - \frac{15}{24}xy + \frac{1}{4}y.
\]

For information of \( n = 5, \ldots , 8 \), the reader is referred to [13].

Now, we are ready to apply Theorem 3.2. The superconvergent points can be located by solving the corresponding systems. These can be done as for the regular pattern. However, the affine mapping from \( \hat{T}_1^{Ch} \) to \( \hat{T}_1^{Rg} \) \((x^{Rg} = 2x^{Ch} + 1, y^{Rg} = 2y^{Ch} - 1)\) simplifies our life. When \( n > 1 \), under this mapping, \( (\psi_{n+1}^{Re} | \hat{T}_1^{\hat{T}_1^{Ch}}) \) and \( (\psi_{n+1}^{Im} | \hat{T}_1^{\hat{T}_1^{Ch}}) \) are the same as \( (\psi_{n+1}^{Re} | \hat{T}_1^{\hat{T}_1^{Rg}}) \) and \( (\psi_{n+1}^{Im} | \hat{T}_1^{\hat{T}_1^{Rg}}) \) up to a constant multiplier. When \( n = 1 \), they are the same up to a constant multiplier and a constant addendum as well.

Therefore, when \( n > 1 \), the function value and derivative superconvergent points in \( \hat{T}_1^{Ch} \) can be obtained from those in \( \hat{T}_1^{Rg} \) (see Remark 3.5), which are listed in TABLE 1 and TABLE 2. When \( n = 1 \), the derivative superconvergent points are also obtained from the regular pattern (TABLE 2). We need to determine only the function value superconvergent points, which are solutions of
Solving this system, we have three superconvergent points: $(0, 0), (0, 1)$ and $(-1, 0)$. The contours $\psi_{n+1}^{Re} = 0$ and $\psi_{n+1}^{Im} = 0$ for $n = 8$ are given in Figure 8. The contours $\frac{\partial \psi_{n+1}^{Re}}{\partial x} = 0$ and $\frac{\partial \psi_{n+1}^{Im}}{\partial x} = 0$ for $n = 8$ are given in Figure 9.

3.3. UNION JACK PATTERN

In the Union Jack pattern, the reference cell is again $\hat{K} = [-1, 1]^2$, which is partitioned into eight triangular elements (see Figure 10).

Let $v_i$ be the linear nodal shape function corresponding to vertex $n_i$, which are defined in (3.23). The nodal shape functions are symmetric corresponding to the geometry of the vertices. The side modes and internal modes are defined as in (3.6) and (3.7), respectively.

The constructions of the periodic basis functions are similar as described in Section 3.1. Here, $\nu_1 + \nu_3 + \nu_7 + \nu_9, \nu_2 + \nu_8, \nu_4 + \nu_6$ and $\nu_5$ are four periodic basis functions.
\[
\begin{align*}
\nu_1 &= \begin{cases} 
-x & \text{in } \hat{T}_1, \\
-y & \text{in } \hat{T}_2, \\
0 & \text{in other } \hat{T}_i; 
\end{cases} \\
\nu_2 &= \begin{cases} 
-x - y & \text{in } \hat{T}_1, \\
-x & \text{in } \hat{T}_3, \\
0 & \text{in other } \hat{T}_i; 
\end{cases} \\
\nu_3 &= \begin{cases} 
x & \text{in } \hat{T}_3, \\
-y & \text{in } \hat{T}_4, \\
0 & \text{in other } \hat{T}_i; 
\end{cases} \\
\nu_4 &= \begin{cases} 
-x + y & \text{in } \hat{T}_2, \\
-x - y & \text{in } \hat{T}_5, \\
0 & \text{in other } \hat{T}_i; 
\end{cases} \\
\nu_5 &= \begin{cases} 
1 + y & \text{in } \hat{T}_1, \hat{T}_3, \\
1 + x & \text{in } \hat{T}_2, \hat{T}_5, \\
1 - x & \text{in } \hat{T}_4, \hat{T}_7, \\
1 - y & \text{in } \hat{T}_6, \hat{T}_8; 
\end{cases} \\
\nu_6 &= \begin{cases} 
x + y & \text{in } \hat{T}_4, \\
x - y & \text{in } \hat{T}_7, \\
0 & \text{in other } \hat{T}_i; 
\end{cases} \\
\nu_7 &= \begin{cases} 
y & \text{in } \hat{T}_5, \\
x & \text{in } \hat{T}_6, \\
0 & \text{in other } \hat{T}_i; 
\end{cases} \\
\nu_8 &= \begin{cases} 
-x + y & \text{in } \hat{T}_6, \\
x - y & \text{in } \hat{T}_8, \\
0 & \text{in other } \hat{T}_i; 
\end{cases} \\
\nu_9 &= \begin{cases} 
y & \text{in } \hat{T}_7, \\
x & \text{in } \hat{T}_8, \\
0 & \text{in other } \hat{T}_i. 
\end{cases}
\end{align*}
\]

It is straightforward to verify that
\[
\text{dim } P_n^w(\hat{K}) = (2n + 1)^2, \quad \text{dim } PP_n^w(\hat{K}) = (2n)^2.
\]  

Theorems 3.1 and 3.2 are valid for the Union Jack mesh. To determine spaces \( \Phi_{n+1}(\hat{K}) \), we may either process an orthogonal decomposition of \( PP_n^w(\hat{K}) \) under the Laplace operator, or carry on a periodic finite element approximation. The first approach is similar as in previous cases. We define \( \Psi_n(\hat{K}) \) as in (3.11). Then by the Gram-Schmidt process, \( PP_n^w(\hat{K}) \) can be decomposed into a sum of \( \Psi_n(\hat{K}) \), as in (3.12). Based on \( \Psi_n(\hat{K}) \), we can construct \( \Phi_n(\hat{K}) \).  

**Remark 3.9.** For the Union Jack pattern, a portion of \( \hat{K} \) in each quadrant can be mapped to the reference cell of the regular pattern by an affine mapping. The corresponding shape functions defined in (3.23) are mapped to the ones of the regular pattern. Similar symmetry results as in Lemma 3.3 hold for the Union Jack pattern. Therefore, we need to work only on superconvergent points in the first element. Superconvergent points in the other elements can be obtained by symmetry. \( \Box \)

Basis functions of \( \Phi_{n+1}(\hat{K}) \) in \( \hat{T}_1 \) are provided for \( n = 1, \ldots, 4 \). For information of \( n = 5, \ldots, 8 \), the reader is referred to [13]. To simplify notations, let \( B_{n+1}^j \) and \( \theta_n \) be as defined in Section 3.2. Thus, the basis functions of \( \Phi_{n+1}(\hat{K}) \) in \( \hat{T}_1 \) are

\[
\begin{align*}
B_1^1 &= \phi_2(x) + \left[ \frac{7}{36}, \frac{11}{36}, \frac{25}{36} \right] \theta_1, \\
B_2^2 &= xy + [-1, 0, 0] \theta_1, \\
B_3^1 &= \phi_2(y) + \left[ \frac{25}{36}, \frac{11}{36}, \frac{43}{36} \right] \theta_1;
\end{align*}
\]

*As described in Remarks 3.3, 3.8, the second approach is more efficient in case that we are interested only in the expression itself (not the structure of the expression).*
\[
B_1^j = \phi_3(x) + \left[ -\frac{3}{80}, 0, 0, \frac{27}{80}, 0, \frac{8}{8} \right] \theta_2,
B_2^j = \phi_2(x)y + \left[ \frac{11}{30}, 0, \frac{11}{30}, 0, \frac{7}{30}, \frac{2}{3} \right] \theta_2,
B_3^j = x\phi_2(y) + \left[ -\frac{31}{40}, 0, 0, -\frac{1}{10}, 0, \frac{3}{8} \right] \theta_2,
B_4^j = \phi_3(y) + \left[ \frac{3}{20}, 0, \frac{33}{80}, 0, \frac{33}{80}, \frac{3}{8} \right] \theta_2;
\]

\[
B_1^j = \phi_4(x) + \left[ \frac{1}{8}, -\frac{13}{54}, \frac{3}{8}, -\frac{33}{224}, \frac{15}{172}, -\frac{3}{31}, \frac{31}{112}, \frac{1}{16}, -\frac{15}{12} \right] \theta_3,
B_2^j = \phi_3(x)y + \left[ -\frac{19}{140}, 0, 0, -\frac{1}{7}, \frac{19}{40}, \frac{3}{10}, 0, \frac{3}{10}, \frac{2}{3} \right] \theta_3,
B_3^j = \phi_2(x)\phi_2(y) + \left[ \frac{439}{5040}, \frac{5040}, \frac{320}{5040}, \frac{336}{168}, \frac{336}{168}, \frac{3}{38}, \frac{3}{8}, -\frac{17}{168} \right] \theta_3,
B_4^j = x\phi_3(y) + \left[ -\frac{19}{140}, 0, 0, -\frac{1}{4}, \frac{19}{40}, 0, \frac{3}{4}, \frac{3}{8} \right] \theta_3,
B_5^j = \phi_4(y) + \left[ -\frac{3}{28}, -\frac{13}{112}, -\frac{3}{4}, -\frac{3}{28}, -\frac{207}{224}, -\frac{1}{12}, \frac{15}{16}, \frac{5}{12} \right] \theta_3;
\]

Choose \( B_{n+1}^j \) as \( \psi \), apply Theorem 3.1, and we can verify the following results for finite elements of degree \( n \) up to 8.

**Proposition 3.5.** Consider element \( \hat{T}_1 \) of the Union Jack mesh. For the Poisson equation, the function value superconvergent points are:

(i) If \( n \) is odd, there is no superconvergent point;

(ii) If \( n \) is even, the vertices and the midpoint of side \( l_{15} \) are the only superconvergent points.

**Proposition 3.6.** Consider element \( \hat{T}_1 \) of the Union Jack mesh. For the Poisson equation, there is no superconvergent point for \( \frac{\partial u}{\partial x} \), nor for \( \frac{\partial u}{\partial y} \).

**Remark 3.10.** Propositions 3.5 and 3.6 agree with the results from the computer-based proof and the conclusion from the symmetry principle. Moreover, our results theoretically confirmed that there are no other superconvergent points. □

Now, we consider harmonic polynomials. Again, we can determine \( \psi^{Re}_{n+1} \) and \( \psi^{Im}_{n+1} \) from the basis functions of \( \Phi_{n+1}(K) \). In the element \( \hat{T}_1 \), we have, for instance

\[
\psi^{Re} = B_1^1 - B_2^1 = x^2 - y^2 + x - y,
\psi^{Im} = B_2^1 - B_1^1 = xy + x;
\psi^{Re} = B_3^1 - 3B_3^2 = x^3 - 3xy^2 + \frac{3}{2}x^2 - \frac{3}{2}xy - x,
\psi^{Im} = B_3^2 - 3B_3^1 = y^3 - 3x^2y + \frac{3}{2}y^2 - \frac{3}{2}xy - 3x^2 + \frac{1}{2}y - \frac{3}{2}x;
\psi^{Re} = B_4^1 - 6B_3^3 + B_4^5 = x^4 - 6x^2y^2 + y^4 + 2x^3 - 9ax^3y - 3xy^2 + 2y^3 - 2x^2 - 4xy^2 + y^2 - x - \frac{35}{3},
\psi^{Im} = B_4^2 - B_4^1 = x^3y - xy^3 + x^3 + x^2y - 2xy^2 + x^2 - xy;
\psi^{Re} = B_5^1 + 10B_3^8 + 5B_3^9 = x^5 - 10x^3y^2 + 5xy^4 + \frac{5}{3}x^4 - \frac{355}{24}x^3y - \frac{35}{24}x^2y^2 + \frac{305}{24}x^3y
- \frac{745}{168}x^3 - \frac{55}{168}x^2 + \frac{535}{56}xy^2 + \frac{125}{28}x^2 + \frac{85}{56}xy - \frac{53}{168}x,
\psi^{Im} = B_6^1 + 10B_3^8 + 5B_3^9 = y^5 - 10x^3y^2 + 5x^4y + \frac{5}{3}y^4 - \frac{355}{24}x^3y - \frac{35}{24}x^2y^2 + \frac{205}{24}x^3y + \frac{5}{28}x^3 + \frac{15}{16}x^2y + \frac{5}{28}x^3 + \frac{1}{84}y - \frac{1}{168}x.
\]

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By Theorem 3.2, the superconvergent points can be located by solving the corresponding systems. These can be done as for the regular pattern. However, as we observed for the Chevron pattern, the affine mapping from \( T_1^{UJ} \) to \( T_1^{Rg} \) maps \((\psi_{n+1}^{Re}|T_1^{UJ})^{Rg}\) and \((\psi_{n+1}^{Im}|T_1^{UJ})^{Rg}\) to \((\psi_{n+1}^{Re}|T_1^{Rg})^{Rg}\) and \((\psi_{n+1}^{Im}|T_1^{Rg})^{Rg}\) up to a constant multiplier when \( n > 1 \). When \( n = 1 \), they coincide with each other up to a constant multiplier and a constant addendum as well.

Thus, when \( n > 1 \), the function value and derivative superconvergent points in \( T_1^{UJ} \) can be obtained from those in \( T_1^{Rg} \), which are listed in TABLE 1 and TABLE 2. When \( n = 1 \), the derivative superconvergent points are also as same as the regular pattern (TABLE 2). Therefore, we need to determine only the function value superconvergent points for \( n = 1 \). Toward this end, we solve

\[
\begin{align*}
\begin{cases}
x^2 - y^2 + x - y = 0, \\
xy + x = 0,
\end{cases}
\end{align*}
\]

and find three superconvergent points: \((0, 0)\), \((0, -1)\) and \((-1, -1)\). The contours \(\psi_{n+1}^{Re} = 0\) and \(\psi_{n+1}^{Im} = 0\) for \( n = 8 \) are given in FIGURE 11. The contours \(\frac{\partial \psi_{n+1}^{Re}}{\partial x} = 0\) and \(\frac{\partial \psi_{n+1}^{Im}}{\partial x} = 0\)
for \( n = 8 \) are given in Figure 12.

### 3.4. Criss-Cross Pattern

In the Criss-Cross Pattern, we partition the reference cell \( \hat{K} = [-1,1]^2 \) into four triangular elements (see Figure 13).

Let \( \nu_i \) be the linear nodal shape function corresponding to vertex \( n_i \).

\[
\begin{align*}
\nu_1 &= \begin{cases} \frac{1}{2}(x+y) & \text{in } \hat{T}_1, \hat{T}_4, \\ 0 & \text{in } \hat{T}_2, \hat{T}_3; \end{cases} \\
\nu_2 &= \begin{cases} \frac{1}{2}(x-y) & \text{in } \hat{T}_1, \hat{T}_2, \\ 0 & \text{in } \hat{T}_3, \hat{T}_4; \end{cases} \\
\nu_3 &= \begin{cases} 1+y & \text{in } \hat{T}_1, \\ 1-x & \text{in } \hat{T}_2, \\ 1-y & \text{in } \hat{T}_3, \\ 1+x & \text{in } \hat{T}_4; \end{cases} \\
\nu_4 &= \begin{cases} \frac{1}{2}(y-x) & \text{in } \hat{T}_3, \hat{T}_4, \\ 0 & \text{in } \hat{T}_1, \hat{T}_2; \end{cases} \\
\nu_5 &= \begin{cases} \frac{1}{2}(x+y) & \text{in } \hat{T}_2, \hat{T}_3, \\ 0 & \text{in } \hat{T}_1, \hat{T}_4. \end{cases}
\end{align*}
\]  

(3.25)

The side modes and internal modes are defined as in (3.6) and (3.7), respectively.

The periodic basis functions can be constructed from the hierarchic basis functions. This time, \( \nu_1 + \nu_2 + \nu_4 + \nu_5 \) and \( \nu_3 \) are periodic.

It is straightforward to verify that

\[
\dim P_n^w(\hat{K}) = n^2 + (n + 1)^2, \quad \dim PP_n^w(\hat{K}) = 2n^2.
\]

(3.26)

Again, Theorems 3.1 and 3.2 are valid for Criss-Cross mesh. To determine spaces \( \Phi_{n+1}(\hat{K}) \), we may either process the orthogonal decomposition of \( PP_n^w(\hat{K}) \), or carry on a periodic finite element approximation. As we mentioned before, the second approach is more efficient when we are interested only in the expressions (rather than the structures) of the basis functions.

**Remark 3.11.** Unlike the Chevron and Union Jack patterns, no portion of the partitioned \( \hat{K} \) for the Criss-Cross pattern "looks like" the regular pattern. In other words, \( \hat{T}_1 \) and \( \hat{T}_2 \) can not be mapped to the reference cell of the regular pattern by an affine mapping, nor
can the other pairs. This leads to the fact that the superconvergence of the Criss-Cross pattern is very much different from that of the regular pattern.

On the other hand, the geometry of the partition in $\hat{K}$ is symmetric, as we can see in definition (3.25). So we need to study only superconvergence in $\hat{T}_1$ and $\hat{T}_2$. Superconvergent points in the other elements can be obtained by symmetry. $\Box$

In the following, basis functions of $\Phi_{n+1}(\hat{K})$ in $\hat{T}_1$ and $\hat{T}_2$ are provided for $n = 1, \ldots, 4$. For cases $n = 5, \ldots, 8$, the reader is referred to [13]. As in Section 3.2, we use $B^j_{n+1}$ to denote the $j^{th}$ basis function of $\Phi_{n+1}(\hat{K})$, and let column vectors $\theta^1_n$ and $\theta^2_n$ consist all nodal shape functions, side modes, and internal modes of order $\leq n$ in $\hat{T}_1$ and $\hat{T}_2$, respectively. For example,

$$\begin{align*}
\theta^1_1 &= [\nu_1, \nu_2, \nu_3]^T, \quad \theta^2_1 = [\nu_2, \nu_5, \nu_3]^T; \\
\theta^2_2 &= [\theta^2_1, \ c_2^{12}, c_2^{23}, c_2^{13}]^T, \quad \theta^2_3 = [\theta^2_1, \ c_2^{25}, c_2^{35}, c_2^{23}]^T; \\
\theta^3_3 &= [\theta^3_2, \ c_3^{12}, c_3^{23}, c_3^{13}, \ell_3^1]^T, \quad \theta^3_4 = [\theta^3_2, \ c_3^{25}, c_3^{35}, c_3^{23}, \ell_3^1]^T; \\
\theta^4_4 &= [\theta^4_3, \ c_4^{12}, c_4^{23}, c_4^{13}, \ell_4^1, \ell_4^2]^T, \quad \theta^4_5 = [\theta^4_3, \ c_4^{25}, c_4^{35}, c_4^{23}, \ell_4^1, \ell_4^2]^T. 
\end{align*}$$

(3.27)

Then, in element $\hat{T}_1$, the basis functions of $\Phi_{n+1}(\hat{K})$ are

$$\begin{align*}
B^1_2 &= \phi_2(x) + \begin{bmatrix} 2 \ 0 \ 5 \ 0 \end{bmatrix} \theta^1_1, \\
B^2_2 &= xy + [-1, 1, 0] \theta^1_1, \\
B^1_3 &= \phi_3(x) + [0, 0, 0, -3/10, 3/10] \theta^1_2.
\end{align*}$$
Figure 15: Contours \( \psi_{n+1}^R = 0 \) (solid) and \( \psi_{n+1}^I = 0 \) (dashed), \( n = 5, \ldots, 8 \).

\[
\begin{align*}
B_2^n &= \phi_2(x)y + [0, 0, 0, 1, \frac{7}{20}, \frac{7}{20}] \theta_1^1, \\
B_3^n &= x\phi_2(y) + [0, 0, 0, 0, -\frac{25}{20}, \frac{20}{20}] \theta_1^2, \\
B_4^n &= \phi_3(y) + [0, 0, 0, 0, \frac{3}{10}, \frac{10}{10}] \theta_2^2; \\
B_1^n &= \phi_4(x) + \left[ \frac{10}{330}, \frac{10}{330}, -\frac{5}{48}, \frac{112}{112}, -\frac{1}{14}, -\frac{1}{14}, 0, \frac{25}{28}, \frac{25}{28}, -\frac{115}{56} \right] \theta_1^3, \\
B_2^n &= \phi_3(x)y + [0, 0, 0, 0, \frac{3}{5}, 1, -\frac{5}{8}, \frac{5}{8}, 0] \theta_3^3, \\
B_3^n &= \phi_2(x)\phi_2(y) + \left[ \frac{1}{504}, \frac{1}{504}, -\frac{91}{365}, \frac{118}{118}, \frac{32}{42}, \frac{42}{42}, 0, \frac{10}{168}, \frac{10}{168}, -\frac{97}{84} \right] \theta_3^4, \\
B_4^n &= x\phi_3(y) + [0, 0, 0, 0, \frac{5}{8}, -\frac{1}{8}, 0, -\frac{3}{8}, \frac{3}{8}, 0] \theta_3^4, \\
B_5^n &= \phi_4(y) + \left[ \frac{10}{330}, \frac{10}{330}, -\frac{5}{48}, \frac{112}{112}, -\frac{1}{14}, -\frac{1}{14}, 0, \frac{25}{28}, \frac{25}{28}, -\frac{115}{56} \right] \theta_3^3; \\
B_1^n &= \phi_5(x) + [0, 0, 0, 0, \frac{121}{1728}, -\frac{121}{1728}, \frac{85}{2041}, \frac{145}{1296}, -\frac{145}{1296}, 0, 0, -\frac{217}{1296}, \frac{217}{1296}, -\frac{13895}{1296}, 0] \theta_4^1, \\
B_2^n &= \phi_4(x)y + [0, 0, 0, 0, \frac{2032}{2032}, \frac{2032}{2032}, -\frac{4064}{4064}, -\frac{1019}{1019}, \frac{1019}{1019}, 1, \frac{61}{61}, \frac{61}{61}, 0, -\frac{2635}{2635}] \theta_4^1, \\
B_3^n &= \phi_3(x)\phi_2(y) + [0, 0, 0, 0, \frac{947}{947}, -\frac{947}{947}, -\frac{756}{756}, \frac{756}{756}, -\frac{56}{56}, 0, 0, -\frac{810}{810}, \frac{810}{810}, -\frac{592}{592} \theta_4^1, \\
B_4^n &= \phi_2(x)\phi_3(y) + [0, 0, 0, 0, -\frac{251}{251}, \frac{251}{251}, \frac{312}{312}, \frac{312}{312}, -\frac{56}{56}, \frac{56}{56}, -\frac{1134}{1134}, 0, 0, -\frac{1134}{1134}, 0, -\frac{324}{324}] \theta_4^4, \\
B_5^n &= x\phi_4(y) + [0, 0, 0, 0, -\frac{251}{251}, \frac{251}{251}, \frac{312}{312}, \frac{312}{312}, -\frac{56}{56}, \frac{56}{56}, -\frac{1134}{1134}, 0, 0, -\frac{1134}{1134}, 0, -\frac{324}{324}, \theta_4^4, \\
B_6^n &= \phi_5(y) + [0, 0, 0, 0, -\frac{1728}{1728}, -\frac{1728}{1728}, 0, -\frac{145}{1296}, -\frac{145}{1296}, 0, -\frac{217}{1296}, -\frac{217}{1296}, -\frac{13895}{1296}, 0] \theta_4^1.
\end{align*}
\]

In element \( T_2 \), the basis functions of \( \Phi_{n+1}(K) \) are

\[
\begin{align*}
B_1^1 &= \phi_2(x) + \left[ \frac{3}{5}, \frac{3}{5}, \frac{3}{5} \right] \theta_1^2, \\
B_2^1 &= xy + [1, -1, 0] \theta_1^2, \\
B_2^2 &= \phi_2(y) + \left[ \frac{3}{5}, \frac{3}{5}, \frac{3}{5} \right] \theta_1^2; \\
B_4^1 &= \phi_3(x) + [0, 0, 0, 0, -\frac{3}{10}, -\frac{3}{10}] \theta_2^2, \\
B_5^2 &= \phi_2(x)y + [0, 0, 0, 0, -\frac{20}{20}, \frac{20}{20}] \theta_2^2.
\end{align*}
\]
\[ B_2^3 = x\phi_2(y) + [0, 0, 0, -1, -\frac{1}{20}, -\frac{1}{112}] \theta_2^2, \]
\[ B_2^3 = \phi_3(y) + [0, 0, 0, -\frac{3}{10}] \theta_2^2. \]

\[ B_2^1 = \phi_4(x) + \left[ \begin{array}{c} \frac{10}{336} \ 10 \ 10 \ -\frac{5}{48} \ -\frac{10}{112} \ -\frac{1}{14} \ -\frac{1}{14} \ 0 \ -\frac{25}{112} \ \frac{25}{112} \ \frac{125}{96} \end{array} \right] \theta_2^2, \]
\[ B_2^2 = \phi_2(x) + [0, 0, 0, -\frac{1}{5}, \frac{1}{5}, 0, \frac{35}{35}, -\frac{35}{35}, 0] \theta_2^2, \]
\[ B_2^3 = x\phi_2(y) + [0, 0, 0, -\frac{1}{5}, \frac{1}{5}, -1, \frac{3}{35}, -\frac{35}{35}, 0] \theta_2^2, \]
\[ B_2^4 = \phi_4(y) + \left[ \begin{array}{c} \frac{10}{336} \ 10 \ -\frac{5}{48} \ -\frac{10}{112} \ -\frac{1}{14} \ -\frac{1}{14} \ 0 \ -\frac{25}{112} \ \frac{25}{112} \ \frac{125}{96} \end{array} \right] \theta_2^2. \]

Take \( B_{n+1}^2 \) as \( \psi_s \), and we can verify the following results for \( n \) up to 8.

**Proposition 3.7.** Consider element \( T_1 \) (or \( T_2 \)) of the Criss-Cross mesh. For the Poisson equation, the function value superconvergent points are:

(i) If \( n \) is odd, there is no superconvergent point;

(ii) If \( n \) is even, the vertices and the midpoint of side \( l_{12} \) (or \( l_{25} \), respectively) are the only superconvergent points.

**Proposition 3.8.** Consider the superconvergent points for \( \frac{\partial u}{\partial x} \) of the Poisson solutions in the Criss-Cross mesh. In \( T_1 \), the superconvergent points are the same as in the regular pattern. In \( T_2 \), there is no superconvergent point.

For \( \frac{\partial u}{\partial y} \), superconvergence can be determined by symmetry. Namely, in \( T_1 \), there is no superconvergent point. In \( T_2 \), the cases are the same as in the regular pattern.

**Remark 3.12.** As in the regular pattern, superconvergence for the Criss-Cross pattern at symmetry points was predicted by the symmetry theory and the computer-based proof. Propositions 3.7 and 3.8 confirm theoretically that there are no other superconvergent points. \( \square \)

As for the Laplace equation, we determine \( \psi_{n+1}^{Re} \) and \( \psi_{n+1}^{Im} \) from the basis functions of \( \Phi_{n+1}(K) \). In the element \( \hat{T}_1 \), we have

\[ \psi_{2}^{Re} = B_2^1 - B_2^2 = x^2 - y^2, \]
\[ \psi_{2}^{Im} = B_2^2 = xy + x; \]
\[ \psi_{3}^{Re} = B_3^1 - 3B_3^2 = x^3 - 3xy^2 - 3x^2 - y, \]
\[ \psi_{3}^{Im} = B_3^2 - 3B_3^3 = y^3 - 3x^2y - 3x^2y - y; \]
\[ \psi_{4}^{Re} = B_4^1 - 6B_4^2 + B_4^3 = x^4 - 6x^2y^2 + y^4 - 9x^2y + 3x^2 - 4x^2 - \frac{2}{15}, \]
\[ \psi_{4}^{Im} = B_4^2 - B_4^3 = x^3y - xy^3 + x^3 - xy^3. \]
\[ \psi^R_3 = B_3^1 - 10B_3^3 + 5B_3^5 = x^5 - 10x^3y^2 + 5xy^4 - \frac{95}{6}x^3y + \frac{35}{6}xy^3 - \frac{605}{28}x^3 - \frac{271}{15}xy^2 - \frac{1}{15}xy - \frac{1}{21}x. \]
\[ \psi^R_5 = B_5^6 - 10B_5^4 + 5B_5^2 = y^5 - 10y^3x^2 + 5yx^4 + \frac{5}{4}y^4 - \frac{65}{4}y^2x^2 + 5x^4 + \frac{25}{8}y^3 - \frac{235}{28}y^2x^2 - \frac{45}{21}x^2 - \frac{1}{21}y. \]

In the element \( T_2 \), we have
\[ \psi^R_2 = B_2^1 - B_2^3 = x^2 - y^2, \]
\[ \psi^R_2 = B_2^1 - B_2^3 = xy - y; \]
\[ \psi^R_3 = B_3^1 - 3B_3^3 = x^3 - 3xy^2 + 3y^2 - x, \]
\[ \psi^R_3 = B_3^1 - 3B_3^3 = y^3 - 3yx^2 + 3yx - y; \]
\[ \psi^R_4 = B_4^1 - 6B_4^3 + B_4^5 = x^4 - 6x^2y^2 + y^4 - x^3 + 9xy^2 - 4y^2 + \frac{2}{15}, \]
\[ \psi^R_4 = B_4^1 - 6B_4^3 + B_4^5 = x^3y - xy^3 - x^2y + y^2; \]
\[ \psi^R_6 = B_6^1 - 10B_6^3 + 5B_6^5 = x^5 - 10x^3y^2 + 5xy^4 - \frac{5}{4}x^4 + \frac{65}{4}x^2y^2 - 5y^4 + \frac{25}{8}x^3 - \frac{235}{28}x^2y^2 + \frac{45}{21}y^2 - \frac{1}{21}x. \]
\[ \psi^R_5 = B_5^6 - 10B_5^4 + 5B_5^2 = y^5 - 10y^3x^2 + 5yx^4 + \frac{25}{8}y^3x - \frac{25}{28}y^3 + \frac{25}{18}yx^2 + \frac{5}{15}yx - \frac{1}{21}y. \]

For cases \( n = 5, \ldots, 8 \), the reader is referred to [13].

By Theorem 3.2, the intersection points of the contours \( \psi^R_{n+1} = 0 \) and \( \psi^R_{n+1} = 0 \) are function value superconvergent points. It can be shown that, for \( k = 1, 2, \ldots, \)
\[ \psi^R_{2k - 1} (-y, -x) = (-1)^k \psi^R_{2k - 1} (x, y), \]
\[ \psi^R_{2k} (-y, -x) = (-1)^{k+1} \psi^R_{2k} (x, y); \]
\[ \psi^R_{2k+1} (-y, -x) = -\psi^R_{2k+1} (x, y), \]
\[ \psi^R_{2k+1} (-y, -x) = -\psi^R_{2k+1} (x, y). \]   (3.28)

Table 3(a). Function Value Superconvergent Points for the Criss-Cross Pattern
(in \( T_1 \), \( n = 1, \ldots, 4 \))

<table>
<thead>
<tr>
<th>( n )</th>
<th>Table 3(a) Function Value Superconvergent Points for the Criss-Cross Pattern (in ( T_1 ), ( n = 1, \ldots, 4 ))</th>
</tr>
</thead>
</table>
| 1     | \begin{align*}
-1.0000000000000000 & \quad -1.0000000000000000 \\
0.0000000000000000 & \quad -0.9173658531054181 \\
-0.0000000000000000 & \quad -1.0000000000000000
\end{align*}& \begin{align*}
-1.0000000000000000 & \quad -1.0000000000000000 \\
-0.0000000000000000 & \quad -0.9173658531054181 \\
-0.0000000000000000 & \quad -1.0000000000000000
\end{align*}|
Therefore, we need to determine only superconvergent points in $\tilde{T}_1$. Superconvergence in $\tilde{T}_2$ are obtained by symmetry. TABLE 3 demonstrates function value superconvergent points in $\tilde{T}_1$ for $n = 1, \ldots, 8$ with 16 digits of accuracy. The contours $\psi_{n+1}^{Re} = 0$ and $\psi_{n+1}^{Im} = 0$ for $n = 1, \ldots, 8$ are given in FIGURE 14 and FIGURE 15.

We now apply Theorem 3.2 in derivative superconvergence to the Laplace equation. For
TABLE 4. x-Derivative Superconvergent Points for the Criss-Cross Pattern
(in $T_1$, $n = 1, \ldots, 8$)

<table>
<thead>
<tr>
<th>$n = 1$</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0000000000000000</td>
<td>-0.5773502691856582</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>2</td>
<td>0.5773502691856582</td>
<td>0.0000000000000000</td>
<td>0.8477332441259476</td>
</tr>
<tr>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>3</td>
<td>0.5773502691856582</td>
<td>-1.0000000000000000</td>
<td>0.8477332441259476</td>
</tr>
<tr>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>4</td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
<td>0.8477332441259476</td>
</tr>
<tr>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
</tr>
<tr>
<td>5</td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
<td>0.8477332441259476</td>
</tr>
<tr>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
<td>-1.0000000000000000</td>
</tr>
</tbody>
</table>

$x$-derivative superconvergence, we need to determine the common zeros of $\frac{\partial \psi_{n+1}^R}{\partial x} = 0$ and $\frac{\partial \psi_{n+1}^I}{\partial x} = 0$ in both $\bar{T}_1$ and $\bar{T}_2$, since (3.28) does not imply any symmetric properties for $x$-derivatives.

We list superconvergent points for $n = 1, \ldots, 8$ in TABLE 4 and TABLE 5 with 16 digits of accuracy. Only the contours $\frac{\partial \psi_{n+1}^R}{\partial x} = 0$ and $\frac{\partial \psi_{n+1}^I}{\partial x} = 0$ for $n = 3, \ldots, 8$ are given in FIGURE 16.

On the other hand, from (3.28), the $y$-derivatives of $\psi_{n+1}^R$ and $\psi_{n+1}^I$ are symmetric to the $x$-derivatives. Thus, the $y$-derivative superconvergent points can be obtained by symmetry.

Remark 3.13. To compare our results with those given in [2], \footnote{Derivative superconvergent points are given for $n = 1, \ldots, 6$ in [2] (TABLE II, III).} we use $\bar{x} = (x + 1)/2$ and
TABLE 5. $x$-Derivative Superconvergent Points for the Criss-Cross Pattern  
(in $T_2$, $n = 1, \ldots, 8$)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x$-coordinates</th>
<th>$y$-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.0000000000000000</td>
<td>0.0773502691896258</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td></td>
<td>0.5243480408583276</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td>2</td>
<td>-0.7500000000000000</td>
<td>-0.3672919579081908</td>
</tr>
<tr>
<td></td>
<td>0.7500000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>-0.2066731091555382</td>
</tr>
<tr>
<td></td>
<td>0.7500000000000000</td>
<td>0.8291726306727321</td>
</tr>
<tr>
<td></td>
<td>0.7500000000000000</td>
<td>0.7941585351606123</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td>3</td>
<td>0.0000000000000000</td>
<td>0.4330127018921193</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td>4</td>
<td>-0.7500000000000000</td>
<td>-0.3672919579081908</td>
</tr>
<tr>
<td></td>
<td>0.7500000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
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<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td>5</td>
<td>0.0000000000000000</td>
<td>0.4330127018921193</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td>6</td>
<td>0.0000000000000000</td>
<td>0.4330127018921193</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td>7</td>
<td>0.0000000000000000</td>
<td>0.4330127018921193</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
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<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
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<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td>8</td>
<td>0.0000000000000000</td>
<td>0.4330127018921193</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
<tr>
<td></td>
<td>0.0000000000000000</td>
<td>0.0000000000000000</td>
</tr>
</tbody>
</table>

The $y$-coordinates of the second and the fourth points in $T_2$ are 0.2834936534 and 0.7165063710, which are accurate in 8 and 7 digits, respectively. By our process, these two points can be located analytically, which are $(4 \pm \sqrt{3})/8$ (after mapped into $T_2$), or 0.2834936490538904 and 0.7165063509461097 in decimals.

References


Figure 16: Contours $\frac{\partial \phi_{n+1}}{\partial x} = 0$ (solid) and $\frac{\partial \phi_{n+1}}{\partial x} = 0$ (dashed), $n = 3, \ldots, 8$.


